

ROBUST OPTIMIZATION MODELS FOR NETWORK REVENUE MANAGEMENT

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By
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Robust Optimization Models for Network Revenue Management

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ROBUST OPTIMIZATION MODELS FOR NETWORK REVENUE MANAGEMENT

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Effective capacity allocation methods play a crucial role in Network Revenue Management. Yet, current methods for determining optimal capacity controls under uncertainty, such as stochastic optimization, often assume a known probability distribution for unknown parameters. This assumption may degrade a model's performance when faced with unexpected data patterns. This thesis explores a novel approach through robust optimization to address stochastic resource allocation problems. We introduce a heuristic based on these robust formulations to derive actionable results. Through extensive simulations focused on seat allocation problems within the revenue management domain, our proposed formulations demonstrate improved worst-case performances. Notably, even under favorable scenarios, our solutions remain comparable to existing methods in the revenue management literature.

Keywords: Robust Optimization, Network Revenue Management, Capacity Allocation.

ÖZET

AĞ GELİR YÖNETİMİ İÇİN SAĞLAMCI OPTİMİZASYON MODELLERİ

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Etkin kapasite ayırma yöntemleri Ağ Gelir Yönetiminde önemli bir rol oynar. Ancak, belirsizlikler altında optimal kapasite kontrollerini belirlemek için mevcut yöntemler, örneğin stokastik optimizasyon, genellikle bilinmeyen parametreler için bilinen bir olasılık dağılımı varsayar. Bu varsayım, beklenmedik veri desenleriyle karşılaşıldığında modelin performansını düşürebilir. Bu tezde, stokastik kaynak tahsis problemlerini ele almak için sağlamcı optimizasyon formülasyonları yoluyla yenilikçi bir yaklaşım sunuyoruz ve bu sağlamcı formülasyonlara dayalı olarak uygulanabilir sonuçlar elde etmek için bir sezgisel yöntem tanıtıyoruz. Gelir yönetimi alanındaki koltuk tahsisi problemlerine odaklanan kapsamlı simülasyonlar aracılığıyla, önerdiğimiz formülasyonların en kötü durum performanslarını inceliyoruz.

Anahtar sözcükler: Sağlamcı Optimizasyon, Ağ Gelir Yönetimi, Kapasite Ayırma.

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Chapter 1

Introduction

Revenue Management (RM) is a critical practice employed by businesses across various industries to optimize pricing strategies and maximize profits. It involves strategically allocating limited resources to different market segments to extract the highest possible revenue. Companies can adjust pricing, distribution, and marketing tactics by analyzing customer behavior, market trends, and competitive dynamics to capitalize on fluctuating demand conditions.

Two primary problems emerge in RM: single-leg and network problems. Single-leg RM optimizes pricing and allocation decisions for individual products or services that use only one resource. It applies to businesses like hotels or airlines when bookings and reservations are controlled per night or flight, respectively. Conversely, network RM extends this approach to interconnected systems of resources or networks, such as multi-leg air travel routes, air cargo, or multiday stays at hotels. In network RM, the strategic allocation of resources considers the interdependencies and synergies among products or services that simultaneously consume more than one resource to maximize overall revenue. This entails not only pricing and inventory allocation decisions for individual segments or resources but also the coordination of capacity across all the resources within the network to exploit revenue opportunities effectively. Network RM thus requires a sophisticated understanding of demand patterns, competitive dynamics, and network structures to optimize revenue generation across the entire network.

Capacity allocation is a critical component of network RM, essential for businesses in industries characterized by perishable inventory or limited capacity, such as airlines, hotels, rental car companies, and entertainment venues, as mentioned above. From a network RM perspective, capacity constraints extend beyond individual assets to include broader limitations within interconnected systems or networks. Airlines, for example, must consider availability of seats on all flights of a multi-flight route, while hotels must manage room availability.

The classical airline RM problem involves maximizing the revenue generated from selling seats on a flight. Airlines face the challenge of setting prices for different seats to balance demand across various customer segments while considering competition, seasonality, and booking patterns. The objective is to sell the right seat to the right customer at the right price and time [1]. This involves dynamic pricing strategies, inventory control, and demand forecasting. Effective RM ensures that airlines optimize their load factor (the percentage of available seats that are filled) and total revenue, considering cancellations, overbooking, and no-shows.

The seat allocation problem, a subset of the broader RM issue, focuses on the distribution of seats among various fare classes. A fare class is a category of airline tickets that is differentiated by multiple factors such as price, flexibility, booking conditions, and availability. For instance, like many other airlines, Turkish Airlines uses a system of fare classes to segment its tickets into EcoFly, ExtraFly and PrimeFly. Airlines must decide how many seats to allocate to different fare classes to maximize revenue while managing capacity constraints. This involves predicting the demand for each fare class and adjusting seat availability dynamically. Factors considered include historical booking data, competitor actions, and remaining time until departure. The goal is to protect enough seats for high-fare passengers who book closer to the departure date while ensuring that seats are not left unsold when the flight takes off. This balancing act requires sophisticated algorithms and real-time data analysis to optimize seat allocation.

As a result, RM extends beyond price adjustments, embracing dynamic inventory management strategies where capacity is strategically allocated across various fare classes or product offerings to optimize revenue potential. For example, hotels may reserve a portion of inventory for higher-priced, last-minute bookings while offering discounted rates for early bookings. This strategic allocation of resources leads the way to maximum profitability by matching supply with demand across different customer segments.

As highlighted, RM depends on knowledge of demand patterns, which often remain unknown to sellers. While firms may have access to historical data to determine demand distributions, the exact demand for products is typically uncertain. To address this challenge, various techniques are employed, including stochastic programming and robust optimization. Stochastic optimization can help decide how much capacity to allocate to different products or services, considering uncertain demand patterns assuming the probability distribution of demand is known. Robust optimization techniques, on the other hand, often involve formulating optimization problems without using distributional information while optimizing the worst-case performance (e.g., max-min revenue or min-max regret) over a set of possible scenarios.

Understanding the demand dynamics for different segments to maximize overall profitability is challenging for sellers due to seasonality and trends, market volatility, complex customer behavior, etc. Consequently, using only an estimate of demand (e.g., mean value) to make

capacity allocation decisions can only partially capture the variability in real-world demand scenarios. Robust optimization is employed in this thesis to handle this. Robust optimization seeks to create solutions that perform well across a range of possible scenarios without explicitly taking into account the probability distribution of uncertainties.

1.1 Aim and Structure of the Thesis

The models and experiments implemented in this thesis have two main purposes. Firstly, they address the uncertainty in demand forecasting within real-life scenarios. This study adopts a robust approach by considering demand as an uncertain parameter originating from an uncertainty set. By incorporating uncertainty into the capacity allocation problem, the thesis aims to develop robust strategies that can effectively handle variations in demand and mitigate the risks associated with inaccurate forecasts.

Secondly, the thesis addresses the inherent conservatism often associated with robust solutions. While robust optimization offers protection against worst-case scenarios, it may lead to overly conservative solutions, limiting the potential for revenue optimization [2]. To overcome this limitation, the study uses different uncertainty sets to compare the conservatism of solutions. By systematically evaluating the performance of various uncertainty sets, the thesis seeks to identify the most suitable approach for balancing robustness and revenue optimization in capacity allocation decisions.

Overall, this thesis makes an effort to contribute to the field of RM by offering robust capacity allocation solutions that effectively address demand uncertainty while considering conservatism. Through extensive computational experiments, it aims to provide insights into the practical implications of different uncertainty sets on decision-making processes, ultimately enhancing the efficiency and effectiveness of capacity allocation strategies in dynamic and uncertain environments.

The remainder of the thesis is structured as follows: Section 2 presents a literature review. In Section 3, we describe our approach and models. Section 4 outlines extensive simulation runs for both independent and dependent data. Finally, Section 5 provides the conclusion.

1.2 Terminology

The table below provides definitions for key terms, concepts, and abbreviations used throughout this master's thesis. This table serves as a reference guide for readers to ensure clarity and understanding of the terminology employed within the thesis.

Term or Abbreviation	Definition
O-D	Origin-Destination
ODF	Origin-Destination Fare Class
Fare Class	Classification indicating the product type sold and the product's fare (unit revenue)
Demand Factor(DF)	Ratio of the average realized demand to the plane capacity
Bid-price	The marginal value of an asset
Uncertainty Set	A collection of possible scenarios or realizations of uncertain parameters
Booking limit	The maximum number of reservations allowed for a specific fare class
Protection level	The minimum number of seats reserved for higher-fare customers
Nesting	A strategy where higher fare classes can access the inventory of lower fare classes.
DLP	Deterministic Linear Program
PNLP	Probabilistic Non-linear Program
DAVN	Displacement Adjusted Virtual Nesting
RM	Revenue Management
NRM	Network Revenue Management
LBH	"low-before-high" arrival pattern of booking requests where demand for the lowest fare class arrives first and arrivals continue in lowest to highest fare class order.

Table 1.1: RM and Robust Terminology

Chapter 2

Problem Definition & Literature Review

While capacity allocation issues are relevant across various sectors, like hotels and car rentals, our attention will be directed towards problems related to allocating seat capacity in airline networks. The exploration of seat inventory management began in the early 1970s, initially focusing on distributing seats on individual flight legs by balancing marginal seat revenues [1]. This research gradually advanced to utilize mathematical programming and network flow analysis to accommodate various passenger itineraries and multiple fare classes for airline networks. We introduce the classical models of network RM below.

The model is taken from [3]. The simplest model of the network revenue management (NRM) problem can be expressed as follows: A network has k resources (flights or legs) and n products to be offered to customers. We also define K and N as sets of resources and products, respectively. Let $\mathbf{c} := (c_1, \dots, c_k) \in \mathbb{Z}_+^k$ denote the vector of capacities. The incidence matrix A shows which products (or fare classes) use which resources and is defined as:

$$A = [a_{ij}], i = 1, \dots, k \text{ and } j = 1, \dots, n, \text{ where}$$
$$a_{ij} = \begin{cases} 1 & \text{if product } j \text{ uses resource } i \\ 0 & \text{otherwise.} \end{cases}$$

$a_{i\cdot}$ is denoted as i^{th} row of incidence matrix and $a_{\cdot j}$ is denoted as j^{th} column. At the onset of the sales horizon, denoted as $t = T$, and at departure, denoted as $t = 0$, for any given time-to-go t , we denote the state of the system as (t, x) where $x \in \mathbb{Z}_+^k$ represents the vector indicating the remaining capacity of resources (i.e. remaining seat inventory). The initial state

is represented as (T, c) [4].

The problem is formulated over T discrete time periods. In period t , demand is characterized as the realization of a single random vector $\mu(t) = (\mu_1(t), \mu_2(t), \dots, \mu_n(t))$. If $\mu_j(t) = 0$, this indicates no request for product j occurred. When $\mu(t) = 0$ (all components equal to zero) it indicates the absence of demands for any product at time t . We denote the revenue as r_j and the revenue vector as $r = (r_1, r_2, \dots, r_n)$. Let $u_j(t)$ be 1 if a request from ODF j is accepted at time t , 0 if it is not. This means that time periods are chosen such that there is at most one request in every period. We define the decision vector $u(t) = (u_1(t), u_2(t), \dots, u_n(t))$.

The decision vector $u(t)$ is dependent on the remaining capacity components of vector x and the revenue vector r , $u(t) = u(t, x, r)$. The decision vector $u(t)$ is limited to the set

$$U(x) = \{u \in \{0, 1\}^n \mid Au \leq x\}.$$

Then, we will define the $V_t(x)$ to represent the highest achievable total expected revenue that can be obtained at state (x, t) . Then, $V_t(x)$ must satisfy the Bellman equation[4]:

$$V_t(x) = E \left[\max_{u \in U(x)} \{\mu(t)^T u + V_{t+1}(x - Au)\} \right] \quad (2.1)$$

with the boundary condition $V_{T+1}(x) = 0, \forall x$. u^* is an optimal control if and only if it satisfies:

$$u_j^*(t, x, r_j) = \begin{cases} 1 & r_j \geq V_{t+1}(x) - V_{t+1}(x - a_j), \quad a_j \leq x \\ 0 & \text{otherwise.} \end{cases}$$

This concept is based on the intuitive notion that revenue r_j for product j is accepted only if it surpasses the opportunity cost of the capacity used to fulfill the request [5]. Although this optimality condition is easy to understand, it poses computational issues and implementation challenges.

In managing network problems, capacity availability can be controlled through various approaches. Partitioned booking limits involve dividing the total available inventory into distinct partitions, each with its own booking limit. These partitions may be based on factors such as customer segments, booking channels, or time periods [1]. By allocating specific booking limits to each partition, businesses can tailor their capacity allocation strategies to different market segments or demand scenarios, thereby optimizing revenue generation while maintaining control over inventory availability. One drawback of this policy is that it divides the seat capacity of the airline's network across all the fare-classes. When seats for a particular booking class are fully booked, higher-class booking requests for this class will be rejected even when there are seats available in other classes, potentially resulting in lost revenue. Intuitively, seats assigned to less profitable booking classes should also be accessible to more profitable classes, a concept known as nesting [6].

Nested booking limits, on the other hand, involve establishing hierarchical booking limits that are contingent upon the fulfillment of higher-level constraints. This approach allows businesses to prioritize reservations based on their importance or profitability, ensuring that high-fare bookings are accommodated by protecting enough resources from lower-priority ones [1]. In the structure of nested booking limits, the highest fare customers are given priority in capacity allocation, with a dedicated portion of the plane's capacity reserved for them. This hierarchical approach continues downwards, with each fare segment being allocated capacity based on the allocation of higher fare segments. This ensures that higher-value customers are accommodated while allowing for efficient allocation of capacity across different fare segments. In Figure 2.1a, a total of 9 seats are allocated to 3 fare classes (indicated as L for lowest fare, M for the second lowest fare, and H for the Highest fare). Each class gets 3 seats in 2.1a. In Figure 2.1b up to 3 seats are available for L, up to 6 seats for M and up to 9 for H. There are 3 seats protected for H from the other two classes.

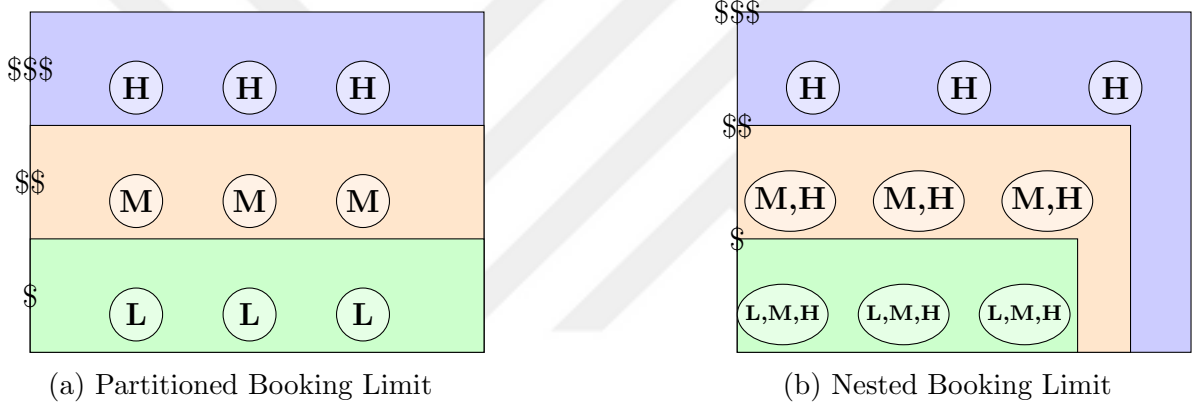


Figure 2.1: Visualization of Partitioned vs Nested Booking Limit Techniques

Applying nested booking limits directly to NRM poses challenges due to the complex interactions and dependencies across interconnected systems. Therefore, bid-price control policies offer much simpler control schemes. In network scenarios, bid-price control entails setting a threshold price, or bid-price, for each resource. Essentially, this bid-price reflects the estimated marginal cost incurred by the network for utilizing the next incremental unit of the resource's capacity and mimics the structure of the optimal control we discussed earlier. Upon receiving a product request, the revenue generated by the request is compared to the sum of the bid prices of the required resources. If the revenue exceeds this total, the request is accepted; otherwise, it is declined [7].

Displacement Adjusted Virtual Nesting (DAVN) scheme was introduced by Smith and Penn of American Airlines to implement nested booking-limit control in a network [8]. In the DAVN scheme, the original fare classes on each leg are substituted with virtual buckets, also known as value buckets. Each bucket corresponds to a range of fare values [9]. By adjusting the prices of each fare class using bid prices, on each leg, the products are arranged in descending

order based on their adjusted price, and then grouped into a specific number of value buckets. Nested booking limits for these buckets are subsequently calculated on each leg, utilizing well-known techniques, such as Deterministic Linear Program [10]. An incoming booking request is approved if there is sufficient capacity available and if, on every leg, the booking limit of the associated bucket has not been surpassed. Details of these procedures are provided in subsequent sections. Analysis of bid-price type controls is beyond the scope of this thesis.

Also, overbooking is a critical component of capacity allocation strategies in industries where no-show rates are common, notably in airlines and hotels. Overbooking involves selling more inventory than available capacity, anticipating cancellations or no-shows [11]. While seemingly dubious, this strategy safeguards sellers against potential losses due to empty seats caused by cancellations and no-shows, ensuring optimal resource utilization. However, it is important to note that within the context of this thesis, overbooking is outside of the scope of consideration.

For the majority of networks with realistic sizes, it's impractical to find an exact solution for the recursive problem in equation 2.1 [3]. Instead, solutions rely on various approximation methods. One such method involves simplifying the network model, often by treating the problem as a static mathematical program.

2.1 Deterministic Linear Programming (DLP)

The dynamic, multi-period problem can be approximated by a deterministic linear program (DLP) [1]. At time $t=0$, DLP simplifies the equation 2.1 into

$$\begin{aligned} & \text{maximize} && \mathbf{r}\mathbf{x} \\ & \text{subject to} && \mathbf{A}\mathbf{x} \leq \mathbf{c}, \\ & && 0 \leq \mathbf{x} \leq E[\mathbf{d}]. \end{aligned} \tag{2.2}$$

where $\mathbf{d}$ represents the vector of demand to come over T periods and $\mathbf{r}$ stands for the revenue vector associated with products (fare classes). The decision variable vector $\mathbf{x}$ denotes the amount of each product to sell. The simplification essentially treats demand to come as deterministic, considering it to be at most equal to its mean $E[\mathbf{d}]$. The DLP determines the amount of products to be sold, hence the capacity of each resource to be used ($\mathbf{A}\mathbf{x}$).

The primary advantage of the DLP model lies in its computational efficiency. Its simplicity and speed make it widely favored in practical applications [3]. However, the weakness of the DLP is its disregard for all distributional information except the mean demand [1]. The effectiveness of the DLP model varies based on factors such as the network type, the sequence of fare product arrivals, and the frequency of re-optimization (the model can be reformulated and

solved at the beginning of any period using the remaining available capacity and mean demand to come). Although DLP simply assumes demand is known (deterministic), various simulation studies conducted by multiple researchers have indicated that frequent re-optimization enhances the performance of DLP bid prices significantly [1]. Bid prices are dual prices that DLP returns at optimality [12]. Bid-price controls stand out from both booking limits and protection levels due to their revenue-based approach rather than being class-based controls. Specifically, in bid-price controls, a threshold price is set (which may vary based on factors such as remaining capacity or time). So, frequently re-optimizing the DLP model enables us to obtain updated bid prices and adjust the thresholds accordingly, leading to improved results.

2.2 Probabilistic Non-Linear Program (PNLP)

The primary limitation of the DLP approximation is its treatment of demand, assuming it equals its mean value, disregarding demand uncertainty [13]. In contrast, the Probabilistic Non-Linear Program (PNLP) model addresses this limitation by incorporating the randomness in demand. It achieves this by integrating the expected sales of product j , denoted as $E[\min\{d_j, x_j\}]$, into the objective function within the context of partitioned allocation.

$$\begin{aligned} & \text{maximize} && \sum_{j \in N} r_j E[\min\{d_j, x_j\}] \\ & \text{subject to} && \mathbf{A}\mathbf{x} \leq \mathbf{c}, \\ & && 0 \leq \mathbf{x}. \end{aligned}$$

where d_j is the random demand in class j and x_j has the same definition as in the DLP case. The term $E[\min\{d_j, x_j\}]$ in the objective function, which represents the expected sales of product j , captures the randomness in demand. It is important to note that solving PNLPP requires more information compared to solving DLP. In DLP, only the mean demands are utilized as inputs, whereas in PNLPP, the distributions of demands are necessary to formulate the objective function.

If demand is discrete, it is possible to transform the model into a linear program, albeit with a significantly large number of variables. This transformation involves assigning a variable to each product and to each unit of capacity. Specifically, we define a variable z_{jd} to represent the units of capacity allocated to product j . Then, the PNLPP (Probabilistic Nonlinear Programming)

model can be expressed as follows:

$$\begin{aligned}
& \text{maximize} && \sum_{j \in N} r_j \sum_{l=1}^{M_j} z_{jl} P(d_j \geq l) \\
& \text{subject to} && \sum_{l=1}^{M_j} z_{jl} = x_j \quad \forall j \in N, \\
& && \sum_{j \in N} a_{ij} x_j \leq c_i \quad \forall i \in K, \\
& && 0 \leq z_{jl} \leq 1 \quad \forall j \in N, \text{ and } l = 1, \dots, M_j.
\end{aligned}$$

where M_j represents an upper bound on the capacity allocated to product j (for instance, it could be the minimum capacity remaining among the resources utilized by product j). These two formulations are equivalent to each other because $E[\min\{d_j, x_j\}] = \sum_{l=1}^{x_j} P(d_j \geq l)$ for discrete distributions [1].

The optimal solutions of PNL and DLP lead to a partitioning of capacity to distinct products. They overlook the possibility of using the same capacity to fulfill demands for different classes since these types of demands arise in distinct scenarios [14]. More precisely, the PNL necessitates a decision at the initial stage regarding whether to allocate capacity for low or high demand but fails to consider the option of reserving the same capacity for both types of demand simultaneously. When the partitioned capacities are used to control bookings, the following undesirable scenario can occur in practice: Suppose $x_1 = 10$ seats and only 5 requests are received in class-1, then $10-5=5$ seats would be unused. If class-2 fare is higher than class-1 and $x_2 = 10$ seats while demand in class 2 is 15, then the excess demand of 5 seats in class-2 cannot be accommodated by the unused capacity of class-1 because of the rigid allocation of seats to each fare class. In partitioned control, each fare class is allocated a fixed number of seats independently of the other fare classes. This means that unused capacity in one fare class cannot be utilized to satisfy excess demand in another fare class. While partitioned control simplifies inventory management by providing clear booking limits for each fare class and easy to implement, it lacks the flexibility to dynamically reallocate capacity based on demand fluctuations. As a result, airlines may miss out on revenue opportunities when demand exceeds the allocated capacity in one fare class but remains unmet due to unused capacity in another fare class.

2.3 Robust Optimization

Many optimization problems in real-life scenarios involve parameters that are not precisely known. Various sources contribute to this uncertainty, including measurement, estimation, and implementation errors in the underlying processes [15]. To address such uncertainty, two

primary classes of techniques are commonly employed: stochastic optimization and robust optimization.

Stochastic optimization covers techniques such as the PNL where all uncertain parameters are presumed to adhere to a known probability distribution. However, stochastic optimization suffers from two primary drawbacks [2]. Firstly, the precise distribution of uncertain parameters is frequently unknown and must be estimated, typically using historical data if accessible. This reliance on estimation can result in significant inaccuracies in the solutions obtained. Secondly, stochastic optimization problems are typically challenging to solve [2]. Because evaluation of the expected value function involves calculation of the corresponding multivariate integrals, which can be done analytically only for special cases. Therefore, typically, one employs a finite discretization of the random data which allows to write the expectation in a form of summation [16]. Note, however, that if a random vector of size s , with each of its elements having 3 possible realizations, leads to the total number of scenarios to be 3^s , i.e., the number of scenarios grows exponentially with the size of the data vector.

On the other hand, robust optimization provides an alternative approach to handle uncertain parameters in optimization problems [15]. Unlike stochastic optimization, it does not rely on assuming a specific probability distribution for uncertain parameters. Instead, robust optimization considers a set of possible values for these parameters and aims to find solutions that perform well across all these scenarios [17], [18]. This approach is advantageous because it does not require precise knowledge of the probability distributions, reducing the reliance on estimation and historical data. Additionally, robust optimization problems are often more tractable than stochastic ones, as they typically involve deterministic optimization techniques rather than complex integrals or estimations [1]. By considering a set of scenarios rather than a probability distribution, robust optimization can effectively handle uncertainty without the computational burden associated with stochastic methods.

The set of possible scenarios in robust optimization is called the uncertainty set. The uncertainty sets accounts for various sources of uncertainty such as fluctuations in demand [19]. By incorporating uncertainty sets such as an interval uncertainty set, optimization problems can effectively address uncertainty without requiring detailed knowledge of probability distributions, making them more adaptable to real-world complexities and reducing the computational burden compared to stochastic methods.

In addition to uncertainty sets, establishing a suitable interval for each uncertain parameter is a significant concern in robust formulations [20]. When historical data are available, they can be utilized to define the bounds of a parameter. Typically, the minimum and maximum observed data points are employed as the lower and upper bounds, respectively. For instance, let $\tilde{\mathbf{d}}$ represent a demand vector belonging to $[\mathbf{l}, \mathbf{u}]$, where $\mathbf{l}$ denotes the lower bound and $\mathbf{u}$

denotes the upper bound. Consequently, the uncertainty set denoted $\mathcal{U}$ takes the following form:

$$\mathcal{U} = \left\{ \tilde{\mathbf{d}} \mid \tilde{\mathbf{d}} \in [\mathbf{l}, \mathbf{u}] \right\}$$

Moreover, if information regarding the standard deviation of the uncertain coefficients is accessible, it can be utilized to establish the bounds of intervals [21]. This proposed approach involves employing the normalized ratio of perturbation to the standard deviation of the historical data to represent uncertainties. Subsequently, the uncertainty set used throughout this thesis is defined as follows:

$$\mathcal{U}_s = \{ \bar{\mathbf{d}} + P\mathbf{u} : \|\mathbf{u}\|_s \leq \epsilon \} \quad (2.3)$$

where $\bar{\mathbf{d}}$ is expectation of demand vector of size $|N|$, where N is the set of fare classes (products), P is square-root of covariance matrix with size $|N| \times |N|$, or other decomposition (details regarding this will be provided in later section), s is the norm of choice and ϵ determines the size of uncertainty set.

Robust optimization plays a vital role in RM strategies, particularly in industries like airlines where uncertainty is inherent [10]. In airline RM, robust optimization techniques are applied to allocate capacities effectively while considering various uncertain factors such as demand fluctuations. By employing robust optimization, airlines can develop strategies that are resilient to different scenarios, ensuring stable performance even when faced with unforeseen events or inaccuracies in demand forecasts [22]. This approach allows airlines to optimize capacity allocation decisions without overly relying on specific probability distributions or historical data, thus enhancing overall revenue generation and proposing tractable solutions [23].

While significant advancements have been achieved in robust optimization, existing methods often assume independence among uncertainties in parameters [24]. Nevertheless, in practical scenarios, correlations may exist among these uncertain parameters. Traditional robust optimization methods that disregard correlation may result in sub-optimal solutions. For instance, in RM, the correlation between factors such as demand for different products, customer preferences, and external factors like economic conditions can impact revenue outcomes [25]. Ignoring or misrepresenting these correlations can lead to sub-optimal decisions and missed revenue opportunities. Therefore, incorporating correlations among uncertain parameters in optimization models enables enhanced decisions, particularly in RM contexts where complex interdependencies exist.

In the RM literature, considering dependencies of demand among fare classes often involves utilizing choice models, which offer a systematic approach to understanding customer behavior, including the number of customers, their arrival order, and their preferences. Choice models

provide a comprehensive framework for analyzing dependencies among various factors influencing consumer decisions, such as pricing, product features, and competitor offerings [26]. In simple terms, a choice model involves predicting which fare class or itinerary a customer chooses when provided a set of alternatives. By leveraging choice models, one can gain insights into customer preferences and tailor pricing and allocation strategies and product assortments to maximize revenue and profitability [7]. Although choice models offer valuable insights into dependency structures in revenue management, this thesis does not delve into their intricacies, focusing instead on alternative methodologies to address dependencies and optimize revenue outcomes.

In this thesis, we address dependencies in RM by employing covariance matrices. As mentioned before, we have the matrix P in the definition of the uncertainty set. P is associated with a covariance matrix. Thiele defines P as the square root of the covariance matrix in her PhD thesis [27]. However, we examine different ways of producing the P matrix by using various decomposition techniques of the covariance matrix. More on this topic will be provided in the subsequent chapter.

Unlike choice models, which analyze customer behavior, covariance matrices focus on statistical relationships among demand parameters in aggregate form. By incorporating covariance matrix, we aim to capture joint variability between fare classes. This allows us to identify demand correlation influencing capacity allocation decisions and develop strategies to optimize resource utilization and revenue generation.

2.4 Robust Network Revenue Management

In this section, we review robust optimization applications in NRM, aiming to provide insights into its theoretical foundations and methods. There are several key methodologies and contributions.

Perakis and Roels (2010) and An et al. (2021) both explore robust formulations for capacity allocation in NRM under polyhedral uncertainty, which is an uncertainty set with ℓ_1 -norm. Perakis and Roels introduce maximin and minimax regret strategies, developing closed-form solutions and LP formulations for airline revenue management problems. They show that the minimax regret policy often matches or outperforms traditional methods such as DLP, particularly in scenarios with correlated or uncertain demands, while the maximin policy ensures minimum revenue across all scenarios [10]. An et al. takes into account scenarios with only known lower and upper bounds on demand. They create a linear programming model for the maximum regret problem and use a generic algorithm to find booking limits [28]. Their

experiments indicate that this robust approach outperforms the DLP in terms of minimax regret.

There is also a data-driven approach to robust optimization in both NRM and the construction of uncertainty sets. Zhang (2023) and Thiele (2004) both address demand uncertainty in NRM by developing robust formulations that use historical data. Zhang constructs a distributionally robust model using statistical goodness-of-fit tests, enabling conic optimization with performance guarantees. This model is extended to dynamic versions, combining robust optimization with approximate dynamic programming. Empirical tests show performance differences under various data conditions [13]. Thiele uses historical data to create robust models for single-leg and network settings, optimizing over the sample average after removing some best-case scenarios to ensure robustness. This approach leads to tractable problems [27]. Additionally, Bertsimas, Gupta, and Kallus propose designing uncertainty sets for robust optimization using hypothesis tests, which provide probabilistic guarantees and are less conservative than traditional robust methods [18]. This new data-centered paradigm adapts traditional robust optimization techniques to high-quality data, maintaining robustness while reducing conservatism.

This thesis makes contributions to the field of NRM by introducing robust models with three distinct uncertainty sets, providing a comprehensive comparison with traditional methods such as DLP and PNLP. We conduct extensive simulation runs to evaluate the impact of various factors, including demand factor (DF), standard deviation (SD), arrival pattern and different types of booking controls on the performance of these robust models. Additionally, we examine the role of demand dependence on the robustness and effectiveness of these models compared to established approaches. This thorough analysis not only provides insights into the practical implications of using robust optimization in volatile demand environments but also offers understanding into improving the reliability and efficiency of RM strategies through the incorporation of advanced uncertainty sets and considerations of demand dependencies.

Unlike other methods in the literature, we compare different uncertainty sets and their conservativeness under various scenarios, including different demand factors, arrival patterns, and booking controls. Additionally, we consider demand dependency by using the square root of the covariance matrix or Cholesky decomposition.

Chapter 3

Formulation

In this chapter, we introduce a new approach to tackle the revenue management capacity allocation problem in the face of demand uncertainty. Our main goal is to enhance the DLP to generate robust solutions. Although the DLP may not be as closely aligned with dynamic programming models [4], it remains widely used in both academic research and practical applications.

DLP refers to using mathematical optimization model to maximize revenue or profit under certain deterministic conditions. In this approach, the model assumes that all parameters, such as demand, capacity and its usage, are known with certainty. The formulation is available in equation 2.2 in section 2.1.

We modify the DLP to incorporate uncertainty into the model. Initially, we redefine the meaning of the decision variable $\mathbf{x}$. To facilitate this adjustment, we introduce a new variable, $\mathbf{y}$. These $\mathbf{y}$ variables represent the percentage of the expected demand of a product that will be fulfilled. Hence, the relationship between the original formulation and the revised one can be expressed as $y_j \bar{d}_j = x_j$ for all $j \in N$. We will use the following variant of the DLP formulation as a building block for our robust formulations.

$$\begin{aligned} & \underset{0 \leq \mathbf{y} \leq 1}{\text{maximize}} && \sum_{j \in N} r_j y_j \bar{d}_j \\ & \text{subject to} && \sum_{j \in N} a_{ij} y_j \bar{d}_j - c_i \leq 0 \quad \forall i \in K. \end{aligned} \tag{3.1}$$

Note that these formulations in (2.2) and (3.1) are equivalent as there is a one-to-one relationship between y_j and x_j for all j .

3.1 Robust Model

We enhance the capacity allocation problem with robust optimization techniques by accounting for worst-case scenarios in realized demand. We want to address the problem in which the demand $\bar{\mathbf{d}}$ is uncertain, but is known to reside in an uncertainty set $\mathcal{U}$ [29]. Notably, as we show below, demand and uncertainty play a crucial role in both the objective function and the constraints. Prior research on robust versions of similar problems has primarily focused on incorporating robustness measures solely in the constraints [10] or solely in the objective function. Note that the demand information only appears in the constraints of the DLP.

We define an uncertainty set using an arbitrary ℓ_s -norm where s indicates the type of the norm used such as ℓ_1 or ℓ_∞ . The uncertainty set based on the norm is denoted as $\mathcal{U}_s$, and defined as in 2.3. Note that a higher ϵ value corresponds to a larger region from which demand realizations are drawn in an uncertainty set. Instead of the vector $\mathbf{d}$ being a random vector with known mean and/or probability distribution, we treat $\mathbf{d}$ as the demand realization that lies in $\mathcal{U}$. This also necessitates defining appropriate objectives to achieve within the uncertainty set and to determine which realizations help achieve that objective. We choose minimization of the revenue within the uncertainty set in the objective function and maximization of resource use in the constraints. Our formulation is:

$$\begin{aligned} & \underset{0 \leq \mathbf{y} \leq 1}{\text{maximize}} \quad \left\{ \underset{\mathbf{d} \in \mathcal{U}}{\text{minimize}} \sum_{j \in N} r_j y_j d_j \right\} \\ & \text{subject to} \quad \max_{\mathbf{d} \in \mathcal{U}} \sum_{j \in N} a_{ij} y_j d_j - c_i \leq 0 \quad \forall i \in K. \end{aligned} \tag{3.2}$$

Where $a_{i \cdot} \in \mathbb{R}^N$ represents the i 'th row of the matrix $A \in \mathbb{R}^{K \times N}$. Here, $\mathbf{d}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^N$, while $\mathbf{c}$ belongs to $\mathbb{R}^K$.

In the robust version of our formulation, we begin by determining the minimum value of $\sum_{j \in N} r_j y_j d_j$ by iterating through all possible demand scenarios in the objective function. Similarly, for the constraint, we find the maximum value of $\sum_{j \in N} a_{ij} y_j d_j$ by iterating through all possible demand scenarios in the constraints. By solving the problem using this approach, we enhance the robustness of the proposed solutions. The robust formulation, while optimizing over demand scenarios, also has to determine the vector $\mathbf{y}$ that maximizes revenues and satisfies the constraints.

Different uncertainty sets lead to different robust counterparts (or formulations) [29]. In the subsequent sections, we will analyze uncertainty sets and the resulting optimization problems.

3.1.1 Uncertainty Set: ℓ_1 -norm

ℓ_1 -norm is a type of norm that is defined as the sum of the absolute values of the vector components. Let us assume that we have vector of u with size n . Then the ℓ_1 -norm of this vector is:

$$\|u\|_1 = |u_1| + |u_2| + \dots + |u_n|$$

We can define the uncertainty set by using ℓ_1 -norm as:

$$\mathcal{U}_1 = \{\bar{d} + Pu : \|u\|_1 \leq \epsilon\} \quad (3.3)$$

where u is the source of uncertainty. In this case, we can write the constraints in the formulation provided in equation (3.3) as follows: $\forall i \in K$ knowing the fact that $\mathbf{d} \in \mathcal{U}_1$:

$$\max_{u: \|u\|_1 \leq \epsilon} \sum_{j \in N} a_{ij} y_j (\bar{d} + Pu)_j - c_i \leq 0, \quad \forall i \in K$$

Now for simplicity we focus on resource i . Observe that the left-hand side of constraint i is:

$$\sum_{j \in N} a_{ij} y_j \bar{d}_j - c_i + \max_{u: \|u\|_1 \leq \epsilon} \sum_{j \in N} a_{ij} y_j (Pu)_j$$

And the maximization part of can be rewritten as:

$$\max_{u: \|u\|_1 \leq \epsilon} \sum_{j \in N} a_{ij} y_j (Pu)_j = \max_{u: \|u\|_1 \leq \epsilon} a_i \cdot \text{diag}(y)(Pu) \quad (3.4)$$

$$= \max_{u: \|u\|_1 \leq \epsilon} [a_i \cdot \text{diag}(y)P] u \quad (3.5)$$

$$= \sum_{k \in N} \max_{u: \sum_{k \in N} |u_k| \leq \epsilon} [a_i \cdot \text{diag}(y)P]_k u_k \quad (3.6)$$

In equation (3.4), we simply rewrite the same equation in a vector form. In equation (3.5), we regroup the matrices and since the elements are seperable in equation (3.6), we separate the elements and sum them. For equation (3.6), we propose a closed form solution to the maximization problem of the constraint set, $\max_{u: \|u\|_1 \leq \epsilon} \sum_{j \in N} a_{ij} y_j (Pu)_j$ for a fixed index of i . Let $k^* = \arg \max_{j \in N} |a_i \cdot \text{diag}(y)P]_j|$. Then, assume that $|u_{k^*}| = \epsilon$. We can indeed prove that this is optimal by using contrapositive. Assume to the contrary that ϵ is not optimal value for $|u_{k^*}|$. Then, there should be a $\beta \in \mathbb{R}^+$ for some $\tilde{k}$, we obtain $|u_{\tilde{k}}| = \beta$. However, it becomes evident that $\sum_{j \in N} a_{ij} y_j (Pu)_j$ could be greater or equal if $|u_{k^*}|$ increased by β , because $|[a_i \cdot \text{diag}(y)P]_{k^*}| \geq |[a_i \cdot \text{diag}(y)P]_{\tilde{k}}|$. This is basically the definition of ℓ_∞ -norm. Finally, we have the following equality for the constraint:

$$\sum_{j \in N} a_{ij} y_j \bar{d}_j - c_i + \max_{u: \|u\|_1 \leq \epsilon} \sum_{j \in N} a_{ij} y_j (Pu)_j = a_i \cdot \text{diag}(y) \bar{d} - c_i + \epsilon \|a_i \cdot \text{diag}(y)P\|_\infty$$

Note that we have minimization problem in the objective function. It is known that $\min[f(x)] = -\max[-f(x)]$. So, by defining minimization problem like this and applying the same steps we can obtain the $\|a_i \text{diag}(y)P\|_\infty$ by the same reasoning. The mathematical model under the uncertainty set $\mathcal{U}_1$ becomes:

$$\begin{aligned} & \underset{0 \leq y \leq 1}{\text{maximize}} && r^T \text{diag}(y) \bar{d} - \epsilon \|r^T \text{diag}(y)P\|_\infty \\ & \text{subject to} && a_i \text{diag}(y) \bar{d} + \epsilon \|a_i \text{diag}(y)P\|_\infty \leq c_i \quad \forall i \in K. \end{aligned} \quad (3.7)$$

In this part, we examine the best possible answer to the problem while making certain assumptions. We focus on a situation where the matrix P follows a diagonal pattern, with the diagonal elements representing the product demand's standard deviations, and all the other entries in the matrix are set to zero. Normally, when the P matrix is not diagonal, we have the model in equation (3.4) above.

By redefining the problem using the ℓ_∞ norm and the maximization operator, we obtain the following non-linear programming model:

$$\begin{aligned} & \underset{0 \leq y \leq 1}{\text{maximize}} && \sum_{j \in N} r_j y_j \bar{d}_j - \epsilon \left\{ \max_{k \in N} \left| \sum_{j \in N} r_j y_j P_{jk} \right| \right\} \\ & \text{subject to} && \sum_{j \in N} a_{ij} y_j \bar{d}_j + \epsilon \left\{ \max_{j \in N} \left| \sum_{k \in N} a_{ik} y_k P_{jk} \right| \right\} \leq c_i \quad \forall i \in K. \end{aligned}$$

Given the specified parameter and decision variable assumptions, we can rework the problem by recognizing that, for every $j \in N$, both $r_j y_j P_{jj}$ and $a_{ij} y_j P_{jj}$ are always non-negative. This insight allows us to eliminate the absolute value operators, leading to the introduction of two new decision variables and the addition of $2N + K$ extra constraints to linearize the problem. This linearized version is noteworthy because it can be effectively solved using Linear Program Solvers such as Gurobi.

$$\begin{aligned} & \underset{0 \leq y \leq 1}{\text{maximize}} && \sum_{j \in N} r_j y_j \bar{d}_j - \epsilon \gamma_{obj} \\ & \text{subject to} && \sum_{j \in N} a_{ij} y_j \bar{d}_j + \epsilon \gamma_{constr_i} \leq c_i \quad \forall i \in K \\ & && \sum_{j \in N} r_j y_j P_{jk} \leq \gamma_{obj} \quad \forall k \in N, \\ & && \sum_{k \in N} a_{ik} y_k P_{kj} \leq \gamma_{constr_i} \quad \forall j \in N, \forall i \in K. \end{aligned}$$

where γ_{obj} is a decision variable to linearize the maximization in the objective function and γ_{constr_i} is a decision variable to linearize the maximization in the constraint i . When diagonal P matrix is considered, the model can be simplified further. The P matrix contains non-zero values exclusively within its diagonal elements, hence, $\sum_{k \in N} a_{ik} y_k P_{kj}$ is equal to $a_{ij} y_j P_{jj}$. Hence,

taking into account the diagonal P matrix, the model transforms into the following:

$$\begin{aligned}
& \underset{0 \leq \mathbf{y} \leq 1}{\text{maximize}} && \sum_{j \in N} r_j y_j \bar{d}_j - \epsilon \gamma_{obj} \\
& \text{subject to} && \sum_{j \in N} a_{ij} y_j \bar{d}_j + \epsilon \gamma_{constr_i} \leq c_i \quad \forall i \in K \\
& && r_j y_j P_{jj} \leq \gamma_{obj} \quad \forall j \in N, \\
& && a_{ij} y_j P_{jj} \leq \gamma_{constr_i} \quad \forall j \in N, \forall i \in K.
\end{aligned}$$

3.1.2 Uncertainty Set: ℓ_1 -norm + ℓ_∞ -norm (Budget Uncertainty Set)

An uncertainty set can also be combination of two norms. In this section, we will consider the budget uncertainty set which contains ℓ_1 -norm and ℓ_∞ -norm. Previously, we had only ϵ to adjust size of the uncertainty set. Now, we have ϵ and ρ to adjust the size. The budget uncertainty set provides more flexible environment to adjust the conservativeness of the results [30]. Now, we will investigate new formulation of our proposed formulation with budget uncertainty set. Budget uncertainty set is defined as:

$$\mathcal{U} = \{\bar{d} + Pu : \|u\|_\infty \leq \rho, \|u\|_1 \leq \epsilon\}$$

After defining $\mathcal{U}$ as a budget uncertainty set, we need to obtain robust counterpart of our problem in (3.3) for both objective function and constraints.

First, we will start by deriving robust counterpart for objective function. To do this, we need dual of minimization problem in the objective function. The dual problem can be formulated as follows:

$$\max_{\substack{0 \leq z \\ 0 \leq w}} \min_u r^T \text{diag}(y) \bar{d} + r^T \text{diag}(y) (Pu) + z(\|u\|_\infty - \rho) + w(\|u\|_1 - \epsilon) \quad (3.8)$$

Then, we can use maximization problems instead of norms to simplify the problem further:

$$= \max_{\substack{0 \leq z \\ 0 \leq w}} \min_u r^T \text{diag}(y) \bar{d} + r^T \text{diag}(y) (Pu) - z\rho + z \max_{\|x\|_1 \leq 1} x^T u - w\epsilon + w \max_{\|t\|_\infty \leq 1} t^T u \quad (3.9)$$

For the maximization problem in equation (3.9), inner maximization problems and the minimization problem can be interchanged as the functions of inner-most maximization terms are continuous convex-concave functions on their domains, and both feasible regions to maximization and minimization problems are bounded sets. Then, we have “max min” = “min max” [31]. We can also express all maximization problems under one problem and obtain the following:

$$= \max_{\substack{0 \leq z \\ 0 \leq w \\ x, t}} r^T \text{diag}(y) \bar{d} - z\rho - w\epsilon + \min_u \{r^T \text{diag}(y) (Pu) + zx^T u + wt^T u\} \quad (3.10)$$

In the equation (3.10), minimization part should be 0 because otherwise that part will tend to $-\infty$ and the outer maximization problem will be meaningless. Therefore, $r^T \text{diag}(y)P + zx^T + wt^T = 0$. Let us define new variables as $\alpha = zx^T$ and $\beta = wt^T$. As a consequence, our maximization problem becomes:

$$\max_{\substack{\|\alpha\|_1 \leq z \\ \|\beta\|_\infty \leq w \\ r^T \text{diag}(y)P + \alpha + \beta = 0}} r^T \text{diag}(y)\bar{d} - z\rho - w\epsilon$$

So robust counterpart of the objective function becomes:

$$r^T \text{diag}(y)\bar{d} - \rho\|\alpha\|_1 - \epsilon\|r^T \text{diag}(y)P + \alpha\|_\infty$$

Then, we will obtain the robust counterpart of the constraint. The constraints are in the following form:

$$\max_{\substack{u: \|u\|_\infty \leq \epsilon \\ \|u\|_1 \leq \epsilon}} a_i \text{diag}(y)(\bar{d} + Pu) \leq c_i, \forall i \in K$$

Then we can do the subsequent calculations, using Lagrangian function, for the maximization part and obtain the robust counterpart. To do this, we will define 2 dual variables z and w for ℓ_∞ -norm and ℓ_1 -norm respectively. The reasonings behind the transformation of norms and everything are the same with the previous case. Then, the Lagrangian problem becomes:

$$\min_{\substack{0 \leq z \\ 0 \leq w}} \max_u a_i \text{diag}(y)\bar{d} + a_i \text{diag}(y)(Pu) + z(\rho - \|u\|_\infty) + w(\epsilon - \|u\|_1) \quad (3.11)$$

Then, by the substitution steps applied before, the equation becomes:

$$= \min_{\substack{0 \leq z \\ 0 \leq w}} \max_u a_i \text{diag}(y)\bar{d} + a_i \text{diag}(y)(Pu) + z(\rho - \|u\|_\infty) + w(\epsilon - \|u\|_1) \quad (3.12)$$

$$= \min_{\substack{0 \leq z \\ 0 \leq w}} \max_u a_i \text{diag}(y)\bar{d} + a_i \text{diag}(y)(Pu) + z\rho - z \max_{\|x\|_1 \leq 1} x^T u + w\epsilon - w \max_{\|t\|_\infty \leq 1} t^T u \quad (3.13)$$

$$= \min_{\substack{0 \leq z \\ 0 \leq w \\ x, t}} a_i \text{diag}(y)\bar{d} + z\rho + w\epsilon + \max_u a_i \text{diag}(y)(Pu) - zx^T u - wt^T u \quad (3.14)$$

In the equation (3.14) maximization part should be 0 due to the same reason explained before. Therefore, $a_i \text{diag}(y)P - zx^T - wt^T = 0$. Let us define new variables as $\alpha = zx^T$ and $\beta = wt^T$. As a consequence, our minimization problem becomes:

$$\min_{\substack{\|\alpha\|_1 \leq z \\ \|\beta\|_\infty \leq w \\ a_i \text{diag}(y)P - \alpha - \beta = 0}} a_i \text{diag}(y)\bar{d} + z\rho + w\epsilon$$

Then, the constraint with the robust counterpart becomes following (i index is added to α since each constraint requires a unique dual variable):

$$a_i \text{diag}(y)\bar{d} + \rho\|\alpha_i\|_1 + \epsilon\|a_i \text{diag}(y)P - \alpha_i\|_\infty \leq c_i, \forall i \in K$$

As a result of the robust counterparts obtained, our problem becomes:

$$\begin{aligned}
& \underset{0 \leq y \leq 1}{\text{maximize}} && r^T \text{diag}(y) \bar{d} - \rho \|\alpha\|_1 - \epsilon \|r^T \text{diag}(y) P + \alpha\|_\infty \\
& \text{subject to} && a_i \text{diag}(y) \bar{d} + \rho \|\alpha_i\|_1 + \epsilon \|a_i \text{diag}(y) P - \alpha_i\|_\infty \leq c_i \quad \forall i \in K \\
& && \alpha \text{ and } \alpha_i \text{ free } \forall i \in K
\end{aligned}$$

The budget uncertainty set

$$\mathcal{U} = \{\bar{d} + Pu : \|u\|_\infty \leq \rho, \|u\|_1 \leq \epsilon\}$$

makes sense if a certain relationship between ρ and ϵ is satisfied [32] because when $\rho > \epsilon$, it is the same case with ℓ_1 -norm case. We can explain this visually in 2 dimensions. Consider the case where $k = 2$, where k is the dimension of u . In the figure 3.1(a), $\rho = 1.2$ and $\epsilon = 1$, so, it can be observed that this only red area is the feasible region which is the same when we only consider ℓ_1 -norm (the green area represents ρ and the red represents ϵ). Moreover, in the figure 3.1(c), when $\rho = \frac{1}{2}\epsilon$, feasible region is only green area, which is the same when we only consider ℓ_∞ -norm. Therefore, the problem is only interesting when $\epsilon > \rho > \frac{1}{k}\epsilon$ satisfied.

