EXTREMAL PROBLEMS ON BERGMAN SPACES A^1_α AND BESOV SPACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF
MASTER OF SCIENCE
IN
MATHEMATICS

by
Alper Balcı
May 2022

Extremal Problems on Bergman Spaces A_α^1 and Besov Spaces
By Alper Balcı
May 2022

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Hakkı Turgay Kaptanoğlu (Advisor)

Alexander Goncharov

Özcan Yazıcı

Approved for the Graduate School of Engineering and Science:

Ezhan Karaşan
Director of the Graduate School

ABSTRACT

EXTREMAL PROBLEMS ON BERGMAN SPACES A^1_α AND BESOV SPACES

Alper Balcı
M.S. in Mathematics
Advisor: Hakkı Turgay Kaptanoğlu
May 2022

Extremal problems in different function spaces have long been investigated. Ferguson provides a method, using Bergman projections, to solve certain types of the extremal problems in Bergman spaces for $1 < p < \infty$ in his work [3]. Later the method is extended to weighted Bergman spaces for $1 < p < \infty$ in [13]. Now, we extend this method to the $p = 1$ case. The two cases differ in the structure of Bergman projections and dual spaces. First, we define some function spaces, namely weighted Bergman spaces, the Bloch space, Besov spaces, and show the usage of Bergman projection on these spaces. Then, we find some conditions to ensure the existence of unique solutions for extremal problems. Later, we use Bergman projection to find a candidate function for the solution in the $p = 1$ case, and we prove that the candidate function is the solution if it never attains the value 0. Finally, under special conditions, we solve a similar problem in Besov spaces.

Keywords: Weighted Bergman spaces, extremal problems, Besov spaces, Bergman projections, Bloch space.

ÖZET

A^1_α BERGMAN UZAYLARI VE BESOV UZAYLARINDA EKSTREMAL PROBLEMLER

Alper Balcı
Matematik, Yüksek Lisans
Tez Danışmanı: Hakkı Turgay Kaptanoğlu
Mayıs 2022

Ekstremal problemler, uzun süredir farklı fonksiyon uzaylarında incelenmiştir. Ferguson [3]'teki çalışmasında, $1 < p < \infty$ için Bergman uzaylarında, Bergman izdüşümlerini kullanarak özel formdaki ekstremal problemleri çözmek için bir yöntem geliştirmiştir. Sonra, bu yöntem [13]'teki çalışmada yine $1 < p < \infty$ olmak üzere ağırlıklı Bergman uzaylarına genelleştirilmiştir. Şimdi, biz bu yöntemi $p = 1$ durumuna genelliyoruz. Bu iki durum Bergman izdüşümlerinin ve dual uzaylarının yapısında ayrışıyor. Önce fonksiyon uzaylarımız olan Bergman, Bloch ve Besov uzaylarını tanımlıyoruz ve Bergman izdüşümünün kullanımını gösteriyoruz. Ardından, ekstremal problemimizin bir tek çözümünün varlığını sağlamak için gerekli koşulları buluyoruz. Sonrasında, Bergman izdüşümünü kullanarak $p = 1$ durumu için çözümümüze bir aday fonksiyon buluyoruz ve eğer bu fonksiyon hiçbir zaman 0 değerini almıyorsa gerçekten de onun çözüm olduğunu gösteriyoruz. Son olarak, özel şartlar altında, benzer bir problemi Besov uzayında çözüyoruz.

Anahtar sözcükler: Ağırlıklı Bergman uzayları, ekstremal problemler, Besov uzayları, Bergman izdüşümleri, Bloch uzayı.

Acknowledgements

I owe a lot to Professor Hakkı Turgay Kaptanoğlu. He is a very valuable supervisor, teacher, and also friend to me. I thank him for the mathematics he taught me and his continuous support.

I want to thank my family. They raised me and always stood by me. Without them, I would not have been able to produce this work.

I would like to thank Gözde Gökalp, who believed in me and supported me in becoming a mathematician. She motivated me to move forward even in my most difficult times.

I would like to thank Özcan Yazıcı and Alexander Goncharov for being jury members.

Finally, I would like to thank TÜBİTAK for supporting me financially throughout my undergraduate and graduate education.

Contents

Chapter 1

Introduction

Extremal problems and the solutions of the extremal problems play important roles in complex analysis. These problems have long been studied in different function spaces. Various authors studied extremal problems in Hardy spaces such as Rogosinski and Shapiro [11] or Khavinson [9]. More recently, extremal problems have been investigated in Bergman spaces A^p; for example, Hedenmalm [4] studies zero sets of Bergman functions using extremal problems. In this thesis, we will study extremal problems on weighted Bergman spaces and Besov spaces.

In [3], Ferguson uses Bergman projections and the representations of the linear functionals in the dual space. By doing that the author can find solutions to specific types of extremal problem in Bergman spaces for $p > 1$. In [13] this method is generalized to weighted Bergman spaces and to points other than the origin. However, these results can be further generalized to the $p = 1$ case with modifications. The dual space of the Bergman space when $p = 1$ is called the Bloch space, and it plays an important role to make such modifications. Moreover, in some special scenarios, we can solve similar problems in Besov spaces using the same method.

In Chapter 2, we introduce our function spaces, namely, weighted Bergman spaces, Bloch space, and Besov spaces. We give important properties of such spaces to be used in later chapters. In Chapter 3, we calculate Bergman projections of certain functions explicitly. We give the notion of the kernel of a linear functional in the same chapter, and lastly, we define the Möbius expansions of analytic functions. In Chapter 4, we define our extremal problems. We also, provide theorems that ensure the existence and the uniqueness of the solutions both for $p > 1$ and $p = 1$. For $p > 1$ our problem have always a unique solution. On the other hand, for $p = 1$ it depends on the kernel of linear functionals. But, for most of the cases, we show that the problem has a unique solution as well. In the Chapter 5, we provide a candidate for the solution of some extremal problems, and we state under which conditions the given candidate is the solution.

Chapter 2

Analytic Function Spaces on the Unit Disc

In this section we will examine weighted Bergman, Bloch and Besov spaces. The material in this section is standard and can be found in books on Bergman spaces such as [2] and [5]. For this reason, we will omit most proofs.

2.1 Weighted Bergman Spaces

We mostly work with weighted Bergman spaces, so it is important to understand the theory. First, we will introduce some notation.

Notation 2.1.1. Unit disc will be denoted by $\mathbb{D}$ and normalized area measure will be denoted by $dA := \pi^{-1}dx\,dy$. It is a finite measure and the area of unit disc is 1 with respect to this measure. We will also use some standard weights on these spaces. Let $\alpha > -1$, then our standard measures will be denoted as $d\mu_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha dA(z) = (\alpha+1)(1-|x+yi|^2)^\alpha \pi^{-1}dxdy$. With this definition the measure of the unit disc is $\int_{\mathbb{D}} d\mu_\alpha = 1$. Also, L^p_α will denote the space of all measurable functions f on the unit disc such that

$$\|f\|_{L^p_\alpha} := \left\{ \int_{\mathbb{D}} |f(z)|^p d\mu_\alpha(z) \right\}^{1/p} < \infty.$$

Now we are ready to give our definition.

Definition 2.1.2. Let $p > 0$ and $\alpha > -1$. The Bergman space A^p_α consists of all analytic functions f on $\mathbb{D}$ such that the Bergman norm is finite, that is,

$$\|f\|_{A^p_\alpha} := \left\{ \int_{\mathbb{D}} |f(z)|^p d\mu_\alpha(z) \right\}^{1/p} < \infty.$$

The next theorem states that the name "Bergman norm" is not a coincidence.

Theorem 2.1.3. *Bergman norm is a true norm if and only if $p \geq 1$ and A^p_α is a Banach space in this case. Moreover, A^p_α is a Hilbert space if and only if $p = 2$.*

We will work only on the case $p \geq 1$, so we can use normed space ideas. Our next theorem will tell us that growth rate of a function in a Bergman space cannot be arbitrarily large.

Theorem 2.1.4. *Suppose $p > 0, \alpha > -1$ and $f \in A^p_\alpha$. Then for all $z \in \mathbb{D}$,*

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1 - |z|^2)^{(2+\alpha)/p}}.$$

Corollary 2.1.5. *Suppose $p > 0, \alpha > -1$ and $f \in A^p_\alpha$. Suppose also $K \subset \mathbb{D}$ is compact. Then there is a constant C depending on α, p, K such that for all $f \in A^p_\alpha$ and for all $z \in \mathbb{D}$,*

$$\sup_{z \in K} |f(z)| \leq C\|f\|_{A^p_\alpha}.$$

Second corollary is more interesting

Corollary 2.1.6. *A^p_α is a closed subspace of L^p_α.*

Proof. We will follow the proof in [16]. We have to show that A^p_α is complete. To do that take a Cauchy sequence $\{f_k\}$ in A^p_α. There is a function $f \in L^p_\alpha$ such that f_k converges to f in L^p_α. By Corollary 2.1.5 $\{f_k(z)\}$ is uniformly Cauchy on every compact subset of $\mathbb{D}$, and so converges uniformly on every compact subset of $\mathbb{D}$. Since each f_k is analytic, the limit function f coincides almost everywhere with an analytic function. □

Here an interesting question is that which functions belong to these spaces. The answer depends on parameters α and p of course. However, the following theorem tells us a simple set of functions is included, and they are enough to approximate functions in A^p_α.

Theorem 2.1.7. *Polynomials are included and dense in every A^p_α for $p > 0$ and $\alpha > -1$.*

Now, we shift our focus to properties of these functions related to operator theory. From Theorem 2.1.4 we can say that the point evaluation functional $z \mapsto f(z)$ is a bounded linear functional on A^p_α for every $z \in \mathbb{D}$. If we restrict our attention to the Hilbert space case, by the Riesz representation theorem, there is a unique function $h_z \in A^2_\alpha$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{h_z(w)}\, d\mu_\alpha(w).$$

Definition 2.1.8. Let $K_\alpha(z,w) = \overline{h_z(w)}$ where h_z is defined as above. We will call $K_\alpha(z,w)$ the reproducing kernel of A^2_α. It is clear from definition that the reproducing formula

$$f(z) = \int_{\mathbb{D}} f(w)K_\alpha(z,w)\, d\mu_\alpha(w)$$

holds.

The next theorem states that it is possible to find an explicit formula for $K_\alpha(z,w)$.

Theorem 2.1.9. *$K_\alpha(z,w)$ is holomorphic with respect to z and conjugate holomorphic with respect to w. Moreover, the explicit form of $K_\alpha(z,w)$ on $\mathbb{D}$ is*

$$K_\alpha(z,w) = \frac{1}{(1 - z\overline{w})^{2+\alpha}}.$$

Here we will make an important observation. The reproducing formula is valid in particular for polynomials. But polynomials are dense in each A^p_α. This leads us to an important theorem.

Theorem 2.1.10. *$f(z) = \int_{\mathbb{D}} f(w)K_\alpha(z,w)d\mu_\alpha(w)$ for all $f \in A^p_\alpha$ where $p \geq 1$.*

Now, we are ready to introduce one of the most important concepts in Bergman spaces. A^2_α is a closed subspace of the Hilbert space L^2_α, so there exists a unique orthogonal projection from L^2_α to A^2_α.

Definition 2.1.11. The orthogonal projection $P_\alpha : L^2_\alpha \to A^2_\alpha$ is called the Bergman projection.

It is easy to give an explicit formula for $P_\alpha f$.

Theorem 2.1.12. *Let $f \in L^2_\alpha$, then $P_\alpha f(z) = \displaystyle\int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2+\alpha}} d\mu_\alpha(w)$.*

Proof. Let us write $k_{\alpha,z}(w) = \overline{K_\alpha(z,w)}$. We know $P_\alpha f \in A^2_\alpha$ and $f - P_\alpha f \perp A^2_\alpha$. Moreover, note that $K_\alpha(z,w) \in A^2_\alpha$ as a function of z, so

$$\begin{aligned} P_\alpha f(z) &= \langle P_\alpha f, k_{\alpha,z}\rangle = \langle P_\alpha f, k_{\alpha,z}\rangle + \langle f - P_\alpha f, k_{\alpha,z}\rangle \\ &= \langle f, k_{\alpha,z}\rangle = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2+\alpha}} d\mu_\alpha(w). \end{aligned}$$ □

The explicit formula for $P_\alpha f$ enables us to expand projection ideas to bigger families of spaces. In the formula above we can take f from any A^p_α and from Theorem 2.1.10 the formula makes sense in those spaces. However, the question is whether this projection is a bounded linear operator or not. The next theorem answers this question. It is [5, Theorem 1.10].

