ACOUSTIC BLACK HOLES

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

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ABSTRACT

In this thesis, we study acoustic black holes that mimic the kinematics of a black hole and investigate several phenomena such as the Hawking radiation and the super-radiance. The particular model we employ is known as the Draining Bathtub (DBT) model in the literature, defined in (3+1)-dimensions with a sink at the origin, where the effective geometry possesses two horizons, inner one corresponding to the event horizon and the outer one is the ergosphere. This feature of the model offers an opportunity to study rotating Kerr black holes, that also posses an event horizon and an ergoregion that allows an effect called the Penrose process. In the acoustic analogy the same effect can be viewed as super-resonance.

After showing the equivalence of the equation of the motion for the sound propagation in an inviscid, barotropic, irrotational fluid with the propagation of light in curved spacetime, DBT effective metric with an event horizon and ergosphere is given. Differential geometric methods are used to derive the Klein-Gordon PDE from the line equation of the model. The super-resonance effect is studied by calculating the transmission and reflection coefficients of a scattered wave through the solution of the PDE at asymptotic regions. Here the super-resonance equation is also obtained through the Frobenius method applied at the singular point, on the event horizon for the model.

ÖZETÇE

Bu tezde kara deliklerin kinematiği ile benzeşen akustik kara delikler çalışılmış, Hawking ışıması ve süper-ışıma gibi etkiler incelenmiştir. Üzerinde durduğumuz Boşalan Havuz Modeli uzayın merkezindeki bir kuyu ile tanımlanır. Modelin etkin geometrisinde, içte bir olay ufku ve dışarıdan bunu kaplayan bir ergo-küre bulunmaktadır. Bu nitelikleriyle Boşalan Havuz Modeli, benzer biçimde bir olay ufku ve ergo-küresi bulunan dönel Kerr kara deliklerin ve Penrose sürecinin incelenmesine olanak sağlar. Penrose süreci akustik kara delikler için süper-rezonans olayı diye de bilinmektedir. Tezde; öncelikle barotropik, irrotasyonel ve viskoz olmayan bir sıvı içinde yayılan ses dalgalarının sağladığı hareket denklemlerinin, (3+1)-boyutlu etkin uzay-zaman geometrisinde yazılan kütlesiz Klein-Gordon denklemi ile denkliği gösterilmiştir. Daha sonra Boşalan Havuz Modelinin etkin metriği verilerek bunun için kütlesiz Klein-Gordon denklemi çıkartılmıştır. Durağan, küresel simetrik yerleşimler için denklem indirgenmiş; Frobenius yöntemi yardımıyla, gelen bir dalga genliği cinsinden yansıyan ve saçılan dalga genlikleri hesaplanmıştır. Böylece sistem için süper-rezonans şartları belirlenmiştir.

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TABLE OF CONTENTS

List of Figures **vii**

Chapter 1: Introduction **1**

Chapter 2: Acoustic Black Holes **5**

2.1 Definition 5

2.2 Fluid Dynamics 6

Chapter 3: Draining Bathtub Model **13**

3.1 Event Horizon and Ergoregion 15

3.2 Hawking Radiation 19

3.2.1 Analogy 21

Chapter 4: Metric Calculations and Super-resonance **25**

4.1 From the Acoustic Metric to the Klein-Gordon Equation 25

4.2 Frobenius Method 29

4.3 Super-resonance 31

Chapter 5: Conclusion **38**

Bibliography **40**

LIST OF FIGURES

3.1 A schematic of the Draining Bathtub model in cylindrical coordinates, outer (red) circle represent the ergosphere, inner circle (black) is the event horizon. Arrow paths are the trajectories of the flow. 14

3.2 Draining Bathtub Model with a sink (singularity). 14

Chapter 1

INTRODUCTION

The existence of black holes is one of the great predictions of general relativity. After the death of a massive star, with minimal mass of $\sim 3M_{\odot}$, ($M_{\odot}$ is the solar mass), the gravitational force overcomes repulsive forces such as pressure, and cause the star to collapse into a black hole, that is a region forms in space where the gravity is so intense that not even light can escape. Stellar-mass black holes are expected to populate each galaxy and indeed a large number of candidates of such black holes have been reported by astronomical observations. The second process by which black holes may be formed will be the gravitational collapse of the entire central core of a dense cluster of stars. The nuclei of galaxies provide most likely sites for the formation of this type of massive black holes, for which the mass range goes up to $\sim 10^{10}M_{\odot}$. By the no-hair theorem, stable black holes can be characterized by their mass, angular momentum and the net electric charge [1]. According to their electric charge and angular momentum, black holes may be divided into four categories: Schwarzschild black holes, Kerr black holes, Reissner-Nordström black holes and Kerr-Newman black holes. Electrically neutral Schwarzschild and rotating Kerr black holes are the two categories that we address throughout this study. Stationary Schwarzschild black holes with no electric charge will be the simplest case, while the uncharged Kerr black holes carry angular momentum only [1].

