

QUANTUM CORRELATIONS UNDER OPEN QUANTUM  
SYSTEMS AND QUANTUM PHASE TRANSITIONS

by

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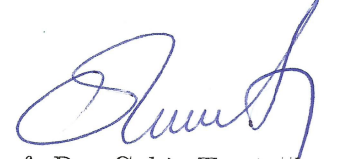
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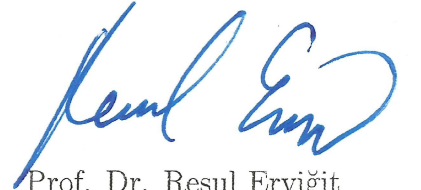
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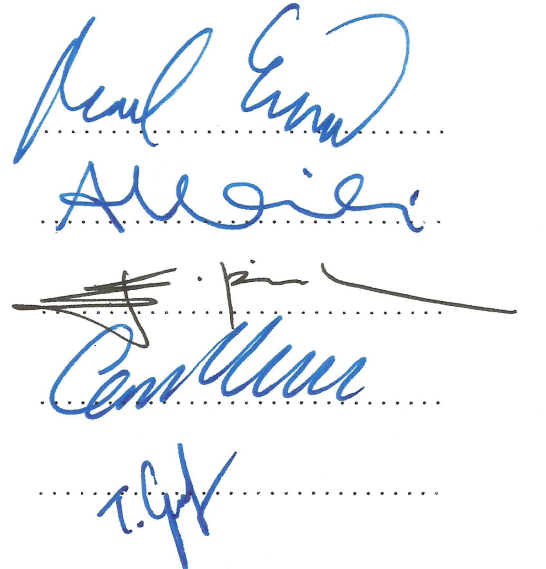


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# **ABSTRACT**

## **Quantum Correlations Under Open Quantum Systems and Quantum Phase Transitions**

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In this thesis, we, first, study the dynamics of different quantumness measures, such as entanglement and beyond (quantum discord and measurement induced disturbance) as well as Bell non-locality under decoherence. We, second, focus the ability of the considered quantum correlation measures to detect the critical points of quantum phase transitions.

First, we have studied the dynamics of quantum correlations such as entanglement, Bell non-locality and quantum discord between identical and unidentical atoms interacting with a single-mode cavity field and subject to cavity decay. The effect of single atom detuning, cavity decay rate and initial preparation of the atoms on the corresponding correlation measures have been investigated under rotating wave approximation by numerically solving the standard Lindblad quantum optical master equation. High quantum discord is found to be induced regardless of the magnitude of the cavity decay rate, while entanglement and Bell non-locality stay zero for an initially product states. Quantum discord is found to increase in a certain time period under dissipation, while entanglement decreases for the initial state that both atoms are in their excited states. For some type of

initial states and identical atom case, cavity decay is shown to drive the system to a stationary state with high entanglement and quantum discord.

Second, we have investigated the dynamics of entanglement and quantum discord between two identical qubits resonantly interacting with a common single mode leaky cavity field beyond the rotating wave approximation (RWA) for general Werner-like initial states. The dynamics have been numerically investigated by using recently derived Lindblad type quantum optical master equation [F. Beaudoin, J.M. Gambetta, A. Blais, Phys. Rev. A **84** (2011) 043832] and compared with the dynamics investigated by *ad hoc* master equation, the so-called standard Lindblad quantum optical master equation. Contrary to previous investigations of the same model in the dissipative regime by using the standard Lindblad quantum optical master equation at ultrastrong coupling regime, we have found that the atom-field steady states are cavity decay rate independent and have a very simple structure determined solely by the overlap of initial atomic state with the decoherence free state which is valid for all coupling regimes. Non-RWA dynamics are found to have remarkable effects on the steady state quantum discord and entanglement that cannot be achieved under RWA conditions, for instance, they can induce steady state entanglement even for the initial states that have no overlap with the decoherence free state.

Finally, we have investigated the quantum phase transitions in the ground states of several critical systems, including transverse field Ising and XY models as well as XY with multiple spin interactions, XXZ and the collective system Lipkin-Meshkov-Glick models, by using different quantumness measures, such as entanglement, quantum discord, measurement-induced disturbance and Bell non-locality. We have basically analyzed the performance of different quantumness measures, especially measurement-induced disturbance and Bell non-locality to detect the critical points of quantum phase transitions. Measurement-induced disturbance is found to detect the first and second order phase transitions present in these critical systems, while it is found to fail to signal the infinite-order phase

transition present in the XXZ model. Remarkably, Bell non-locality is found to detect all the phase transitions, even when quantum and classical correlations are zero for the relevant ground state.

Keywords: Quantum Correlations; Open Quantum Systems; Quantum Phase Transitions.

# ÖZET

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**Doktora, Fizik Bölümü**

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Bu tezde ilk olarak farklı kuantumluk ölçütlerinin, dolaşıklık ve ardı (kuantum geçimsizlik ve ölçüm bağı rahatsızlık) gibi ayrıca Bell lokalsizliğinin, eşevresizlik altında dinamiklerini inceledik. İkinci olarak, ilgilendiğimiz kuantum korelasyon ölçütlerinin kuantum faz geçişlerinin kritik noktalarını tespit etmedeki yeteneklerini çalıştık.

İlk olarak, tek modlu sızdıran kavite ile etkileşen iki özdeş ve özdeş olmayan atomlar arasındaki dolanıklık, kuantum geçimsizliği ve Bell lokalsizliği gibi kuantum korelasyon ölçütlerinin dinamiklerini inceledik. Tek atom bozunumu, kavite bozunma oranı ve atomların ilk hazırlanma durumlarının kuantum korelasyon ölçütlerindeki etkilerini standart Lindblad optikal master denkleminde dönen dalga yaklaşımı (DDY) altında numerik çözümüyle incelenmiştir. Yüksek kuantum geçimsizliğinin kavite bozunum oranı büyüklüğünden bağımsız olarak ortaya çıkar iken, dolaşıklık ve Bell lokalsizliğinin zaman içinde sıfır kaldığı bulunmuştur. Tüm atomların uyarılmış olduğu ilk durumda, kuantum geçimsizliğin enerji kaybı altında belirli zaman aralığında arttığı bulunurken, dolaşıklığın azaldığı gözlenmiştir. Bazı başlangıç ve özdeş atomlar için, kavitedeki bozunumun sistemi yüksek dolanıklık ve kuantum geçimsizliğine sahip kararlı bir duruma sürüklediği ispatlanmıştır.

İkinci kısımda, genel Werner-tipi başlangıç durumları için DDY ardında tek-modlu sızdıran kavite alanı ile rezonans haliyle etkileşen iki özdeş kübit arasındaki dolanıklığı ve kuantum geçimsizliğin dinamiği araştırıldı. Bu dinamik son zamanlarda türetilen Lindblad tipi kuantum optiksel master denklemi kullanılarak [F. Beaudoin, J.M. Gambetta, A. Blais, Phys. Rev. A **84** (2011) 043832] nümerik olarak incelendi ve daha önce dinamiği araştırılan *ad hoc* master denklemiyle, diğer bir adıyla standart Lindblad kuantum optiksel master denklemi, karşılaştırıldı. Şuana kadar ki sızdıran sistemlerde standart Lindblad kuantum optiksel master denklemi kullanılarak yapılan araştırmaların aksine, biz yaptığımız çalışmalarda atom-alan durgun durumlarının kavite bozunum oranından bağımsız olduğunu ve sadece üstüste binen başlangıç durumlarıyla eşevresiz serbest durumlarının belirlediği oldukça basit bir yapı olduğunu gösterilmiştir. DDY olmayan dinamiklerin, DDY şartlarında elde edilemeyen kuantum geçimsizliği ve dolanık üzerinde dikkat çeken etkileri olduğu bulunmuştur. Örneğin, eşevresiz serbest durumlar ile üstüste binmeyen başlangıç durumları için bile durgun durum dolanıklığı oluşturulabilir.

En son olarak, dolaşıklık, kuantum dgeçimsizliği, ölçüm bağı rahatsızlık ve Bell lokalsizliği gibi kuantumluk ölçütlerini kullanarak, Ising, XY, XYT, XXZ ve LMG modellerinin temel durumlarındaki kuantum faz geçişlerini çalıştık. Farklı kuantum korelasyon ölçütlerinin, özellikle ölçüm bağı rahatsızlığının ve Bell lokalsizliğinin, kuantum faz geçişlerinin kritik noktalarını gösterme performansını temel olarak inceledik. Ölçüm bağı rahatsızlık ölçütünün incelediğimiz kritik sistemlerdeki birinci ve ikinci dereeden tüm faz geçişlerinin saptar iken, XXZ modelindeki sonsuz seviyedeki kuantum faz geçişini göstermede başarısız olduğu bulundu. Dikkate şayan bir şekilde, Bell lokalsizliği ölçütünün ilgilendiğimiz temel durumlardaki kuantum ve klasik korelasyonların sıfır olduğu durumlarda bile tüm faz geçişlerinin yakaladığı bulundu.

Anahtar Kelimeler: Kuantum Korelasyonları; Açık Kuantum Sistemleri; Kuantum Faz Geçişleri.

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To My Family . . .

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# CHAPTER 1

## INTRODUCTION

Entanglement, which is a purely quantum mechanical phenomena, states that two or more particles can become somehow quantum correlated so that they always share the same properties even they are separated from each other by large distances. It is a kind of quantum correlation shared by multi-parties and has no classical counterpart. Indeed, entangled states are shown to be the most useful and necessary type of quantum correlation for a set of important quantum information and computation tasks, such as quantum teleportation [1], superdense coding [2] and quantum cryptography [3]. Moreover, the quantum computers will be expected to be directly implemented by the use of the quantum entanglement. On the other hand, the most useful part of entanglement can be determined by the violation of Bell inequalities where this part is often called nonlocal entangled correlations [4, 5, 6]. In fact, the nonlocality as measured by the violation of Bell inequalities is a strong signature of the inseparability of a quantum state and signifies entangled states whose correlations cannot be reproduced by a classical local hidden-variable model. Although, the presence of entanglement does not always imply the violation of Bell inequalities, the violation of Bell inequalities always implies entanglement. Such entangled states violating Bell inequalities are shown to play a central role in some applications in quantum information science, such as to guarantee the safety of device-independent key distribution protocols in quantum cryptography [7, 8, 9]. On the other hand, the absence of entanglement does not always mean the non-existence of all quantum correlations. In fact, there are quantum correlations beyond entanglement that can even exist in some separable mixed states. Several measures have been proposed to characterize and quantify quantum correlations beyond entanglement [10, 11, 12]. One such

a measure of quantum correlations, that is called quantum discord (QD) [10], is shown to be useful in DQC1 (deterministic quantum computation with one quantum bit) [13], Grover search [14] algorithms and remote state preparation [15]. Indeed, QD can be non-zero for some separable mixed states, actually, as shown by Ferraro *et al.* that almost all bipartite quantum states have non-zero quantum discord [16]. Beside original quantum discord, its geometric and entropic versions as well as measurement-induced disturbance have also been introduced very recently as a measure of quantum correlations beyond entanglement for multipartite states [12]. In fact, what these quantumness measures account for a given state and their role in the protocols of quantum information science are still under debate. The general definitions and more explanations for the entanglement, Bell nonlocality and quantumness of correlations beyond entanglement have been discussed in details in Chapter 2.

Maintaining entanglement and other types of quantum correlations is not easy, when the system is in contact with an environment which leads to decoherence. Therefore, the study of the dynamics of different quantumness measures under environmental-induced-noise and finding the conditions for which the quantum correlations last longer or preserved are the most important aspects for the protocols of quantum information and computation processing [17, 18, 19, 20, 21, 22]. In fact, as shown by many groups, entanglement is very fragile and can cease to exist in a finite time under dissipation and dephasing processes. This phenomena is named by Yu and Eberly as entanglement sudden death (ESD) [17]. On the other hand, the quantum correlations beyond entanglement, such as QD, are more robust against the environmental-induced-noise; they decay exponentially or suffer only instant death [18, 19]. In fact, quantum discord compared to entanglement has remarkable behavior under dissipative and decohering systems. Exhibiting noise-free-area for a finite time interval [20], increasing under local memoryless noise acting on only one party of the two qubit system initially in a fully classical state [21] and being maximal for a separable state in the long

time limit under classical memoryless noise(s) [22] are the remarkable examples of these novel behaviors of QD, while entanglement dies very quickly or is not induced for those situations.

The main investigation of quantum optics is the description of the light-matter interaction. The single-mode spin-boson model, the so-called quantum Rabi model, is the simplest possible physical model that describes the interaction of a two-level atom (a qubit) with a quantized single mode electromagnetic field (a harmonic oscillator) [23]. Indeed, this simple model has been used to understand a wide variety of phenomena in quantum optics and condensed matter systems, such as quantum dots, trapped ions, superconducting (artificial) qubits, optical and microwave cavity QED, among others [24, 25, 26, 27, 28, 29, 30, 31, 32]. Despite its old age and numerous investigations over the last few decades, the spectrum of the Rabi model has been given only recently in a closed form as the solution of a polynomial equation with a single variable [33]. On the other hand, Rabi model is quite simplified and can be solved analytically when the excitation non-conserving terms, the so-called counter-rotating terms that are responsible for simultaneously exciting or de-exciting the atom and field, are ignored [34]. This approximation is known as the rotating wave approximation (RWA) which reduces Rabi model to Jaynes-Cummings model and has been widely used in quantum optical settings. The RWA is valid only at weak atom-field coupling strengths and nearly resonant conditions. Indeed, the typical quantum optics experiments are performed at microwave to visible light range of electromagnetic wave spectrum where the atom-field coupling strength is several orders of magnitude smaller than the transition frequencies of the atom and the field, so that RWA is inherently justified for those parameters. Despite being an approximation, the analytical achievements of RWA have successfully explained many novel phenomena and experimental results, such as Rabi oscillations [35], squeezing [36], non-classical [37] and Fock [38] states, collapse and revivals of atomic inversion [39] and entanglement between atoms as well as atom and field sys-

tems [40]. However, with the recent developments in the area of circuit and cavity QED systems [24, 25, 26, 27, 28, 29, 30, 31, 32], ultra and deep strong light-matter couplings became experimentally achievable which makes it necessary to use full Rabi model to explain all the dynamical and statistical properties of these systems.

The discussion of the role of counter-rotating terms on the atomic dynamics has received plenty of attention, recently [41, 42, 43, 44, 45, 46]. Although the effects of these terms are extremely small and averages to zero over a very short time scale for the weak coupling regime, they, nonetheless, cause significant shifts in the dynamics of the system at intermediate to deep interaction strength region. Generation of photons [41] and entanglement [42] from zero excitation initial state, bifurcation in the phase space [43], a fine structure in optical Stern-Gerlach effect [44], Bloch-Siegert shift [45] and chaos in the Jaynes-Cummings model [46] are the remarkable examples of these novel effects and none of them are observed under RWA. On the other hand, the significance of the counter-rotating terms, especially for multipartite and dissipative systems, is still under debate and one of the main focus of this thesis will go along this direction.

Since any realistic quantum system inevitably interacts with the outside world, the environmental-induced-noise leads to qubit relaxation and dephasing as well as cavity relaxation in circuit and cavity QED systems with atom-field interactions. The standard Lindblad quantum optical master equation [47] is shown to be the general form of a Markovian master equation that can describe the irreversible and non-unitary processes of the atom-field system. While deriving the standard dissipators in this master equation, the qubit-field interaction is neglected and therefore separate interactions of the involved parts with the environment are taken into account. For weak atom-field coupling regime where RWA can be safely applied, this master equation is shown to work well and can be used to predict the experimental results in cavity and circuit QED systems. On the other hand, the main error is to use the standard Lindblad master equation without any



formal proof to describe the dephasing and relaxation processes at strong-coupling regime where RWA is no longer valid [41, 48, 49, 50, 51, 52, 53, 54, 55, 56]. In fact, this master equation is the root of unphysical effects at ultrastrong coupling regime, since qubit-field coupling becomes influential [48]; even at zero temperature, the standard master equation with Rabi model produces spurious qubit flipping and creates excess photons in the atom-field system. It was shown in Ref. [48] that it is possible to correct the predictions of Rabi model in the dissipative case by carefully deriving a new Lindblad type dissipator which involves transitions between the eigenstates of the Rabi model. The ameliorated Lindblad master equation given in Ref. [48] is expected to be a new avenue in the studies of strong light-matter interactions including dissipative and multipartite systems and to help understanding the role of counter-rotating terms under dissipation and dephasing processes.

In Chapter 3, we have first investigated the dissipative dynamics of quantum correlations such as quantum discord, entanglement and Bell nonlocality between two qubits in a cavity loss mechanism by using standard Lindblad quantum optical master equation. Very recently, entanglement, Bell nonlocality and quantum discord have been studied in the same setting [57, 58, 59, 60, 61, 62]. However, most of those studies assume identical atoms, or the atom-cavity resonant case, or only one excitation in the system. Here, we have investigated the effect of atom-cavity detuning and cavity decay rate on the corresponding correlation measures for a system which is initially prepared in a separable or a maximally entangled state. We will identify the atoms as identical or unidentical depending on the type of detuning. As we will show in Chapter 3, being the atoms identical or unidentical have a pronounced effect on the dynamics of correlations. The most important results of Chapter 3 as follows. Depending on the initial preparation of the system and the type of the detunings between the frequency of the cavity mode and the transition frequency of the qubits, the induced quantum discord is found to increase for a long time under cavity decay loss mechanism for certain

conditions, while there is no increase in entanglement and Bell nonlocality. In some situations, the quantum discord is induced regardless of the magnitude of cavity decay rate even while the atoms remain separable at all times. Also, the preservation of induced entanglement and quantum discord was observed under strong dissipation for identical atoms and the type of initial states which has a non-vanishing overlap with the decoherence free state.

In Chapter 4, we have extended the setting of Chapter 3 by studying a model of two identical atoms resonantly and strongly interacting with a single-mode leaky cavity field beyond rotating wave approximation. The dynamics of atom-field system and the quantum correlations, entanglement and quantum discord, have been investigated at ultrastrong coupling regime by using recently derived Lindblad type master equation that can describe the cavity losses at strong atom-field coupling regime where RWA is not valid [48]. Under RWA, the dynamics of atom-field system under losses have been investigated thoroughly and the atom-field steady states have a very interesting structure determined solely by the overlap between the initial state and the decoherence free state [63, 64, 65, 66]. On the other hand, the same model has also been investigated under non-RWA conditions by using the standard Lindblad master equation [49, 50, 51, 52]. Since the master equation with standard dissipators cannot explain properly the effect of dissipation at the strong coupling regime, the reported role of the counter-rotating terms should be reconsidered. By using the ameliorated Lindblad master equation given in Ref. [48] and the structure of atom-field steady states under RWA, we provide a clear picture for the role of counter-rotating terms on atom-field steady states and also give their analytical expressions for the general class of Werner-like initial states. We also reexamined some of the main results obtained by the standard Lindblad master equation at strong atom-field coupling regime and revised them by using the results obtained by the ameliorated Lindblad master equation. The role of counter-rotating terms on the dynamics of entanglement and quantum discord shows that the counter-rotating terms can, peculiarly, re-

verse the initial state dependence of the steady state entanglement and quantum discord contrary to the RWA case. For RWA dynamics, the overlap of initial state with the decoherence free state is crucial to obtain a long-lived quantum correlated state under losses [63, 64, 65, 66]. In this chapter, we showed that the counter-rotating term contributions break down this description and lead to long-lived quantum correlated state without this overlap.

Beyond the investigation of the dynamics of quantum correlations under open quantum systems, another fundamental question in quantum information community is the relation between quantum phase transitions (QPT) and quantum correlation measures. Unlike classical phase transitions which occurs at a critical temperature, quantum phase transitions occur at absolute zero temperature as a tuning parameter (a coupling constant or an external parameter) of the system Hamiltonian is changed and are solely due to the quantum fluctuations [67]. The superfluid-Mott insulator transition in a Bose-Einstein condensate [68], superconductor-insulator transitions [69] and the phase transition in spin models at  $T = 0$  [70] are the remarkable examples of the QPTs. Although the quantum and classical phase transitions are both characterized by the specific behavior of correlation length in the system, the quantum correlations, mostly, have no classical counterparts. Several attempts have been made to find a universal quantum correlation measure that can unambiguously signal all the QPTs and to search for a relation between QPTs and quantum correlation measures [71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92]. Indeed, different measures of quantum correlations, such as entanglement [71, 72, 73, 74, 75, 76, 77, 78], quantum discord [79, 80, 81, 82, 83, 84, 85, 86, 87], fidelity [88, 89] and Bell inequalities [90, 91, 92] have been shown to provide the signatures of QPTs. Although not always, the extremal points (minimums or maximums) as well as the behavior of the derivatives of the correlation measures can signal the existence of a critical point (CP). In general, at the first-order QPT point [93], QD and entanglement (measured by concurrence and entanglement of

formation (EoF)) are found to be discontinuous at the critical point [79, 82, 83], while at the second-order QPT, the divergence or discontinuity behavior of the first and second derivatives of QD and entanglement could spotlight the critical points [79, 80, 81]. On the other hand, QD and entanglement are found to show local extreme points at the CPs of infinite-order QPTs [82, 83, 84].

On the other hand, entanglement may fail to measure the quantum correlations of a state when it reaches to a certain level of mixture, especially for distant spin pairs in a spin system and at finite temperatures. In addition, QPTs might involve transitions from unentangled to unentangled ground states. On the other hand, quantum discord, mostly, does not vanish for mixed separable states, actually almost all quantum states have non-zero quantum discord [16]. This feature makes QD more reliable and useful measure compared to entanglement to signal the QPTs. Indeed, QD is able to detect the critical points of QPTs, especially between far neighbors and also at finite temperatures where entanglement is fully absent near the critical points [81, 82, 83, 85]. Moreover, QD can provide more conventional information about the critical points of QPTs than that of the conventional measures (thermodynamical quantities, such as entropy, specific heat, magnetization, etc.); as shown recently QD, in contrast to entanglement and some of the thermodynamics quantities, can spotlight the QPTs, even at finite temperatures [82, 83]. It should be pointed out here that Bell nonlocality as measured by the Clauser-Horne-Shimony-Holt-type (CHSH) inequality or its derivative has been used to detect the quantum phase transitions only in a few model systems, including Heisenberg XXZ [91] and Kitaev-Castelnovo-Chamon models [92]. In these studies, it was shown that CHSH-Bell inequality, although not violated for some cases, exhibits the existence of QPTs present in the models, even for separable states. Moreover, in Ref. [90], the authors investigated only the behavior of the CHSH-Bell function near the Ising-like critical point in the infinite XY model and found a non-ergodic behavior of the nonlocality measure near the phase transition point. However, making a general conclusion about the

ability of the CHSH inequality to detect the CPs of QPTs is hard. On the other hand, although a number of studies indicate a strong relation between the QPT and a discontinuity in QD and entanglement or in their derivatives, so far there is no single correlation measure that can indicate all the QPTs or shows universal behavior at the CPs of QPTs.

In Chapter 5, we have searched for new quantum information theory based indicator of quantum phase transitions, such as measurement-induced disturbance (MID) and the CHSH-Bell function, and compare them with the well known CP-detectors, quantum discord, classical correlations (CC) and entanglement in the ground states of several critical systems, including transverse field Ising and XY models as well as the XY model with multiple spin interactions, the XXZ model and the collective system Lipkin-Meshkov-Glick (LMG) model. The focus of this chapter will be two fold. On one hand, we will compare and contrast the behavior of different quantumness measures as a function of the tuning parameter of the system Hamiltonians, especially for the regions close to the critical points of the QPTs of the system. On the other hand, we will study different critical systems that exhibit first and infinite order QPTs as well as several continuous phase transitions belonging to different universality classes to see whether or not the corresponding correlation measures (particularly MID and the CHSH-Bell function) can signal the abrupt change of the ground state. We will assume both finite and infinite chains as well as the finite size effects of the derivatives of the correlations at the second order QPT points. In fact, by considering a number of models that exhibit several QPTs obeying different universality classes, with different correlation measures, we will shed some light on the search for "universal" correlation measure that can account for all the QPTs. We have shown that although none of the considered correlation measures show a universal behavior for signaling the CPs of QPTs, remarkably, the CHSH-Bell function is found to detect all CPs of QPTs present in the considered models, even when the relevant ground state has neither quantum nor classical correlations.

The thesis is structured as follows. In Chapter 2, we briefly review the quantum correlation measures, entanglement and beyond as well as Bell nonlocality and give their analytical expressions for a general X-structured bipartite density matrix. In Chapters 3 and 4, we analyze the dissipative dynamics of quantum correlations in a cavity loss mechanism under and beyond RWA, respectively. In Chapter 5, the performance of different quantum correlations measures to detect QPTs in several critical systems is discussed. The summary of the important results obtained in this thesis is summarized in Chapter 6.

## CHAPTER 2

### QUANTUMNESS OF CORRELATIONS IN BIPARTITE STATES

#### 2.1 Introduction

In this thesis, we will use different correlation measures, such as entanglement, quantum discord, measurement-induced disturbance and Bell non-locality, to characterize the quantumness of correlations in bipartite qubit systems. Moreover, in the model systems that will be analyzed in detail in next chapters, we will deal with X-structured bipartite density matrices given as:

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \quad (2.1)$$

Such states are important, since they are experimentally produced and include mixed Werner states as well as Bell diagonal states. Therefore, in the following, we will first briefly describe what these correlation measures signify for the "quantumness" of correlations, and then we will give their analytic expressions in terms of density matrix elements for a general two-qubit X-structured density matrix.

#### 2.2 Entanglement

Entanglement has evolved after the paper of EPR [94] as a "spooky action at a distance"; influence of one system to the other instantaneously, even at large distances. Entanglement is broadly accepted as the most useful and necessary

type of quantum correlation for a set of important quantum information and computation tasks. Several measures have been introduced to quantify the entanglement type quantum correlations [95, 96, 97]. Among them, entanglement of formation (EoF) or concurrence (an entanglement monotone) are the most promised ones [96, 97]. Although, they are sometimes disagree on the value of correlations for a given entangled state, they both give zero for a separable state. Originally, EoF is defined as [96, 97]

$$EoF(\rho_{AB}) = \min \sum_j p_j S(\rho_{A(B)}), \quad (2.2)$$

where the minimization is taken over all ensembles of pure states  $\{p_j, |\Psi_j\rangle\}$  for a density matrix  $\rho_{AB} = \sum_j p_j |\Psi_j\rangle \langle \Psi_j|$ . Here  $p_j$  is the probability amplitude,  $\rho_{A,B} = Tr_{B,A} \rho_{AB}$  are the reduced density matrices and  $S(\rho) = -Tr(\rho \log_2 \rho)$  is the von-Neumann entropy. EoF quantifies how many singlets, at least for pure states and asymptotically, are needed per copy to prepare by using the local operators and classical communications. In fact, for a two-qubit system, EoF is shown to be a function of concurrence [96, 97]:

$$EoF(\rho_{AB}) = H \left[ \frac{1 + \sqrt{1 - C(\rho_{AB})^2}}{2} \right], \quad (2.3)$$

where  $H[x] = -x \log_2 x - (1 - x) \log_2 (1 - x)$  is the binary entropy function and  $C(\rho_{AB})$  is the concurrence. When EoF or concurrence is zero, it means that a given bipartite state can be written in terms of the tensor product of states of the subsystems, A and B, i.e.,  $\rho_{AB} = \sum_j \rho_A^j \otimes \rho_B^j$ . They are both normalized to one for maximally entangled (Bell) states. The concurrence is defined as [96, 97]:

$$C(\rho_{AB}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (2.4)$$

where the quantities  $\lambda_i (i = 1, 2, 3, 4)$  are the eigenvalues of the matrix  $R = \rho_{AB}(\sigma_y^A \otimes \sigma_y^B) \rho_{AB}^* (\sigma_y^A \otimes \sigma_y^B)$  in descending order and  $\rho_{AB}^*$  is the conjugate of  $\rho_{AB}$ .



For the X-structured density matrix (2.1), the concurrence can be obtained easily

$$C(\rho_{AB}) = 2 \max\{0, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\}. \quad (2.5)$$

### 2.3 Bell Non-locality

After EPR [94], the idea of entanglement remained just as a concept (theoretical). However, in 1964, Bell inequalities was introduced by John Bell and moved the concept of entanglement from the philosophical debates to the experimental verifications [4]. In fact, the violation of the Bell inequalities describe the part of entanglement (often called non-local correlations) that cannot be produced by local operations on the systems with classical communication between the parts (a local hidden variable model or classical systems). Although the concept of entanglement was considered as the property of states that cannot be exhibited by classical systems, it is entirely true for pure entangled states. On the other hand, as shown by Werner that all entangled mixed states do not necessarily violate the Bell inequalities [5]. Although the interplay between entanglement and non-locality seems to be understood, it is true for pure states, but complicated for mixed states.