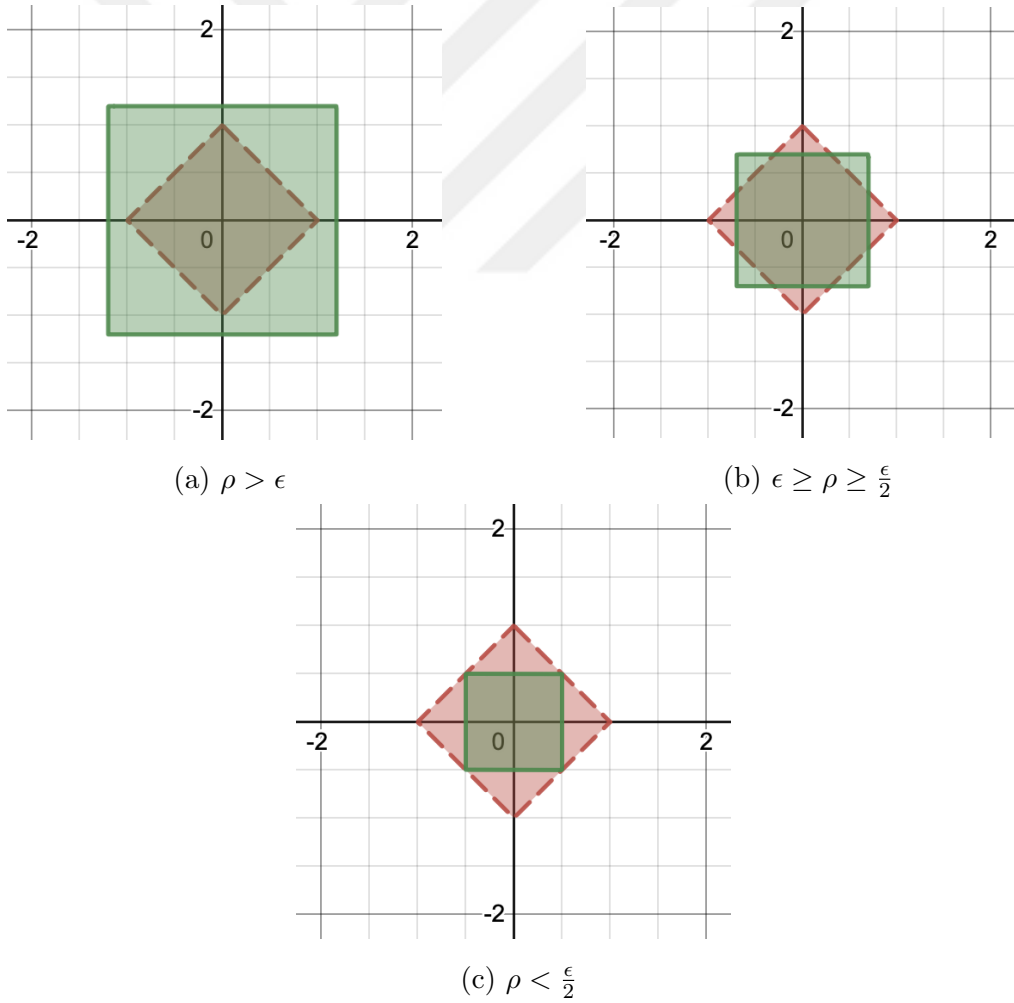


Figure 3.1: Relationship Between ρ and ϵ

After defining $\mathcal{U}$ as a budget uncertainty set and obtaining robust counterpart of the budget

uncertainty set for both objective function and constraints, our problem becomes as follows:

$$\begin{aligned}
& \underset{0 \leq y \leq 1}{\text{maximize}} && r^T \text{diag}(y) \bar{d} - \rho \|\alpha\|_1 - \epsilon \|r^T \text{diag}(y) P + \alpha\|_\infty \\
& \text{subject to} && a_i \text{diag}(y) \bar{d} + \rho \|\alpha_i\|_1 + \epsilon \|a_i \text{diag}(y) P - \alpha_i\|_\infty \leq c_i \quad \forall i \in K \\
& && \alpha \text{ and } \alpha_i \text{ free } \forall i \in K
\end{aligned}$$

Define π_{obj1} , π_{objinf} , $\pi_{constr1,i}$, $\pi_{constrinf,i}$ to linearize the problem. π_{obj1} is defined to linearize the ℓ_1 -norm in the objective function, π_{objinf} is defined to linearize the ℓ_∞ -norm in the objective function, $\pi_{constr1,i}$ is defined to linearize the ℓ_1 -norm in the constraint i , $\pi_{constrinf,i}$ is defined to linearize ℓ_∞ -norm in the constraint i . Then, the problem becomes:

$$\begin{aligned}
& \underset{0 \leq y \leq 1}{\text{maximize}} && r^T \text{diag}(y) \bar{d} - \rho \pi_{obj1} - \epsilon \pi_{objinf} \\
& \text{subject to} && \sum_{j \in N} a_{ij} y_j \bar{d}_j + \rho \pi_{constr1,i} + \epsilon \pi_{constrinf,i} \leq c_i \quad \forall i \in K \\
& && \sum_{j \in N} \omega_j \leq \pi_{obj1} \\
& && \omega_j \geq \alpha_j && \forall j \in N \\
& && \omega_j \geq -\alpha_j && \forall j \in N \\
& && (r^T \text{diag}(y) P + \alpha)_j \leq \pi_{objinf} && \forall j \in N \\
& && -(r^T \text{diag}(y) P + \alpha)_j \leq \pi_{objinf} && \forall j \in N \\
& && \sum_{j \in N} \tau_j \leq \pi_{constr1,i} && \forall i \in K \\
& && \tau_j \geq \alpha_{ij} && \forall j \in N, \forall i \in K \\
& && \tau_j \geq -\alpha_{ij} && \forall j \in N, \forall i \in K \\
& && (a_i \text{diag}(y) P - \alpha_i)_j \leq \pi_{constrinf,i} && \forall j \in N, \forall i \in K \\
& && -(a_i \text{diag}(y) P - \alpha_i)_j \leq \pi_{constrinf,i} && \forall j \in N, \forall i \in K \\
& && \alpha \text{ and } \alpha_i \text{ free } \forall i \in K
\end{aligned}$$

3.1.3 Uncertainty Set: ℓ_2 -norm + ℓ_∞ -norm

Now, we will investigate new formulation of our proposed formulation with box + ellipsoidal uncertainty sets. It is defined as:

$$\mathcal{U} = \{\bar{d} + Pu : \|u\|_\infty \leq \rho, \|u\|_2 \leq \Omega\}$$

First, we will start by deriving robust counterpart for objective function. To do this, we need dual of minimization problem in the objective function. The dual problem can be formulated as follows:

$$= \max_{\substack{0 \leq z \\ 0 \leq w}} \min_u r^T \text{diag}(y) \bar{d} + r^T \text{diag}(y) (Pu) + w(\|u\|_\infty - \rho) + z(\|u\|_2 - \Omega) \quad (3.15)$$

Then, we can use maximization problems instead of norms to simplify the problem further:

$$(3.16)$$

$$= \max_{\substack{0 \leq z \\ 0 \leq w}} \min_u r^T \text{diag}(y) \bar{d} + r^T \text{diag}(y) (Pu) - w\rho + w \max_{\|x\|_1 \leq 1} x^T u - z\Omega + z \max_{\|t\|_2 \leq 1} t^T u \quad (3.17)$$

For the maximization problem in equation (3.16), inner maximization problems and the minimization problem can be interchanged as the functions of inner-most maximization terms are continuous convex-concave functions on their domains, and both feasible regions to maximization and minimization problems are bounded sets. Then, we have “max min” = “min max” [31]. We can also express all maximization problems under one problem and obtain the following problem:

$$= \max_{\substack{0 \leq z \\ 0 \leq w \\ x, t}} r^T \text{diag}(y) \bar{d} - w\rho - z\Omega + \min_u r^T \text{diag}(y) (Pu) + wx^T u + zt^T u \quad (3.18)$$

In the equation (3.17), minimization part should be 0 because otherwise that part will tend to $-\infty$ and the outer maximization problem will be meaningless. Therefore, $r^T \text{diag}(y) P + wx^T + zt^T = 0$. Let us define new variables as $\alpha = wx^T$ and $\beta = wz^T$. As a consequence, our maximization problem becomes:

$$\max_{\substack{\|\alpha\|_2 \leq w \\ \|\beta\|_1 \leq z \\ r^T \text{diag}(y) P + \alpha + \beta = 0}} r^T \text{diag}(y) \bar{d} - z\rho - w\Omega$$

So robust counterpart of the objective function becomes:

$$r^T \text{diag}(y) \bar{d} - \Omega \|\alpha\|_2 - \rho \|r^T \text{diag}(y) P + \alpha\|_1$$

Then, we will obtain robust counterpart for constraint. The constraints are in the following form:

$$\max_{\substack{u: \|u\|_\infty \leq \rho \\ \|u\|_2 \leq \Omega}} a_i \text{diag}(y) (\bar{d} + Pu) - c_i \leq 0, \forall i \in K$$

Then we can do the subsequent calculations, using Lagrangian function, for the maximization part and obtain the robust counterpart. To do this, we will define 2 dual variables w and z for ℓ_∞ -norm and ℓ_2 -norm respectively. Then, the Lagrangian problem becomes:

$$= \min_{\substack{0 \leq z \\ 0 \leq w}} \max_u a_i \text{diag}(y) \bar{d} + a_i \text{diag}(y) (Pu) + w(\rho - \|u\|_\infty) + z(\Omega - \|u\|_2) \quad (3.19)$$

We can use maximization problems instead of norms to simplify the problem further:

$$= \min_{\substack{0 \leq z \\ 0 \leq w}} \max_u a_i \text{diag}(y) \bar{d} + a_i \text{diag}(y) (Pu) + w\rho - w \max_{\|x\|_\infty \leq 1} x^T u + z\Omega - z \max_{\|t\|_2 \leq 1} t^T u$$

Then, by substituting the min and max function because of the previous reasoning, we can obtain:

$$= \min_{\substack{0 \leq z \\ 0 \leq w \\ x, t}} a_i \text{diag}(y) \bar{d} + w\rho + z\Omega + \max_u a_i \text{diag}(y)(Pu) - wx^T u - zt^T u$$

In the last equation maximization part should be 0. Therefore, $a_i \text{diag}(y)P - zx^T - wt^T = 0$. Let us define new variables as $\alpha = wx^T$ and $\beta = zt^T$. As a consequence, our minimization problem becomes:

$$\min_{\substack{\|\alpha\|_2 \leq w \\ \|\beta\|_1 \leq z \\ a_i \text{diag}(y)P - \alpha - \beta = 0}} a_i \text{diag}(y) \bar{d} + z\rho + w\Omega$$

Then, the constraint with the robust counterpart becomes following (i index is added to α since each constraint requires different dual variable):

$$a_i \text{diag}(y) \bar{d} + \Omega \|\alpha_i\|_2 + \rho \|a_i \text{diag}(y)P - \alpha_i\|_1 \leq c_i, \forall i \in K$$

As a result of robust counterparts found, our problem becomes as follows:

$$\begin{aligned} & \underset{0 \leq y \leq 1}{\text{maximize}} && r^T \text{diag}(y) \bar{d} - \Omega \|\alpha\|_2 - \rho \|r^T \text{diag}(y)P + \alpha\|_1 \\ & \text{subject to} && a_i \text{diag}(y) \bar{d} + \Omega \|\alpha_i\|_2 + \rho \|a_i \text{diag}(y)P - \alpha_i\|_1 \leq c_i \quad \forall i \in K \\ & && \alpha \text{ and } \alpha_i \text{ free } \forall i \in K \end{aligned}$$

Now, we will linearize the ℓ_1 -norm in the new formulation of our proposed formulation with box + ellipsoidal uncertainty sets. However, before that, to have a meaningful uncertainty set, we need to determine border points for ρ and Ω [32]. We can visually show why this boundary is needed. In figure 3.2 (a), $\rho = 2$ and $\Omega = 3$, it can be observed that this only red area is the feasible region which is the same when we only consider ℓ_∞ -norm (the red area represents ρ and the blue area represents Ω). Moreover, in the figure 3.2 (c), when $\rho = 2$ and $\Omega = 1$, feasible region is only blue area, which is the same when we only consider ℓ_2 -norm. Therefore, the problem is only interesting when $\|u\|_\infty \leq \|u\|_2 \leq \sqrt{n}\|u\|_\infty$ satisfied.

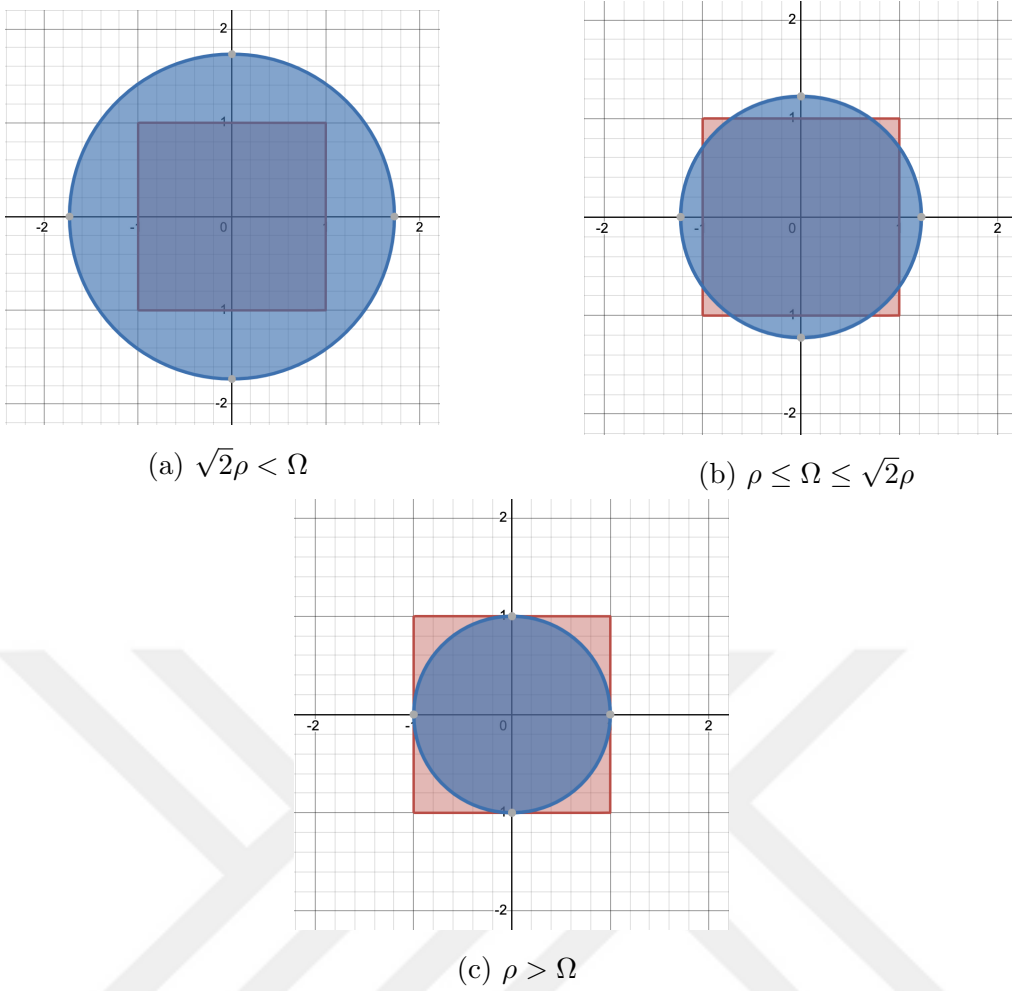


Figure 3.2: Relationship Between ρ and Ω

So, the relationship between ℓ_∞ and ℓ_2 is:

$$\|u\|_\infty \leq \|u\|_2 \leq \sqrt{n}\|u\|_\infty$$

where n is the size of u . So, the relationship between ρ and Ω is as follows:

$$\rho \leq \Omega \leq \sqrt{n}\rho$$

Moreover, as a result of robust counterparts found before, our problem becomes:

$$\begin{aligned} & \underset{0 \leq y \leq 1}{\text{maximize}} && r^T \text{diag}(y) \bar{d} - \Omega \|\alpha\|_2 - \rho \|r^T \text{diag}(y) P + \alpha\|_1 \\ & \text{subject to} && a_i \text{diag}(y) \bar{d} + \Omega \|\alpha_i\|_2 + \rho \|a_i \text{diag}(y) P - \alpha_i\|_1 \leq c_i \quad \forall i \in K \\ & && \alpha \text{ and } \alpha_i \text{ free } \forall i \in K \end{aligned}$$

We define π_{obj1} to linearize the ℓ_1 -norm in the objective function, and $\pi_{constr1,i}$ to linearize the

ℓ_1 -norm in constraint i . Then, the problem becomes:

$$\begin{aligned}
& \underset{0 \leq y \leq 1}{\text{maximize}} && r^T \text{diag}(y) \bar{d} - \Omega(\alpha^T \alpha)^{1/2} - \rho \pi_{obj1} \\
& \text{subject to} && a_i \text{diag}(y) \bar{d} + \Omega(\alpha_i^T \alpha_i)^{1/2} + \rho \pi_{constr1,i} \leq c_i \quad \forall i \in K \\
& && \sum_{j \in N} w_j \leq \pi_{obj1} \\
& && (r^T \text{diag}(y) P + \alpha)_j \leq w_j && \forall j \in N \\
& && -(r^T \text{diag}(y) P + \alpha)_j \leq w_j && \forall j \in N \\
& && \sum_{j \in N} \tau_{ij} \leq \pi_{constr1,i} && \forall i \in K \\
& && (P \text{diag}(y) a_i^T - \alpha_i)_j \leq \tau_{ij} && \forall j \in N, \forall i \in K \\
& && -(P \text{diag}(y) a_i^T - \alpha_i)_j \leq \tau_{ij} && \forall j \in N, \forall i \in K \\
& && \alpha \text{ and } \alpha_i \text{ free } \forall i \in K
\end{aligned}$$

Unlike the other two robust models, the robust counterpart of this model is not linear; it is a second-order cone program, which can be solved using Gurobi. The non-linearity stems from the ℓ_2 -norm, which is inherently non-linear. Consequently, its robust counterpart also is nonlinear, as the ℓ_2 -norm is equal to itself in this context.

3.2 Booking Limit Calculation

As a consequence of the discussed models, we derive values for y within the range of zero to one. Thus, we require an algorithm to translate these y values into booking limits for our capacity allocation problem. To address this need, we propose an algorithm. This algorithm accepts y values, expected demand, and capacity as inputs, and computes booking limits for each product. The steps for the algorithm are outlined below.

The proposed algorithm aims to address the challenge of translating optimized values of y , derived from a Linear or Second-order Cone Programming Model, into practical booking limits for a capacity allocation problem within a multi-leg context. Initially, the algorithm solves the Linear or Second-order Cone Programming Model to ascertain the optimal allocation of resources, represented by the values of y . These values indicate the expected demand for each product, considering the constraints of the system. Following this, the algorithm calculates the product of the expected demand and a coefficient, ϕ_i , which signifies the utilization factor of each resource. This calculation enables the determination of partitioned booking limits for each product, considering the constraints imposed by available resources.

Algorithm 1: Finding Booking Limits for Multi-Leg Model

1. Step 1: The counterpart of the robust problem is solved to find the optimal vector $\mathbf{y}^*$.
2. Step 2: For each product $j \in N$, $\bar{d}_j y_j^*$ is calculated.
3. Step 3: Multiplier ϕ_i is found for each resource $i \in K$ by using the above condition as follows: $\phi_i = \frac{c_i}{\sum_{j \in a_i} \bar{d}_j y_j^*}$ where $a_i \cdot$ is the i^{th} row of A matrix. As a consequence, we obtain ϕ vector to be used. The reason behind using this multiplier is to enhance robustness. By introducing this multiplier, capacity is distributed in a more balanced way across different fare classes, allowing the model to distribute any capacity leftover proportionally across all products.
4. Step 4: Partitioned booking limits, b , is determined to be $b_j = \text{round}(\frac{\sum_{i=1}^K \phi_i a_{ij}}{\sum_{i=1}^K a_{ij}} \bar{d}_j y_j^*)$ where a_{ij} represents i^{th} row of j^{th} column. It is rounded up if decimal point is greater than 0.5 and rounded down otherwise.
5. Step 5: There are two possible outcomes:
 - If the summation of booking values, represented as $\sum_{j \in N} b_j$, for leg i , falls short of the leg's capacity c_i , we increment the booking limit of the highest fare class that exclusively utilizes this leg. This increment continues until the capacity of the leg is reached.
 - If the total sum of booking values, denoted as $\sum_{j \in N} b_j$, for leg i surpasses the leg's capacity c_i , we reduce the booking limit of the lowest fare class that solely relies on this leg. This reduction continues until the total number of seats allocated is at the leg capacity.

Additionally, if the total booking values for a leg fall short of its capacity, the algorithm incrementally increases the booking limit for the highest fare class exclusively utilizing that leg until capacity is reached. Conversely, if the total booking values exceed the leg's capacity, the algorithm decreases the booking limit for the lowest fare class relying solely on that leg to ensure compliance with capacity constraints. By iteratively adjusting booking limits based on resource and capacity constraints, the algorithm ensures efficient utilization of resources while maintaining compliance with capacity limitations. These two situations might happen due to rounding in Step 4.

Chapter 4

Computational Experiments

In our pursuit of evaluating the efficiency and effectiveness of our models, we have conducted computational experiments that involve various network structures and parameters crucial for defining uncertainty sets. Rather than directly comparing the objective function values of each optimization model, we focus on comparing the revenues generated through booking control simulations. Additionally, we explore the impact of demand factors, arrival pattern, standard deviation and different types of booking controls on our models' performances.

We compare our three proposed robust models against PNLP and DLP. This comparison enables us to assess the performance of our models in comparison to both a stochastic model (PNLP) and a deterministic model (DLP). By analyzing how our models' revenues compare against these established models, we gain insights into their effectiveness and suitability for real-world applications across various decision-making scenarios.

To conduct these comparisons, we randomly generate demands for each product within each instance and calculate the resulting revenue based on the booking controls obtained by different models. For each simulation run we take 2000 samples. To aid in visualizing our findings, we utilize percentile graphs and histograms. To create a percentile graph, we randomly divide the 2000 samples into 20 groups of size 100 each. In each group, we computed the percentiles, and took the averages of the percentiles of the groups to have a good estimate.

Yet, before these, we will state important notes on how to construct covariance matrix and how to decompose it to obtain P matrix, especially for dependent data cases.

4.1 Covariance Matrix and Decomposition

We construct the P matrix in different ways, depending on the data structure. If we assume the data is independent, we obtain the P matrix by placing the standard deviations on the diagonal and setting the off-diagonal elements to zero. If the data is dependent, we use different decomposition techniques to construct the P matrix accordingly.

As mentioned in literature review, Thiele defines P as square root of covariance matrix [27]. To compute the square root of a matrix, one can use the matrix square root property, which states that if A is a square matrix, then its square root, denoted as $A^{1/2}$, exists if and only if A is positive definite. To compute the square root of a positive definite matrix A , you can use methods like diagonalization.

Diagonalization: If $A = Q\Lambda Q^{-1}$, where Q is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues, then the square root of A can be computed as: $A^{1/2} = Q\Lambda^{1/2}Q^{-1}$. Here, $\Lambda^{1/2}$ is the diagonal matrix with the square roots of the eigenvalues on the diagonal. After obtaining $A^{1/2}$, we set $P = A^{1/2}$.

We will also consider different decomposition of the covariance matrix for the dependent cases such as Cholesky decomposition and LDL^T decomposition.

Cholesky decomposition: If $A = LL^T$, where L is a lower triangular matrix, then the square root of A can be computed as: $A^{1/2} = L$.

Throughout the simulation runs, we also explore these two decompositions of the covariance matrix techniques to evaluate their impact on the results. By employing different decomposition techniques, we aim to understand how the choice of decomposition method influences the performance of our models and their ability to handle uncertainty. This analysis provides valuable insights into the robustness and reliability of our approach under different decomposition schemes.

4.2 Computational Experiments with Independent Data

As our first step, we focus on independent demand examples, where our P matrix takes the form of a diagonal matrix. This simplifies our model, as previously noted. We consider three distinct examples: one sourced from [4] (featuring 6 products and 2 resources), one that we created (featuring 3 resources and 5 products), and another from [33] (with 18 products and 3 resources).

4.2.1 Experiments on a Network with 2 Resources and 6 Products

Example 1: Let's introduce the parameters for the example from [4], which follows a Poisson demand distribution. We have 2 resources and 6 products (each resource has a capacity of 90.), represented by the following A matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with the following parameters

Product	O-D	Fare	Flights Used	Exp. Demand
1	A-B	150	1	30
2	A-B	100	1	60
3	B-C	120	2	20
4	B-C	80	2	80
5	A-C	250	1, 2	30
6	A-C	170	1, 2	40

Table 4.1: Parameters

Since we're considering a diagonal P matrix with Poisson demand distribution, the variance is equal to the expectation for each product. Therefore, we can construct the P matrix using the square root of the expected demands, resulting in the following form:

$$P = \begin{bmatrix} 5.47 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7.74 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.47 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.47 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.32 \end{bmatrix}$$

As the first part of our analysis, we need to determine the proper parameters for our uncertainty sets through experimentation. We solve each model 2000 times with different parameters and obtain average revenues. Then, we identify the ones that yield favorable (the highest average revenue) results.

ϵ	ℓ_1	ϵ, ρ	Budget
$\epsilon = 0.8$	19368.065	$\epsilon = 0.9, \rho = 0.7$	19253.615
$\epsilon = 1$	19264.675	$\epsilon = 1.1, \rho = 0.9$	19213.875
$\epsilon = 1.2$	19512.325	$\epsilon = 1.5, \rho = 1.1$	19474.965
$\epsilon = 1.5$	19462.245	$\epsilon = 1.5, \rho = 1.4$	19450.825
$\epsilon = 1.8$	19032.095	$\epsilon = 2, \rho = 1.9$	18947.43
$\epsilon = 2.1$	18965.51	$\epsilon = 2.7, \rho = 2.2$	18681.12
$\epsilon = 3$	18674.595	$\epsilon = 3.5, \rho = 3.1$	18640.59
$\epsilon = 4.1$	18617.02	$\epsilon = 4.6, \rho = 4.2$	18560.13
$\epsilon = 4.9$	18571.225	$\epsilon = 5.4, \rho = 5.0$	18688.245
$\epsilon = 6.5$	18608.55	$\epsilon = 7.0, \rho = 6.6$	18606.74

ρ, Ω	$\ell_2 + \ell_\infty$
$\rho = 0.7, \Omega = 0.8$	19237.43
$\rho = 0.9, \Omega = 1.1$	18983.97
$\rho = 1.1, \Omega = 1.6$	18604.31
$\rho = 1.4, \Omega = 1.6$	18647.43
$\rho = 1.9, \Omega = 1.9$	18436.81
$\rho = 2.2, \Omega = 2.5$	18702.78
$\rho = 3.1, \Omega = 3.7$	18343.185
$\rho = 4.2, \Omega = 4.8$	12347.02
$\rho = 5.0, \Omega = 5.6$	12317.57
$\rho = 6.6, \Omega = 7.2$	12332.92

Table 4.2: Different Parameter Values of the Robust Models

The Table 4.2 illustrates that for the ℓ_1 -norm uncertainty set, the best simulation results are achieved when $\epsilon = 1.2$. Conversely, for the Budget uncertainty set, optimal results are obtained with $\epsilon = 1.5$ and $\rho = 1.1$. Lastly, for the $\ell_2 + \ell_\infty$ -norm uncertainty set, the most favorable outcomes occur when $\Omega = 0.8$ and $\rho = 0.7$. Therefore, we will continue with these parameters as they have demonstrated superior performance for our models. We use these parameters for other examples also since we have not observed a meaningful difference in the results of the other examples when we changed the parameter selections. Additionally, it is worth noting that as the parameters increase, the simulation results generally decrease, as anticipated. This trend aligns with expectations since these parameters contribute to the objective function as a form of cost with negative coefficients.

Now, we will compare our results with PNLP and DLP to observe our models' performances. For this purpose as mentioned before we will use percentile graphs and consider LBH arrival pattern first, demand factor (DF) is 1.44 for this example. In the network setting, ordering products is challenging since they may use one or multiple resources. Therefore, we order products by their expected gain. This involves multiplying the expected demand vector by the price vector and then sorting the resulting vector in ascending order. The product with the highest expected revenue is considered the highest fare class, and this ordering continues until all products are ranked [1]. We test performance of partitioned booking limits since the problem is small (unless otherwise is stated).

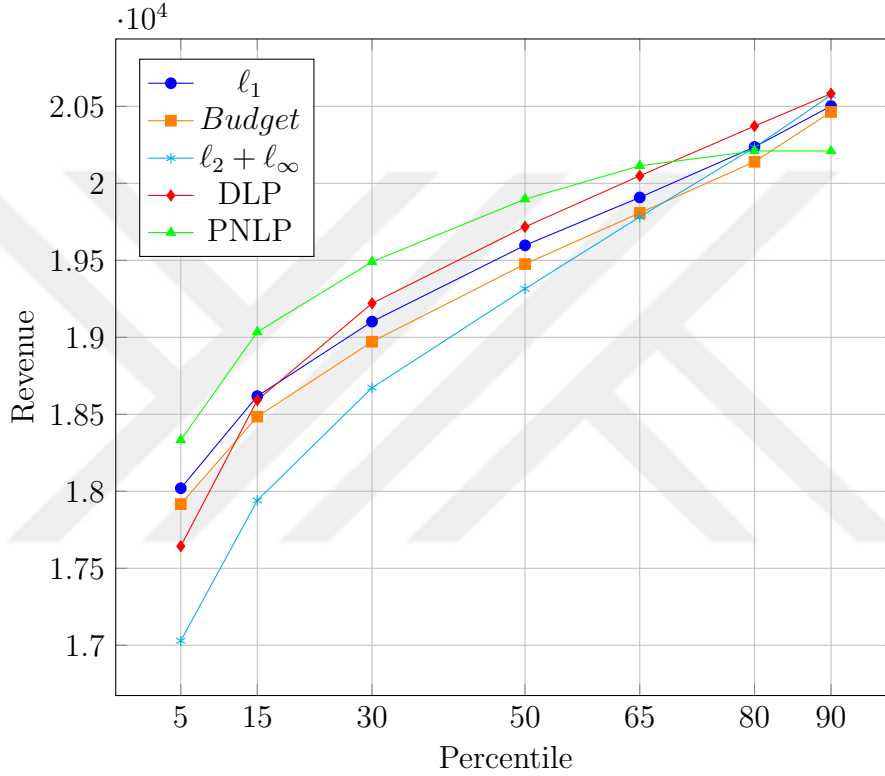


Figure 4.1: Revenue Percentiles With DF of 1.44

Model	Booking Limits						Avg. Cap. Unused	
	P1	P2	P3	P4	P5	P6	R1	R2
ℓ_1	32	31	21	42	27	0	4.1	3.3
Budget Uncertainty	34	30	22	42	26	0	5.4	3.7
$\ell_2 + \ell_\infty$	33	25	22	36	32	0	7.3	6.3
DLP	30	30	20	40	30	0	4.2	3.8
PNLP	28	35	19	44	27	0	2.3	2.3

Table 4.3: Booking limits and average capacity unused of Example 1

From Figure 4.1, we observe that our models generally perform slightly below the DLP, while PNLP outperforms our models across most percentiles. This superior performance is primarily

because DLP and PNLP allocate the highest number of seats to products with high expected demand, reducing the likelihood of empty seats and enhancing revenue potential. Nevertheless, as previously mentioned, robust models try to maximize the worst-case scenarios. Therefore, their performances at lower percentile are better as expected, except PNLP. However, it is important to note that PNLP relies on distribution information to achieve these results, whereas our models do not, making our models more versatile in scenarios lacking such information. Additionally, the DLP begins to outperform our models after the 30th percentile, likely due to the high demand factor of 1.44.

Example 2: We investigate the effect of demand factor variations on our models by adjusting this parameter. We use the same structure in Example 1, we just change the demand factor. We consider the following demand vector: $[30, 25, 20, 35, 25, 30]$, resulting in a demand factor of 91%, which is classified as a high demand factor [34].

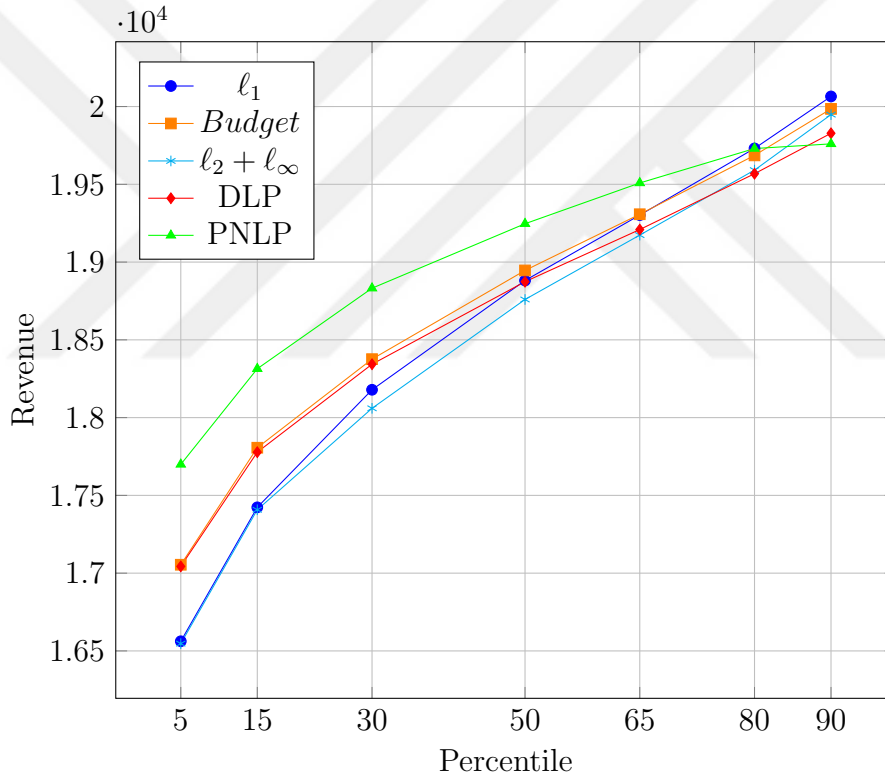


Figure 4.2: Revenue Percentiles With DF of 0.91

It's evident that our Budget uncertainty set model outperforms the DLP or quite close to it across all percentiles (both lower percentiles and higher percentiles) when the demand factor is decreased. In this scenario, the DLP's performance deteriorates because it relies solely on expected demand to make allocations. Consequently, with a significant decrease in expected demand to adjust demand factor, particularly at the lowest fare level, the DLP allocates too many seats, resulting in a surplus of empty seats. It is also noteworthy that the ℓ_1 -norm model surpasses all other models after the 50th percentile along with the budget uncertainty set, except

Model	Booking Limits						Avg. Cap. Unused	
	P1	P2	P3	P4	P5	P6	R1	R2
ℓ_1 -norm	33	20	21	32	27	10	7.5	6.5
Budget Uncertainty	34	17	22	29	24	15	6.4	4.9
$\ell_2 + \ell_\infty$	33	22	22	33	27	8	8.1	7.5
DLP	30	25	20	35	25	10	6.1	6.1
PNLP	28	18	19	27	23	21	2.7	2.7

Table 4.4: Booking limits and average capacity unused of Example 2

PNLP. However, PNL P requires demand distribution information, as mentioned earlier, which is a significant limitation of this model.

Additionally, we observe that the average capacity left in the DLP is higher compared to the Budgeted uncertainty case. This discrepancy arises due to the DLP’s inability to accurately account for the drastic decrease in expected demand, leading to inefficient allocation of resources.

Example 3: We investigate the performance under a medium demand factor, which falls between 80-84%. Let’s consider the following mean demand vector: $[11, 41, 6, 61, 11, 21]$, resulting in an 83% demand factor.

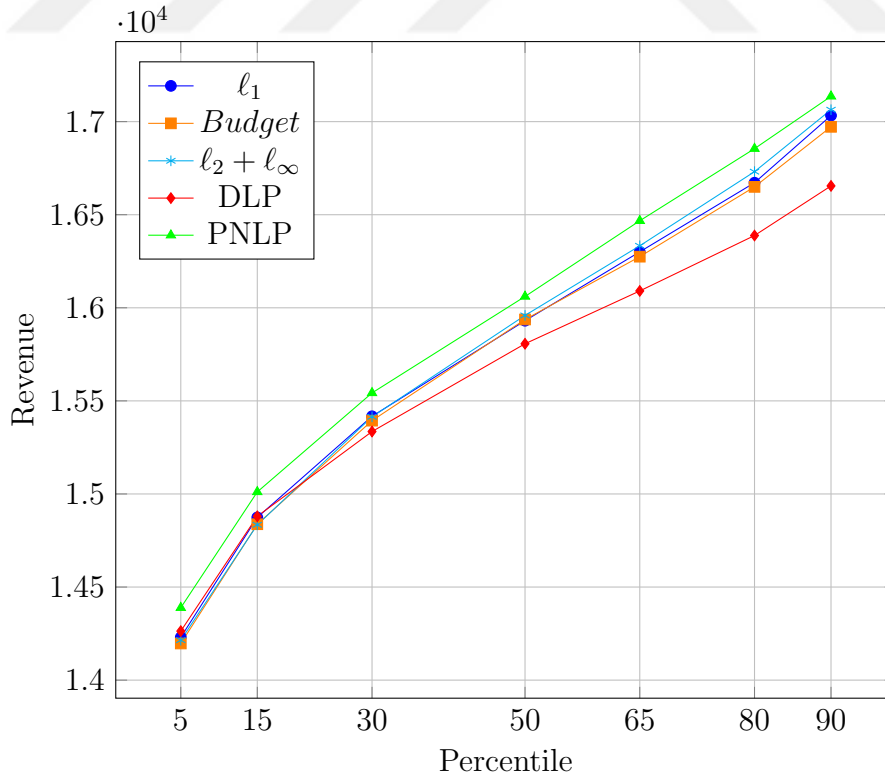


Figure 4.3: Revenue Percentiles With DF of 0.83

Model	Booking Limits						Avg. Cap. Unused	
	P1	P2	P3	P4	P5	P6	R1	R2
ℓ_1 -norm	12	43	7	48	12	23	10.6	6.7
Budget Uncertainty	13	43	7	49	11	23	10.8	6.1
$\ell_2 + \ell_\infty$	12	44	7	49	12	22	10.6	6.1
DLP	11	41	6	52	11	21	12.9	4.5
PNLP	14	44	5	53	12	20	11.0	4.3

Table 4.5: Booking limits and average capacity unused of Example 3

As the demand factor decreases, DLP’s performance consistently deteriorates because it relies heavily on expected demand to determine booking limits, as mentioned before. Conversely, robust models consistently outperform DLP across all percentiles due to their adaptability to varying demand conditions. However, robust models exhibit inferior performance compared to PNLP, primarily because PNLP allocates more seats to the most expensive product and third most expensive product, which has high expected demand, resulting in higher revenue opportunities. Also, all models’ worst-case performances are close to each other since as the demand factor decreases, revenue opportunities decreases, resulting similar worst-case performances.

Example 4: We examine how our models compare to DLP and PNLP under low-demand conditions, which can occur during extreme events such as a global pandemic [35]. We will consider the following demand vector: $[11, 25, 6, 41, 11, 21]$, resulting in a 64% demand factor.

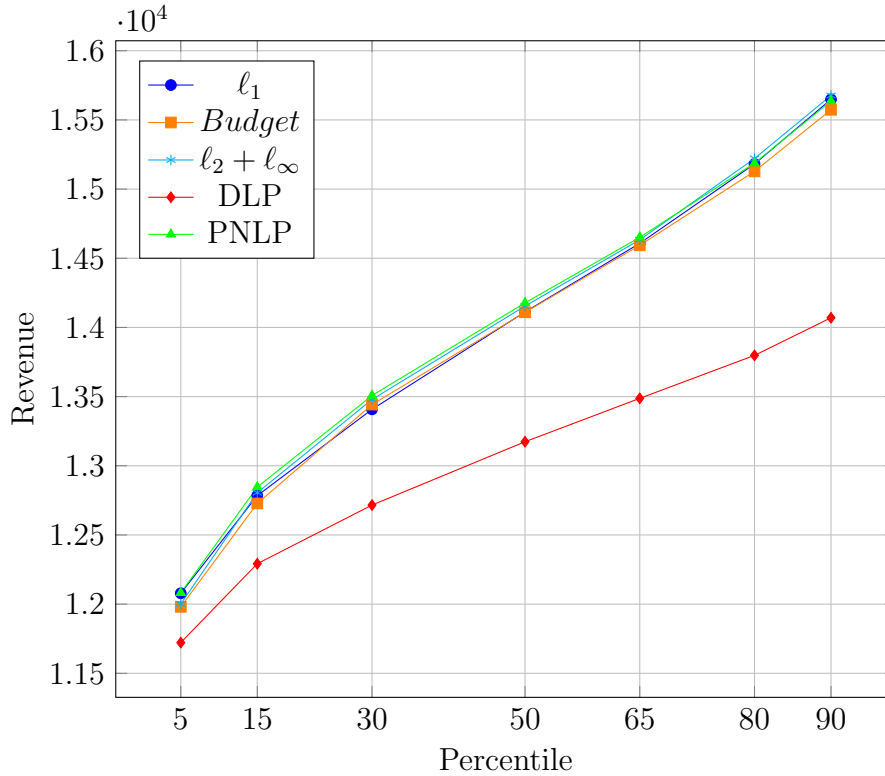


Figure 4.4: Revenue Percentiles With DF of 0.64

Model	Booking Limit	Avg. Capacity Unused
ℓ_1 -norm	[17, 33, 7, 43, 14, 26]	[22.9, 14.2]
Budget Uncertainty	[18, 34, 7, 45, 12, 26]	[23.3, 13.8]
$\ell_2 + \ell_\infty$	[17, 33, 7, 43, 14, 26]	[22.7, 13.9]
DLP	[11, 25, 6, 41, 11, 21]	[28.3, 17.6]
PNLP	[18, 32, 8, 42, 16, 24]	[23.1, 14.4]

Table 4.6: Booking limits and average capacity unused of Eaxmple 4

As seen in Figure 4.4, DLP performs poorly in scenarios with low demand factors because it works with only mean demand and booking limits are less than capacity. If you normalize DLP just like robust, it would do better. This poor performance arises because DLP allocates fewer seats for high fare customers, resulting in fewer opportunities to maximize profits and ultimately leading to lower revenue. The model bases its seat allocations directly on expected demand, so when the actual number of customers exceeds expectations, it cannot accommodate them even if there is available capacity. In contrast, other models exhibit similar booking limits and revenues because the limited opportunities to generate revenue are fully utilized by all models.

Example 5: Until now, our simulation runs have utilized the LBH arrival pattern. However, we are interested in investigating the impact of different arrival patterns on our models. To this end, we will consider a mixed arrival pattern, where customers arrive in a random order but not necessarily in the LBH sequence. Furthermore, we will examine how the demand factor affects the robustness of our models under this mixed arrival pattern.

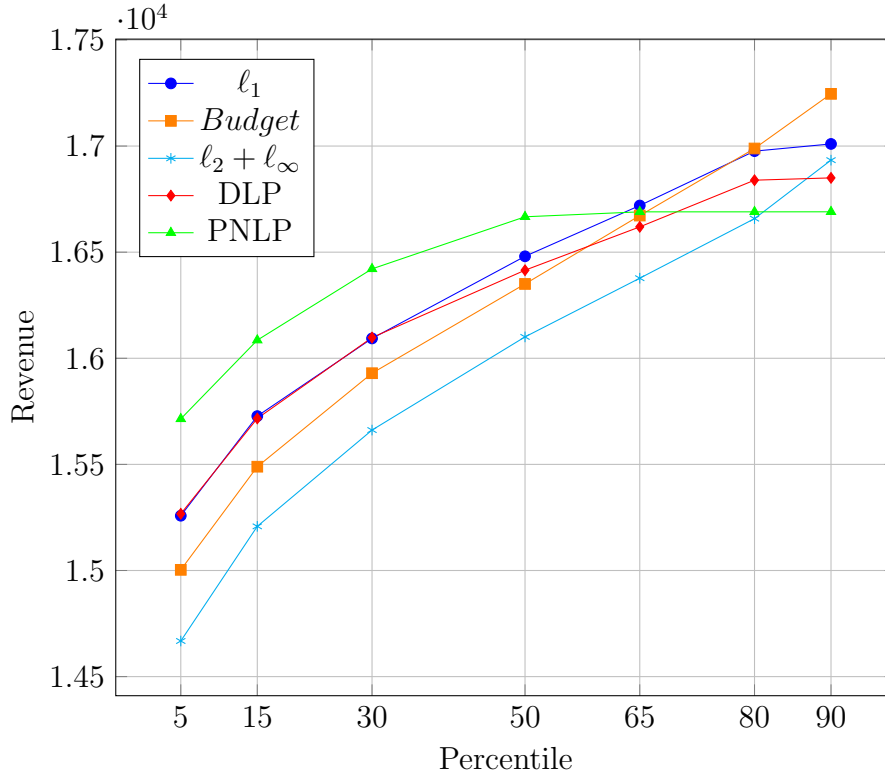


Figure 4.5: Revenue Percentiles Under Mix Arrival Pattern With DF of 1.44

Model	Booking Limits						Avg. Cap. Unused	
	P1	P2	P3	P4	P5	P6	R1	R2
ℓ_1 -norm	32	31	21	42	27	0	2.8	1.8
Budget Uncertainty	34	30	22	42	26	0	4.8	3.0
$\ell_2 + \ell_\infty$	33	25	22	36	32	0	4.2	2.9
DLP	30	30	20	40	30	0	2.1	1.7
PNLP	28	35	18	45	27	0	1.0	0.6

Table 4.7: Booking limits and average capacity unused of Example 5

With a demand factor as high as 1.44, our robust models, notably the ℓ_1 -norm set model, exhibit similar performance as DLP. Surprisingly, they even surpass the performance of PNLP, which utilizes demand distribution information, beyond the 65th percentile. This is because robust models allocate seats more evenly than other models as it can be observed in Table 4.18. As a result, when the arrival pattern is mixed, they have more potential to generate revenues. Therefore, while they offer similar worst-case scenario performance, they also offer better best-case scenario performance.

Example 6: Let us also analyze the performance of our models under scenarios with fairly high, medium, and low demand factors. We consider 91%, 83% and 64% demand factors and expected demand vectors for these are $[30, 25, 20, 35, 25, 30]$, $[11, 41, 6, 61, 11, 21]$ and $[11, 25, 6, 41, 11, 21]$ respectively.