General Relativity (GR) describes the universe at large scales, thus providing a relativistic theory of gravitation, while the Standard Model (SM) of Electroweak and

Strong Interactions is the theoretical framework for quantum field theoretic models of subatomic particles, where gravity does not play any role. Attempts to combine GR and SM have not been successfully accomplished so far. However, quantum field theory in curved spacetimes or the so called semiclassical approximation, allows us to work in areas where matter fields are quantized and gravity is treated at the classical level.

Hawking's semiclassical approach to black holes in 1974 uncovered a great achievement that now is called the "Hawking effect". By applying the techniques of quantum field theory in curved spacetimes to the formation of a Schwarzschild black hole by gravitational collapse, Hawking proved that black holes should emit particles with a characteristic temperature. The experimental evidence for the Hawking radiation by black holes would be a great accomplishment since its confirmation means confirming the semiclassical approximation and may take us several steps closer to an understanding of quantum gravity. Unfortunately, the Hawking radiation from a black hole with the size of one solar mass ($\sim 10^{-7}K$) is completely suppressed by the cosmic microwave background radiation ($\sim 2.7K$), making its astrophysical observation nearly impossible. Moreover, the derivation of Hawking radiation faces a difficulty called the trans-Planckian problem[1].

Problems with the derivation and the setback to actual observation of the Hawking radiation led people to search alternative ways by way of analogies. The analogy initiated by Unruh's calculations showed the equivalence of velocity perturbations of a perfect barotropic, irrotational Newtonian fluid with respect to a background solution with the Klein-Gordon field propagating in a 4-dimensional pseudo-Riemannian manifold, in which the speed of sound plays the role of speed of light.

[1] Tracing back a quanta backwards from infinity where it is detected, it is shown that near the horizon, quanta should attain an arbitrarily large frequencies, well beyond the Planck scale, with respect to a local free fall observer. So that, at such frequencies physics may change and affect the process beyond our understanding. This uncertainty is called the trans-Planckian problem.

Unruh [2] considered a hydrodynamic analogy to create sonic horizons that will be equivalent to the event horizon in black holes and hope to gain some insight on the Hawking effect or any other effects that will result from background curvature rather than the dynamical features of black holes. Besides the event horizon, one other feature that black holes may display is the ergoregion: the region outside of the event horizon for rotating black holes where the particles are forced to co-rotate with the black hole and could gain negative energies. The presence of such negative energies makes it possible in principle to extract energy from rotating black holes through a process proposed by Penrose [3], (whose acoustic wave analogue was proposed earlier by Zel'dovich [4]) known as super-resonance.

The possibility of studying the kinematics of black holes in the laboratory through such analogies opened up a wide research area, where several proposals for analogue black holes have been proposed. These proposals are mainly based on the variations of the simple idea of a moving medium and the co-propagating waves that are dragged by it. Solid state models are considered by Reznik [5], [6], where electromagnetic waves are used to create an effective horizon at a surface with singular electric permittivity and magnetic permeability with specifically chosen initial state of the field. Optical waveguide models were also considered by Leonhardt and Piwnicki[7], where an optically active medium moves faster than the speed of light traveling in the medium, keeping in mind that the medium slows down the speed of light in vacuum c by the refractive index n^{-1}. Quantum liquids or Bose-Einstein condensates or water waves are other examples of analog black holes that have been considered to gain insight into the black holes.

This thesis is my personal effort to understand acoustic black holes and study important features of black holes such as horizons, super-resonance, and Hawking radiation in analogue models.

Chapter 2 is devoted to the mathematical demonstration of the equivalence between the sound waves propagating in a fluid medium to the light propagating in a curved spacetime. The massless Klein-Gordon equation is derived in this chapter.

Draining Bathtub model introduced by Visser is studied in Chapter 3. Two important features of the black holes, namely the event horizon and the ergoregion are determined for the model and the Hawking radiation is analyzed.

In Chapter 4, I used differential geometric techniques based on Hodge duals and wedge products to transform the line equation for the Draining Bathtub model to a partial differential equation (PDE). Then using the Frobenius method, I calculated the singularities of the indicial equation of that PDE and reached a super-resonance condition. I also obtained the same result by analyzing the PDE at the asymptotic limits.