Like entanglement measures, several kind of Bell inequalities have been introduced recently. However, unlike entanglement measures, the non-locality measures do not agree with the violation limits of the Bell inequalities. Therefore, I will proceed with the most promise one, introduced by Clauser, Horne, Shimony and Horne, termed CHSH-Bell inequality [6].

For two-qubit case the CHSH-type Bell inequality can be written in the following form [6]

$$CHSH(\rho_{AB}) = \max_{\{O_i, O_j\}} | |\langle O_A O_B \rangle - \langle O_A O'_B \rangle| + \langle O'_A O_B \rangle + \langle O'_A O'_B \rangle | \leq 2, \quad (2.6)$$

where  $\langle O_A O_B \rangle = Tr(\rho_{AB} O_A O_B)$  is the correlation function and  $O_S = \mathbf{O}_S \cdot \boldsymbol{\sigma}_S$ ,

wherein  $\mathbf{O}_S = (\sin \theta_S \cos \phi_S, \sin \theta_S \sin \phi_S, \cos \theta_S)$  is the unit vector,  $\sigma_S = (\sigma_1^S, \sigma_2^S, \sigma_3^S)$  is the Pauli matrices vector and  $O'_S = O_S(\theta'_S, \phi_S)$ . The idea on its definition is as follows: for two particles, A and B, one can choose one direction,  $O_i$  and  $O_j$  to measure simultaneously and independently the observable  $O_S$ . If the measured parts takes two direction, any local and realistic theory gives the Bell inequality. So, the maximum information over all directions defines the Bell measurement given by Eq. (2.6).

In fact by using the Horodecki criteria [98], the maximum of the CHSH-Bell function for bipartite X-states can be obtained easily and it is given as [57, 99]:

$$CHSH(\rho_{AB}) = \max\{B_1, B_2\}, \quad (2.7)$$

where  $B_1 = 2\sqrt{k_1^2 + 4k_2^2}$  and  $B_2 = 4\sqrt{k_2^2 + k_3^2}$ ,  $k_1 = (\rho_{11} + \rho_{44} - \rho_{22} - \rho_{33})$ ,  $k_2 = (|\rho_{14}| + |\rho_{23}|)$  and  $k_3 = (|\rho_{14}| - |\rho_{23}|)$ . As mentioned above when the value of CHSH function exceeds 2 for a given state, it is violated, so no classical local model can reproduce all correlations possessed by the quantum objects A and B.

## 2.4 Quantum Correlations Beyond Entanglement

### 2.4.1 Quantum Discord

The absence of entanglement does not always imply the non-existence of quantumness of correlations. In fact, there exist more-general-than-entanglement-type quantum correlations that can even exist for some mixed separable states. Recently, Ollivier and Zurek introduced the notion, called quantum discord (QD), as a measure of such more-general-than-entanglement-type quantum correlations for bipartite states [10]. Indeed, QD can be non-zero for some mixed separable states. Originally, QD is defined as the difference between the quantum versions of two classically equivalent definitions of mutual information. QD can be given

as

$$QD(\rho_{AB}) = I(\rho_{AB}) - CC(\rho_{AB}), \quad (2.8)$$

where

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

is the quantum mutual information and measures the total correlations. Here  $\rho_{A,B} = \text{Tr}_{B,A} \rho_{AB}$  are the reduced density matrices and  $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$  is the von-Neumann entropy.  $CC(\rho_{AB})$  in Eq. (2.8) is the measure of classical correlations between two subsystems defined as the maximum information extractable from the measurement, for example, on B about A. It is given as [10, 11]:

$$CC(\rho_{AB}) = \max_{\{\hat{\Pi}_i\}} \{S(\hat{\rho}_A) - \sum_i p_i S(\hat{\rho}_{A|i})\}, \quad (2.9)$$

where  $\hat{\rho}_{A|i} = \frac{\text{Tr}_B(\hat{\Pi}_i \hat{\rho}_{AB} \hat{\Pi}_i)}{\text{Tr}_{AB}(\hat{\Pi}_i \hat{\rho}_{AB} \hat{\Pi}_i)}$  and  $\hat{\Pi}_i$  is a set of projectors defined as  $\hat{\Pi}_i = \hat{I} \otimes |i\rangle\langle i|$  ( $i = 1, 2$ ) where  $|1\rangle = \cos \theta |+_z\rangle + e^{i\phi} \sin \theta |-_z\rangle$  and  $|2\rangle = \sin \theta |+_z\rangle - e^{i\phi} \cos \theta |-_z\rangle$  are the orthogonal states and  $|\pm_z\rangle$  are the eigenstates of Pauli z-operator for subsystem B. The projector operators are used to measure subsystem B, corresponding to the outcome  $i$  with probability  $p_i = \text{Tr}_{AB}(\hat{\Pi}_i \hat{\rho}_{AB} \hat{\Pi}_i)$ . Because of the complex maximization procedure in the definition of classical correlations, obtaining an analytic expression of  $CC(\rho_{AB})$  for general states is not an easy task. However, for X-states the analytic expression of  $CC(\rho_{AB})$  and also QD can be obtained [58]. According to the results in Ref. [58], QD and CC are given as:

$$QD(\rho_{AB}) = \min\{Q_1, Q_2\}, \quad CC(\rho_{AB}) = \max\{CC_1, CC_2\}, \quad (2.10)$$

where  $CC_j = H[\rho_{11} + \rho_{22}] - D_j$  and  $Q_j = H[\rho_{11} + \rho_{33}] + \sum_{k=1}^4 \lambda^k \log_2 \lambda^k + D_j$  with  $\lambda^k$  being the eigenvalues of  $\rho_{AB}$  and  $H[x]$  is the binary entropy defined in Sec. 2.1. Here  $D_1 = H[\tau]$  where  $\tau = \left(1 + \sqrt{[1 - 2(\rho_{33} + \rho_{44})]^2 + 4(|\rho_{14}| + |\rho_{23}|)^2}\right) / 2$  and  $D_2 = -\sum_{k=1}^4 \rho_{kk} \log_2 \rho_{kk} - H[\rho_{11} + \rho_{33}]$ . In general, QD and CC are not symmetric

quantities, i.e., their value can depend on the party which projective measurements are performed on. Indeed, for general X states, the order of measurements performed are important since  $S(\rho_A) \neq S(\rho_B)$ . However for the considered model systems in the present thesis, we mostly have  $\rho_{22} = \rho_{33}$  which ensures  $S(\rho_A) = S(\rho_B)$ ; irrespective of whether the measurement is performed locally on the subsystem A or B.

On the other hand, several discord-like quantum correlation measures have been introduced recently based on the Hilbert-Schmidt and entropy distance between a given state and zero-discord state [12]. In fact, they generally give the same qualitative information about a given state as QD can do, so these correlation measure are out of the scope of this thesis.

#### 2.4.2 Measurement Induced Disturbance

Along similar lines, the so-called measurement-induced disturbance was proposed very recently by Luo in order to quantify the more-general-than-entanglement-type quantum correlations [100]. If the mutual information,  $I(\rho_{AB})$ , is accepted to quantify all kind of correlations for a bipartite state, then MID is just the mutual information difference between a given state,  $\rho_{AB}$ , and the closest classical state to  $\rho_{AB}$ . The idea on the definition of MID can be summarized as follows. For a given state  $\rho_{AB}$ , any complete local projective measurements  $\{\Pi_i^A \otimes \Pi_j^B\}$  on the subsystems  $A$  and  $B$  will induce a classical state  $\Pi(\rho_{AB})$ . If the measurements  $\{\Pi_i\}$  are chosen as the spectral resolutions of the reduced states;  $\rho_A = \sum_i p_i^A \Pi_i^A$  and  $\rho_B = \sum_i p_i^B \Pi_i^B$  (where  $p_i^{A,B}$  are the probability amplitudes), the state  $\Pi(\rho_{AB}) = \sum_{i,j} (\Pi_i^A \otimes \Pi_j^B) \rho_{AB} (\Pi_i^A \otimes \Pi_j^B)$  will be the closest classical state to  $\rho_{AB}$ . Indeed, the measurements induced by the spectral resolution of the reduced states will leave the marginal states invariant; in the sense of the least disturbing measurements. Then MID is defined as:

$$MID(\rho_{AB}) = I(\rho_{AB}) - I(\Pi(\rho_{AB})). \quad (2.11)$$

MID is easily computable and can be applied to bipartite higher dimensions. For the density matrix in Eq. (2.1), the closest classical state would be  $\Pi(\rho_{AB}) = \text{diag}\{\rho_{11}, \rho_{22}, \rho_{33}, \rho_{44}\}$ . Since the reduced states are not affected by this measurements and the closest classical state is diagonal, MID reduces to the entropy distance,  $S(x||y) = -\text{tr}(x \log_2 y) - S(x)$ , between the given state and the closest classical state to  $\rho_{AB}$ :

$$\begin{aligned} \text{MID}(\rho_{AB}) &= S(\rho_{AB} || \Pi(\rho_{AB})) \\ &= \sum_{k=1}^4 \lambda^k \log_2 \lambda^k - \sum_{k=1}^4 \rho_{kk} \log_2 \rho_{kk}, \end{aligned} \quad (2.12)$$

where  $\lambda^k$  are the eigenvalues of  $\rho_{AB}$ .

Although the ideas on the definitions of the considered correlation measures are quite different, EoF, MID and QD are equal to each other for pure states and CHSH Bell inequality is always violated for entangled pure states. However, the relation is complicated for mixed states; MID and QD may give non-zero values for some mixed separable states. On the other hand, MID can be nonzero, even maximal for purely classical states (where  $QD = 0$ ). Therefore, MID is sometimes regarded as an unfaithful and non-refined measure of nonclassical correlations in bipartite states [101, 102, 103]. CHSH Bell inequality is generally violated for mixed states having high entanglement.

# CHAPTER 3

## QUANTUM CORRELATIONS BETWEEN IDENTICAL AND UNIDENTICAL ATOMS IN A DISSIPATIVE ENVIRONMENT

### 3.1 Introduction

As discussed in the previous section, entanglement, non-locality and the quantum correlations beyond entanglement are all different quantifiers of quantum correlations for mixed states but are somehow related features for pure states. Since any realistic system inevitably interacts with the environment and such interaction drives the coherent state to an incoherent one, protecting the lifetime of the correlations from detrimental environmental noise is one of the fundamental obstacle. Therefore, it would be very desirable to investigate the dynamics of different quantum correlations under open quantum systems. Toward that goal, in the present chapter, we will investigate the dynamics of quantum correlations such as quantum discord, entanglement and Bell non-locality between two qubits interacting with a single mode leaky cavity field (dissipation). Our aim is to investigate the effect of atom-cavity detuning and cavity decay rate on the dynamics of corresponding correlation measures for a system which is initially prepared in a product or a maximally quantum-correlated state. The dynamics of the atom-field system will be investigated by numerically solving the standard Lindblad quantum optical master equation under rotating wave approximation (RWA) which is valid only at weak coupling and nearly resonant conditions. The extension of the same system beyond RWA will be investigated in the next chapter.

### 3.2 The Model

Here we consider two two-level atoms with excited states  $|e_A\rangle, |e_B\rangle$  and ground states  $|g_A\rangle, |g_B\rangle$  interacting with a single-mode cavity field of frequency  $\omega_r$ . The Hamiltonian of this system under RWA is given as [104]:

$$H = \frac{\hbar\omega_A}{2}(|e_A\rangle\langle e_A| - |g_A\rangle\langle g_A|) + \frac{\hbar\omega_B}{2}(|e_B\rangle\langle e_B| - |g_B\rangle\langle g_B|) + \omega_r a^\dagger a \\ + \hbar g_1(a |e_A\rangle\langle g_A| + a^\dagger |g_A\rangle\langle e_A|) + \hbar g_2(a |e_B\rangle\langle g_B| + a^\dagger |g_B\rangle\langle e_B|), \quad (3.1)$$

where  $\omega_A$  and  $\omega_B$  are the transition frequencies for the atoms  $A$  and  $B$ ,  $a$  and  $a^\dagger$  are the annihilation and creation operators for the field and  $g_1$  and  $g_2$  are the coupling constants to the cavity mode for the atoms  $A$  and  $B$ , respectively.

When maximum two-photon is considered to be present in the cavity (i.e., the total excitation is equal to two) and if there is no dissipation in the cavity modes, the closed form of the total state vector at any time may be written as [104, 105]:

$$|\Psi(t)\rangle = C_1 |e_A, e_B, 0\rangle + C_2 |e_A, g_B, 1\rangle + C_3 |g_A, e_B, 1\rangle + C_4 |g_A, g_B, 2\rangle, \quad (3.2)$$

where  $C_i$  ( $i = 1, 2, 3, 4$ ) are the time-dependent probability amplitudes. For example,  $|g_A, g_B, 2\rangle$  represents that the atoms are in their ground states while there are two photons inside the cavity. For this case, the dynamics of the atoms and single mode cavity field can be solved by using time-dependent Schrödinger equation and recently many studies have been reported for this case [104, 105, 106]. On the other hand, when cavity decay is taken into account and since the cavity decay only changes the number of photons inside the cavity, the atoms and the cavity mode can be found in any of the states [104, 107]:  $|e_A, e_B, 0\rangle, |e_A, g_B, 1\rangle, |e_A, g_B, 0\rangle, |g_A, e_B, 1\rangle, |g_A, e_B, 0\rangle, |g_A, g_B, 2\rangle, |g_A, g_B, 1\rangle, |g_A, g_B, 0\rangle$ . Then the density matrix of the system of two atoms interacting with a single mode cavity field will

evolve according to the master equation [104, 108]:

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] - \frac{\kappa}{2} (a^\dagger a \rho - 2a \rho a^\dagger + \rho a^\dagger a). \quad (3.3)$$

We should stress here that the details for the derivation of the master equation can be found in Ref. [109]. Here the first term in Eq. (3.3) is generated by the Hamiltonian given in Eq. (3.1) and represents the coherent evolution, while the other term denotes the incoherent dynamics;  $\kappa$  is the rate of loss of photons from the cavity mode. By setting  $\{|1\rangle \equiv |e_A, e_B, 0\rangle, |2\rangle \equiv |e_A, g_B, 1\rangle, |3\rangle \equiv |e_A, g_B, 0\rangle, |4\rangle \equiv |g_A, e_B, 1\rangle, |5\rangle \equiv |g_A, e_B, 0\rangle, |6\rangle \equiv |g_A, g_B, 2\rangle, |7\rangle \equiv |g_A, g_B, 1\rangle, |8\rangle \equiv |g_A, g_B, 0\rangle\}$  as a basis and considering the Eqs. (3.1) and (3.3), the differential equations governing the dynamics of two atoms and the single mode cavity field can be obtained easily. In our analysis we will restrict ourselves for the initial states:  $\rho(0) = |e_A, e_B, 0\rangle \langle e_A, e_B, 0|$ ,  $\rho(0) = |e_A, g_B, 1\rangle \langle e_A, g_B, 1|$  and  $\rho(0) = |\Psi(0)\rangle \langle \Psi(0)|$ , where  $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|g_A, e_B, 1\rangle \pm |e_A, g_B, 1\rangle)$ . It is straightforward to obtain a closed set of first order coupled differential equations from the master equation for the considered initial states. We note for these initial states the diagonal elements and some of the off-diagonal elements form a closed evolution. They can be listed as:

$$\begin{aligned} \dot{\rho}_{1,1} &= ig_1(\rho_{1,4} - \rho_{4,1}) + ig_2(\rho_{1,2} - \rho_{2,1}), \\ \dot{\rho}_{2,2} &= i\sqrt{2}g_1(\rho_{2,6} - \rho_{6,2}) + ig_2(\rho_{2,1} - \rho_{1,2}) - \kappa\rho_{2,2}, \\ \dot{\rho}_{3,3} &= ig_1(\rho_{3,7} - \rho_{7,3}) + \kappa\rho_{2,2} \\ \dot{\rho}_{4,4} &= ig_1(\rho_{4,1} - \rho_{1,4}) + i\sqrt{2}g_2(\rho_{4,6} - \rho_{6,4}) - \kappa\rho_{4,4}, \\ \dot{\rho}_{5,5} &= ig_2(\rho_{5,7} - \rho_{7,5}) + \kappa\rho_{4,4}, \\ \dot{\rho}_{6,6} &= i\sqrt{2}g_1(\rho_{6,2} - \rho_{2,6}) + i\sqrt{2}g_2(\rho_{6,4} - \rho_{4,6}) - 2\kappa\rho_{6,6}, \\ \dot{\rho}_{7,7} &= ig_1(\rho_{7,3} - \rho_{3,7}) + ig_2(\rho_{7,5} - \rho_{5,7}) + \kappa(2\rho_{6,6} - \rho_{7,7}), \\ \dot{\rho}_{8,8} &= \kappa\rho_{7,7}, \\ \dot{\rho}_{1,2} &= -i\delta_2\rho_{1,2} + ig_1(\sqrt{2}\rho_{1,6} - \rho_{4,2}) + ig_2(\rho_{1,1} - \rho_{2,2}) - \frac{1}{2}\kappa\rho_{1,2}, \end{aligned}$$



$$\begin{aligned}
\dot{\rho}_{1,4} &= -i\delta_1\rho_{1,4} + ig_1(\rho_{1,1} - \rho_{4,4}) + ig_2(\sqrt{2}\rho_{1,6} - \rho_{2,4}) - \frac{1}{2}\kappa\rho_{1,4}, \\
\dot{\rho}_{1,6} &= -i(\delta_1 + \delta_2)\rho_{1,6} + ig_1(\sqrt{2}\rho_{1,2} - \rho_{4,6}) + ig_2(\sqrt{2}\rho_{1,4} - \rho_{2,6}) - \kappa\rho_{1,6}, \\
\dot{\rho}_{2,4} &= -i(\delta_1 - \delta_2)\rho_{2,4} + ig_1(\rho_{2,1} - \sqrt{2}\rho_{6,4}) + ig_2(\sqrt{2}\rho_{2,6} - \rho_{1,4}) - \kappa\rho_{2,4}, \\
\dot{\rho}_{2,6} &= -i\delta_1\rho_{2,6} + i\sqrt{2}g_1(\rho_{2,2} - \rho_{6,6}) + ig_2(\sqrt{2}\rho_{2,4} - \rho_{1,6}) - \frac{3}{2}\kappa\rho_{2,6}, \\
\dot{\rho}_{3,5} &= -i(\delta_1 - \delta_2)\rho_{3,5} - ig_1\rho_{7,5} + ig_2\rho_{3,7} + \kappa\rho_{2,4}, \\
\dot{\rho}_{3,7} &= -i\delta_1\rho_{3,7} + ig_1(\rho_{3,3} - \rho_{7,7}) + ig_2\rho_{3,5} + \frac{\kappa}{2}(2\sqrt{2}\rho_{2,6} - \rho_{3,7}), \\
\dot{\rho}_{4,6} &= -i\delta_2\rho_{4,6} + ig_1(\sqrt{2}\rho_{4,2} - \rho_{1,6}) + i\sqrt{2}g_2(\rho_{4,4} - \rho_{6,6}) - \frac{3}{2}\kappa\rho_{4,6}, \\
\dot{\rho}_{5,7} &= -i\delta_2\rho_{5,7} + ig_1\rho_{5,3} + ig_2(\rho_{5,5} - \rho_{7,7}) + \frac{\kappa}{2}(2\sqrt{2}\rho_{4,6} - \rho_{5,7}), \tag{3.4}
\end{aligned}$$

where  $\delta_1 = \omega_A - \omega_r$  and  $\delta_2 = \omega_B - \omega_r$  are called the detunings and describes the frequency difference between the energy levels of atoms and the cavity mode,  $\rho_{i,j} = \langle i | \rho | j \rangle$  are the density matrix elements of the atom-field density matrix and  $\rho_{i,j} = \rho_{j,i}^*$ . We will consider two particular detuning types: for  $\delta_1 = \delta_2 = \delta$  the atoms will be called identical since the transition frequencies of the atoms are the same and the atoms will be called unidentical for  $\delta_1 = -\delta_2 = \delta$  case for which the individual atom transition frequencies are different [105]. In practice, the detunings might be slightly different due to broadening, however, in this chapter, we will consider relatively large values of  $\delta$ . One should note that it is quite difficult to obtain an analytic solution for this set of coupled differential equations, therefore we will solve these equations numerically for the considered initial states.

The density matrix of the two atoms is obtained by taking a partial trace over the cavity degrees of freedom:  $\rho_{AB}(t) = Tr_c\{\rho\} = \sum_{n=0}^2 \langle n | \rho | n \rangle$  and used to calculate the quantum correlations as measured by CHSH-Bell inequality, concurrence and quantum discord between two atoms interacting with a single mode cavity field. For the considered initial states whose dynamics is given by Eq. (3.4), the reduced density matrix of the atoms in the two-qubit standard basis  $\{|1\rangle \equiv |e_A, e_B\rangle, |2\rangle \equiv |e_A, g_B\rangle, |3\rangle \equiv |g_A, e_B\rangle, |4\rangle \equiv |g_A, g_B\rangle\}$  can be calculated

as:

$$\rho_{AB}(t) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & e & 0 \\ o & e^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \quad (3.5)$$

where  $a = \rho_{1,1}$ ,  $b = \rho_{2,2} + \rho_{3,3}$ ,  $c = \rho_{4,4} + \rho_{5,5}$ ,  $d = \rho_{6,6} + \rho_{7,7} + \rho_{8,8}$  and  $e = \rho_{2,4} + \rho_{3,5}$ . It should be noted that the density matrix in Eq. (3.5) has an X-form and the dynamics preserve its form. The concurrence, CHSH-Bell inequality and QD for this type of states can be calculated easily by using Eqs. (2.5), (2.7) and (2.10), respectively.

### 3.3 Results

#### 3.3.1 Effects of Cavity Damping-Initially Product States

In this section, we will examine the effect of cavity decay rate,  $\kappa$ , on the quantum correlations as measured by concurrence, CHSH-Bell inequality and quantum discord. To do this we will solve the differential equations in Eq. (3.4) numerically for the initial states:  $\rho(0) = |e_A, e_B, 0\rangle \langle e_A, e_B, 0|$ ,  $\rho(0) = |e_A, g_B, 1\rangle \langle e_A, g_B, 1|$  and  $\rho(0) = |\Psi(0)\rangle \langle \Psi(0)|$ , where  $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|g_A, e_B, 1\rangle \pm |e_A, g_B, 1\rangle)$  and we will set  $g_1 = g_2 = g$  (symmetric coupling),  $\delta_1 = \delta_2 = 5g$  for identical atoms and  $\delta_1 = -\delta_2 = 5g$  for unidentical atoms. One should note that the detunings are chosen somewhat large because at higher detunings the qubit-cavity energy exchange is low [64, 104, 105] and a high-degree of cavity-induced quantum correlations can be created [64].

We first consider the dynamics of concurrence and quantum discord for initially separable state  $\rho(0) = |e_A, e_B, 0\rangle \langle e_A, e_B, 0|$  and display  $C$  and QD versus dimensionless time,  $gt$  in Figs. 3.1 and 3.2 for unidentical and identical atoms, respectively. For this initial state neither non-zero concurrence nor violation in

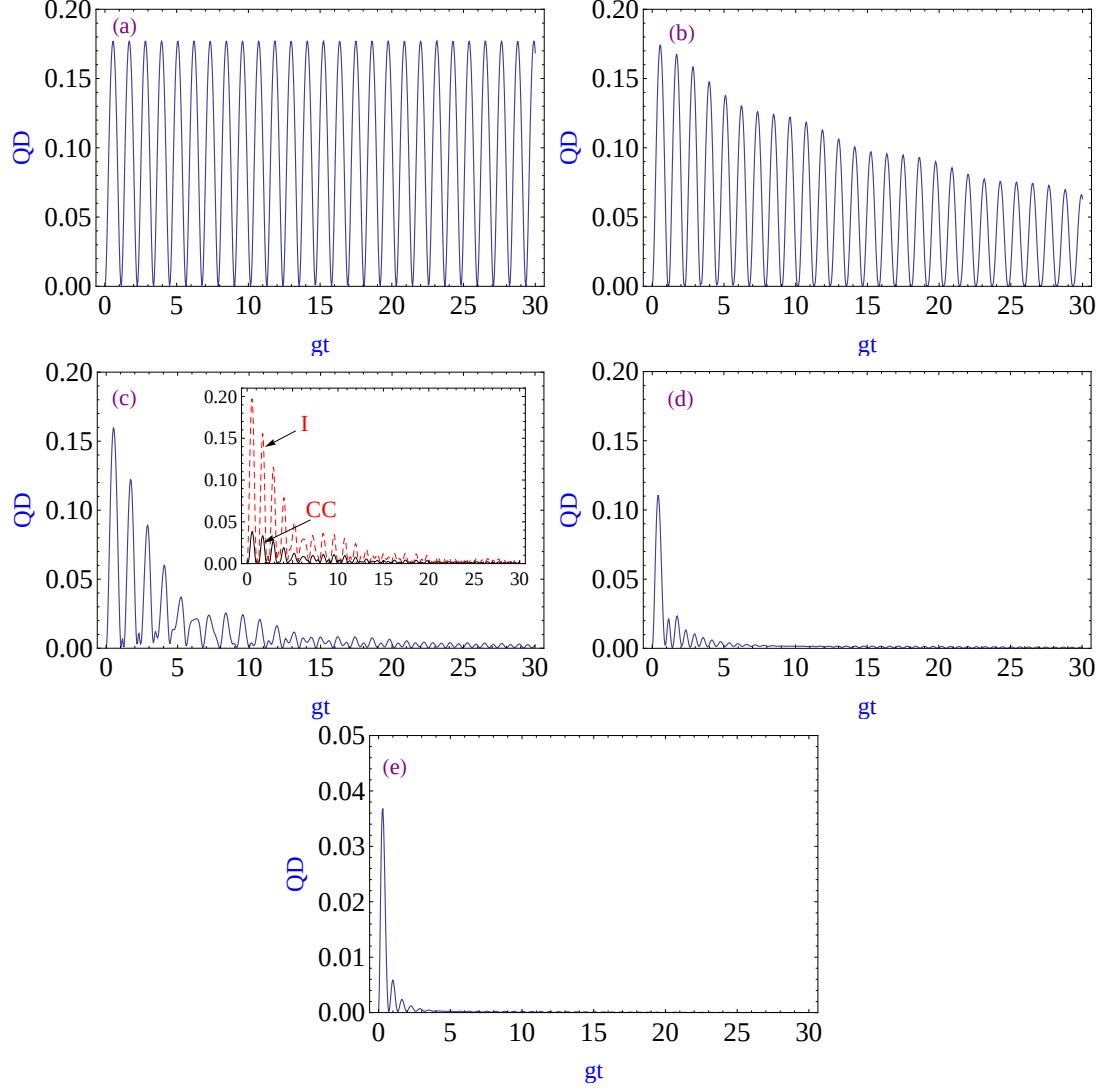


Figure 3.1: Quantum discord versus  $gt$  for unidentical atoms ( $\delta_1 = -\delta_2 = 5g$ ) and  $\rho(0) = |e_A, e_B, 0\rangle \langle e_A, e_B, 0|$  initial state for the decay rates: (a)  $\kappa = 0$ , (b)  $\kappa = 0.02g$ , (c)  $\kappa = 0.2g$ , (d)  $\kappa = 2.0g$  and (e)  $\kappa = 20g$ . Note that under these conditions entanglement and Bell non-locality are not induced, thus these are not plotted here and the inset in (c) represents the total  $I$  and classical  $CC$  correlations versus  $gt$  for  $\kappa = 0.2g$ .