Theorem 2.1.13. *Let $\alpha, \beta > -1$ and $1 \leq p < \infty$. Then P_β is a bounded operator from L^p_α to A^p_α if and only if $\alpha + 1 < (\beta + 1)p$. Moreover if $f \in A^p_\alpha$, then $P_\beta f = f$.*

Note that if $p > 1$ we can take $\alpha = \beta$ in this theorem and obtain that $P_\alpha : L^p_\alpha \to A^p_\alpha$ is bounded. However, we need to take $\beta > \alpha$ to obtain a bounded map $P_\beta : L^1_\alpha \to A^1_\alpha$. In this thesis we will especially focus on the $p = 1$ case. This theorem states we should be careful about projection ideas as $p = 1$ case can be different. Lastly, we will look the dual space of A^p_α, the space of bounded linear functionals on A^p_α.

Theorem 2.1.14. *Suppose $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $\alpha > -1$. Then the dual space of A^p_α is $(A^p_\alpha)^* = A^q_\alpha$ with equivalent norms, that is, for any $\Phi \in (A_\alpha)^*$, there exists a unique function in A^q_α such that for any $f \in A^p_\alpha$ we have*

$$\Phi(f) = \langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(w)\overline{g(w)}\, d\mu_\alpha(w).$$

The next result is a key for most of our calculations. It is contained in [5, Theorem 1.16].

Theorem 2.1.15. *Suppose $f \in A^p_\alpha$ and $g \in L^q_\alpha$, where $1 < p < \infty$ and $p^{-1}+q^{-1} = 1$. Then*

$$\int_{\mathbb{D}} f\overline{P_\alpha g}\, d\mu_\alpha = \int_{\mathbb{D}} f\,\overline{g}\, d\mu_\alpha.$$

Theorem 2.1.14 gives us the dual space of A^p_α for $p > 1$. However, to work on the case $p = 1$, we should also look at the dual space of A^1_α. It can be identified with the Bloch space. At this point, it is wise to introduce our second function space.

2.2 Bloch Space

Definition 2.2.1. The Bloch space $\mathcal{B}$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Here $\|f\|_{\mathcal{B}}$ is called the "Bloch norm" even though it is not an actual norm. However, $\|f\| = |f(0)|+\|f\|_{\mathcal{B}}$ is a true norm. Let us denote bounded holomorphic functions by H^∞. Our next theorem tells at least bounded holomorphic functions belongs to the Bloch space. It is from [16, Proposition 5.1].

Theorem 2.2.2. *$H^\infty \subset \mathcal{B}$. Moreover, $\|f\|_{\mathcal{B}} \le \|f\|_\infty$ for all $f \in H^\infty$.*

An interesting observation is that $\log(1-z) \in \mathcal{B}$ but $\log(1-z) \notin H^\infty$. Hence, the relation $H^\infty \subset \mathcal{B}$ is proper. Now we will introduce two operators.

Notation 2.2.3. To ease our writing we will use the following two operators. Suppose f and g are analytic on $\mathbb{D}$. Define

$$D_\alpha f(z) = f(z) + \frac{zf'(z)}{2+\alpha},$$

$$I_\alpha g(z) = \frac{2+\alpha}{1+\alpha}(1-|z|^2)D_\alpha g(z).$$

The next two results show us the duality between A^1_α and $\mathcal{B}$. The second result is from [6].

Theorem 2.2.4. *Suppose $\alpha > -1$, then $P_\alpha : L^\infty \to \mathcal{B}$ is bounded. Moreover, if $f \in \mathcal{B}$, then $P_\alpha I_\alpha f(z) = f(z)$.*

Theorem 2.2.5. *The dual space of A^1_α can be identified with $\mathcal{B}$. For any linear continuous functional $\Phi : A^1_\alpha \to \mathbb{C}$, there exists a unique element $g \in \mathcal{B}$ such that for every $f \in A^1_\alpha$*

$$\Phi(f) = \int_{\mathbb{D}} f \overline{I_\alpha g} \, d\mu_\alpha.$$

Our last theorem is analogous to Theorem 2.1.15.

Theorem 2.2.6. *Suppose $f \in A^1_\alpha$ and $g \in L^\infty$. Then*

$$\int_{\mathbb{D}} f \overline{I_\alpha P_\alpha g} \, d\mu_\alpha = \int_{\mathbb{D}} f \overline{g} \, d\mu_\alpha.$$

Proof. The proof is a straightforward calculation. We will follow the proof in [13].

$$\begin{aligned}
\int_{\mathbb{D}} f \overline{I_\alpha P_\alpha g} \, d\mu_\alpha &= \int_{\mathbb{D}} f(z) \frac{2+\alpha}{1+\alpha} (1-|z|^2) \overline{D_\alpha P_\alpha g(z)} \, d\mu_\alpha(z) \\
&= \int_{\mathbb{D}} f(z) \frac{2+\alpha}{1+\alpha} (1-|z|^2) \overline{D_\alpha \int_{\mathbb{D}} \frac{g(w)}{(1-z\overline{w})^{2+\alpha}} \, d\mu_\alpha(w)} \, d\mu_\alpha(z) \\
&= \int_{\mathbb{D}} f(z) \overline{D_\alpha \int_{\mathbb{D}} \frac{g(w)}{(1-z\overline{w})^{2+\alpha}} d\mu_\alpha(w)} (\alpha+2)(1-|z|^2)^{\alpha+1} dA(z) \\
&= \int_{\mathbb{D}} f(z) \overline{D_\alpha \int_{\mathbb{D}} \frac{g(w)}{(1-z\overline{w})^{2+\alpha}} \, d\mu_\alpha(w)} \, d\mu_{\alpha+1}(z) \\
&= \int_{\mathbb{D}} f(z) \int_{\mathbb{D}} \frac{\overline{g(w)}}{(1-w\overline{z})^{3+\alpha}} \, d\mu_\alpha(w) \, d\mu_{\alpha+1}(z) \\
&= \int_{\mathbb{D}} \overline{g(w)} \int_{\mathbb{D}} \frac{f(z)}{(1-w\overline{z})^{3+\alpha}} \, d\mu_{\alpha+1}(z) \, d\mu_\alpha(w) = \int_{\mathbb{D}} f(w) \overline{g(w)} \, d\mu_\alpha(w).
\end{aligned}$$

We used Fubini's theorem and Theorem 2.1.13 on the last line. □

Before closing this chapter, we will introduce our last function spaces.

2.3 Besov Spaces

The Bergman space A^p_α is meaningful only when $\alpha > -1$. If we try to expand our function spaces to cover the parameter range less than or equal to -1, we should look at Besov spaces. We will use q instead of α to indicate that we work in Besov spaces and not in Bergman spaces. Interestingly, the reproducing kernel $K_q(z, w)$ can be generalized for $q \leq -1$, and it is the starting point for us to define Besov spaces. First we give a notation.

Notation 2.3.1. Pochhammer symbol is $(a)_b := \dfrac{\Gamma(a+b)}{\Gamma(a)}$ where Γ is the gamma function.

Definition 2.3.2. For $q > -2$, we let

$$K_q(z, w) = \frac{1}{(1 - z\overline{w})^{2+q}} = \sum_{k=0}^{\infty} \frac{(2+q)_k}{k!} (z\overline{w})^k.$$

For $q \leq -2$, we define

$$K_q(z, w) = \sum_{k=0}^{\infty} \frac{k!}{(-q)_k} (z\overline{w})^k.$$

We denote the coefficient of $(z\overline{w})^k$ by $c_k(q)$ in either case.

In the previous section, we saw the same explicit formula for $q > -1$, but with this definition we use it up to -2. For $q \leq -2$ the other formula is valid. We will not discuss how we generalize $K_q(z, w)$ for $q \in \mathbb{R}$ in this thesis. Instead we will use it to define Besov spaces. Now, we introduce more notation.

Notation 2.3.3. Let $q \in \mathbb{R}$, we will define $d\nu_q(z) = (1 - |z|^2)^q dA(z)$. We also use the following notation to shorten our calculations $d_k(s, t) = \dfrac{c_k(s+t)}{c_k(s)}$, where c_k is as defined in Definition 2.3.2.

Here $d\nu_q$ is the non-normalized version of our standard weighted $d\mu_\alpha$. However, the biggest difference is that it is no longer required that $q > -1$ which means that ν_q is not required to be finite either. Before giving the definition of Besov spaces, lastly we introduce two familiar but slightly different operators.

Notation 2.3.4. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function on $\mathbb{D}$ with $a_k \in \mathbb{C}$. Then we define

$$I_s^t f(z) := (1 - |z|^2)^t D_s^t f(z),$$

$$D_s^t f(z) := \sum_{k=0}^{\infty} a_k\, d_k(s,t) z^k.$$

It can be shown that D_α of Notation 2.2.3 is equal to D_α^1 of Notation 2.3.4, and $I_\alpha = \dfrac{2+\alpha}{1+\alpha} I_\alpha^1$. The multiplicative term $\dfrac{2+\alpha}{1+\alpha}$ is added earlier for normalization purposes but here we do not normalize ν_q. Now we are ready to define Besov spaces.

Definition 2.3.5. Let $q, s, t \in \mathbb{R}$ and $p > 0$. The Besov space B_q^p consists of all analytic functions f on the unit disc such that

$$\int_{\mathbb{D}} |I_s^t f|^p \, d\nu_q < \infty$$

for some s, t satisfying the relation $q + pt > -1$.

The next theorem shows us the parameter s in Definition 2.3.5 can be chosen as any real number and finding only one candidate for the parameter t is sufficient. It is in [7].

Theorem 2.3.6. *Let $q \in \mathbb{R}$ and $p \geq 1$. If an analytic function f satisfies $\int_{\mathbb{D}} |I_{s_1}^{t_1} f|^p \, d\nu_q < \infty$ condition for a pair $s_1, t_1 \in \mathbb{R}$ with $q + pt_1 > -1$, then $\int_{\mathbb{D}} |I_s^t f|^p \, d\nu_q < \infty$ for all $s, t \in \mathbb{R}$ satisfying the relation $q + pt > -1$.*

By this theorem, we can finally understand the functions that belong to B_q^p. Clearly I_s^0 is the identity operator. Hence if $q > -1$, B_q^p consists of the same functions as those in A_q^p, because if we take $t = 0$ the definition of B_q^p gives the exact definition of the Bergman space (normalization of the measure in A_q^p does not affect the elements in these sets). Hence we can see Besov spaces as the generalized version of Bergman spaces or we can see Bergman spaces as a subfamily of Besov spaces. Also by this setting it is possible to embed Besov spaces in the Bergman spaces. An analytic function f belongs to B_q^p if and only if $D_s^t f \in A^p_{q+pt}$ for $q + pt > -1$. The operator D_s^t is an invertible operator that preserves analyticity and the inverse operator is $(D_s^t)^{-1} = D_{s+t}^{-t}$. The next theorem enables us to define norms on Besov spaces.

Theorem 2.3.7. *Suppose $q, s, t \in \mathbb{R}$ and $p \geq 1$. If $q+pt > -1$ and $f \in B_q^p$, then*

$$\|f\|_{B_q^p} := \left(\int_{\mathbb{D}} |D_s^t f|^p d\nu_{q+pt} \right)^{1/p}$$

makes B_q^p space a normed space.

There are infinitely many choices for a norm and even though the space is the same B_q^p, the norm defined on this space may be different for the different choices of s, t. However, all these norms are equivalent. One can consult [7] or [6] to learn more about Besov spaces.

Chapter 3

Bergman Projections and Kernels of Functionals

In this section we will compute Bergman projections of some important functions. Moreover, we will find representations of functionals in the dual spaces.