A few concluding remarks are given in Chapter 5

Chapter 2

ACOUSTIC BLACK HOLES

2.1 Definition

The lack of experimental opportunities in the study of Black Holes has been an unfortunate setback. In 1981 Unruh presented a model to make black holes more accessible [2]. What started with Unruh was an attempt to find experimental areas and hopefully to test the theories of black holes, opening a promising research area. Basic idea underlying this method was explained by Unruh using a story of a blind fish that is also a physicist living in a river, where there is also a drastic waterfall. At some point in the waterfall, the water velocity over the waterfall exceeds the velocity of the sound waves traveling in the water. If a blind fish is falling from the waterfall and screaming help after passing that particular point, its sound will not be able to get to other side to an outside observer. Sound will travel with the same speed inside the water relative to the falling fish, but the flowing water will not let the sound propagate any further than that one particular point, thus creating an equivalent of an apparent horizon for sound waves. This analogy by Unruh, started as a way to explain analogue black holes turned out to be more than a simple explanation [8]. By looking at the equation of propagation of sound waves in moving fluids, we can see the analogy to the propagation of scalar fields in general relativity given by the Klein-Gordon equation satisfied by a massless scalar field

$$\frac{1}{\sqrt{-g}}\sum_{i,j}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{ij}\frac{\partial}{\partial x^j}\phi\right)=0. \tag{2.1}$$

Here ϕ is the potential for small perturbations, the metric components $g_{ij} = \rho^2 (\delta_{ij} - \upsilon^i \upsilon^j)$, ρ is the density of the fluid and υ^i are the components of the background fluid flow velocity vector.

This mathematical equivalence, shown in detail in section 2.2, would help to illustrate some of the features of black holes and may provide a chance to study them experimentally.

Before beginning to analyze the equivalence it is important to note that acoustic metric determines the fluid dynamics and need not satisfy the Einstein's field equations. Analogue black holes provide an effective metric and capture the kinematic of black holes, which is related to how the fields move in curved spacetime. Therefore; the analogy can not tell us anything about the dynamical aspects of black holes [9].

There are shortcomings and constraints in the analogy which we will examine later, but the opportunity to observe the kinematics of a black hole in the laboratory allows the examination of essential features such as horizons [10].

2.2 Fluid Dynamics

Under the right conditions, the equation describing the propagation of a massless scalar field in curved (3+1)-dimensional Lorentzian spacetime is equivalent to the equation describing acoustic fluctuations of the velocity vector field of a fluid. The conditions imposed on the fluid medium can be formulated as follows: The fluid should be

(i) locally irrotational; $\vec{\upsilon}$ is defined as gradient of a velocity potential $\vec{\upsilon} = -\vec{\nabla}\psi$,

(ii) barotropic; barotropic equation of state is satisfied and ρ is a function of pressure only,

(iii) inviscid; external forces $\vec{F}$ are gradient-derived from a potential $\vec{F} = -\rho\vec{\nabla}\Phi$.

To state the analogy we start with three main equations; namely the continuity equation, the zero-viscosity Euler equation and the barotropic equation [11]:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{2.2}$$

$$\rho \left[\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = -\vec{\nabla} p - \vec{F}, \tag{2.3}$$

$$p = p(\rho), \tag{2.4}$$

where p is the pressure, ρ is the density field and $\vec{F}$ shows the net external force.

Now we can define the enthalpy h, which depends on the pressure only as

$$\vec{\nabla} h = \frac{\vec{\nabla} p}{\rho}. \tag{2.5}$$

If we consider the Euler equation (2.3), substituting the identity $\frac{1}{2}\vec{\nabla}\,(\vec{v} \cdot \vec{v}) = \vec{v} \times (\vec{\nabla} \times \vec{v}) + (\vec{v} \cdot \vec{\nabla})\vec{v}$, with the assumptions we had, the equation becomes

$$\partial_t \vec{v} = \frac{-\vec{\nabla} p}{\rho} - \vec{\nabla} \Phi - \frac{\vec{\nabla} v^2}{2} + \vec{v} \times (\vec{\nabla} \times \vec{v}), \tag{2.6}$$

or in terms of potentials

$$-\partial_t \vec{\nabla} \psi - \vec{\nabla} \psi \times (\vec{\nabla} \times \vec{\nabla} \psi) + \vec{\nabla} h + \vec{\nabla} (\frac{(\vec{\nabla} \psi)^2}{2} + \Phi) = 0. \tag{2.7}$$

Finally integrating this over a closed surface will gives us a Bernoulli type of equation [11]:

$$-\partial_t \psi + h + (\frac{(\vec{\nabla} \psi)^2}{2} + \Phi) = 0. \tag{2.8}$$