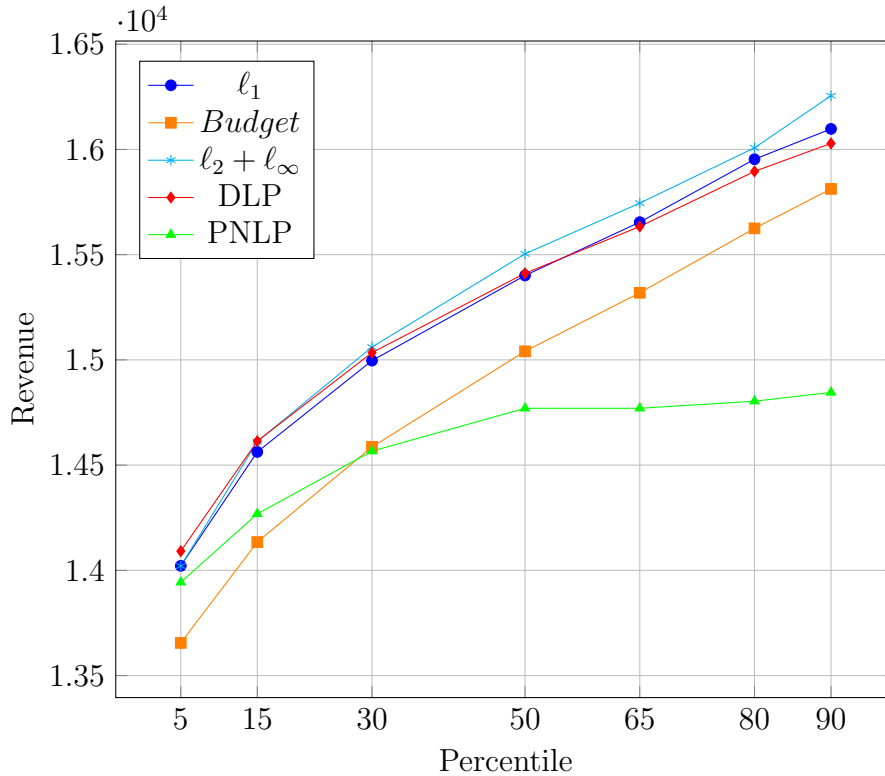


Figure 4.6: Revenue Percentiles Under Mix Arrival Pattern, DF=0.91

Model	Booking Limits						Avg. Cap. Unused	
	P1	P2	P3	P4	P5	P6	R1	R2
ℓ_1 -norm	33	20	21	32	27	10	3.7	2.8
Budget Uncertainty	34	17	22	29	24	15	4.2	2.7
$\ell_2 + \ell_\infty$	33	22	22	33	27	8	4.3	3.7
DLP	30	25	20	35	25	10	3.6	3.4
PNLP	28	18	19	27	23	21	0.8	0.7

Table 4.8: Booking limits and average capacity unused of Example 6, DF=0.91

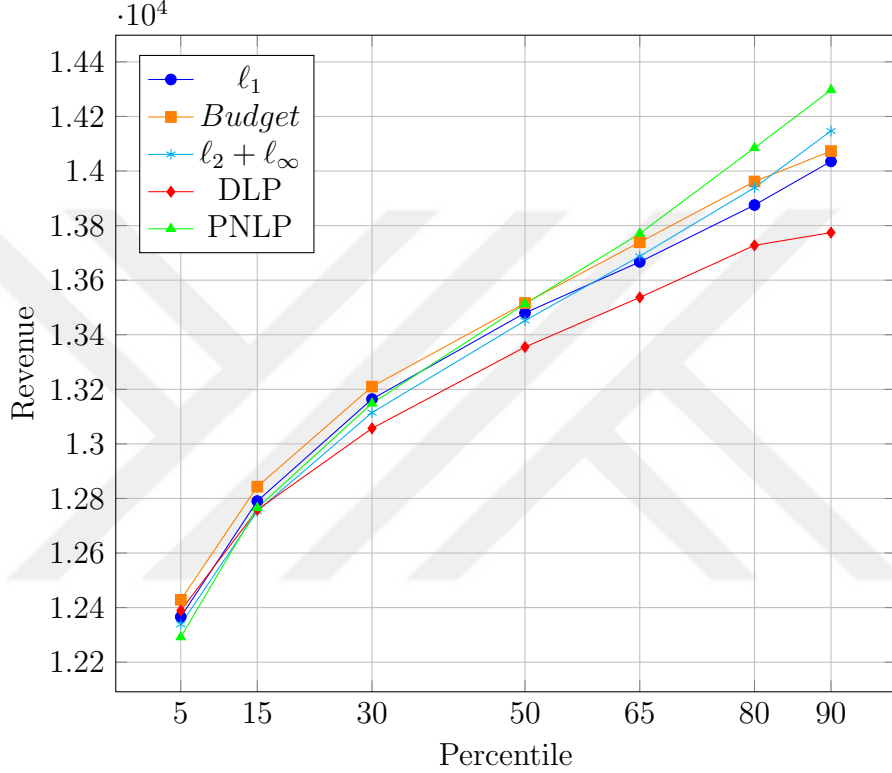


Figure 4.7: Revenue Percentiles Under Mix Arrival Pattern, DF=0.83

Model	Booking Limits						Avg. Cap. Unused	
	P1	P2	P3	P4	P5	P6	R1	R2
ℓ_1 -norm	12	43	7	48	12	23	5.0	1.2
Budget Uncertainty	13	43	7	49	11	23	4.9	0.8
$\ell_2 + \ell_\infty$	12	44	7	49	12	22	6.4	1.9
DLP	11	41	6	52	11	21	8.1	0.7
PNLP	14	44	5	53	12	20	8.0	1.3

Table 4.9: Booking limits and average capacity unused of Example 6, DF=0.83

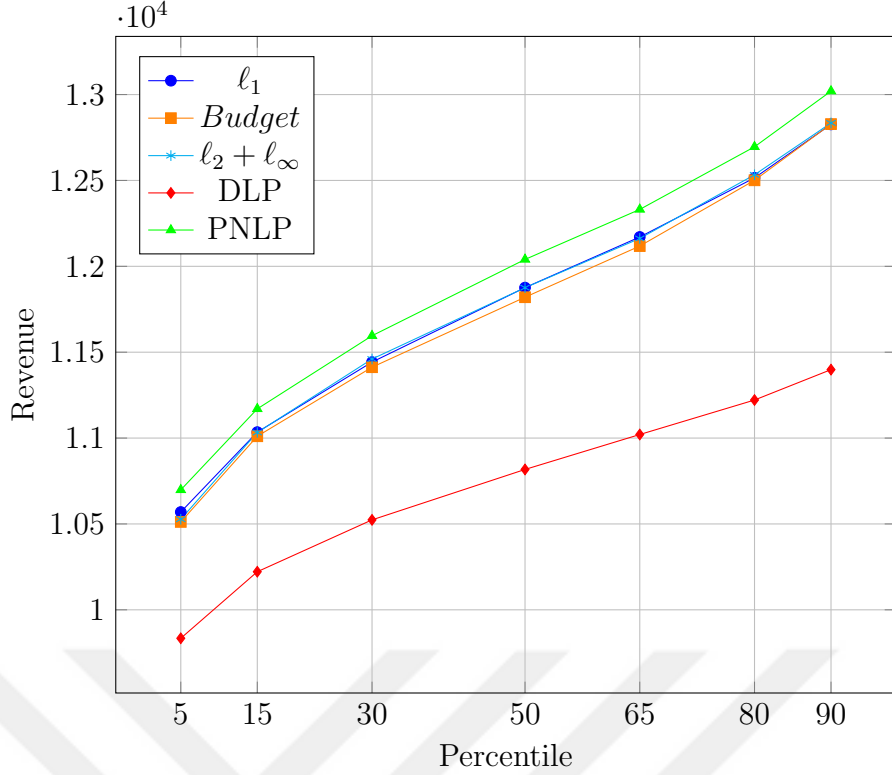


Figure 4.8: Revenue Percentiles Under Mix Arrival Pattern, DF=0.64

Model	Booking Limits						Avg. Cap. Unused	
	P1	P2	P3	P4	P5	P6	R1	R2
ℓ_1 -norm	17	33	7	43	14	26	14.6	5.5
Budget Uncertainty	17	33	7	43	14	26	14.7	5.6
$\ell_2 + \ell_\infty$	17	33	7	43	14	26	14.5	5.6
DLP	11	25	6	41	11	21	23.6	12.7
PNLP	17	33	7	43	15	25	12.9	4.4

Table 4.10: Booking limits and average capacity unused of Example 6, DF=0.64

As expected, DLP's performance decreases with lower demand factors. While PNLP performs well under LBH arrival patterns, its effectiveness diminishes with mixed arrivals and decreasing demand factors due to its conservative seat allocation strategy. In mixed arrival scenarios, PNLP's preference for low fare customers leads to revenue loss for high fare products. In contrast, robust models and DLP outperforms PNLP at every percentile with high DFs. However, as DF decreases revenues get close to each other, except DLP, because there are limited opportunities to generate revenues. DLP is worse than other models since it directly depends on expected demand.

In summary, across this network setup, we observe a consistent trend similar to the previous example. With a high demand factor and LBH arrival pattern, PNLP generally performs

well, while robust models and DLP show similar performance. However, as the demand factor decreases, DLP's performance is more affected due to its direct dependency on expected demand. PNLP's performance also decreases with decreasing demand factor, although not as significantly as DLP, while robust models maintain similar performance levels. When the arrival pattern changes, particularly with high demand factors, robust models and DLP begin to outperform other models as they distribute capacity more evenly across products.

4.2.2 Experiments on a Network with 3 Resources and 5 Products

We first test our methods and benchmark them against PNLP and DLP using a small airline network. There are 5 possible products and 3 resources on this network. Capacity of each resource is 100. Expected demand vector is $[60, 75, 70, 45, 40]$ and standard deviations are $[15, 20, 15, 25, 15]$ and it is assumed that demand is normally distributed.

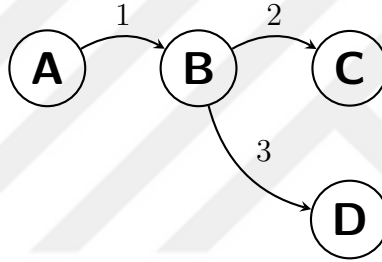


Figure 4.9: Network Structure of the Example

Product	O-D	Fare	Flights Used	Exp. Demand	Std. Dev.
1	A-B	130	1	60	15
2	B-C	120	2	75	20
3	B-D	100	3	70	15
4	A-C	240	1, 2	45	25
5	A-D	210	1, 3	40	15

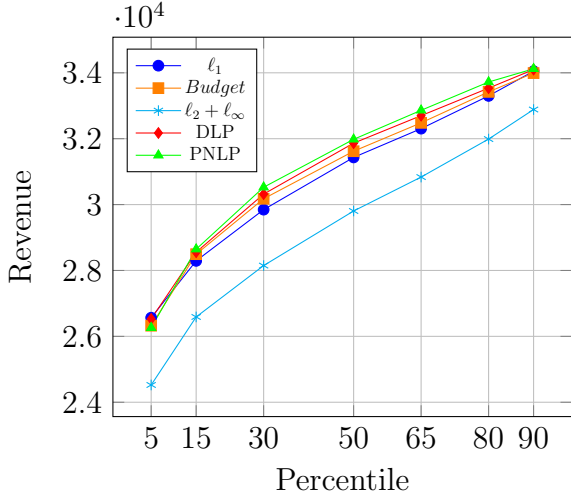
Table 4.11: Parameters

The A matrix for this product is as follows:

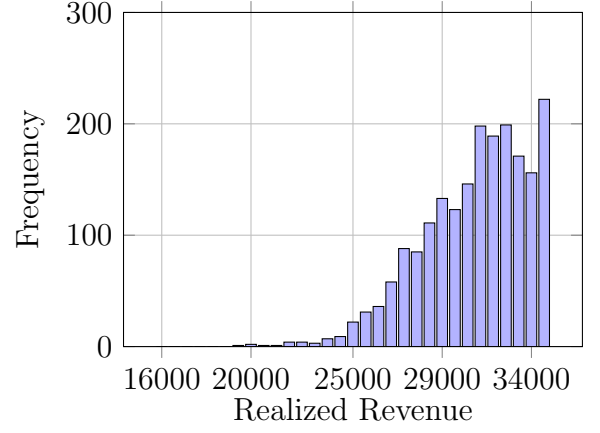
$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Example 7: We test performance of partitioned booking limits since the problem is small (unless otherwise is stated). Unlike previous example, we will analyze the revenue distributions.

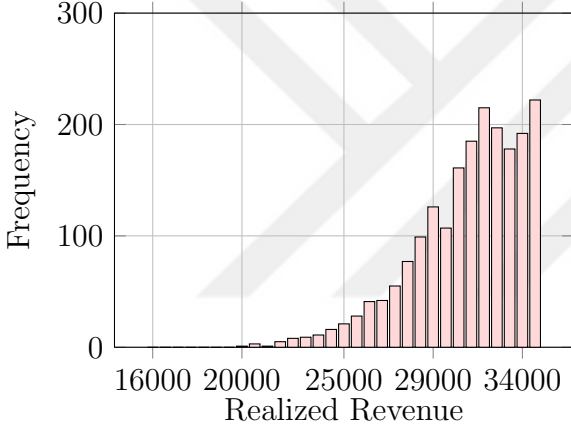
We split revenues into bins and observe their behaviors under changing conditions. The DF is 0.96, the arrival pattern is LBH.



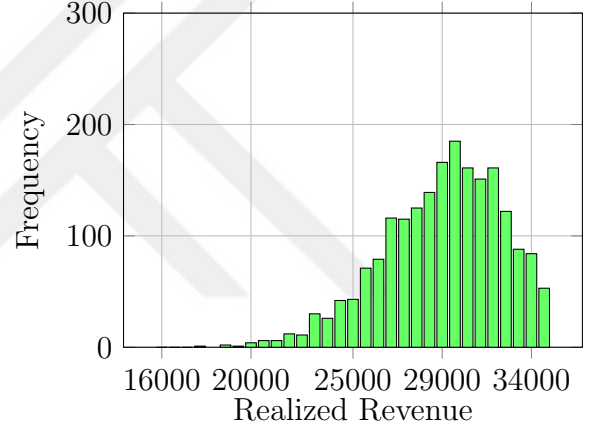
(a) Percentiles and Revenues



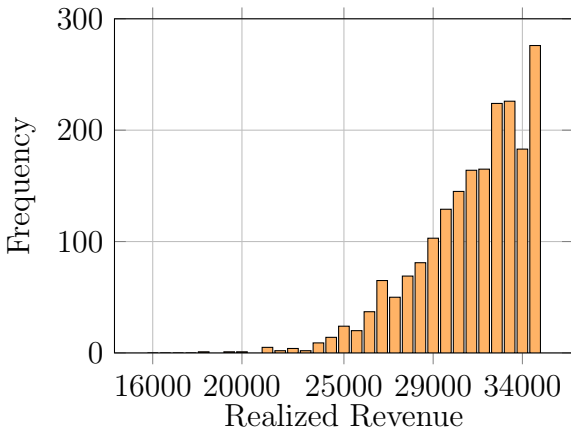
(b) Histogram for ℓ_1 -norm



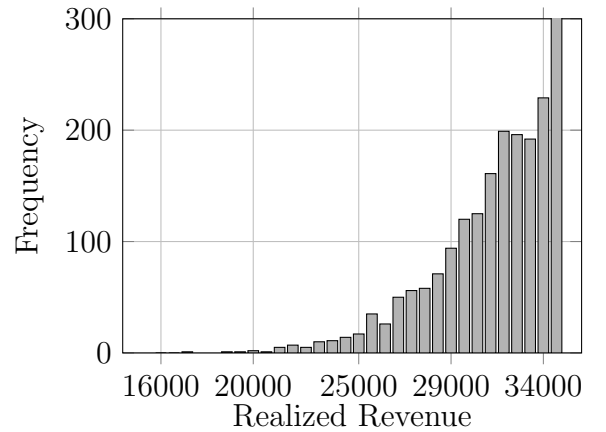
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNLP

Figure 4.10: Performance of different models in Example 7

From the histograms in Figure 4.10, one can observe that robust models, especially ℓ_1 and

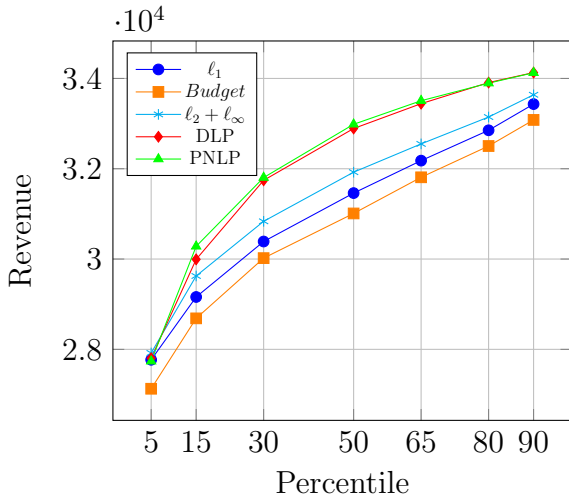
Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	52	82	70	18	30	6.3	13.8	7.9
Budget Uncertainty	51	81	70	19	30	6.2	12.7	8.4
$\ell_2 + \ell_\infty$	69	89	80	11	20	13.1	16.8	13.0
DLP	45	75	70	25	30	6.2	10.8	8.4
PNLP	43	72	71	28	29	6.4	10.1	8.2

Table 4.12: Booking limits and average capacity unused of Example 7

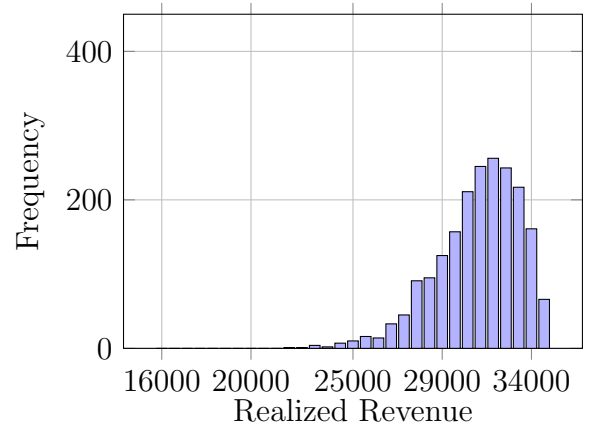
budget uncertainty set models, exhibit a smaller range and more evenly distributed, less skewed outcomes. In contrast, DLP and PNLP's revenue distributions appear highly skewed. Therefore, one can claim that the robust models provide more consistent performance with less variability in outcomes. It is worth noting that DLP and PNLP tend to perform better in best-case (highest revenue) scenarios, but are also prone to worse worst-case (lowest revenue) performance. In contrast, robust models offer better performance in worst-case scenarios, providing more stable and reliable outcomes.

In Table 4.12., One can observe booking limits and average number of seats unsold for each model. Both from the Table and Figure 4.10. (a), it can be observed while all models have similar booking limits and average unused capacity, $\ell_2 + \ell_\infty$ model is more conservative and allocates more seats for low-priced products. As a result, it underperforms and its revenues are lower than the other models' revenues.

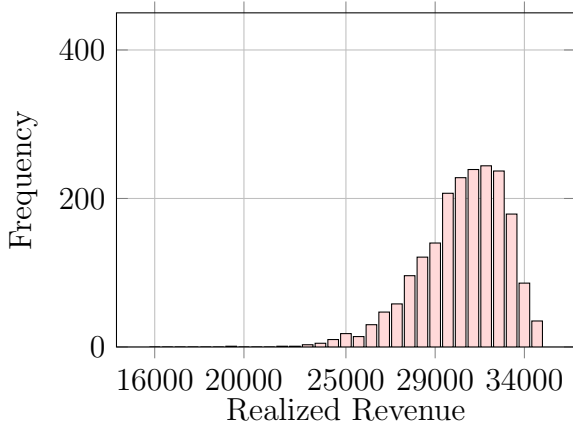
Example 8: We use the same network we use in Example 7. All parameters are the same as in Example 7 except that the standard deviations of the two lowest priced products are decreased by 10 units, resulting following SD vector: [15, 10, 5, 25, 15]



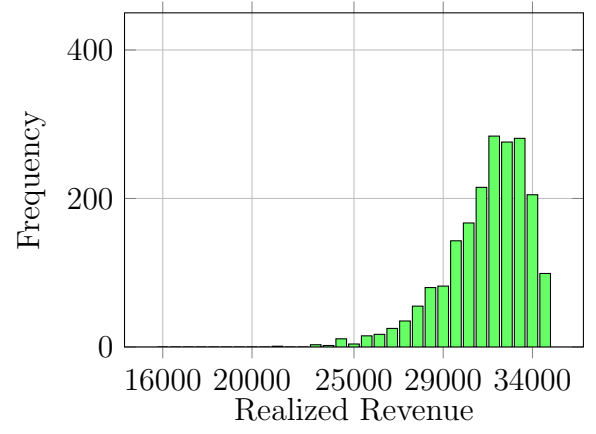
(a) Percentiles and Revenues



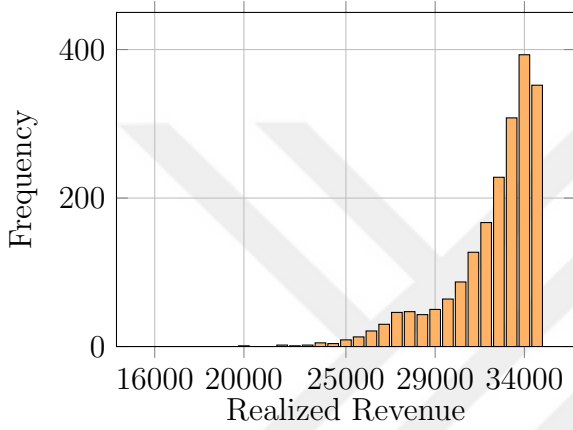
(b) Histogram for ℓ_1 -norm



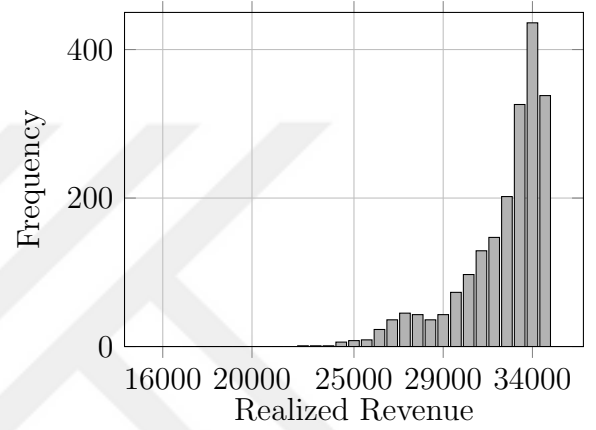
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNL P

Figure 4.11: Performance of different models in Example 8

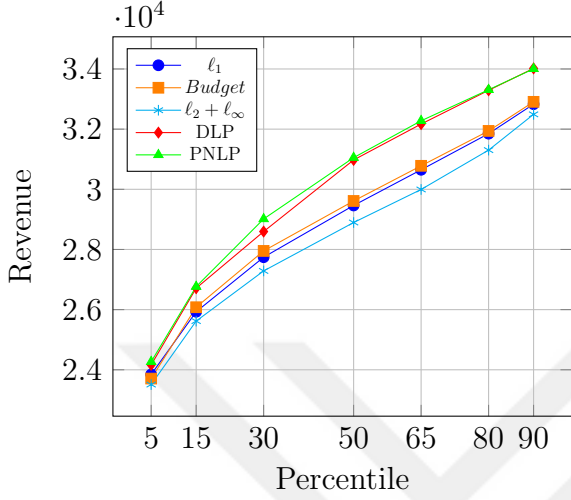
Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	61	85	76	15	24	8.5	11.5	7.4
Budget Uncertainty	64	86	78	14	22	9.9	12.4	8.9
$\ell_2 + \ell_\infty$	58	83	75	17	25	7.4	10.2	6.5
DLP	45	75	70	25	30	6.2	6.8	4.1
PNLP	44	73	71	27	29	6.0	5.9	4.6

Table 4.13: Booking limits and average capacity left of Example 8

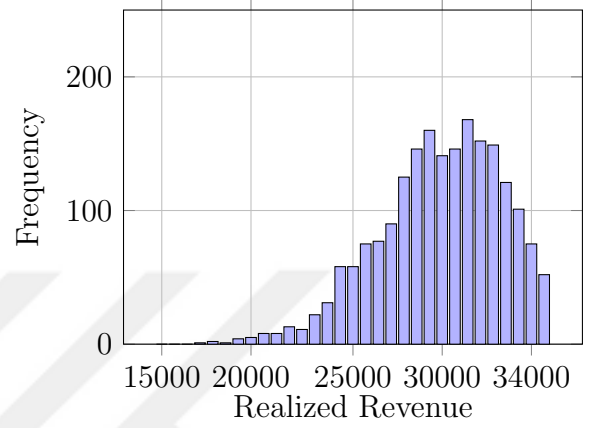
When we decrease the standard deviation of the two lowest-priced products, the performance of our models is affected. The range of revenues of all three robust models, as well as DLP, get wider. Therefore, we see that robust models and DLP are susceptible to variability. In contrast, PNL P is less impacted by such changes, highlighting its ability to better manage situations with reduced demand variability for lower-priced products. As the SDs are decreased, robust models allocate more seats for low-priced products, resulting less revenue because they miss the opportunity to sell seats to high-priced products. This suggests that robust models may be

less effective in environments where low-priced product demand is more predictable. It is also worth mentioning that PNLP's performance in worst-case scenarios have improved, whereas the robust models' worst-case performance has decreased.

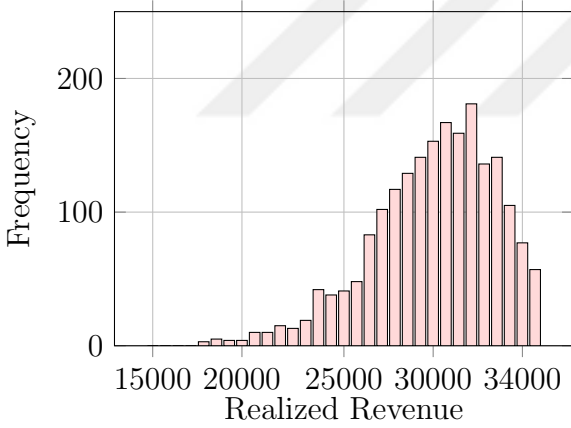
Example 9: We repeat Example 7 with new SDs: increase the standard deviation of 2 highest priced products by 10 units, resulting following SD vector: [15, 20, 15, 35, 25]



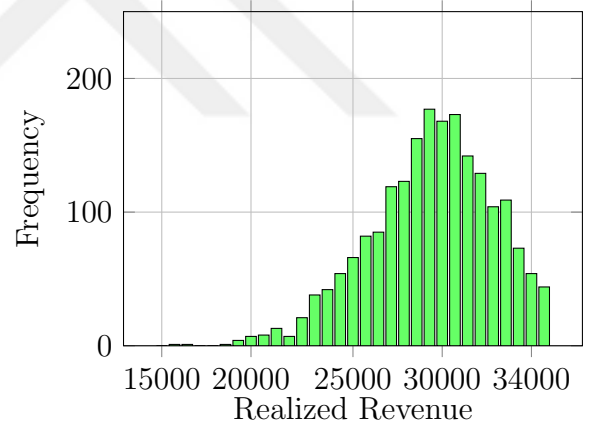
(a) Percentiles and Revenues



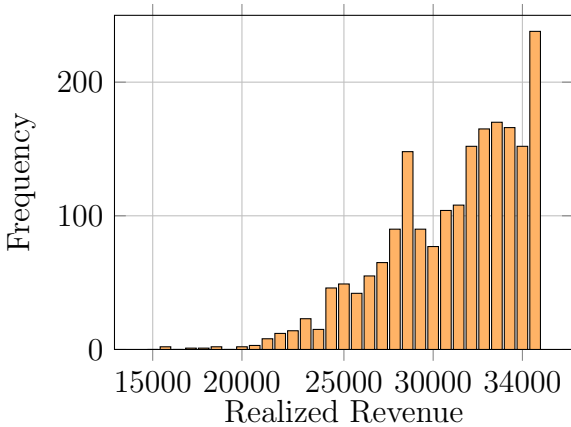
(b) Histogram for ℓ_1 -norm



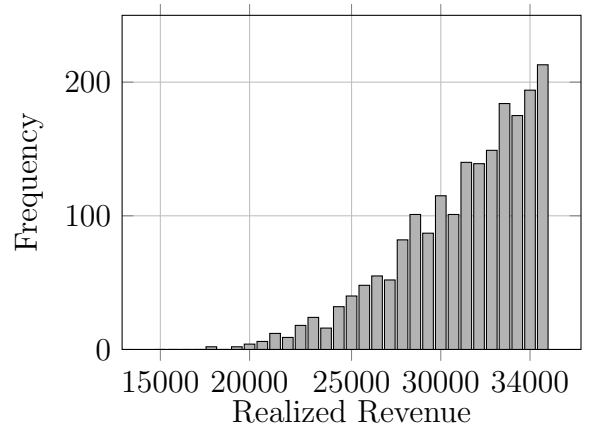
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNLP

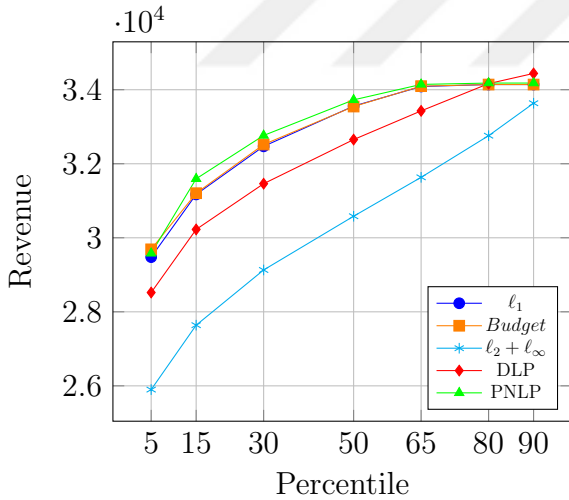
Figure 4.12: Performance of different models in Example 9

Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	67	86	81	14	19	14.4	17.1	14.9
Budget Uncertainty	66	86	80	14	20	13.8	16.7	14.7
$\ell_2 + \ell_\infty$	71	90	81	10	19	16.3	18.8	14.9
DLP	45	75	70	25	30	11.0	12.8	11.0
PNLP	47	74	73	26	27	10.6	12.1	11.4

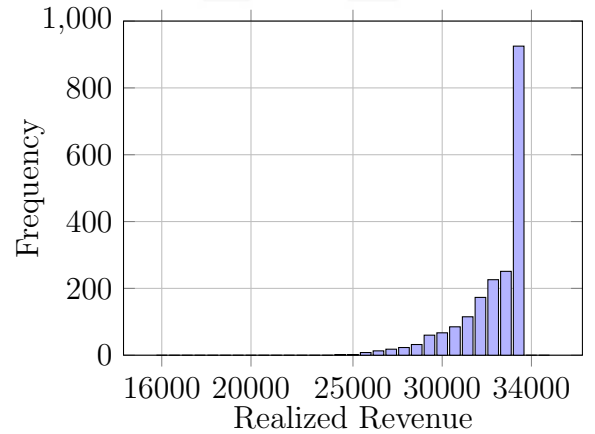
Table 4.14: Booking limits and average capacity unused of Example 9

When we increase the standard deviation of the two highest-priced products, all models exhibit longer left tails, as expected because the variation has increased. The difference is that robust models tend to have more symmetric distributions. In contrast, DLP and PNLN revenue distributions are highly skewed. This again suggests that the robust models' best-case performance is less than that of DLP and PNLN. However, the ℓ_1 and Budget uncertainty set models outperform DLP in worst-case scenarios while showing similar worst-case performance to PNLN.

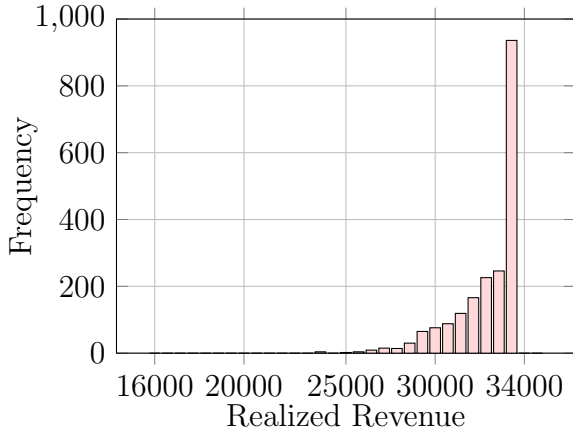
Example 10: We increase the expected demand of each product in Example 7 by 10 units, leading to a DF of 1.13, resulting following expected demand vector: $[70, 85, 80, 55, 50]$.



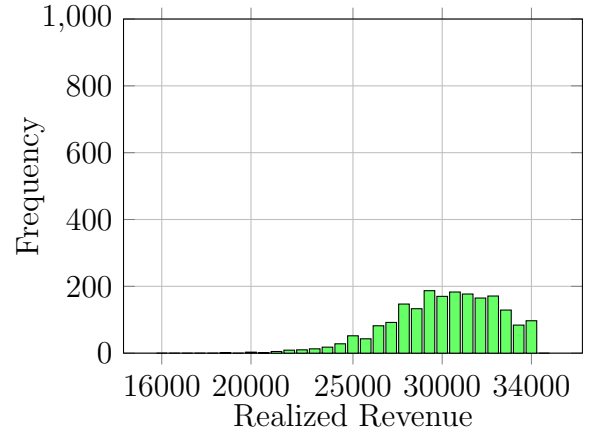
(a) Percentiles and Revenues



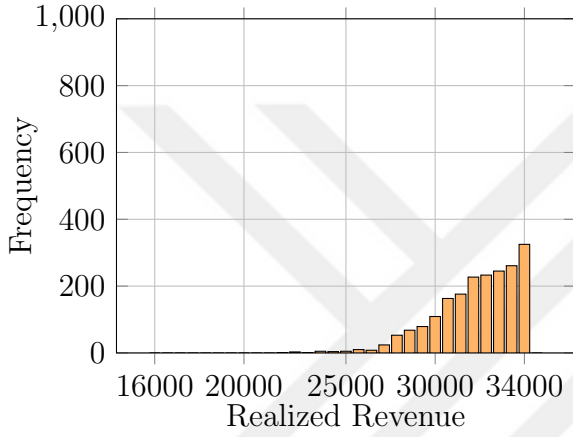
(b) Histogram for ℓ_1 -norm



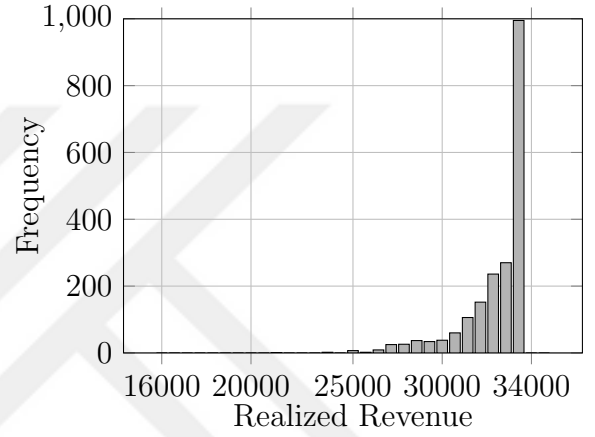
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNL P

Figure 4.13: Performance of different models in Example 10

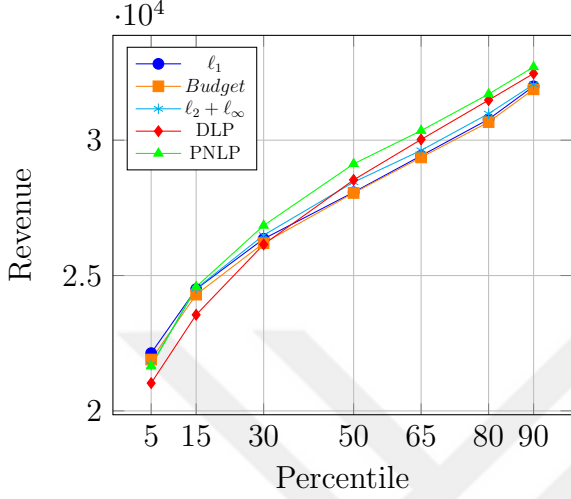
Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	47	80	67	20	33	2.1	6.4	2.6
Budget Uncertainty	47	80	67	20	33	2.1	6.2	2.6
$\ell_2 + \ell_\infty$	80	99	81	1	19	12.2	16.6	6.6
DLP	65	85	80	15	20	4.6	8.2	6.1
PNLP	46	74	72	26	28	2.2	4.9	3.2

Table 4.15: Booking limits and average unused capacity of Example 10

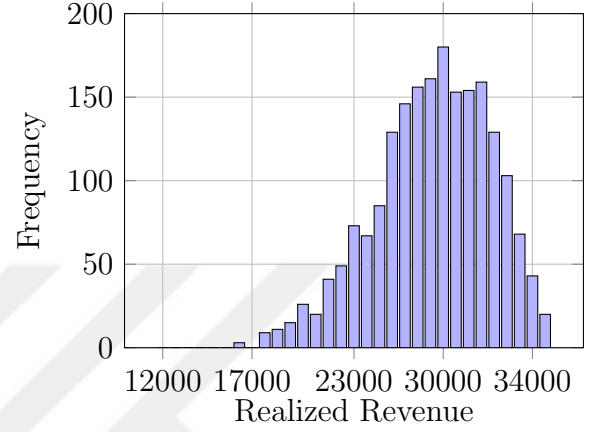
When the demand factor increases, we notice that both ℓ_1 -norm and budget uncertainty set models exhibit significantly narrower ranges compared to DLP and $\ell_2 + \ell_\infty$ models. PNL P also demonstrates relatively narrower range in this example. In the graphs, it becomes evident that the observed the highest revenue value is elevated due to the high demand factor, and fewer empty seats. This observation suggests that the ℓ_1 -norm and budget uncertainty set models, along with PNL P, are better at managing higher demand factors, resulting in a more

concentrated distribution of high revenue outcomes. Crucially, our models do not use demand distribution information underscoring the superiority of our models over PNL P.

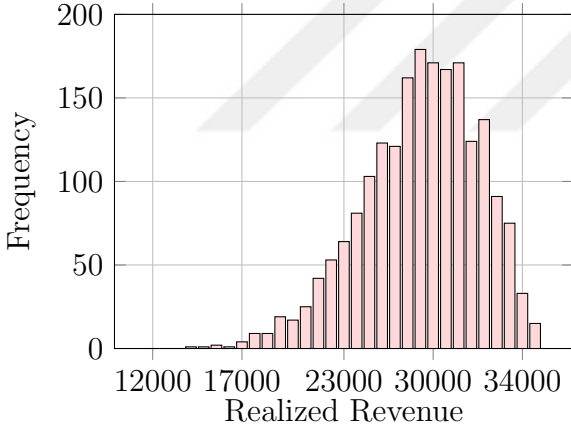
Example 11: Decrease the demand factor by decreasing expected demand of each product by 10 units. Demand Factor is: 0.8 under LBH, and expected demand vector is $[50, 65, 60, 35, 30]$.



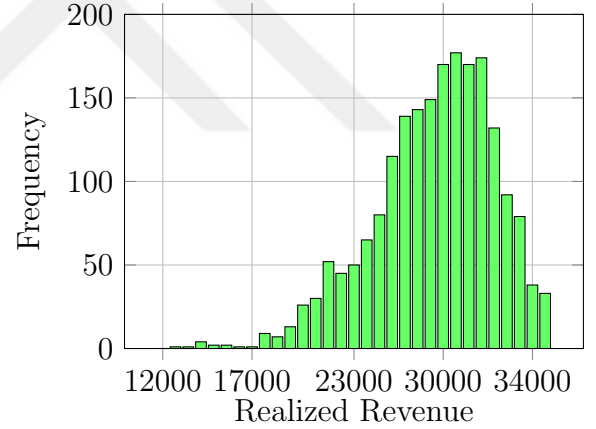
(a) Percentiles and Revenues



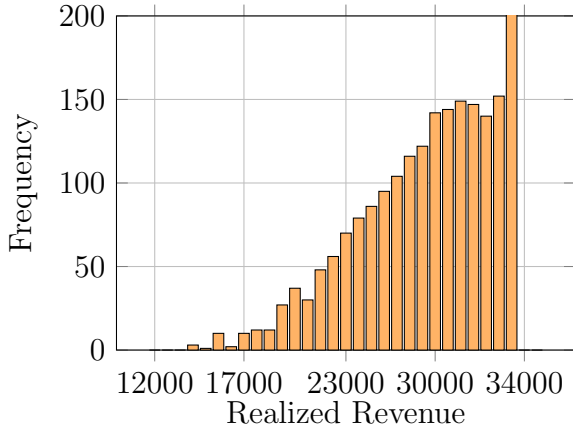
(b) Histogram for ℓ_1 -norm



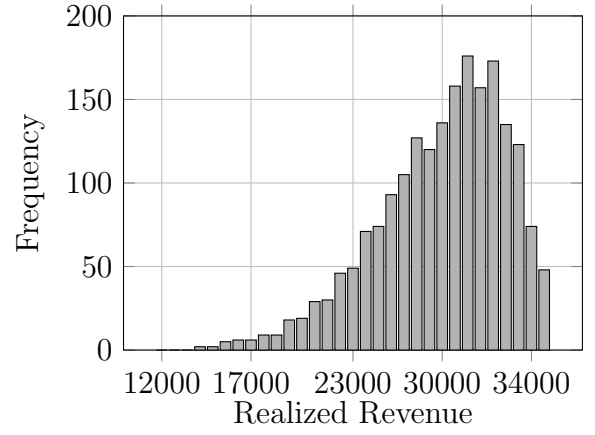
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNL

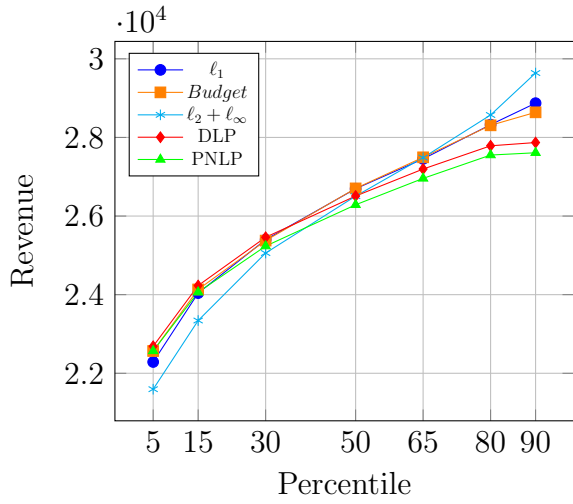
Figure 4.14: Performance of different models in Example 11

Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	55	82	73	18	27	15.6	21.8	18.7
Budget Uncertainty	56	83	73	17	27	15.8	22.3	19.0
$\ell_2 + \ell_\infty$	50	80	70	20	30	15.4	21.3	18.1
DLP	35	65	60	35	30	16.6	17.1	21.6
PNLP	42	70	72	30	28	14.5	17.2	19.2

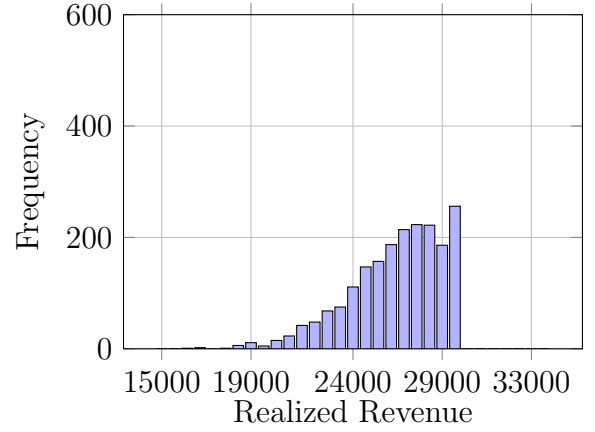
Table 4.16: Booking limits and average capacity unused of Example 11

Decreasing the demand factor leads to longer left tails in all models except for the one utilizing the ℓ_1 uncertainty set, which exhibits the shortest tail and a bell-shaped graph. While other models display greater variability, the ℓ_1 uncertainty set model demonstrates a narrower revenue range. This observation suggests that the ℓ_1 uncertainty set model is better suited for handling lower demand factors, providing better worst-case scenarios. Its best-case scenario performance is similar to other models, except for DLP. Although DLP has the best best-case scenario, it has a long left tail, leading to poor worst-case performance.

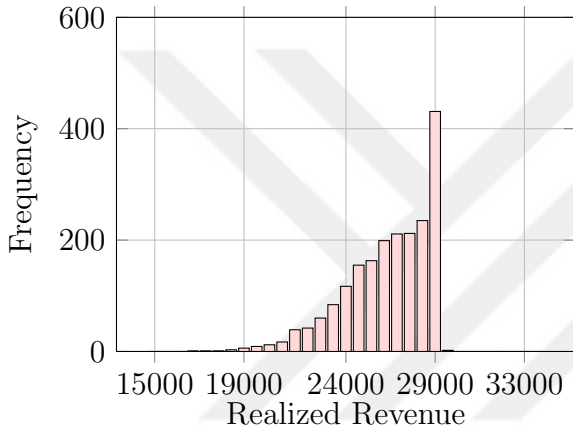
Example 12: We change the arrival pattern. Until now, we have considered LBH arrival pattern. Now, we use a random arrival pattern where customers in the same fare class are considered identical. Imagine a deck of cards, where each card represents a customer and the suit of the card represents the fare class. The number of cards of each suit is determined by a random variable reflecting the demand for each fare class. When a customer arrives, we draw a card at random from the deck, which determines the fare class of the arriving customer. We call this the “Mixed Arrival Pattern”. Now, we use exactly the same parameters in Example 7 under mixed arrival pattern until Example 16.



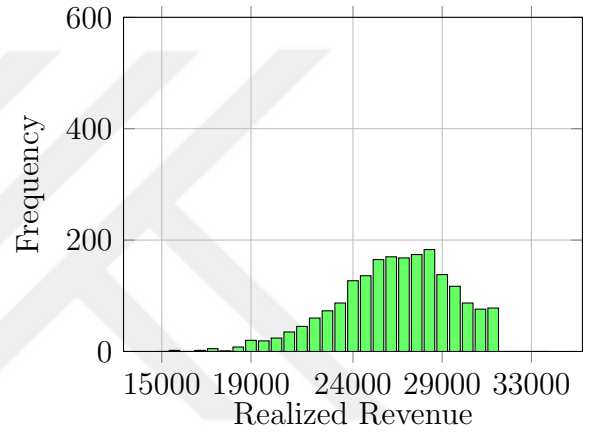
(a) Percentiles and Revenues



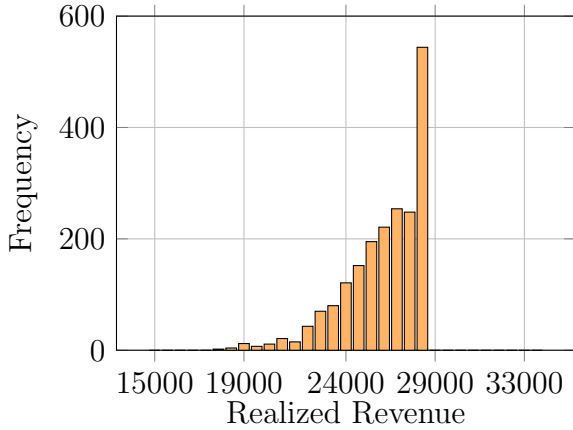
(b) Histogram for ℓ_1 -norm



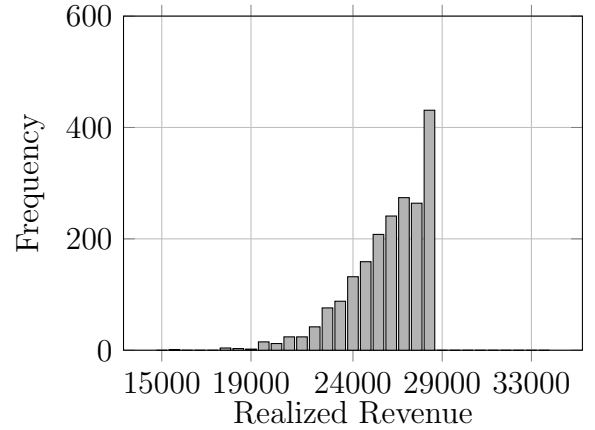
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNLP

Figure 4.15: Performance of different models in Example 12

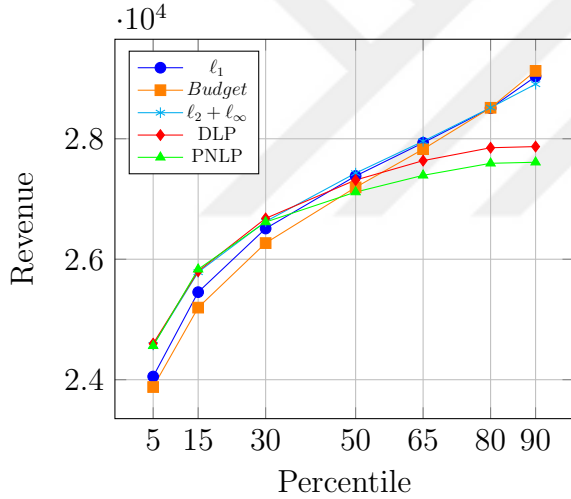
When the arrival pattern is mixed, the revenue histograms of robust models are comparable to those of DLP and PNLP, except for the $\ell_2 + \ell_\infty$ model. While the other four models have similar best- and worst-case performance, the $\ell_2 + \ell_\infty$ model achieves a superior best-case scenario, reaching revenues up to \$32,000. The other two robust models also show slightly improved

Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	52	82	70	18	30	4.6	13.0	6.5
Budget Uncertainty	51	81	70	19	30	3.5	11.8	6.4
$\ell_2 + \ell_\infty$	69	89	80	11	20	12.0	16.6	12.5
DLP	45	75	70	25	30	3.2	8.7	6.2
PNLP	43	72	71	28	29	3.4	8.2	6.5

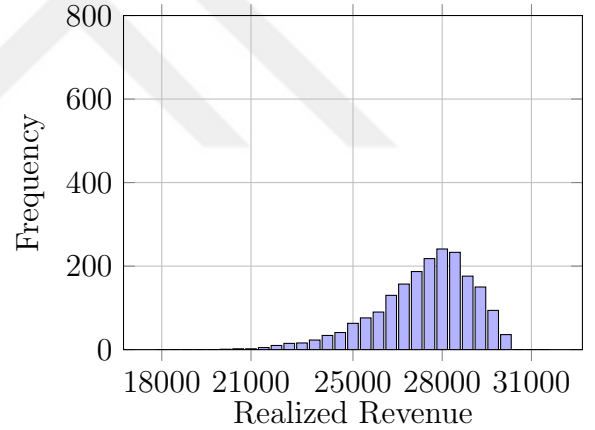
Table 4.17: Booking limits and average capacity unused of Example 12

best-case scenario performance. This extended range allows robust models to outperform DLP and PNLP in the percentile graph, indicating their ability to capture higher revenue outcomes effectively. This is because robust models distribute less seats to high-priced products and more seats to lower-priced products.