Bell inequality ever occurs for unidentical qubits, so their dynamics are not plotted in Fig. 3.1. However, this does not indicate the loss of quantum correlations; QD induced by the atom-field interaction oscillates nearly between 0 and 0.18 for  $\kappa = 0$  (Fig. 3.1(a)). A similar result is found by Sun *et al.*, in Ref. [110] where the authors studied the entanglement and quantum discord between two qubits which are independently coupled to a common environment modeled as an Ising spin chain. It can be seen from the analysis of Figs. 3.1(b)-3.1(e) that QD is cre-

ated independently of the cavity decay rate, but its maximum value depends on the decay rate  $\kappa$  inversely and the effect of cavity decay on the dynamics of QD is to create a damped oscillatory behavior; damping in QD is proportional to the cavity decay rate. Also, in order to understand the nature of non-zero quantum discord, we have plotted the total correlation,  $I$ , and classical correlation,  $CC$ , for  $\kappa = 0.2g$  in the inset of Fig. 3.1(c). It should be noted that for this initial state, the total as well as classical correlations are also induced and  $I$  is always greater than  $CC$  as a consequence non-zero quantum discord is present. This inset also demonstrates that  $I$  and  $CC$  are also damped oscillatory as QD. Correlations in the system are gradually lost to the reservoir because of the cavity decay as found in the system studied in Ref. [59].

We have calculated the dynamics of CHSH inequality, concurrence and quantum discord for the same initial state  $\rho(0) = |e_A, e_B, 0\rangle \langle e_A, e_B, 0|$  for the identical atoms ( $\delta_1 = \delta_2 = 5g$ ) case to understand the effect of the type of detuning for the same leaky cavity and plot the resulting QD and  $C$  as a function of  $gt$  in Figs. 3.2(a)-3.2(e) and insets of Figs. 3.2(a)-3.2(c), respectively. As can be seen from the insets of Figs. 3.2(a)-3.2(c), non-zero entanglement, although small, is created which decays to zero as cavity decay rate is increased and the induced entanglement has no dynamics for higher values of  $\kappa$  ( $\kappa = 2g$  and  $\kappa = 20g$ ). Bell inequality is found to be not violated for any value of  $\kappa$ , which is expected because entanglement is low and generally Bell inequality violation is shown to be a hallmark of high entanglement [60, 61, 111], so CHSH is not displayed in Fig. 3.2. One should note the difference in the dynamics of QD in the leaky cavity for unidentical and identical atom cases (Figs. 3.1 and 3.2). While QD has an oscillatory decay type dynamics for the unidentical atoms (Fig. 3.1), it increases until the time  $t = t_{max}$  and then decays for the identical atoms. Also, for the same decay parameters, QD is found to have a much longer lifetime for the identical atoms. The higher leakage in the cavity decreases  $t_{max}$  and the lifetime of QD after the time  $t = t_{max}$ . On the other hand,  $\kappa$  has no appreciable

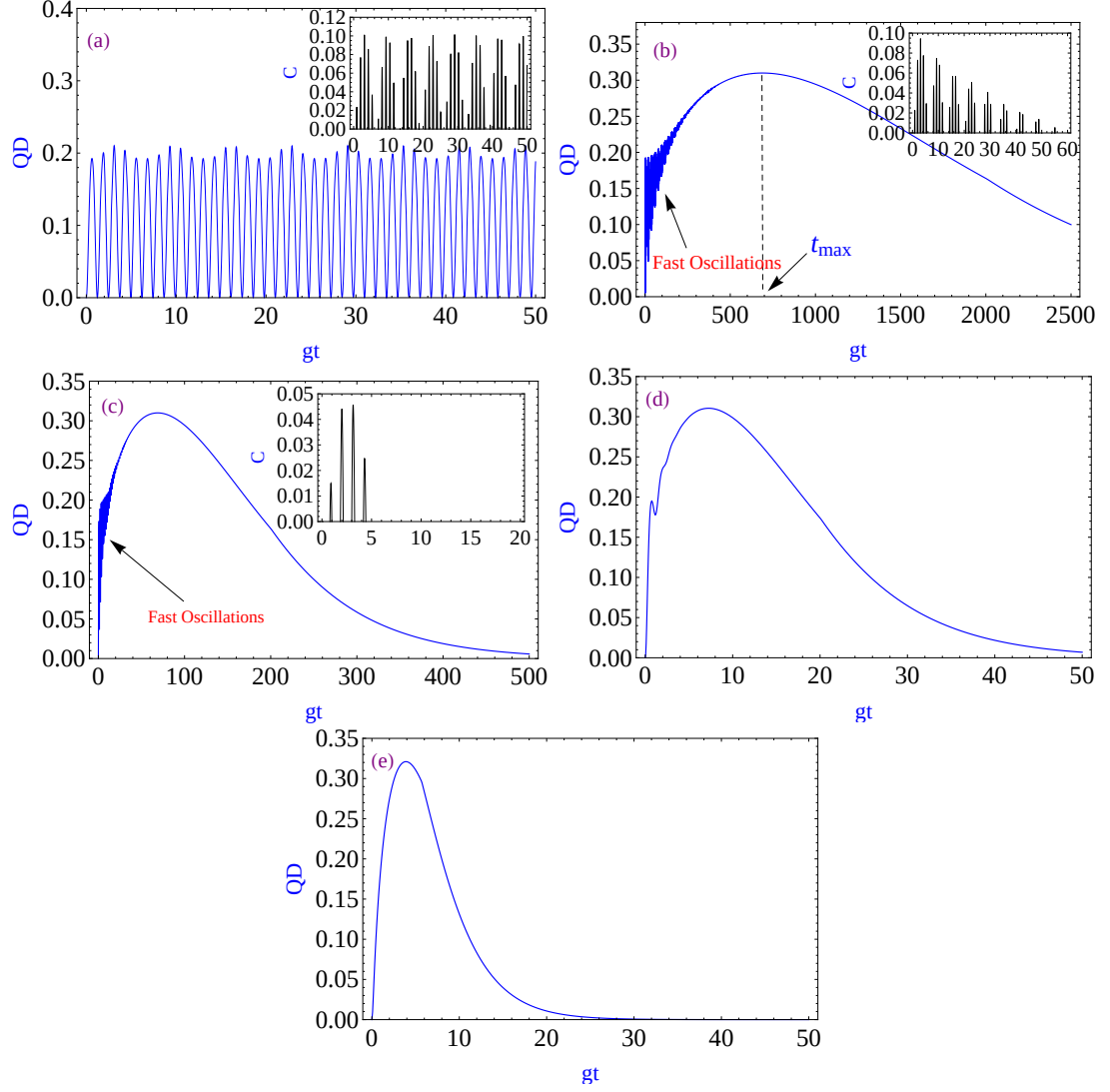


Figure 3.2: Quantum discord and concurrence (insets in (a), (b) and (c)) versus  $gt$  for identical atoms ( $\delta_1 = \delta_2 = 5g$ ) and  $\rho(0) = |e_A, e_B, 0\rangle \langle e_A, e_B, 0|$  initial state for the decay rates: (a)  $\kappa = 0$ , (b)  $\kappa = 0.02g$ , (c)  $\kappa = 0.2g$ , (d)  $\kappa = 2.0g$  and (e)  $\kappa = 20g$ . Note that under these conditions Bell non-locality is not induced, and for  $\kappa = 2g$  and  $\kappa = 20g$ ,  $C$  has no dynamics as well, thus they are not plotted here. The insets are plotted under the same conditions as QD.

effect on the maximum value of QD. Also note that the fast oscillations in QD are gradually smoothed for larger values of  $\kappa$  and longer times as can be seen from Figs. 3.2(b)-3.2(e).

Now we consider another initial separable state of the form  $|\Psi(0)\rangle = |e_A, g_B, 1\rangle$  which has one excitation in the field and the other excitation in one of the atoms to see the effect of initial partition of the excitation on the dynamics of the quantum correlations. The time dependent of CHSH-inequality, concurrence and quantum

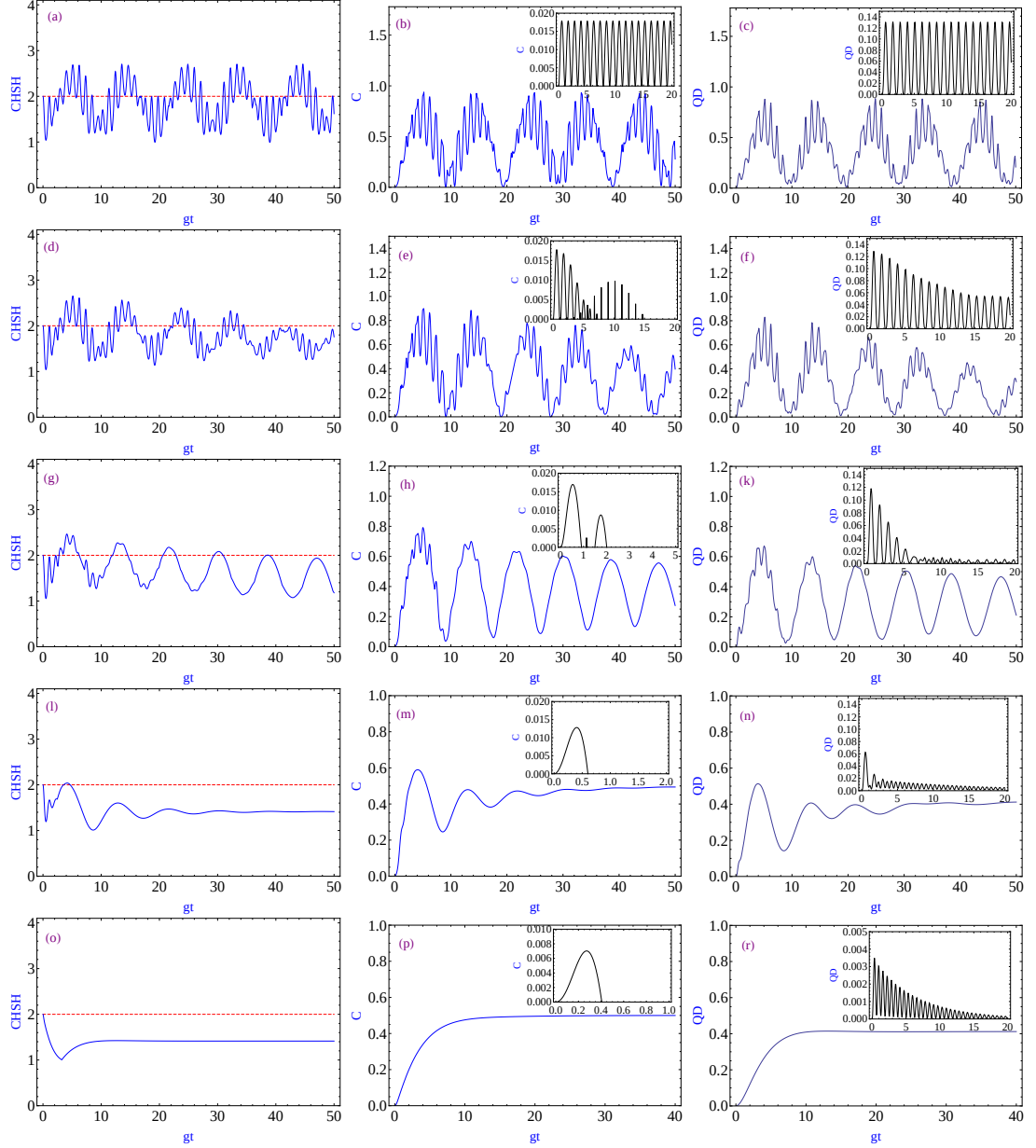


Figure 3.3: CHSH inequality (left plots), concurrence (middle plots) and quantum discord (right plots) versus  $gt$  for  $\rho(0) = |e_A, g_B, 1\rangle \langle e_A, g_B, 1|$  initial state and  $\kappa = 0$  ((a), (b), (c)),  $\kappa = 0.02g$  ((d), (e), (f)),  $\kappa = 0.2g$  ((g), (h), (k)),  $\kappa = 2g$  ((l), (m), (n)) and  $\kappa = 20g$  ((o), (p), (r)). Here the main subfigures are for identical atoms with  $\delta_1 = \delta_2 = 5g$ , while the insets are for unidentical atoms with  $\delta_1 = -\delta_2 = 5g$  under the same decay rate parameters. Note that for unidentical atoms CHSH-Bell inequality is not violated, so they are not reported here.

discord are displayed in the first, second and third columns of Fig. 3.3, respectively. In these figures, insets show the dynamics for unidentical atoms ( $\delta_1 = -\delta_2 = 5g$ ) while main sub-figures are for the identical atoms ( $\delta_1 = \delta_2 = 5g$ ). Inspection of sub-figures demonstrates the pronounced effect of qubits being iden-

tical or unidentical. For identical atoms, the atom-field interaction induces high values of  $C$  and QD and Bell inequality is also violated at the high values of entanglement for a high-quality cavity (Figs. 3.3(a), 3.3(b) and 3.3(c)). In the case of unidentical atoms,  $C$  and QD are also created, but their maximum are small; 0.018 and 0.13, respectively, and CHSH Bell-inequality is not violated. When unidentical atoms interact with a low-quality cavity, entanglement can cease to exist very quickly and QD shows a fast oscillatory decaying behavior (see the insets in Figs. 3.3(e)-3.3(r)) which is very similar to what we have found for  $\rho(0) = |e_A, e_B, 0\rangle \langle e_A, e_B, 0|$  initial state. On the other hand, this situation is completely and significantly different in the case of identical atoms. Although CHSH-Bell inequality goes below 2 very quickly for  $0 < \kappa \leq 2.0g$  and no longer is violated for  $\kappa > 2.0g$ , the dynamics of entanglement and quantum discord are very similar that both approach to constant values in an oscillatory manner; the oscillations die down as the cavity decay rate is increased. In effect, cavity decay entangles initially separable state and the resulting entanglement as well as quantum discord are robust. In many studies, the dissipation of cavity fields were found to play a constructive role in the generation of quantum correlations as measured by concurrence and quantum discord [62, 63, 64, 112, 113, 114, 115]. In fact, the long-lived quantum correlated state stems from the non-zero overlap of initial state with the dark state which will be discussed in details in the next chapter. On the other hand, we can study the steady state of the dynamics to understand this situation by solving  $\dot{\rho} = 0$  in Eq. (3.3). For  $g_1 = g_2 = 1$ ,  $\kappa > 0$  and  $\delta_1 = \delta_2 = \delta$ , the  $\dot{\rho} = 0$  equation has a solution independent of  $\kappa$  and  $\delta$  and the atom-atom reduced density matrix is given as:  $\rho_{ss} = \frac{1}{4}(|e_A, g_B\rangle \langle e_A, g_B| + |g_A, e_B\rangle \langle g_A, e_B|) - \frac{1}{4}(|e_A, g_B\rangle \langle g_A, e_B| + |g_A, e_B\rangle \langle e_A, g_B|) + \frac{1}{2}|g_A, g_B\rangle \langle g_A, g_B|$ . The concurrence for this state is 0.50 and quantum discord of it can be calculated from Eq. (2.10) to be 0.412 which are seen clearly from Figs. 3.3(p) and 3.3(r). Also, we want to emphasize that the dynamics of the considered correlation measures between identical atoms for the initial state  $|\Psi(0)\rangle = |e_A, g_B, 1\rangle$  is the same as

that for  $|\Psi(0)\rangle = |g_A, e_B, 1\rangle$ , since we assume symmetric coupling (i.e.,  $g_1 = g_2$ ) between the atoms and the single mode cavity field.

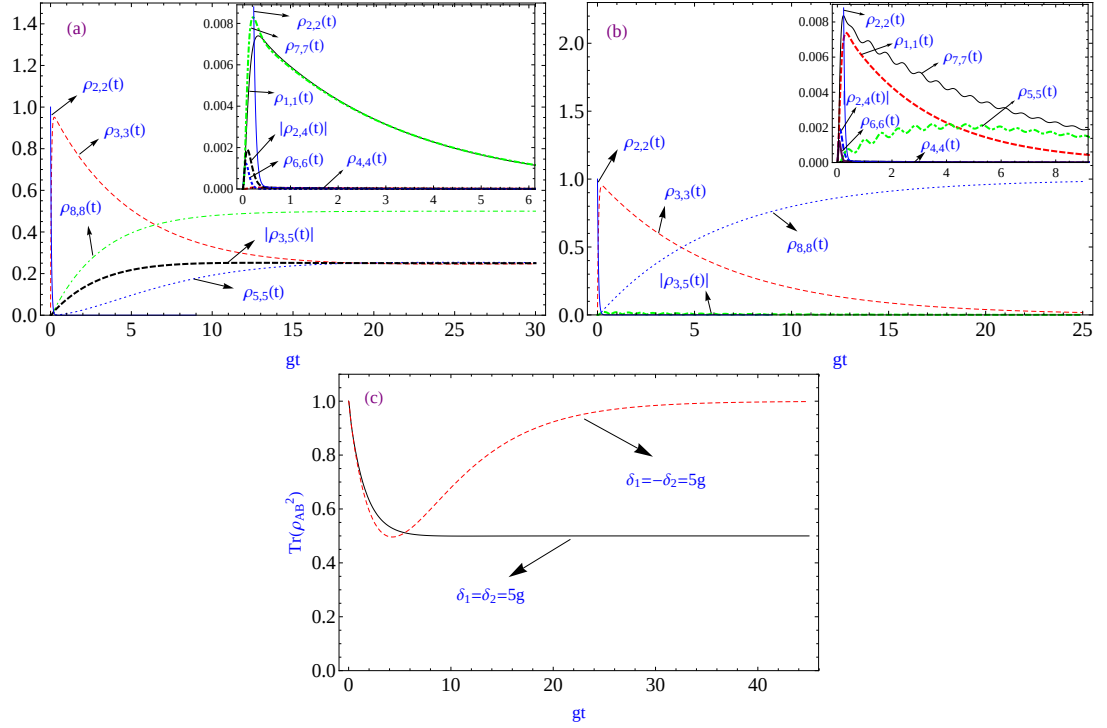


Figure 3.4: The populations  $\rho_{i,i}(t)$  ( $i = 1, 2, \dots, 8$ ) and the absolute value of the coherences  $\rho_{2,4}(t), \rho_{3,5}(t)$  given by Eq. (3.4) versus  $gt$  for the initial state  $\rho(0) = |e_A, g_B, 1\rangle \langle e_A, g_B, 1|$  and  $\kappa = 20g$  for identical ( $\delta_1 = \delta_2 = 5g$ ) (a) and unidentical ( $\delta_1 = -\delta_2 = 5g$ ) (b) atoms. (c) shows the atomic purity ( $\text{Tr} \rho_{AB}^2$ ) for the density matrix Eq. (3.5) versus  $gt$  for identical and unidentical atoms. Here the insets are for small time regions.

To further elucidate the asymptotic constant entanglement and quantum discord for the initial state  $\rho(0) = |e_A, g_B, 1\rangle \langle e_A, g_B, 1|$ , we display the time dependence of the density matrix elements of the atom-field system given by Eq. (3.4) and the purity of the atomic subsystem for the density matrix Eq. (3.5) for identical and unidentical detuning cases in Figs. 3.4(a)-3.4(c) for  $\kappa = 20g$ . As can be seen from Fig. 4(a), all populations, except  $\rho_{3,3}, \rho_{5,5}$  and  $\rho_{8,8}$ , become zero for a short time while  $\rho_{8,8}$  goes to  $1/2$  and  $\rho_{3,3} = \rho_{5,5}$  go to  $1/4$  asymptotically. The key mechanism for the creation of the entanglement and QD in this case seems to be the coherence  $\rho_{3,5}$  generated by the interaction of the atom with the dissipative cavity. The absolute value of  $\rho_{3,5}$  also reaches to  $1/4$  asymptotically as can be seen from the Fig. 3.4(a). The evolved state is a trapping state which



has the property that the probabilities of the different decay channels interfere destructively and the state is robust against the cavity loss mechanism [116, 117]. Generation or protecting entanglement by population trapping has been discussed by a number of groups before [63, 64, 116, 117, 118]. The magnitude of  $\kappa$  has no effect on the long time limit of  $\rho_{3,3}$ ,  $\rho_{5,5}$  and  $\rho_{8,8}$ , it only affects how long other population components can be nonzero. The dynamics of atomic purity (which is equal to 1 for a pure state and 1/4 for a maximally mixed state in four dimensional Hilbert space) displayed in Fig. 3.4(c) as the solid line for  $\delta_1 = \delta_2 = 5g$  indicates that the state approaches to a partially mixed state which has high entanglement and quantum discord as indicated before in Figs. 3.3(p) and 3.3(r). For unidentical atoms, the short time dynamics is similar to the identical detuning case that all states are populated for a short time, but the asymptotic state is now  $\rho(t \rightarrow \infty) = |g_A, g_B, 0\rangle \langle g_A, g_B, 0|$  as can be seen from Fig. 3.4(b). The atomic purity (dotted line in Fig. 3.4(c)) indicates that the state of the atoms firstly goes to a mixed state in a short time and then approaches to the pure state  $\rho = |g_A, g_B\rangle \langle g_A, g_B|$  which is the reason of the loss of correlations between unidentical atoms in the asymptotic limit.

### 3.3.2 *Effects of Cavity Decay-Maximally Quantum-Correlated Initial State*

In this subsection, we will investigate the effect of initial correlations in the atom-atom part of the state on the dynamics of quantum correlations.

We take  $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|g_A, e_B\rangle + |e_A, g_B\rangle) \otimes |1\rangle$  as the initial state which is the maximally entangled Bell state for the atomic system. QD and Bell inequality violation are also maximal for this initial state. We display the considered correlation measures as a function of  $gt$  for several cavity decay rates for unidentical (Fig. 3.5) and identical (Fig. 3.6) atoms. At  $\kappa = 0$ , all correlations show oscillatory behavior; for the unidentical atoms CHSH,  $C$  and QD oscillate nearly in the ranges  $[2\sqrt{2}, 2.2]$ ,  $[1.0, 0.63]$  and  $[1.0, 0.58]$ , respectively (Fig. 3.5). On

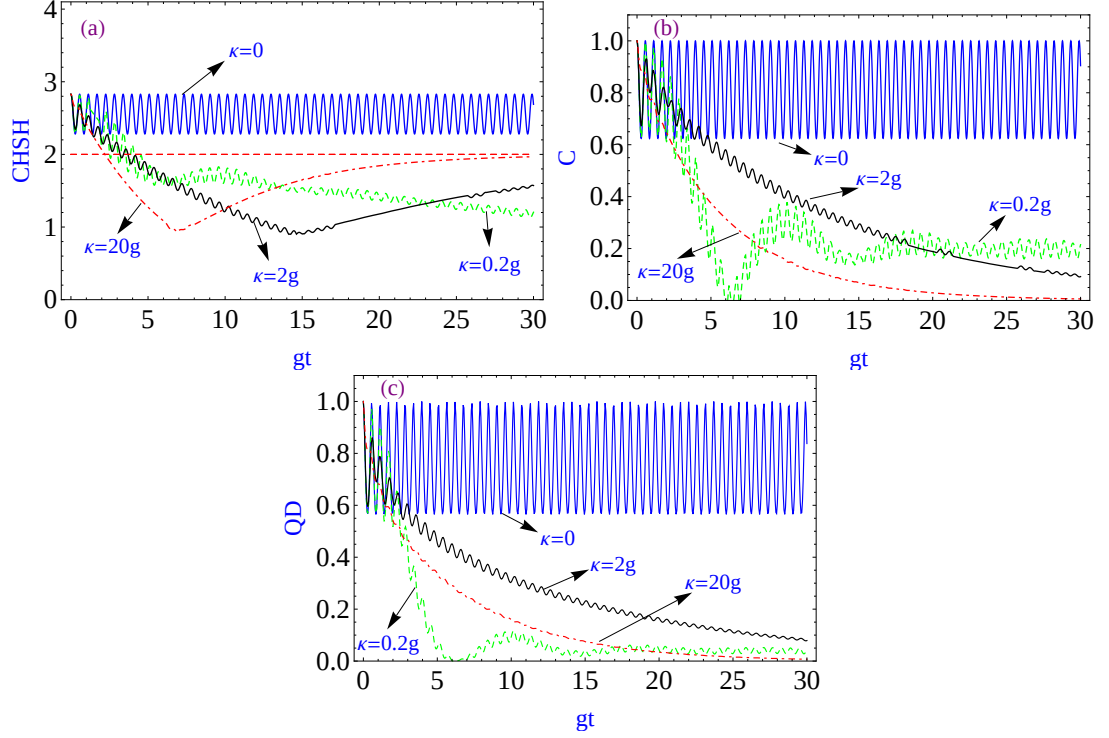


Figure 3.5: CHSH inequality (a), concurrence (b) and quantum discord (c) versus  $gt$  for unidentical atoms,  $\delta_1 = -\delta_2 = 5g$  and  $\rho(0) = |\Psi(0)\rangle \langle \Psi(0)|$  where  $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|g_A, e_B, 1\rangle + |e_A, g_B, 1\rangle)$ . Here all subfigures include plots for  $\kappa = 0$ ,  $\kappa = 0.2g$ ,  $\kappa = 2g$  and  $\kappa = 20g$ .

the other hand, CHSH and  $C$  for  $\kappa = 0$  have periodic sudden deaths and births for the identical detunings of the atoms (Figs. 3.6(a) and 3.6(b)), while QD oscillates nearly between 0.3 and 1 (Fig. 3.6(c)). For small values of  $\kappa$  ( $\kappa = 0.2g$ ), the effect of cavity decay on the dynamics of correlations can be described as under-damped oscillations for both identical and unidentical detuning cases. As  $\kappa$  is increased to  $2g$ , the oscillation becomes over-damped and finally at large  $\kappa$ 's ( $\kappa = 20g$ ) damping becomes critical, i.e.  $C$  and QD show exponential decay for the unidentical atom cases. For the identical atoms at  $\kappa = 2g$  and  $\kappa = 20g$  concurrence has sudden death, while QD decays exponentially. One should note that Bell non-locality is the most fragile quantum correlation considered here; it suffers sudden death in a short time as long as  $\kappa > 0$ .

We should stress here that the dynamics correlations for  $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|g_A, e_B\rangle - |e_A, g_B\rangle) \otimes |1\rangle$  have a very interesting structure for the identical and unidentical qubit cases. For unidentical detuning case all correlations are lost in the time

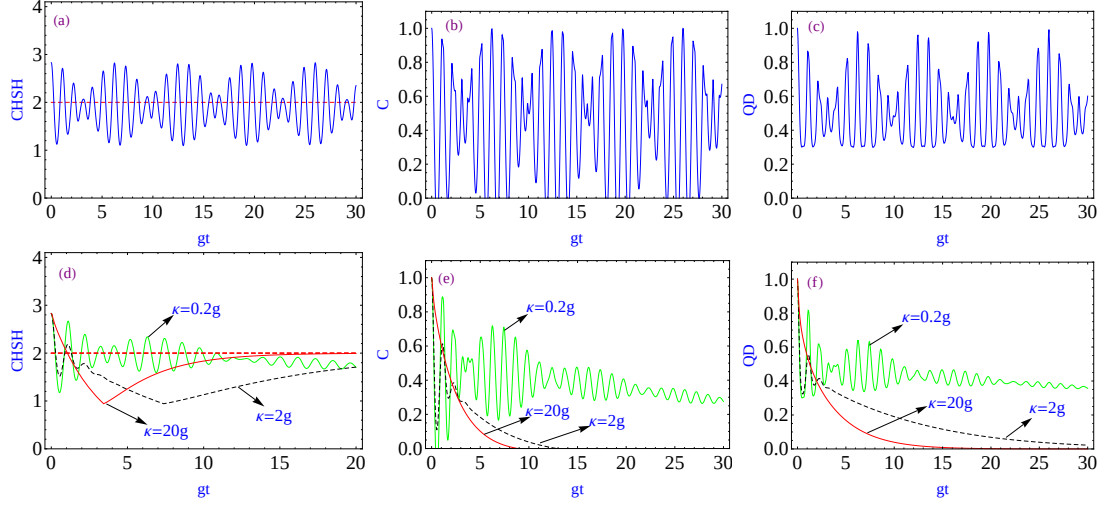


Figure 3.6: CHSH inequality (a) and (d), concurrence (b) and (e) and quantum discord (c) and (f) versus  $gt$  for identical atoms,  $\delta_1 = \delta_2 = 5g$  and  $\rho(0) = |\Psi(0)\rangle \langle\Psi(0)|$  where  $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|g_A, e_B, 1\rangle + |e_A, g_B, 1\rangle)$ . Here (a), (b) and (c) are for  $\kappa = 0$ , while (d), (e) and (f) include plots for  $\kappa = 0.2g$ ,  $\kappa = 2g$  and  $\kappa = 20g$ .

evolution under cavity losses, while for the identical atom cases the correlations become intact in the time evolution. In fact, for the identical atoms case the initial state is directly driven to the decoherence free state  $\frac{1}{\sqrt{2}}(|g_A, e_B\rangle - |e_A, g_B\rangle) \otimes |0\rangle$  where such transition does not induce a change in the dynamics of correlations. As we will discuss in the next chapter, the decoherence free state has an important role in the dynamics; the overlap of initial atomic state with the decoherence free state has a crucial role for obtaining a long-lived entangled state.