Now we introduce small (first order in ϵ) perturbations in ρ, p and ψ and discard terms with ϵ^2 and higher, upon substituting in the equations above:

$$\rho = \rho_0 + \varepsilon\rho_1, \tag{2.9}$$

$$p = p_o + \varepsilon p_1, \tag{2.10}$$

$$\psi = \psi_0 + \varepsilon\psi_1. \tag{2.11}$$

Now, linearize the enthalpy and use Taylor expansion

$$\begin{aligned} h = h_0 + \frac{\partial h}{\partial p}(p - p_0) = h_0 + \frac{1}{r_0}(p - p_0), \\ h = h_0 + \varepsilon\frac{p_1}{\rho_0}; \end{aligned} \tag{2.12}$$

and put all into the Euler equation (2.7):

$$-\partial_t(\psi_0 + \varepsilon\psi_1) + h_0 + \varepsilon\frac{p_1}{\rho_0} + (\frac{(\vec{\nabla}\psi_0 + \varepsilon\psi_1)^2}{2} + \Phi) = 0. \tag{2.13}$$

The solution to this equation gives two distinct equations, one for zeroth order in ε and one for first order in ε;

$$-\partial_t\psi_0 + h_0 + (\frac{(\vec{\nabla}\psi_0)^2}{2} + \Phi) = 0, \tag{2.14}$$

$$-\partial_t\psi_1 + \frac{p_1}{\rho_0} + \vec{\nabla}\psi_1 \cdot \vec{\nabla}\psi_0 = -\partial_t\psi_1 + \frac{p_1}{\rho_0} - \vec{\nabla}\psi_1 \cdot \vec{v}_0 = 0. \tag{2.15}$$

The first equation is the unperturbed Euler equation and from the second one we get

$$p_1 = \rho_0(\vec{\nabla}\psi_1 \cdot \vec{v}_0 + \partial_t\psi_1). \tag{2.16}$$

Now, using the barotropic feature of the fluid, we apply the Taylor expansion to the density function around the pressure p_0;

$$\rho = \rho_0 + \varepsilon\rho_1,$$
$$\rho = \rho(p_0) + \frac{d\rho}{dp}(p - p_0), \tag{2.17}$$

and inserting p_1 into the Taylor expansion of the density, we obtain

$$\rho_1 = \frac{\partial\rho}{\partial p}\rho_0(\vec{\nabla}\psi_1 \cdot \vec{v}_0 + \partial_t\psi_1). \tag{2.18}$$

Now apply the perturbation to continuity equation (2.2):

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0,$$
$$\frac{\partial(\rho_0 + \varepsilon\rho_1)}{\partial t} + \vec{\nabla}\cdot((\rho_0 + \varepsilon\rho_1)(\vec{v}_o + \varepsilon\vec{v}_1)) = 0, \tag{2.19}$$

discarding the terms with ϵ^2 and higher, we get two equations;

$$\frac{\partial\rho_0}{\partial t} + \vec{\nabla}\cdot(\rho_0\vec{v}_0) = 0, \tag{2.20}$$
$$\frac{\partial\rho_1}{\partial t} + \vec{\nabla}\cdot(\rho_1\vec{v}_0 + \rho_0\vec{v}_1) = 0. \tag{2.21}$$

Substituting ρ_1 from (2.18) into the last equation we obtain

$$\partial_t\left(\frac{\partial\rho}{\partial p}\rho_0(\vec{\nabla}\psi_1 \cdot \vec{v}_0 + \partial_t\psi_1)\right) + \vec{\nabla}\cdot\left(\frac{\partial\rho}{\partial p}\rho_0(\vec{\nabla}\psi_1 \cdot \vec{v}_0 + \partial_t\psi_1)\vec{v}_0 + \rho_0\vec{v}_1\right) = 0. \tag{2.22}$$

Knowing the speed of sound $\frac{1}{c} = \sqrt{\frac{d\rho}{dp}}$, this equation resembles the Klein-Gordon equation for a massless scalar field

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi_1\right) = 0, \tag{2.23}$$

where g is the determinant of the inverse metric $g^{\mu\nu}$; $g = \det(g_{\mu\nu})$. From the relation between the equations (2.22) and (2.23), we can finally calculate the corresponding effective spacetime metric which governs the propagation of sound waves in the fluid:

$$g^{\mu\upsilon} \equiv \frac{1}{c\rho_0} \begin{pmatrix} -1 & -\upsilon_0^j \\ -\upsilon_0^i & (c^2\delta^{ij} - \upsilon_0^i\upsilon_0^j) \end{pmatrix}. \tag{2.24}$$

Inverting this result we have

$$g_{\mu\nu} \equiv \frac{\rho_0}{c} \begin{pmatrix} -(c^2 - \upsilon_0^2) & -\upsilon_0^j \\ -\upsilon_0^i & \delta_{ij} \end{pmatrix}, \tag{2.25}$$

where the determinant of the metric

$$-g = \frac{\rho_0}{c^2} > 0. \tag{2.26}$$

One of the important properties of the effective metric is that although the fluid is non-relativistic and is given in a flat Minkowski background, that is in the 3+1 real vector space setting with signature (-,+,+,+), the effective metric is Lorentzian but with non-zero curvature [12].