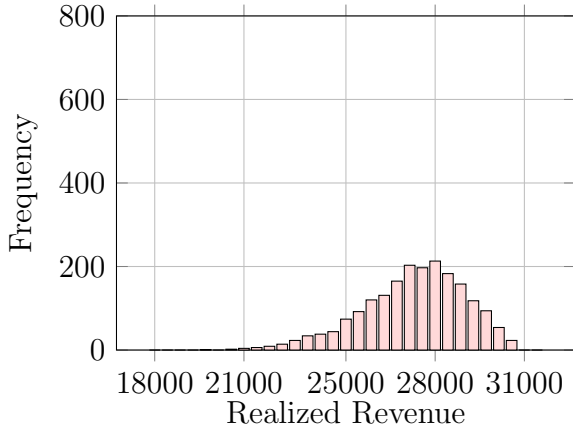
Example 13: Decrease the SD of the 2 lowest priced products by 10 units under mixed arrival pattern., resulting SD vector of $[15, 10, 5, 25, 15]$.



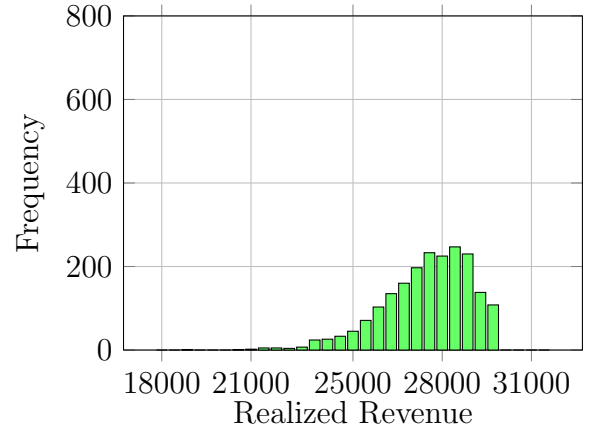
(a) Percentiles and Revenues



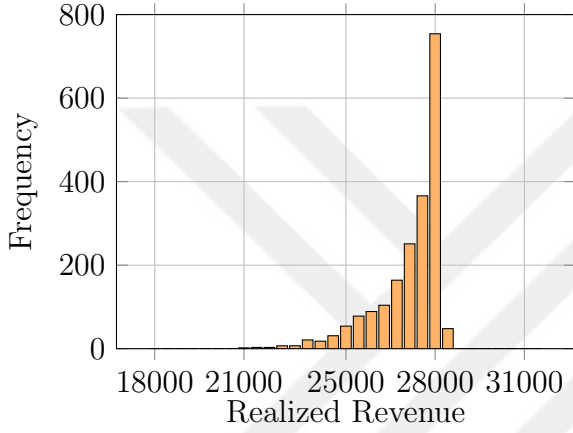
(b) Histogram for ℓ_1 -norm



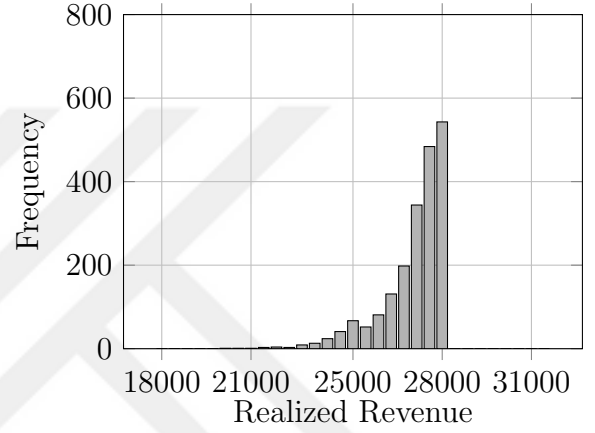
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNLP

Figure 4.16: Performance of different models in Example 13

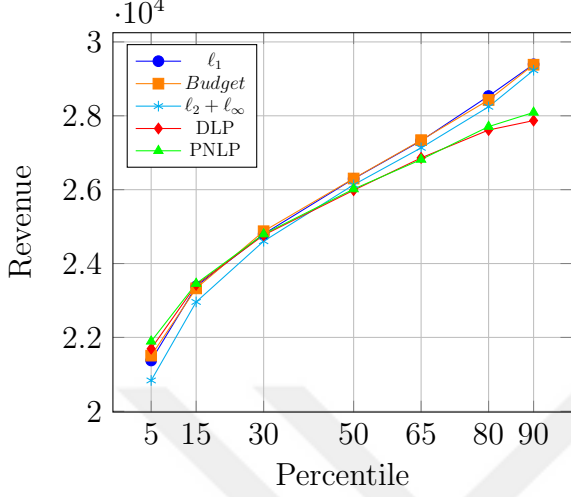
Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	61	85	76	15	24	7.0	11.3	6.9
Budget Uncertainty	64	86	78	14	22	9.1	12.7	8.4
$\ell_2 + \ell_\infty$	58	83	75	17	25	5.3	9.7	5.1
DLP	45	75	70	25	30	3.0	5.0	2.5
PNLP	44	73	71	27	29	2.9	4.5	2.4

Table 4.18: Booking limits and average capacity unused of Example 13

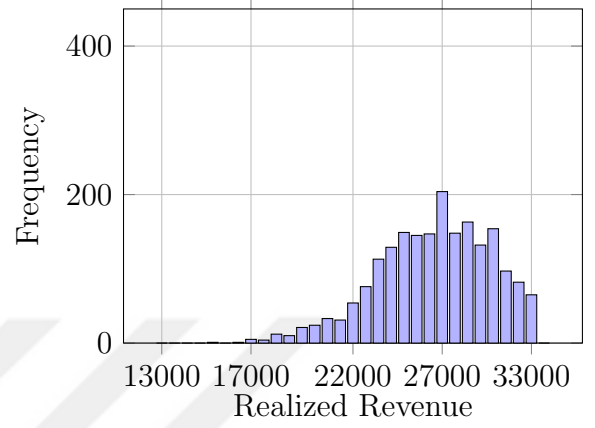
With the decreased standard deviation of the two lowest-priced products, the distribution graphs undergo noticeable changes. While the left tails maintain a similar shape and extend to nearly the same point, the right tails differ significantly. The right tails of the robust models extend further than DLP and PNLP, indicating superior best-case scenarios. The robust models' graphs resemble a bell shape, whereas the DLP and PNLP graphs follow an exponential pattern. Although PNLP and DLP may have numerous instances at high revenue

points, the extended reach of the robust models to the right of the distribution makes them superior to DLP and PNLP on percentile graphs.

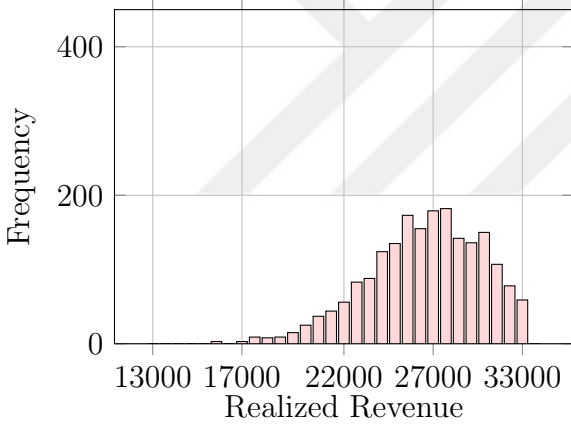
Example 14: Increase the standard deviation of 2 highest priced products by 10 units, all the other parameters are the same with the initial example. SD vector is $[15, 20, 15, 35, 25]$.



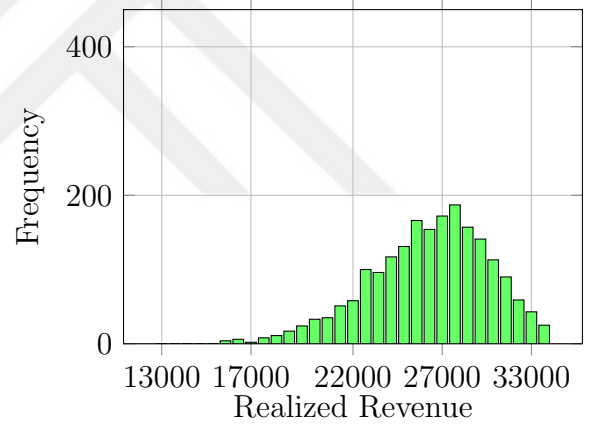
(a) Percentiles and Revenues



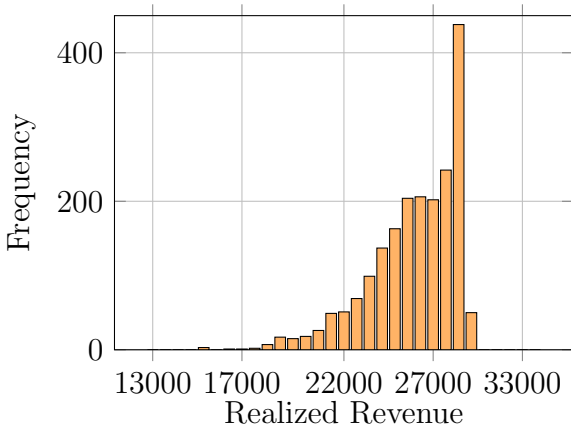
(b) Histogram for ℓ_1 -norm



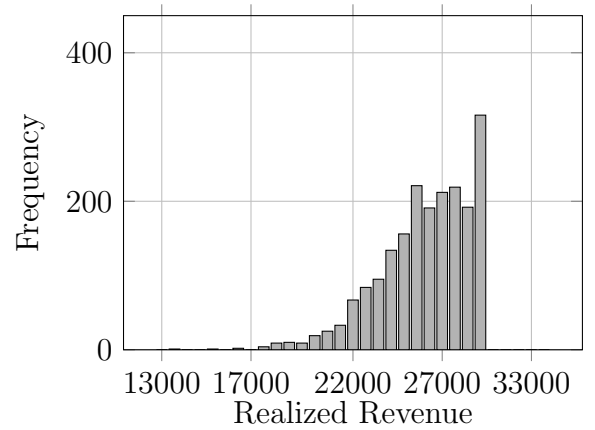
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNLP

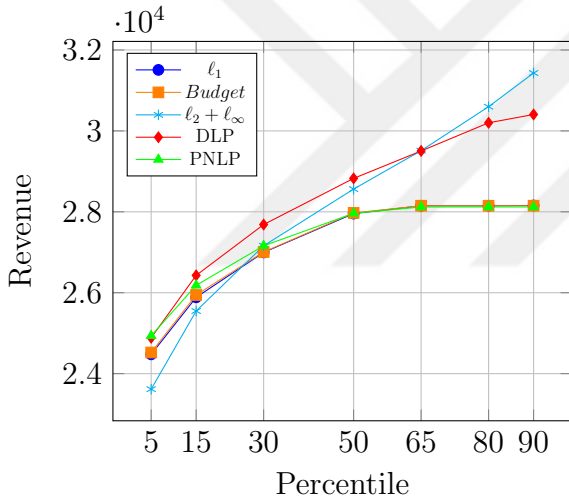
Figure 4.17: Performance of different models in Example 14

Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	67	86	81	14	19	12.5	16.4	13.7
Budget Uncertainty	66	86	80	14	20	12.8	16.2	14.5
$\ell_2 + \ell_\infty$	71	90	81	10	19	15.3	19.4	14.0
DLP	45	75	70	25	30	7.3	10.8	8.7
PNLP	47	74	73	26	27	7.2	11.2	9.6

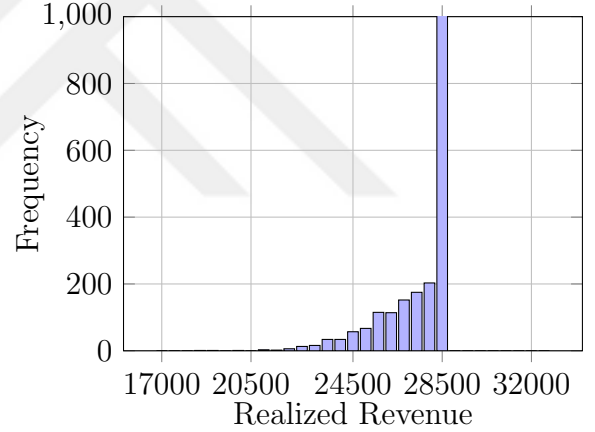
Table 4.19: Booking limits and average capacity unused of Example 14

Similar to the previous scenario, the worst-case revenues are achieved at approximately the same value. However, the robust models exhibit better best-case performance, making them superior to DLP and PNLP on percentile graphs.

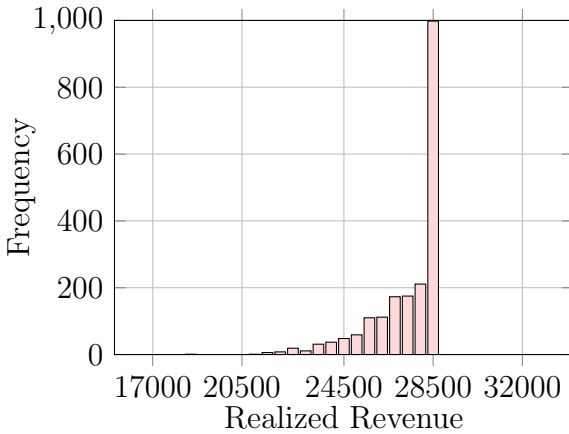
Example 15: We increase the expected demand of each product by 10 units, leading DF of 1.13 under mixed arrival pattern, resulting expected demand vector of $[70, 85, 80, 55, 50]$.



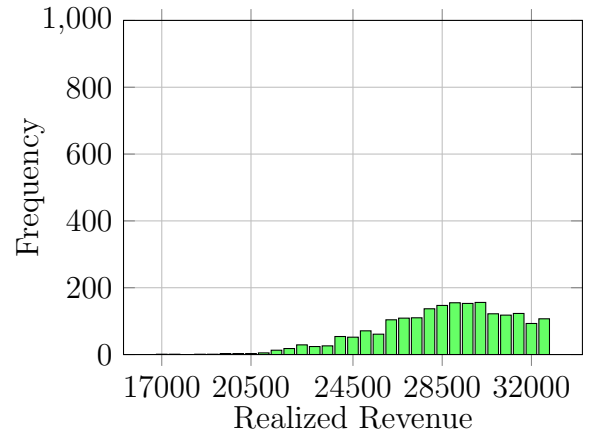
(a) Percentiles and Revenues



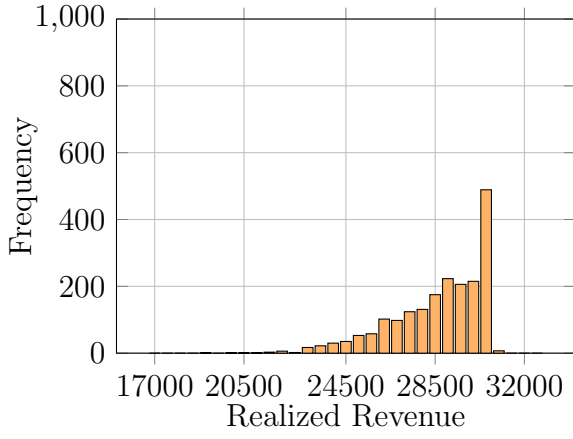
(b) Histogram for ℓ_1 -norm



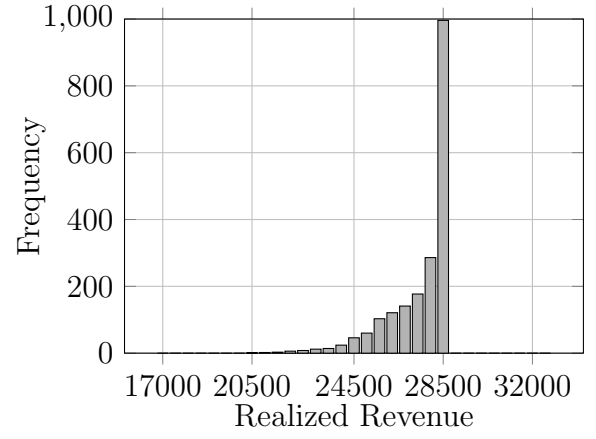
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNL

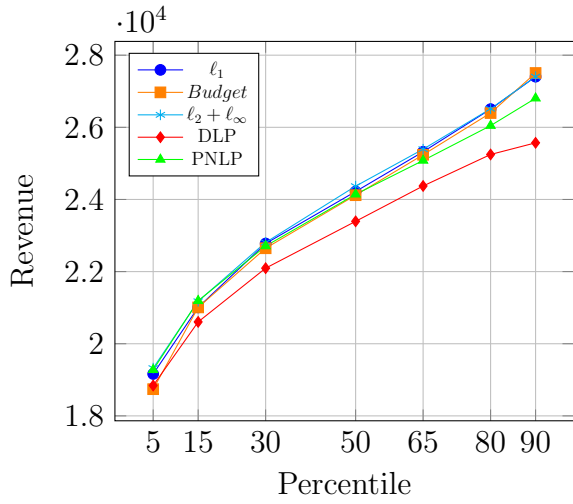
Figure 4.18: Performance of different models in Example 15

Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	47	80	67	20	33	0.8	6.1	1.6
Budget Uncertainty	47	80	67	20	33	0.9	6.0	1.5
$\ell_2 + \ell_\infty$	80	99	81	1	19	10.9	16.2	6.1
DLP	65	85	80	15	20	3.7	8.0	6.3
PNLP	46	74	72	26	28	1.0	4.0	2.7

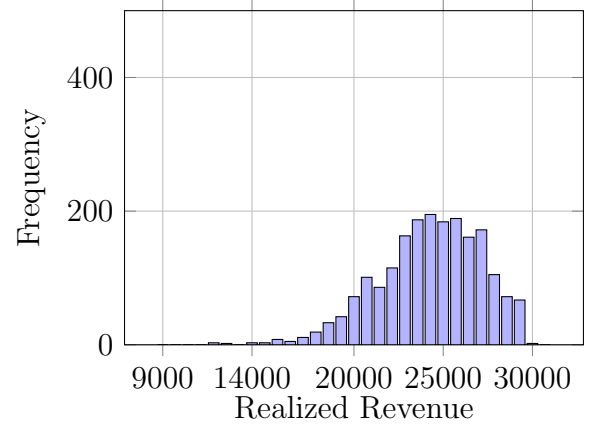
Table 4.20: Booking limits and average capacity unused of Example 15

When we increase the demand factor, the distribution graphs generally maintain the same structure across models, except for $\ell_2 + \ell_\infty$ and DLP. The left tails of PNL and two robust models are relatively short, indicating better worst-case scenarios. In contrast, DLP and $\ell_2 + \ell_\infty$ have worse worst-case, but also better best-case. Most revenue distributions have at a revenue of \$28,500, while DLP and the $\ell_2 + \ell_\infty$ model extend to \$32,000. That is because demand factor is high and $\ell_2 + \ell_\infty$ model allocates highest number of seats for low fares. Even though their revenue is lower than highest priced products, they generate more revenues due to number. These two models exhibit relatively more evenly distributed revenues, and their extensions to further points on the right side of the graphs render them superior on percentile graphs.

Example 16: Decrease the demand factor by decreasing expected demand of each product by 10 units. Demand Factor is: 0.8 like in Example 11 under mixed arrival pattern with expected demand of [50, 65, 60, 35, 30].

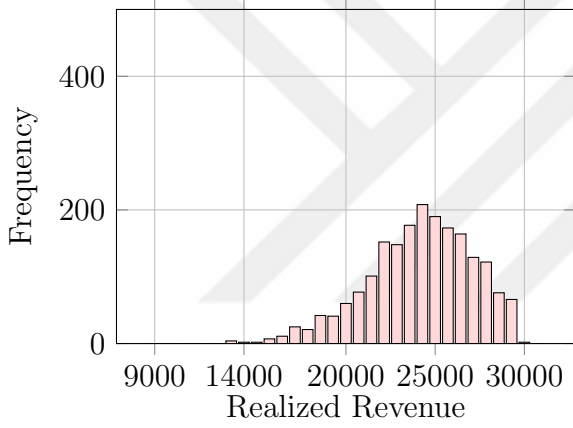


(a) Percentiles and Revenues

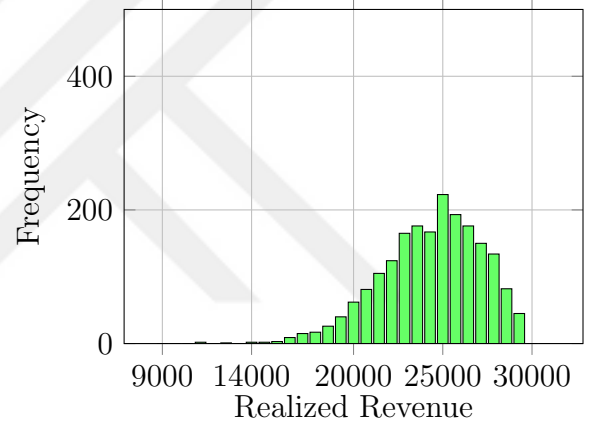


(b) Histogram for ℓ_1 -norm

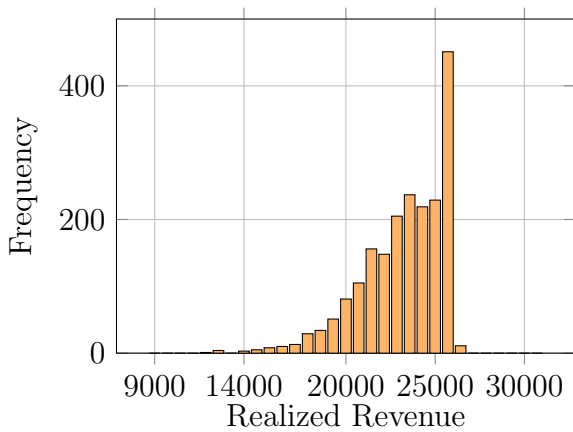
Figure 4.19: Performance of different models in Example 16



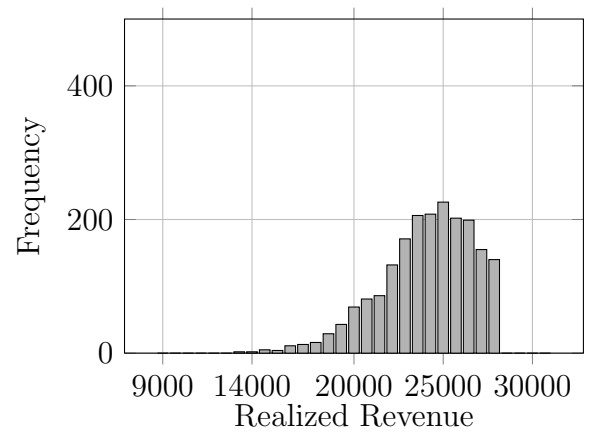
(c) Histogram for Budget Uncertainty Set



(d) Histogram for $\ell_2 + \ell_\infty$



(e) Histogram for DLP



(f) Histogram for PNLP

Figure 4.19: Performance of different models in Example 16

As the demand factor decreases, robust models tend to achieve better best-case scenarios,

Model	Booking Limits					Avg. Cap. Unused		
	P1	P2	P3	P4	P5	R1	R2	R3
ℓ_1	55	82	73	18	27	12.4	21.1	15.2
Budget Uncertainty	56	83	73	17	27	12.2	21.7	15.6
$\ell_2 + \ell_\infty$	50	80	70	20	30	10.8	19.1	14.8
DLP	35	65	60	35	30	8.8	13.6	18.2
PNLP	42	70	72	30	28	9.0	15.1	15.9

Table 4.21: Booking limits and average capacity unused of Example 16

reaching up to \$30,000 in revenue. Conversely, DLP and PNLN exhibit worse best-case scenarios because these models allocate more seats to high-fare customers. With a low demand factor, many of these allocated seats remain empty, resulting in a loss of revenue. Also, all models have similar worst-case scenarios.

In conclusion, our comprehensive analysis demonstrates that robust models, particularly those utilizing ℓ_1 and budget uncertainty sets, consistently provide more stable and reliable performance across various scenarios. These models exhibit less variability and more symmetric revenue distributions compared to DLP and PNLN, which tend to perform better in best-case scenarios but suffer in worst-case scenarios. When demand variability is low, robust models and DLP show increased sensitivity, whereas PNLN remains stable. On the other hand, high demand variability affects all models, but robust models maintain better worst-case performance. High demand factors narrow the revenue distribution ranges for robust models, showcasing their effective management of increased demand. Conversely, decreased demand factors highlight the ℓ_1 uncertainty set model's consistent performance and superior worst-case scenario handling. In mixed arrival patterns, robust models match or exceed the performance of DLP and PNLN, with the $\ell_2 + \ell_\infty$ model excelling in best-case scenarios. Ultimately, the robust models' ability to manage demand variability and provide consistent outcomes underscores their suitability for unpredictable environments.

4.2.3 Experiments on a Network with 18 Products and 3 Resources

So far, we have examined smaller examples from [4] and created by us. Now, we will shift our focus to a larger example from [33], featuring 18 products and 3 resources.

Example 17: Demand has a negative binomial distribution with given parameters in table 4.22 below. Before delving into the analysis, let's define the parameters of this example and create a visualization to enhance our understanding.

As mentioned earlier, our example comprises 3 resources, each with a capacity of 200 seats.

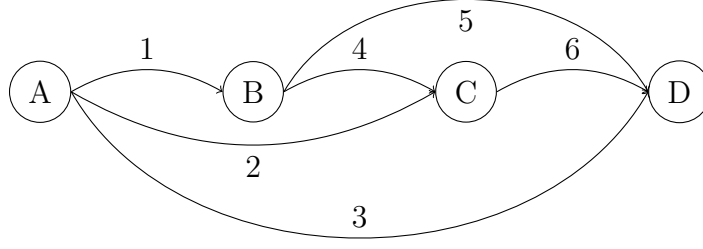


Figure 4.20: Network of de Boer's example

OD	Fare class 3				Fare class 2				Fare class 1			
	Exp.	SD	Fare	Pro. no.	Exp.	SD	Fare	Pro. no.	Exp.	SD	Fare	Pro. no.
A-B	50	9.01	75	13	40	7.75	125	7	30	18.17	250	1
A-C	40	7.75	130	14	25	6.12	170	8	20	14.83	400	2
A-D	30	6.71	200	15	24	5.66	320	9	20	14.83	460	3
B-C	30	6.71	100	16	20	5.48	150	10	20	14.83	330	4
B-D	30	6.71	160	17	20	5.16	200	11	20	9.31	420	5
C-D	50	9.01	80	18	40	7.75	110	12	30	13.42	235	6

Table 4.22: Parameters of de Boer's example

Leveraging the defined parameters, we applied our proposed models to tackle the multi-leg problem across different scenarios. Subsequently, employing Algorithm 1, we derived partitioned booking limits. These limits were then utilized to conduct simulations based on a low-before-high arrival pattern, with 2,000 simulations performed for each scenario. Below, we present the results obtained from these simulations.

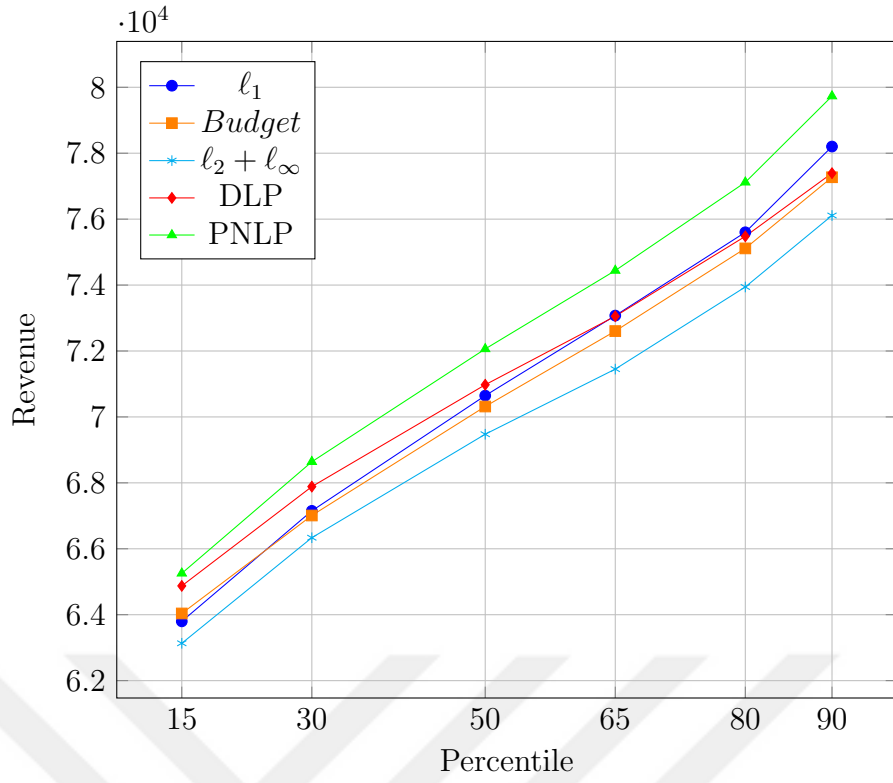


Figure 4.21: Revenue Percentiles Under LBH With DF=0.89

Model	Booking Limits								
	P1	P2	P3	P4	P5	P6	P7	P8	P9
ℓ_1 -norm	36	22	19	22	22	33	45	12	26
Budget Uncertainty	36	18	16	23	22	33	47	15	27
$\ell_2 + \ell_\infty$	34	22	6	23	22	33	45	22	27
DLP	30	20	20	20	20	30	40	25	24
PNLP	40	22	17	27	22	35	40	18	21

Model	Booking Limits								
	P10	P11	P12	P13	P14	P15	P16	P17	P18
ℓ_1 -norm	22	22	43	40	0	0	33	0	35
Budget Uncertainty	23	22	44	41	0	0	34	0	36
$\ell_2 + \ell_\infty$	22	22	44	44	0	0	34	0	46
DLP	20	20	40	41	0	0	30	1	45
PNLP	19	16	36	42	0	0	23	15	38

(a) Booking Limits

Model	Average Capacity Unused		
	R1	R2	R3
ℓ_1 -norm	33.3	38.8	28.9
Budget Uncertainty	31.9	37.3	28.3
$\ell_2 + \ell_\infty$	30.0	37.2	27.3
DLP	26.4	32.4	23.4
PNLP	29.8	30.5	20.3

(b) Average Capacity Unused

Table 4.23: Booking limits and average capacity unused of Example 17

From Figure 4.21, we observe a performance similar to the previous example. In Figure 4.1, our models were mostly dominated by PNLP and DLP. Similarly, in Figure 4.21, PNLP dominates our models; however, after the 65th percentile, the ℓ_1 -norm model slightly surpasses the DLP. It's worth noting that since PNLP requires demand distribution information, our models perform notably well in this regard without relying on any demand distribution information. This is evident from the small gap between PNLP and our robust models.

Example 18: Let us consider a scenario with a medium demand factor of 81%. We will use the following expected demand vector, obtained by decreasing the expectation of low fare classes: [25, 25, 25, 25, 25, 30, 40, 15, 14, 20, 20, 30, 35, 30, 30, 30, 35].

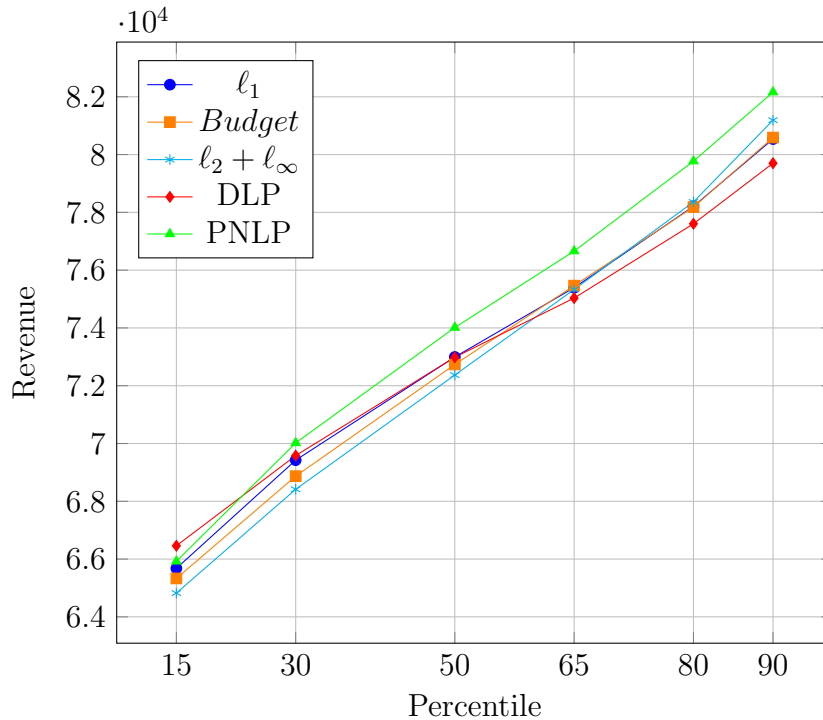


Figure 4.22: Revenue Percentiles Under LBH With DF=0.81

Model	Booking Limits									
	P1	P2	P3	P4	P5	P6	P7	P8	P9	
ℓ_1 -norm	30	28	24	27	27	33	45	17	15	
Budget Uncertainty	31	28	25	29	28	33	46	14	16	
$\ell_2 + \ell_\infty$	29	28	28	28	28	34	46	13	16	
DLP	25	25	25	25	25	30	40	15	14	
PNLP	41	26	23	28	27	38	43	9	12	

Model	Booking Limits									
	P10	P11	P12	P13	P14	P15	P16	P17	P18	
ℓ_1 -norm	22	22	33	39	0	2	9	7	37	
Budget Uncertainty	22	22	33	40	0	0	12	4	39	
$\ell_2 + \ell_\infty$	23	23	34	40	0	0	13	0	37	
DLP	20	20	30	35	0	21	10	0	35	
PNLP	18	16	29	34	0	12	15	14	29	

(a) Booking Limits

Model	Average Capacity Unused		
	R1	R2	R3
ℓ_1 -norm	38.5	35.2	32.4
Budget Uncertainty	42.7	37.4	34.3
Ellips+Box	43.4	41.1	37.6
DLP	29.7	29.3	25.4
PNLP	39.6	30.8	30.3

(b) Average Capacity Unused

Table 4.24: Booking limits and average capacity unused of Example 18

In Table 4.22, we observe that as the demand factor decreases, our robust models begin to perform better. While the DLP leads until the 50th percentile, after that point, all three of our models dominate the DLP. Although PNLp outperforms our models, we believe that not requiring demand distribution information compensates for the small difference between PNLp and our models.

Example 19: Let us examine a scenario with a low demand factor of 72%, obtained using the following demand vector: [25, 30, 30, 25, 20, 25, 40, 15, 14, 20, 20, 30, 25, 20, 20, 20, 20,

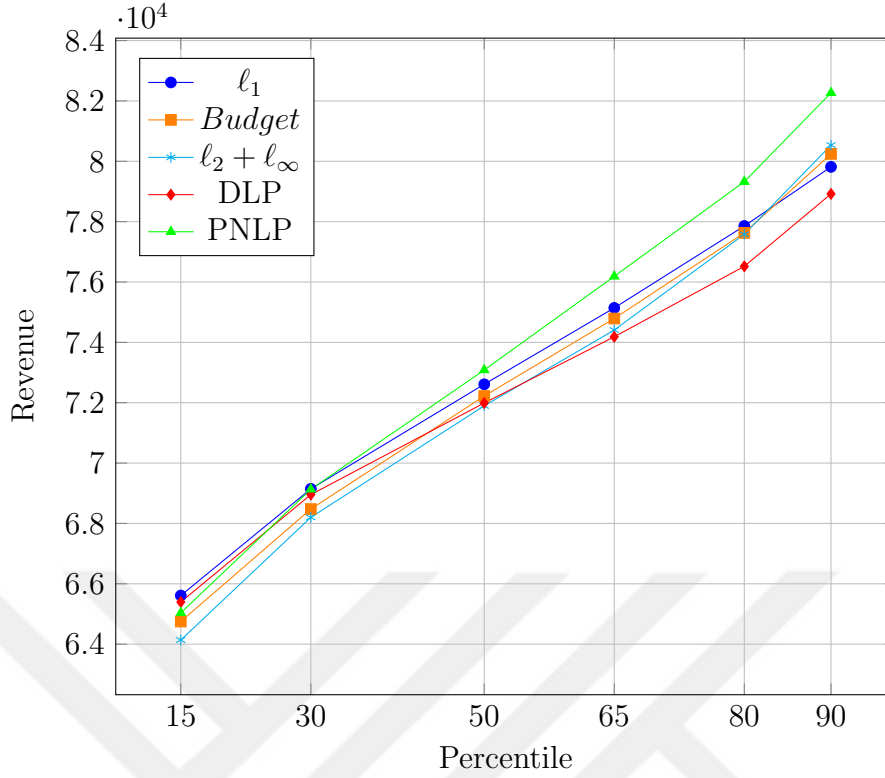


Figure 4.23: Revenue Percentiles Under LBH With DF=0.72

Model		Booking Limits								
		P1	P2	P3	P4	P5	P6	P7	P8	P9
Budget Uncertainty	ℓ_1 -norm	30	33	29	27	22	27	45	14	15
	$\ell_2 + \ell_\infty$	29	34	34	28	23	28	46	13	16
	DLP	25	30	30	25	20	25	40	15	14
	PNLP	39	35	32	29	22	34	43	9	12

Model	Booking Limits								
	P10	P11	P12	P13	P14	P15	P16	P17	P18
ℓ_1 -norm	22	22	33	28	0	6	0	10	36
Budget Uncertainty	22	22	33	29	0	0	0	11	39
$\ell_2 + \ell_\infty$	23	23	34	28	0	0	0	6	36
DLP	20	20	30	25	0	20	0	6	35
PNLP	18	16	31	21	0	9	7	11	33

(a) Booking Limits

As expected, we observe a decrease in the performance of DLP as the demand factor decreases. This is because our robust models are designed to handle worst-case scenarios more effectively, whereas DLP relies directly on expected demand. Although DLP may have fewer

Model	Average Capacity Unused		
	R1	R2	R3
ℓ_1 -norm	39.2	35.7	31.7
Budget Uncertainty	43.9	39.0	34.7
Ellips+Box	44.0	43.4	38.0
DLP	33.5	32.3	28.2
PNLP	42.0	33.1	31.9

(b) Average Capacity Unused

Table 4.25: Booking limits and average capacity unused of Example 19

empty seats on average, these empty seats often occur in high fare classes, leading to significant profit loss. This emphasizes the importance of robustness in our models, which helps mitigate the impact of low-demand scenarios more effectively. Additionally, the differences between the three scenarios are minimal, mainly because there are many products that can substitute for one another, resulting in relatively stable revenues across scenarios.

Example 20: We consider the cases where there is no arrival pattern (mix arrival pattern). First DF=89%.

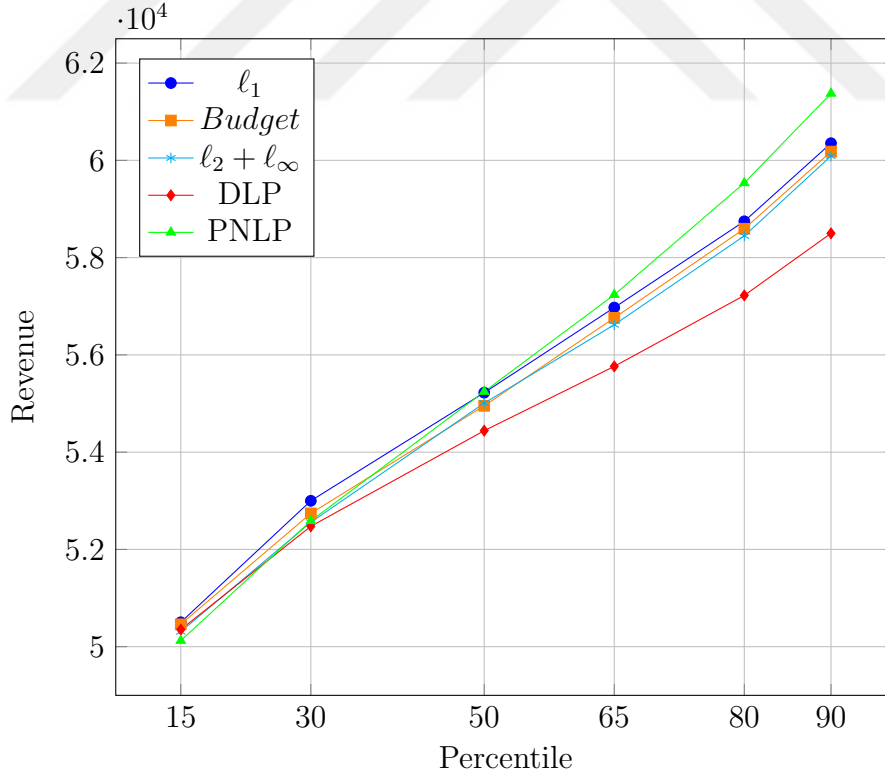


Figure 4.24: Revenue Percentiles Under Mix Arrival Pattern With DF=0.89

Under the mixed arrival pattern with an 89% demand factor, our models slightly outperform PNL and DLP until the 50th percentile, then they stay below PNL while remaining above

Model	Booking Limits									
	P1	P2	P3	P4	P5	P6	P7	P8	P9	
ℓ_1 -norm	36	22	19	22	22	33	45	12	26	
Budget Uncertainty	36	18	16	23	22	33	47	15	27	
$\ell_2 + \ell_\infty$	34	22	6	23	22	33	45	22	27	
DLP	30	20	20	20	20	30	40	25	24	
PNLP	40	22	17	27	22	35	40	18	21	

Model	Booking Limits									
	P10	P11	P12	P13	P14	P15	P16	P17	P18	
ℓ_1 -norm	22	22	43	40	0	0	33	0	35	
Budget Uncertainty	23	22	44	41	0	0	34	0	36	
$\ell_2 + \ell_\infty$	22	22	44	44	0	0	34	0	46	
DLP	20	20	40	41	0	0	30	1	45	
PNLP	19	16	36	42	0	0	23	15	38	

(a) Booking Limits

Model	Average Capacity Unused		
	R1	R2	R3
ℓ_1 -norm	19.1	17.5	12.0
Budget Uncertainty	19.3	18.0	13.0
$\ell_2 + \ell_\infty$	20.1	20.5	15.5
DLP	13.3	12.5	10.7
PNLP	20.8	15.6	10.1

(b) Average Capacity Unused

Table 4.26: Booking limits and average capacity unused of Example 20

DLP. This is an expected scenario since our models are robust and should perform better at worse case scenarios. As in previous examples, our models perform better under the mixed arrival pattern relative to the others. However, again the structure of the graph is not much different due to the high number of products.

Example 21: We consider a medium demand factor case with DF=81% under mixed arrival pattern.

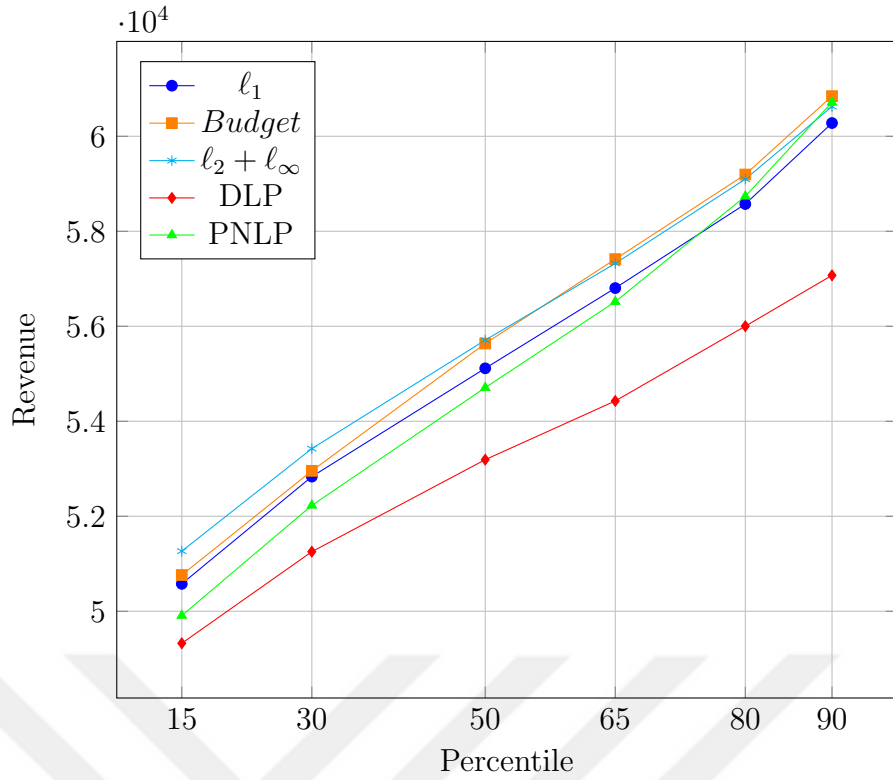


Figure 4.25: Revenue Percentiles Under Mix Arrival Pattern With DF=0.81

Model	Booking Limits								
	P1	P2	P3	P4	P5	P6	P7	P8	P9
ℓ_1 -norm	30	28	24	27	27	33	45	17	15
Budget Uncertainty	31	28	25	29	28	33	46	14	16
$\ell_2 + \ell_\infty$	29	28	28	28	28	34	46	13	16
DLP	25	25	25	25	25	30	40	15	14
PNLP	41	26	23	28	27	38	43	9	12

Model	Booking Limits								
	P10	P11	P12	P13	P14	P15	P16	P17	P18
ℓ_1 -norm	22	22	33	39	0	2	9	7	37
Budget Uncertainty	22	22	33	40	0	0	12	4	39
$\ell_2 + \ell_\infty$	23	23	34	40	0	0	13	0	37
DLP	20	20	30	35	0	21	10	0	35
PNLP	18	16	29	34	0	12	15	14	29

(a) Booking Limits

As anticipated, the performance of DLP declines with decreasing demand factor, consistent with previous observations. Robust models consistently outperform DLP and PNLP across various percentiles under medium demand factor scenarios, highlighting their superior worst-case scenario outcomes. This advantage stems from the robust models' consideration of standard deviation, enabling them to maintain more stable results even as expected demand decreases.

Model	Average Capacity Unused		
	R1	R2	R3
ℓ_1 -norm	24.5	11.5	14.8
Budget Uncertainty	24.7	12.7	15.9
$\ell_2 + \ell_\infty$	22.1	11.6	16.2
DLP	12.9	7.1	10.0
PNLP	27.5	9.4	14.0

(b) Average Capacity Unused

Table 4.27: Booking limits and average capacity unused of Example 21

Example 22: We examine the case with DF=72%, which represents a low demand factor under mixed arrival pattern.