# CHAPTER 4

## DISSIPATIVE DYNAMICS OF QUANTUM CORRELATIONS AT ULTRASTRONG COUPLING REGIME

### 4.1 Introduction

The results of Chapter 3 are valid only for weak coupling regime where rotating wave approximation can be performed safely. On the other hand, with the recent developments in the area of circuit and cavity QED systems [24, 25, 26, 27, 28, 29, 30, 31, 32], ultra and deep strong light-matter couplings became experimentally achievable which makes it necessary to use full Hamiltonian, the so-called Rabi Hamiltonian to explain all the dynamical and statistical properties of these systems. Therefore, the extension of the setting investigated in the previous chapter to strong-light matter interaction case is due. Therefore, in the present chapter, we consider a model of two cold-trapped atoms resonantly and strongly interacting with a single-mode leaky cavity field. Under RWA, the dynamics of atom-field system under losses have been investigated thoroughly by using the standard Lindblad master equation and the atom-field steady states were found to have a very interesting structure determined solely by the overlap between the initial state and the decoherence free state. On the other hand, the same model has also been investigated by many groups in literature under non-RWA conditions by using the standard Lindblad master equation. However, this master equation at ultrastrong coupling regime is the root of unphysical effects; the standard Lindblad master equation with Rabi model produces spurious qubit flipping and excess excitations in the atom-field system. Recently, Beaudoin, Gambetta and Blais in Ref. [48] derived a new Lindblad type master equation

that can properly work in ultrastrong coupling regime. By using the ameliorated master equation given in Ref. [48], we study the dynamics of atom-field system and the quantum correlations between atoms, such as quantum discord and entanglement at ultrastrong coupling regime for a general class of Werner-like initial states. The main aim of this chapter will be a search of a simple structure for atom-field steady states that is valid for all atom-field coupling regime. Beyond of this, we also reexamined some of the main results obtained by the standard Lindblad master equation at strong atom-field coupling regime and revised them by using the results obtained by the ameliorated Lindblad master equation.

## 4.2 Hamiltonian and the master equation

The Rabi model describes the interaction of a two-level atom with a quantized single mode electromagnetic field (a harmonic oscillator). The Hamiltonian can be given as (with  $\hbar = 1$  throughout) [23]:

$$H_R = \frac{\omega_a}{2}\sigma_z + \omega_r a^\dagger a + g(\sigma_+ + \sigma_-)(a + a^\dagger), \quad (4.1)$$

where  $\omega_a$  ( $\omega_r$ ) is the transition frequency for the atom (cavity field),  $\sigma_z$  is the qubit energy operator,  $\sigma_\pm$  are the spin-flip operators and  $g$  is the atom-field coupling constant.

The terms  $a^\dagger\sigma_+$  and  $a\sigma_-$  that appear in the interaction part of Rabi Hamiltonian are called counter-rotating terms and they describe "virtual processes";  $a\sigma_-$  describes a process in which a photon is annihilated in the cavity mode as the atom makes a transition from its excited to the ground state, while  $a^\dagger\sigma_+$  describes the creation of a photon in the cavity mode as the atom makes an upward transition. So, the counter-rotating terms do not conserve the total number of excitations in the system. For example, even from initially zero excitation state,  $|g0\rangle$  where the atom and the field are in their ground states, the virtual processes in the atom-field system lead to non-zero mean excitation number in the time

evolution (Fig. 4.1). On the other hand, the total energy of the system is conserved during these virtual processes, since  $\frac{d}{dt} \langle H_R \rangle = 0$ , so no physical law is violated (Fig. 4.1).

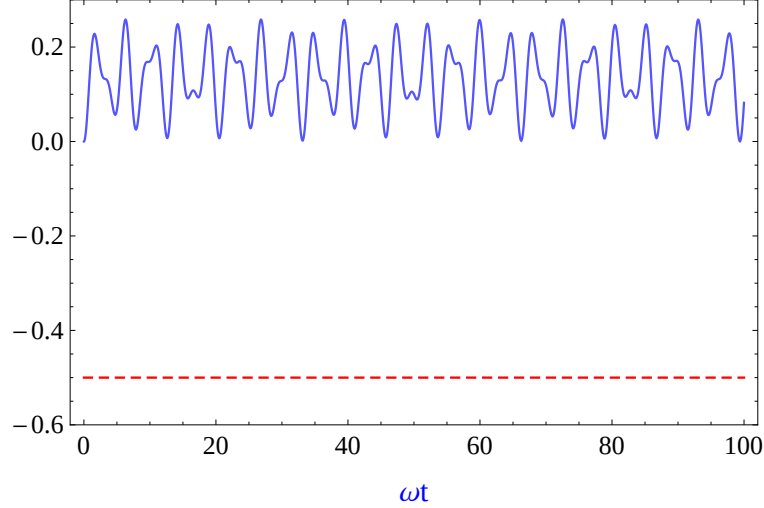


Figure 4.1: Time evolution of the mean excitation number in the whole system,  $\langle a^\dagger a + \sigma_+ \sigma_- \rangle_t$  (solid) and the mean energy  $\langle H_R \rangle_t$  (dashed) versus  $\omega t$  for vacuum  $\rho(0) = |g0\rangle \langle g0|$  initial state,  $\omega_a = \omega_r = \omega$  and  $g = 0.35\omega$ . Here  $\langle \dots \rangle_t = \text{Tr}(\rho \dots)$ .

The virtual processes have observable effects when  $g$  is comparable with  $\omega_a$  and  $\omega_r$ , i.e., at strong atom-field coupling regime, however when the weak coupling,  $g \ll \omega_a, \omega_r$ , and the nearly resonant,  $|\Delta| = |\omega_a - \omega_r| \ll \omega_a, \omega_r$ , conditions are simultaneously satisfied, these effects become negligible and the counter-rotating terms can be neglected in the interaction part of the Rabi Hamiltonian. This approximation is known as the rotating wave approximation (RWA) and the Rabi model reduces to the well known Jaynes-Cummings model given by the Hamiltonian [34]:

$$H_{JC} = \frac{\omega_a}{2} \sigma_z + \omega_r a^\dagger a + g(\sigma_+ a + \sigma_- a^\dagger). \quad (4.2)$$

In opposition to the Rabi Hamiltonian, the total number of excitations in  $H_{JC}$  is a conserved quantity which makes  $H_{JC}$  analytically accessible.

In cavity QED systems, there would be photon loss from the cavity due to the imperfections in the cavity mirrors. At absolute zero temperature ( $T = 0$ ) and

under Markov approximation, this can be represented by the standard Lindblad quantum optical master equation [47]:

$$\dot{\rho} = -i[H, \rho] + \kappa D[a]\rho, \quad (4.3)$$

where  $D[m]\rho = \frac{1}{2} (2m\rho m^\dagger - \rho m^\dagger m - m^\dagger m \rho)$ ,  $\kappa$  is the photon leakage rate,  $m$  is the appropriate dissipator operator and  $\rho$  is the atom-field density matrix. The first term in Eq. (4.3) describes the unitary Hamiltonian dynamics, while the second part accounts for the dissipation. At the weak-coupling regime where RWA can be applied, the dissipator part of the master equation is well known and involves field annihilation operator,  $a$ , which is responsible for the leakage of photons from the cavity. While deriving the standard dissipators in this master equation, the qubit-field interaction is neglected and therefore separate interactions of the involved parts with the environment are taken into account. Indeed, at weak-coupling regime where RWA can be applied, this master equation can be safely used and the dissipation can bring the system to the ground state,  $|g0\rangle$ , of  $H_{JC}$ .

Although it has been widely investigated by many groups, recently [41, 49, 50, 51, 52, 53, 54, 55, 56], using the above dissipator in the dissipative Rabi model in an *ad hoc* manner has been shown to lead to unphysical consequences [48]; at  $T = 0$ , the dissipation will generate excess excitations when even no energy is added to the system and consequently the dissipation will drive the system out of the ground state of  $H_R$ . In fact, the main failure of this master equation at strong coupling regime is to neglect the qubit-field coupling when deriving the dissipator part of the master equation. In Ref. [48], the qubit-field coupling have been included in the derivation of master equation and the new Lindblad master equation that describes the losses of the cavity field at  $T = 0$  and ultrastrong-

coupling regime under Markov approximation can be given as:

$$\dot{\rho} = -i[H_R, \rho] + \sum_{j,k>j} \Gamma_{\kappa}^{jk} D[|j\rangle\langle k|] \rho, \quad (4.4)$$

where  $\Gamma_{\kappa}^{jk} = 2\pi d(\omega_k - \omega_j) \alpha^2(\omega_k - \omega_j) |\langle j|(a + a^\dagger)|k\rangle|^2$  are the relaxation coefficients and are related to the spectral density of the bath  $d(\omega_k - \omega_j)$ , and the system-bath coupling constant  $\alpha(\omega_k - \omega_j)$  at the respective transition frequencies. For a cavity that couples to the momentum quadratures of fields in one-dimensional output waveguides, the relaxation coefficients can be simplified as [119],  $\Gamma_{\kappa}^{jk} = \kappa \frac{\omega_k - \omega_j}{\omega_r} |\langle j|(a + a^\dagger)|k\rangle|^2$ . Here  $|j\rangle$  and  $\omega_j$  are the eigenstates (dressed basis) and the eigenvalues of the Rabi Hamiltonian, respectively, i.e.,  $H_R |j\rangle = \omega_j |j\rangle$  and the eigenstates are labeled according to increasing energy, i.e., label  $|j\rangle$  such that  $\omega_k > \omega_j$  for  $k > j$ . Indeed, as shown in Ref. [48], the new dissipator drives the atom-field system to the true ground state. Also, note that the standard dissipator of Eq. (4.3) can be obtained from that of Eq. (4.4) in the limit  $g \rightarrow 0$ .

Since the strong atom-field coupling regime became relevant with the development of the recent technology [24, 25, 26, 27, 28, 29, 30, 31, 32] and a number of investigations of dissipative Rabi model with standard dissipator have been reported in the literature at strong coupling regime [49, 50, 51, 52, 53, 54, 55, 56], a reexamination of these studies in the light of ameliorated master equation (4.4) is due. Toward that goal, in the following, we will analyze the dynamics of entanglement and quantum discord between two identical qubits (A and B) interacting with a common single mode leaky cavity field. The Hamiltonians for non-RWA and RWA cases can be found by just replacing  $\sigma_z \rightarrow \sigma_z^A + \sigma_z^B$  and  $\sigma_{\pm} \rightarrow \sigma_{\pm}^A + \sigma_{\pm}^B$  in Eqs. (4.1) and (4.2), respectively. We will solve the master equations (4.3) and (4.4) numerically for the considered Hamiltonians and the initial states where the atoms are one of the four Werner-like states and the cavity



field is in its vacuum:

$$\begin{aligned}\rho_{\Psi_{\alpha}^{\pm}}(0) &= \left[ \frac{1-r}{4} I_{AB} + r |\Psi_{\alpha}^{\pm}\rangle \langle \Psi_{\alpha}^{\pm}| \right] \otimes |0\rangle \langle 0|, \\ \rho_{\Phi_{\alpha}^{\pm}}(0) &= \left[ \frac{1-r}{4} I_{AB} + r |\Phi_{\alpha}^{\pm}\rangle \langle \Phi_{\alpha}^{\pm}| \right] \otimes |0\rangle \langle 0|,\end{aligned}\quad (4.5)$$

where  $I_{AB}$  is the  $4 \times 4$  identity,  $r$  ( $0 \leq r \leq 1$ ) is the purity of the initial states,  $|\Phi_{\alpha}^{\pm}\rangle = \alpha |eg\rangle \pm \sqrt{1-\alpha^2} |ge\rangle$  and  $|\Psi_{\alpha}^{\pm}\rangle = \alpha |ee\rangle \pm \sqrt{1-\alpha^2} |gg\rangle$  are the Bell-like states and  $\alpha$  ( $0 \leq \alpha \leq 1$ ) is called the degree of correlations. Indeed, such states cover a wide range of states including pure product ( $\alpha = 0$  or  $1$ ) and Bell ( $\alpha = \frac{1}{\sqrt{2}}$ ) states at  $r = 1$  as well as mixed states for  $0 \leq r < 1$ .

A brief outline of the numerical procedure to solve the master equations Eqs. (4.3) and (4.4) with Rabi Hamiltonian should be given. Due to excitation number being a conserved quantity in the Jaynes-Cummings model, the Hamiltonian is block-diagonal in the Hilbert space of  $C^2 \otimes C^2 \otimes F^{\infty}$  (where  $C^2$  and  $F^{\infty}$  indicate the qubit and  $\infty$ -dimensional Fock space for the field, respectively), and diagonalizing blocks corresponding to total excitations of the system gives the analytic solution of the problem. Therefore, an exact closed set of differential equations can be obtained from the master equation (4.3) with Jaynes-Cummings Hamiltonian. In the full-model, excitation conservation is no longer valid and one should diagonalize an infinite dimensional Hamiltonian. In practice, for the considered initial state we have done a convergence study of the Fock space dimension of the cavity field. In this work, we have considered basis vectors of type  $|i, j, n\rangle$  where  $i, j = e, g$  and  $n = 0, 1, 2, \dots, N$ . The converge criterion was considered as the absolute value of the system density matrix elements was smaller than  $10^{-10}$  which was found to be satisfied approximately for  $N = 40$  for the largest atom-field coupling constant and the smallest cavity decay considered in the present work; for the results reported in the remainder of the text, we have taken into account all the basis vectors  $|i, j, n\rangle$  where  $i, j = e, g$  and  $n = 0, 1, 2, \dots, 50$ . Also, for all the considered cases the basic properties of the atom-atom density matrix,

such as positivity, hermiticity and trace preservation have been all monitored during the computational process.

The atomic reduced density matrix can be obtained by taking a partial trace of atom-field density matrix over the cavity degrees of freedom. The above initial states in the standard basis  $\{|1\rangle \equiv |ee\rangle, |2\rangle \equiv |eg\rangle, |3\rangle \equiv |ge\rangle, |4\rangle \equiv |gg\rangle\}$  have an X-structure in the atomic space with non-zero elements only in its main- and anti-diagonals:

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \quad (4.6)$$

It was shown that the standard Lindblad master equation (4.3) for both RWA and non-RWA Hamiltonians preserves the X-structure of the density matrix [49, 51, 52]. We have checked that the X-structure of the atomic reduced density matrix remains intact also under the ameliorated master equation (4.4). By using the above reduced density matrix, the entanglement and quantum discord between atoms can be calculated easily by using their analytical expressions given in Chapter 2.

In the following, our main concerns will be three-fold. Since numerous studies have been recently based on the standard master equation (4.3) at strong-coupling regime [49, 50, 51, 52, 53, 54, 55, 56], our first thought is to reexamine some of the main results obtained in these studies by using the new Lindblad master equation (4.4). Since the standard dissipator cannot describe the losses at strong atom-field coupling regime, the role of counter-rotating terms on the the dissipative dynamics of atom-field system are far from being understood. As the second issue, we will provide a comprehensive picture for the atom-field steady states and give their analytical expressions for the Werner-like initial states. Our final focus is to analyze the role of counter-rotating terms on the dynamics of entan-

glement and quantum discord. Without loss of generality, in our calculations we will consider the resonant case,  $\omega_a = \omega_r = \omega$ .

### 4.3 Dynamics of quantum correlations at ultrastrong coupling regime

We start our qualitative analysis by examining the main disagreements between the two master equations at the strong coupling regime. In Fig. 4.2, we display the effect of cavity decay rate,  $\kappa$ , on the dynamics of QD and EoF for the vacuum ( $|gg0\rangle$ ) initial state at  $g = 0.35\omega$ . The results in top plots are obtained by the use of the standard master equation, Eq. (4.3), while the bottom plots display the results for the new dissipator, Eq. (4.4), in the case of two-qubit Rabi Hamiltonian. Since vacuum state contains no initial excitations to start with and the Jaynes-Cummings Hamiltonian conserves the total excitation number in the atom-field system, no type of correlations will be induced in the time evolution under RWA. On the other hand, the counter-rotating terms in the Rabi Hamiltonian will generate virtual excitations and consequently non-zero correlations between the atoms which can be trapped at a non-zero value under cavity losses for both cases. If we study the effect of cavity losses on the dynamics of correlations by using the standard dissipator (Figs. 4.2(a) and 4.2(b)) as was done in Refs. [52, 53, 54, 55], the results seem to be quite surprising; as  $\kappa$  increases, although the disentanglement time and the maximum of EoF decreases, the steady state QD, peculiarly, increases with  $\kappa$  and is saturated in the maximum possible value ( $QD_{max} = 1/3$ ) in the absence of entanglement for a very bad quality cavity case ( $\kappa/\omega > 20$ ) [52]. In fact, for the very bad quality cavity case, the steady atomic density matrix is saturated and can be nearly written as an equal weighted sum,  $\rho_{AB}(\kappa/\omega > 20) = \frac{1}{3}(|ee\rangle\langle ee| + |gg\rangle\langle gg| + |\Phi^+\rangle\langle\Phi^+|)$ , where  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)$  is the Bell state, and the  $g$ -dependence of this atomic steady state becomes negligible as shown in Ref. [52]. Although this atomic state contains no entanglement, it is the maximally discordant separa-

ble mixed state [120, 121]. Although these results seem to indicate a surprising possibility of dissipation enhancement of quantum discord [52] (as well as atomic energy as found in Refs. [53, 54, 55]) to its maximally possible value for a separable state, its source is the error in the dissipator which creates excess excitations in the steady state. Considering the role of  $\kappa$  on the dynamics of QD (Fig. 4.2(c)) and EoF (Fig. 4.2(d)) for the new dissipator demonstrates that the steady states are  $\kappa$ -independent; QD and EoF reaches 0.069 and 0.0086 in the long time limit, respectively, regardless of the magnitude of  $\kappa$ . In fact,  $\kappa$  only determines how fast the correlations reach their steady state values. This is quite understandable, since  $\kappa$  is just a parameter that determines how fast the photons can escape from the cavity mirrors, it should control the speed of reaching the steady state rather than its magnitude in the long time limit. Bath decay dependent steady state shifts of several properties of the system at strong-coupling regime have been recently investigated by using the standard dissipator [52, 53, 54, 55, 56] and a more general master equation approach, the so-called hierarchical equation method without employing Born and Markov approximations [122]. The simple example shown in Fig. 4.2 indicates that some of the previously obtained results with Rabi model need to be reexamined. As shown in Fig. 4.2(b) for the standard dissipator case, the atomic steady states become separable as  $\kappa > 0.05\omega$  [52], while the steady states can possess entanglement for the new dissipator case (Fig. 4.2(d)).

Now, we will consider the steady states of RWA and non-RWA dynamics. The steady state density matrix under cavity losses and RWA dynamics investigated by standard master equation (4.3) are known to be  $\kappa$ -independent [63, 64]. As shown in the bottom plots of Fig. 4.2, they are also  $\kappa$ -independent in the case of non-RWA dynamics with the ameliorated dissipator. Therefore, in the following, we will set  $\kappa = 0.1\omega$ .

The dynamics under RWA with losses given by Eq. (4.3) have been investigated by many groups which indicate that the atom-field steady states have a simple structure. We follow the ideas developed in Refs. [63, 64, 65, 66] to

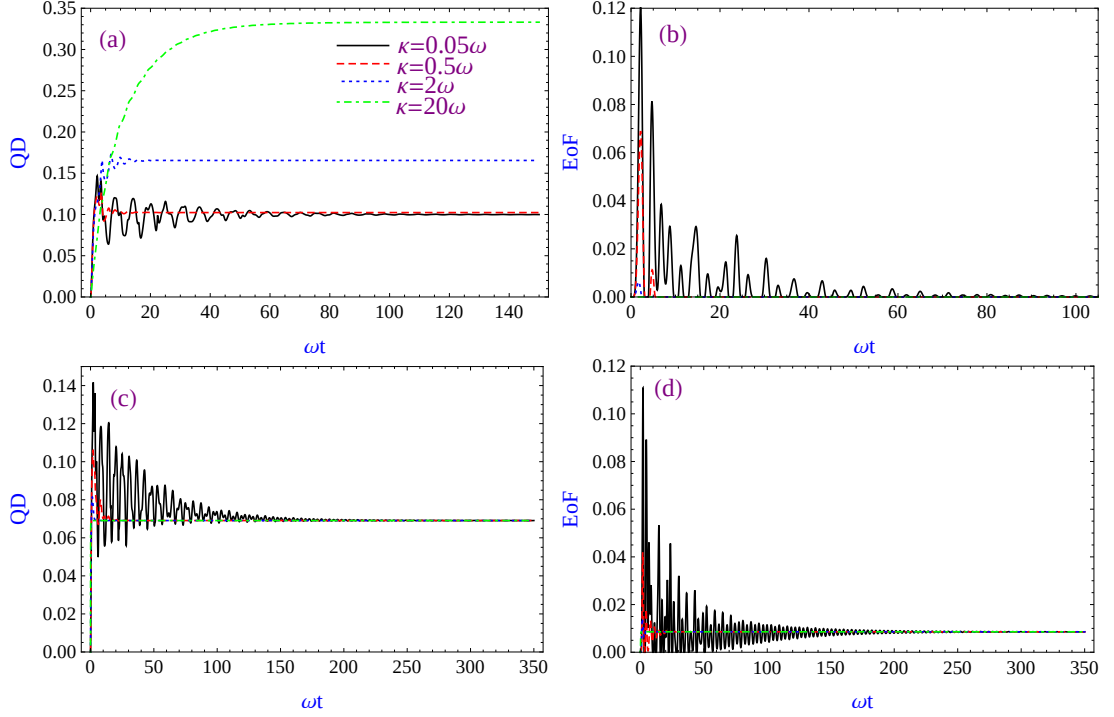


Figure 4.2: The effect of cavity decay rate,  $\kappa$ , on quantum discord ((a) and (c)) and entanglement ((b) and (d)) versus  $\omega t$  for  $|gg0\rangle$  initial state,  $\kappa = 0.05\omega$  (black, solid),  $\kappa = 0.5\omega$  (red, dashed),  $\kappa = 2\omega$  (blue, dotted), and  $\kappa = 20\omega$  (green, dot-dashed) at strong-coupling regime with  $g = 0.35\omega$ . The top figures are obtained by using the standard dissipator Eq. (4.3), while the bottom plots are for the ameliorated dissipator Eq. (4.4).

determine those states. The cavity decay tends to throw out any initial excitations from the cavity mirror and consequently it drives the atom-field system to the ground state ( $|gg0\rangle$ ) of two-qubit  $H_{JC}$ . On the other hand, there exist a highly entangled subradiant state (decoherence free state) for this model due to the interaction of the atoms with a common environment which leads to an indirect qubit-qubit interaction. This subradiant state for this model can be simply determined by finding the eigenvector of two-qubit  $H_{JC}$  for zero eigenvalue which is the maximally entangled (Bell) state in the atomic subsystem given as  $|\Phi^-0\rangle = \frac{1}{\sqrt{2}}(|eg0\rangle - |ge0\rangle)$ . If an initial atomic state has a non-vanishing overlap with the subradiant state,  $b = \langle \Phi^- | \rho_{AB}(0) | \Phi^- \rangle$ , the amount of overlap determined by  $b$  would be trapped in the subradiant state, while the remaining part,  $1 - b$ , would decay to the ground state. Therefore, depending on the specific initial state [123], the atom-field steady states would be a combination of  $|gg0\rangle$

and  $|\Phi^-0\rangle$  which can be written as:

$$\rho^{SS} = (1-b)|gg0\rangle\langle gg0| + b|\Phi^-0\rangle\langle\Phi^-0|, \quad (4.7)$$

This, indeed, explains why the steady states under RWA with losses are  $\kappa$ - and  $g$ -independent. By using Eq. (4.7) or by studying the solution of the master equation (4.3), for example by the method of solving first order differential equations via eigenvalues and eigenvectors or the pseudomode approach [63, 64], we can determine the analytic form of the steady states for the Werner-like initial states (4.5). For  $\rho_{\Psi_\alpha^\pm}(0)$  type initial states, the steady states can be found as:

$$\rho_{\Psi_\alpha^\pm}^{SS} = \left(\frac{3+r}{4}\right)|gg0\rangle\langle gg0| + \left(\frac{1-r}{4}\right)|\Phi^-0\rangle\langle\Phi^-0|. \quad (4.8)$$

For  $r = 1$ , i.e., for initially pure Bell-like states,  $|\Psi_\alpha^\pm\rangle$ ,  $\rho_{\Psi_\alpha^\pm}(0)$  has no overlap with the subradiant state and all the excitations are lost in time evolution regardless of what  $\alpha$  is, while for  $0 \leq r < 1$ , there would be a non-vanishing overlap term with the subradiant state determined by  $r$  and the atom-field system will decay to a trapping state which is superposition of  $|gg0\rangle$  and  $|\Phi^-0\rangle$ . On the other hand, for  $\rho_{\Phi_\alpha^\pm}(0)$  type initial states, the steady states can be determined as:

$$\begin{aligned} \rho_{\Phi_\alpha^\pm}^{SS} = & \left(\frac{3-r}{4} \pm r\alpha\sqrt{1-\alpha^2}\right)|gg0\rangle\langle gg0| \\ & + \left(\frac{1+r}{4} \mp r\alpha\sqrt{1-\alpha^2}\right)|\Phi^-0\rangle\langle\Phi^-0|. \end{aligned} \quad (4.9)$$

One should note that  $\rho_{\Phi_\alpha^\pm}(0)$  have always a non-vanishing term (except for  $\rho_{\Phi_\alpha^\pm}(0)$  at  $\alpha = \frac{1}{\sqrt{2}}, r = 1$ ) with the subradiant state, so the steady states are highly  $\alpha$ - and  $r$ -dependent.

$|\Phi^-0\rangle$  is also the subradiant state for two-qubit  $H_R$  and since  $H_{JC}$  is the limiting case of  $H_R$  and the master equation (4.4) can describe the cavity losses at ultrastrong coupling regime, we conjecture that such a simple structure for the atom-field steady states should exist at strong coupling regime also. Below we

will provide numerical evidence for our conjecture.

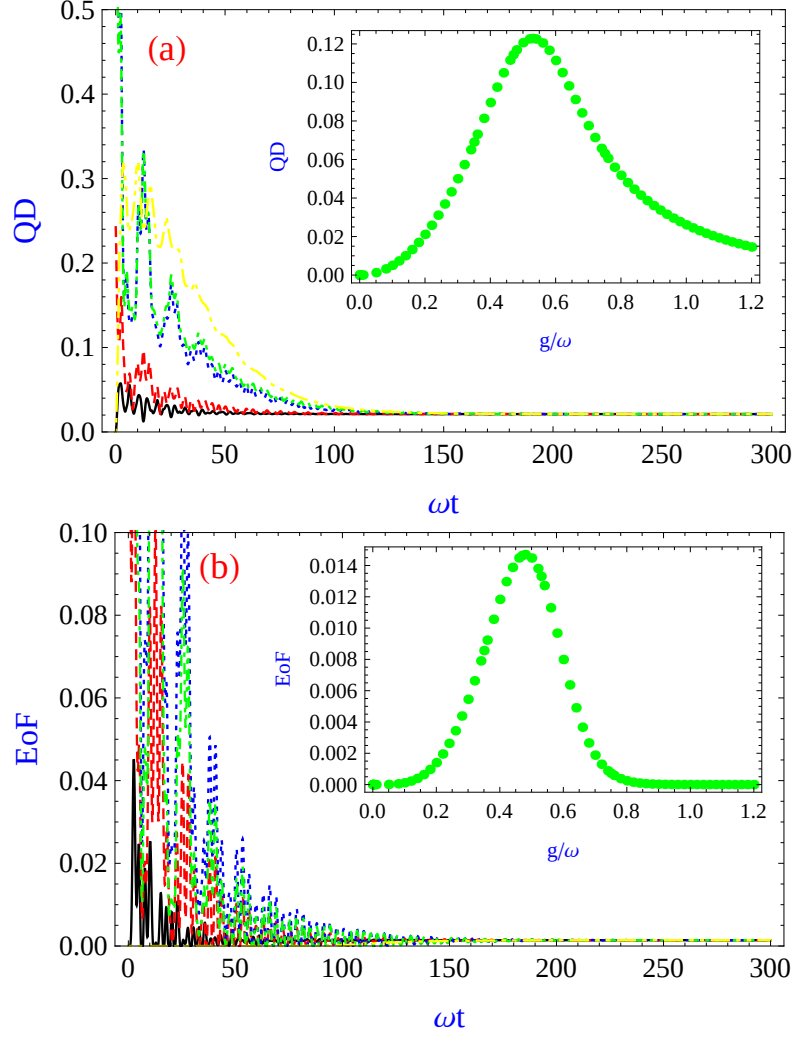


Figure 4.3: Effects of degree of correlations,  $\alpha$ , on QD (a) and EoF (b) versus  $\omega t$  for  $\rho_{\Psi_{\alpha}^{+}}(0)$  initial state with  $r = 1$ ,  $g = 0.2\omega$ ,  $\kappa = 0.1\omega$  and  $\alpha = 0$  (black, solid),  $\alpha = 0.2$  (red, dashed),  $\alpha = 1/\sqrt{2}$  (blue, dotted),  $\alpha = 0.8$  (green, dot-dashed) and  $\alpha = 1$  (yellow, dot-dot-dashed). Here the results are obtained by using two-qubit  $H_R$  and the master equation, Eq. (4.4). Inset: QD and EoF of the ground state versus  $g/\omega$ . Around  $g/\omega \approx 1.3$ , the ground state of the Rabi model becomes almost two-fold degenerate. So, we consider only  $g/\omega \in [0, 1.2]$  range in the insets.