Now we can see that the propagation of sound waves in the fluid is equivalent to the propagation of a scalar field in a curved spacetime which is described by the Klein-Gordon equation. As a result, studying the propagation of sound waves in such fluids offers a chance to study different phenomena generic in the Lorentzian spacetimes.

Finally, we give the line element for the analogy in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\rho_0}{c}\left[-c^2dt^2 + (dx^i - \upsilon_0^i dt)\delta_{ij}(dx^j - \upsilon_0^j dt)\right], \tag{2.27}$$

where $x^{\mu,\nu} \equiv (t, x^i)$ and μ, ν run from 0 to 3 while i, j run from 1 to 3.

We should stress that the three assumptions (irrotational, inviscid and barotropic fluid) leading to this analogous formulation are of critical importance. Any failure in the barotropic condition will lead to a failure in the irrotational condition over time. If we further assume that fluid has some rotation, we can not define $\vec{v}$. The definition of the analogue effective metric relies on the velocity field being irrotational whose failure would be highly problematic. The introduction of a new scalar field with rotation, also satisfying the conditions for the analogy is also a very difficult task to accomplish [13].

The second assumption we made during the calculation was that the density and the velocity of the fluid are constant classical fields. Fluid equation of motion is not applicable at high frequencies or short wavelengths. Sound waves do not exist at wavelengths shorter than the interatomic spacing. This implies that the length scales of the system should be long enough so that the microscopic features of the fluid can be safely ignored. As a result, this analogy can not investigate short scale behavior.

The third assumption we mention is regarding the fluctuations. To derive the analogy all fluctuations are considered to be very small, considering the potential problems that could arise from the back reaction. For black holes the back reaction refers to quantum effects of the matter fields on spacetime structure and dynamics, whereas in acoustic analogy, the back reaction is caused by the non-linearity of the hydrodynamical equations, that is linear perturbations will backreact on the flow [14].

The effective metric for the acoustic black hole analogy is calculated from the background density and the evolution of the background caused by the continuity equation (2.2) and zero viscosity Euler equation (2.3). The metric describing the black holes evolves from the Einsteins field equations. Therefore this big difference in the evolution of the backgrounds only allows the analogy to map the propagation of sound waves on a fixed background fluid flow to a fixed background space time. We can not link the background behavior of the fluid to the background behavior of a spacetime.

As mentioned before, the analogy refers only to the kinematic aspects of black holes. But it is important to know that analogue models of gravity can demonstrate the quantum physics of the black holes as well as the classical physics such as trapping of waves or positive and negative norm mixing at the horizon which correspond to non-trivial Bogoliubov coefficients in the calculations of Hawking radiation [15], [16]. This part would be examined in detail in section 3.2.

Chapter 3

DRAINING BATHTUB MODEL

A spacetime geometry, having two distinct horizons and one ergoregion in between is very important for the analogy to mimic the actual Kerr black holes. One of the methods that gives two horizons, where the ergoregion is containing the event horizon is the draining bathtub model that was first introduced by Visser [9]. Visser's model is defined by a 2-dimensional fluid flow over a plane with a sink (or a source) at the origin. It is possible to immerse this model in three dimensions by adding a third dimension that is perpendicular to the plane of the fluid flow. As mentioned before, we are studying fluids that are locally irrotational, barotropic and inviscid. From the continuity equation (2.2), for a time-independent density function ρ, we obtain the condition that the radial component of the velocity vector field should satisfy,

$$\rho v_{\hat{r}} \sim \frac{1}{r}. \tag{3.1}$$

Furthermore the vorticity free condition $\vec{\nabla} \times \vec{v} = 0$ gives us the condition for the tangential component,

$$v_{\hat{\phi}} \sim \frac{1}{r}. \tag{3.2}$$

If we consider the conservation of angular momentum $L = V \rho v_{\hat{\phi}} r$ where V is the volume of the draining bathtub taken as constant, we obtain another condition for the tangential component, $\rho v_{\hat{\phi}} \sim \frac{1}{r}$. Comparing these two, we see that density ρ should also be constant.