3.1 Bergman Projection of Functions

Proposition 3.1.1. *Let* $\alpha > -1$ *and* $m, n \in \mathbb{Z}^+ \cup \{0\}$. *Then*

$$P_\alpha(z^m \overline{z}^n) = \frac{(1+m-n)_n z^{m-n}}{(2+\alpha+m-n)_n}$$

for $m \geq n$, *and equals* 0 *otherwise.*

Proof. Let us first observe that if $b \in \mathbb{Z}^+$, $(a)_b = (a)(a+1)\cdots(a+b-1)$. Moreover, if $|z| < 1$ and $a > 0$, then $\dfrac{1}{(1-z)^a} = \displaystyle\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k$ which is a binomial expansion. Then

$$
\begin{aligned}
P_\alpha(z^m\overline{z}^n) &= \int_{\mathbb{D}} w^m\overline{w}^n(1-z\overline{w})^{-(2+\alpha)}d\mu_\alpha(w)\\
&= \int_{\mathbb{D}} w^m\overline{w}^n\sum_k \frac{(2+\alpha)_k}{k!}z^k\overline{w}^k d\mu_\alpha(w)\\
&= \sum_k \frac{(2+\alpha)_k}{k!}z^k\int_{\mathbb{D}} w^m\overline{w}^{n+k}d\mu_\alpha(w)\\
&= \sum_k \frac{(1+\alpha)_{k+1}}{k!}z^k\int_0^1\int_0^{2\pi} e^{i\theta(m-(n+k))}r^{m+n+k}(1-r^2)^\alpha\pi^{-1}d\theta dr\\
&= \sum_k \frac{(1+\alpha)_{k+1}}{k!}z^k\int_0^{2\pi} e^{(i\theta(m-(n+k)))}(2\pi)^{-1}d\theta\int_0^1 r^{m+n+k}(1-r^2)^\alpha 2dr\\
&= \sum_k \frac{(1+\alpha)_{k+1}}{k!}z^k\delta(m,n+k)\int_0^1 r^{m+n+k}(1-r^2)^\alpha 2dr\\
&= \frac{(1+\alpha)_{m-n+1}}{(m-n)!}z^{m-n}\int_0^1 u^m(1-u)^\alpha du\\
&= \frac{\Gamma(2+\alpha+m-n)}{\Gamma(1+\alpha)\Gamma(1+m-n)}\frac{\Gamma(m+1)\Gamma(\alpha+1)}{\Gamma(m+\alpha+2)}z^{m-n}\\
&= \frac{(1+m-n)_n z^{m-n}}{(2+\alpha+m-n)_n}.
\end{aligned}
$$

We exchange the order of integration and summation using Fubini's theorem. For $z\in\mathbb{D}$ it is clear that the expression on the second line is absolutely convergent. We used the relation between the beta function and the gamma function when evaluating the integral. Also $\delta(\,\cdot\,,\cdot)$ stands for the Kronecker delta function. Lastly, the result is obviously 0 if $m<n$. $\square$

The theorems below can be found in [13] for $p>1$. We prove them for $p=1$.

Proposition 3.1.2. *Let $g\in A^1_\alpha$, and $a\in\mathbb{D}$. Then we have*

$$
P_\alpha\left(\frac{z^n}{(1-\overline{a}z)^{2+\alpha+n}}\overline{g(z)}\right) = \frac{1}{(2+\alpha)_n}\sum_{j=0}^n\binom{n}{j}(2+\alpha)_j\overline{g^{(n-j)}(a)}\frac{z^j}{(1-\overline{a}z)^{2+\alpha+j}}.
$$

The result holds also when $g\in A^p_\alpha$ with $p>1$.

Proof. Let's first define the step function. $u(t)=1$ for $t\geq 0$ and 0 otherwise. Moreover, note that $(a)_k = \dfrac{1}{a-1}\dfrac{1}{a-2}\cdots\dfrac{1}{a-(-k)}$ for negative integer k. In the calculations, some of the functions may not be defined at some points. However, all such terms also have a step function that takes the value 0 as a multiplicative term, and treating these terms as 0 causes no problem. Let's write $g(z) = \sum_{k=0}^{\infty} c_k z^k$ and $y_r = (2+\alpha+n)_r \overline{a}^r z^r$. We first look at the left hand side of the equation. In the first line, we interchange summation and integration, and then we interchange the order of double summation. We justify the interchanges at the end of the proof.

$$
\begin{aligned}
P_\alpha\left(\frac{z^n}{(1-\overline{a}z)^{2+\alpha+n}}\overline{g(z)}\right) &= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2+\alpha+n)_i}{i!}\overline{a}^i\overline{c_k}P_\alpha(z^{i+n}\overline{z}^k) \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(2+\alpha+n)_i}{i!}\overline{a}^i\overline{c_k}P_\alpha(z^{i+n}\overline{z}^k)
\end{aligned}
$$

substituting $i = r + k$, and using Proposition 3.1.1, we obtain

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\overline{c_k}\,\overline{a}^k\sum_{r=-k}^{\infty}\frac{(2+\alpha+n)_r}{(k+r)!}\overline{a}^r(1+r+n)_k z^{r+n}u(r+n) \\
&= z^n\sum_{k=0}^{\infty}\overline{c_k}\,\overline{a}^k\Big(\sum_{r=-k}^{\infty}\frac{(1+r+n)_k}{(k+r)!}u(r+n)[(2+\alpha+n)_r\overline{a}^r z^r]\Big) \\
&= z^n\sum_{k=0}^{\infty}\overline{c_k}\,\overline{a}^k\sum_{r=\max(-n,-k)}^{\infty}y_r\frac{(1+r+n)_k}{(k+r)!} \\
&= z^n\sum_{k=0}^{\infty}\overline{c_k}\,\overline{a}^k\sum_{r=-n}^{\infty}y_r\frac{(1+r+n)_k}{(k+r)!}u(r+k).
\end{aligned}
$$

Now let us look at the right hand side:

$$
\begin{aligned}
&\frac{1}{(2+\alpha)_n}\sum_{j=0}^{n}\binom{n}{j}(2+\alpha)_j\overline{g^{(n-j)}(a)}\frac{z^j}{(1-\overline{a}z)^{2+\alpha+j}} \\
&= \sum_{j=0}^{n}\frac{z^j n!}{j!(n-j)!(2+\alpha+j)_{n-j}}\sum_{k=0}^{\infty}\overline{c_k}(k+1-(n-j))_{n-j}\overline{a}^{k-(n-j)}\sum_{t=0}^{\infty}\frac{z^t\overline{a}^t(2+\alpha+j)_t}{t!} \\
&= \sum_{k=0}^{\infty}\overline{c_k}\,\overline{a}^k\Big(\sum_{j=0}^{n}\frac{z^j n!(k+1-(n-j))_{n-j}}{j!(n-j)!(2+\alpha+j)_{n-j}}\overline{a}^{-(n-j)}\Big)\Big(\sum_{t=0}^{\infty}\frac{(2+\alpha+j)_t}{t!}z^t\overline{a}^t\Big) \\
&= z^n\sum_{k=0}^{\infty}\overline{c_k}\,\overline{a}^k\sum_{j=0}^{n}\sum_{t=0}^{\infty}\frac{(k+1-(n-j))_{n-j}n!}{t!j!(n-j)!}\frac{(2+\alpha+j)_t}{(2+\alpha+j)_{n-j}}\overline{a}^{t+j-n}z^{t+j-n} \\
&= z^n\sum_{k=0}^{\infty}\overline{c_k}\,\overline{a}^k\sum_{j=0}^{n}\sum_{t=0}^{\infty}\frac{(k+1-(n-j))_{n-j}n!}{t!j!(n-j)!}y_{t+j-n}.
\end{aligned}
$$

Now we replace j by $n-j$. By doing that, we sum the same terms but in reverse order. Also, we can change the order of summation if one sum contains only finitely many terms. If we do those, the right hand side of our equation is

$$= z^n \sum_{k=0}^{\infty} \overline{c_k}\,\overline{a}^k \Big(\sum_{t=0}^{\infty} \sum_{j=0}^{n} \frac{n!}{t!j!(n-j)!} (k+1-j)_j y_{t-j} \Big)$$

Now we put $t = j + r$. Then right hand side becomes

$$= z^n \sum_{k} \overline{c_k}\,\overline{a}^k \sum_{r=-n}^{\infty} y_r \sum_{j=0}^{n} \frac{n!}{(j+r)!j!(n-j)!} (k+1-j)_j u(r+j).$$

If we look at the expressions of the left hand side and the right hand side, it is obvious that if we can show

$$\frac{(1+r+n)_k}{(k+r)!} u(r+k) = \sum_{j=0}^{n} \frac{n!}{(j+r)!j!(n-j)!} (k+1-j)_j u(r+j)$$

for any $n \geq 0$, $k \geq 0$ and $r \geq -n$, then the proof will be completed. This equality is true for $r < -k$. To see that, observe that the left hand side is 0 for $r < -k$. But also, $(k+1-j)_j u(r+j) = 0$, because either $j \geq -r > k$ which makes $(k+1-j)_j = 0$ or $-r > j$ which makes $u(r+j) = 0$. Hence it is enough to show

$$\frac{(1+r+n)_k}{(k+r)!} = \sum_{j=0}^{n} \frac{n!}{(j+r)!j!(n-j)!} (k+1-j)_j u(r+j)$$

holds for $n \geq 0$, $k \geq 0$ and $r \geq -\min\{n,k\}$. Writing the Pochhammer symbol in terms of factorials, this is equivalent to showing that

$$\frac{(r+k+n)!}{(r+n)!(r+k)!} = \sum_{j=0}^{\min\{n,k\}} \frac{n!k!}{j!(n-j)!(k-j)!} \frac{1}{(j+r)!} u(r+j).$$

In the last equation nothing will change if we exchange n and k. Hence, without loss of generality, we can assume $k \geq n$. Let's say $k = n+s$ for a positive s. Let's also say $r+n = m \geq 0$, multiply both sides of the above equation by $\frac{m!}{n!}$, and also replace j with $n-j$ as before. By doing that our job is simplified to showing

$$\binom{m+n+s}{n} = \sum_{j=0}^{n} \binom{m}{j} \binom{n+s}{n-j} u(m-j)$$

for every positive m,s,n.

Now we state a combinatorics question. Let's say there are $m+s+n$ items and we want to choose n of them. Obviously we have $\binom{m+n+s}{n}$ way to choose them. Now we seperate our items into two groups as m items in the first group and $n+s$ items in the second group. Now we can choose n items from these two groups. Let's say we choose j items from the first group and $n-j$ items from the second one. Hence we have $\binom{m}{j}\binom{n+s}{n-j}$ ways to choose. However, if $m<j$, there is no way to do this. Hence, if we sum $\binom{m}{j}\binom{n+s}{n-j}u(m-j)$ for $0\leq j\leq n$, we must obtain the number of ways of choosing. Hence we end up with $\binom{m+n+s}{n}=\sum_{j=0}^{n}\binom{m}{j}\binom{n+s}{n-j}u(m-j)$, which is the desired result.