We start to analyze the dynamics of correlations at strong-coupling regime by using the ameliorated master equation (4.4) for the type of initial states which have no overlap with the subradiant state. Thus, in Fig. 4.3 we display QD and EoF versus  $\omega t$  for the initial state  $\rho_{\Psi_{\alpha}^{+}}(0)$  with  $r = 1$ ,  $g = 0.2\omega$  and  $\kappa = 0.1\omega$  and for several  $\alpha$ -values. In the case of RWA, the steady state is the vacuum which has no types of correlations. On the other hand, as shown in Fig. 4.3, the

cavity decay at strong-coupling regime drives the atom-field to a state which can carry appreciably high quantum discord ( $QD = 0.021$ ) and entanglement ( $EoF = 0.0014$ ) independent of what  $\alpha$  is. In fact, we have observed that for the  $\rho_{\Psi_{\alpha}^{\pm}}(0)$  type of initial states with  $r = 1$ , the atom-field steady state is the ground state of two-qubit  $H_R$  regardless of  $g$ . One can observe that the most important difference between the RWA and non-RWA dynamics at the long-time limit is the existence of non-zero QD and EoF created by the virtual processes; for RWA dynamics the necessary condition for the long-lived correlated state is the overlap of initial state with the subradiant state [63, 64, 65, 66], while this can be achieved without the overlap under non-RWA conditions because of the entangled nature of the ground state. In the insets of Fig. 4.3, we display the  $g$ -dependence of the ground state QD and EoF. Both QD and EoF as a function of  $g/\omega$  display a peculiar structure; they have a single maximum at the intermediate values of  $g/\omega$ . Furthermore,  $g/\omega$  dependence of EoF is almost Gaussian. For small  $g/\omega$  where the RWA regime is valid, the QD and EoF are nearly zero, since the ground state is close to vacuum state, while the correlations are high between  $0.2 < g/\omega < 0.7$ . Note that for the  $g/\omega$  regions where EoF is absent, still non-zero QD exists especially for  $g/\omega > 0.8$ . Also note that the ground state QD and EoF show picks at different  $g/\omega$  values; QD is maximum nearly at  $g/\omega \approx 0.53$  while EoF is at  $g/\omega \approx 0.48$ . On the other hand, the decrease in both QD and EoF as a function of  $g/\omega$  at very strong coupling regime might be an indication of the inability of the Rabi model to account for the possible nonlinear effects at large  $g$  values which may require higher order terms in the description of atom-field interactions [124].

A special attention should also be given to the initial Bell  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)$  and Bell-like  $|\Psi_{\alpha}^{\pm}\rangle = \alpha|ee\rangle \pm \sqrt{1 - \alpha^2}|gg\rangle$  states at strong coupling regime, since the entanglement dynamics have been extensively analyzed by using the standard dissipator (4.3) under losses [49, 50, 51, 52]. For  $\kappa \geq 0.1\omega$ , entanglement between atoms becomes zero or useless (smaller than  $10^{-4}$ ) for a wide range of  $g$ , especially between  $0.2 < g/\omega < 1$  under standard dissipator case [49, 50, 51, 52]. Contrary



to those findings, one can see that such states can contain appreciably high EoF for  $0.2 < g/\omega < 0.7$  independent of  $\kappa$  as shown in the inset of Fig. 2(b), when one uses the ameliorated master equation.

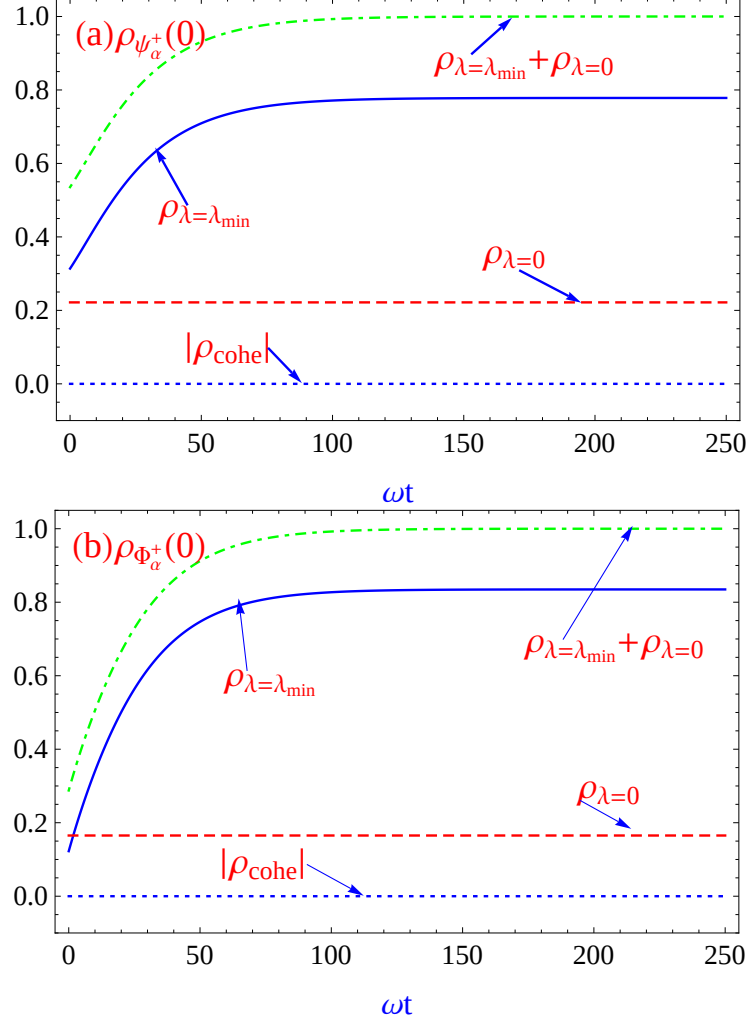


Figure 4.4: The population of the ground ( $\rho_{\lambda=\lambda_{min}}$ ) and the subradiant ( $\rho_{\lambda=0}$ ) states and the sum of the corresponding populations as well as the absolute of the coherence ( $|\rho_{cohe}|$ ) between the corresponding dressed basis as a function of  $\omega t$  for  $\rho_{\Psi_{\alpha}^{+}}(0)$  (a) at  $g = 0.2\omega$ ,  $r = 0.113$  and  $\alpha = 0.365$  and for  $\rho_{\Phi_{\alpha}^{+}}(0)$  (b) at  $g = 0.35\omega$ ,  $r = 0.456$  and  $\alpha = 0.863$ . Note that the population of dark state is trapped at  $b = \frac{1-r}{4}|_{r=0.113} \cong 0.222$  and  $b = \frac{1+r}{4} - r\alpha\sqrt{1-\alpha^2}|_{r=0.456, \alpha=0.863} \cong 0.165$  for  $\rho_{\Psi_{\alpha}^{+}}(0)$  and  $\rho_{\Phi_{\alpha}^{+}}(0)$ , respectively, while the population of the ground state goes to  $1-b$  for each initial state in the long-time limit. The results are here obtained by using ameliorated dissipator Eq. (4.4) at  $\kappa = 0.1\omega$ .

Now we consider the dynamics of initial states that have non-zero overlap with the dark state,  $|\Phi^{-}0\rangle$ . Before discussing the role of counter rotating terms on the steady state quantum correlations as a function of initial state parameters, we

should stress that a detailed numerical analysis demonstrates that the atom-field steady states at strong coupling regime for  $\rho_{\Psi_{\alpha}^{\pm}}(0)$  and  $\rho_{\Phi_{\alpha}^{\pm}}(0)$  initial states are in the form of Eqs. (4.8) and (4.9), respectively, with just  $|gg0\rangle$  replaced by the ground state,  $|\widetilde{gg0}\rangle$ , of two-qubit  $H_R$  and can be written as:

$$\begin{aligned}\rho_{\Psi_{\alpha}^{\pm}}^{SS} &= \left(\frac{3+r}{4}\right) |\widetilde{gg0}\rangle \langle \widetilde{gg0}| + \left(\frac{1-r}{4}\right) |\Phi^{-0}\rangle \langle \Phi^{-0}|. \\ \rho_{\Phi_{\alpha}^{\pm}}^{SS} &= \left(\frac{3-r}{4} \pm r\alpha\sqrt{1-\alpha^2}\right) |\widetilde{gg0}\rangle \langle \widetilde{gg0}| \\ &+ \left(\frac{1+r}{4} \mp r\alpha\sqrt{1-\alpha^2}\right) |\Phi^{-0}\rangle \langle \Phi^{-0}|.\end{aligned}\quad (4.10)$$

Since the dark state of two-qubit  $H_{JC}$  is also the dark state of two-qubit  $H_R$ , the form of the atom-field steady states for non-RWA Hamiltonian are in expected form based on the same arguments that lead to Eqs. (4.8) and (4.9). More precisely, the overlap of atomic initial state with the subradiant state is trapped in the subradiant state, while the remaining part decays to the ground state of the open quantum system under consideration. This assertion is proven numerically in Fig. 4.4 for the considered initial states where we display the population of the ground and dark states and the coherence between them as a function of  $\omega t$  for  $\rho_{\Psi_{\alpha}^{+}}(0)$  at  $g = 0.2\omega$ ,  $r = 0.113$  and  $\alpha = 0.365$  and for  $\rho_{\Phi_{\alpha}^{+}}(0)$  at  $g = 0.35\omega$ ,  $r = 0.456$  and  $\alpha = 0.863$ . As shown from the red-dashed line in these figures, the population of the dark state remains intact in the time evolution, while the population of the ground state reaches to a certain value determined by  $\alpha$  and  $r$  to normalize the population sum of the ground and dark state in the steady states. Also note that no coherence between the ground and dark state is induced in the time evolution. Indeed, the simple illustration in Fig. 4.4 agrees with the Eqs. (4.8) and (4.9). Also note that in the small  $g$  limit  $|\widetilde{gg0}\rangle$  reduces to  $|gg0\rangle$ . Therefore, the overall role of the counter-rotating term contributions stems from the ground state of two-qubit Rabi Hamiltonian. We have checked the form of Eq. (4.10) extensively by numerical means and found that it holds for all the initial states considered in the present work.

The initial purity-dependence of steady state QD and EoF displayed in Fig. 4.5 for the RWA case shows an interesting feature; steady state values of both correlation measures show an inverse dependence on  $r$ . Since  $r$  is a measure of initial mixedness of the state and the dissipation dynamics lead to an increase of the mixedness of the initial state, this finding might seem to be contradictory, but it stems from the fact that as  $r$  decreases the overlap between the initial state and the maximally correlated dark state increases and the final steady state would contain a larger fraction of  $|\Phi^-\rangle$  which leads to higher QD and EoF. At the strong-coupling regime, ground state is  $g$ -dependent which makes QD and EoF dependent on  $g$  also. In fact, the counter-rotating contributions complicate the simple RWA picture and can induce separable atomic steady states for a wide range of  $r$ . Similar to the RWA case, the steady state correlations can display inverse  $r$  dependence for a wide range, but contrary to RWA case, the correlations can also be linearly  $r$  dependent at strong-coupling regime. Moreover, a generic result can be obtained from the comparison of the magnitudes of QD and EoF in Fig. 4.5; for the regions where EoF is zero (especially for strong couplings), appreciably high QD can still exist in the steady states.

To further elucidate the role of initial state and virtual processes on the steady state quantum correlations, we display the corresponding correlation measures versus  $\alpha$  for  $\rho_{\Phi_a^\pm}(0)$  with  $r = 1$  in Fig. 4.6 and correlations measures versus  $r$  for the same initial states with  $\alpha = 0.2$  in the insets of Fig. 4.6. If we concisely analyze the steady state QD and EoF dependence on the initial state parameters under RWA (red, dashed line) signifies that the magnitude of QD and EoF directly depends on the amount of overlap with the subradiant state. On the other hand, strong virtual processes can stir the simple RWA picture and interestingly can reverse the initial state parameter dependence of the steady QD compared to the RWA case. The  $\alpha$  and  $r$  dependence of steady state EoF seem to be qualitatively similar to that of QD with a difference as previously indicated in Fig. 4.5(b); EoF can suffer death in the steady states for a wide range of  $\alpha$  and  $r$  under non-RWA

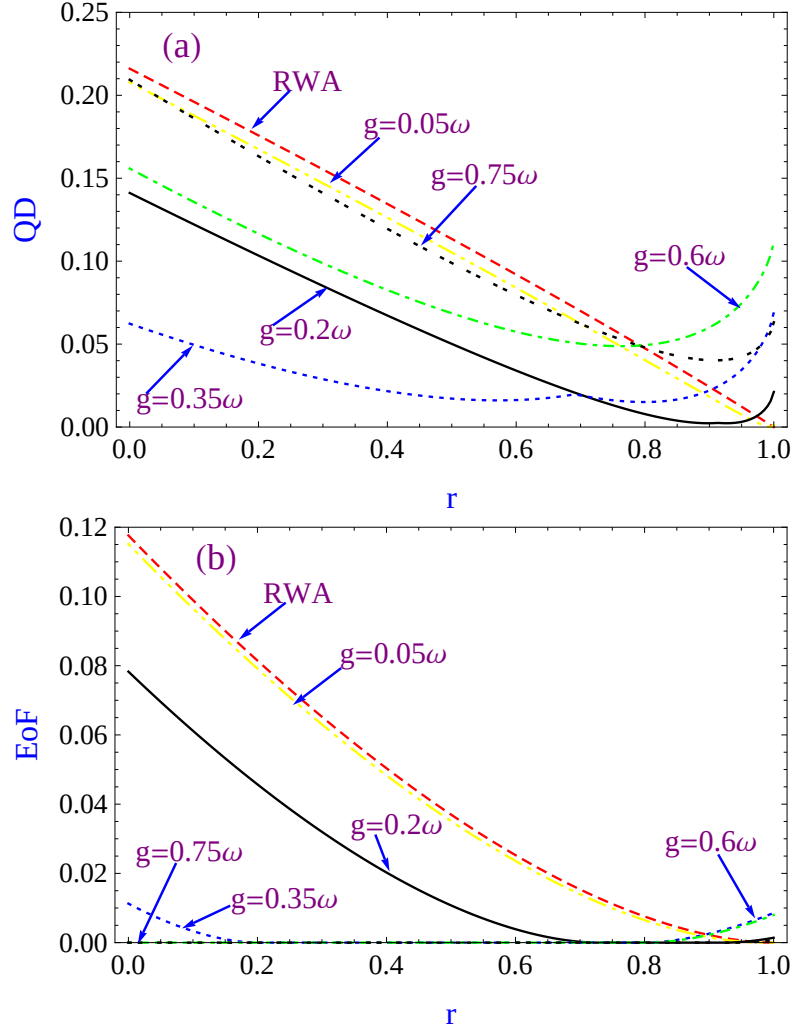


Figure 4.5: Steady state QD (a) and EoF (b) versus initial purity,  $r$ , for  $\rho_{\Psi_{\alpha}^{\pm}}(0)$  initial states under RWA (red, dashed) and non-RWA dynamics with  $g = 0.05\omega$  (yellow, dot-dot-dashed),  $g = 0.2\omega$  (black, solid),  $g = 0.35\omega$  (blue, dotted),  $g = 0.6\omega$  (green, dot-dashed) and  $g = 0.75\omega$  (black, long-dotted). Note that for  $g < 0.01\omega$ , the results obtained by the new master equation (4.4) coincide with the RWA case (red, dashed) given by Eq. (4.8) and the steady states for both cases are  $\alpha$  independent.

dynamics, especially at  $g = 0.6\omega$  and  $g = 0.75\omega$ .

Contrary to the numerous investigations of the role of counter-rotating terms on dissipative settings by using standard dissipators, the results of the present chapter are quite simple and general. More importantly, it represents the exact role of counter-rotating terms on the dynamics of the atom-field system and quantum correlations. The developed ideas can be expanded for local or global environment cases including multipartite and other decoherence channels, such as dephasing and spontaneous emissions of qubits and fields. For instance, for a

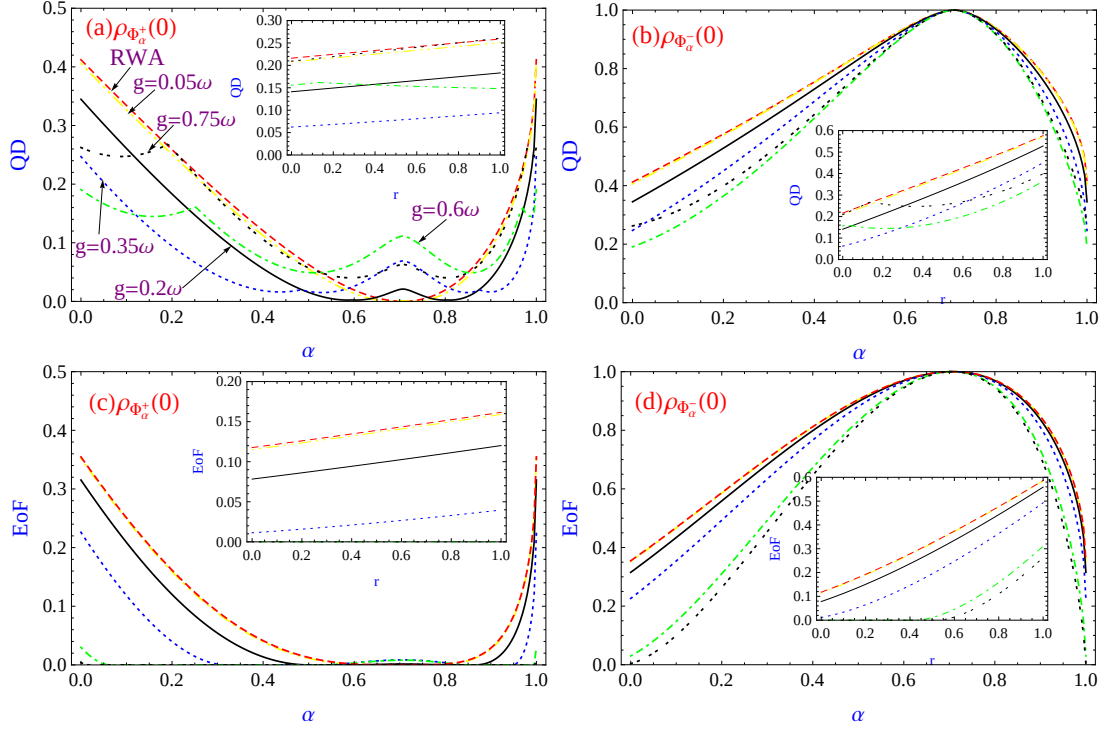


Figure 4.6: Steady state QD (a) and (b) and EoF (c) and (d) versus degree of correlations,  $\alpha$ , for  $\rho_{\Phi_{\alpha}^{+}}(0)$  (a) and (c) and  $\rho_{\Phi_{\alpha}^{-}}(0)$  (b) and (d) initial states with  $r = 1$  under RWA (red, dashed) and non-RWA dynamics with  $g = 0.05\omega$  (yellow, dot-dot-dashed),  $g = 0.2\omega$  (black, solid),  $g = 0.35\omega$  (blue, dotted),  $g = 0.6\omega$  (green, dot-dashed) and  $g = 0.75\omega$  (black, long-dotted). Inset: Steady state QD and EoF versus  $r$  for  $\rho_{\Phi_{\alpha}^{+}}(0)$  (a) and (c) and  $\rho_{\Phi_{\alpha}^{-}}(0)$  (b) and (d) initial states with  $\alpha = 0.2$  and the same  $g$  values used in the subfigures. Note that for  $g < 0.01\omega$ , the results obtained by the new master equation (4.4) coincide with the RWA case (red, dashed) given by Eq. (4.9).

model system with two qubits locally trapped inside leaky cavity fields [125], dark state is no longer exists. Therefore, it is expected that the atom-field steady states under cavity decay would always be the ground state of the system Hamiltonian. However, a more detailed discussion for this subject goes beyond the aim of this chapter.

# CHAPTER 5

## CORRELATION AND NONLOCALITY MEASURES AS INDICATORS OF QUANTUM PHASE TRANSITIONS IN SEVERAL CRITICAL SYSTEMS

### 5.1 Introduction

Beyond the characterization and quantification of quantumness of correlations as well as the dynamics of quantum correlations under open quantum systems as discussed in the previous chapters, another fundamental concept in quantum information and computation theory is to find a relation between quantum phase transitions (QPT) and quantum correlation measures. Since QPTs are due to the quantum fluctuations that occur at absolute zero temperature, they are supposed to show the scaling of the quantum correlations. Therefore, at least one measure of quantum correlations should be able to display a scaling behavior at and around the phase transition point as a function of the external parameter or the coupling constant. In fact, it was shown by many groups that entanglement and quantum discord are the promise indicators of QPTs. However, so far there is no single correlation measure that can unambiguously indicate all QPTs or shows a universal behavior at the critical points (CP) of QPTs. Consequently, the main motivation of the present chapter is to search for new quantum information theory based indicator of quantum phase transitions, such as measurement-induced disturbance (MID) and CHSH-Bell function, and compare them with the well known CP-detectors, quantum discord, classical correlations (CC) and entanglement. Towards that goal, we will investigate different correlation measures, such as EoF, QD and its classical counterpart as well as the CHSH-Bell function and the quantumness measure, the so-called, measurement-induced disturbance in the

ground states of several critical systems, including transverse field Ising and XY models as well as XY model with multiple spin interactions, XXZ model and the collective system Lipkin-Meshkov-Glick (LMG) model.

Before starting our qualitative analysis, we should stress that the Hamiltonians of all model systems exhibit  $Z_2$  symmetry; the Hamiltonians are invariant under  $\pi$  rotation around a given spin axis. This symmetry leads to the two-qubit reduced density matrix at any sites  $i$  and  $j$  in the standard basis  $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$  (here  $|1\rangle$  and  $|0\rangle$  are the eigenstates of Pauli z-matrix for +1 and -1 eigenvalues, respectively) has an  $X$ -structure, so one can calculate the considered correlation measures analytically as discussed in Chapter 2.

## 5.2 XYT model and its variants

The anisotropic spin 1/2 XY chain with three spin interaction under external transverse field (XYT model) can be defined by the following Hamiltonian [85, 126]:

$$\begin{aligned}
H_{XYT} = & - \sum_{j=1}^N \left\{ \left( \frac{1+\gamma}{2} \right) \sigma_j^x \sigma_{j+1}^x + \left( \frac{1-\gamma}{2} \right) \sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z \right\} \\
& - \sum_{j=1}^N \alpha \left( \sigma_{j-1}^x \sigma_j^z \sigma_{j+1}^x + \sigma_{j-1}^y \sigma_j^z \sigma_{j+1}^y \right), \tag{5.1}
\end{aligned}$$

where  $\sigma_j^{x,y,z}$  are the spin 1/2 Pauli matrices on site  $j$ ,  $N$  is the total number of spins in the chain,  $\gamma$  is the degree of anisotropy and  $\lambda$  and  $\alpha$  are the strengths of the external field and the three spin interaction, respectively.

The XYT model reduces to the transverse field Ising model for  $\alpha = 0$  and  $\gamma = 1$  and to the anisotropic XY model for  $\alpha = 0$ . It is exactly solvable and can be diagonalized by mapping it into spinless free fermion model with single

orbitals [85, 126]. We first introduce the Jordan-Wigner transformation:

$$\begin{aligned}\sigma_j^z &= 1 - 2c_j^\dagger c_j, \\ \sigma_j^x &= \prod_{i<j} (1 - 2c_i^\dagger c_i) (c_j^\dagger + c_j), \\ \sigma_j^y &= -i \prod_{i<j} (1 - 2c_i^\dagger c_i) (c_j^\dagger - c_j),\end{aligned}\tag{5.2}$$

where  $c_j$  is the mapped spinless creation operator and  $c_j^\dagger$  is its hermitian conjugate. Under the Jordan-Wigner transformation, the XYT Hamiltonian can be written as

$$\begin{aligned}H &= - \sum_{j=1}^N \{ [c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + \gamma(c_j^\dagger c_{j+1}^\dagger - c_j c_{j+1})] \\ &\quad + 2\alpha(c_{j-1}^\dagger c_{j+1} + c_{j+1}^\dagger c_{j-1}) + \lambda(1 - 2c_j^\dagger c_j) \}.\end{aligned}\tag{5.3}$$

Then we apply two successive transformations. First, the Fourier transformation of the fermionic operators given as  $c_j = -\frac{1}{\sqrt{N}} \sum_k c_k e^{2\pi i k j / N}$ , where  $k = -M, \dots, M$  and  $M = (N-1)/2$  for odd  $N$ . Then we apply the Bogoliubov transformation given as  $d_k = \cos \frac{\theta_k}{2} c_k - i \sin \frac{\theta_k}{2} c_{-k}^\dagger$  which diagonalizes the Hamiltonian in momentum space:

$$H = \sum_k 2\omega_k (d_k^\dagger d_k - 1/2),\tag{5.4}$$

where  $\omega_k = \sqrt{\epsilon_k^2 + (\gamma \sin(x_k))^2}$ ,  $\epsilon_k = \lambda - \cos(x_k) - 2\alpha \cos(2x_k)$  is the energy spectrum with  $x_k = \frac{2\pi k}{N}$

Because of symmetries of the Hamiltonian of the system, the structure of two point reduced density matrix reduces to the X-state. Since Hamiltonian of XYT model is real, then  $\rho_{ij} = \rho_{ij}^*$ . Because of translational invariance of  $H_{XYT}$ , the two-point density matrix of the system is position independent, i.e.,  $\rho_{ij} = \rho_{ii+r}$  for integer  $r$ . Furthermore, the density matrix commutes with  $\sigma_i^z \sigma_j^z$  because of the global rotation symmetry ( $Z_2$ -symmetry) of  $H_{XYT}$ . As a result



of these symmetries, the two qubit reduced density matrix of the system has an X-structure with the following matrix elements [85]:

$$\begin{aligned}
\rho_{11} &= \frac{1}{4} \left( 1 + 2 \langle \sigma^z \rangle + \langle \sigma_i^z \sigma_{i+r}^z \rangle \right), \\
\rho_{22} &= \rho_{33} = \frac{1}{4} \left( 1 - \langle \sigma_i^z \sigma_{i+r}^z \rangle \right), \\
\rho_{23} &= \rho_{32} = \frac{1}{4} \left( \langle \sigma_i^x \sigma_{i+r}^x \rangle + \langle \sigma_i^y \sigma_{i+r}^y \rangle \right), \\
\rho_{14} &= \rho_{41} = \frac{1}{4} \left( \langle \sigma_i^x \sigma_{i+r}^x \rangle - \langle \sigma_i^y \sigma_{i+r}^y \rangle \right), \\
\rho_{44} &= \frac{1}{4} \left( 1 - 2 \langle \sigma^z \rangle + \langle \sigma_i^z \sigma_{i+r}^z \rangle \right).
\end{aligned} \tag{5.5}$$

The magnetization,  $\langle \sigma^z \rangle$ , and the two-points correlation functions,  $\langle \sigma_i^{x,y,z} \sigma_{i+r}^{x,y,z} \rangle$ , can be calculated by the methods given in Ref. [127, 128]. They have a general form for the diagonal Hamiltonian (5.4) at absolute zero temperature  $T = 0$ :

$$\begin{aligned}
\langle \sigma^z \rangle &= \frac{1}{N} \sum_k \frac{\epsilon_k}{\omega_k}, \\
\langle \sigma_i^x \sigma_{i+r}^x \rangle &= \begin{vmatrix} G_{-1} & G_{-2} & \cdots & G_{-r} \\ G_0 & G_{-1} & \cdots & G_{-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{r-2} & G_{r-3} & \cdots & G_{-1} \end{vmatrix}, \\
\langle \sigma_i^y \sigma_{i+r}^y \rangle &= \begin{vmatrix} G_1 & G_0 & \cdots & G_{-r+2} \\ G_2 & G_1 & \cdots & G_{-r+3} \\ \vdots & \vdots & \ddots & \vdots \\ G_r & G_{r-1} & \cdots & G_1 \end{vmatrix}, \\
\langle \sigma_i^z \sigma_{i+r}^z \rangle &= \langle \sigma^z \rangle^2 - G_r G_{-r},
\end{aligned} \tag{5.6}$$

where  $G_k = -\sum_k (\cos(x_k r) \epsilon_k + \gamma \sin(x_k r) \sin(x_k)) / N \omega_k$ . In the following, we will restrict ourselves to the ground state correlations (where the temperature  $T = 0$ ) between the nearest-neighbor sites ( $r = 1$ ). The transverse magnetization  $\langle \sigma^z \rangle$  and the two-point correlation functions between nearest-neighbor sites at

$T = 0$  for the XYT model then reduce to:

$$\begin{aligned}\langle \sigma^z \rangle &= \frac{1}{N} \sum_k \frac{\epsilon_k}{\omega_k}, \\ \langle \sigma_i^x \sigma_{i+1}^x \rangle &= G_{-1}, \quad \langle \sigma_i^y \sigma_{i+1}^y \rangle = G_1, \\ \langle \sigma_i^z \sigma_{i+1}^z \rangle &= \langle \sigma^z \rangle^2 - G_1 G_{-1}.\end{aligned}\tag{5.7}$$

Now, we discuss the critical behavior of correlation measures for the transverse field Ising, XY and XYT models.