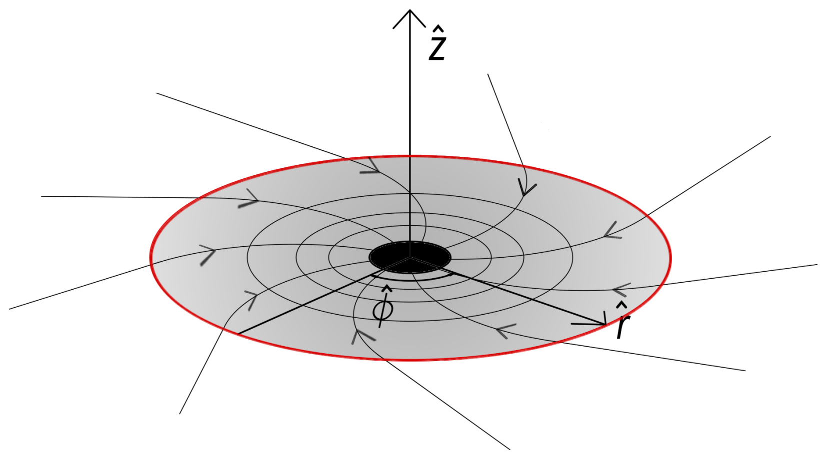

Figure 3.1: A schematic of the Draining Bathtub model in cylindrical coordinates, outer (red) circle represent the ergosphere, inner circle (black) is the event horizon. Arrow paths are the trajectories of the flow.

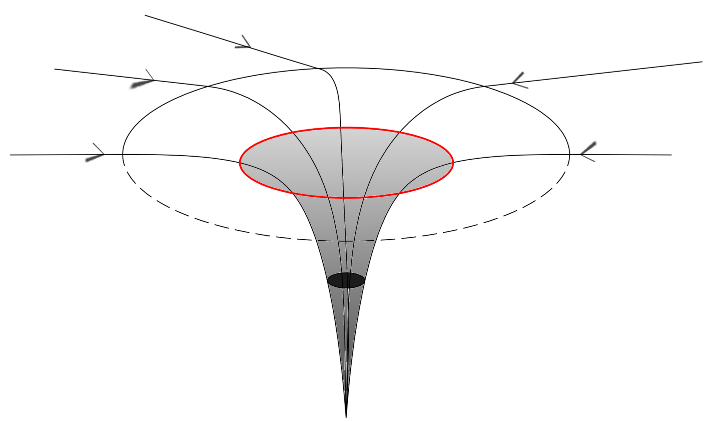

Figure 3.2: Draining Bathtub Model with a sink (singularity).

So, we have the velocity and the velocity potential given by

$$\vec{v} = v_{\hat{\phi}} + v_{\hat{r}} = \frac{-ca\hat{r} + \Omega a^2 \hat{\phi}}{r} = -\vec{\nabla}\psi, \tag{3.3}$$

$$\psi(r, \phi) = -calog(r/a) + \Omega a^2 \phi, \tag{3.4}$$

where a is a length scale associated with radius of the vortex horizon and Ω is the constant rotation frequency.

It is clear from the above formula that velocity potential is not a continuous function. For every rotation of 2π radians we have a discontinuity. That is why velocity potential is defined patch-wise on overlapping regions with a vortex core at r=0 [17]. Consequently, the line element for the (3+1) draining bathtub model should be taken as (rather than (2.27))

$$ds^2 = -\left(\frac{c^2r^2 - a^2c^2 - a^4\Omega^2}{r^2}\right)dt^2 + \frac{2ca}{r}dtdr - 2\Omega a^2 dtd\phi + dr^2 + r^2d\phi^2 + dz^2. \quad (3.5)$$

3.1 Event Horizon and Ergoregion

One of the essential features of black holes is event horizon. In general relativity it is defined as a surface where trajectories of all particles, that are moving in time, are pointing inwards, away from the boundary. Thus, in the other side of the horizon, there is no future time for any particle to exist. In other words it can be defined as point of no return, an outer trap for light.

Fluid analogy of the event horizon is a sonic horizon which works under the same principle but for the sound waves. At the surface now, the fluid velocity is pointing inwards and the magnitude of the velocity normal to the surface is equal to the speed of sound [9]. Inside the horizon normal velocity is increasing in the direction of the flow, causing the radial component of the fluid to exceed the local speed of sound. In particular, the surface is behaving as an outer trap in acoustic geometry. Since the propagation speed is required to be greater than the velocity of the fluid or local speed of sound, sound waves cannot overcome the fluid velocity and would be trapped inside [13]. It is an important opportunity that this effect might be created in laboratory.