Now as we promised, we justify that we can change the order of projection and summation. We have

$$P_\alpha\Big(\frac{z^n}{1-\overline{a}z^{2+\alpha+n}}\overline{g(z)}\Big)=P_\alpha\Big(\sum_{i=0}^{\infty}\frac{(2+\alpha+n)_i}{i!}\overline{a}^iz^{i+n}\overline{g(z)}\Big)$$
$$=\int_{\mathbb{D}}\sum_{i=0}^{\infty}\frac{(2+\alpha+n)_i}{i!}\frac{\overline{a}^iw^{i+n}\overline{g(w)}}{(1-z\overline{w})^{2+\alpha}}\,d\mu_\alpha(w),$$

but

$$\int_{\mathbb{D}}\sum_{i=0}^{\infty}\Big|\frac{(2+\alpha+n)_i}{i!}\frac{\overline{a}^iw^{i+n}\overline{g(w)}}{(1-z\overline{w})^{2+\alpha}}\,d\mu_\alpha(w)\Big|<\infty,$$

because

$$\int_{\mathbb{D}}\frac{|g(w)||w|^n}{|a-z\overline{w}|^{2+\alpha}}\sum_{i=0}^{\infty}\frac{(2+\alpha+n)_i}{i!}|\overline{a}w|^id\mu_\alpha(w)<\infty$$

since $a,z\in\mathbb{D}$ are fixed and $g\in A^1_\alpha$. Therefore,

$$P_\alpha\Big(\frac{z^n}{1-\overline{a}z^{2+\alpha+n}}\overline{g(z)}\Big)=\sum_{i=0}^{\infty}\frac{(2+\alpha+n)_i}{i!}\overline{a}^iP_\alpha(z^{i+n}\overline{g(z)}).$$

But we also have

$$P_\alpha\Big(\sum_{k=0}^{\infty}c_kz^{i+n}\overline{z}^k\Big)=\int_{\mathbb{D}}\sum_{k=0}^{\infty}\frac{c_kw^{i+n}\overline{w}^k}{1-z\overline{w}^{2+\alpha}}\,d\mu_\alpha(w)=\lim_{R\to1^-}\int_{R\mathbb{D}}\sum_{k=0}^{\infty}\frac{c_kw^{i+n}\overline{w}^k}{1-z\overline{w}^{2+\alpha}}\,d\mu_\alpha(w).$$

Since the Taylor series of $g(z)$ is uniformly convergent on every compact subset of $\mathbb{D}$, for any $0<R<1$, we can find N_R such that $|\sum_{k=N_R}^{\infty}c_k\overline{w}^k|<1-R$. However, since N_R increases without bound as $R \to 1^-$ and the integral is 0 if $k > i+n$,

$$\lim_{R\to 1^-} \sum_{k=0}^{N_R} \int_{R\mathbb{D}} \frac{c_k w^{i+n}\overline{w}^k}{1-z\overline{w}^{2+\alpha}}\, d\mu_\alpha(w) = \lim_{R\to 1^-} \sum_{k=0}^{i+n} \int_{R\mathbb{D}} \frac{c_k w^{i+n}\overline{w}^k}{1-z\overline{w}^{2+\alpha}}\, d\mu_\alpha(w)$$
$$= \int_{\mathbb{D}} \sum_{k=0}^{i+n} \frac{c_k w^{i+n}\overline{w}^k}{(1-z\overline{w})^{2+\alpha}}\, d\mu_\alpha(w) = \sum_{k=0}^{i+n} P_\alpha(z^{i+n}\overline{z}^k) = \sum_{k=0}^{\infty} P_\alpha(z^{i+n}\overline{z}^k).$$

Now the only remaining job is showing that double summation can change its order. This can be shown by showing the elements present in the double summation are absolutely summable. We have already proved

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{(2+\alpha+n)_i}{i!}\overline{a}^i\overline{c_k}P_\alpha(z^{i+n}\overline{z}^k)$$

$$=\sum_{j=0}^{n} \frac{z^j n!}{j!(n-j)!(2+\alpha+j)_{n-j}} \sum_{k=0}^{\infty} \overline{c_k}(k+1-(n-j))_{n-j}\overline{a}^{k-(n-j)} \sum_{t=0}^{\infty} \frac{(2+\alpha+j)_t}{t!} z^t\overline{a}^t.$$

If we change $\overline{a}$ with $|a|$ and z with $|z|$ we see that

$$\sum_{j=0}^{n} \frac{|z|^j n!}{j!(n-j)!(2+\alpha+n)_{n-j}} \sum_{k=0}^{\infty} \frac{|c_k||a|^{k-n+j}(k+1-(n-j))_{n-j}}{(1-|az|)^{2+\alpha+j}}$$
$$\le C\sum_{k=0}^{\infty}(|c_k|^{1/k}|a|)^k(k+1)_n$$

for a constant C. $g(z) = \sum_{k=0}^{\infty} c_k z^k$ has radius of convergence at least 1, so there is a positive integer K such that if $k > K$ we have $|c_k|^{1/k} < 1 + \dfrac{1-|a|}{2|a|}$. That is, $|c_k||a|^k \le \left(\dfrac{1+|a|}{2}\right)^k$. If we apply the ratio test, we find $\sum_{k=0}^{\infty}(|c_k|^{1/k}|a|)^k(k+1)_n < \infty$ since $a \in \mathbb{D}$. Since A^1_α contains all A^p_α with $p \ge 1$, this proposition is also true when g is taken from A^p_α. $\square$

Corollary 3.1.3. *Let f be a polynomial of degree N and $g \in A^1_\alpha$. Then $P_\alpha(f\overline{g})$ is a polynomial of degree at most N. This result is true even when $g \in A^p_\alpha$ with $p > 1$ just as Proposition 3.1.2 is.*

Proof. Take $a = 0$ in Proposition 3.1.2. Then it is clear that $P_\alpha(z^n\overline{g})$ is polynomial of degree at most n. Since P_α is a linear operator, the result follows. $\square$

3.2 Kernels of Some Functionals

Definition 3.2.1. If $1 < p < \infty$, $\Phi \in (A^p_\alpha)^*$ is given and $p^{-1} + q^{-1} = 1$, then there is a function $k_\Phi \in A^q_\alpha$ such that $\Phi(f) = \int_{\mathbb{D}} f(z)\overline{k_\Phi(z)}d\mu_\alpha(z)$ by Theorem 2.1.14. We call k_Φ is the kernel of the functional Φ.

If $p = 1$ and $\Phi \in (A^1_\alpha)^*$ is given, then there is a function $k_\Phi \in \mathcal{B}$ such that $\Phi(f) = \int_{\mathbb{D}} f(z)\overline{I_\alpha k_\Phi(z)}d\mu_\alpha(z)$ by Theorem 2.2.5. We again call k_Φ the kernel of the functional Φ.

Even though we used the same word "kernel" in this definition, one should not confuse it with the reproducing kernel. Here we define the kernel of a functional and it is a way to represent the functional in the dual space.

Here, we define kernels of functionals significantly differently for $p > 1$ and $p = 1$ cases. Hence, the results related to kernels can be expected to be different. In [13], [3] some important results of these kernels were studied for $p > 1$. In this thesis we focus on the case $p = 1$.

Proposition 3.2.2. *Let $f \in A^p_\alpha$ for $p \geq 1$, $a \in \mathbb{D}$ and $n = 0, 1, 2, \ldots$. The kernel of the functional $\Phi : f \mapsto f^{(n)}(a)$ is*

$$k_\Phi(z) = \frac{(\alpha+2)_n z^n}{(1-\overline{a}z)^{2+\alpha+n}}.$$

Proof. This result is quite unexpected since the definitions of the kernels for $p = 1$ and $p > 1$ are different. Since definitions are different, we will seperate the proof into two parts. First we will do the proof for $p > 1$ by using ideas in [13], [3]. Then we will do the proof for $p = 1$. Suppose $p > 1$. If we think of a as a variable in the unit disc, we obtain

$$f^{(n)}(a) = \frac{d^n}{da^n}f(a) = \frac{d^n}{da^n}\int_{\mathbb{D}} K_\alpha(a,z)f(z)d\mu_\alpha(z) = \int_{\mathbb{D}} f(z)\frac{\partial^n K_\alpha(a,z)}{\partial a^n}\,d\mu_\alpha(z),$$

hence the kernel k_Φ is equal to the expression $\overline{\dfrac{\partial^n K_\alpha(a,z)}{\partial a^n}}$. So

$$k_\Phi(z) = \overline{\frac{\partial^n K_\alpha(a,z)}{\partial a^n}} = \overline{\frac{(2+\alpha)(3+\alpha)\ldots(n+1+\alpha)}{(1-a\overline{z})^{2+\alpha+n}}\overline{z}^n} = \frac{(\alpha+2)_n z^n}{(1-\overline{a}z)^{2+\alpha+n}}.$$

Next let us consider the case $p = 1$. If we put $\beta = \alpha + 1$, then

$$
\begin{aligned}
&\int_{\mathbb{D}} f(z) I_\alpha\Big(\frac{(\alpha+2)_n z^n}{(1-\overline{a}z)^{2+\alpha+n}}\Big) d\mu_\alpha(z) \\
&= \int_{\mathbb{D}} f(z) \Big[\frac{(\beta+1)_n z^n}{(1-\overline{a}z)^{\beta+1+n}} + \frac{z(\beta+1)_n}{\beta+1}\Big(\frac{z^n}{(1-\overline{a}z)^{\beta+1+n}}\Big)'\Big] d\mu_\beta(z) \\
&= \int_{\mathbb{D}} f(z) \frac{(\beta+1)_n z^n}{(1-\overline{a}z)^{\beta+1+n}} \Big[1 + \frac{1}{\beta+1}\Big(n + \overline{a}z\frac{\beta+1+n}{1-\overline{a}z}\Big)\Big] d\mu_\beta(z) \\
&= \int_{\mathbb{D}} f(z) \frac{(\beta+2)_n}{(1-\overline{a}z)^{\beta+2+n}} z^n d\mu_\beta(z).
\end{aligned}
$$

The last line is very similar to the kernel of the case $p > 1$. The proof for $p > 1$ in reverse order will enable us to reach our goal. So,

$$
\begin{aligned}
\int_{\mathbb{D}} f(z) \frac{(\beta+2)_n}{(1-\overline{a}z)^{\beta+2+n}} z^n \, d\mu_\beta(z) &= \int_{\mathbb{D}} f(z) \frac{\partial^n K_\beta(a,z)}{\partial a^n} \, d\mu_\beta(z) \\
&= \frac{d^n}{da^n} \int_{\mathbb{D}} K_\beta(a,z) f(z) \, d\mu_\beta(z) \\
&= f^{(n)}(a),
\end{aligned}
$$

and hence we have proved that even for $p = 1$, the kernel is $k_\Phi(z) = \dfrac{(\alpha+2)_n z^n}{(1-\overline{a}z)^{2+\alpha+n}}$. In the last line we have used Theorem 2.1.13. In our case $\beta > \alpha$ so the operator P_β is bounded on A^1_α and for a function $f \in A^1_\alpha$ we have $P_\beta f(a) = \displaystyle\int_{\mathbb{D}} K_\beta(a,z) f(z) \, d\mu_\beta(z) = f(a)$. $\square$

3.3 Möbius Expansions

In this section, we will define an expansion for analytic functions. It is analogous to the Taylor expansion and the same as the Taylor expansion for $a = 0$.

Definition 3.3.1. Let $a \in \mathbb{D}$ and $\alpha > -1$. We will denote $w := \dfrac{z-a}{1-\overline{a}z}$. Any analytic function $f(z)$ where $z \in \mathbb{D}$ can be uniquely written as

$$
f(z) = \frac{1}{(1-\overline{a}z)^{2+\alpha}}\Big(c_0 + c_1 w + \frac{c_2}{2!} w^2 + \cdots + \frac{c_N}{N!} w^N + \cdots\Big).
$$

We will call this expansion the Möbius expansion at the point a and the coefficients c_k the Möbius coefficients at a.

Note that $w = 0$ if and only if $z = a$. It is easy to see these coefficients are uniquely determined when we notice $z = \dfrac{w+a}{1+\overline{a}w}$ and

$$c_k = \frac{d^k}{dw^k}\Big((1-\overline{a}z)^{2+\alpha} f(z)\Big)\Big|_{w=0} = \frac{d^k}{dw^k}\Big(\big(1-\overline{a}\frac{w+a}{1+\overline{a}w}\big)^{2+\alpha} f\big(\frac{w+a}{1+\overline{a}w}\big)\Big)\Big|_{w=0}.$$

The next proposition will be vital for upcoming chapters.