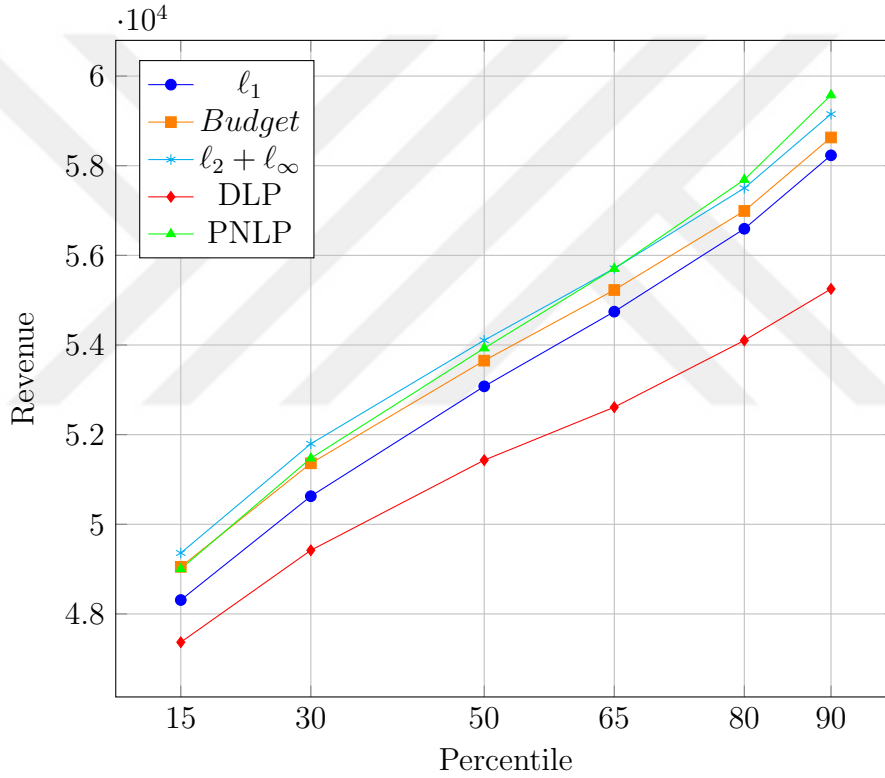


Figure 4.26: Revenue Percentiles Under Mix Arrival Pattern With DF=0.72

As the demand factor decreases, the performance of DLP continues to decline. Conversely, the performance of the other models converges, as they are limited by the same revenue-generating opportunities and optimize these to achieve the best possible revenue. Consequently, their performance lines become closer to each other.

To sum up, in line with previous examples, PNLP and DLP perform well under high demand factors and LBH arrival patterns, with robust models slightly underperforming. As the demand factor decreases, DLP's performance declines, while other models remain relatively

Model	Booking Limits									
	P1	P2	P3	P4	P5	P6	P7	P8	P9	
ℓ_1 -norm	30	33	29	27	22	27	45	14	15	
Budget Uncertainty	31	34	29	29	22	28	46	15	16	
$\ell_2 + \ell_\infty$	29	34	34	28	23	28	46	13	16	
DLP	25	30	30	25	20	25	40	15	14	
PNLP	39	35	32	29	22	34	43	9	12	

Model	Booking Limits									
	P10	P11	P12	P13	P14	P15	P16	P17	P18	
ℓ_1 -norm	22	22	33	28	0	6	0	10	36	
Budget Uncertainty	22	22	33	29	0	0	0	11	39	
$\ell_2 + \ell_\infty$	23	23	34	28	0	0	0	6	36	
DLP	20	20	30	25	0	20	0	6	35	
PNLP	18	16	31	21	0	9	7	11	33	

(a) Booking Limits

Model	Average Capacity Unused		
	R1	R2	R3
ℓ_1 -norm	22.4	12.2	16.0
Budget Uncertainty	23.3	11.1	16.1
$\ell_2 + \ell_\infty$	20.3	10.4	14.2
DLP	13.4	7.7	11.5
PNLP	23.2	10.3	17.5

(b) Average Capacity Unused

Table 4.28: Booking limits and average capacity unused of Example 22

stable. Transitioning to a mixed arrival pattern leads to decreased performance for both DLP and PNL. However, due to the large number of products in this example, changes in the graphs are minimal since there are many substitutes for each product, resulting in consistent revenues across models and scenarios.

4.2.4 Experiments With Wrong Demand Information

In real-life scenarios, obtaining precisely accurate data can be challenging. Sometimes, the data may be overestimated, while other times it may be underestimated. To assess the performance of our models under these conditions, we examine cases where the expected demand and/or the standard deviation are not accurately estimated. First, let us focus on the cases where the expected demand is underestimated while the arrival pattern is LBH. In these scenarios, we calculate the booking limits with the estimated demand vector, however, we use demand values

that are multiples of the estimates in the simulation runs.

Example 23: We examine the example with 2 resources and 6 products (each resource has a capacity of 90.) with underestimated demand. Parameters are:

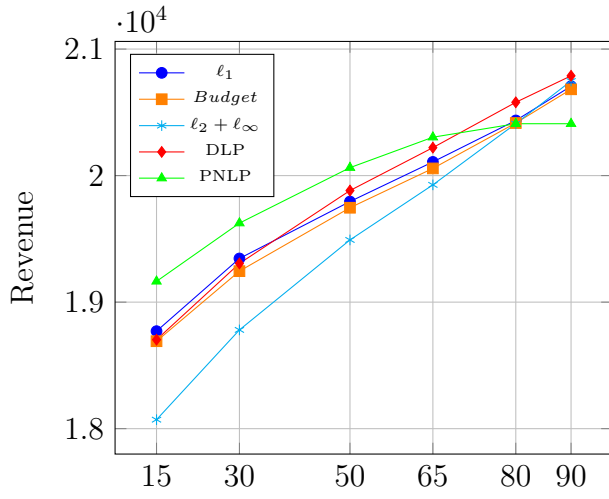
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Product	O-D	Fare	Flights Used	Exp. Demand
1	A-B	150	1	30
2	A-B	100	1	60
3	B-C	120	2	20
4	B-C	80	2	80
5	A-C	250	1, 2	30
6	A-C	170	1, 2	40

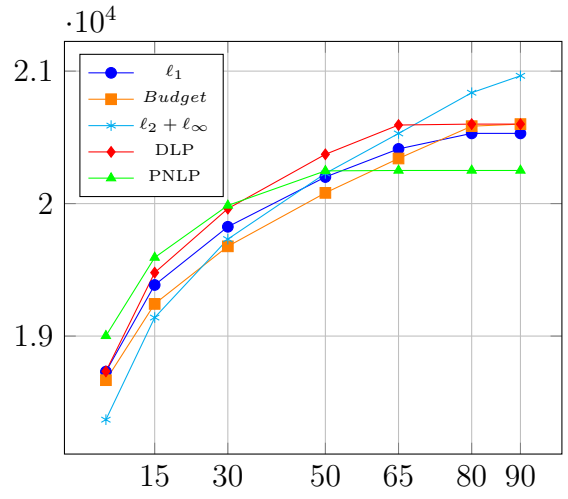
Table 4.29: Parameters

We can construct the P matrix using the square root of the expected demands, resulting in the following form:

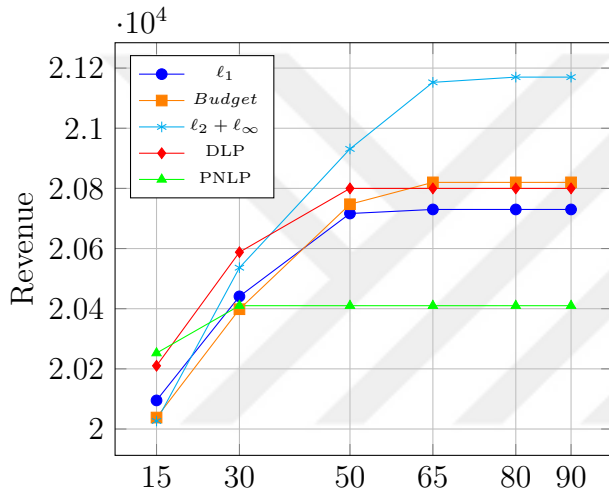
$$P = \begin{bmatrix} 5.47 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7.74 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.47 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.47 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.32 \end{bmatrix}$$



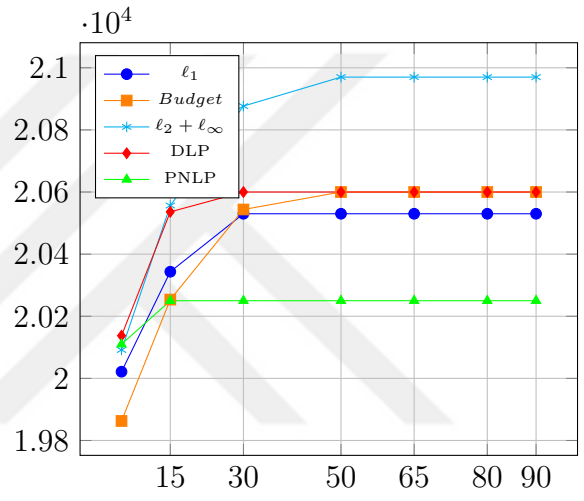
(a) Correct Demand Estimation



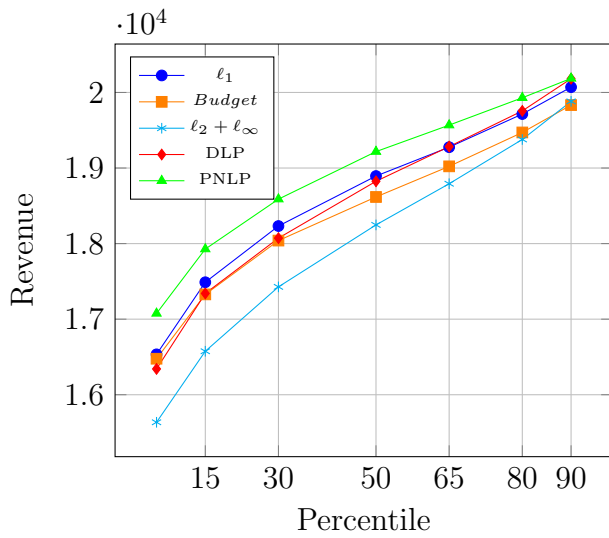
(b) Correct demand is 10% higher than estimates



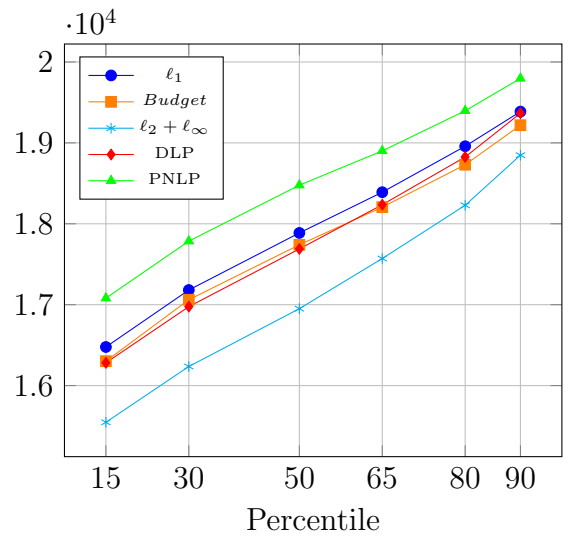
(c) Correct demand is 20% higher than estimates



(d) Correct demand is 30% higher than estimates



(e) Correct demand is 90% of estimated demand



(f) Correct demand is 80% of estimated demand

Figure 4.27: Revenue Percentiles With Underestimated and Overestimated Expected Demand

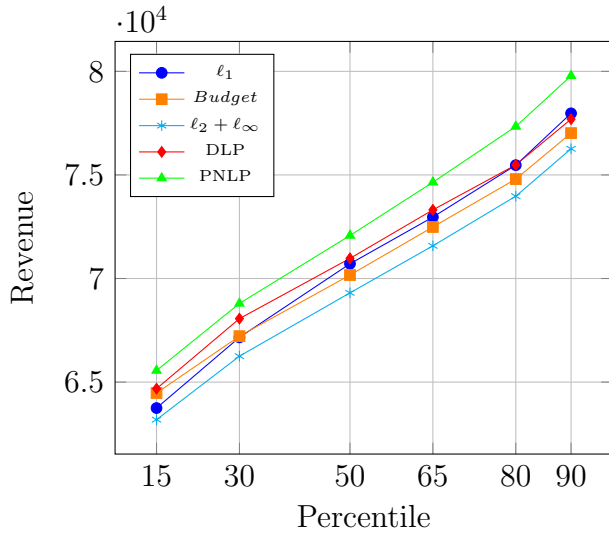
Figure 4.27 demonstrates that our $\ell_2 + \ell_\infty$ model outperforms all other models by allocating more seats to the highest fare product. PNLP performs particularly poorly as the underestimation ratio increases, allocating 40 seats for the lowest fare product, resulting in missed revenue opportunities. In contrast, while DLP's performance is consistent with expectations, it still underperforms compared to our robust model. Also, as the degree of demand Overestimation decreases, we observe that the results of the models converge towards each other. This trend occurs because regardless of how many seats each model allocates to each product, the insufficient demand fails to fulfill these allocations, resulting in similar total revenues across the models.

Example 24: We consider the example with 18 products and 3 resources. Parameters of this example are:

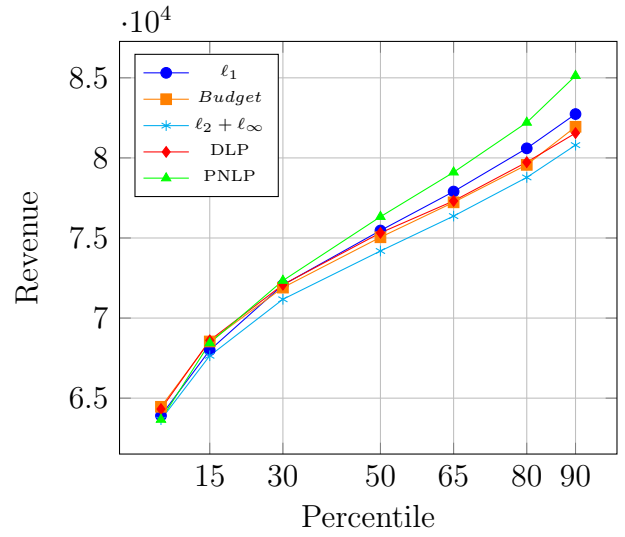
OD	Fare class 3				Fare class 2				Fare class 1			
	Exp.	SD	Fare	Pro. no.	Exp.	SD	Fare	Pro. no.	Exp.	SD	Fare	Pro. no.
A-B	50	9.01	75	13	40	7.75	125	7	30	18.17	250	1
A-C	40	7.75	130	14	25	6.12	170	8	20	14.83	400	2
A-D	30	6.71	200	15	24	5.66	320	9	20	14.83	460	3
B-C	30	6.71	100	16	20	5.48	150	10	20	14.83	330	4
B-D	30	6.71	160	17	20	5.16	200	11	20	9.31	420	5
C-D	50	9.01	80	18	40	7.75	110	12	30	13.42	235	6

Table 4.30: Parameters of de Boer's example

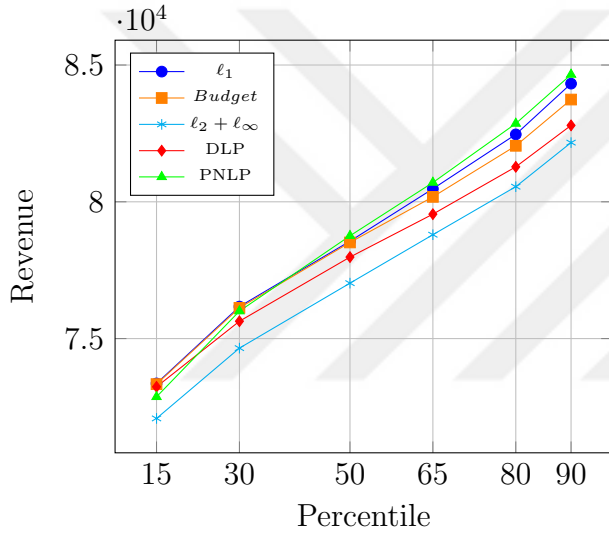
We assess the performance of our models with overestimated or underestimated demand information, this time considering a larger example. We conduct calculations for both expectation and standard deviation overestimation and underestimation. Let us start by examining the scenarios of expectation overestimation and underestimation.



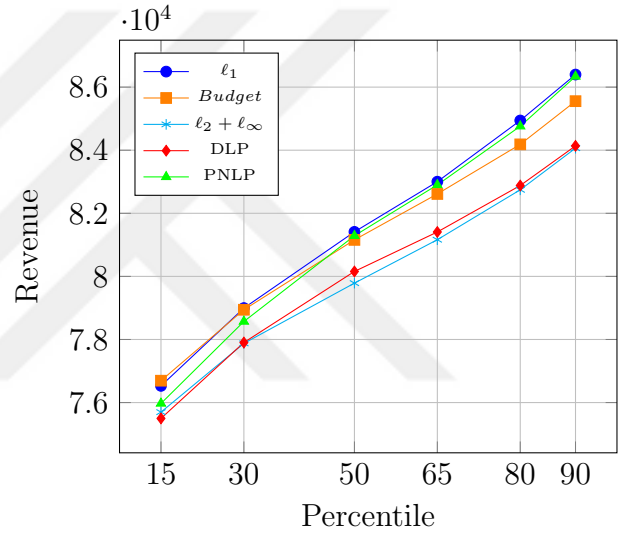
(a) Correct Demand Estimation



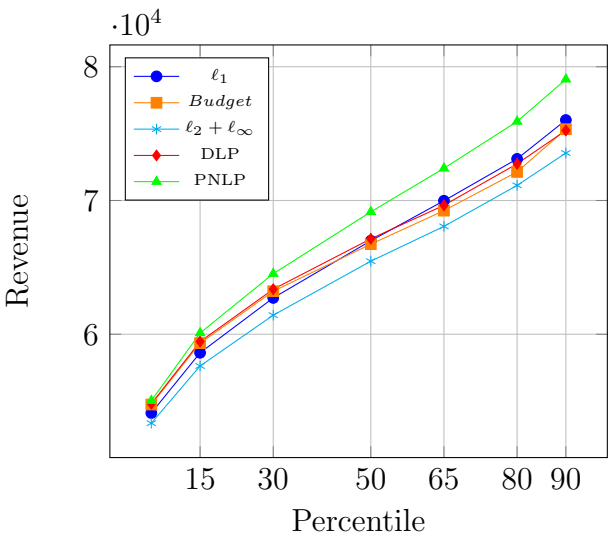
(b) Correct demand is 10% higher than estimates



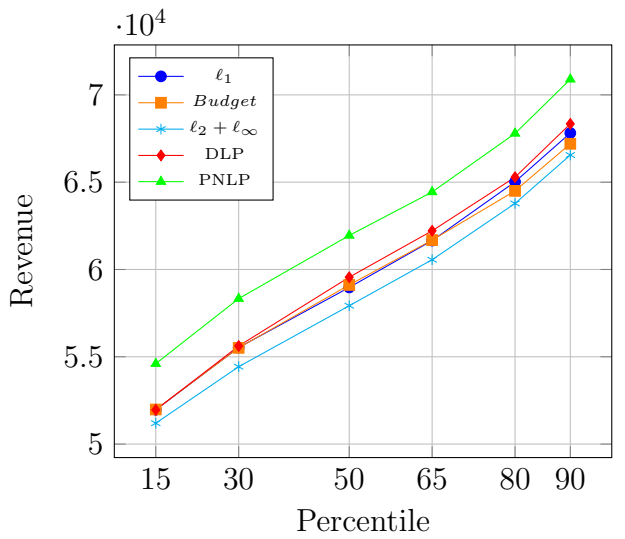
(c) Correct demand is 20% higher than estimates



(d) Correct demand is 30% higher than estimates



(e) Correct demand is 90% of estimated demand

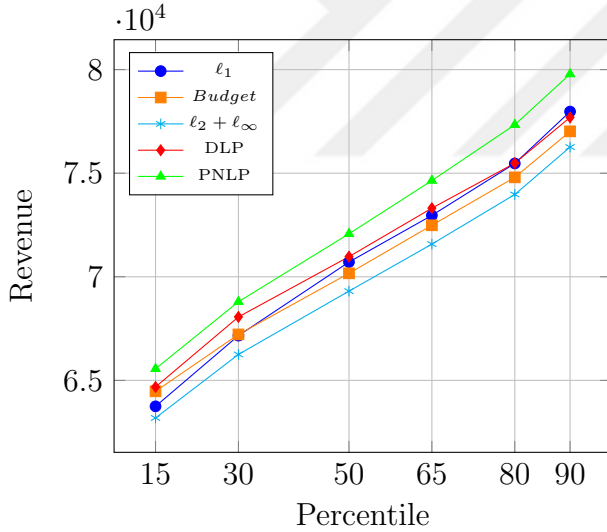


(f) Correct demand is 80% of estimated demand

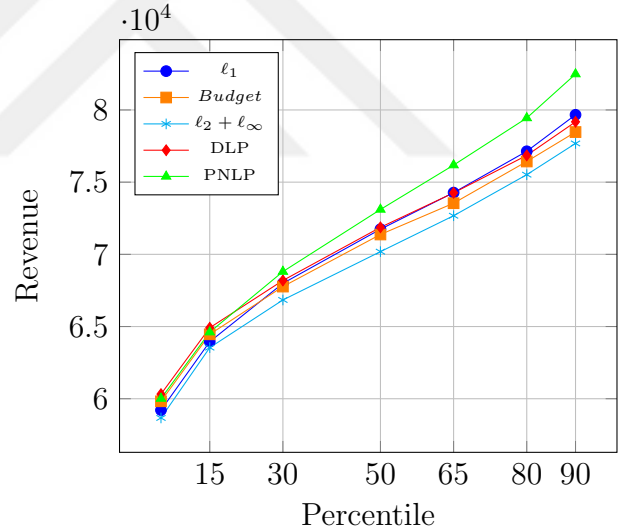
Figure 4.28: Revenue Percentiles With Underestimated and Overestimated Expected Demand

From Figure 4.28-d, we observe that our model with the ℓ_1 norm outperforms all other models, albeit with minor superiorities at certain points when demand is most underestimated. This highlights its capability to handle worst-case scenarios effectively. Additionally, PNLP shows improved performance compared to the smaller example, likely due to the increased number of products and reduced volatility. Conversely, DLP's performance consistently deteriorates compared to other models, especially as the multiplicative factor increases. This decline is attributed to DLP allocating fewer seats to high-fare customers. When this allocation is made under the assumption of a lower demand vector, DLP misses the opportunity to capture more demand when there are more customers than expected, leading to suboptimal performance in higher demand conditions. As previously noted in the Topaloglu example [4], as the extent of demand overestimation diminishes, we observe a convergence of results among the models. This trend arises from the fact that despite differences in seat allocations made by each model, the overall shortage in demand prevents these allocations from being fully utilized, leading to comparable total revenues across the models.

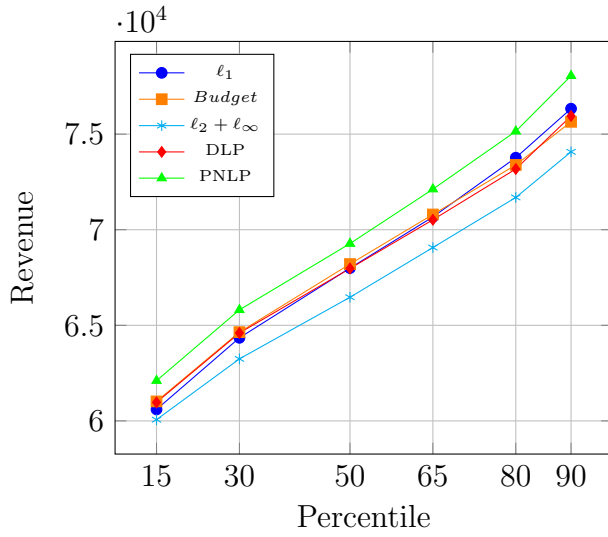
Example 25: We examine the cases where standard deviation is both underestimated and overestimated. We use again the previous example, the one with 18 products and 3 resources.



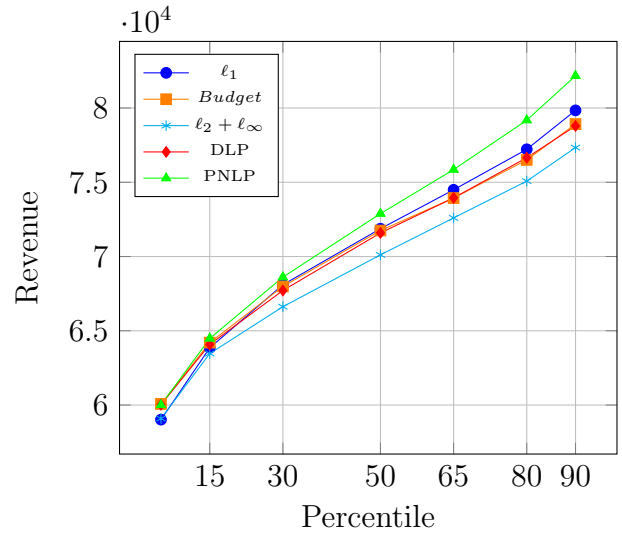
(a) Correct Estimates



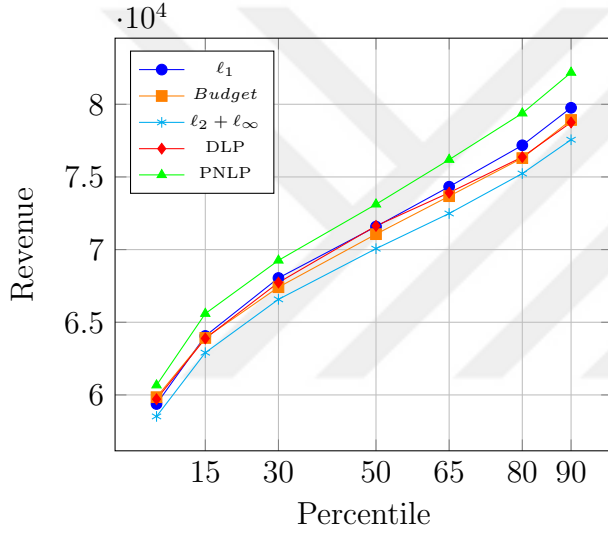
(b) Correct SD is 10% higher than estimates



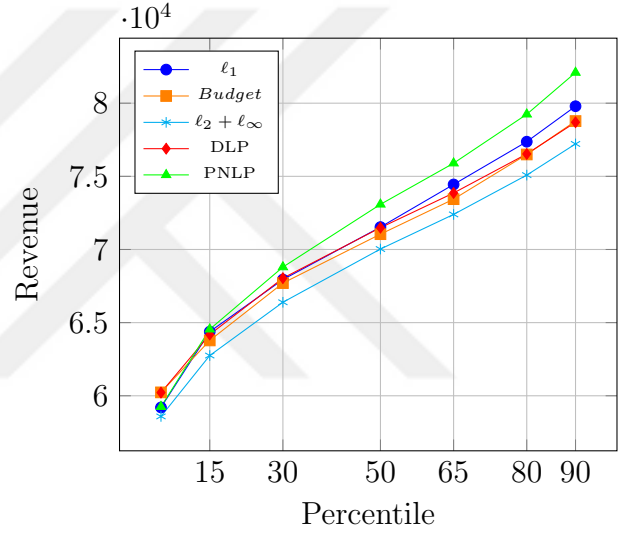
(c) Correct demand is 20% higher than estimates



(d) Correct demand is 30% higher than estimates



(e) Correct SD is 90% of estimated demand



(f) Correct SD is 80% of estimated demand

Figure 4.29: Revenue Percentiles With Overestimated Standard Deviation of Demands

As observed, even when the standard deviation is overestimated (or underestimated), and the degree of overestimation (or underestimation) increases, the overall picture remains consistent across each case, as it did before. This consistency is attributed to the numerous simulation runs conducted, which ultimately converge towards a common outcome.

4.2.5 Displacement Adjusted Virtual Nesting (DAVN)

In the process of capacity allocation, it's crucial to distinguish between the mathematical models employed to calculate allocations and the methods used to control the actual availability of seats

for each fare class or origin-destination-flight itinerary (ODF) [36].

The initial step involves optimization, where network flow algorithms and mathematical programming techniques determine the optimal seat allocations for each ODF within a flight network [37]. However, equally important is the implementation of these seat allocations as booking limits, which directly controls seat inventories. This aspect of implementation is vital for maximizing revenue through the reservations inventory control process, alongside the optimization step.

There are many other booking controls such as: nested booking limits, partitioned booking limits, displacement-adjusted virtual nesting (DAVN), and fixed bid prices. Until now, we have only considered one allocation type, which is partitioned booking limit control. Next, for the scope of this thesis, we will also examine the performances of our robust model with DAVN allocation and compare their performances with DLP and PNLP.

The Displacement Adjusted Virtual Nesting scheme was introduced by Smith and Penn of American Airlines to apply nested booking-limit control in a network [8]. The steps for DAVN is outlined below.

4.2.5.1 Steps to Follow for DAVN

To implement DAVN, begin by determining a set of bid prices, which represent the optimal values of dual variables. Then, proceed with the following steps:

- Let $\{\pi_l \mid l \in K\}$ be set of bid prices for each capacity constraint. Calculate the displacement adjusted leg revenues

$$\bar{r}_{ODF}^l = r_{ODF} - \sum_{l' \neq l: ODF \in S_{l'}} \pi_{l'} \quad (4.1)$$

Where S_l is the set of products (ODF) that utilize leg l .

- In the next step, clustering, or indexing, is performed. Here, the displacement-adjusted revenue values on each resource are grouped into a specified number $\bar{n} + 1$ of virtual classes, or buckets. The number of virtual classes, $\bar{n} + 1$, is a design parameter, typically around 10, and may vary across resources [1].
- Cluster booking classes whose adjusted leg revenues are close to each other into a practical number of leg buckets on each leg. These buckets are arranged from highest to lowest, with bucket 1 on leg l containing the most profitable booking classes for that leg, followed by bucket 2, and so on. Let B_i^l be the set of booking classes (odf's) in bucket i on leg l . N_l be number of buckets defined for leg l [6].

- On each leg l , the process recursively organizes all booking classes into predefined leg buckets as follows. Initially, let $k = 0$ denote the number of already defined buckets B_i^l ($1 \leq i \leq k$). The adjusted leg revenues of all booking solutions, constrained to be non-negative, are clustered into at most $N_{max} - k$ additional buckets. This clustering begins from 0 and extends to

$$UB_{k+1}^l = \max \left\{ \bar{r}_{ODF}^l \geq 0 \mid odf \in S_l \setminus \bigcup_{i=1}^k B_i^l \right\}$$

The range is divided into $N_{max} - k$ equal segments, and booking classes falling into the last sub-interval are grouped into bucket B_{k+1}^l . Thus,

$$B_{k+1}^l = \left\{ odf \in S_l \mid \frac{N_{max} - k - 1}{N_{max} - k} UB_{k+1}^l \leq \bar{r}_{ODF}^l \leq UB_{k+1}^l \right\}$$

This bucket encompasses the booking class with the highest adjusted leg revenue not yet clustered. Increment k by one, and halt the recursion when $k = N_{max}$, or when no booking classes with non-negative adjusted leg revenue remain unclustered. Subsequently, let $N_l = k + 1$ denote the total number of buckets on leg l , after assigning all booking classes with negative adjusted leg revenue into the lowest bucket $B_{N_l}^l = \{odf \in S_l \mid \bar{r}_{ODF}^l < 0\}$, if such a set is not empty. Otherwise, $N_l = k$. This fundamental algorithm, although potentially distinct from practical airline procedures, ensures no empty buckets and minimizes the number of buckets on a leg when adjusted leg revenues are closely grouped [6].

- We define a booking limit b_i^l for each leg bucket i on leg l , ensuring $b_i^l \geq b_{i+1}^l$ for $i = 1, \dots, N_l - 1$, and setting $b_1^l = C$ because, in the absence of overbooking, the sale of booking classes in the highest bucket should only be restricted by capacity. In our study, we have explored two methods for calculating booking limits: one based on LP seat allocations, originally proposed by [36], and the other being our robust model employing the ℓ_1 -norm.

– LP-Based Booking-Limit Calculation Method [36]

In this context, the solution corresponds to the DLP model previously discussed. We set $b_1^l = c_l$ and define p_i^l as the protection level on leg l for buckets 1 to i from bookings for bucket $i + 1$ or lower. Based on the LP solution, we express p_i^l as $p_i^l = \sum_{j=1}^i \sum_{odf \in B_j^l} x_{odf}^*$, where $i = 1, \dots, N_l - 1$. Then, we compute $b_{i+1}^l = c_l - p_i^l$ for $i = 1, \dots, N_l - 1$.

- Seats on leg l to booking classes assigned to bucket i are no longer available for sale after b_i^l seats have been sold on this leg, irrespective of the customer.

Now, we will utilize the previous two examples to demonstrate the performance of our models alongside the other two models.

Example 26: We begin with the example from [4] (the one with 2 resources 6 products) with LBH arrival pattern. To begin, we need to determine the maximum number of buckets. We will consider three different bucket numbers (2,3, and 4) to assess the effect of the number of buckets on the models' performances. Let us start with 4 buckets.

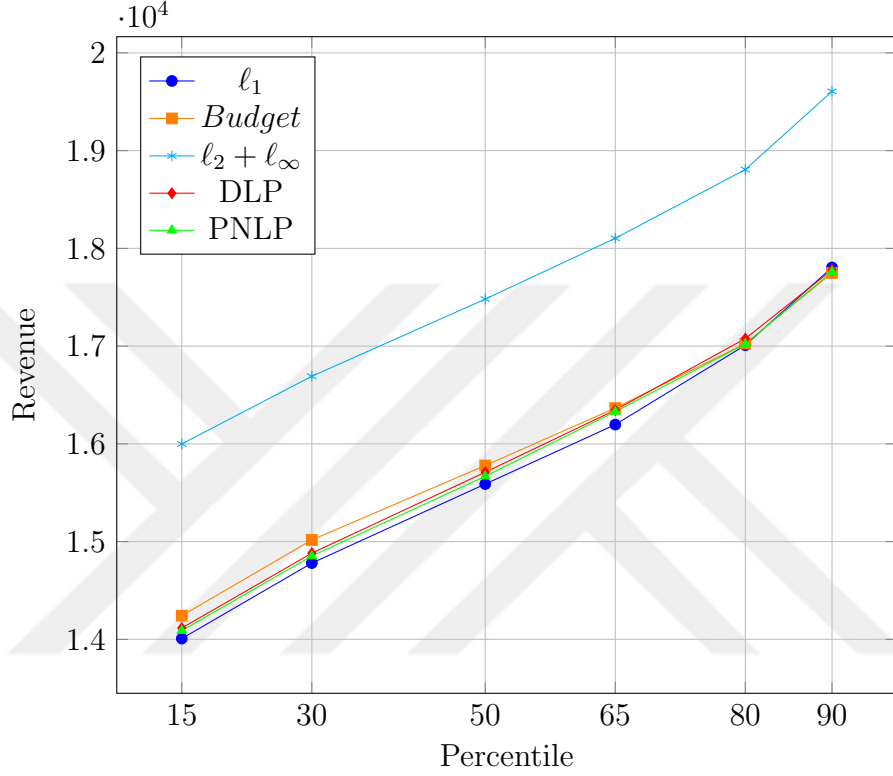


Figure 4.30: Revenue Percentiles of Example 26

Model	Leg	Booking Limit	Buckets		Shadow Price
			1	2	
ℓ_1 -norm	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	79.2
Budget Uncertainty	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	80
$\ell_2 + \ell_\infty$	1	[90, 25]	[1, 5]	[2, 6]	87.6
	2	[90, 59]	[5]	[3, 4, 6]	70.7
DLP	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	80
PNLP	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	80

Table 4.31: Booking limits, buckets, and shadow prices of Example 26

From Figure 4.30 and the shadow prices, we observe that the model with $\ell_2 + \ell_\infty$ norm outperforms the other four models. This is primarily due to the shadow prices. While the shadow prices of the other four models are nearly identical, the model with $\ell_2 + \ell_\infty$ norm exhibits lower shadow prices. Consequently, it optimally allocates all capacity for the highest fare, resulting in the highest overall revenue.

Example 27: Let us proceed with the scenario where only 3 buckets are considered.

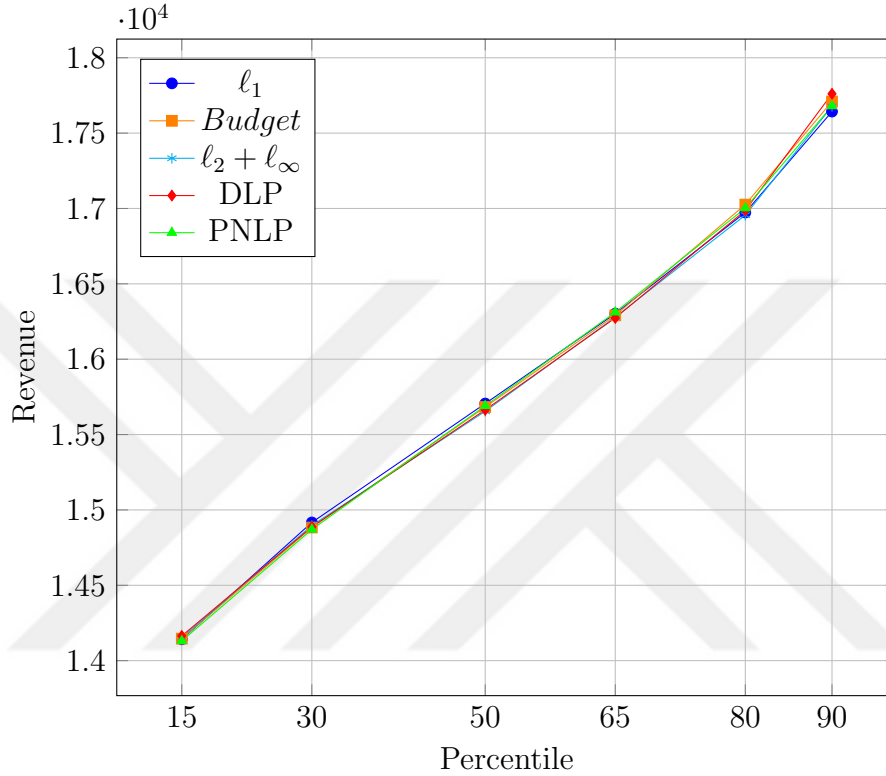


Figure 4.31: Revenue Percentiles of Example 27

Model	Leg	Booking Limit	Buckets		Shadow Price
			1	2	
ℓ_1 -norm	1	[90.0, 25.0]	[1, 5]	[2, 6]	100.0
	2	[90.0, 36.0]	[3, 5]	[4, 6]	79.23
Budget Uncertainty	1	[90.0, 25.0]	[1, 5]	[2, 6]	100.0
	2	[90.0, 36.0]	[3, 5]	[4, 6]	80.0
$\ell_2 + \ell_\infty$	1	[90.0, 25.0]	[1, 5]	[2, 6]	87.63
	2	[90.0, 36.0]	[3, 5]	[4, 6]	70.74
DLP	1	[90.0, 25.0]	[1, 5]	[2, 6]	100.0
	2	[90.0, 36.0]	[3, 5]	[4, 6]	80.0
PNLP	1	[90.0, 25.0]	[1, 5]	[2, 6]	99.98
	2	[90.0, 36.0]	[3, 5]	[4, 6]	80.0

Table 4.32: Booking limits, buckets, and shadow prices of Example 27

In Figure 4.31, we have solely modified the maximum number of buckets, resulting in the same bucketing and capacity allocation. Despite the lower shadow prices of the $\ell_2 + \ell_\infty$ norm model, it cannot yield a significant difference at this point since the maximum number of buckets has been altered. This parameter exclusively impacts bucketing, as explained earlier.

Example 28: Let us consider the scenario where there are only 2 buckets.

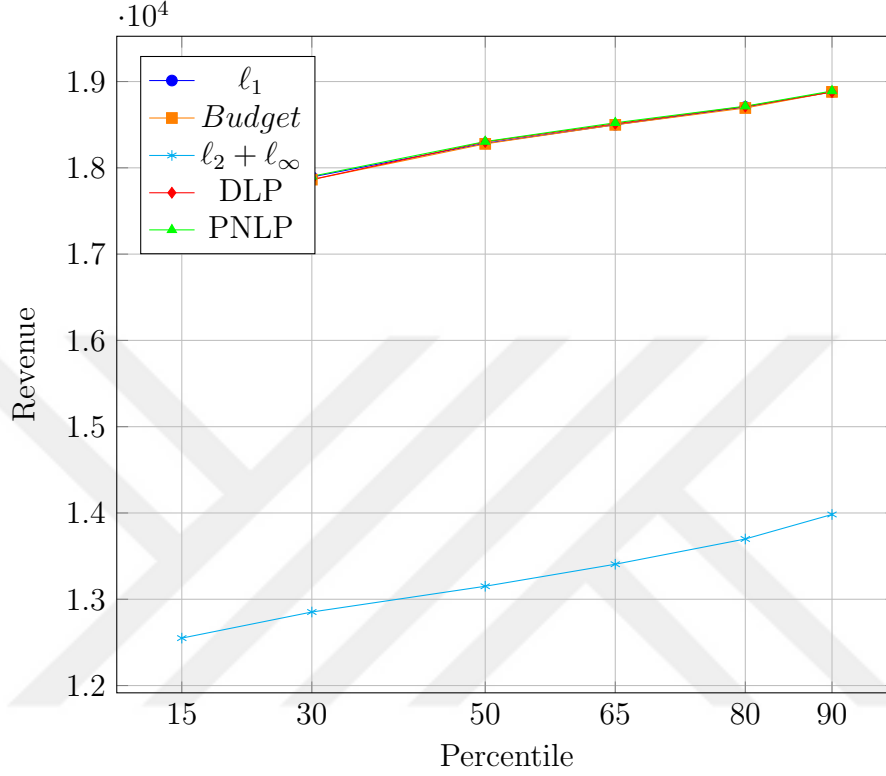


Figure 4.32: Revenue Percentiles of Example 28

Model	Leg	Booking Limit	Buckets		Shadow Price
			1	2	
ℓ_1 -norm	1	[90.0]	[1, 2, 5, 6]	[]	100.0
	2	[90.0, 33.0]	[3, 4, 5]	[6]	79.23
Budget Uncertainty	1	[90.0]	[1, 2, 5, 6]	[]	100.0
	2	[90.0, 33.0]	[3, 4, 5]	[6]	80.0
$\ell_2 + \ell_\infty$	1	[90.0]	[1, 2, 5, 6]	[]	87.63
	2	[90.0, 0.0]	[3, 5, 6]	[4]	70.74
DLP	1	[90.0]	[1, 2, 5, 6]	[]	100.0
	2	[90.0, 33.0]	[3, 4, 5]	[6]	80.0
PNLP	1	[90.0]	[1, 2, 5, 6]	[]	99.98
	2	[90.0, 33.0]	[3, 4, 5]	[6]	80.0

Table 4.33: Booking limits, buckets, and shadow prices of Example 28

When the maximum number of buckets is 2, our model with $\ell_2 + \ell_\infty$ norms is outperformed by

the other models. This can be attributed to shadow prices. In the case of the other four models, only product 5 exhibits negative adjusted leg revenue and is therefore bucketed separately, as it falls outside the interval of the others. However, for the model with $\ell_2 + \ell_\infty$ norms, both product 3 and product 5 have negative adjusted leg revenue. Additionally, product 3 has the second lowest revenue overall, resulting in a lower total revenue for this model.

Example 29: As mentioned previously, we utilized the ℓ_1 -norm model to solve single-leg problems after bucketing. Now, we explore the effect of solving these single-leg problems with DLP. We again consider a maximum number of buckets of 4, 3, and 2.

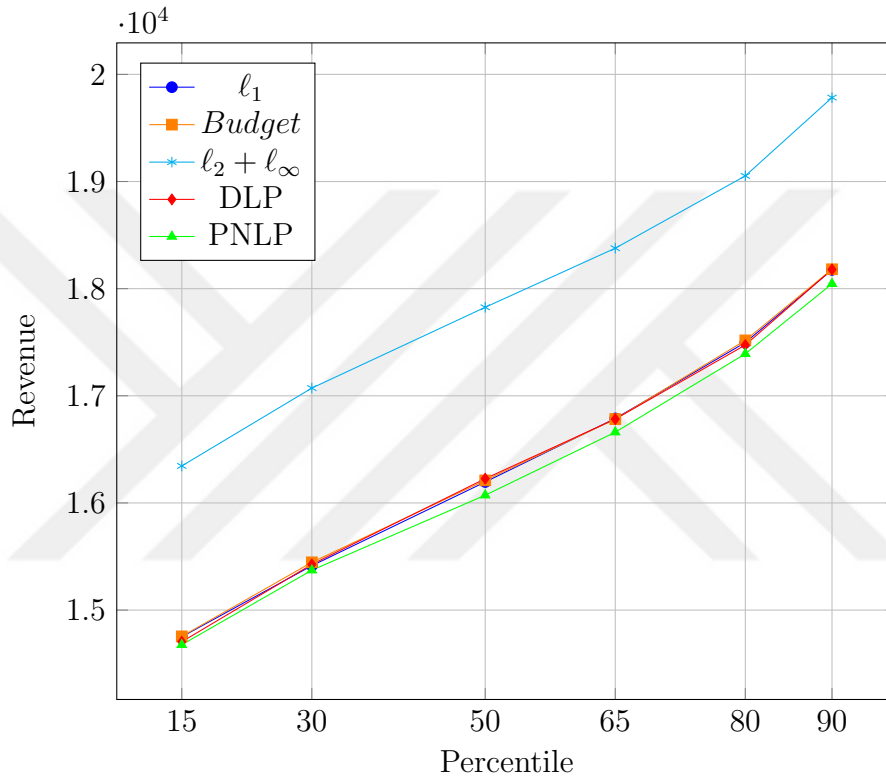


Figure 4.33: Revenue Percentiles of Example 29 with 4 Buckets

Model	Leg	Booking Limit	Buckets		Shadow Price
			1	2	
ℓ_1 -norm	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	79.2
Budget Uncertainty	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	80
$\ell_2 + \ell_\infty$	1	[90, 25]	[1, 5]	[2, 6]	87.6
	2	[90, 59]	[5]	[3, 4, 6]	70.7
DLP	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	80
PNLP	1	[90, 25]	[1, 5]	[2, 6]	100
	2	[90, 36]	[3, 5]	[4, 6]	80

Table 4.34: Booking limits, buckets, and shadow prices of Example 29 with 4 buckets

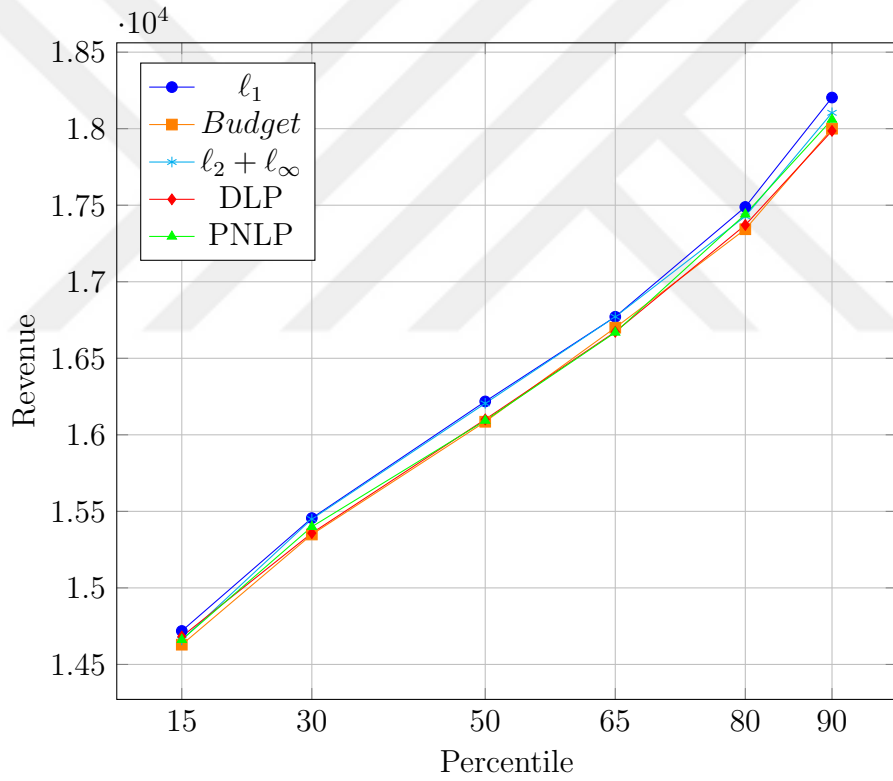


Figure 4.34: Revenue Percentiles of Example 29 with 3 Buckets

Model	Leg	Booking Limit	Buckets		Shadow Price
			1	2	
ℓ_1 -norm	1	[90.0, 25.0]	[1, 5]	[2, 6]	100.0
	2	[90.0, 36.0]	[3, 5]	[4, 6]	79.23
Budget Uncertainty	1	[90.0, 25.0]	[1, 5]	[2, 6]	100.0
	2	[90.0, 36.0]	[3, 5]	[4, 6]	80.0
$\ell_2 + \ell_\infty$	1	[90.0, 25.0]	[1, 5]	[2, 6]	87.63
	2	[90.0, 36.0]	[3, 5]	[4, 6]	70.74
DLP	1	[90.0, 25.0]	[1, 5]	[2, 6]	100.0
	2	[90.0, 36.0]	[3, 5]	[4, 6]	80.0
PNLP	1	[90.0, 25.0]	[1, 5]	[2, 6]	99.98
	2	[90.0, 36.0]	[3, 5]	[4, 6]	80.0

Table 4.35: Booking limits, buckets, and shadow prices of Example 29 with 3 buckets

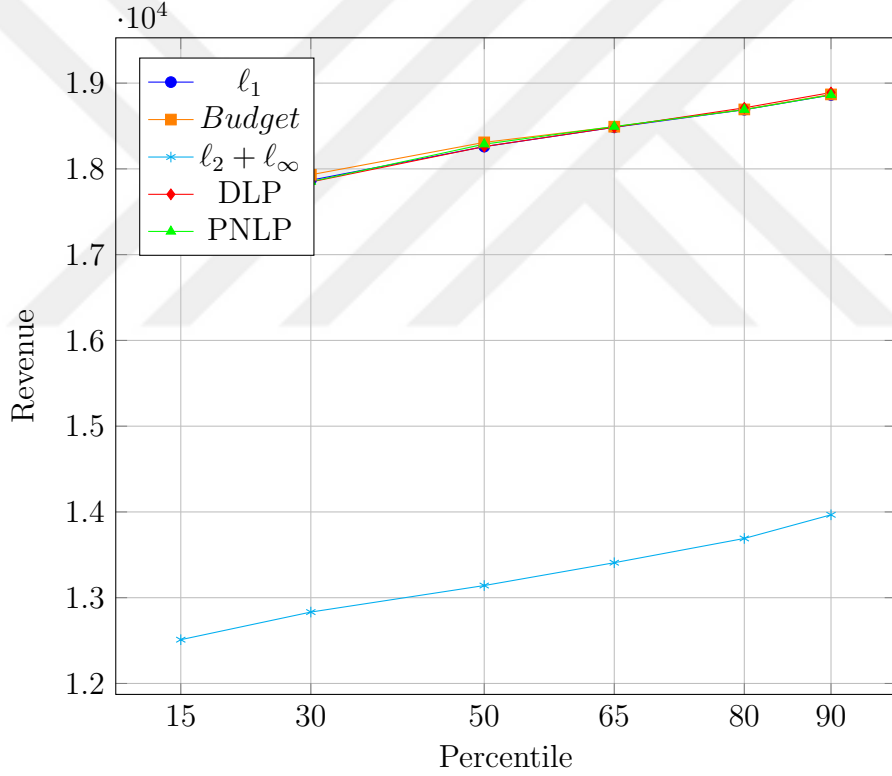


Figure 4.35: Revenue Percentiles of Example 29 with 2 Buckets

As observed, for this small example, solving the single-leg problems with DLP or the ℓ_1 -norm model does not create much of a difference. The same pattern observed when using the ℓ_1 -norm model persists. Thus, we can conclude that for small instances such as this one, the method used to solve the single-leg problem when applying DAVN is not significantly influential.