If we express the line equation we postulated for the draining bathtub model (3.5), with a new set of coordinates,

$$dt = d\tau + \frac{ardr}{cr^2 - ca^2}, \tag{3.6}$$

$$d\phi = d\varphi + \frac{\Omega a^2 adr}{r\left(cr^2 - ca^2\right)}, \tag{3.7}$$

then

$$ds^2 = -\left(c^2 - \frac{a^2c^2 + a^4\Omega^2}{r^2}\right)\left(d\tau + \frac{ardr}{cr^2 - ca^2}\right)^2 + \frac{2ca}{r}\left(d\tau + \frac{ardr}{cr^2 - ca^2}\right)dr$$

$$- 2\Omega a^2\left(dt\tau + \frac{ardr}{cr^2 - ca^2}\right)\left(d\varphi + \frac{\Omega a^2 adr}{r\left(cr^2 - ca^2\right)}\right) + dr^2 + r^2\left(d\varphi + \frac{\Omega a^2 adr}{r\left(cr^2 - ca^2\right)}\right)^2 + dz^2.$$

Simplification of this equation leads to

$$ds^2 = -\left(1 - \frac{(c^2a^2 + \Omega^2 a^4)}{c^2r^2}\right)d\tau^2 + \left(1 - \frac{a^2}{r^2}\right)^{-1} dr^2 - \frac{2\Omega a^2 d\varphi d\tau}{c} + r^2 d\varphi^2 + dz^2. \tag{3.8}$$

The final form of the line element above makes the location of the event horizon and the ergosphere manifest. By definition, event horizon is calculated from the coefficient of g_{rr} as

$$r_H = a. \tag{3.9}$$

Since we defined a as the radius of the vortex horizon, this result was expected.

The other important feature of a black hole that we can create in the lab is called the ergoregion. The boundary of the ergoregion is called the ergosphere. If we consider the time translation Killing vector

$$K^\mu \equiv (\partial/\partial t)^\mu = (1, 0, 0, 0)^\mu, \tag{3.10}$$

we have for our metric

$$g_{\mu\nu}(\partial/\partial t)^{\mu}(\partial/\partial t)^{\nu} = g_{tt} = -\left[c^2 - v^2\right]. \tag{3.11}$$

When $\|\vec{v}\| > c$, this quantity changes sign, so that any region with a supersonic flow is an ergoregion. In general relativity ergoregion corresponds to the region surrounding any spinning black hole, where spatial points move with superluminal velocity relative to a fixed star. Analogue of the ergoregion in the fluid case would be that: magnitude of the flow velocity is greater than the speed of sound everywhere inside the region and would be equal to local speed of sound on the surface. We can deduce from this definition that the event horizon must be either contained within or coincide with the ergosphere. For stationary black holes the ergosphere, which thus has a very little effect after all, actually coincides with the event horizon. For the draining bathtub model on the other hand, the radius of the ergoregion, calculated from the coefficient of g_{tt}, becomes

$$r_e = \frac{\sqrt{c^2a^2 + a^4\Omega^2}}{c} \geq a. \tag{3.12}$$

The presence of ergoregion in rotating black holes implies an important effect called the frame dragging. Rotating black holes or supermassive rotating stars will drag the reference frame in the direction of the rotation for observes near by. This dragging effect causes the ergoregion to cover the event horizon. However, as happens in our analogy, there are other systems with a flow velocity for which the ergoregion lies outside of the event horizon. This makes the draining bathtub model a great choice to examine such effects [13]. Presence of an ergoregion makes an important process called the Penrose process possible, which basically amounts to extraction of energy by outside agents from the rotational energy of a black hole. Analogue of the Penrose process here that allows particles to carry more energy coming out from a black hole is called super-radiance, that will be discussed in detail in sec. 4.3.

3.2 Hawking Radiation

According to classical theory, black holes are created by gravitational collapse and they will rapidly settle down to a stationary axisymmetric equilibrium state, which is characterized by mass M, angular momentum J and charge Q of the black hole. This result is also known as the "no hair theorem". In the general case, collapse phase is ignored and black holes are represented by the stationary, axisymmetric Kerr-Newman metric. Quantum field theory calculations show us that outside of a rotating black hole, pairs of particles may be created. A cartoon picture of the scenario (that should not be taken literally as warned by Hawking) is given in terms of quantum vacuum fluctuations and leads to a picture of virtual particles being first created and then annihilated in pairs without any observable effect in vacuum. However, in the vicinity of an event horizon this picture may change completely. One of the virtual particles may fall in through the horizon leaving the other one with a positive energy to escape away from the black hole to large distances [15], [18].

Particles with positive energy, emitted from a black hole carry out part of the rotational energy of the black hole and eventually making it reach a non-rotating state. That is the reason why particle emission is also called "black hole evaporation". Since this picture is theorized for rotating black holes, one expects that there should not be any particle creation in the vicinity of a Schwarzschild black hole (a static black hole with no charge and zero angular momentum). But there may have been some radiation during the collapse phase that is a time dependent formation phase, after which the black hole becomes static. However, Hawking's calculations in 1974 showed that this will not be the case. The emission rate by Hawking effect due to particle creation does not drop to zero over time, but reaches a steady non-zero value [18], [19].