Proposition 3.3.2. *Let $a \in \mathbb{D}$ and f be an analytic function on $\mathbb{D}$ such that for $n > N$ its Möbius coefficients at a are $c_n = 0$. If $g \in A^1_\alpha$, also $P_\alpha(f\overline{g})$ is an analytic function such that for $n > N$ its Möbius coefficients are $d_n = 0$.*

Proof. First, note that $\dfrac{z-a}{1-\overline{a}z} = \Big(\dfrac{z}{1-\overline{a}z}\Big)^0(-a) + (1-|a|^2)\dfrac{z}{1-\overline{a}z}$. It is also clear that $\dfrac{z}{1-\overline{a}z} = \Big(\dfrac{z-a}{1-\overline{a}z}\Big)^0 \dfrac{a}{1-|a|^2} + \dfrac{1}{1-|a|^2}\dfrac{z-a}{1-\overline{a}z}$. These two observations indicate that linear combinations of

$$\Big(\frac{z-a}{1-\overline{a}z}\Big)^0, \Big(\frac{z-a}{1-\overline{a}z}\Big)^1, \ldots, \Big(\frac{z-a}{1-\overline{a}z}\Big)^N$$

are the same as those of

$$\Big(\frac{z}{1-\overline{a}z}\Big)^0, \Big(\frac{z}{1-\overline{a}z}\Big)^1, \ldots, \Big(\frac{z}{1-\overline{a}z}\Big)^N.$$

However, from Theorem 3.1.2 it is clear that $P_\alpha(f\overline{g})$ is a linear combination of $\Big(\dfrac{z}{1-\overline{a}z}\Big)^0, \Big(\dfrac{z}{1-\overline{a}z}\Big)^1, \ldots, \Big(\dfrac{z}{1-\overline{a}z}\Big)^N$. Hence it is a linear combination of $\Big(\dfrac{z-a}{1-\overline{a}z}\Big)^0, \Big(\dfrac{z-a}{1-\overline{a}z}\Big)^1, \ldots, \Big(\dfrac{z-a}{1-\overline{a}z}\Big)^N$. So $P_\alpha(f\overline{g})$ has the form claimed. $\square$

Our next proposition shows Möbius expansion at the point a and Taylor series expansion around a are related to each other.

Proposition 3.3.3. *Let $\alpha > -1$ is given. Let f_1 and f_2 be two analytic functions on unit disc such that their first $N+1$ Möbius coefficients agree with each other. Then $f_1(a) = f_2(a), f_1'(a) = f_2'(a), \ldots, f_1^{(N)}(a) = f_2^{(N)}(a)$. Likewise, for two analytic functions g_1 and g_2 on unit disc if we have $g_1(a) = g_2(a), g_1'(a) = g_2'(a), \ldots, g_1^{(N)}(a) = g_2^{(N)}(a)$, then their first $N+1$ Möbius coefficients are the same.*

Proof. Take an analytic function f on unit disc and call its Möbius coefficients at a by c_n. Both statements in the proposition clearly hold for $N = 0$ because $f(a) = \dfrac{c_0}{(1-|a|^2)^{2+\alpha}}$ and $c_0 = f(a)(1-|a|^2)^{2+\alpha}$. Suppose the proposition holds up to $N-1$. We will prove it also holds for N. It is easy to check that if $w := \dfrac{z-a}{1-\overline{a}z}$, $\left.\dfrac{d^k}{dz^k}w^m\right|_{z=a} = 0$ if $k < m$ and $\left.\dfrac{d^k}{dz^k}w^m\right|_{z=a} = \dfrac{1}{(1-|a|^2)^m}$ if $k = m$. Hence if we take Nth derivative of f, we obtain

$$\begin{aligned} f^{(N)}(a) &= \left.\Big(\frac{1}{(1-\overline{a}z)^{2+\alpha}}\Big(c_0 + c_1 w + \frac{c_2}{2!}w^2 + \cdots + \frac{c_N}{N!}w^N + \cdots\Big)\Big)^{(N)}\right|_{z=a} \\ &= h_1(a, c_0, \ldots, c_{N-1}) + \frac{1}{(1-|a|^2)^{2+\alpha}}\Big(h_2(a, c_0, \ldots, c_{N-1}) + \frac{c_N}{(1-|a|^2)^N N!}\Big) \quad (1) \end{aligned}$$

where the functions h_1 and h_2 are known functions for a given N. If for two functions the first N Möbius coefficients are the same, then their Nth derivatives at the point a will be equal to each other, because the expression (1) is valid for both. Likewise, if two functions agree up to the Nth derivatives at the point a, then they also agree up to $(N-1)$th derivatives at the point a. From induction hypothesis, this shows that the first $N-1$ Möbius coefficients of both functions are the same. But, this also shows that h_1 and h_2 values are the same for both functions. By (1) it should be also true that their Nth Möbius coefficients are the same. $\square$

Chapter 4

Extremal Problems

4.1 Definition and Properties of Extremal Problems

In this chapter, we define our extremal problems and prove some important theorems related to them.

Definition 4.1.1. Let $1 \leq p < \infty$, $\alpha > -1$ and $\Phi \in (A^p_\alpha)^*$ are given. Suppose we want to find $g \in A^p_\alpha$ with $\Phi(g) = 1$ such that

$$\|g\|_{A^p_\alpha} = \inf\{\|h\|_{A^p_\alpha} : h \in A^p_\alpha, \Phi(h) = 1\}.$$

This problem will be called an extremal problem and a solution $G \in A^p_\alpha$ of this problem will be called an extremal function.

The following proposition shows that there is an equivalent problem. Moreover, the solutions of the original problem and the equivalent problem can be obtained from one another.

Proposition 4.1.2. *Suppose that $F \in A^p_\alpha$ is a solution of the following problem. For a given $\Phi \in (A^p_\alpha)^*$ we find $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} = 1$ such that*

$$\Phi(f) = \sup\{\Phi(h) : \|h\|_{A^p_\alpha} = 1, \Phi(h) > 0\} = \|\Phi\|.$$

Then $F/\Phi(F)$ is an extremal function for our original extremal problem. Likewise, if a solution of our original extremal problem is G, then $G/\|G\|_{A^p_\alpha}$ is a solution of this new problem.

Proof. We will follow the proof given in [13]. If F solves the new problem, let's show $\dfrac{F}{\Phi(F)}$ is a solution of the original extremal problem.

$$\Big\|\frac{F}{\Phi(F)}\Big\|_{A^p_\alpha} = \frac{1}{|\Phi(F)|} = \frac{1}{\|\Phi\|} = \frac{1}{\sup_{h\in A^p_\alpha}|\Phi(h)|/\|h\|_{A^p_\alpha}}$$

$$= \inf_{h\in A^p_\alpha}\Big\|\frac{h}{\Phi(h)}\Big\|_{A^p_\alpha} = \inf\{\|h\|_{A^p_\alpha} : h\in A^p_\alpha, \Phi(h)=1\}.$$

Hence if we can find a solution for this new problem, we can find a solution for our original problem. The converse is also true. Suppose G is a solution for our original problem. Then

$$\Phi\Big(\frac{G}{\|G\|_{A^p_\alpha}}\Big) = \frac{1}{\inf_{\Phi(G)=1}\|G\|_{A^p_\alpha}} = \sup_{\Phi(G)=1}\frac{1}{\|G\|_{A^p_\alpha}} = \sup_{\Phi(G)=1}\Phi\Big(\frac{G}{\|G\|_{A^p_\alpha}}\Big)$$

$$= \sup_{\|h\|_{A^p_\alpha}=1}\Phi(h),$$

so $\dfrac{\Phi(G)}{\|G\|_{A^p_\alpha}}$ is a solution for the new problem. □

The most obvious question that comes to mind for a given extremal problem is whether there is a solution or not. To answer this question first we should give some definitions.

Definition 4.1.3. A Banach space X is called strictly convex if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$ we have $\dfrac{1}{2}\Big\|\dfrac{x_1}{\|x_1\|}+\dfrac{x_2}{\|x_2\|}\Big\| < 1$.

Definition 4.1.4. A Banach space X is called uniformly convex if for all $x_1, x_2 \in X$ such that $\|x_1\| = \|x_2\| = 1$ the following statement is true. For every $\epsilon > 0$, there is a $0 < \delta < 1$ such that $\|\frac{1}{2}(x_1+x_2)\| > 1-\delta$ implies $\|x_1 - x_2\| < \epsilon$.

The proposition below states that being uniformly convex is stronger than being strictly convex.

Proposition 4.1.5. *If a Banach space is uniformly convex, then it is also strictly convex.*

Proof. One can consult [2]. $\square$

Our next two theorems enable us to talk about unique solutions of extremal problems on Bergman spaces. They are from [2].

Theorem 4.1.6. *In a convex subset of a strictly convex Banach space, there is at most one element of minimal norm.*

Theorem 4.1.7. *Each closed convex subset of a uniformly convex Banach space contains a unique element of minimal norm.*

We will apply these theorems when the Banach space is L^p_α and the convex subset is the subspace A^p_α. For all $p \geq 1$ it is known that L^p_α spaces are strictly convex. Thus if we have a solution for our extremal problem, then it is unique. Also, it is known that if $1 < p < \infty$, the space L^p_α is uniformly convex, but L^1_α is not (one can consult [2]). Hence in A^p_α for $1 < p < \infty$ our extremal problem has a unique solution. We will especially investigate the case $p = 1$ in this thesis. It is clear that if a solution exists, it is unique. So, the question is whether a solution exists or not. Our answer will depend on a property of functionals we are dealing with, and as we can see, majority of most useful functionals have this property. We will give a different proof for an existence theorem in [13]. First we give some notation.

Notation 4.1.8. Let $C(\overline{\mathbb{D}})$ denote the Banach space of all complex functions on $\overline{\mathbb{D}}$ with $\|.\|_\infty$ norm. Let $M(\overline{\mathbb{D}})$ denote the Banach space of all regular complex Borel measure in $\overline{\mathbb{D}}$ normed by $\|\nu\| = |\nu|(\overline{\mathbb{D}})$, where $|\nu|$ is total variation of ν.

Theorem 4.1.9. *$M(\overline{\mathbb{D}}) = C(\overline{\mathbb{D}})^*$ isometrically. So, for every $\Psi \in C(\overline{\mathbb{D}})^*$, there is a unique $\nu \in M(\overline{\mathbb{D}})$ such that $\Psi(f) = \int_{\mathbb{D}} f d\nu$ for every $f \in C(\overline{\mathbb{D}})$.*

The theorem below is from [12, Theorem 8.2].

Theorem 4.1.10. *Let $\{f_n\} \subset A^1_\alpha$, $\lambda \in M(\overline{\mathbb{D}})$ and $f_n \, d\mu_\alpha \to d\lambda$ weak* in $M(\overline{\mathbb{D}}) = C(\overline{\mathbb{D}})^*$. Then there exists an $f \in A^1_\alpha$ such that $d\lambda = f d\mu_\alpha$.*

Proof. The proof works for any region with smooth boundary. In our case, $\partial\mathbb{D}$ is smooth. Let $C_c^\infty(\overline{\mathbb{D}})$ denote the infinitely differentiable functions of compact support in $\mathbb{D}$. By hypothesis, for every $\Phi \in C(\overline{\mathbb{D}})$, $\int_{\mathbb{D}} f_n \Phi d\mu_\alpha \to \int_{\mathbb{D}} \Phi d\lambda$. In particular, this works with $\Phi(z) = (\alpha+1)^{-1}(1-|z|^2)^{-\alpha}\dfrac{\partial\psi}{\partial\bar{z}} \in C(\overline{\mathbb{D}})$, where $\psi \in C_c^\infty(\overline{\mathbb{D}})$. Thus for all n,

$$\int_{\mathbb{D}} f_n \Phi \, d\mu_\alpha = \int_{\mathbb{D}} f_n(z)(1-|z|^2)^{-\alpha}\frac{\partial\psi}{\partial\bar{z}}(1-|z|^2)^\alpha d\mu(z) = \int_{\mathbb{D}} f_n(z)\frac{\partial\psi}{\partial\bar{z}} d\mu(z),$$

which is equal to

$$2i\int_{\mathbb{D}} d(f_n\psi \, dz) = 2i\int_{\partial\mathbb{D}} f_n\psi \, dz = 0$$

by the Stokes theorem and since ψ has compact support in $\mathbb{D}$. So

$$\int_{\mathbb{D}}(1-|z|^2)^{-\alpha}\frac{\partial\psi}{\partial\bar{z}} d\lambda(z) = \int_{\mathbb{D}} \frac{\partial\psi}{\partial\bar{z}} d\kappa(z) = 0.$$

By the Weyl lemma [10, Page 243], κ has the form $d\kappa(z) = f(z)\, d\mu(z)$ for some $f \in H(\mathbb{D})$. Then $d\lambda(z) = f(z)(1-|z|^2)^\alpha d\mu(z) = (1+\alpha)^{-1} f(z)\, d\mu_\alpha(z)$. But by hypothesis $\lambda \in M(\overline{\mathbb{D}})$, so $\|\lambda\| < \infty$ and $\int_{\mathbb{D}} |f(z)| d\mu_\alpha(z) < \infty$, which means $f \in A_\alpha^1$. □

The lemma below is also from [12] and [8].