### 5.2.1 Transverse field Ising model

As a first investigation, we set  $\alpha = 0$  and  $\gamma = 1$  in the XYT Hamiltonian which reduces to the well known transverse field Ising Hamiltonian [84]:

$$H_{Ising} = - \sum_{j=1}^N \{ \sigma_j^x \sigma_{j+1}^x + \lambda \sigma_j^z \}.\tag{5.8}$$

The system will undergo a second-order quantum phase transition at the critical value  $\lambda_c = 1$  where the global phase flip symmetry breaks and the correlation length diverges [70]. The aim of this section is two-fold; first we want to analyze the behavior of correlation measures (EoF, QD, CC, MID and CHSH-Bell function) near the critical point. Second, we want to check the ability of the correlation measures to detect the QPT and investigate the finite-size scaling behavior of the derivatives of the correlations at the critical point,  $\lambda_c$ .

In Fig. 5.1, the correlation measures as a function of  $\lambda$  for a chain with  $N = 2001$  sites in the transverse field Ising model are displayed. When the parameter  $\lambda = 0$ , all spins are oriented in either positive or negative z-direction and the ground state is doubly degenerate given by the density matrix:

$$\rho = \frac{1}{2} |++\dots+\rangle \langle ++\dots+| + \frac{1}{2} |--\dots- \rangle \langle --\dots-|,\tag{5.9}$$

where  $|\pm\rangle = (|1\rangle \pm |0\rangle)/\sqrt{2}$ . After taking a partial trace of  $\rho$  overall spins except

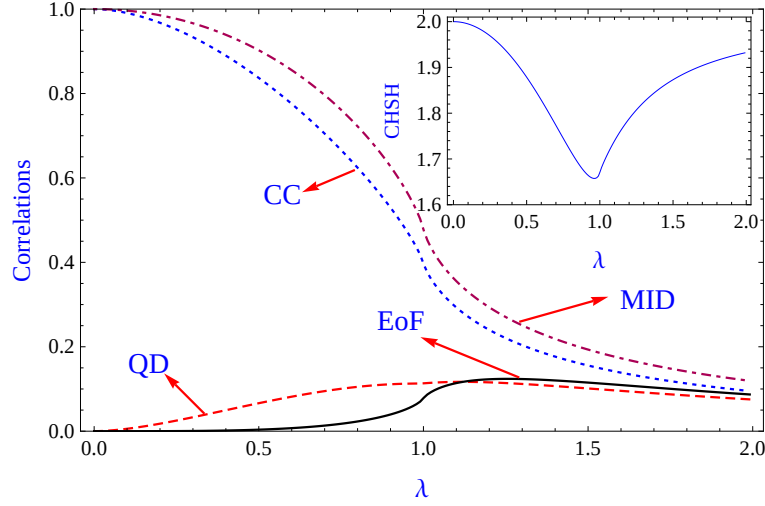


Figure 5.1: (Color online) Correlation measures as a function of  $\lambda$  for nearest-neighbor spins in the transverse field Ising model ( $\gamma = 1$  and  $\alpha = 0$ ) for a finite chain with  $N = 2001$  sites.

the nearest neighbors, the two-point reduced density matrix in the standard basis would be

$$\rho_{ii+1} = \begin{pmatrix} 0.25 & 0 & 0 & 0.25 \\ 0 & 0.25 & 0.25 & 0 \\ 0 & 0.25 & 0.25 & 0 \\ 0.25 & 0 & 0 & 0.25 \end{pmatrix}, \quad (5.10)$$

which contains  $EoF = QD = 0$ , but  $MID = CC = 1$  as can be seen from Fig. 5.1. On the other hand, for  $\lambda \rightarrow \infty$  all spins are oriented along the positive z-direction and the ground state density matrix is equal to  $\rho = |\uparrow\uparrow \dots \uparrow\rangle \langle \uparrow\uparrow \dots \uparrow|$  which is a product state that contains no type of correlations between any parties. The transition between the paramagnetic and ordered ferromagnetic phases occurs at  $\lambda_c = 1$ .

As the magnitude of the external field increases from zero, QD and EoF increase and they attain their maximum value near the critical point [79, 84]. On the other hand, CC and MID are monotonous functions of  $\lambda$ ; the magnitudes of CC and MID decrease as  $\lambda$  increases. Similar to EoF and QD, CC and MID increase when approaching the critical point from right for  $\lambda > 1$ . CHSH Bell

function, although below 2, reaches its minimum value in a region that is very close to  $\lambda_c = 1$  (see inset in Fig. 5.1). Fig. 5.1 also shows that  $\lambda$ -dependence of MID is, peculiarly, very close to that of CC and both of them are equal to 1 for  $\lambda = 0$  where  $EoF = QD = 0$ . One should note that although MID is easier to calculate than QD, it shows an inconsistency to reflect the quantum correlations compared to QD. Its being a true quantum correlation measure as defined in Chapter 2, is questioned, recently [101, 102, 103]. It is claimed that lack of any optimization procedure over the projective measurements in the definition of MID makes it a weak correlation measure for some states [101, 102, 103].

The QPT at  $\lambda_c = 1$  is a second-order phase transition, so only the derivative of the correlation measures can signal the presence of the QPT as can be seen in Fig. 5.2. Indeed, the first derivative of EoF (Fig. 5.2(a)), CC (Fig. 5.2(c)), MID (Fig. 5.2(d)) and CHSH (Fig. 5.2(d)) show pronounced minimums or maximums at some values of  $\lambda$  which tends to the quantum critical point,  $\lambda_c = 1$ , as the system size approaches infinity ( $N \rightarrow \infty$ ). The insets in Fig. 5.2 show that these correlations exhibit a logarithmic divergence at  $\lambda_c$ . Peculiarly, the first derivative of QD behaves differently; it shows an inflection point around  $\lambda_c$  (see Fig. 5.2(b)), while its second derivative shows a pronounced maximum which has a quadratic logarithmic divergence at  $\lambda_c$  (see Fig. 5.2(f) and Ref. [79]). It is remarkable to note that the behavior of QD is rather different from the other correlation measures considered in this chapter.

### 5.2.2 Transverse field XY model

The second model that we will consider in this chapter is the transverse field XY model which can be obtained by setting  $\alpha = 0$  in Eq. (5.1):

$$H_{XY} = - \sum_{j=1}^N \left\{ \left( \frac{1+\gamma}{2} \right) \sigma_j^x \sigma_{j+1}^x + \left( \frac{1-\gamma}{2} \right) \sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z \right\}. \quad (5.11)$$

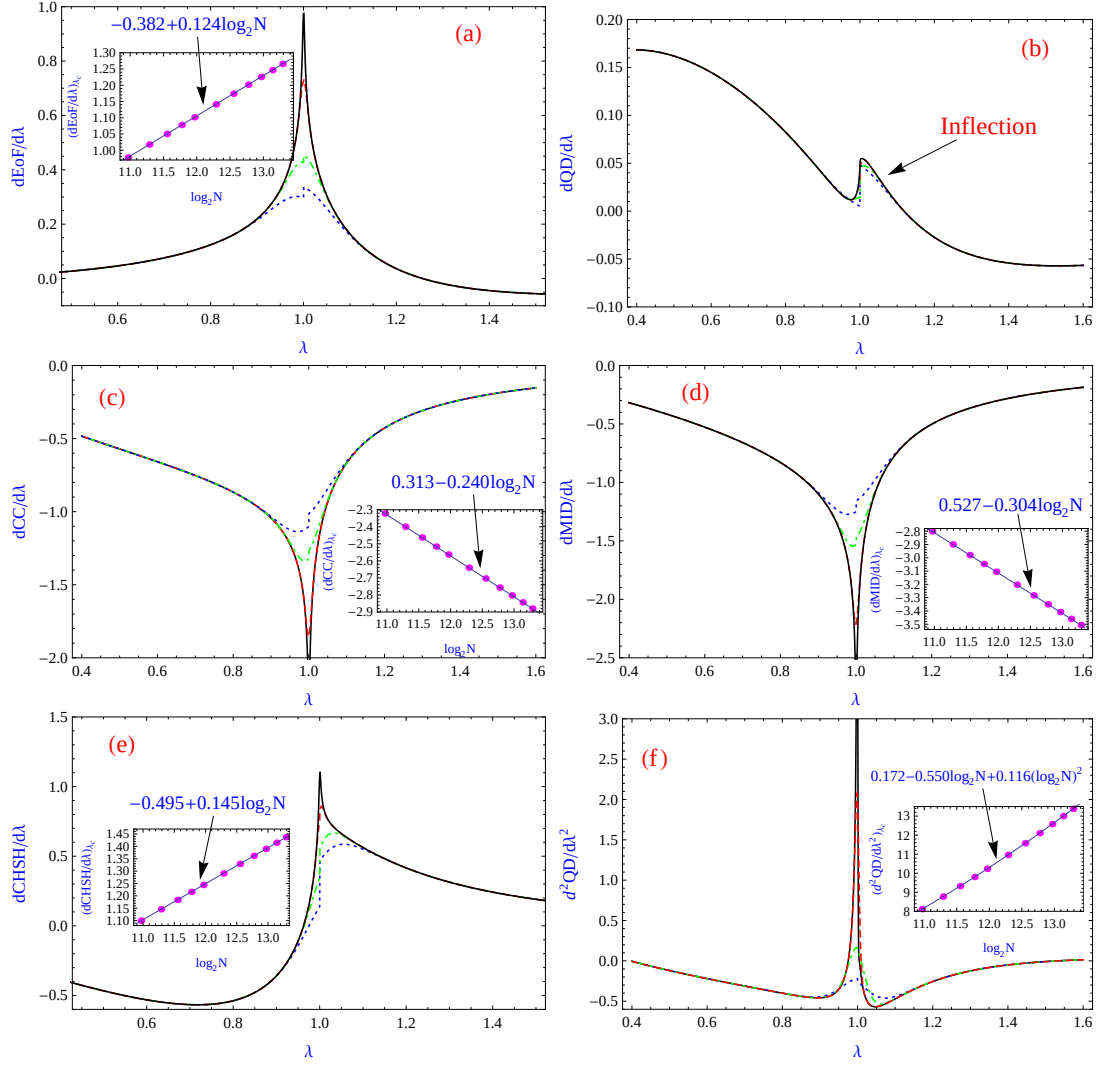


Figure 5.2: (Color online) The first derivative of correlation measures ((a)-(e)) and the second derivative of QD (f) for nearest-neighbor spins as a function of  $\lambda$  in the transverse field Ising chain ( $\gamma = 1$  and  $\alpha = 0$ ) for lattice sizes  $N = 51$  (blue-dotted),  $N = 101$  (green-dotdashed),  $N = 501$  (red-dashed) and  $N = 2001$  (black-solid). Insets: Finite size scaling behavior of the derivative(s) of the correlations at the critical point  $\lambda_c = 1$ . Here filled circles are the numerical data, while the lines are the fitting curves fitted to  $y = ax + b$  for (a)-(e) and to  $y = ax^2 + bx + c$  for (f).

The XY model has interesting quantum phase transitions belonging to two different universality classes [129]: **(i)** For non-zero values of anisotropy,  $\gamma$ , the model exhibits a second-order quantum phase transition at the critical point  $\lambda_c = 1$ . In the interval  $0 < |\gamma| \leq 1$ , this quantum phase transition has a universality class which is the same as that of the transverse field Ising model. **(ii)** For  $\lambda \in (0, 1)$ , the model exhibits again a second order phase transition, the so-called

anisotropy transition, at the critical point  $\gamma_c = 0$ . This critical point separates a ferromagnetic ordered along  $x$ -direction and the ferromagnetic ordered along the  $y$ -direction.

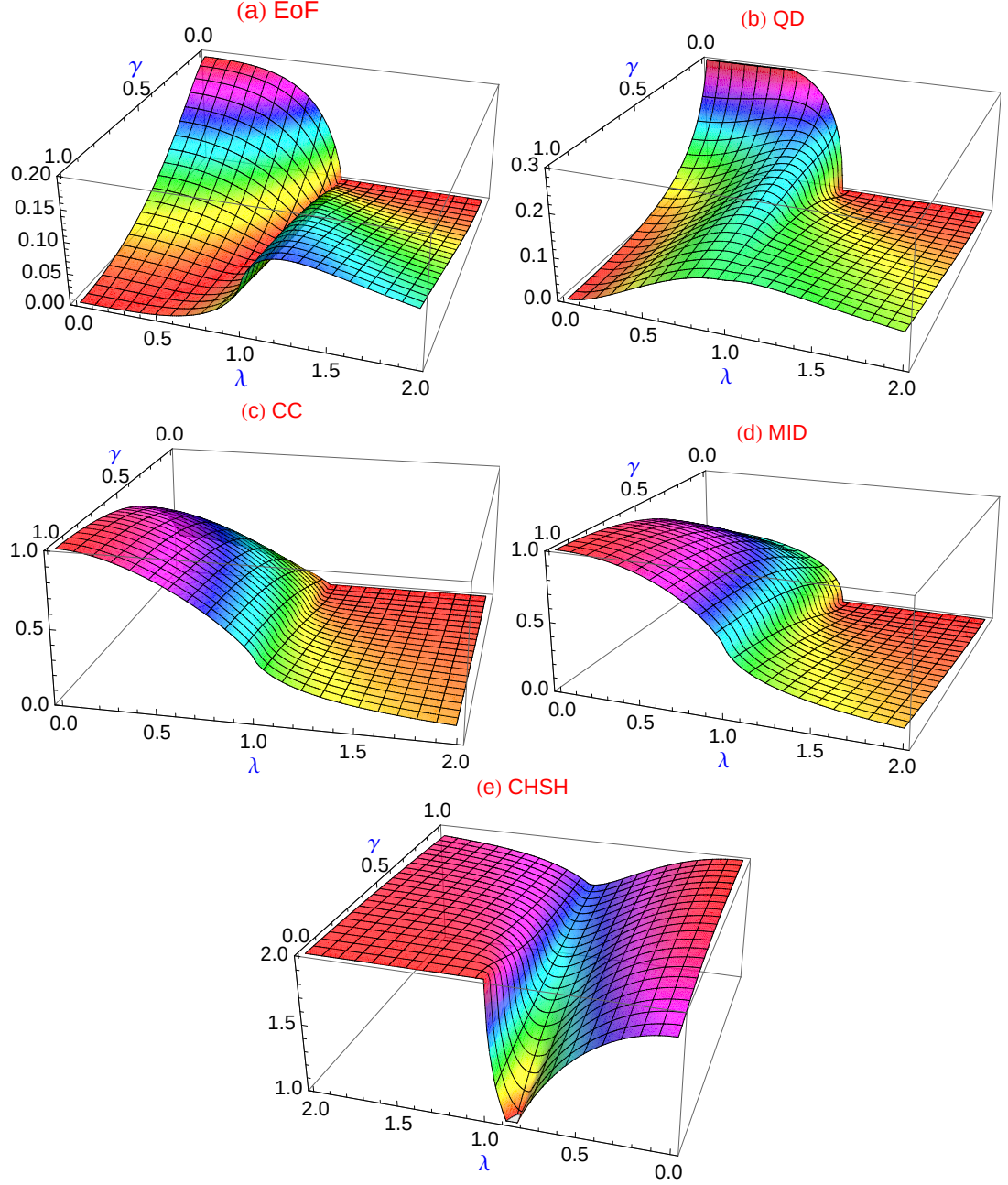


Figure 5.3: (Color online) EoF (a), QD (b), CC (c), MID (d), CHSH (e) in the XY spin chain ( $\alpha = 0$ ) with  $N = 2001$  sites as a function of  $\gamma$  and  $\lambda$ .

In Fig. 5.3, we display EoF (a), QD (b), CC (c), MID (d) and CHSH-function (e) for nearest-neighbor spins in the XY spin chain with  $N = 2001$  as a function of  $\gamma$  and  $\lambda$  at zero temperature. All figures demonstrate that there is a

pronounced difference in the correlation measures between the regions  $0 < \lambda < 1$  and  $1 < \lambda < 2$  where the QPT occurs at the critical line  $\lambda_c = 1$ . When  $\lambda < 1$ , EoF and QD decrease with  $\gamma$ . Contrarily, CC and MID increase. On the other hand, when  $\lambda > 1$ , the correlations (MID, QD, EoF and CC) increase with anisotropy. Actually, MID and CC are increasing functions of  $\gamma$  in the considered range of  $\lambda$ . CHSH-Bell function is always below 2, i.e., Bell inequality is not violated, and its dependence on  $\gamma$  is weak for the regions far from the critical point  $\lambda_c = 1$ . Near the critical point, CHSH attains its minimum. The minimum decreases when anisotropy decreases and becomes more sharp when  $\gamma \rightarrow 0$  ( $XX$  model).

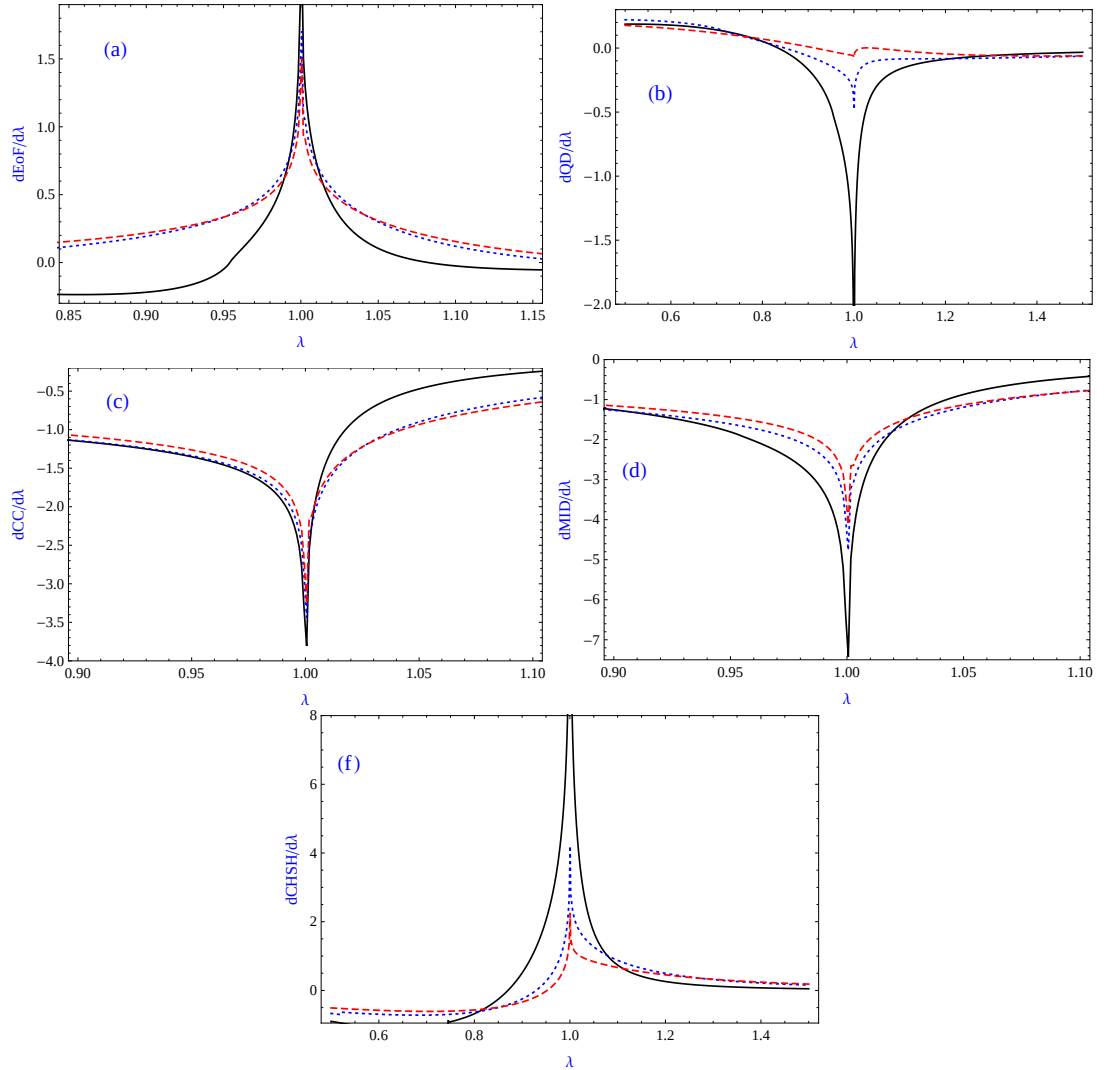


Figure 5.4: (Color online) The first  $\lambda$ -derivative of correlations, EoF (a), QD (b), CC (c), MID (d) and CHSH (e) as a function of  $\lambda$  in the anisotropic XY model ( $\alpha = 0$ ) for different values of anisotropies  $\gamma = 0.3$  (black-solid),  $\gamma = 0.7$  (blue-dotted) and  $\gamma = 0.9$  (red-dashed) and  $N = 2001$ .

As mentioned above, the XY model undergoes a second-order Ising like quantum phase transition at  $\lambda_c = 1$ . Thus, in Fig. 5.4, we display the first derivative of correlations as a function of  $\lambda$  for different values of  $\gamma$  and  $N = 2001$ . As depicted in these subfigures, the first derivative of all correlation measures are divergent at  $\lambda_c = 1$ , signaling the presence of the QPT. It seems that the decrease in  $\gamma$  sharpens the singularities to some degree. One should note that the derivative of QD has a different character for XY model compared to that for the Ising model discussed above. For a finite chain Ising model ( $\gamma = 1$  and  $N = 2001$ ), the first derivative of QD is found to exhibit an inflection point around  $\lambda_c$ , while its second derivative is divergent at  $\lambda_c$  (see Figs. 5.2(b) and 5.2(f)). As shown in Fig. 5.4(b), for anisotropic XY model, its first derivative is already divergent at  $\lambda_c$ .

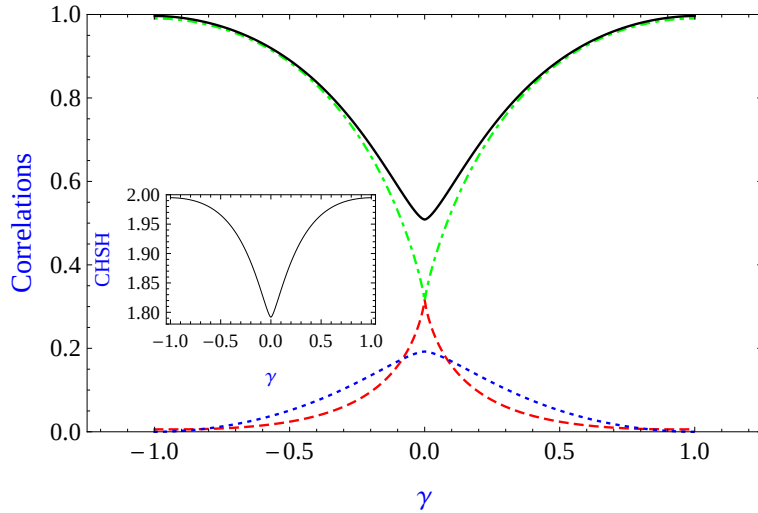


Figure 5.5: (Color online) QD (red-dashed), EoF (blue-dotted), CC (green-dotdashed), MID (black-solid) and CHSH-Bell function (inset) as a function of  $\gamma$  for the anisotropic XY model ( $\alpha = 0$ ) with  $N = 2001$  and  $\lambda = 0.1$ .

In Fig. 5.5, we plot EoF, QD, MID, CC and CHSH-Bell function (inset) as a function of anisotropy  $\gamma$  for  $N = 2001$  and  $\lambda = 0.1$ . All correlation measures are symmetric according to  $\gamma \rightarrow -\gamma$ . EoF and QD are maximum at  $\gamma_c$ , while MID, CC and CHSH-function reach their minimums. It is safe to say that all the considered correlation measures precisely detect the anisotropy phase transition by reaching their maximums or minimums. Especially, the behavior of QD and



CC are more drastic, showing extremum at  $\gamma_c$  with a cusplike behavior arising from the minimization procedure over the complete set of projectors in their definitions. In fact, the first  $\gamma$  derivative of EoF, MID and CHSH show an inflection around the critical point  $\gamma_c = 0$ , while the first derivative of QD and CC are discontinuous at  $\gamma_c$ . On the other hand, second derivative of all correlations diverge at the anisotropy transition point.

### 5.2.3 *XYT model*

Now, we consider the general Hamiltonian, Eq. (5.1), and analyze the correlation measures in the ground state of the XYT model. The model has a rich quantum phase diagram (see Refs. [126, 130]). Differently from the transverse field XY model, the QPTs are caused not only by the external field and anisotropy, but also by the three spin interactions. Indeed, the competition between  $\lambda$ ,  $\gamma$  and  $\alpha$  may induce QPTs belonging to different universality classes, for example, ferromagnetic, chiral and different spin liquid phases [126, 130].

We start the discussion of the correlation-phase diagram of XYT model with an investigation of the effect of three-spin interaction strength and the external magnetic field while the anisotropy is held constant. In Fig. 5.6, we display the correlations as functions of  $\lambda$  and  $\alpha$  at  $\gamma = 0.5$  and  $N = 2001$  in the XYT model. The dramatic changes in the correlations show the phase transitions in the model. The region between  $\alpha \in [0, 0.5)$  and  $\lambda \in (0.9, 2.0)$  indicates the spin-saturated phase (ferromagnetic), while the regions between  $\alpha \in (0.5, 1.5)$  and  $\lambda \in [0, 2]$  as well as  $\alpha \in (1.5, 2]$  and  $\lambda \in [0, 2]$  are the two kinds of incommensurate phases with weak spontaneous magnetization (see Refs. [85, 89, 131] for details).