Upon examining the overestimation and underestimation of expected demand and standard

Model	Leg	Booking Limit	Buckets		Shadow Price
			1	2	
ℓ_1 -norm	1	[90.0]	[1, 2, 5, 6]	[]	100.0
	2	[90.0, 33.0]	[3, 4, 5]	[6]	79.23
Budget Uncertainty	1	[90.0]	[1, 2, 5, 6]	[]	100.0
	2	[90.0, 33.0]	[3, 4, 5]	[6]	80.0
$\ell_2 + \ell_\infty$	1	[90.0]	[1, 2, 5, 6]	[]	87.63
	2	[90.0, 0.0]	[3, 5, 6]	[4]	70.74
DLP	1	[90.0]	[1, 2, 5, 6]	[]	100.0
	2	[90.0, 33.0]	[3, 4, 5]	[6]	80.0
PNLP	1	[90.0]	[1, 2, 5, 6]	[]	99.98
	2	[90.0, 33.0]	[3, 4, 5]	[6]	80.0

Table 4.36: Booking limits, buckets, and shadow prices of Example 29 with 2 buckets

deviation, we observed consistent results across all instances and cases. There were no notable differences or patterns observed. Therefore, we have decided not to go into further details regarding this aspect of the analysis.

Example 30: We now analyze the example from [33], which comprises 18 products and 3 resources. Again, we will consider the LBH arrival pattern. Due to the larger size of this example compared to the previous one, we will evaluate it with maximum numbers of buckets set to 6, 5, and 4. Let us begin with 6 buckets.

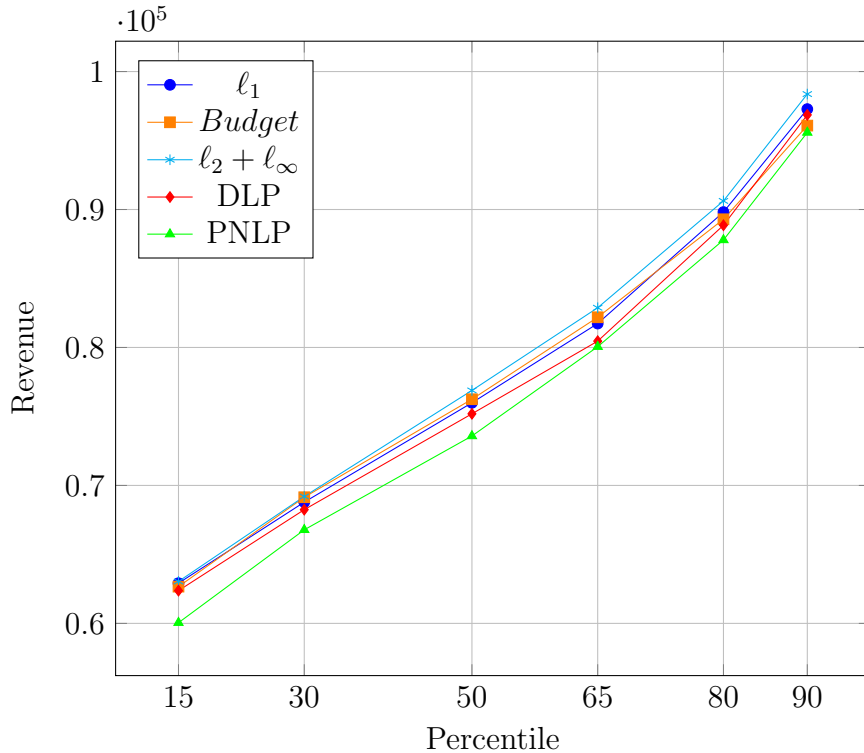


Figure 4.36: Revenue Percentiles of Example 30

In contrast to the previous example, we observe no significant disparities in simulation run results across different models. The ℓ_1 -norm model, Budget uncertainty set model, and DLP exhibit identical bucketings due to their similar shadow prices, resulting in almost identical total revenue. However, the model with $\ell_2 + \ell_\infty$ -norm slightly outperforms the other models. This can be attributed to its allocation of zero seats for the lowest fare customers, instead allocating more seats for other products with higher revenue potential.



Model	Leg	Booking Limit	Buckets						Shadow Price
			1	2	3	4	5	6	
ℓ_1 -norm	1	[200, 173, 148, 77, 77]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75
	2	[200, 106, 52, 37, 37]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	95
	3	[200, 157, 123, 96, 68, 34]	[3, 5]	[6]	[9]	[11, 12, 18]	[17]	[15]	80
Budget Uncertainty	1	[200, 173, 148, 77, 77]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75
	2	[200, 106, 52, 37, 37]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	95
	3	[200, 157, 123, 96, 68, 34]	[3, 5]	[6]	[9]	[11, 12, 18]	[17]	[15]	80
$\ell_2 + \ell_\infty$	1	[200, 163, 129, 101, 27]	[2, 3]	[1]	[9]	[7, 8]	[13, 14, 15]	[]	57.51
	2	[200, 105, 76, 27, 0]	[2, 3, 4, 5]	[9]	[10, 11]	[8, 15, 16, 17]	[14]	[]	61.28
	3	[200, 165, 105, 38]	[3, 5]	[6, 9]	[11, 12]	[15, 17, 18]	[]	[]	63.93
DLP	1	[200, 173, 148, 77, 77]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75
	2	[200, 106, 52, 37, 37]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	80
	3	[200, 156, 122, 95, 34]	[3, 5]	[6]	[9]	[11, 12, 17, 18]	[15]	[]	80
PNLP	1	[200, 163, 129, 101, 27]	[2, 3]	[1]	[9]	[7, 8]	[13, 14, 15]	[]	60.17
	2	[200, 105, 51, 51]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 15, 17]	[]	[]	69.83
	3	[200, 165, 105, 38]	[3, 5]	[6, 9]	[11, 12]	[15, 17, 18]	[]	[]	73.7

Table 4.37: Booking limits, buckets, and shadow prices of Example 30

Example 31: We investigate the scenario with a maximum number of 5 buckets.

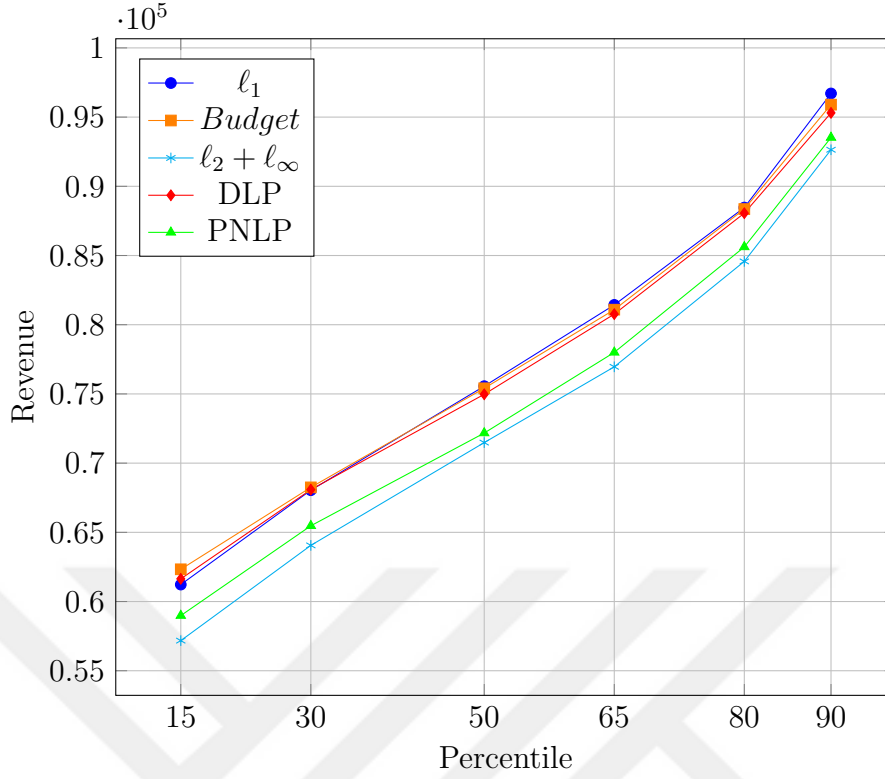


Figure 4.37: Revenue Percentiles of Example 31

In Figure 4.37, we observe that the ℓ_1 -norm model, Budget uncertainty set model, and DLP yield revenues that are close to each other, despite having slightly different bucketings. The discrepancy arises from the 17th product, which is placed in the last bucket on leg 3 for the ℓ_1 -norm model and Budget uncertainty set model, while it is in the second last bucket for DLP. However, since we consider nested booking limits, this difference does not have a significant impact. On the other hand, the model with $\ell_2 + \ell_\infty$ -norm no longer outperforms the others. Unlike the previous case, more products are in the last bucket of leg 2, and the model assigns a booking limit of 0 to that bucket, resulting in revenue loss. Also, note that although we consider 5 buckets, there are 6 buckets in some cases. The reason behind this is that we assign products with negative adjusted leg revenues into one additional bucket (if there is any).

Model	Leg	Booking Limit	Buckets						Shadow Price
			1	2	3	4	5	6	
ℓ_1 -norm	1	[200, 173, 148, 77, 77]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 106, 52, 37, 37]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	95.0
	3	[200, 157, 123, 96, 68, 34]	[3, 5]	[6]	[9]	[11, 12, 18]	[17]	[15]	80.0
Budget Uncertainty	1	[200, 173, 148, 77, 77]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 106, 52, 37, 37]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	95.0
	3	[200, 157, 123, 96, 68, 34]	[3, 5]	[6]	[9]	[11, 12, 18]	[17]	[15]	80.0
$\ell_2 + \ell_\infty$	1	[200, 163, 129, 101, 27]	[2, 3]	[1]	[9]	[7, 8]	[13, 14, 15]	[]	57.51
	2	[200, 105, 76, 27, 0]	[2, 3, 4, 5]	[9]	[10, 11]	[8, 15, 16, 17]	[14]	[]	61.28
	3	[200, 165, 105, 38]	[3, 5]	[6, 9]	[11, 12]	[15, 17, 18]	[]	[]	63.93
DLP	1	[200, 173, 148, 77, 77]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 106, 52, 37, 37]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	80.0
	3	[200, 156, 122, 95, 34]	[3, 5]	[6]	[9]	[11, 12, 17, 18]	[15]	[]	80.0
PNLP	1	[200, 163, 129, 101, 27]	[2, 3]	[1]	[9]	[7, 8]	[13, 14, 15]	[]	60.17
	2	[200, 105, 51, 51]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 15, 17]	[]	[]	69.83
	3	[200, 165, 105, 38]	[3, 5]	[6, 9]	[11, 12]	[15, 17, 18]	[]	[]	73.7

Table 4.38: Booking limits, buckets, and shadow prices of Example 31

Example 32: We investigate the scenario with a maximum number of 4 buckets.

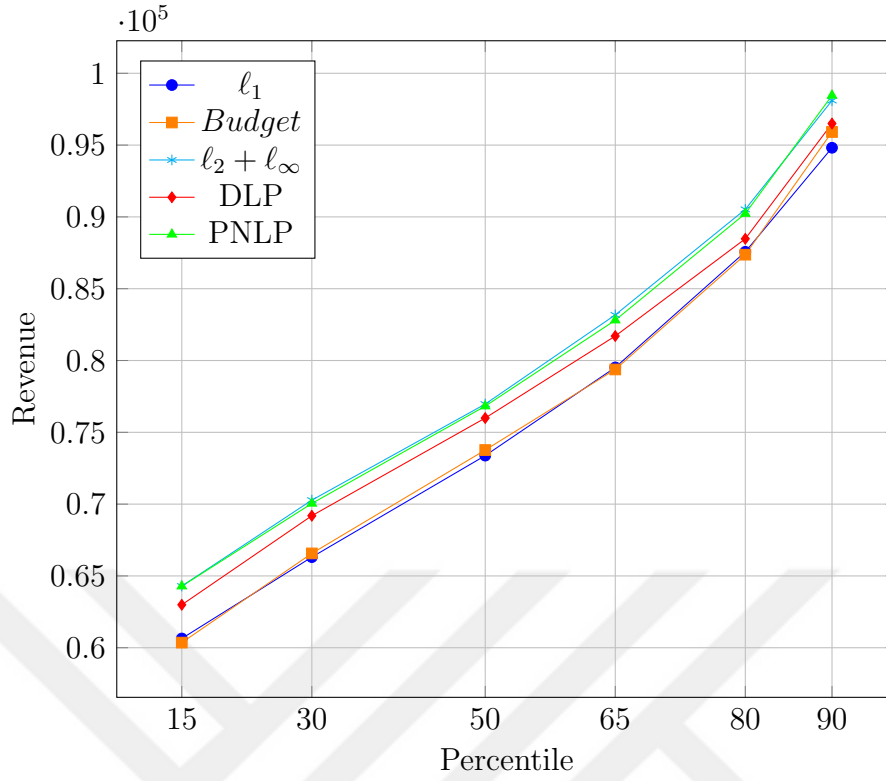


Figure 4.38: Revenue Percentiles of Example 32

In this figure, it is evident that the ℓ_1 -norm model and Budget uncertainty set model exhibit the same bucketing. They place the 13th and 18th products in the second last bucket, assigning relatively high numbers of booking limits to these buckets. However, since the demands for these products are higher than others and their revenue is lower, they result in the lowest total revenue. Conversely, the other two models, PNLP and the model with $\ell_2 + \ell_\infty$ -norm, hold exactly the same bucketing and almost identical booking limits. As a result, they yield very close simulation runs, and they achieve the highest revenue. This is because they place the lowest fare products in the last buckets, allowing other products to generate more revenue.

Model	Leg	Booking Limit	Buckets				Shadow Price
			1	2	3	4	
ℓ_1 -norm	1	[200, 151, 79, 79]	[1, 2, 3]	[7, 9]	[8, 13]	[14, 15]	75.0
	2	[200, 113, 36, 36]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 16, 17]	[15]	95.0
	3	[200, 168, 135, 66]	[3, 5]	[6]	[9, 11, 12, 18]	[15, 17]	80.0
Budget Uncertainty	1	[200, 151, 79, 79]	[1, 2, 3]	[7, 9]	[8, 13]	[14, 15]	75.0
	2	[200, 113, 36, 36]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 16, 17]	[15]	95.0
	3	[200, 168, 135, 66]	[3, 5]	[6]	[9, 11, 12, 18]	[15, 17]	80.0
$\ell_2 + \ell_\infty$	1	[200, 160, 98]	[2, 3]	[1, 9]	[7, 8, 13, 14, 15]	[]	57.51
	2	[200, 105, 27]	[2, 3, 4, 5]	[9, 10, 11]	[8, 13, 14, 15, 16]	[]	61.28
	3	[200, 164, 104]	[3, 5]	[6, 9]	[11, 12, 15, 17, 18]	[]	63.93
DLP	1	[200, 117, 39, 37]	[1, 2, 3]	[7, 9]	[8, 13, 14]	[15]	75.0
	2	[200, 113, 36, 36]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 16, 17]	[15]	80.0
	3	[200, 156, 95, 34]	[3, 5]	[6, 9]	[11, 12, 17, 18]	[15]	80.0
PNLP	1	[200, 174, 105]	[1, 2, 3]	[7, 9]	[8, 13, 14, 15]	[]	60.17
	2	[200, 105, 27]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 15, 16, 17]	[]	69.83
	3	[200, 164, 104]	[3, 5]	[6, 9]	[11, 12, 15, 17, 18]	[]	73.7

Table 4.39: Booking limits, buckets, and shadow prices of Example 32

Example 33: Up until now, we have utilized the ℓ_1 -norm model to solve single-leg problems for each leg in bucketed cases. Now, we will switch to using DLP to solve the single-leg problems and examine the results, following the same structure as before. First, we examine the case with 6 buckets.

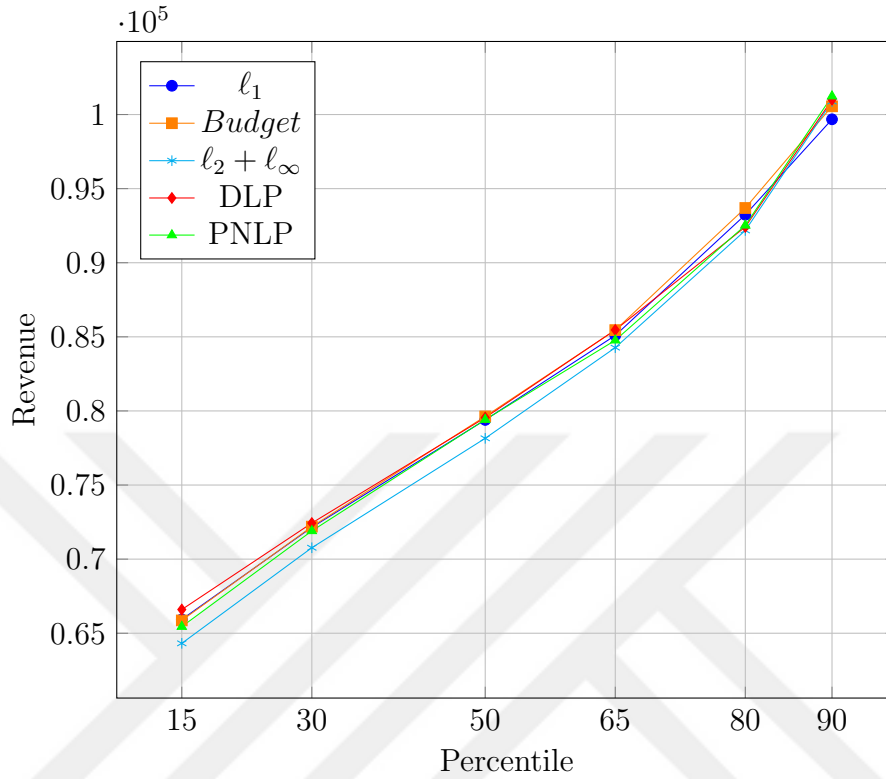


Figure 4.39: Revenue Percentiles of Example 33

If one compares the booking limits between this case and the previous one, where we solved single-leg problems using the ℓ_1 -norm model, it becomes evident that DLP assigns more seats to lower buckets. This results in higher total revenue. The trends observed in Figures 4.39 and 4.36 are similar. Therefore, using DLP instead of the ℓ_1 -norm model to solve single-leg problems does not yield significantly different results other than more total revenue.

Model	Leg	Booking Limit	Buckets						Shadow Price
			1	2	3	4	5	6	
ℓ_1 -norm	1	[200, 160, 130, 66, 66]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 120, 76, 30, 30]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	95.0
	3	[200, 160, 130, 106, 60, 30]	[3, 5]	[6]	[9]	[11, 12, 18]	[17]	[15]	80.0
Budget Uncertainty	1	[200, 160, 130, 66, 66]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 120, 76, 30, 30]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	95.0
	3	[200, 160, 130, 106, 60, 30]	[3, 5]	[6]	[9]	[11, 12, 18]	[17]	[15]	80.0
$\ell_2 + \ell_\infty$	1	[200, 160, 130, 106, 41]	[2, 3]	[1]	[9]	[7, 8]	[13, 14, 15]	[]	57.51
	2	[200, 120, 96, 56, 0]	[2, 3, 4, 5]	[9]	[10, 11]	[8, 15, 16, 17]	[14]	[]	61.28
	3	[200, 160, 106, 46]	[3, 5]	[6, 9]	[11, 12]	[15, 17, 18]	[]	[]	63.93
DLP	1	[200, 160, 130, 66, 66]	[2, 3]	[1]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 120, 76, 30, 30]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 17]	[15]	[]	80.0
	3	[200, 160, 130, 106, 30]	[3, 5]	[6]	[9]	[11, 12, 17, 18]	[15]	[]	80.0
PNLP	1	[200, 160, 130, 106, 41]	[2, 3]	[1]	[9]	[7, 8]	[13, 14, 15]	[]	60.17
	2	[200, 120, 76, 76]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16]	[14, 15, 17]	[]	[]	69.83
	3	[200, 160, 106, 46]	[3, 5]	[6, 9]	[11, 12]	[15, 17, 18]	[]	[]	73.7

Table 4.40: Booking limits, buckets, and shadow prices of Example 33

Example 34: We investigate the scenario with a maximum number of 5 buckets.

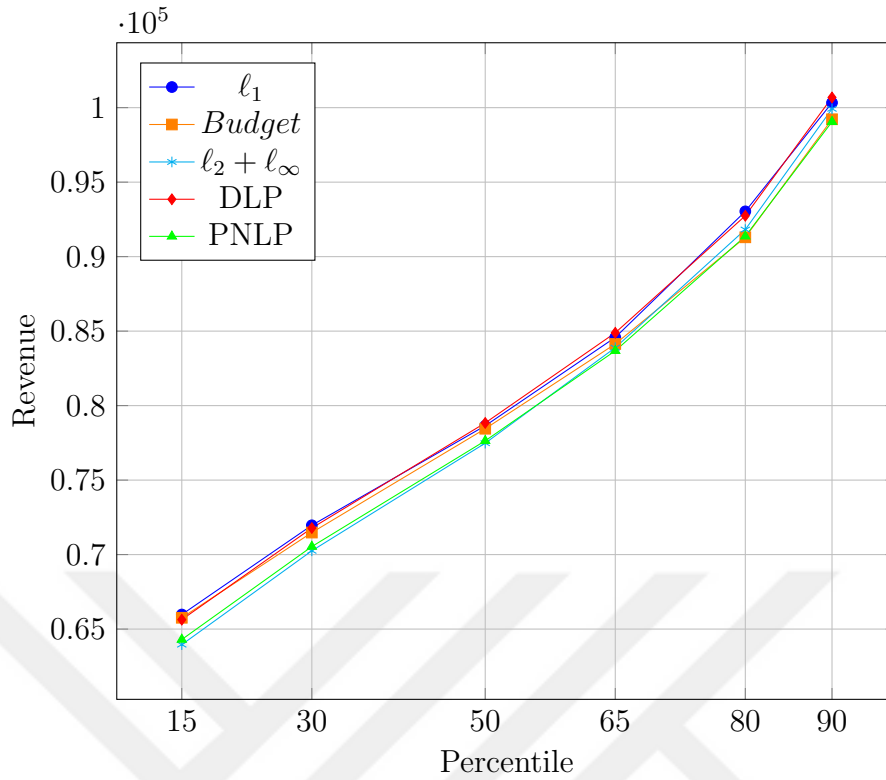


Figure 4.40: Revenue Percentiles of Example 34

When we have a maximum number of 5 buckets, previously, the $\ell_2 + \ell_\infty$ model yielded the lowest total revenue among all the models, along with the PNLP. Here, we observe the same trend, with the only difference being that since the DLP assigns more seats to lower classes, the results of all models are closer to each other.

Model	Leg	Booking Limit	Buckets					Shadow Price
			1	2	3	4	5	
ℓ_1 -norm	1	[200, 130, 66, 66]	[1, 2, 3]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 120, 76, 70]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16, 17]	[14, 15]	[]	95.0
	3	[200, 160, 130, 46, 30]	[3, 5]	[6]	[9, 11, 12]	[17, 18]	[15]	80.0
Budget Uncertainty	1	[200, 130, 66, 66]	[1, 2, 3]	[7, 9]	[8, 13]	[14, 15]	[]	75.0
	2	[200, 120, 76, 70]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16, 17]	[14, 15]	[]	95.0
	3	[200, 160, 130, 46, 30]	[3, 5]	[6]	[9, 11, 12]	[17, 18]	[15]	80.0
$\ell_2 + \ell_\infty$	1	[200, 160, 106, 41]	[2, 3]	[1, 9]	[7, 8]	[13, 14, 15]	[]	57.51
	2	[200, 120, 76, 70]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16, 17]	[14, 15]	[]	61.28
	3	[200, 160, 106, 16]	[3, 5]	[6, 9]	[11, 12, 17]	[15, 18]	[]	63.93
DLP	1	[200, 160, 130, 66, 30]	[2, 3]	[1]	[7, 9]	[8, 13, 14]	[15]	75.0
	2	[200, 120, 76, 70]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16, 17]	[14, 15]	[]	80.0
	3	[200, 160, 130, 46]	[3, 5]	[6]	[9, 11, 12]	[15, 17, 18]	[]	80.0
PNLP	1	[200, 160, 130, 66]	[2, 3]	[1]	[7, 9]	[8, 13, 14, 15]	[]	60.17
	2	[200, 120, 76, 70]	[2, 3, 4, 5]	[9, 10]	[8, 11, 16, 17]	[14, 15]	[]	69.83
	3	[200, 160, 106, 16]	[3, 5]	[6, 9]	[11, 12, 17]	[15, 18]	[]	73.7

Table 4.41: Booking limits, buckets, and shadow prices of Example 34

Example 35: We examine the scenario with a maximum number of 4 buckets.

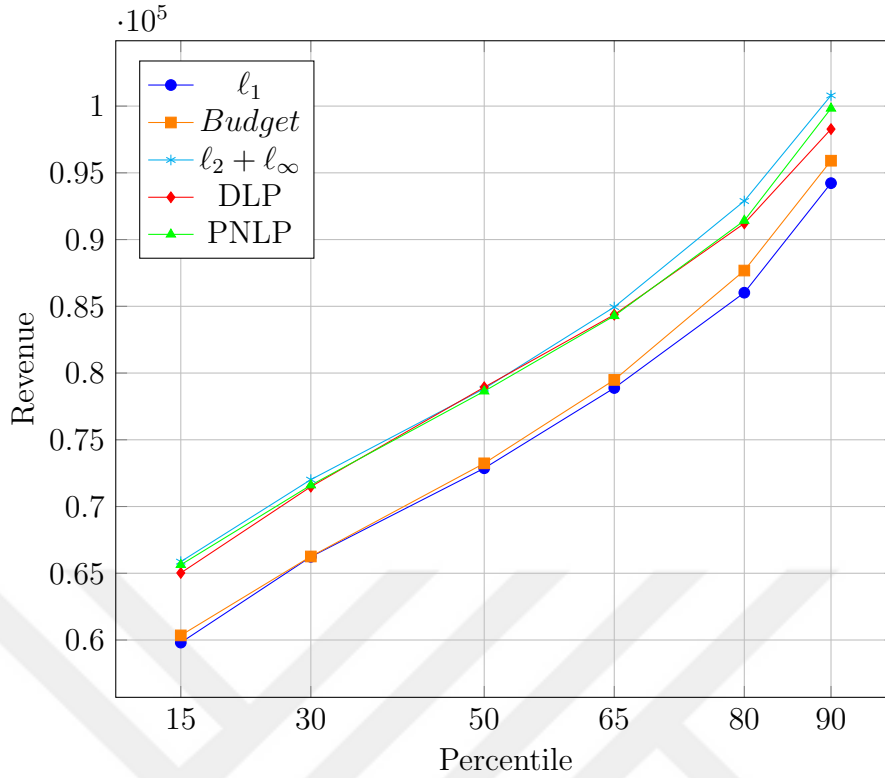


Figure 4.41: Revenue Percentiles of Example 35

In the case where we have a maximum of 4 buckets, we observe the same previous situation where our ℓ_1 -norm and Budget Uncertainty set models result in lower total revenue. However, it is noteworthy that when we assign booking limits to single-leg problems using the ℓ_1 -norm model, the lines on the graph are closer to each other because we assign more seats to lower buckets. Here, since DLP assigns fewer seats to lower buckets, the difference between the lines is larger. So, we can claim that when we have a smaller number of buckets, our model tends to perform better.

To conclude, in the example involving six products and two resources, using DAVN instead of segmented booking limits shows that the model employing the $\ell_2 + \ell_\infty$ uncertainty set performs better, especially with more buckets. This advantage arises from the lower shadow price associated with the $\ell_2 + \ell_\infty$ model compared to similar models, resulting in improved allocation of buckets and higher revenue compared to four other models. However, implementing DAVN does not significantly change the graph's structure in the large-scale example, mainly due to the large number of products, which reduces sensitivity to changes and maintains system stability.

Model	Leg	Booking Limit	Buckets				Shadow Price
			1	2	3	4	
ℓ_1 -norm	1	[200, 130, 66, 66]	[1, 2, 3]	[7, 9]	[8, 13]	[14, 15]	75.0
	2	[200, 120, 56, 30]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 16, 17]	[15]	95.0
	3	[200, 160, 130, 60]	[3, 5]	[6]	[9, 11, 12, 18]	[15, 17]	80.0
Budget Uncertainty	1	[200, 130, 66, 66]	[1, 2, 3]	[7, 9]	[8, 13]	[14, 15]	75.0
	2	[200, 120, 56, 30]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 16, 17]	[15]	95.0
	3	[200, 160, 130, 60]	[3, 5]	[6]	[9, 11, 12, 18]	[15, 17]	80.0
Ellips+Box	1	[200, 160, 106]	[2, 3]	[1, 9]	[7, 8, 13, 14, 15]	[]	57.51
	2	[200, 120, 56]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 15, 16, 17]	[]	61.28
	3	[200, 160, 106]	[3, 5]	[6, 9]	[11, 12, 15, 17, 18]	[]	63.93
DLP	1	[200, 130, 66, 30]	[1, 2, 3]	[7, 9]	[8, 13, 14]	[15]	75.0
	2	[200, 120, 56, 30]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 16, 17]	[15]	80.0
	3	[200, 160, 106, 30]	[3, 5]	[6, 9]	[11, 12, 17, 18]	[15]	80.0
PNLP	1	[200, 130, 66]	[1, 2, 3]	[7, 9]	[8, 13, 14, 15]	[]	60.17
	2	[200, 120, 56]	[2, 3, 4, 5]	[9, 10, 11]	[8, 14, 15, 16, 17]	[]	69.83
	3	[200, 160, 106]	[3, 5]	[6, 9]	[11, 12, 15, 17, 18]	[]	73.7

Table 4.42: Booking limits, buckets, and shadow prices of Example 35

4.2.6 Deviation From Highest Possible Revenue

In this part of the thesis, we will examine the scenarios to determine how much each algorithm deviates from the maximum attainable revenue. To achieve this, we will first find the booking limits of each model. Then, we will generate random demands, prioritizing the highest fare class to the lowest fare class, contrary to the LBH pattern. This approach will allow us to calculate the highest possible revenue that can be generated. For comparison purposes, we will utilize confidence intervals, computing them for each algorithm. We will begin our analysis with the example provided by Topaloglu [4].

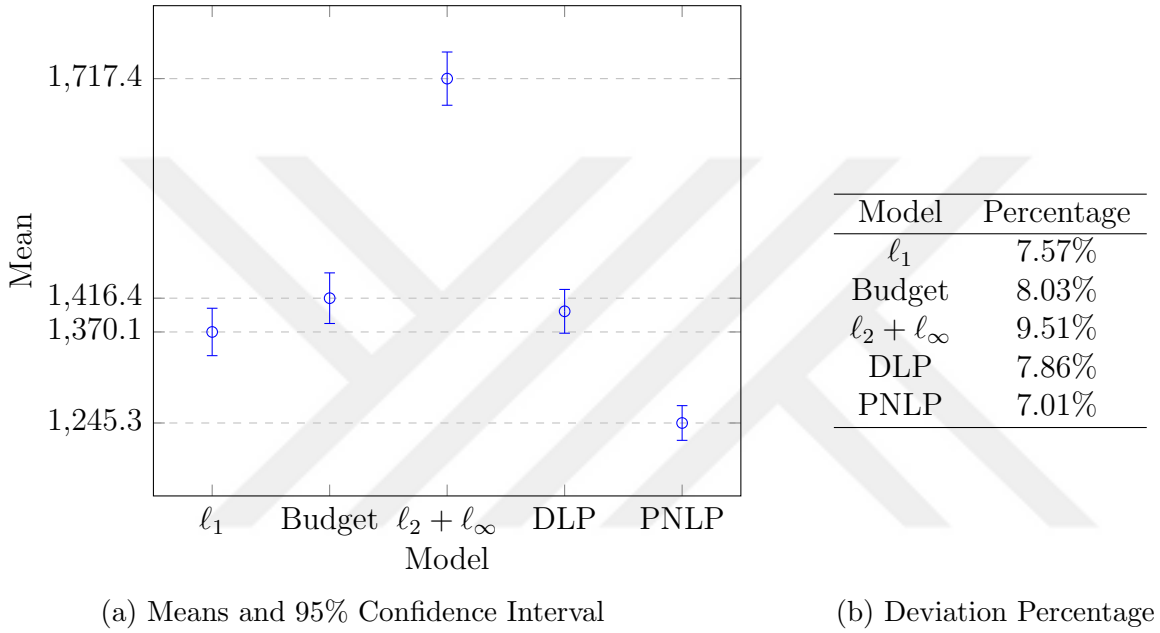


Figure 4.42: Confidence intervals and percentages

Based on Figure 4.42 (a), PNLP demonstrates the least deviation from maximum revenue at very high demand factors, followed closely by our ℓ_1 model. Doumpos (2016) suggest that models with higher mean and standard deviation are more robust. Figures 4.42 (a) and (b) indicate that robust models exhibit higher means and standard deviations, implying their greater robustness.

Next, we will consider the following demand vector: [30, 25, 20, 35, 25, 30], resulting in a demand factor of 91%, which is classified as a high demand factor [34].

When the demand factor decreases from very high to high, DLP performs remarkably better than other models, in terms of lower standard deviation. This improvement is due to its booking limits. Previously, DLP allocated 30 seats for the highest fare product, equal to its expected demand, but when actual demand fell short, it resulted in many empty seats and revenue loss. However, comparing models in terms of robustness according to [38], robust models still appear more robust than others.

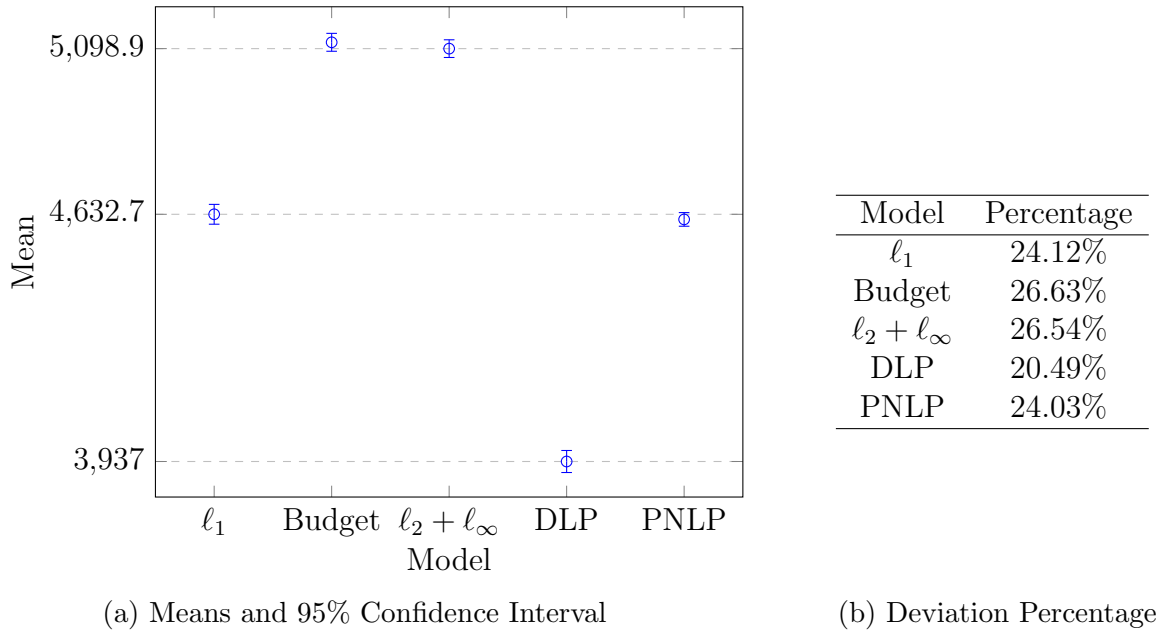


Figure 4.43: Confidence intervals and percentages

Now, we'll investigate the performance under a medium demand factor, which falls between 80-84%. Let's consider the following demand vector: [11, 41, 6, 61, 11, 21], resulting in an 83% demand factor.

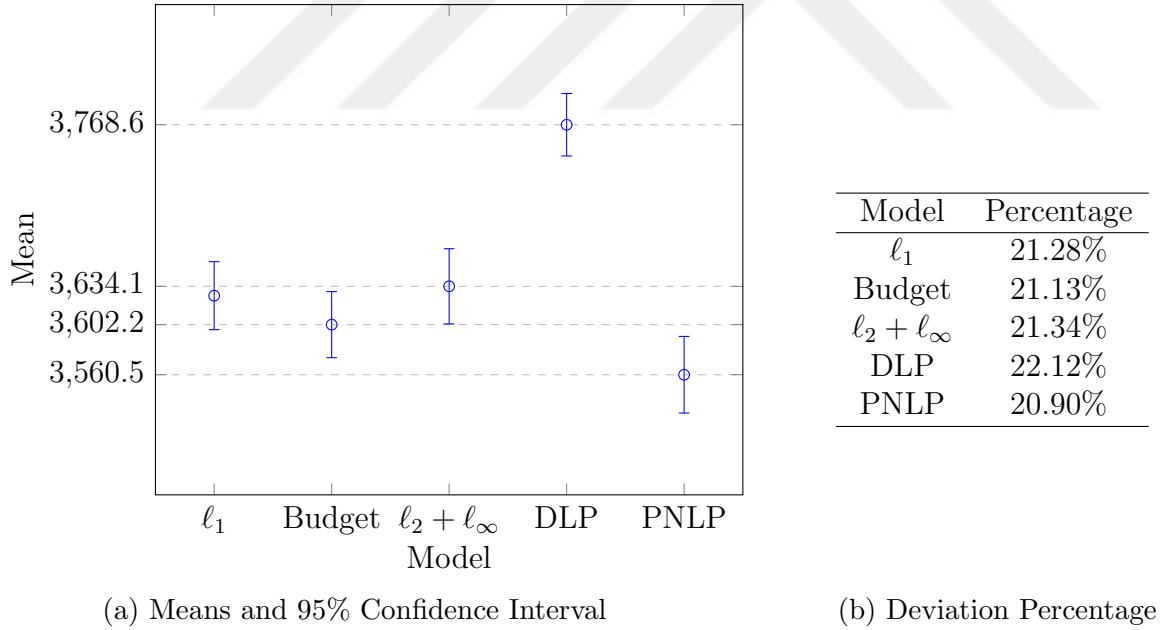
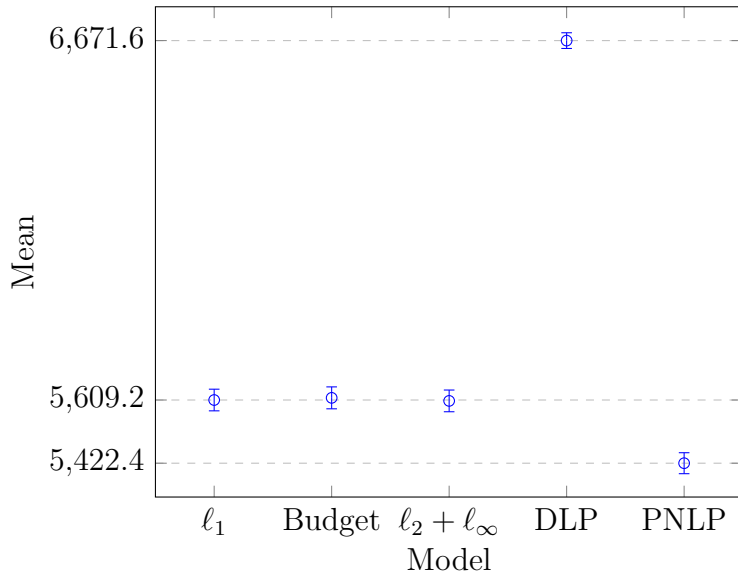


Figure 4.44: Confidence intervals and percentages

In the Figure 4.44, it appears that DLP is more robust under medium demand factor.

Lastly, we will examine how our models compare to DLP and PNLP under low-demand conditions, which can occur during extreme events such as a global pandemic [35]. We will consider the following demand vector: [11, 25, 6, 41, 11, 21], resulting in a 64% demand factor.



(a) Means and 95% Confidence Interval

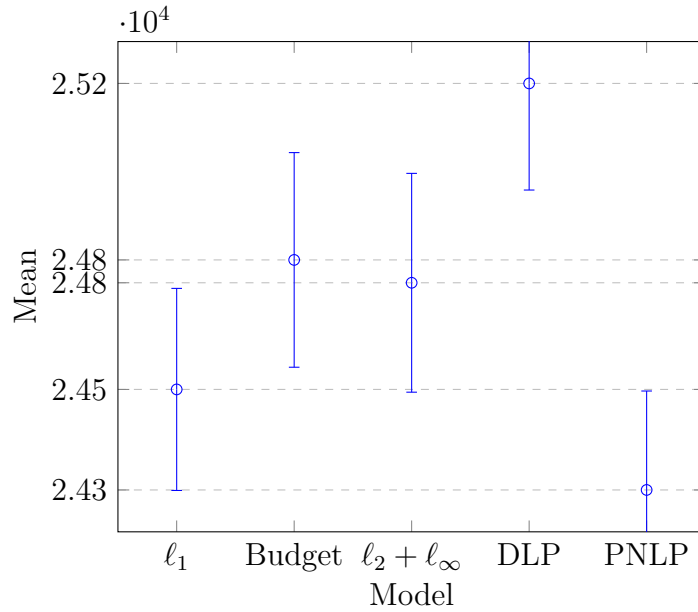
Model	Percentage
ℓ_1	32.12%
Budget	32.17%
$\ell_2 + \ell_\infty$	32.10%
DLP	38.19%
PNLP	31.05%

(b) Deviation Percentage

Figure 4.45: Confidence intervals and percentages

As expected, the performance of DLP decreases as the demand factor decreases. The other models perform similarly and their performance converges as the demand factor decreases. This convergence occurs because the opportunity to generate demand becomes limited, leading all models to fully utilize available demand. Consequently, the results of these models become more alike.

Next, we will examine the de Boer's example in [33]. We will do the same steps we did before and observe the results. First, we consider the base case in Example 17, demand factor is 0.89.



(a) Means and 95% Confidence Interval

Model	Percentage
ℓ_1	30.82%
Budget	31.15%
$\ell_2 + \ell_\infty$	31.09%
DLP	31.77%
PNLP	30.47%

(b) Deviation Percentage

Figure 4.46: Confidence intervals and percentages

Let us consider a scenario with a medium demand factor of 81%. We will use the following expected demand vector, obtained by decreasing the expectation of low fare classes: [25, 25, 25, 25, 30, 40, 15, 14, 20, 20, 30, 35, 30, 30, 30, 30, 35].

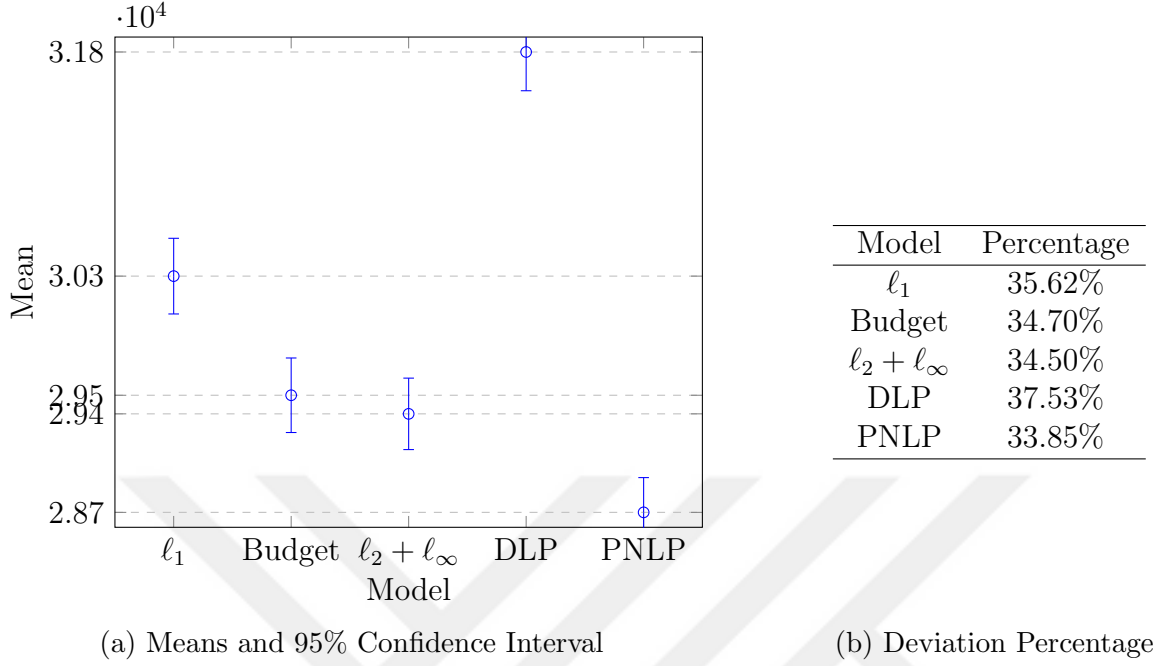


Figure 4.47: Confidence intervals and percentages

Finally, let us examine a scenario with a low demand factor of 72%, obtained using the following demand vector: [25, 30, 30, 25, 20, 25, 40, 15, 14, 20, 20, 30, 25, 20, 20, 20, 35].