Lemma 4.1.11. *Let $f_n, f \in A_\alpha^1$ and $\alpha > -1$. Then $f_n \to f$ weak* in A_α^1 if and only if $\left\{\|f_n\|_{A_\alpha^1}\right\}$ is bounded and $f_n(z) \to f(z)$ for all $z \in \mathbb{D}$.*

Now we are ready to state when our extremal problem has a unique solution.

Theorem 4.1.12. *Let $\Phi \in (A_\alpha^1)^*$. By Theorem 2.2.5, there is a unique k_Φ such that for all $g \in A_\alpha^1$, $\Phi(g) = \int_{\mathbb{D}} g(z) I_\alpha k_\Phi(z) d\mu_\alpha(z)$. Let's also suppose $I_\alpha k_\Phi \in C(\overline{\mathbb{D}})$. Then our extremal problem in Definition 4.1.1 has a unique solution. That is, there exists a unique $f \in A_\alpha^1$ such that $\Phi(f) = 1$ and*

$$\|f\|_{A_\alpha^1} = \inf\{\|g\|_{A_\alpha^1} : g \in A_\alpha^1, \Phi(g) = 1\}.$$

Proof. We adapt the proof of [1, Theorem 2.2] to our case. Let's take a sequence $\{g_n\} \subset A_\alpha^1$ with $\Phi(g_n) = 1$ such that

$$\lim_{n\to\infty} \|g_n\|_{A_\alpha^1} = \inf\{\|g\|_{A_\alpha^1} : g \in A_\alpha^1, \Phi(g) = 1\}.$$

This shows that $\{g_n \, d\mu_\alpha\} \subset M(\overline{\mathbb{D}}) \subset C(\overline{\mathbb{D}})^*$ is a bounded sequence. By the Banach-Alaoglu theorem, there is a subsequence $\{g_{n_m}\}$ such that $g_{n_m} d\mu_\alpha \to d\lambda \in M(\overline{\mathbb{D}})$ weak*, that is, $\lim_{m\to\infty} \int_{\mathbb{D}} g_{n_m} h d\mu_\alpha = \int_{\mathbb{D}} h d\lambda$, for all $h \in C(\overline{\mathbb{D}})$. By Theorem 4.1.10, there is an $f \in A^1_\alpha$ such that $d\lambda = f \, d\mu_\alpha$. Hence $g_{n_m} \to f$ weak* and by Lemma 4.1.11 $g_{n_m}(z) \to f(z)$ for all $z \in \mathbb{D}$. In particular, with $h = I_\alpha k_\Phi$ we have

$$\lim_{m\to\infty} \int_{\mathbb{D}} g_{n_m} I_\alpha k_\Phi \, d\mu_\alpha = \lim_{m\to\infty} \Phi(g_{n_m}) = 1 = \int_{\mathbb{D}} f I_\alpha k_\Phi \, d\mu_\alpha = \Phi(f).$$

By Fatou's lemma, $\|f\|_{A^1_\alpha} \leq \lim_{m\to\infty} \|g_{n_m}\|_{A^1_\alpha} < \infty$. So

$$\|f\|_{A^1_\alpha} = \inf\{\|g\|_{A^1_\alpha} : g \in A^1_\alpha, \Phi(g) = 1\}.$$

Thus f solves our extremal problem. The uniqueness follows from the strict convexity of A^1_α. $\square$

We will now give some properties of the extremal function. First, we give some notation. Then, we will prove three useful lemmas two of which are from [13].

Notation 4.1.13. Let $z \in \mathbb{C}$. Then sgn $z := \frac{z}{|z|}$ for $z \neq 0$ and sgn $0 := 0$.

Lemma 4.1.14. *Suppose $f \in A^p_\alpha$, $\Phi \in (A^p_\alpha)^*$ for $1 \leq p < \infty$ and $\Phi(f) = 1$. Then f solves the extremal problem in Definition 4.1.1 if and only if $\|f\|_{A^p_\alpha} \leq \|f+h\|_{A^p_\alpha}$ for all h satisfying $\Phi(h) = 0$.*

Proof. Suppose f solves the extremal problem. For any h satisfying $\Phi(h) = 0$ it is obvious that $\Phi(f + h) = \Phi(f) + \Phi(h) = \Phi(f) = 1$. Since f solves our problem it is clear that $\|f\|_{A^p_\alpha} \leq \|f+h\|_{A^p_\alpha}$. Now let's look at the other direction. For any g such that $\Phi(g) = 1$, we will prove $\|f\||_{A^p_\alpha} \leq \|g\|_{A^p_\alpha}$. To do that, let's observe that $\Phi(g - f) = 0$. From our assumption $\|f\|_{A^p_\alpha} \leq \|f + g - f\|_{A^p_\alpha} = \|g\|_{A^p_\alpha}$. $\square$

Lemma 4.1.15. *Suppose $\int_{\mathbb{D}} |f| \, d\mu = \int_{\mathbb{D}} f g \, d\mu$, $g \in L^\infty$, $|g| \leq 1$ and $f(z) \neq 0$ almost everywhere. Then $g = \operatorname{sgn} \overline{f}$ almost everywhere.*

Proof. First we prove $|g| = 1$ almost everywhere. Let $U_n := \{z : |g(z)| < 1 - \frac{1}{n}\}$ where $n = 1, 2, \ldots$. We have

$$\int_{\mathbb{D}} |f|\, d\mu = \int_{\mathbb{D}\setminus U_n} |f|\, d\mu + \int_{U_n} |f|\, d\mu \geq \Big| \int_{\mathbb{D}\setminus U_n} fg\, d\mu \Big| + \Big| \int_{U_n} fg\, d\mu \Big|$$
$$\geq \Big| \int_{\mathbb{D}} fg\, d\mu \Big| = \int_{\mathbb{D}} |f|\, d\mu,$$

and hence

$$\int_{U_n} |f|\, d\mu = \Big| \int_{U_n} fg\, d\mu \Big| \leq \int_{U_n} |f||g|\, d\mu \leq \Big(1 - \frac{1}{n}\Big) \int_{U_n} |f|\, d\mu$$

for every n. Hence the measure of each U_n should be 0 and also measure of $\bigcup_{n=1}^{\infty} U_n$ is 0. This proves that $|g| = 1$ almost everywhere. Now let's define h by $g = h \operatorname{sgn} \overline{f}$ for some $h \in L^{\infty}$. So $|g| = |h||\operatorname{sgn} \overline{f}|$ and $|h| = 1$ almost everywhere. Now take any $A \subset \mathbb{D}$. We have

$$\int_A |f|\, d\mu + \int_{\mathbb{D}\setminus A} |f|\, d\mu \geq \Big| \int_A |f| h\, d\mu \Big| + \Big| \int_{\mathbb{D}\setminus A} |f| h\, d\mu \Big| \geq \Big| \int_{\mathbb{D}} |f| h\, d\mu \Big| = \int_{\mathbb{D}} |f|\, d\mu.$$

So for any measurable $A \subset \mathbb{D}$ we have $\displaystyle\int_A |f|\, d\mu = \int_A |f| h\, d\mu$. Hence $\displaystyle\int_A |f|(1 - h) d\mu = 0$. Now, suppose the measure of the set $\{z \in \mathbb{D} : h(z) \neq 0\}$ does not equal 0. Then it is clear that there exist $0 < \theta < \pi/2$ such that the set $B := \{z \in \mathbb{D} : \operatorname{Arg}\{1 - h(z)\} \leq \theta\}$ has positive measure. From simple geometry it can be seen that if $z \in B$, $\operatorname{Re}\{1 - h(z)\} \geq 2\cos^2 \theta$ so we have

$$\Big| \int_B |f|(1 - h) \Big|^2 = \Big(\int_B |f| \operatorname{Re}\{1 - h\}\, d\mu \Big)^2 + \Big(\int_B |f| \operatorname{Im}\{1 - h\}\, d\mu \Big)^2$$
$$\geq 4\cos^4 \theta \Big| \int_B |f|\, d\mu \Big|^2 > 0$$

which contradicts the above observation. Hence $|h| = 1$ almost everywhere and $g = \operatorname{sgn} \overline{f}$ almost everywhere. □

Lemma 4.1.16. *Suppose that X is a closed subspace of L^1_α, $f \in L^1_\alpha$ and $f \neq 0$ almost everywhere. Then $\displaystyle\int_{\mathbb{D}} (\operatorname{sgn} \overline{f}) h d\mu_\alpha = 0$ if and only if $\|f\|_{L^p_\alpha} \leq \|f + h\|_{L^p_\alpha}$ for every $h \in X$.*

Proof. ($\Rightarrow$): Without loss of generality we can assume $\|f\|_{L^1_\alpha} = 1$. Let $g = \operatorname{sgn} \overline{f}$; so $g \in L^{\infty}$ and $\|g\|_{L^{\infty}} = \|f\|_{L^1_\alpha} = 1$. Then for each $h \in X$ we have

$$\|f\|_{L^1_\alpha} = 1 = \int_{\mathbb{D}} fg d\mu_\alpha = \int_{\mathbb{D}} (f + h) g d\mu_\alpha \leq \int_{\mathbb{D}} |(f + h) g| d\mu_\alpha.$$

If we apply the Hölder inequality to the last expression we obtain

$$\int_{\mathbb{D}} |(f+h)g| d\mu_\alpha \leq \|f+h\|_{L^1_\alpha} \|g\|_{L^\infty} = \|f+h\|_{L^1_\alpha}.$$

($\Leftarrow$): The condition $\|f\|_{L^1_\alpha} \leq \|f+h\|_{L^1_\alpha}$ for all $h \in X$ along with $f \neq 0$ almost everywhere implies that $f \notin X$. By the Hahn-Banach theorem there is an $M : L^1_\alpha \to \mathbb{C}$ with $\|M\| = 1$ such that $Mf = \|f\|_{L^1_\alpha}$ and $Mh = 0$ for any $h \in X$. By the Riesz representation theorem, there is a $g \in L^\infty$ such that $Mf = \int_{\mathbb{D}} fg\, d\mu_\alpha = \|f\|_{L^1_\alpha}$, where $\|M\| = \|g\|_{L^\infty} = 1$ and $Mh = \int_{\mathbb{D}} hg\, d\mu_\alpha = 0$. Then $\|f\|_{L^1_\alpha} = \int_{\mathbb{D}} |f|\, d\mu_\alpha = \int_{\mathbb{D}} fg\, d\mu_\alpha$. If $f \neq 0$ almost everywhere, this condition is equivalent to $g = \operatorname{sgn} \overline{f}$ by Lemma 4.1.14. Finally, $Mh = 0$ yields $\displaystyle\int_{\mathbb{D}} (\operatorname{sgn} \overline{f}) h d\mu_\alpha = 0$. □

Now we are ready to give some properties of our extremal functions. They are from [13] and [15]

Theorem 4.1.17. *Suppose* $1 < p < \infty$, $\Phi \in (A^p_\alpha)^*$, *and* $\Phi(f) = 1$, *where* $f \in A^p_\alpha$. *The following are equivalent:*

(i) f *solves the extremal problem in Definition 4.1.1.*

(ii) For any $h \in A^p_\alpha$, *we have*

$$\int_{\mathbb{D}} |f|^{p-1} (\operatorname{sgn} \overline{f}) h d\mu_\alpha = \|f\|^p_{A^p_\alpha} \Phi(h).$$

(iii)

$$k_\Phi = P_\alpha \Big(\frac{|f|^{p-1} \operatorname{sgn} f}{\|f\|^p_{A^p_\alpha}} \Big).$$

The proof can be found in [13]. In this thesis we will be interested in the case $p = 1$. Surprisingly, a similar theorem can be proven. In [13] the theorem below is stated without proof. We provide a proof here.