All transitions are second-order, thus one can expect that the derivative of correlation measures could signal these QPTs. In Fig. 5.7, the  $\alpha$ -derivative of correlation measures as a function of  $\alpha$  are displayed at  $\lambda = 2.0$  and  $\gamma = 0.5$  in XYT model with different lattice sizes. The first derivative of all correlation measures diverge at  $\alpha_{c,1} = 0.5$  and  $\alpha_{c,2} = 1.5$  showing the presence of QPTs. These

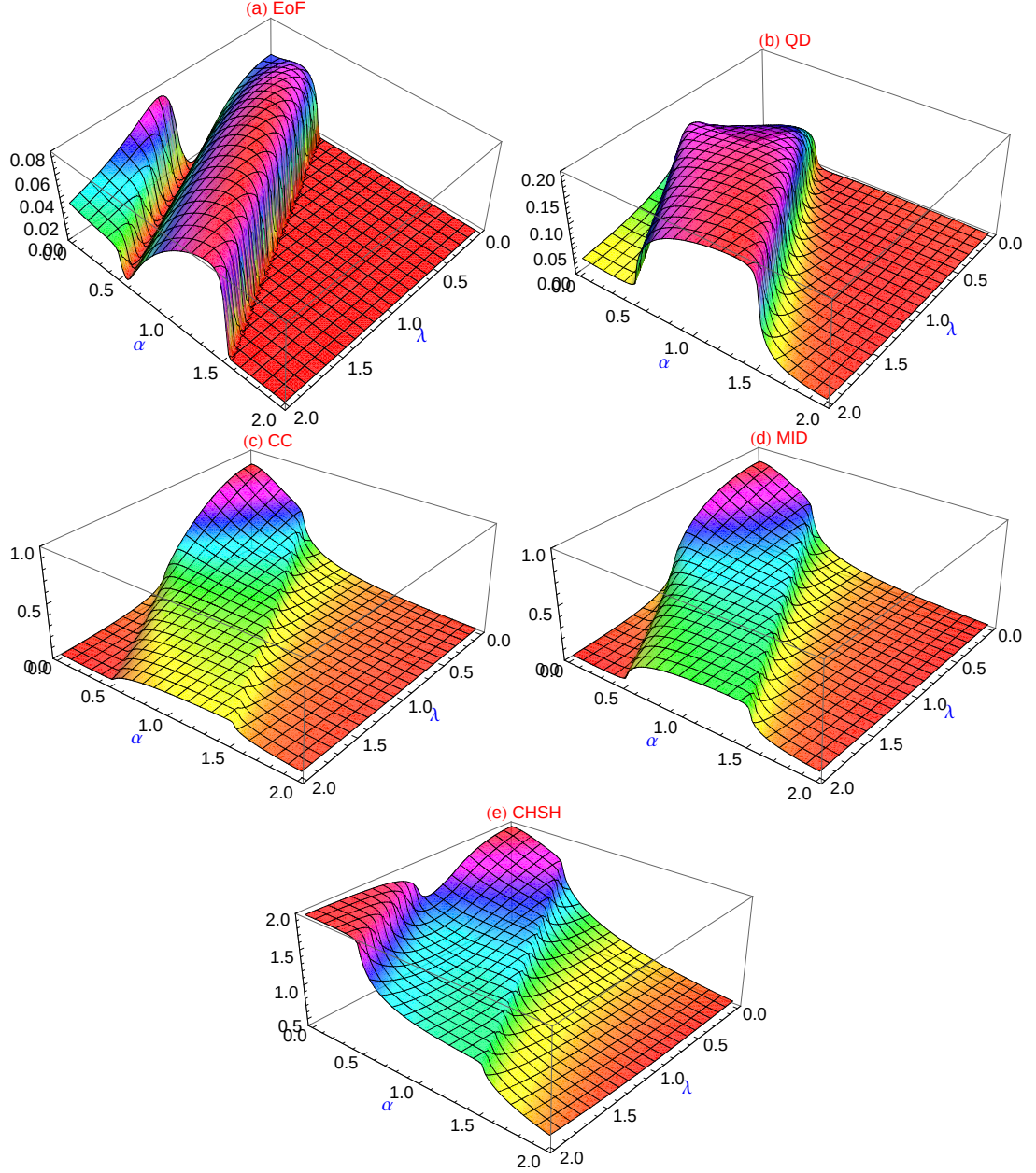


Figure 5.6: EoF (a), QD (b), CC (c), MID (d) and CHSH (e) as a function of  $\lambda$  and  $\alpha$  at  $\gamma = 0.5$  and  $N = 2001$  in the XYT model.

extremum points indicate that there exists three different phases as a function of  $\alpha$  in the region  $\alpha \in [0, 2]$  for the XYT model. Note that the first derivative of all the correlations exhibit a logarithmic divergence at  $\alpha_{c,1}$  and  $\alpha_{c,2}$  as  $N$  approaches to infinity (see insets in Fig. 5.7).

As shown in Fig. 5.6, for a fixed  $\alpha$ , the external magnetic field can also lead to QPTs in XYT model. For instance, for  $\alpha = 0.2$  and  $\alpha = 1.0$ , there are second-order quantum phase transitions at  $\lambda_c = 1.4$  and  $\lambda_c = 1.0$ , respectively.

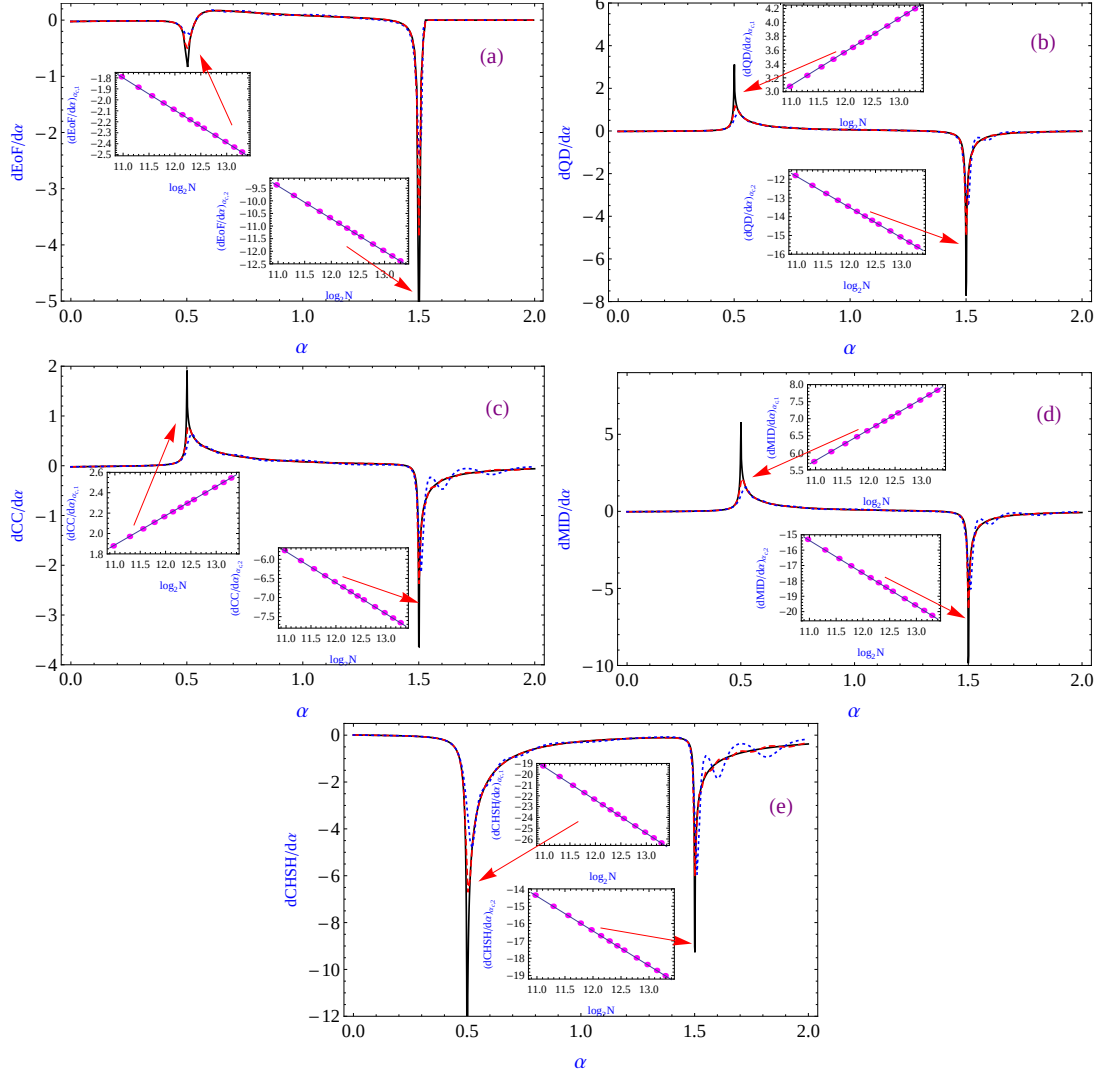


Figure 5.7: (Color online) The  $\alpha$ -derivative of correlation measures as a function of  $\alpha$  at  $\lambda = 2$  and  $\gamma = 0.5$  in the XYT chain with  $N = 51$  (blue-dotted),  $N = 101$  (red-dashed) and  $N = 2001$  (black). Insets show the finite size scaling behavior of correlation measures at the QPTs  $\alpha_{c,1} = 0.5$  and  $\alpha_{c,2} = 1.5$  where the filled circles are numerical data, while the lines are the fitting curves fitted to  $y = ax + b$ .

The divergent behavior of the first derivative of correlations with respect to  $\lambda$  signals these phase transitions as shown in Figs. 5.8(a) and 5.8(b). They diverge logarithmically as the system size increases, which is similar to the QPTs in the line of  $\alpha$ . In Ref. [85], the authors studied QPTs in XYT model and they only focused on the critical point  $\alpha_c = 0.5$  for  $\gamma = 0.5$  and  $\lambda = 0$ . It was shown that the first  $\alpha$ -derivative of QD and CC are capable of detecting the QPT. Different from Ref. [85], we have shown that QD and CC as well as MID, CHSH and EoF

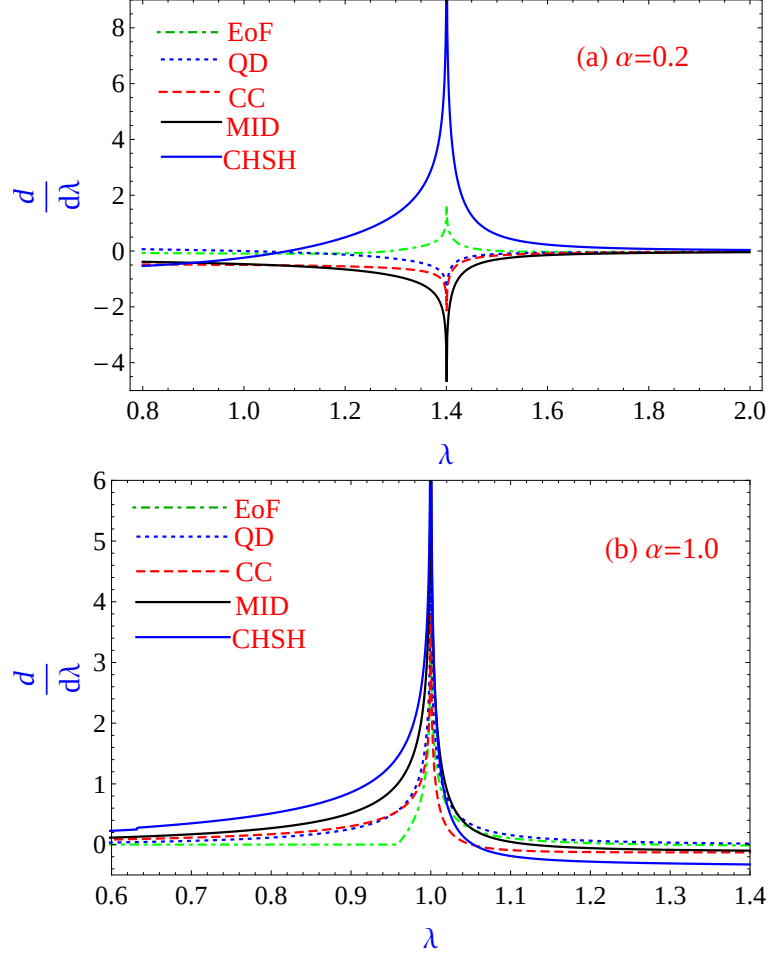


Figure 5.8: (Color online) First  $\lambda$ -derivative of correlation measures as a function of  $\lambda$  in the XYT model with  $N = 2001$ ,  $\gamma = 0.5$  and  $\alpha = 0.2$  (a) and  $\alpha = 1.0$  (b).

can signal all the phase transitions present in the XYT model for  $\gamma = 0.5$  and  $\lambda > 0$ .

### 5.3 XXZ model

The next model that we tackle in this subsection is the one dimensional anisotropic Heisenberg XXZ chain given by the Hamiltonian [84]:

$$H_{XXZ} = \sum_{i=1}^N S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z, \quad (5.12)$$

where  $\Delta$  is the anisotropy parameter,  $S_j^\mu = \sigma_j^\mu/2$  and  $N$  is the number of particles in the chain. The model has two critical points that separate three different phases: (i) At  $\Delta_c = -1$ , there is a first order QPT that separate the ferromagnetic

phase, where all the spins are pointing in the same direction for  $\Delta < -1$ , and the gapless phase, where the correlation decays polynomially for  $-1 < \Delta < 1$ . (ii) At  $\Delta_c = 1$ , there is an infinite order QPT that separates the gapless phase and the anti-ferromagnetic phase for  $\Delta > 1$ .

The Hamiltonian (5.12) has a spin-flip symmetry that leads to zero magnetization  $\langle \sigma^z \rangle = 0$  and it is symmetric with respect to the rotation in  $xy$  plane which leads to  $\langle \sigma_i^x \sigma_j^x \rangle = \langle \sigma_i^y \sigma_j^y \rangle$ . Due to these symmetries, the nearest neighbor reduced density matrix would be in an X-state form as in Eq. (2.1) with matrix elements

$$\begin{aligned}\rho_{11} = \rho_{44} &= \frac{1 + \langle \sigma_i^z \sigma_{i+1}^z \rangle}{4}, \\ \rho_{22} = \rho_{33} &= \frac{1 - \langle \sigma_i^z \sigma_{i+1}^z \rangle}{4}, \\ \rho_{23} = \rho_{32} &= \frac{\langle \sigma_i^x \sigma_{i+1}^x \rangle}{2}, \quad \rho_{14} = 0.\end{aligned}\tag{5.13}$$

The XXZ model cannot be diagonalized, but the two point correlation functions at zero temperature,  $T = 0$ , and in the thermodynamic limit,  $N \rightarrow \infty$ , can be obtained by using Bethe ansatz technique [132]. The spin-spin correlation functions between nearest-neighbor spin sites for  $-1 < \Delta < 1$  are given by [133]:

$$\begin{aligned}\langle \sigma_i^z \sigma_{i+1}^z \rangle &= 1 - \frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{dx}{\sinh x} \frac{x \cosh x}{\cosh^2(\Phi x)} + \frac{2 \cot(\pi \Phi)}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\sinh x} \frac{\sinh((1 - \Phi)x)}{\cosh \Phi x}, \\ \langle \sigma_i^x \sigma_{i+1}^x \rangle &= \frac{\cos \pi \Phi}{\pi^2} \int_{-\infty}^{\infty} \frac{dx}{\sinh x} \frac{x \cosh x}{\cosh^2(\Phi x)} - \frac{1}{\pi \sin(\pi \Phi)} \int_{-\infty}^{\infty} \frac{dx}{\sinh x} \frac{\sinh((1 - \Phi)x)}{\cosh \Phi x},\end{aligned}\tag{5.14}$$

where  $\phi = \frac{1}{\pi} \cos^{-1} \Delta$ . For  $\Delta > 1$ , the correlation functions are given by [134]

$$\begin{aligned}\langle \sigma_i^z \sigma_{i+1}^z \rangle &= 1 + 2 \int_{-\infty+i/2}^{\infty+i/2} \frac{dx}{\sinh(\pi x)} \left( \cot(\nu x) \coth(\nu) - \frac{x}{\sin^2(\nu x)} \right), \\ \langle \sigma_i^x \sigma_{i+1}^x \rangle &= \int_{-\infty+i/2}^{\infty+i/2} \frac{dx}{\sinh(\pi x)} \left( \frac{x}{\sin^2(\nu x)} \cosh \nu - \frac{\cot(\nu x)}{\sinh \nu} \right),\end{aligned}\tag{5.15}$$

where  $\nu = \cosh^{-1} \Delta$ . And for  $\Delta \leq -1$ ,  $\langle \sigma_i^z \sigma_{i+1}^z \rangle = 1$  and  $\langle \sigma_i^x \sigma_{i+1}^x \rangle = 0$  [91]. The

derivation of the two-point correlation functions for XXZ model is heavily long, but the details can be found in Refs. [133, 134].

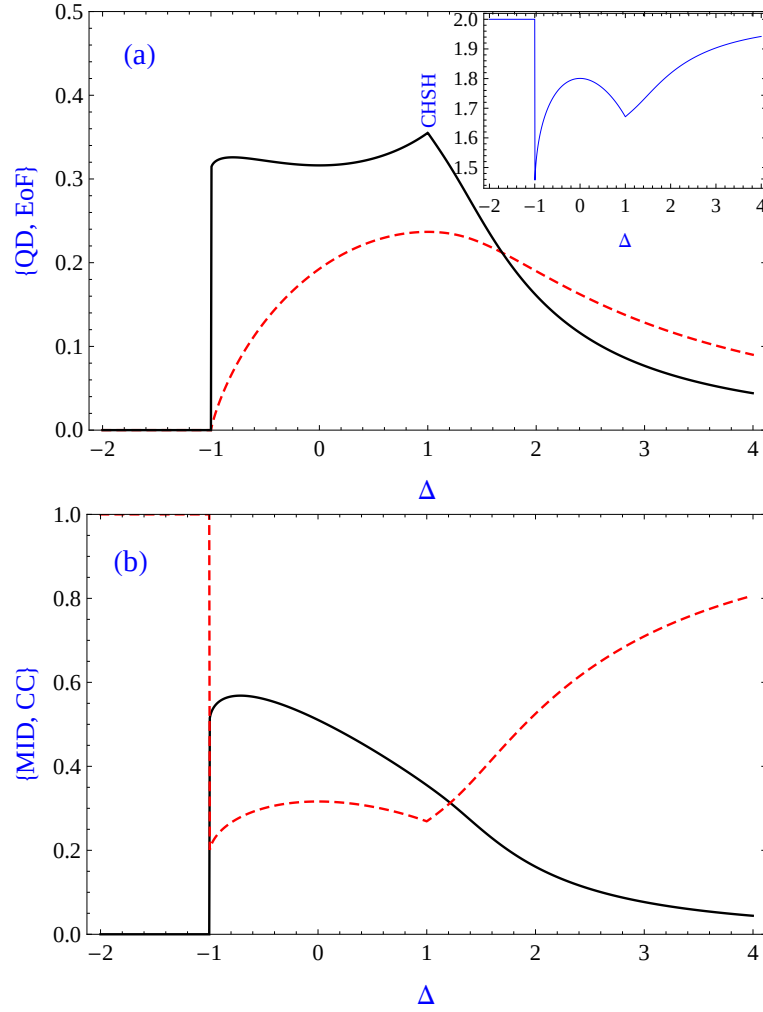


Figure 5.9: (Color online) (a) shows EoF (red-dashed), QD (black-solid) and CHSH (inset), while (b) shows CC (red-dashed) and MID (black-solid) for nearest-neighbor spins as a function of  $\Delta$  in the XXZ chain.

In Fig. 5.9, we display the correlation measures as a function of anisotropy parameter  $\Delta$  in the XXZ chain at  $T = 0$  and  $N \rightarrow \infty$ . At the first order critical point,  $\Delta_c = -1$ , all correlation measures, except EoF, are discontinuous showing the presence of the QPT, while this jump behavior is not observed in EoF. At the infinite order QPT,  $\Delta_c = 1$ , QD, CC and CHSH function show abrupt cusps. The cusp behavior in the considered correlation measures is due to the level-crossing in the model which involves transition from the gapless to antiferromagnetic phases. On the other hand, EoF is less sensitive to this level crossing;

it just reaches its maximum at  $\Delta_c = 1$  which is also the case for QD [84]. These cusplike behavior at infinite order QPT point implies that the first derivative of QD, CC and CHSH are discontinuous at  $\Delta_c = 1$ , while their second derivatives are divergent. Surprisingly, MID (black line in Fig. 5.9(b)) does not exhibit any abrupt cusp or an extremum point at  $\Delta_c = 1$ . It is continuous at the phase transition point and shows no sign of the infinite order phase transition. By using the correlation functions in Refs. [133, 134], we have also calculated MID for next-nearest neighbors, and have observed that MID cannot signal the infinite order phase transition for this case as well. Note that the infinite order phase transition cannot be revealed by the ground state energy of the system. Nonetheless, some of the quantum information tools (QD, CC and CHSH) can indicate this critical point with a cusplike behavior. In fact, around this critical point, there is a competition between the correlation functions  $\langle \sigma_i^z \sigma_{i+1}^z \rangle$  and  $\langle \sigma_i^x \sigma_{i+1}^x \rangle$  as a function of  $\Delta$  and they encounter a crossover at the critical point,  $\Delta_c = 1$  [79]. Similarly, the constituents in the definitions of these correlation measures (we mean  $Q_1$  and  $Q_2$  in Eq. (2.10) for QD, and similar for the others) exhibit the same behavior at and around this critical point. Therefore, these cusplike behavior can be understood due to the extremization procedure in the definitions of QD, CC and CHSH. Since MID has no extremization in its definition, it has no indication of this phase transition point. Although EoF is governed by a single non-zero variable (since  $\rho_{14} = 0$  for XXZ model), it reaches its maximum at this critical point. However, this is not the generic behavior of entanglement at the infinite order phase transition which may depend on the specific symmetries of the model [70]. It is remarkable to note that between  $0 < \Delta < 1$ , QD and EoF increase with  $\Delta$ , while MID is a decreasing function of anisotropy in this region. On the other hand, they are all decreasing function of  $\Delta$  for  $\Delta > 1$  and reach zero for  $\Delta \rightarrow \infty$ . We should note that while MID and CC behave similarly for Ising and XY models as functions of tuning parameters, as discussed previously, they exhibit completely different dependence on  $\Delta$  for the XXZ model. See also

Ref. [91], where the authors showed that CHSH-Bell function, although is not violated, is found to be capable to detect the presence of the QPT at  $\Delta_c = 1$  for next- and next-next-nearest neighbor spins where entanglement is absent for such distances in the XXZ model around the critical point.

## 5.4 LMG model

Finally, we analyze the ground state correlations of the model introduced by Lipkin, Meshkov and Glick (LMG) [135]. The LMG model, originally, describes a two-level Fermi system with one-level below the Fermi level, while the other level just above. The level below the Fermi level is assumed to be filled with  $\Gamma$  nucleons (here  $\Gamma$  can also be thought of as a degeneracy). In its spin 1/2 representation, the LMG model can be regarded as a chain of  $N$  spin particles with infinite-ranged interactions, where each spin is subject to an external transverse magnetic field  $\lambda$ . Indeed, the Hamiltonian of the LMG model can be written as [78, 79, 136]:

$$H_{LMG} = -\frac{1}{N} (S_x^2 - S_y^2) + \lambda S_z, \quad (5.16)$$

where  $S_z = \sum_{m=1}^N \frac{1}{2} (c_{+m}^\dagger c_{+m} - c_{-m}^\dagger c_{-m})$  and  $S_x + iS_y = \sum_{m=1}^N c_{+m}^\dagger c_{-m}$ , with  $c_{+m}^\dagger$  and  $c_{-m}^\dagger$  being the particle creation operators in the upper ( $+m$ ) and lower ( $-m$ ) levels, respectively. The model has a second-order quantum phase transition at  $\lambda_c = 1$  between a symmetric phase ( $\lambda > 1$ ) and a broken phase ( $\lambda < 1$ ) [79, 137]. The ground state is unique and fully polarized in the direction of the field for  $\lambda > 1$ , while it is twofold degenerate for  $\lambda < 1$ .

The ground state of the LMG model for  $N \rightarrow \infty$  can be obtained by using Hartree-Fock (HF) approach. Indeed, HF ground state provides the exact description of the critical point. The pairwise ground state density matrix for general modes ( $+m$ ) and ( $-n$ ) in the HF ground state of the LMG model can be



found as [79]

$$\rho_{m,-n} = \begin{pmatrix} \langle M_{+m} M_{-n} \rangle & 0 & 0 & 0 \\ 0 & \langle M_{+m} N_{-n} \rangle & \langle c_{+m}^\dagger c_{-n} \rangle & 0 \\ 0 & \langle c_{-n}^\dagger c_{+m} \rangle & \langle N_{+m} M_{-n} \rangle & 0 \\ 0 & 0 & 0 & \langle N_{+m} N_{-n} \rangle \end{pmatrix}, \quad (5.17)$$

where  $M_k = 1 - N_k$  and  $N_k = c_k^\dagger c_k$ . The matrix elements can be given as

$$\begin{aligned} \langle M_{+m} M_{-n} \rangle &= (1 - \delta_{mn}) \sin^2 \alpha \cos^2 \alpha, \\ \langle M_{+m} N_{-n} \rangle &= \delta_{mn} \cos^2 \alpha + (1 - \delta_{mn}) \cos^4 \alpha, \\ \langle N_{+m} M_{-n} \rangle &= \delta_{mn} \sin^2 \alpha + (1 - \delta_{mn}) \sin^4 \alpha, \\ \langle N_{+m} N_{-n} \rangle &= (1 - \delta_{mn}) \sin^2 \alpha \cos^2 \alpha, \\ \langle c_{+m}^\dagger c_{-n} \rangle &= \delta_{mn} \cos \alpha \sin \alpha, \end{aligned} \quad (5.18)$$

where  $\delta_{mn}$  is the Kronecker-Delta function and  $\alpha = \frac{1}{2} \cos^{-1} \lambda$  for  $\lambda < 1$  and  $\alpha = 0$  for  $\lambda \geq 1$  (see Refs. [79] and [136], for the detailed calculations of Eqs. (5.17) and (5.18)). One should note that the exact ground state of the LMG model, instead of the HF ground state studied here, can be obtained and highly entangled for  $\lambda > 1$  [138, 139, 140].

In Fig. 5.10, we display the calculated correlations and their first derivatives with respect to  $\lambda$  between the modes  $(+m)$  and  $(-m)$  as well as between the modes  $(+m)$  and  $(-n)$  ( $m \neq n$ ) in the HF ground state of the LMG model. CC, QD, MID and EoF are all equal for the density matrix of the ground state for the  $(+m, -m)$  modes, because for those modes the ground state is pure and these measures are equal for the pure states. The value of correlations range from 1 for  $\lambda = 0$  to zero for  $\lambda > 1$  (see Fig. 5.10(a)). Also, CHSH Bell function is violated for  $\lambda < 1$  (Fig. 5.10(b)). The derivatives of correlation measures are discontinuous at  $\lambda_c = 1$ , signaling the QPT (see the insets in Figs. 5.10(a) and 5.10(b)). On the other hand, for  $m \neq n$ , there are no correlations (MID,

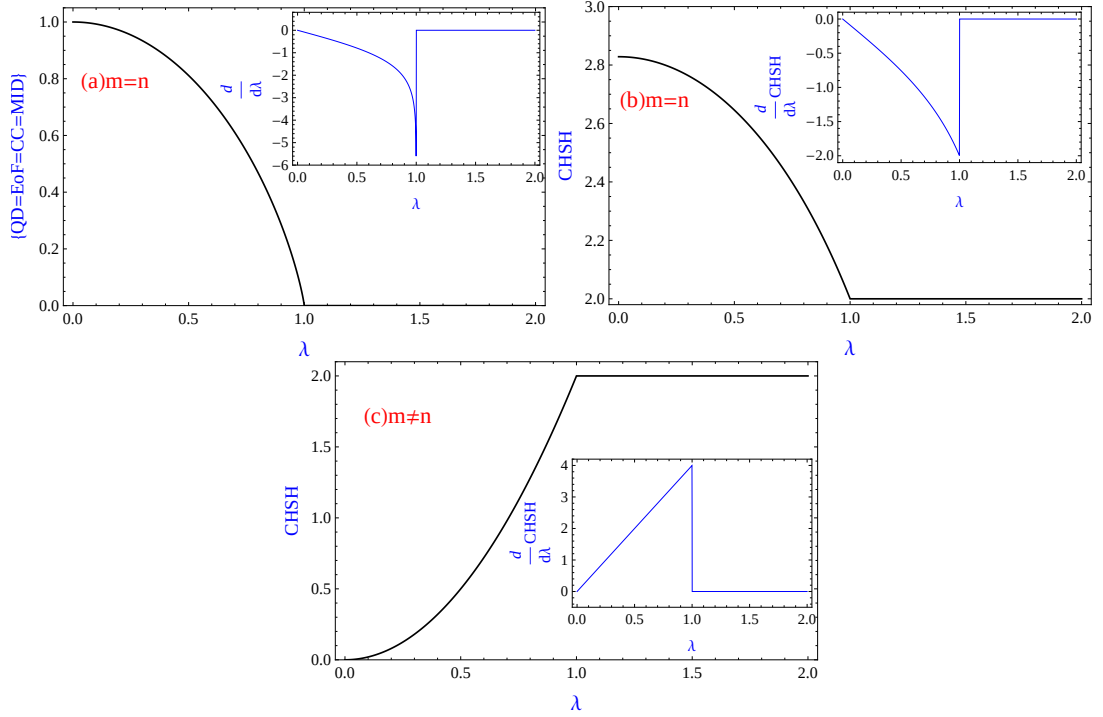


Figure 5.10: Correlations ( $QD = EoF = CC = MID$ ) (a) and CHSH-Bell function (b) between the modes  $(+m)$  and  $(-m)$  as a function of  $\lambda$  in the HF ground state of the LMG model. (c) shows CHSH function for the modes  $m \neq n$ . Insets are the corresponding first order  $\lambda$ -derivative of correlation measures as a function of  $\lambda$ . Note that for  $m \neq n$ , there is no bipartite correlations, i.e.,  $QD = EoF = CC = MID = 0$ .