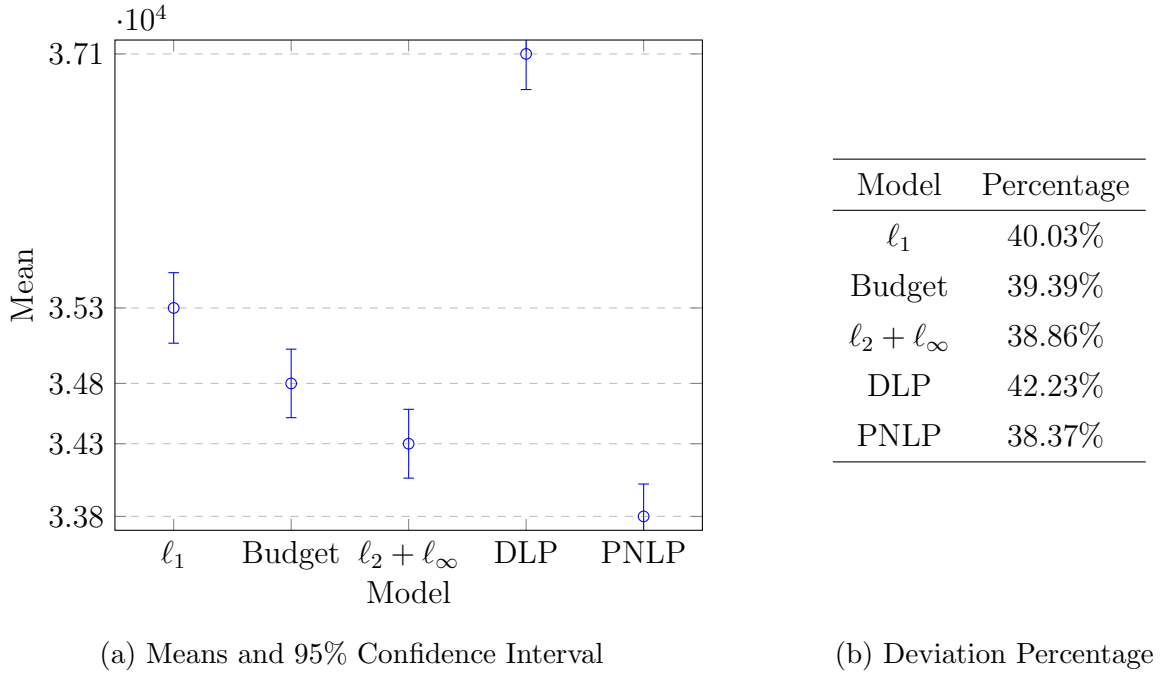


Figure 4.48: Confidence intervals and percentages

When we analyze all three different demand factor scenarios for de Boer's example, we

observe that PNLP dominates other models regardless of the demand factor. In contrast, DLP consistently performs the worst. This is due to DLP's allocation strategy, which reserves many seats for low-fare products and fewer seats for high-fare products, leading to a significant deviation from the highest possible revenue.

4.3 Computational Experiments with Dependent Data

In this section, we explore scenarios where product dependence is acknowledged, leading to a non-diagonal P matrix. We discuss a concise example comprising three products and two resources. Our assumptions entail Poisson distributed demand, chosen for its computational convenience. The P matrix is constructed using the square root of the covariance matrix, after employing a different decomposition such as Cholesky decomposition. The example's parameters are as follows: the price vector is $[120, 100, 200]$, the expected demand is $[35, 40, 15]$, and the resource capacity is 55 units for each resource.

4.3.1 PNLP in Dependent Case

To formulate the linear form of the PNLP for the dependent case in the small example with 3 products and 2 resources, we consider a scenario where product two (N_2) and three (N_3) are dependent on each other, while product one is independent. Both N_2 and N_3 follow Poisson distributions with parameters λ_2 and λ_3 , respectively. Here we define an additional random variable to connect product 2 and 3. N_4 represents a poisson distribution with parameter λ_4 for this purpose. The goal is to find a linear approximation of the PNLP model, enabling us to optimize booking limits efficiently while considering the dependency between product 2 (D_2) and product 3 (D_3), where $D_2 = N_2 + N_4$ and $D_3 = N_3 + N_4$. Realize that D_2 and D_3 have also Poisson distribution. Therefore, we have the following PNLP model when the random variables have discrete distribution:

$$\begin{aligned}
& \text{maximize} && r_1 \sum_{d=1}^{M_1} z_{1d} P(d_1 \geq d) + r_2 \sum_{d=1}^{M_2} z_{2d} P(d_2 \geq d) + r_3 \sum_{d=1}^{M_3} z_{3d} P(d_3 \geq d) \\
& \text{subject to} && \sum_{d=1}^{M_j} z_{jd} = y_j \quad \forall j \in N, \\
& && \sum_{j \in N} a_{ij} y_j \leq c_i \quad \forall i \in K, \\
& && 0 \leq z_{jd} \leq 1 \quad \forall j \in N, \text{ and } d = 1, \dots, M_j.
\end{aligned}$$

Expectation of independent product is simple. To calculate $E(D_2)$, where D_2 follows a

Poisson distribution with parameter $\lambda_2 + \lambda_4$, we need to find the expected value of D_2 .

Given that N_4 follows a Poisson distribution, then, we can use this expected value of $D_2 = N_2 + N_4$:

$$E(D_2) = E(\lambda_2 + \lambda_4) = \lambda_2 + \lambda_4$$

$$\text{So, } E(D_2) = \lambda_2 + \lambda_4.$$

The same calculations are valid for D_3 as well and the variance is also equal to this because the products follow Poisson distributions, and we can determine the variances of each random variable directly. To compute the covariances between each pair of random variables, we utilize the following formula:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

For the independent product (product one), the covariance with other products will be zero, except for the diagonal element. For the dependent products (product 2 and 3), we calculate the covariances using the given formula. For this purpose, we need to calculate the expected value of $D_2 D_3$. As mentioned before, N_2 , N_3 , and N_4 are independent Poisson random variables with parameters λ_2 , λ_3 , and λ_4 respectively. We define $D_2 = N_2 + N_4$ and $D_3 = N_3 + N_4$. We want to find the expected value of the product $D_2 \times D_3$:

$$\begin{aligned} E[D_2 \times D_3] &= E[(N_2 + N_4) \times (N_3 + N_4)] \\ &= E[N_2 \times N_3 + N_2 \times N_4 + N_3 \times N_4 + N_4^2] \\ &= E[N_2 \times N_3] + E[N_2 \times N_4] + E[N_3 \times N_4] + E[N_4^2] \end{aligned}$$

Using the properties of the Poisson distribution, we have:

$$\begin{aligned} E[D_2 \times D_3] &= (\lambda_2 \times \lambda_3) + (\lambda_2 \times \lambda_4) + (\lambda_3 \times \lambda_4) + \text{Var}[N_4] + E[N_4]^2 \\ &= \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 + \lambda_4 + \lambda_4^2 \end{aligned}$$

Therefore, $E[D_2 \times D_3] = \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 + \lambda_4 + \lambda_4^2$. To find the covariance of D_2 and D_3 , we use the formula:

$$\text{Cov}(D_2, D_3) = E[D_2 \times D_3] - E[D_2] \times E[D_3]$$

We've already calculated $E[D_2 \times D_3]$ as:

$$E[D_2 \times D_3] = \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 + \lambda_4 + \lambda_4^2$$

and $E[D_2]$, $E[D_3]$ as:

$$E[D_2] = E[N_2 + N_4] = E[N_2] + E[N_4] = \lambda_2 + \lambda_4$$

$$E[D_3] = E[N_3 + N_4] = E[N_3] + E[N_4] = \lambda_3 + \lambda_4$$

Substituting these values into the covariance formula:

$$\begin{aligned} \text{Cov}(D_2, D_3) &= (\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 + \lambda_4 + \lambda_4^2) - (\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4) \\ &= \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 + 2\lambda_4^2 - (\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 + \lambda_4^2) \\ &= \lambda_4 \end{aligned}$$

So, the covariance of D_2 and D_3 is $\text{Cov}(D_2, D_3) = \lambda_4$.

Example 36: By using the calculations above, one can generate the covariance matrix easily. We have the following parameters: $\lambda_1 = 40$; N_2, N_3 and N_4 has $\lambda_2 = 40, \lambda_3 = 15$, and $\lambda_4 = 15$. To streamline analysis, matrices A and P are calculated as follows:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \text{Cov. Matrix} &= \begin{bmatrix} 35 & 0 & 0 \\ 0 & 55 & 15 \\ 0 & 15 & 30 \end{bmatrix} \end{aligned}$$

When we are constructing the P matrix, we have used square root of covariance matrix and obtain the following matrix:

$$P = \begin{bmatrix} 5.92 & 0 & 0 \\ 0 & 7.32 & 1.18 \\ 0 & 1.18 & 5.35 \end{bmatrix}$$

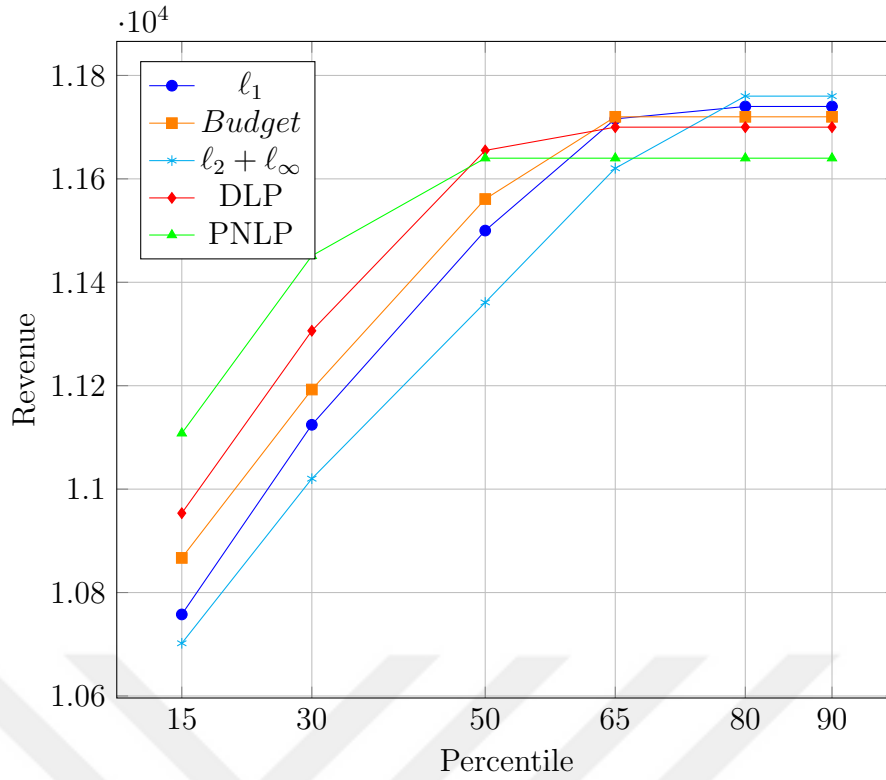


Figure 4.49: Revenue Percentiles of Example 36 With DF=1.09

Model	Booking Limits			Avg. Cap. Unused	
	P1	P2	P3	R1	R2
ℓ_1	37	37	18	3.4	0.0
Budget Uncertainty	36	36	19	2.9	0.0
$\ell_2 + \ell_\infty$	38	38	17	4.2	0.0
DLP	35	35	20	2.4	0.1
PNLP	32	32	23	1.5	0.2

Table 4.43: Booking limits and average capacity unused of Example 36

In the graph above, the P matrix is derived using the square root of the covariance matrix and DF is 1.09. Notably, towards the end of the simulation, the robust models outperform other models. This superiority can be attributed to its allocation of more seats to lower fare classes. Despite having a higher average capacity left empty, the robust models achieve higher sales volumes due to high demand for lower-fare products. Consequently, their total revenue surpass that of other models. However, unexpectedly, robust model have worse worst-case scenario performances. that is because they allocate more seats for low-priced products and as a result have wider range of revenues.

Example 37: Let us decrease the demand factor to 1 by decreasing demand factor for the highest fare product, resulting expected demand vector of $[35, 30, 15]$. We have the following result:

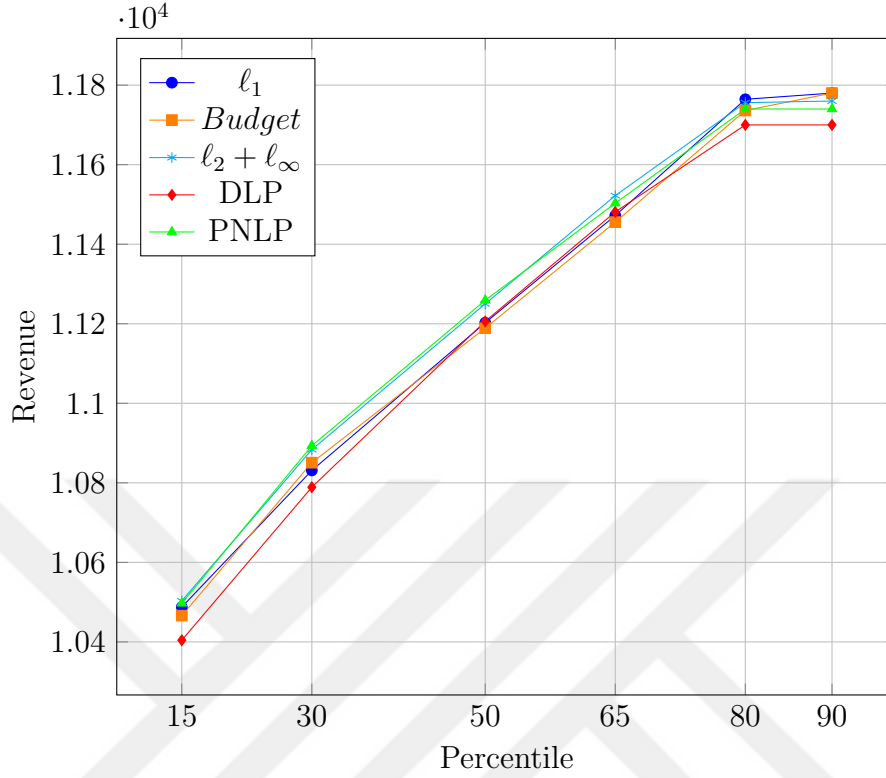


Figure 4.50: Revenue Percentiles of Example 37 With Decreased Expected Demand of Highest Fare Product, DF=1

Model	Booking Limits			Avg. Cap. Unused	
	P1	P2	P3	R1	R2
ℓ_1	39	39	16	5.2	0.4
Budget Uncertainty	39	39	16	5.3	0.5
$\ell_2 + \ell_\infty$	38	38	17	4.8	0.6
DLP	35	35	20	4.1	1.7
PNLP	37	37	18	4.4	0.9

Table 4.44: Booking limits and average capacity unused of Example 37

In this scenario, the decrease in expected demand for the highest fare product prompted models to reduce their allocations for this product category to mitigate the risk of empty seats. With fewer products available, this decrease significantly impacted the results. Consequently, all models ended up with similar seat allocations for each product, resulting in similar overall revenue.

Example 38: We decrease the demand for the lower fare product while maintaining the same demand factor as before which is 1 and resulting expected demand vector is 35, 40, 5].

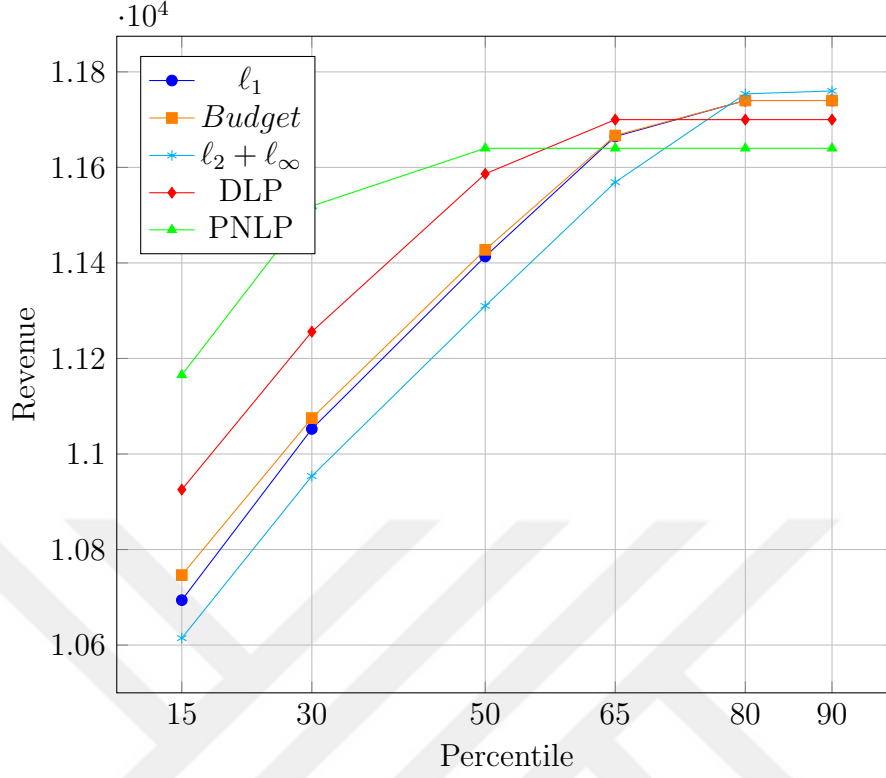


Figure 4.51: Revenue Percentiles of Example 38 With Decreased Expected Demand of Lowest Fare Product, DF=1

Model	Booking Limits			Avg. Cap. Unused	
	P1	P2	P3	R1	R2
ℓ_1	37	37	18	3.6	0.3
Budget Uncertainty	37	37	18	3.5	0.4
$\ell_2 + \ell_\infty$	38	38	17	4.3	0.4
DLP	35	35	20	2.5	0.2
PNLP	32	32	23	1.3	0.3

Table 4.45: Booking limits and average capacity unused of Example 38

In this case, the booking limits of the model remain relatively stable, showing minimal changes compared to the first scenario. However, with the decrease in demand factor, the performance of our ℓ_1 -norm and Budget Uncertainty Set models exhibits a slight decline. While in the first scenario, these models outperformed DLP at the 65th percentile, in this scenario, they surpass DLP at the 70th percentile.

Example 39: We do the same procedure by using Cholesky decomposition to construct P matrix. Parameters are the same with the Example 36.

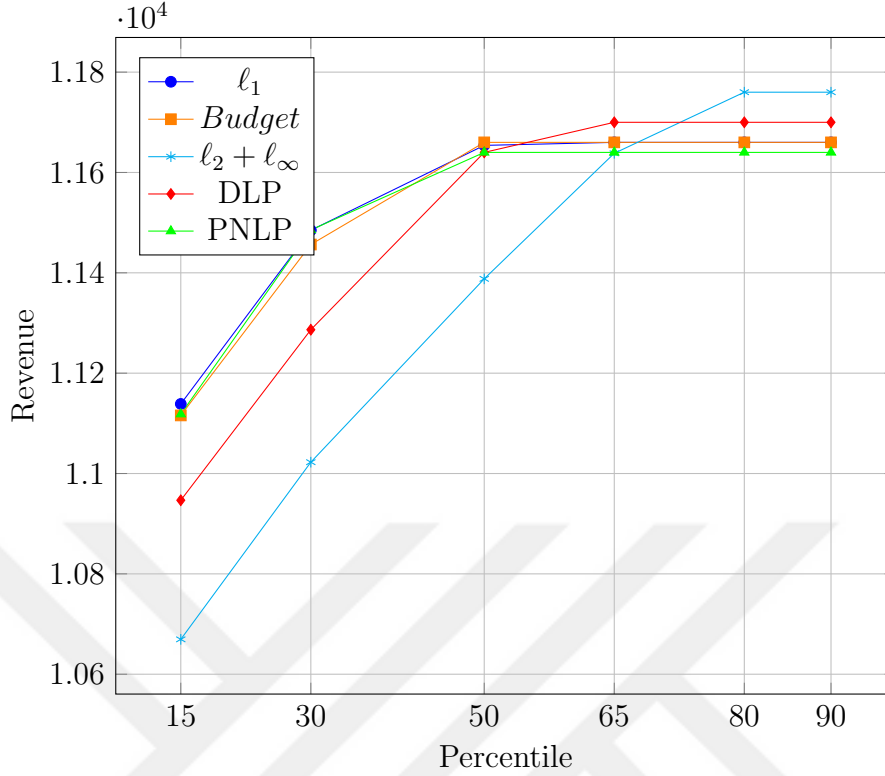


Figure 4.52: Revenue Percentiles of Example 39 With DF=1.09

Model	Booking Limits			Avg. Cap. Unused	
	P1	P2	P3	R1	R2
ℓ_1	33	33	22	1.5	0.1
Budget Uncertainty	33	33	22	1.6	0.2
$\ell_2 + \ell_\infty$	38	38	17	4.2	0.0
DLP	35	35	20	2.5	0.1
PNLP	32	32	23	1.4	0.2

Table 4.46: Booking limits and average capacity unused of Example 39

Here, we observe a consistent graph structure for PNLP, DLP, and $\ell_2 + \ell_\infty$ models when the P matrix is obtained using the square root of the covariance matrix. The main difference lies in the initial percentiles, representing worst-case scenarios, where the ℓ_1 -norm and Budget Uncertainty set models start with higher revenue due to their adjusted booking limits. These two models decreased their allocation for single-capacity user products and increased allocation for high-priced products, which uses two of the resources. This adjustment provides higher overall revenue, resulting in better worst-case scenario performance.

Example 40: We decrease the demand factor by decreasing the expected demand of highest fare product, having the same expected demand vector in Example 37 which is [35, 30, 15].

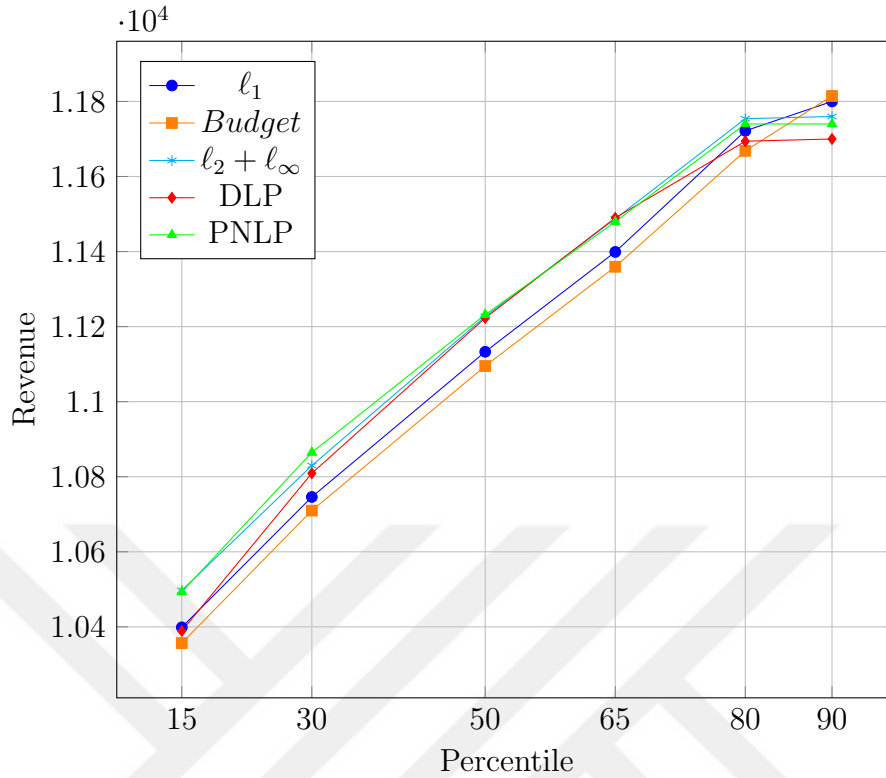


Figure 4.53: Revenue Percentiles of Example 40 With Decreased Expected Demand of Highest Fare Product, DF=1

Model	Booking Limits			Avg. Cap. Unused	
	P1	P2	P3	R1	R2
ℓ_1	40	40	15	6.0	0.3
Budget Uncertainty	41	41	14	6.4	0.2
$\ell_2 + \ell_\infty$	38	38	17	4.9	0.6
DLP	35	35	20	4.0	1.8
PNLP	37	37	18	4.6	0.9

Table 4.47: Booking limits and average capacity unused of Example 40

In this scenario, we observe a nearly identical graph structure as in Example 37, indicating that the method of finding the P matrix does not significantly affect the outcome in this scenario.

Example 41: We decrease the demand factor by decreasing the expected demand of lowest fare product, leading to the same expected demand vector in Example 38 that is $[35, 40, 5]$.

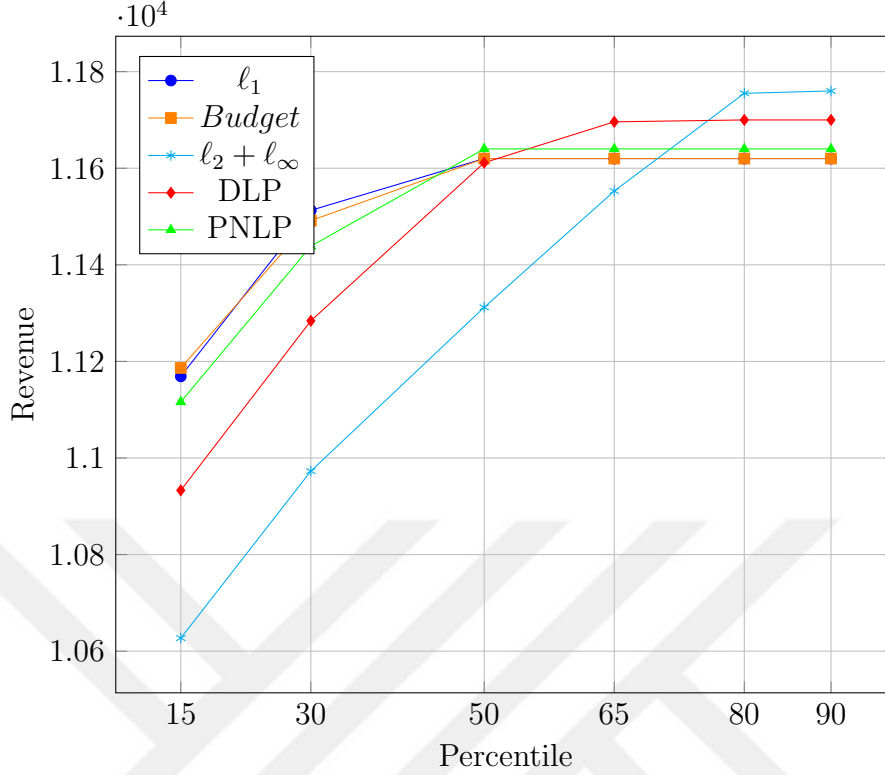


Figure 4.54: Revenue Percentiles of Example 41 With Decreased Expected Demand of Lowest Fare Product, DF=1

Model	Booking Limits			Avg. Cap. Unused	
	P1	P2	P3	R1	R2
ℓ_1	31	31	24	1.2	0.4
Budget Uncertainty	31	31	24	1.2	0.3
$\ell_2 + \ell_\infty$	38	38	17	4.2	0.5
DLP	35	35	20	2.4	0.2
PNLP	32	32	23	1.4	0.3

Table 4.48: Booking limits and average capacity unused of Example 41

The graph demonstrates a trend similar to the initial scenario, suggesting that the reduction in demand for the lowest fare product is not notably impacted by the method employed to derive the P matrix. This adjustment benefits the ℓ_1 -norm model and the Budget Uncertainty model in the lower percentiles, which represents worst-case scenarios, and the graph is similar to the one in Example 39. We observe this behavior when we use square root of covariance matrix as P. So, it can be suggested that structure of graphs do not change when decomposition technique is changed. However, behaviors of the graphs are changed, and Cholesky decomposition works better for robust models.

Chapter 5

Conclusion

This thesis has addressed two objectives: managing demand forecasting uncertainty and mitigating robust optimization solutions' conservatism. The study developed strategies to handle demand variations and reduce risks from inaccurate forecasts by treating demand as an uncertain parameter within an uncertainty set. Additionally, it explored various uncertainty sets to balance robustness with revenue optimization, identifying approaches that minimize conservatism without sacrificing performance. This research contributes to RM by providing robust, practical capacity allocation solutions that enhance decision-making. Extensive computational experiments have validated these findings, offering insights for effective capacity allocation.

In the example featuring six products and two resources with a high demand factor and an LBH arrival pattern, our models exhibit performance closely aligned with the PNLP and DLP methods.

- Robust models achieve comparable effectiveness without relying on probability distribution information, highlighting their practical applicability and robustness by simplifying computational requirements.
- Under mixed arrival pattern, PNLP's performance reduces at high demand factors due to its lower allocation of seats to high-priced products, resulting in revenue loss.
- Robust models and DLP surpass PNLP after the 65th percentile at high DFs, but converge as DF decreases, with DLP performing worse due to its direct dependence on expected demand.

In the example consisting of 5 products and 3 resources, we have following findings through distributional graphs:

- Robust models, particularly the ℓ_1 and budget uncertainty set models, demonstrate shorter tails and more evenly distributed outcomes compared to DLP and PNLP.

- DLP and PNLP tend to excel in best-case scenarios, while robust models perform better in worst-case scenarios.
- Decreasing the standard deviation of the two lowest-priced products results in longer tails for all models except for PNLP, which exhibits a shorter tail.
- Robust models and DLP become more susceptible to variability with reduced uncertainty in low-priced products, while PNLP remains stable and better manages predictable demand.
- Increasing the standard deviation of the two highest-priced products results in longer tails for all models, with robust models having more symmetric distributions.
- The ℓ_1 and budget uncertainty set models outperform DLP in worst-case scenarios.
- Decreasing the demand factor leads to longer tails in all models except for the ℓ_1 uncertainty set, which shows the shortest tail and a bell-shaped graph, indicating better handling of lower demand factors.
- Robust models do not rely on demand distribution information, highlighting their superiority over PNLP.
- In mixed arrival patterns, the $\ell_2 + \ell_\infty$ model demonstrates superior best-case scenarios, reaching revenues up to \$32,000.
- When the demand factor increases, PNLP and two robust models show relatively short left tails, indicating better worst-case scenarios.
- As the demand factor decreases, DLP underperforms in best-case scenarios, generating lower revenues with longer tails on the left side.

In the larger-scale example consisting eighteen products and three resources, our models display a graph structure comparable to previous scenarios.

- Robust models consistently outperform DLP and closely trail PNLP as the demand factor decreases.
- The observed graph structure remains largely unchanged under mixed arrival pattern or LBH, i.e., performances are more balanced due to additional substitution possibilities.

When we consider the scenarios where we observe wrong demand information, i.e., when expected demand is underestimated or overestimated:

- The $\ell_2 + \ell_\infty$ norm model consistently outperforms all other models in high percentiles when demand is underestimated since it allocates more seats for high-fare products.

- Overestimated demand leads to similar performance across all models, as they are constrained by the actual achievable demand and work to maximize revenue based on available data.
- When we apply the same procedure to bigger example, we again observe minimal changes in graph structure due to high number of products.
- These findings highlight the $\ell_2 + \ell_\infty$ norm model's advantage in underestimated demand scenarios and the importance of accurate demand forecasting in revenue management strategies.

We switch to the DAVN from partitioned booking limit to observe the effect of different booking limit controls on robust models.

- For the example in [4], the $\ell_2 + \ell_\infty$ uncertainty set demonstrates superior performance, especially when the number of buckets is high due to its lower shadow price, leading to improved bucket allocation and enhanced revenue outcomes.
- Other models share similar shadow prices, resulting in less effective performance compared to the $\ell_2 + \ell_\infty$ model.
- For the example in [33], implementing DAVN does not significantly alter the graph structure due to the high number of products.
- The larger number of products in the large-scale example makes the system less sensitive to changes, maintaining stability despite the introduction of DAVN.

When we examine an example with 3 products and 2 resources where there is a correlation between demands:

- Robust models consistently outperform DLP and PNLP after the 75th percentile, regardless of the demand factor and decomposition technique.
- Poorer worst-case scenario performances are observed when using P obtained via square root of covariance matrix.
- Employing Cholesky decomposition significantly enhances worst-case performances of robust models, particularly the ℓ_1 norm and budget uncertainty set.
- Cholesky decomposition allows robust models to allocate more seats to the highest fare class, thereby increasing overall revenue.
- Also, PNLP requires additional steps to linearize the model with dependent data, making our models preferable due to their simplified handling of demand dependence.

These findings underscore the robustness and enhanced performance capabilities of our models in handling dependent demand scenarios.

For future research, we can expand our approach by incorporating more than two norms into our uncertainty sets, which can enable us to observe the conservativeness of the uncertainty sets with three or more norms. Additionally, we have only considered one correlated demand example with a Poisson demand distribution. Conducting more experiments with various demand distributions and different example sizes would provide a more comprehensive understanding of correlated demand scenarios. Furthermore, exploring other covariance matrix decompositions, such as LDL^T decomposition, could yield interesting insights and results.

To sum up, robust models generally perform comparably against PNL and DLP in worst-case scenarios and even have better or similar performances as the demand factor (DF) decreases, despite not using probability distribution information. Moreover, robust models present narrower revenue ranges and light tails, especially when DF and standard deviation (SD) are high, proving their robustness. Therefore, robust models can be used in such cases to achieve results within a narrower range, decreasing variability.

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Appendix A

Histograms

A.1 Histograms of the Network with 2 Resources and 6 Products

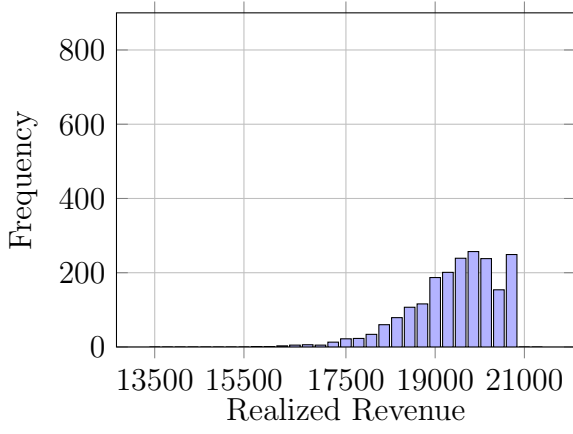
Example 42: Let us restate the parameters of the network example with 2 resources and 6 products (each resource has a capacity of 90), represented by the following A matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 5.47 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7.74 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.47 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.47 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.32 \end{bmatrix}$$

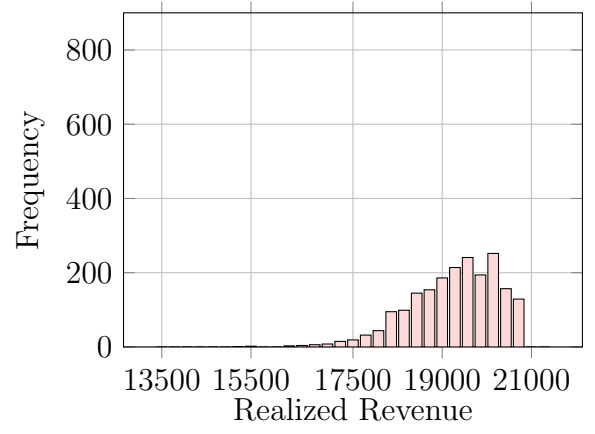
Product	O-D	Fare	Flights Used	Exp. Demand
1	A-B	150	1	30
2	A-B	100	1	60
3	B-C	120	2	20
4	B-C	80	2	80
5	A-C	250	1, 2	30
6	A-C	170	1, 2	40

Table A.1: Parameters

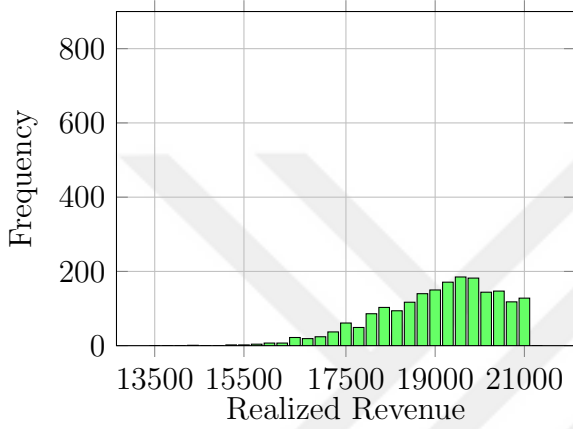
Since we're considering a diagonal P matrix with Poisson demand distribution, the standard deviation is equal to the expectation for each product. We basically repeat the Example 1 and obtain the following histograms:



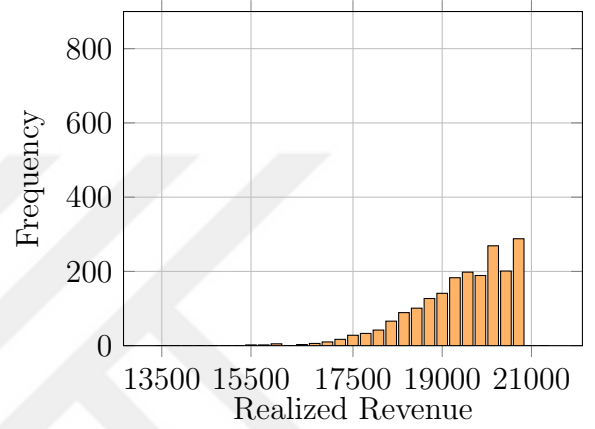
(a) Histogram for ℓ_1 -norm



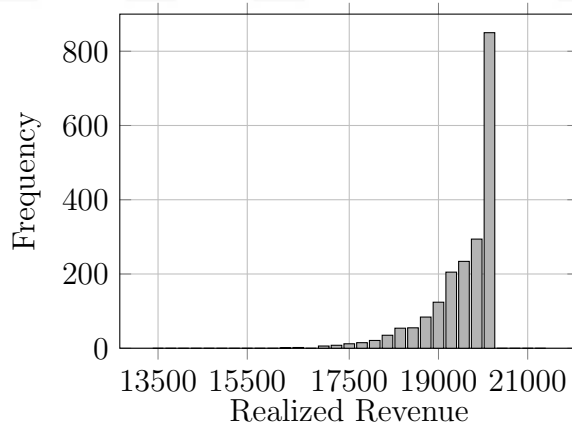
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



(d) Histogram for DLP



(e) Histogram for PNL P

Figure A.1: Performance of different models in Example 42

From the histograms above, we observe that PNL P has a narrower range, making it better in worst-case scenarios, and it peaks at its best-case scenario. Although the other four models exhibit a wider range, they achieve better best-case scenarios, with the $\ell_2 + \ell_\infty$ model reaching a revenue of \$21,000. This suggests that while PNL P offers more consistent performance with less variability, robust models, particularly the $\ell_2 + \ell_\infty$ model, have the potential to generate higher revenues under favorable conditions.

Example 43: We use the same parameters in Example 2 and generate histograms.

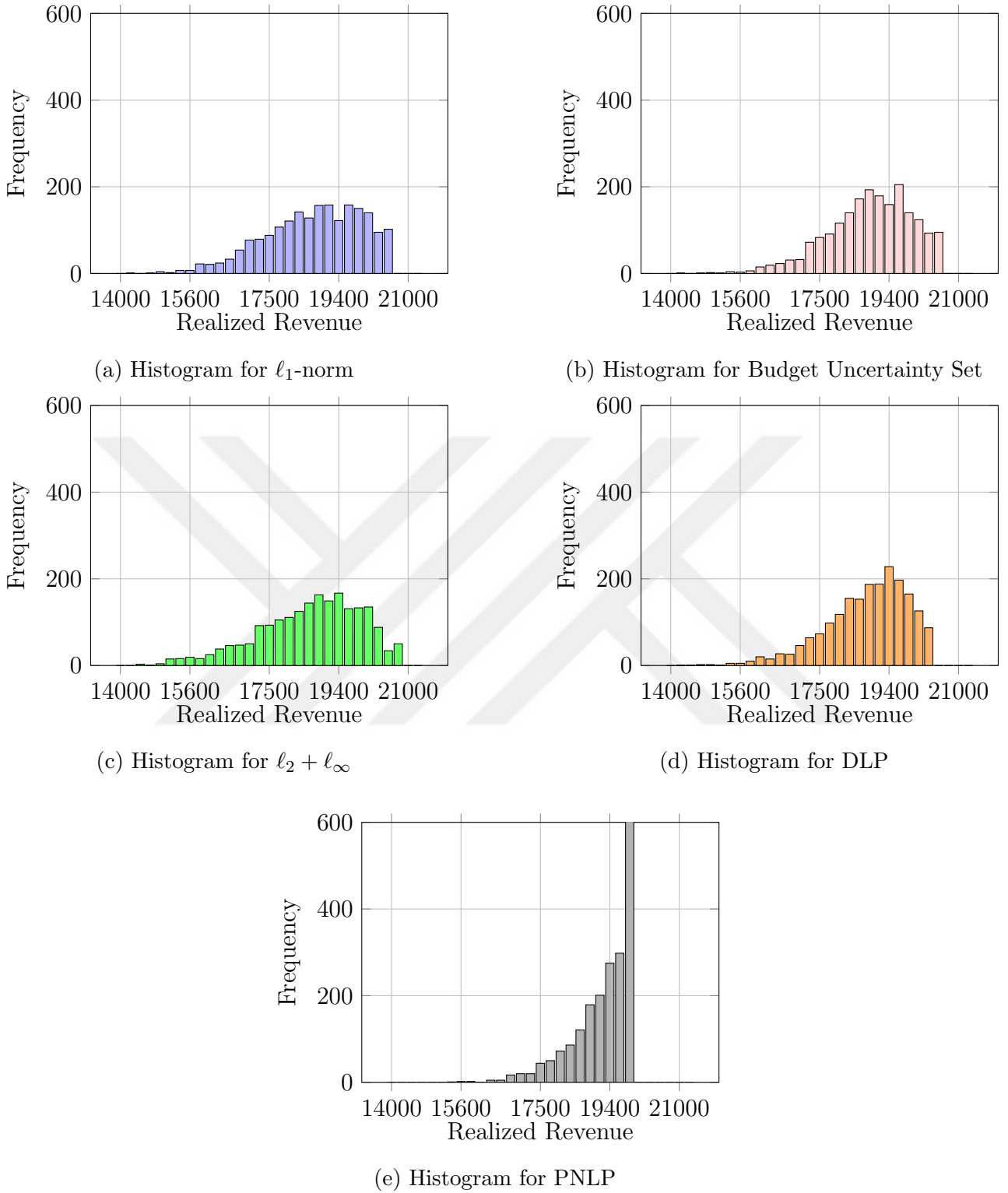


Figure A.2: Performance of different models in Example 43

In this example, we observe a similar structure to the previous one, primarily due to the high demand factors present in both cases. This consistency in the structure is attributed to the elevated demand levels, which lead to comparable performance patterns across the models. Consequently, the reasoning behind the performance of the models remains consistent, with PNLP showing outcomes in a narrower range and the robust models, especially the $\ell_2 + \ell_\infty$

model, demonstrating higher potential revenues under favorable conditions.

Example 44: We use the same parameters in Example 3 and generate histograms.

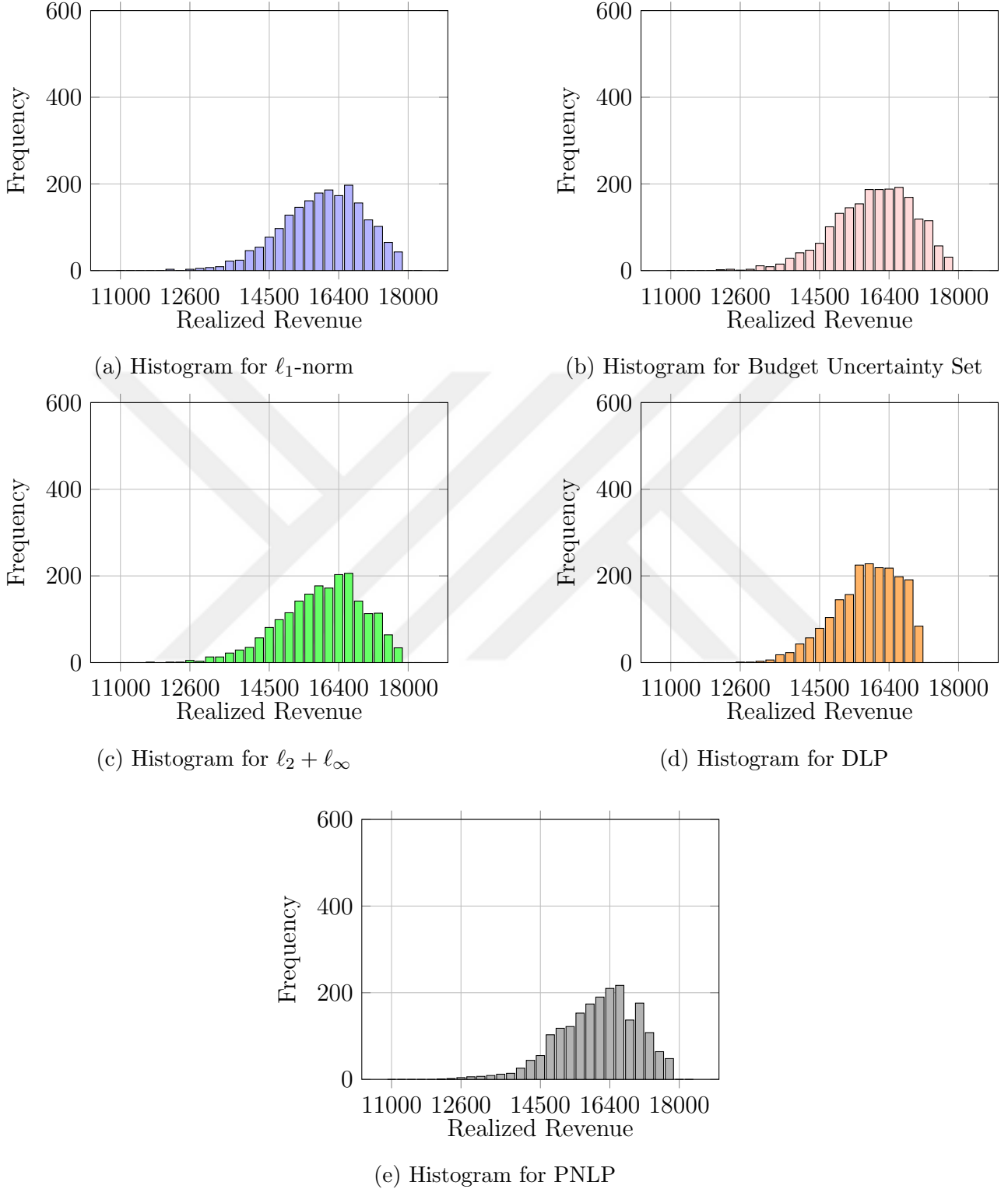


Figure A.3: Performance of different models in Example 44

In this example, robust models and PNLP exhibit similar performance, staying within the same range and displaying comparable graph shapes. However, it is important to note that while PNLP leverages distribution information, robust models achieve their performance without relying on this information.

Example 45: We use the same parameters in Example 4 and generate histograms.