Theorem 4.1.18. *Suppose* $\Phi \in (A^1_\alpha)^*$, *and* $\Phi(f) = 1$ *where* $f \in A^1_\alpha$. *The following are equivalent:*

(i) f *solves the extremal problem in Definition 4.1.1.*

(ii) For any $h \in A^1_\alpha$, we have

$$\int_{\mathbb{D}} (\operatorname{sgn} \overline{f}) h d\mu_\alpha = \|f\|_{A^1_\alpha} \Phi(h).$$

(iii)

$$k_\Phi = P_\alpha\Big(\frac{\operatorname{sgn} f}{\|f\|_{A^1_\alpha}}\Big).$$

Proof. (ii) $\Leftrightarrow$ (iii): Suppose (ii) holds. Then for any $h \in A^1_\alpha$,

$$\Phi(h) = \int_{\mathbb{D}} h \frac{\operatorname{sgn} \overline{f}}{\|f\|_{A^1_\alpha}} d\mu_\alpha = \int_{\mathbb{D}} h \overline{I_\alpha P_\alpha\Big(\frac{\operatorname{sgn} f}{\|f\|_{A^1_\alpha}}\Big)} d\mu_\alpha,$$

where the last line is immediate from Theorem 2.2.6. The converse is obvious if we read these equalities in reverse order.
(i) $\Leftrightarrow$ (ii): Note that $\operatorname{null} \Phi$ is a closed subspace of A^1_α and $f \neq 0$ almost everywhere. By Lemma 4.1.14 and Lemma 4.1.16 it is clear that condition (i) is equivalent to say that for any $h \in A^1_\alpha$ with $\Phi(h) = 0$, $\int_{\mathbb{D}} (\operatorname{sgn} \overline{f}) h d\mu_\alpha = 0$. Since $\Phi(f) = 1$ it is clear that for any $h \in A^1_\alpha$, $\Phi(h - \Phi(h) f) = 0$. So, for any $h \in A^1_\alpha$ it is true that $\int_{\mathbb{D}} (\operatorname{sgn} \overline{f})(h - \Phi(h) f) d\mu_\alpha = 0$ and this means it is equivalent to (ii). $\square$

4.2 Generalized Extremal Problems

Now we will generalize the problem given in the previous section. We still use terms extremal problems and extremal functions though.

Definition 4.2.1. Let $1 \leq p < \infty$, $\alpha > -1$ and $\Phi_1, \ldots, \Phi_N \in (A^p_\alpha)^*$ are given. We will try to solve the following problem. Find $g \in A^p_\alpha$ with $\Phi_1(g) = a_1$, $\Phi_2(g) = a_2$, ..., $\Phi_N(g) = a_N$ such that

$$\|g\|_{A^p_\alpha} = \inf\{\|h\|_{A^p_\alpha} : h \in A^p_\alpha, \Phi_1(h) = a_1, \Phi_2(h) = a_2, \ldots, \Phi_N(h) = a_N\}$$

where $a_k \in \mathbb{C}$.

Our next goal is to solve this version of the extremal problem. However first we should prove the following useful lemma.

Lemma 4.2.2. *Let V be a vector space over complex numbers and let $\Psi, \Phi_1, \ldots, \Phi_N$ be linear functionals on V. Then $\bigcap_{j=1}^N \operatorname{null} \Phi_j \subset \operatorname{null} \Psi$ if and only if there are complex numbers c_k such that $\Psi(v) = \sum_{k=1}^N c_k \Phi_k(v)$ for every $v \in V$.*

Proof. If there are such $c_k \in \mathbb{C}$, it is clear that $\bigcap_{j=1}^N \operatorname{null} \Phi_j \subset \operatorname{null} \Psi$. Let's prove the other direction. If $\Psi = 0$, take all $c_k = 0$. Hence we can assume $\Psi \neq 0$. We will use induction. First we prove this theorem holds for $N = 1$. Suppose $\operatorname{null} \Phi_1 \subset \operatorname{null} \Psi$. Take an element $v_1 \in V$ such that $\Psi(v_1) \neq 0$ then we know $\Phi_1(v_1) \neq 0$ too. For an arbitrary $v \in V$ we have $\Phi_1(v - \frac{\Phi_1(v)}{\Phi_1(v_1)} v_1) = \Phi_1(v) - \frac{\Phi_1(v)}{\Phi_1(v_1)} \Phi_1(v_1) = 0$. Since $\operatorname{null} \Phi_1 \subset \operatorname{null} \Psi$, then $\Psi(v - \frac{\Phi_1(v)}{\Phi_1(v_1)} v_1) = 0$ too. Hence, if we define $c_1 = \frac{\Psi(v_1)}{\Phi_1(v_1)}$, we obtain that for any $v \in V$, $\Psi(v) = c_1 \Phi_1(v)$. So, for $N = 1$ the theorem holds.

Now suppose the theorem holds for $N-1$ functionals; we will prove it also holds for N functionals. If for any $1 \leq k \leq N$, $\operatorname{null} \Phi_k \subset \bigcap_{\substack{j=1 \\ j \neq k}}^N \operatorname{null} \Phi_j$, then obviously this theorem holds; just take all Φ_j's except Φ_k and apply this theorem for $N-1$ elements. So we can assume for any $1 \leq k \leq N$, $\operatorname{null} \Phi_k$ is not subset of $\bigcap_{\substack{j=1 \\ j \neq k}}^N \operatorname{null} \Phi_j$. Let us take $v_k \in V$ such that $v_k \in \bigcap_{\substack{j=1 \\ j \neq k}}^N \operatorname{null} \Phi_j \setminus \operatorname{null} \Phi_k$ for each $1 \leq k \leq N$. Now we take an arbitrary element $v \in V$ and compute

$$\Phi_k\left(v - \frac{\Phi_1(v)}{\Phi_1(v_1)} v_1 - \frac{\Phi_2(v)}{\Phi_2(v_2)} v_2 - \cdots - \frac{\Phi_N(v)}{\Phi_N(v_N)} v_N\right)$$

$$= \Phi_k(v) - \frac{\Phi_1(v)}{\Phi_1(v_1)} \delta(1,k) \Phi_1(v_1) - \frac{\Phi_2(v)}{\Phi_2(v_2)} \delta(2,k) \Phi_2(v_2) - \cdots - \frac{\Phi_N(v)}{\Phi_N(v_N)} \delta(N,k) \Phi_N(v_N)$$

$$= \Phi_k(v) - \frac{\Phi_k(v)}{\Phi_k(v_k)} \Phi_k(v_k) = 0.$$

Hence, this element is in the intersection of the null spaces of the functionals Φ_k, $k = 1, \ldots, N$, and hence in the null space of Ψ. If we define $c_k = \frac{\Psi(v_k)}{\Phi_k(v_k)}$ we see that

$$\Psi\left(v - \frac{\Phi_1(v)}{\Phi_1(v_1)} v_1 - \frac{\Phi_2(v)}{\Phi_2(v_2)} v_2 - \cdots - \frac{\Phi_N(v)}{\Phi_N(v_N)} v_N\right) = \Psi_k(v) - \sum_{k=1}^N c_k \Phi_k(v) = 0.$$

and so theorem holds for N functionals too and we are done. $\square$

Now we are ready to give one of the most important theorems in this thesis. Its initial form will be given for $1 < p < \infty$, however we will prove that the theorem can be generalized to $p = 1$ case too.

Theorem 4.2.3. *Let $1 < p < \infty$ and $\Phi_1, \Phi_2, \dots, \Phi_N \in (A^p_\alpha)^*$ be linearly independent. For $F \in A^p_\alpha$ we have*

$$\|F\|_{A^p_\alpha} = \inf\left\{\|f\|_{A^p_\alpha} : \Phi_1(f) = \Phi_1(F), \dots \Phi_N(f) = \Phi_N(F)\right\}$$

if and only if $P_\alpha(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of the kernels of $\Phi_1, \dots, \Phi_N$.

This theorem is from [13, Theorem 3.2.6] and the proof can be found there. The proof however, does not touch the case $p = 1$. In fact this theorem can be generalized even for $p = 1$ if we consider the different meaning of the kernel for $p = 1$.

Theorem 4.2.4. *Let $\Phi_1, \Phi_2, \dots, \Phi_N \in (A^1_\alpha)^*$ be linearly independent. For $F \in A^1_\alpha$ we have*

$$\|F\|_{A^1_\alpha} = \inf\left\{\|f\|_{A^p_\alpha} : \Phi_1(f) = \Phi_1(F), \dots, \Phi_N(f) = \Phi_N(F)\right\}$$

if and only if $P_\alpha(\operatorname{sgn} F)$ is a linear combination of the kernels of $\Phi_1, \dots, \Phi_N$.

Proof. First suppose $P_\alpha(\operatorname{sgn} F)$ is a linear combination of $k_{\Phi_1}, \dots, k_{\Phi_N}$. Let $h \in A^1_\alpha$ be such that $\Phi_j(h) = 0$ for $1 \le j \le N$. Then by Theorem 2.2.6 and Definition 3.2.1,

$$\|F\|_{A^1_\alpha} = \int_{\mathbb{D}} F\overline{\operatorname{sgn} F}d\mu_\alpha = \int_{\mathbb{D}} F\overline{I_\alpha P_\alpha(\operatorname{sgn} F)}d\mu_\alpha = \int_{\mathbb{D}} (F+h)\overline{I_\alpha P_\alpha(\operatorname{sgn} F)}d\mu_\alpha$$
$$= \int_{\mathbb{D}} (F+h)\overline{(\operatorname{sgn} F)}d\mu_\alpha \le \int_{\mathbb{D}} |(F+h)|d\mu_\alpha \le \|F+h\|_{A^1_\alpha}.$$

So, we have $\|F\|_{A^1_\alpha} \le \|F+h\|_{A^1_\alpha}$. If we call $f = F + h$, $\Phi_j(f) = \Phi_j(F)$ so F is extremal. Next suppose $\|F\|_{A^1_\alpha} = \inf\left\{\|f\|_{A^p_\alpha} : \Phi_1(f) = \Phi_1(F), \dots, \Phi_N(f) = \Phi_N(F)\right\}$. Let $h \neq 0$ and $h \in A^1_\alpha$ be such that $\Phi_1(h) = \cdots = \Phi_N(h) = 0$. Since there are only finitely many functionals Φ_j such a function h exists. Then $\|F\|_{A^1_\alpha} \le \|F+h\|_{A^1_\alpha}$ by the extremality of F. Applying Lemma 4.1.16, for $X = \bigcap_{n=1}^N \operatorname{null}\Phi_j$, we have $\int_{\mathbb{D}} \overline{(\operatorname{sgn} F)}hd\mu_\alpha = 0$ and so $\int_{\mathbb{D}} \overline{I_\alpha P_\alpha(\operatorname{sgn} F)}hd\mu_\alpha = 0$. Define $\Psi(f) = \int_{\mathbb{D}} f\overline{I_\alpha P_\alpha(\operatorname{sgn} F)}d\mu_\alpha$ for $f \in A^1_\alpha$. By Lemma 4.2.2 $\Psi = \sum_{j=1}^N c_j\Phi_j$. Thus $P_\alpha(\operatorname{sgn} F)$, which is the kernel of Ψ, is a linear combination of $k_{\Phi_1}, \dots, k_{\Phi_N}$. □

Chapter 5

Solving Extremal Problems

5.1 Bergman Projection Methods on Extremal Problems

In this section we will find the solutions of certain extremal problems explicitly. We will choose our functionals as $\Phi_k : f \to f^{(k)}(a)$ which are evaluations of kth derivatives at $a \in \mathbb{D}$. First we will prove two fundamental theorems for the cases $1 < p < \infty$ and $p = 1$. The first one is from [13], however here it is more general.

Theorem 5.1.1. *Suppose $a \in \mathbb{D}$ and $1 < p < \infty$. Also suppose $F \in A^p_\alpha$ and the Möbius coefficients of $F^{p/2}$ at a are c_n. If $c_n = 0$ for every $n > N$ and $F^{p/2-1} \in A^1_\alpha$, then*

$$\|F\|_{A^p_\alpha} = \inf\{\|f\|_{A^p_\alpha} : f(a) = F(a), \ldots, f^{(N)}(a) = F^{(N)}(a)\}.$$

That means F is a solution for our extremal problem.