QD, CC and EoF) between the modes. In fact, the density matrix Eq. (5.17) is diagonal for  $m \neq n$  and contains no pairwise classical and quantum correlations. However, the diagonal elements vary with  $\lambda$  and the pairwise density matrix changes between maximally mixed state for  $\lambda = 0$  and the fully polarized state for  $\lambda > 1$ . Although this transition does not contain any bipartite correlations, remarkably, CHSH Bell function changes with  $\lambda$  and the discontinuity (jump) in its first derivative at  $\lambda_c = 1$  exhibits the QPT (see Fig. 5.10(c)).

As a further discuss, it is remarkable to note that the CHSH-Bell function is found to exhibit all the phase transitions present in the considered critical models, even when the relevant ground state is pairwise quantum and classical uncorrelated. This result is, indeed, very desirable for the quantum information community, because the definition or analytic forms of quantum correlation measures for mixed states are mostly given for bipartite systems. Although the

unbroken bipartite state of the entire system is expected to provide the exact description of the critical behavior as well as its scaling in finite systems, sometimes the bipartite density matrix can lose its coherence, and thus, QD, CC and EoF can become insufficient to detect the CPs of QPTs, such as for the  $(+m, -n)$  mode case of the LMG model in HF ground state, for very far neighbor case and at finite temperatures. On the other hand, the CHSH-Bell function does not freeze until the bipartite state is frozen. Moreover, the definition and also the analytic form (for some special cases) of CHSH-Bell function for multipartite mixed states are available. We believe that the definition and the non-ergodic behavior of the non-locality measure make it more sophisticated CP-detector than the other well known CP-detectors and MID. However, its ability to detect the CPs of QPTs in different critical systems and particularly at finite temperatures needs strict investigations to make a direct connection between CHSH function as a CP-detector and experiments. Also, it is interesting to note that the bipartite states (except the LMG model for the  $(+m, -m)$  mode case) are mixed and do not violate the CHSH inequality, although they contain entanglement. This indicates that the correlations possessed by the two-point ground states are highly local. On the other hand, the quantum correlation measures are not expected to signal the classical phase transitions, since the two-point density matrix can lose its coherence when the temperature rises. Nonetheless, very recently, QD was found to be able to indicate the critical temperature,  $T_c$ , of the classical phase transitions in the derived model from the two-dimensional square Ising model, while entanglement could not [141].

## CHAPTER 6

### CONCLUSIONS

The scope of this thesis was two fold. On one hand, we analyzed the dynamics of quantumness of correlations under dissipation (Chapters 3 and 4). On the other hand, we considered the ability of the quantum correlations to detect the critical points of quantum phase transitions (Chapter 5). The most important results of these chapters can be summarized as follows.

We have, first, investigated the dynamics of quantum correlations as quantified by entanglement, Bell nonlocality and quantum discord for two qubits embedded in a common leaky cavity for four different initial states with double excitations. The dynamics have been studied by using standard Lindblad master equation under rotating wave approximation and the qubits were considered as identical or unidentical based on the detuning of their transition frequency with respect to the frequency of the field mode. The main findings of this chapter can be summarized as: Qubits being identical or not have profound influence on the dynamics of the considered correlations; starting from the same initial state, the correlations show very different behavior depending on whether the detunings of the individual qubits satisfy  $\delta_1 = \delta_2 = \delta$  or  $\delta_1 = -\delta_2 = \delta$ . For unidentical qubits initially in their excited states and interacting with a high or low quality cavity, quantum discord is found to be induced for all considered cavity decay rates and it decays in an oscillatory manner for a leaky cavity although the qubits remain unentangled for all times. For the same initial state and the leaky cavity, QD is found to increase in the long time for identical qubits, while entanglement is either "never created" or decays to zero in a short time. One of the most interesting finding in this chapter is the constant high entanglement and quantum discord state which is obtained in the long-time limit of the dynamics of  $\rho(0) =$

$|e_A, g_B\rangle \langle e_A, g_B| \otimes |1\rangle \langle 1|$  initial state for the identical qubit case. Here the final state is independent of the cavity decay rate (if  $\kappa > 0$ ) and the magnitude of the detuning and has high concurrence ( $C = 0.5$ ) and quantum discord ( $D = 0.412$ ). From an analysis of the time dependence of the density matrix elements it is seen that the considered initial state leads to a trapping state which happen to have high degree of quantum correlations. On the other hand, for the same initial state, the correlations between the unidentical atoms are found to be destroyed quickly even for a small cavity decay rate. For the atomic Bell-type initial state, we have found ESD for the identical qubits interacting with a low quality cavity, while concurrence decays exponentially for the unidentical qubits. Quantum discord is found to have asymptotic decay behavior independent of the type of atoms for this initial state and leaky cavity. On the other hand, CHSH-Bell inequality violations survive only for a finite time for  $\kappa > 0$  for both identical and unidentical qubits. In fact, Bell nonlocality as manifested by CHSH inequality is the most fragile correlation measure under dissipative environment and is violated only at the high values of entanglement.

We have, second, studied the same model of Chapter 3, but beyond rotating wave approximation. The dynamics of atom-field system and quantum correlations between atoms, such as quantum discord and entanglement at ultrastrong-coupling regime have been investigated for general Werner-like initial states by using recently derived Lindblad type master equation which offers quantum jumps among the eigenstates of the Hamiltonian of open quantum system under consideration. Contrary to previous investigations of the same model in the dissipative regime by using the standard Lindblad quantum optical master equation at ultrastrong coupling regime, the atom-field steady states are found to be cavity decay rate independent and have a very simple structure determined solely by the overlap of initial atomic state with the subradiant state; the amount of overlap between initial and dark states remains intact in the time evolution, while the other part decays to the ground state of the atom-field system. This result

is found to hold for all atom-field coupling strengths and coincides at the weak-coupling regime with the results under RWA studied by the standard Lindblad master equation. The presence of counter-rotating terms are found to break down some of the recently predicted results for RWA dynamics, for example, they lead to quantum-correlated atomic steady states without the overlap of an initial state with the dark state which is crucial for RWA conditions. Moreover, the virtual processes can reverse the initial state parameter dependence of steady state QD and EoF contrary to the RWA case which is strongly initial-dark state overlap dependent. However, we should stress here that the strong coupling regime seems to have quantum advantages compared to RWA case only for initial states which lead to low ( $QD < 0.05$ ) and zero discordant atomic steady states for the RWA case. Specifically, the strong virtual processes can induce separable atomic steady states.

We have, finally, investigated the quantum phase transitions of the critical systems, such as transverse field Ising, XY, XYT, XXZ and LMG models by using different quantumness measures, such as entanglement of formation, quantum discord, as well as its classical counterpart, measurement induced disturbance and CHSH-Bell function. Different aspects of quantum correlations of a given state are compared and contrasted when the tuning parameters of the system's Hamiltonians are changed, specifically at different phase regions. The ability of correlation measures to detect different order QPTs and their finite-size scaling behavior at different continuous phase transition points have been also investigated. The most important findings of the present chapter can be summarized as follows. The correlation measures that we consider reflect somewhat different aspects of the quantum as well as classical properties of the ground state of the examined models and it is not possible to find a universal behavior of these measures for signaling the quantum phase transition critical point. Furthermore, for individual measures, the dependence on the tuning parameter near the critical points is not universal across the order of the transition. In particular, for the

first order QPT observed in XXZ model, all the considered measures, except EoF, show a discontinuity behavior at the CP, while at the infinite-order QPT observed in the same model, MID has no indication of the QPT. On the other hand, EoF has its maximum and QD, CC and CHSH have a cusp-like structure, showing the infinite-order phase transition. The number of second-order QPTs investigated in the present chapter is much higher than that of the first or infinite-order QPTs. None of the considered measures was found to behave exactly the same at CP for the investigated seven second-order phase transitions. Except for the LMG model for  $(+m, -n)$  modes with  $m \neq n$ , the first or second derivatives of all correlation measures are found to be discontinuous or divergent at the critical point of the relevant models, signaling the CPs of QPTs. Remarkably, the CHSH function is the sole indicator of the phase transition between  $(+m)$  and  $(-n)$  modes for  $m \neq n$  in the HF ground state of the LMG model. Moreover, the tuning parameter dependence of MID is found to be completely different to that of QD in some phase regions. Interestingly, for Ising and XY models, the classical correlations and MID as functions of tuning parameters are nearly the same. In fact, the main motivation of this chapter was to search for new quantum information theory based indicators of quantum phase transitions, such as MID and the CHSH-Bell function, and compare them with the well known CP-detectors, QD, CC and EoF. It is found that the finite-size scaling behaviors of MID and CHSH-Bell function in Ising and XYT models exhibit a logarithmic scaling which is similar to that of EoF, QD and CC. On the other hand, MID cannot signal the infinite-order phase transition present in the XXZ model, although the others can. Above all, first and foremost, the CHSH-Bell function is found to exhibit all the phase transitions present in the considered critical models, even when the relevant ground state is pairwise quantum and classical uncorrelated.

## REFERENCES

- [1] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W.K. Wootters, Phys. Rev. Lett. **70** (1993) 1895.
- [2] C.H. Bennett, J.S. Wiesner, Phys. Rev. Lett. **69** (1992) 2881.
- [3] A.K. Ekert, Phys. Rev. Lett. **67** (1991) 661.
- [4] J.S. Bell, Physics **1** (1964) 195.
- [5] R.F. Werner, Phys. Rev. A **40** (1989) 4277.
- [6] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, Phys. Rev. Lett. **23** (1969) 880.
- [7] A. Acin, N. Gisin, L. Masanes, V. Scarani, Int. J. Quantum Inf. **2** (2004) 23.
- [8] A. Acin, N. Gisin, L. Masanes, Phys. Rev. Lett. **97** (2006) 120405.
- [9] N. Gisin, R. Thew, Nature Photon **1** (2007) 165.
- [10] H. Ollivier and W.H. Zurek, Phys. Rev. Lett. **88** (2001) 017901.
- [11] L. Henderson and V. Vedral, J. Phys. A **34** (2001) 6899.
- [12] K. Modi, A. Brodutch, H. Cable, T. Paterek, V. Vedral, arXiv: 1112.6238 [quant/ph].
- [13] E. Knill, R. Laflamme, Phys. Rev. Lett. **81** (1998) 5672.
- [14] J. Cui, H. Fan, J. Phys. A: Math. Theor. **43** (2010) 045305.
- [15] B. Dakic, et al., arXiv:1203.1629 [quant/ph].
- [16] A. Ferraro, L. Aolita, D. Cavalcanti, F.M. Cucchietti, A. Acin, Phys. Rev. A **81** (2010) 052318.
- [17] T. Yu, J.H. Eberly, Phys. Rev. Lett. **93** (2004) 140404.
- [18] F.F. Fanchini, T. Werlang, C.A. Brasil, L.G.E. Arruda and A.O. Caldeir, Phys. Rev. A **81** (2010) 052107.
- [19] T. Werlang, S. Souza, F.F. Fanchini and C.J.V. Boas, Phys. Rev. A **80** (2009) 024103.
- [20] L. Mazzola, J. Piilo and S. Maniscalco, Phys. Rev. Lett. **104** (2010) 200401.
- [21] F. Ciccarello and V. Giovannetti, Phys. Rev. A **85** (2012) 010102; Phys. Rev. A **85** (2012) 022108.
- [22] F. Altintas, A. Kurt, R. Eryigit, Phys. Lett. A **377** (2012) 53.



- [23] I.I. Rabi, Phys. Rev. **49** (1936) 324.
- [24] For a brief review see E. Solano, Physics **4** (2011) 68.
- [25] D. Englund et al., Nature (London) **450** (2007) 857.
- [26] T. Niemczyk et al., Nature Phys. **6** (2010) 772.
- [27] A.T. Sornborger, A.N. Cleland, M.R. Geller, Phys. Rev. A **70** (2004) 052315.
- [28] D. Liebfried, R. Blatt, C. Monroe, D. Wineland, Rev. Mod. Phys. **75** (2003) 281.
- [29] A. Wallraff et al., Nature (London) **431** (2004) 162.
- [30] P.F. Diaz et al., Phys. Rev. Lett. **105** (2010) 237001.
- [31] A. Federov et al., Phys. Rev. Lett. **105** (2010) 060503.
- [32] J. Casanova et al., Phys. Rev. Lett. **105** (2010) 263603.
- [33] D. Braak, Phys. Rev. Lett. **107** (2011) 100401.
- [34] E.T. Jaynes, F.W. Cummings, Proc. IEEE **51** (1963) 89.
- [35] M. Brune, F.S. Kaler, A. Maali, J. Dreyer, E. Hagley, J.M. Raimond, S. Haroche, Phys. Rev. Lett. **76** (1996) 1800.
- [36] J.R. Kuklinski, J.L. Madajczyk, Phys. Rev. A **37** (1988) 3175.
- [37] M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, N. Zagury, Phys. Rev. A **45** (1992) 5193.
- [38] M. Weidinger, B.T.H. Varcoe, R. Heerlein, H. Walther, Phys. Rev. Lett. **82** (1999) 3795.
- [39] G. Rempe, H. Walther, N. Klein, Phys. Rev. Lett. **58** (1987) 353.
- [40] S.J.D. Phoenix, P.L. Knight, Phys. Rev. A **44** (1991) 6023.
- [41] T. Werlang, A.V. Dodonov, E.I. Duzzioni, C.J.V. Boas, Phys. Rev. A **78** (2008) 053805.
- [42] H.T. Ng, K. Burnett, New J. Phys. **10** (2008) 123014.
- [43] S.J.D. Phoenix, J. Mod. Opt. **38** (1991) 695.
- [44] V.E. Lembessis, Phys. Rev. A **78** (2008) 043423.
- [45] F. Bloch, A. Siegert, Phys. Rev. **57** (1940) 522.
- [46] P.W. Milonni, J.R. Ackerhalt, H.W. Galbraith, Phys. Rev. Lett. **50** (1983) 966.
- [47] D. Walls, G. Milburn, *Quantum Optics* (Springer, Berlin, 1994).

- [48] F. Beaudoin, J.M. Gambetta, A. Blais, Phys. Rev. A **84** (2011) 043832.
- [49] Z. Ficek, J. Jing, Z.G. Lu, Phys. Scr. **T140** (2010) 014005.
- [50] J. Jing, Z.G. Lu, H.R. Ma, H. Zheng, J. Phys. B: At. Mol. Opt. Phys. **41** (2008) 135502.
- [51] J. Jing, Z.G. Lu, Z. Ficek, Phys. Rev. A **79** (2009) 044305.
- [52] F. Altintas, R. Eryigit, Phys. Lett. A **376** (2012) 1791.
- [53] J. Seke, Physica A **193** (1993) 587.
- [54] J. Seke, Quantum Opt. **4** (1992) 151.
- [55] J. Seke, Physica A **240** (1997) 635.
- [56] M. Bina, G. Romero, J. Casanova, J.J.G. Ripoll, A. Lulli, E. Solano, Eur. Phys. J. Special Topics **203** (2012) 207.
- [57] L. Mazzola, B. Bellomo, R.L. Franco and G. Compagno, Phys. Rev. A **81** (2010) 052116.
- [58] C.Z. Wang, C.X. Li, L.Y. Nie and J.F. Li, J. Phys. B: At. Mol. Opt. Phys. **44** (2011) 015503.
- [59] R.C. Ge, M. Gong, C.F. Li, J.S. Xu and G.C. Guo, Phys. Rev. A **81** (2010) 064103.
- [60] B. Bellomo, R.L. Franco and G. Compagno, Adv. Sci. Lett. **2** (2009) 459.
- [61] B. Bellomo, R.L. Franco and G. Compagno, Phys. Rev. A **78** (2008) 062309.
- [62] F. Francica, F. Plastina and S. Maniscalco, Phys. Rev. A **82** (2010) 052118.
- [63] S. Maniscalco, F. Francica, R.L. Zaffino, N.L. Gullo and F. Plastina, Phys. Rev. Lett. **100** (2008) 090503.
- [64] F. Francica, S. Maniscalco, J. Piilo, F. Plastina, K.A. Suominen, Phys. Rev. A **79** (2009) 032310.
- [65] Z. Ficek, R. Tanas, Phys. Rev. A **77** (2008) 054301.
- [66] C.H. Fleming, N.I. Cummings, C. Anastopoulos, B.L. Hu, J. Phys. A: Math. Theor. **45** (2012) 065301.
- [67] S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press, (Cambridge, 1999).
- [68] M. Greiner et al., Nature (London) **415** (2002) 39.
- [69] V.F. Gantmakher and V.T. Dolgoplov, Phys. Usp. **53** (2010) 1.
- [70] L. Amico, R. Fazio, A. Osterloh and V. Vedral, Rev. Mod. Phys. **80** (2008) 517.

- [71] L.A. Wu, M.S. Sarandy and D.A. Lidar, Phys. Rev. Lett. **93** (2004) 250404.
- [72] A. Osterloh, L. Amico, G. Falci and R. Fazio, Nature **416** (2002) 608.
- [73] P. Smacchia, et. al., Phys. Rev. A **84** (2011) 022304.
- [74] R. Orus and T.C. Wei, Phys. Rev. B **82** (2010) 155120.
- [75] R. Eryigit, Y. Gunduc and R. Eryigit, Phys. Lett. A **358** (2006) 363.
- [76] C.C. Rulli and M.S. Sarandy, Phys. Rev. A **81** (2010) 032334.
- [77] T.J. Osborne and M.A. Nielsen, Phys. Rev. A **66** (2002) 032110.
- [78] L.A. Wu, M.S. Sarandy, D.A. Lidar and L.J. Sham, Phys. Rev. A **74** (2006) 052335.
- [79] M.S. Sarandy, Phys. Rev. A **80** (2009) 022108.
- [80] B.Q. Liu, B. Shao, J.G. Li, J. Zou and L.A. Wu, Phys. Rev. A **83** (2011) 052112.
- [81] J. Maziero, H.C. Guzman, L.C. Celeri, M.S. Sarandy and R.M. Serra, Phys. Rev. A **82** (2010) 012106.
- [82] T. Werlang, C. Trippe, G.A.P. Ribeiro and G. Rigolin, Phys. Rev. Lett. **105** (2010) 095702.
- [83] T. Werlang, G.A.P.R. Ribeiro and G. Rigolin, Phys. Rev. A **83** (2011) 062334.
- [84] R. Dillenschneider, Phys. Rev. B **78** (2008) 224413.
- [85] Y.C. Li and H.Q. Lin, Phys. Rev. A **83** (2011) 052323.
- [86] C.C. Rulli and M.S. Sarandy, Phys. Rev. A **84** (2011) 042109.
- [87] J. Maziero, L.C. Celeri, R.M. Serra and M.S. Sarandy, Phys. Lett. A **376** (2012) 1540.
- [88] P. Zanardi and N. Paunkovic, Phys. Rev. E **74** (2006) 031123.
- [89] W.W. Cheng and J.M. Liu, Phys. Rev. A **82** (2010) 012308.
- [90] J. Batle and M. Casas, Phys. Rev. A **82** (2010) 062101.
- [91] L. Justino and T.R. de Oliveira, Phys. Rev. A **85** (2012) 052128.
- [92] D.L. Deng, C. Wu, J.L. Chen, S.J. Gu, S. Yu and C.H. Oh, arXiv: 1111.4341 [quant-ph].

- [93] The order of the QPTs can be determined by looking at the non-analyticities of the derivatives of the ground state energy. If the first derivative of the ground state energy with respect to the tuning parameter presents a finite discontinuity then the QPT is of the first order. Similarly, if the first derivative of the ground state energy is continuous, while its second derivative exhibits a finite discontinuity or divergence, then the QPT is second order (also called continuous QPT). On the other hand, if any finite-order derivatives of the ground state energy is a continuous function with respect to the relevant tuning parameter, then the QPT is called an infinite order.
- [94] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47** (1935) 777.
- [95] M.B. Plenio and S. Virmani, Quant. Inf. Compt. **7** (2007) 1-51.
- [96] W.K. Wootters, Phys. Rev. Lett. **80** (1998) 2245.
- [97] T. Yu and J.H. Eberly, Quant. Inf. Compt. **7** (2007) 459.
- [98] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **200** (1995) 340.
- [99] L. Derkaz and L. Jackdoczyk, Phys. Rev. A **72** (2005) 042321.
- [100] S. Luo, Phys. Rev. A **77** (2008) 022301.
- [101] D. Girolami, M. Paternostro and G. Adesso, J. Phys. A: Math. Theor. **44** (2011) 352002.
- [102] S. Wu, U.V. Poulsen and K. Molmer, Phys. Rev. A **80** (2009) 032319.
- [103] S. Campbell, L. Mazzola and M. Paternostro, Int. J. Quant. Inf. **9** (2011) 1685.
- [104] P.K. Pathak and G.S. Agarwal, Phys. Rev. A **70** (2004) 043807.
- [105] M.S. Kim and G.S. Agarwal, Phys. Rev. A **57** (1998) 3059.
- [106] Z. Deng, Opt. Commun. **54** (1985) 222.
- [107] S. Mahmood and M.S. Zubairy, J. Mod. Optic **37** (1990) 243.
- [108] G.S. Agarwal and S.D. Gupta, Phys. Rev. A **42** (1990) 1737.
- [109] C.A. Brasil, F.F. Fanchini and R.J. Napolitano, arXiv: 1110.2122 [quant-ph].
- [110] Z. Sun, X.M. Lu and L. Song, J. Phys. B: At. Mol. Opt. Phys. **43** (2010) 215504.
- [111] F. Altintas and R. Eryigit, J. Phys. A: Math. Theor. **43** (2010) 415306.
- [112] F.F. Fanchini, K.L. Castelano and A.O. Caldeira, New. J. Phys. **12** (2010) 073009.
- [113] J.S. Zhang, L. Chen, A. Aty and A.X. Chen arXiv:1101.5429v1.

- [114] J.B. Yuan, L.M. Kuang and J.Q. Liao, J. Phys. B: At. Mol. Opt. Phys. **43** (2010) 165503.
- [115] Z. He, J. Zou, B. Shao and S.Y. Kong, J. Phys. B: At. Mol. Opt. Phys. **43** (2010) 115503.
- [116] B.M. Garraway, Phys. Rev. A **55** (1997) 4636.
- [117] B. Bellomo, R.L. Franco, S. Maniscalco, G. Compagno, Phys. Rev. A **78** (2008) 060302.
- [118] F.Q. Wang, Z.M. Zhang, R.S. Liang, Phys. Rev. A **78** (2008) 042320.
- [119] A. Ridolfo, M. Leib, S. Savasta, M.J. Hartmann, Phys. Rev. Lett. **109** (2012) 193602.
- [120] F. Galve, G.L. Giorgi and R. Zambrini, Phys. Rev. A **83** (2011) 012102.
- [121] A. Al-Qasbi and D.F.V. James, Phys. Rev. A **83** (2011) 032101.
- [122] J. Ma, Z. Sun, X. Wang, F. Nori, Phys. Rev. A **85**, 062323 (2012).
- [123] For the most general initial states, one would also have coherence terms between the ground and subradiant states which leads to a very long-lived state rather than a stationary state (see Ref. [66] for details). In the present study, the considered initial conditions do not have such coherence terms, i.e.,  $\langle gg | \rho_{AB}(0) | \Phi^- \rangle = 0$ .
- [124] L.H. Du, X.F. Zhou, Z.W. Zhou, X. Zhou, G.C. Guo, Phys. Rev. A **86**, 014303 (2012).
- [125] B. Bellomo, R. Lo Franco, G. Compagno, Phys. Rev. Lett. **99**, 160502 (2007).
- [126] X. Liu, M. Zhong, H. Xu and P. Tong, J. Stat. Mech. **1** (2012) 01003.
- [127] E. Barouch and B.M. McCoy, Phys. Rev. A **2** (1970) 1075.
- [128] E. Barouch and B.M. McCoy, Phys. Rev. A **3** (1971) 786.
- [129] M. Zhang and P. Tong, J. Phys. A: Math. Theor. **43** (2010) 505302.
- [130] I. Titvinidze and G.I. Japaridze, Eur. Phys. J. B **32** (2003) 383.
- [131] A.A. Zvyagin, Phys. Rev. B **80** (2009) 144408.
- [132] M. Shiroishi and M. Takahashi, J. Phys. Soc. Jpn. **74** (2005) Suppl. pp. 47-52.
- [133] G. Kato, M. Shiroishi, M. Takahashi and K. Sakai, J. Phys A: Math. Gen. **36** (2003) L337.
- [134] M. Takahashi, G. Kato and M. Shiroishi, J. Phys. Soc. Jpn. **73** (2004) 245.
- [135] H.J. Lipkin, N. Meshkov and A.J. Glick, Nucl. Phys. **62** (1965) 188.

- [136] P. Ring and P. Schuck, *The Nuclear Many-Body Problem*, Springer, (New York, 1980).
- [137] P. Ribeiro, J. Vidal and R. Mosseri, Phys. Rev. E **78** (2008) 022106.
- [138] R. Orus, S. Dusuel and J. Vidal, Phys. Rev. Lett. **101** (2008) 025701.
- [139] J. Vidal, G. Palacios and R. Mosseri, Phys. Rev. A **69** (2004) 022107.
- [140] S. Dusuel and J. Vidal, Phys. Rev. B **71** (2005) 224420.
- [141] Z.Y. Sun, L. Li, N. Li, K.L. Yao, J. Liu, B. Luo, G.H. Du, H.N. Li, EPL **95** (2011) 30008.

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### Publications

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2. **Ferdi Altintas** and Resul Eryigit, "Dynamics of entanglement and Bell non-locality for two stochastic qubits with dipole-dipole interaction", Journal of Physics A: Mathematical and Theoretical, **43** (2010) 415306.
3. **Ferdi Altintas**, "Geometric measure of quantum discord in non-Markovian environments", Optics Communications, **283** (2010) 5264.
4. **Ferdi Altintas** and Resul Eryigit, "Quantum correlations between identical and unidentical atoms in a dissipative environment", Journal of Physics B: Atomic, Molecular and Optical Physics, **44** (2011) 125501.
5. **Ferdi Altintas** and Resul Eryigit, "Entanglement dynamics of two qubits under the influence of external kicks and Gaussian pulses", Journal of Physics A: Mathematical and Theoretical, **44** (2011) 405302.
6. **Ferdi Altintas** and Resul Eryigit, "Creation of quantum correlations between two atoms in a dissipative environment from an initial vacuum state", Physics Letters A, **376** (2012) 1791.
7. **Ferdi Altintas** and Resul Eryigit, "Correlation and nonlocality measures as indicators of quantum phase transitions in several critical systems", Annals of Physics, **327** (2012) 3084.

8. **Ferdi Altintas**, Arzu Kurt and Resul Eryigit, "Classical memoryless noise-induced maximally discordant mixed separable steady states", Physics Letters A, **377** (2012) 53.
9. **Ferdi Altintas** and Resul Eryigit, "Control and manipulation of entanglement between two coupled qubits by fast pulses", Quantum Information Processing (2013, Accepted).
10. **Ferdi Altintas** and Resul Eryigit, "Dissipative dynamics of quantum correlations in strong coupling regime", Physical Review A (2013, Accepted).