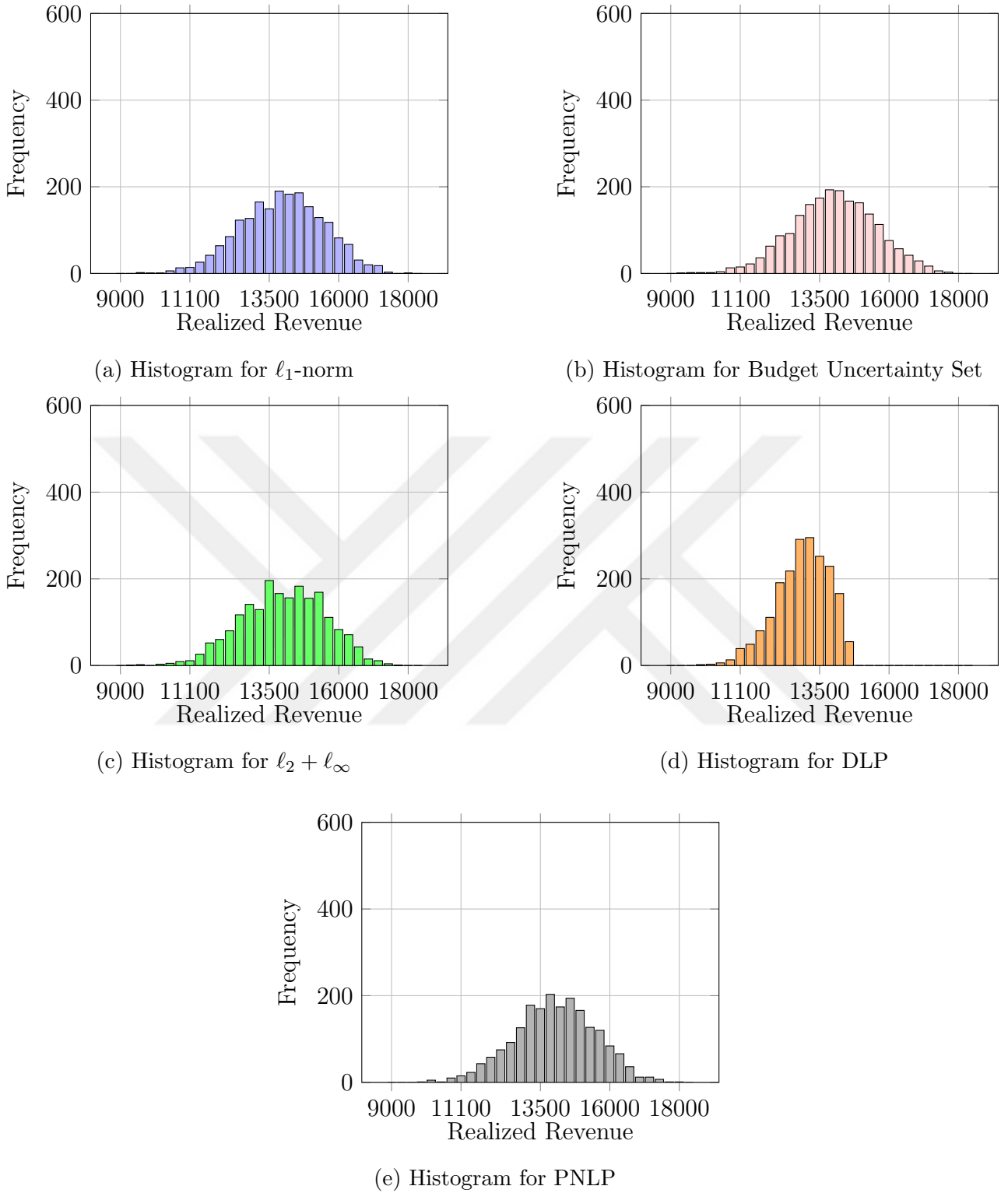
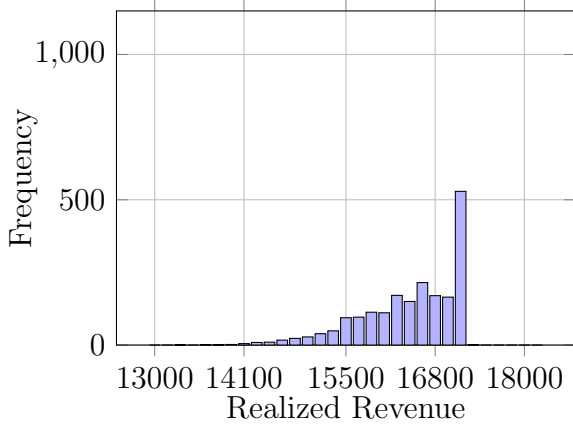


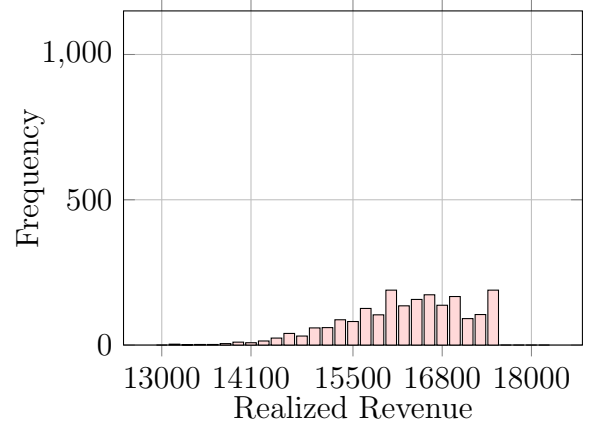
Figure A.4: Performance of different models in Example 45

In this example, DLP's performance declines significantly due to the decreased demand factor. However, the other models maintain similar performance levels, remaining within the same range and exhibiting comparable distribution patterns. This suggests that while DLP struggles with lower demand factors, robust models and PNLP are more resilient, maintaining consistent performance across varying demand levels.

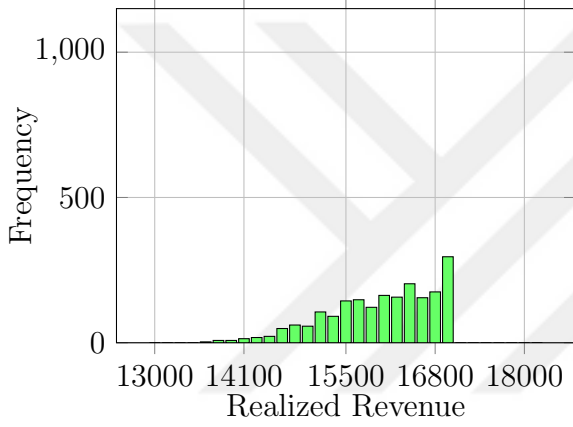
Example 46: We use the same parameters in Example 5, where we use mixed arrival pattern, and generate histograms.



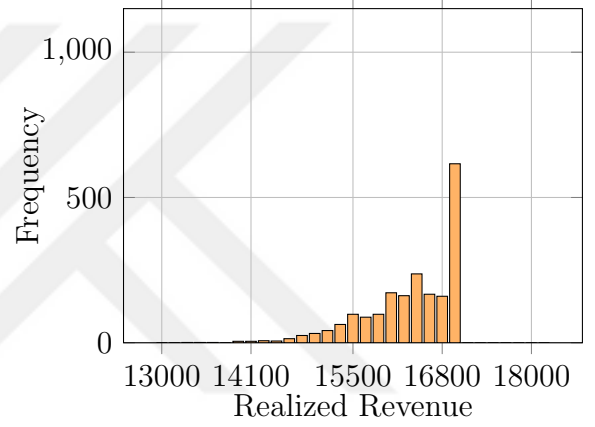
(a) Histogram for ℓ_1 -norm



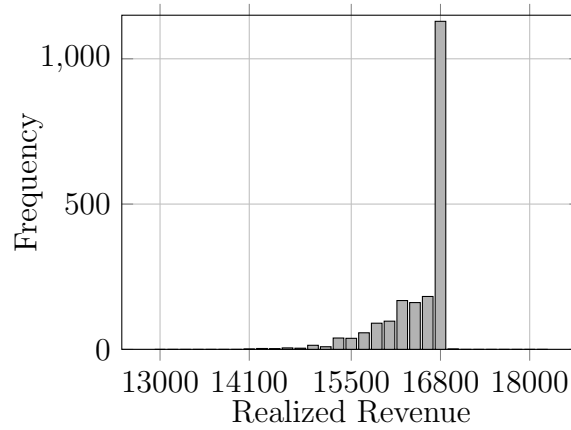
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



(d) Histogram for DLP

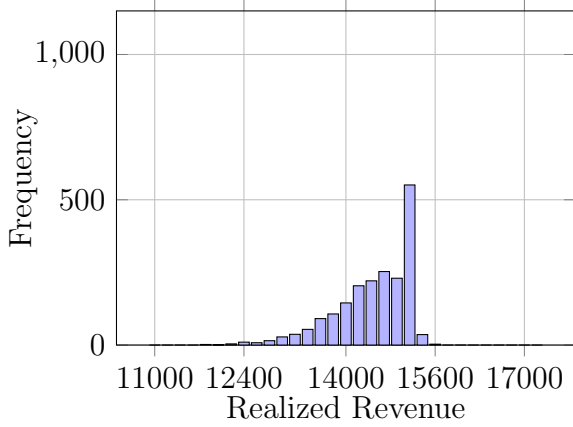


(e) Histogram for PNL

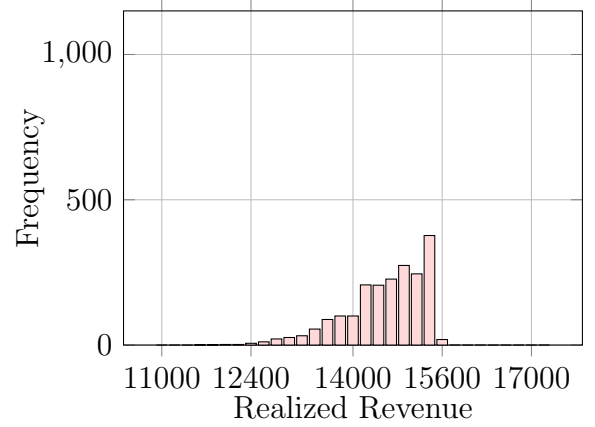
Figure A.5: Performance of different models in Example 46

In this scenario, it is evident that the best-case scenario performances of each model have decreased. However, the overall graph structure is similar to that in Example 42. The key difference is that only the Budget uncertainty set model exhibits better best-case performance, whereas in Example 42, all robust models demonstrated superior performance, indicating the arrival pattern has a more pronounced impact on the best-case performances of our models.

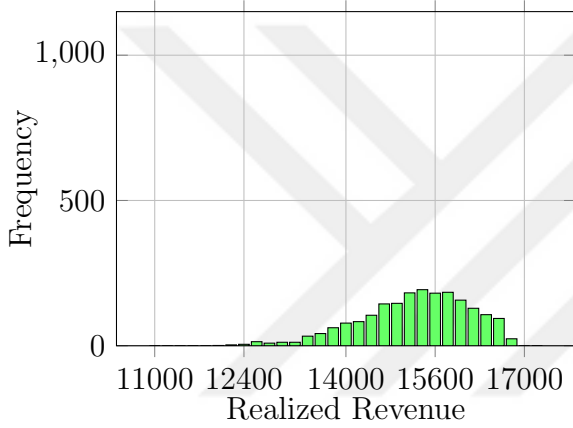
Example 47: We utilize the same parameters in Example 6 and generate histograms for high, medium and low demand factors.



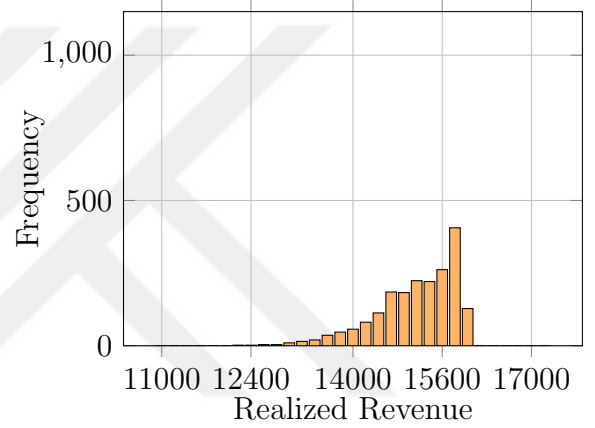
(a) Histogram for ℓ_1 -norm



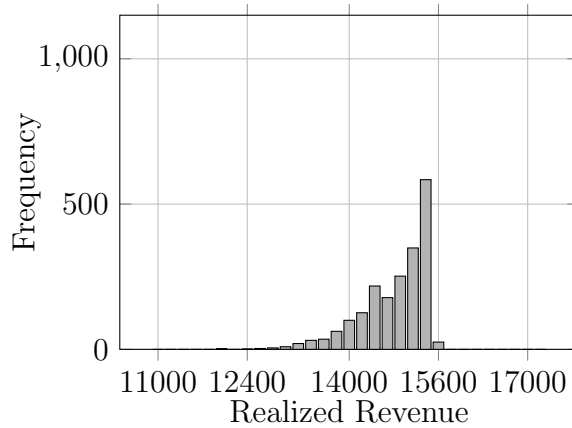
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



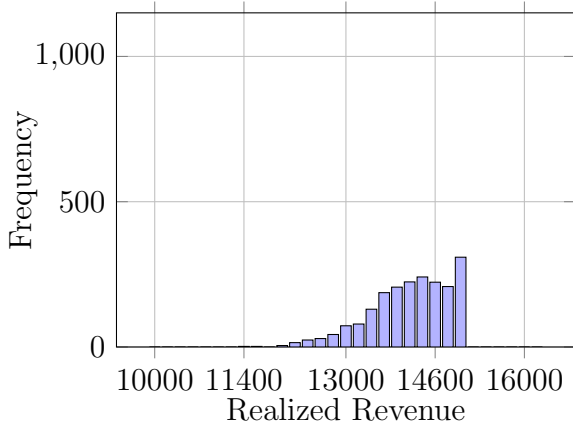
(d) Histogram for DLP



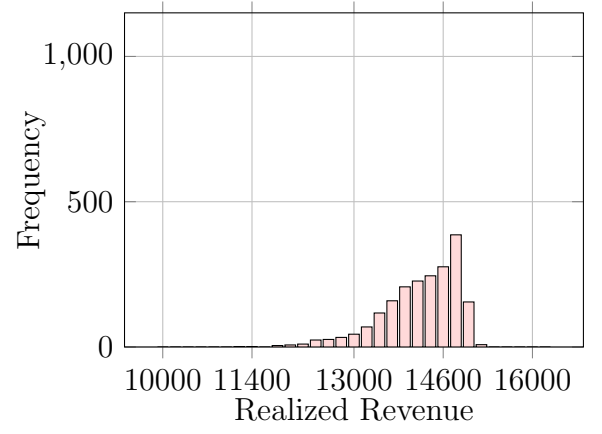
(e) Histogram for PNLP

Figure A.6: Performance of different models in Example 47 with DF of 0.88

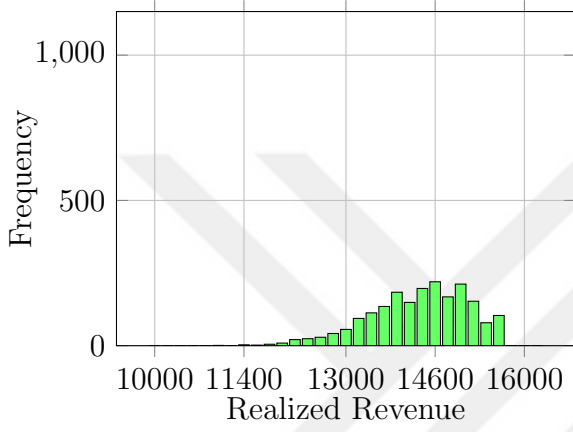
Previously, when the DF was very high, DLP performed better. However, as DF decreases, the best-case performance of DLP also diminishes. On the other hand, the best-case performance of the $\ell_2 + \ell_\infty$ model improves. The other graph structures remain similar to each other. It is important to emphasize that robust models achieve these performances without using distribution information.



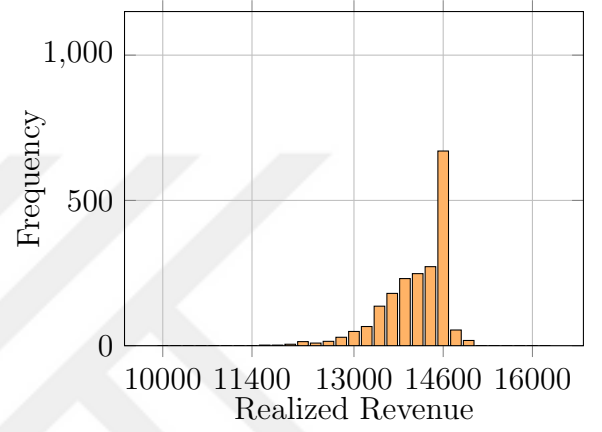
(a) Histogram for ℓ_1 -norm



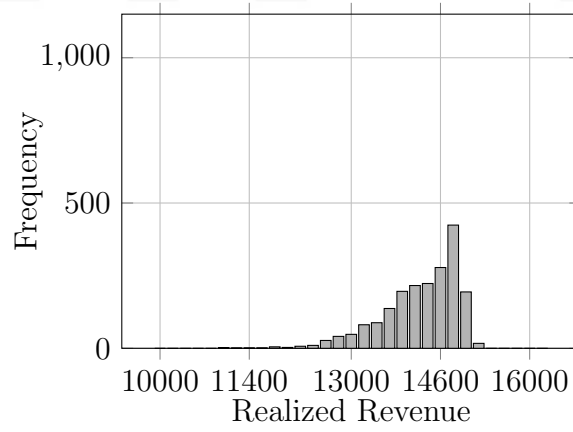
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



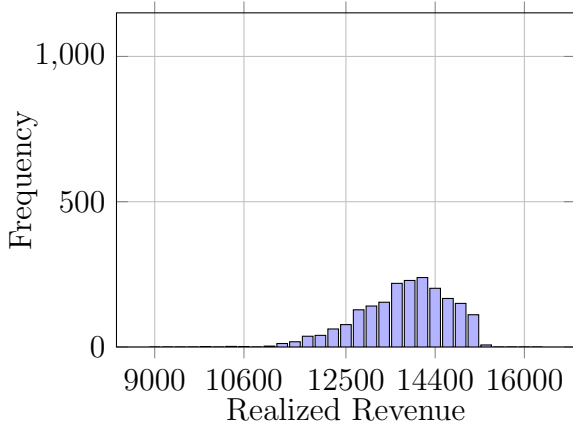
(d) Histogram for DLP



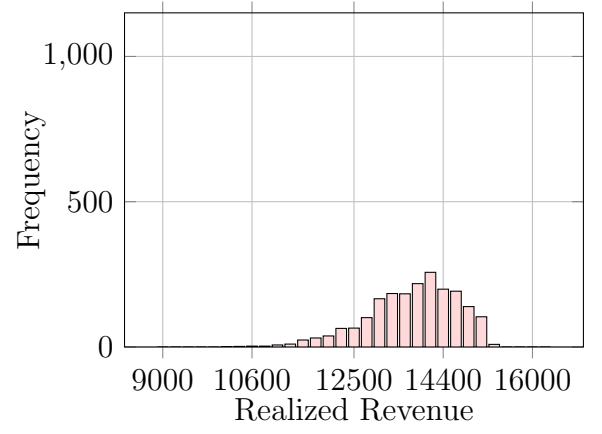
(e) Histogram for PNL P

Figure A.7: Performance of different models in Example 47 with DF of 0.83

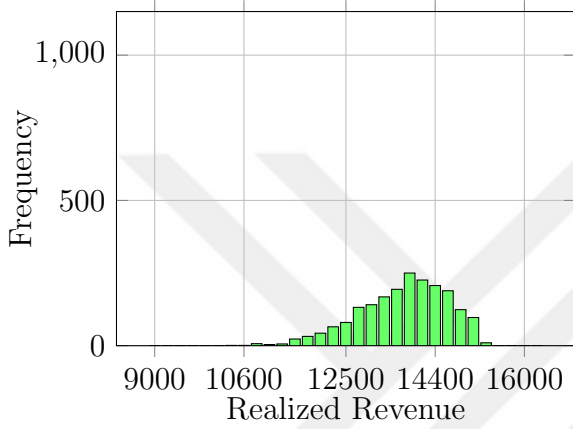
In this example, we observe the same trends as before: the performance of DLP continues to decrease, while the other models perform similarly.



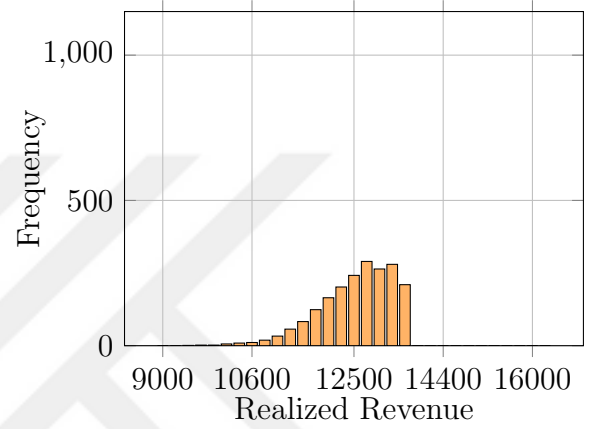
(a) Histogram for ℓ_1 -norm



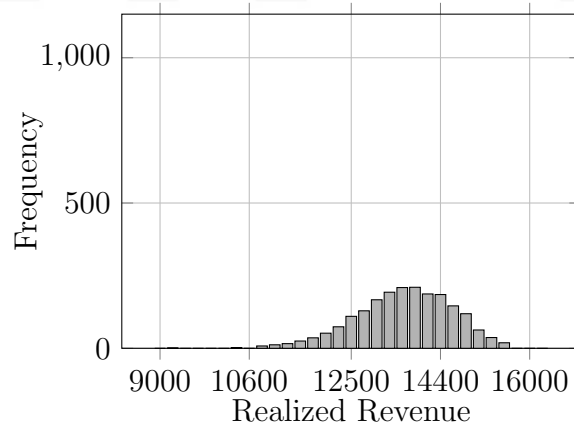
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



(d) Histogram for DLP



(e) Histogram for PNL P

Figure A.8: Performance of different models in Example 47 with DF of 0.64

In this scenario, the histograms look very similar to each other due to the low DF. With limited opportunities to generate revenue, all models perform similarly and exhibit similar graph structures.

A.2 Histograms of the Network with 3 Resources and 18 Products

Example 48: Let us restate the parameters of the network example with 3 resources and 18 products (each resource has a capacity of 200):

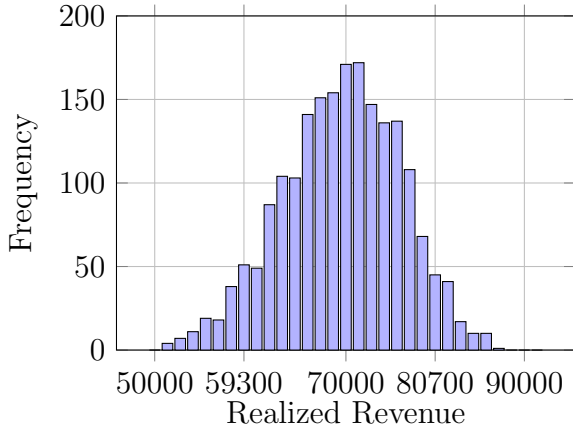
OD Number	OD	Fare class 3		Fare class 2		Fare class 1	
		Exp.	SD	Exp.	SD	Exp.	SD
1	A-B	50	9.01	40	7.75	30	18.17
2	A-C	40	7.75	25	6.12	20	14.83
3	A-D	30	6.71	24	5.66	20	14.83
4	B-C	30	6.71	20	5.48	20	14.83
5	B-D	30	6.71	20	5.16	20	9.31
6	C-D	50	9.01	40	7.75	30	13.42

Table A.2: Expectation and Standard Deviation Values for de Boer's Example

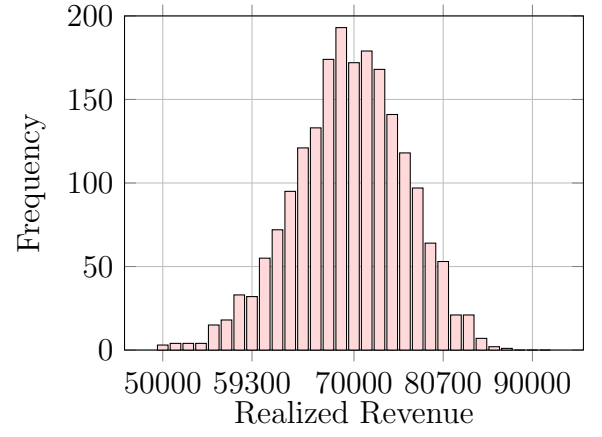
OD Number	Origin-Destination	Fare class 3	Fare class 2	Fare class 1
1	A-B	75	125	250
2	A-C	130	170	400
3	A-D	200	320	460
4	B-C	100	150	330
5	B-D	160	200	420
6	C-D	80	110	235

Table A.3: Parameters of de Boer's example

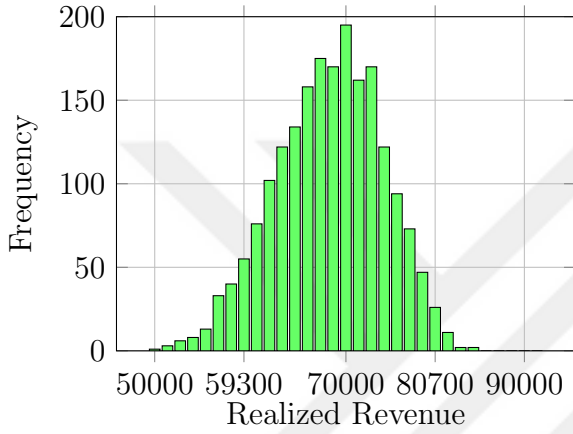
We use the parameters and obtain the histograms for example 17.



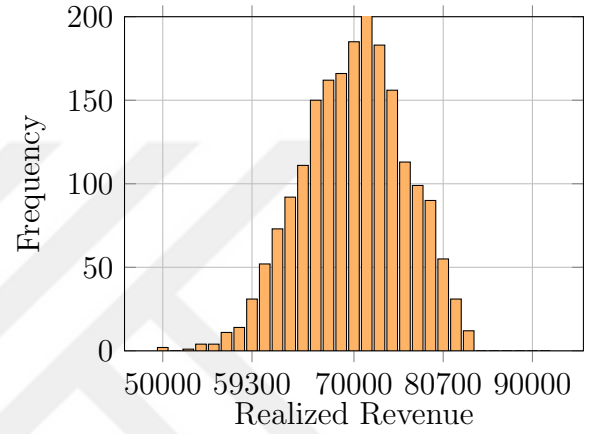
(a) Histogram for ℓ_1 -norm



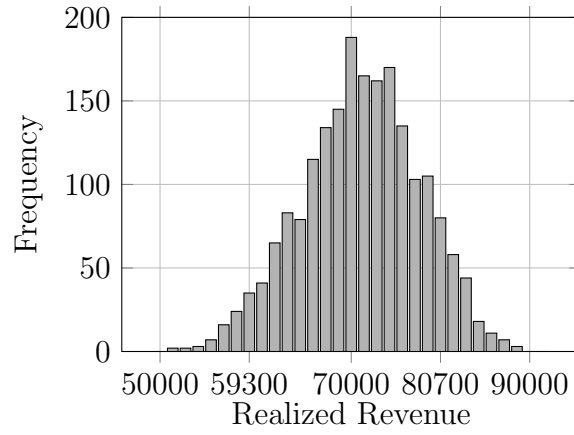
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



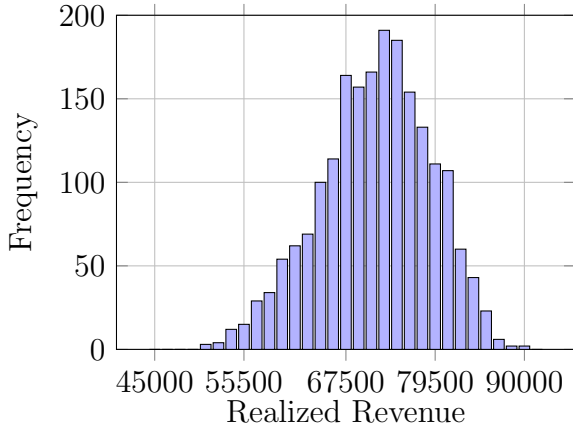
(d) Histogram for DLP



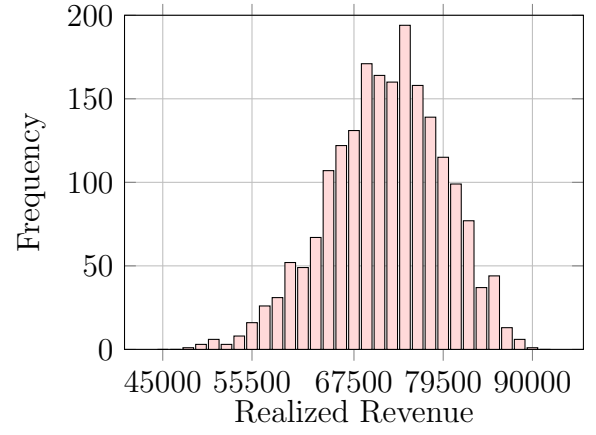
(e) Histogram for PNL

Figure A.9: Performance of different models in Example 48

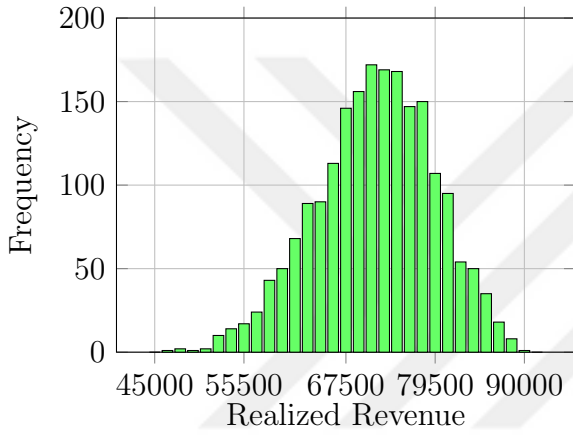
Example 49: Histograms for example 18



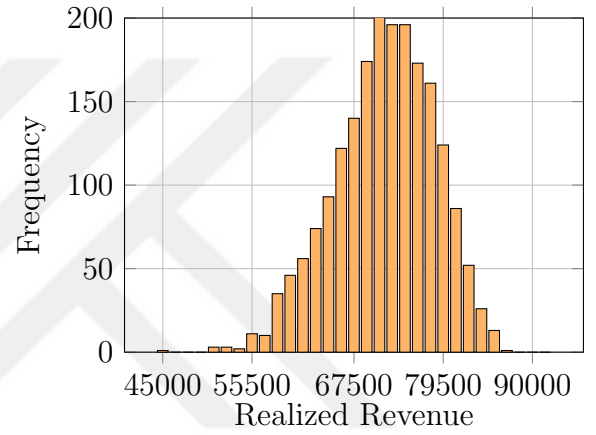
(a) Histogram for ℓ_1 -norm



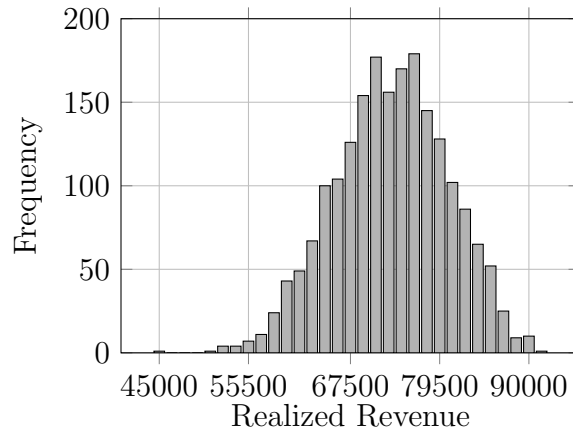
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



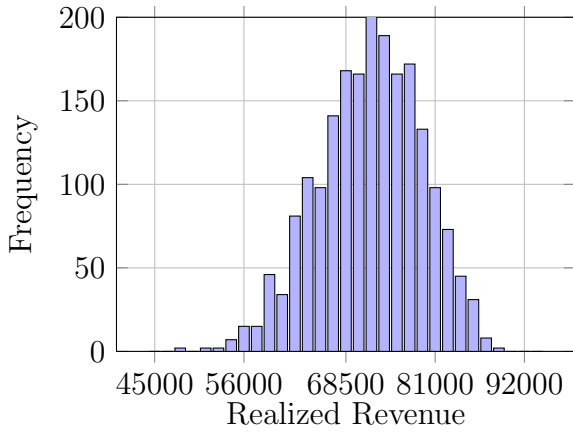
(d) Histogram for DLP



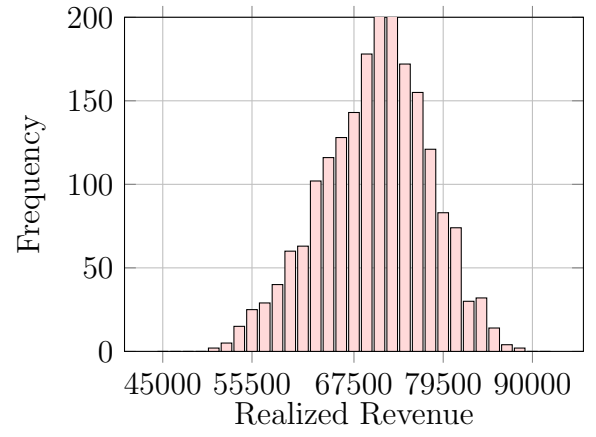
(e) Histogram for PNL

Figure A.10: Histograms of Example 43

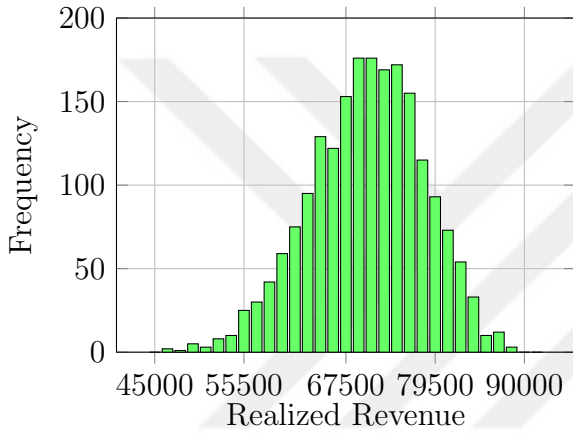
Example 50: Histogram for example 19



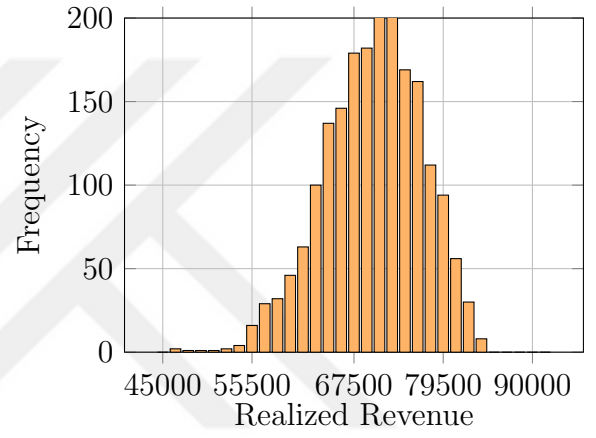
(a) Histogram for ℓ_1 -norm



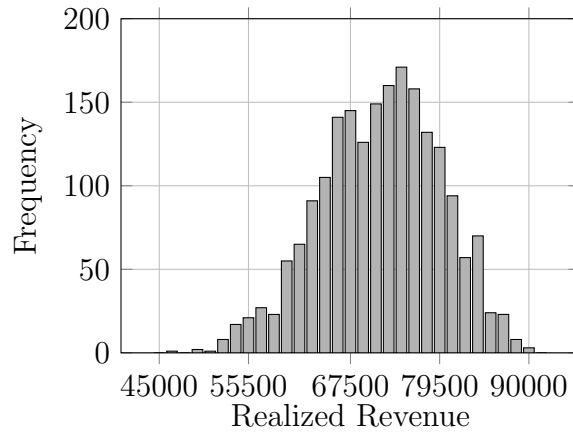
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



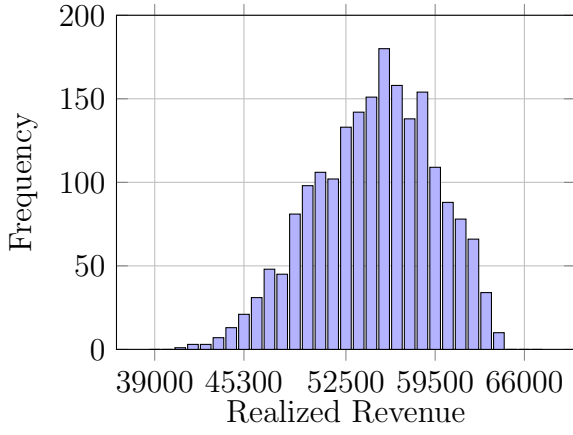
(d) Histogram for DLP



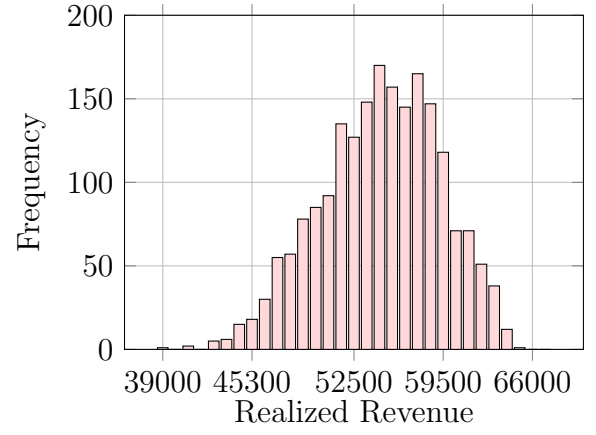
(e) Histogram for PNL

Figure A.11: Performance of different models in Example 50

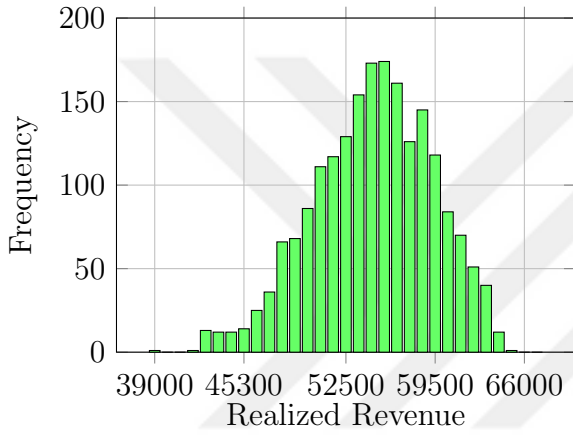
Example 51: histogram for example 20, where the arrival pattern is mixed.



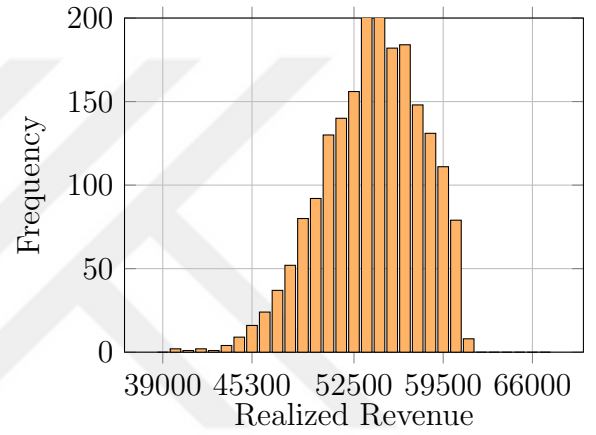
(a) Histogram for ℓ_1 -norm



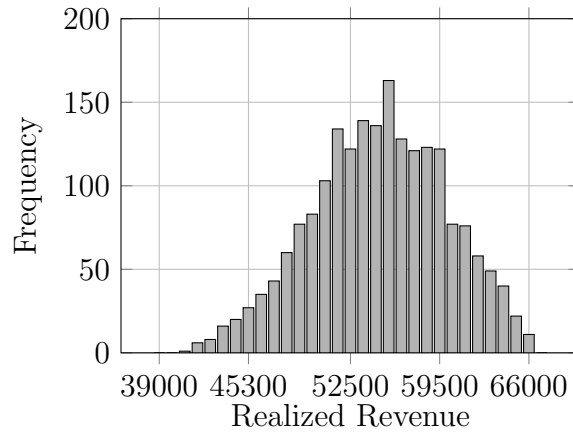
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



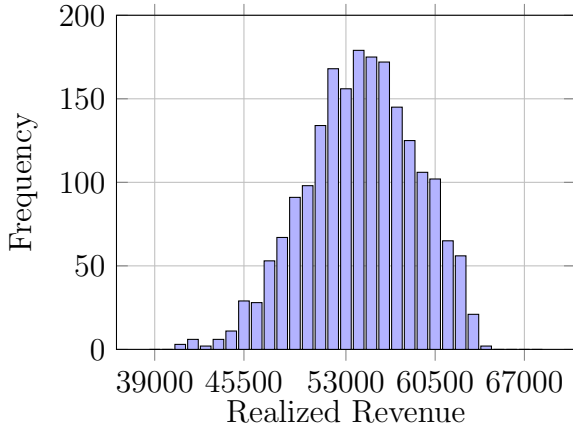
(d) Histogram for DLP



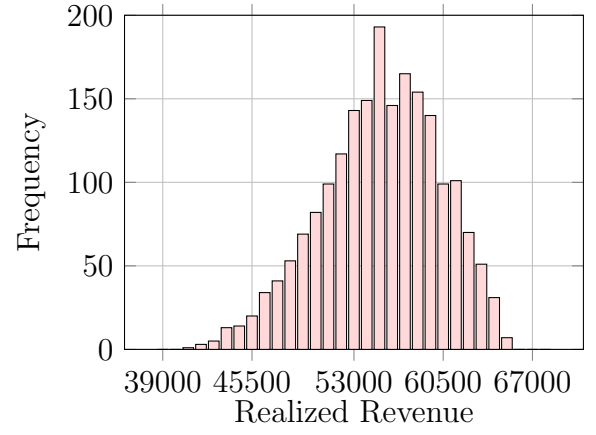
(e) Histogram for PNL

Figure A.12: Performance of different models in Example 51

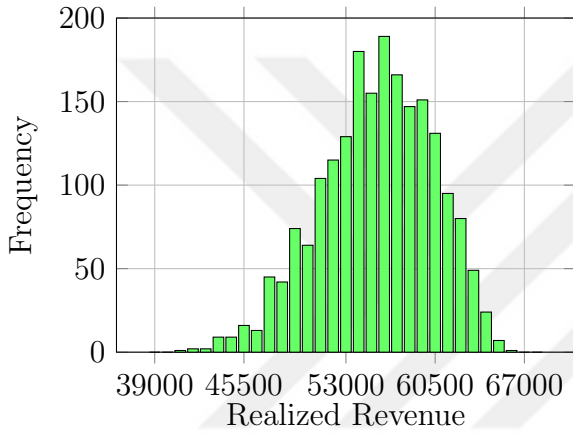
Example 52: histogram for example 21, where the arrival pattern is mixed.



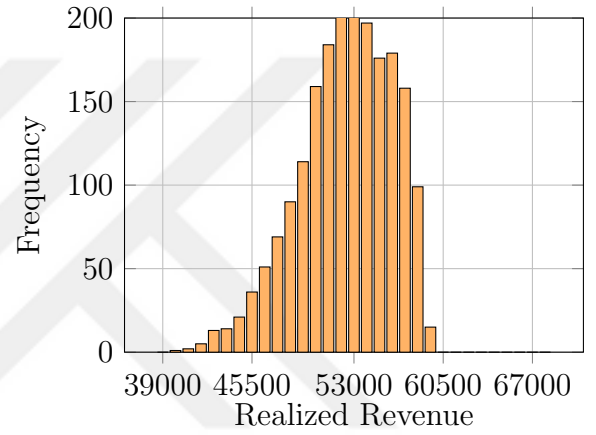
(a) Histogram for ℓ_1 -norm



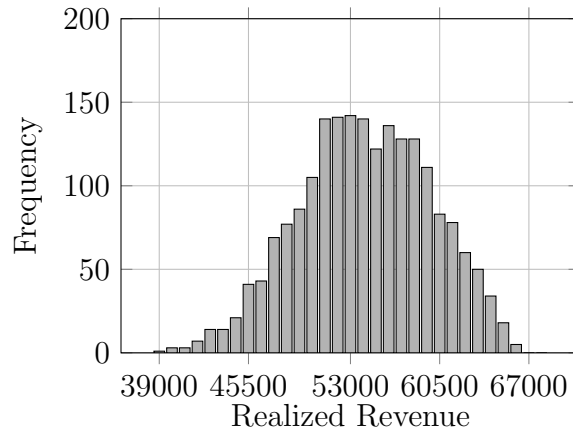
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



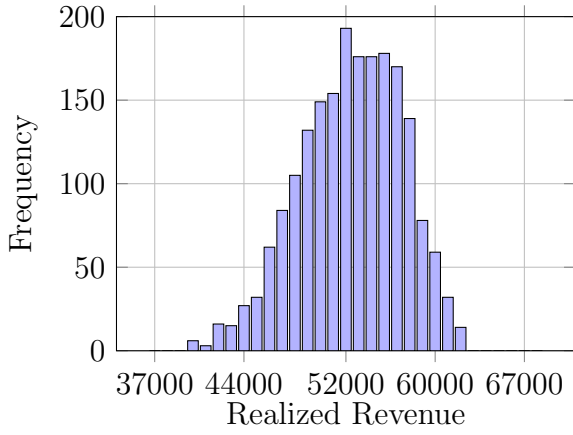
(d) Histogram for DLP



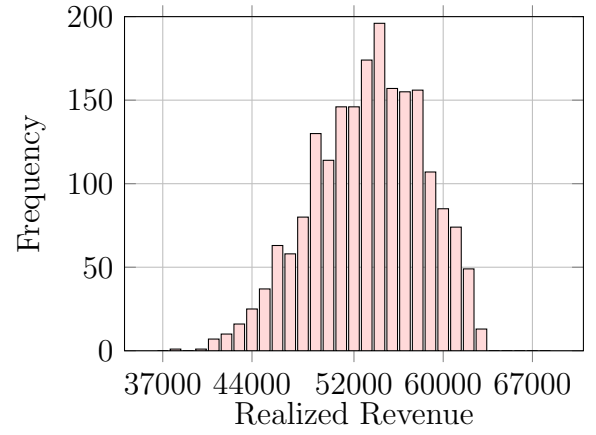
(e) Histogram for PNL

Figure A.13: Performance of different models in Example 52

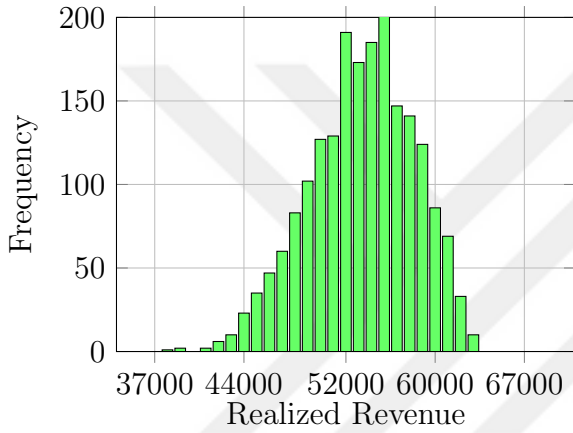
Example 53: histogram for example 22, where the arrival pattern is mixed.



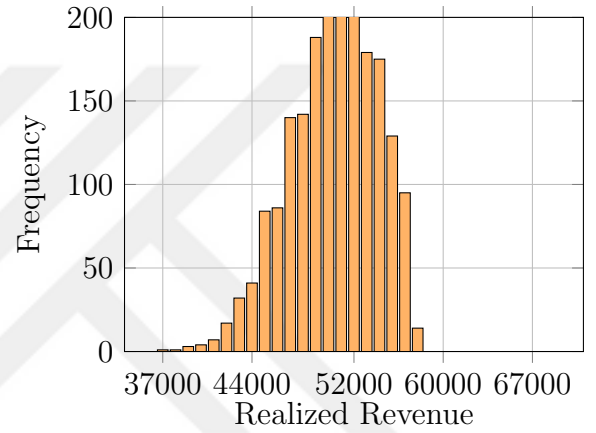
(a) Histogram for ℓ_1 -norm



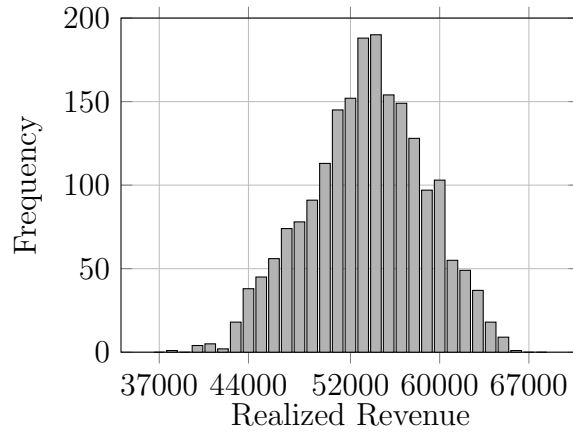
(b) Histogram for Budget Uncertainty Set



(c) Histogram for $\ell_2 + \ell_\infty$



(d) Histogram for DLP



(e) Histogram for PNL

Figure A.14: Performance of different models in Example 53

When we observe the histograms across all the examples, we notice a consistent bell-shaped structure throughout the graphs. This consistency is due to the high number of products, which provide many substitutes for one another. Consequently, differences in booking limits between

models do not significantly impact revenues. The noteworthy exception is the best-case scenario performance of DLP, which decreases when the arrival pattern changes from LBH to mixed. In figures A.12, A.13, and A.14, it can be observed that the graphs for DLP are truncated on their right tails, indicating worse best-case scenarios—a trend observed throughout the thesis. Overall, there are no significant differences in revenue distribution between the robust models, stochastic model, and deterministic model when the network is larger.