Proof. Observe that $|F|^{p-1} \operatorname{sgn} F = F^{p/2}\overline{F}^{p/2-1}$. It is also true that $F^{p/2} \in A^1_\alpha$. We know $P_\alpha(|F|^{p-1} \operatorname{sgn} F) = P_\alpha(F^{p/2}\overline{F}^{p/2-1})$. Proposition 3.3.2 shows us in fact, $P_\alpha(F^{p/2}\overline{F}^{p/2-1})$ has the same form as $F^{p/2}$, that is, for $n > N$ its Möbius coefficients are zero. Hence for suitable $b_k \in \mathbb{C}$ we have

$$P_\alpha(|F|^{p-1} \operatorname{sgn} F) = \sum_{k=0}^{N} b_k \frac{(z-a)^k}{(1-\overline{a}z)^{2+\alpha+k}}.$$

But the proof of Proposition 3.3.2 also indicates that the right side is in the linear span of $\dfrac{(\alpha+2)_n z^n}{(1-\overline{a}z)^{2+\alpha+n}}$. However, by Proposition 3.2.2 it is a linear combination of $k_{\Phi_1}, k_{\Phi_2}, \ldots, k_{\Phi_N}$ where $\Phi_k : f \to f^{(k)}(a)$. Now, applying Theorem 4.2.3 gives us the desired result. □

In the theorem above we implicitly assumed there is an analytic function $F^{p/2}$. For even integer p, it exists obviously. However, for other p values it may not be the case. However, if F has no zeros in $\mathbb{D}$ then we can define $F^{p/2}$ analytically for some branch of the power function. Now we will also look for the case $p = 1$.

Theorem 5.1.2. *Suppose $a \in \mathbb{D}$ and $h(z)$ is an analytic function on $\mathbb{D}$ such that its Möbius coefficients at a are 0 after the Nth term. If $\frac{1}{h(z)} \in A^1_\alpha$, then $F(z) := h^2(z)$ satisfies*

$$\|F\|_{A^1_\alpha} = \inf\{\|f\|_{A^1_\alpha} : f(a) = F(a), \ldots, f^{(N)}(a) = F^{(N)}(a)\}.$$

Proof. $P_\alpha(\operatorname{sgn} F) = P_\alpha(h\overline{h}^{-1})$, so it has the same form as $h(z)$ by Proposition 3.3.2. The rest of the proof is the same as that of Theorem 5.1.1 only with the difference that we use Theorem 4.2.4 instead of Theorem 4.2.3. □

The theorem below gives us a candidate for a solution.

Theorem 5.1.3. *Let $a \in \mathbb{D}$ and $1 \le p < \infty$. Suppose*

$$G(z) = \frac{1}{(1-\overline{a}z)^{2+\alpha}}\left(b_0 + b_1 w + \cdots + \frac{b_N}{N!}w^N\right)$$

where $w := \dfrac{z-a}{1-\overline{a}z}$, that is, Möbius coefficients of $G(z)$ at a are 0 except for finitely many terms. If G has no zeros in $\mathbb{D}$ and also $G^{1-2/p} \in A^1_\alpha$, then $F := G^{2/p}$ satisfies

$$\|F\|_{A^p_\alpha} = \inf\{\|f\|_{A^p_\alpha} : f(a) = F(a), \ldots, f^{(N)}(a) = F^{(N)}(a)\}.$$

In other words, F is the solution of the extremal problem in Definition 4.2.1.

Proof. Since $G(z) \neq 0$ for all $z \in \mathbb{D}$, we can define $F := G^{2/p}$. Hence $F^{p/2} = G$, and Theorem 5.1.1 and Theorem 5.1.2 give us the desired result. The solution is unique because of strict convexity of L^p_α, and Theorem 4.1.6. □

5.2 Examples of Solutions for $p = 1$

In [13], the solutions of certain type of extremal problems are found with similar methods. However, the method there does not work for $p = 1$. In this thesis, we generalize it to $p = 1$ and examples below show that we can use Theorem 5.1.3 to solve similar problems.

Example 5.2.1. Let $a \in \mathbb{D}$, $c_0 \in \mathbb{C}$ and $c_0 \neq 0$. Suppose that $F \in A^1_\alpha$ is the solution of the following extremal problem: $F(a) = c_0$ and

$$\|F\|_{A^1_\alpha} = \inf\big\{\|f\|_{A^1_\alpha} : f(a) = c_0\big\}.$$

By Theorem 5.1.3 our candidate for this function is

$$\Big(\frac{b_0}{(1-\overline{a}z)^{2+\alpha}}\Big)^2.$$

To ensure $F(a) = c_0$, we should choose $b_0 = c_0^{1/2}(1-|a|^2)^{2+\alpha}$ for some branch of the square root, so our candidate is $F(z) = c_0\big(\dfrac{1-|a|^2}{1-\overline{a}z}\big)^{2(2+\alpha)}$. It is clear that $G(z) = c_0^{1/2}\Big(\dfrac{(1-|a|^2)^{2+\alpha}}{(1-\overline{a}z)^{2+\alpha}}\Big)$ has no zeros in $\mathbb{D}$ and $G(z)^{-1} = \Big(\dfrac{1}{c_0^{1/2}}\dfrac{(1-\overline{a}z)^{2+\alpha}}{(1-|a|^2)^{2+\alpha}}\Big)$ belongs to A^1_α. This justifies the use of Theorem 5.1.3. So

$$F(z) = c_0\bigg(\frac{1-|a|^2}{1-\overline{a}z}\bigg)^{2(2+\alpha)}$$

is the solution of our extremal problem. In [14], the same problem is solved and the same answer is found by another method.

Example 5.2.2. Let $a \in \mathbb{D}$ and $c_1 \in \mathbb{D}$. Suppose that we try to find $F \in A^1_\alpha$ such that $F(a) = 1$, $F'(a) = c_1$ and $\|F\|_{A^1_\alpha}$ is minimum. From Theorem 5.1.3 our candidate for the extremal function is

$$F(z) = \bigg(\frac{1}{(1-\overline{a}z)^{2+\alpha}}\Big(b_0 + b_1\frac{z-a}{1-\overline{a}z}\Big)\bigg)^2.$$

Here $F(a) = 1$ so $b_0 = (1-|a|^2)^{2+\alpha}$. Now, taking the derivative gives $F'(a) = \dfrac{2(b_1 + (1-|a|^2)^{2+\alpha}(2+\alpha)\overline{a})}{(^1-\overline{a}z)^{3+\alpha}} = c_1$. From there we obtain $b_1 = (1-|a|^2)^{2+\alpha}\Big[\dfrac{c_1(1-|a|^2)}{2} - (2+\alpha)\overline{a}\Big]$. Hence if the function

$$G(z)^{-1} = \bigg(\frac{1}{(1-\overline{a}z)^{2+\alpha}}\Big(b_0 + b_1\frac{z-a}{1-\overline{a}z}\Big)\bigg)^{-1}$$

$$= \left(\frac{1}{(1-\overline{a}z)^{2+\alpha}} \Big((1-|a|^2)^{2+\alpha} + (1-|a|^2)^{2+\alpha} \big[\frac{c_1(1-|a|^2)}{2} - (2+\alpha)\overline{a} \big] \frac{z-a}{1-\overline{a}z} \Big) \right)^{-1}$$

is in A^1_α (it also means that $G(z)$ has no zeros in $\mathbb{D}$), then

$$F(z) = \left(\frac{1}{(1-\overline{a}z)^{2+\alpha}} \Big(b_0 + b_1 \frac{z-a}{1-\overline{a}z} \Big) \right)^2$$

$$= \left(\frac{1}{(1-\overline{a}z)^{2+\alpha}} \Big((1-|a|^2)^{2+\alpha} + (1-|a|^2)^{2+\alpha} \big[\frac{c_1(1-|a|^2)}{2} - (2+\alpha)\overline{a} \big] \frac{z-a}{1-\overline{a}z} \Big) \right)^2$$

is indeed our solution.

In the next section we will see that similar methods work for Besov spaces.

5.3 Extremal Functions for Besov Spaces

This method can be used in Besov spaces in some special cases. Suppose that we work in B^p_q for $1 < p < \infty$ and $q \leq 1$. If the functionals Φ_k are chosen as $\Phi_k : F \to F^{(k)}(0)$ we can still solve extremal problems.

Example 5.3.1. Suppose we try to solve following extremal problem: Let $1 < p < \infty$, $q \leq -1$ and $s, t \in \mathbb{R}$. Also, suppose $s > -2$ and denote $\alpha := q + pt > -1$. Find $F \in B^p_q$ such that $F(0) = 1$ and $F'(0) = c_1$ and the norm of F which is defined as $\|F\|_{B^p_q} := \|D^t_s F\|_{B^p_{q+pt}} = \frac{1}{\alpha+1}\|D^t_s F\|_{A^p_\alpha}$ is minimum. If we use Notation 2.3.3, it is easy to see $d_k(s,t) = \dfrac{(2+s+t)_k}{(2+s)_k}$. The direct conclusion is that $F(0) = 1$ if and only if $(D^t_s F)(0) = 1$ and that $F'(0) = c_1$ if and only if $(D^t_s F)'(0) = \dfrac{2+s+t}{2+s} c_1$. Hence, if we call $G := D^t_s F$, then our question transforms to the following one. Find $G \in A^p_\alpha$ such that $G(0) = 1$, $G'(0) = \dfrac{2+s+t}{2+s} c_1$ and $\|G\|_{A^p_\alpha}$ is minimum. It is the same question as in Example 5.2.2 where the only difference is $p > 1$ now. Similar calculations like those in Example 5.2.2 show that if $H^{1-2/p} \in A^1_\alpha$ and $H(z)$ never attains 0, then we can find a solution for this problem, where

$$H(z) := 1 + \frac{p(2+s+t)}{2(2+s)} c_1 z.$$

A straightforward calculation gives us $F = (D^t_s)^{-1}(G) = (D^t_s)^{-1}(H^{2/p})$. We find

$$F(z) = D^{-t}_{s+t} \Big[\Big(1 + \frac{p(2+s+t)}{2(2+s)} c_1 z \Big)^{2/p} \Big].$$

The example above shows us this method can be used to find extremal functions on Besov spaces. It is restrictive though if we consider evaluations at a point other than 0, since then computations become much harder, and we will not touch that case in this thesis. However, if a relation between Möbius coefficients and D_s^t can be found, then this method would be promising to solve such questions.

Bibliography

[1] C. Bénéteau and D. Khavinson, *A survey of linear extremal problems in analytic function spaces*, in *Complex Analysis and Potential Theory*, pp. 33–46, CRM Proc. Lecture Notes, vol. 55, Amer. Math. Soc., Providence, 2012.

[2] P. Duren and A. Schuster, *Bergman Spaces*, Math. Surveys Monogr., vol. 100, Amer. Math. Soc., Providence, 2014.

[3] T. Ferguson, *Solution of extremal problems in Bergman spaces using the Bergman projection*, Comput. Methods Funct. Theory **14** (2014), 35–61.

[4] H. Hedenmalm, *A factorization theorem for square area-integrable analytic functions*, Walter de Gruyter, Berlin/New York, 1991.

[5] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Springer, New York, 2000.

[6] H. T. Kaptanoğlu, *Bergman projections on Besov spaces on balls*, Illinois J. Math. **49** (2005), 385–403.

[7] H.T. Kaptanoğlu and A.E. Üreyen, *Analytic properties of Besov spaces via Bergman projections*, Contemp. Math., **455** (2008), 169–182.

[8] H.T. Kaptanoğlu, *Carleson measures for Besov spaces on the ball with applications*, J. Func. Analy., **250** (2007), 483–520.

[9] S. Ya. Khavinson, *On an extremal problem in the theory of analytic function*, Russian Math. Surv, **4** (1949) 158–159.

[10] R. Narasimhan, *Complex Analysis in One Variable*, Birkhäuser, Boston, 1985

[11] W.W. Rogosinski, and H.S Shapiro, *On certain extremum problems for analytic functions*, Acta Math., **90** (1953), 287–318.

[12] H. S. Shapiro, *The Schwarz Function and its Generalization to Higher Dimensions*, Univ. Arkansas Lecture Notes Math. Sci., vol. 9, Wiley, New York, 1992.

[13] R. Özbek, *Extremal Problems and Bergman Projections*, M.S thesis, Bilkent University, Ankara (2017).

[14] D. Vukotić, *A sharp estimate for A^p_α functions in $\mathbb{C}^n$*, Proc. Amer. Math. Soc. **117** (1993), 753–756.

[15] D. Vukotić, *Linear extremal problems for Bergman spaces*, Expo. Math. **14** (1996), 313–352.

[16] K. Zhu, *Operator Theory in Function Spaces*, Math. Surveys Monogr., vol. 138, American Mathematical Soc., 2007.