Hawking's approximation in the analogy in 1975, was to replace the Minkowski metric in the matter field equation with the classical spacetime metric, since quantum effects of the gravitational field are supposed not to have any effect on the particle creation and annihilation processes. Then the matter field is decomposed into its positive frequency and negative frequency parts in order to interpret field operators in terms of particle creation and annihilation. Furthermore the spacetime is considered to be spherically symmetric, asymptotically flat, containing an isolated mass at the origin to form a Schwarzschild black hole which is a static black hole that has neither electric charge nor angular momentum. After the Schwarzschild black hole settles down to its stationary final state, the surface gravity becomes constant over its event horizon. Quantum fields propagating in this background in the distant past are specified and considered to be the vacuum. Although the results do not change for any other choice of a non-singular state, here field operators in distant past are considered to provide the initial data for the field equations. By solving the field equations, it is possible to calculate field operators for later events and see how quantum states and expectation values of field operators propagate [20].

Hawking calculated the expected number of particles emitted by a black hole during spherical gravitational collapse and showed that there should be a steady rate of emission of particles [18]. The main result of his analysis is that emission of the particles depends only on the final quasi-stationary state of the black hole and is exactly identical to blackbody thermal radiation at a specific temperature $T_H = \hbar\kappa/2\pi k$ where κ is the surface gravity of a black hole[1].

Consequently, black holes are not completely black but they should emit particles in the form of thermal radiation. The presence of an event horizon with the subsequent

[1]The super-resonance effect that will be explained in sec. 4.3 occurs when an incident wave on rotating black hole is scattered with an increased amplitude. This effect may be considered as the stimulated emission of particles, considering the amplification of a wave corresponds to an increase of the number of emitted particles. Therefore in a stationary system with stimulated emission, one can seek a correlation for spontaneous emission, that is nothing but the Hawking radiation.

quantum instabilities on the surface causes this thermal radiation or evaporation of the black hole to occur at a characteristic temperature. In the simplest case of Schwarzschild black holes, we have the Hawking temperature given by

$$T_H = \frac{\hbar c^3}{8\pi k G M}, \tag{3.13}$$

where G is a the gravitational constant, c is the speed of light in vacuum and M is the mass of a black hole.

The actual observation of Hawking radiation seems not to be possible. Astrophysical detection of thermal flux at Hawking temperature (3.13) corresponds to temperatures $\sim 6 \times 10^{-8} K$ for one solar mass black holes, that gets totally overwhelmed by cosmic microwave background radiation at $T_{CMB} \sim 3K$. Radiation from solar-mass black holes, therefore, would be $\sim 5 \times 10^7$ times colder than microwave background, and hence could not be observed today [21].

There is one significant difficulty with derivation of Hawking radiation known as trans-Planckian problem that should again be noted. For Hawking quanta to reach an observer at infinity where they are detected; field modes of the quanta near the black hole horizon must have arbitrarily high frequencies beyond the Planck scale with respect to any local free fall observer. At that point, a quantum theory of gravity should be considered, in contrast with Hawking's derivation that relies only on the analysis of quantum fields in the region exterior to the black hole.

3.2.1 Analogy

Observing the extremely subtle effect of the Hawking radiation and appreciating the difficulties involved in the derivation considered so far led Unruh to consider an analogy with black holes. We already discussed this analogy that the sound waves propagating in a fluid is equivalent to propagation of a scalar field in curved-spacetimes, and derived the analogue of an event horizon as the acoustic horizon for sound waves.

One of the great expectations of the Unruh analogy with black holes is the possible detection of Hawking radiation given that the alternative derivation does not rely on gravity but follows from the kinematics of black holes and realization of these features of event horizons in laboratory environment. Analogy at a quantum level is also established by Unruh as the quantized scalar field in gravitational black holes are analogous to phonons, the quanta of sound waves in acoustic black holes [2]. Thus phonons may be emitted from an analogue black hole horizon similar to the emission of photons from a black hole. Simply following the derivation done by Hawking for black holes, Unruh was able to calculate the analogue temperature for acoustic black holes as

$$T = \frac{\hbar}{4\pi kc}\frac{d(c^2 - \upsilon^2)}{dr} \tag{3.14}$$

where c is the velocity of the waves in fluid at rest and υ is the velocity of the fluid.

For a typical acoustic black hole, the estimated Hawking temperature is still very low to be detected. However, it may not be impossible in the near future to be able to perform such experiments and observe Hawking radiation in analog models provided a few important conditions could be met.

The creation of an event horizon relative to an effective metric analogous to black holes in classical fluids such as water, as demonstrated by Unruh and Schutzhold in 2003, may not be enough to give Hawking radiation, which relies on quantum effects [22]. Although the mode fixing that underlies the effect of Hawking radiation in analogue models is a classical phenomenon, the commutation relations for the field variables has to match those of the quantum scalar field in the corresponding curved spacetime.

In addition to that, the creation of Hawking radiation analogy requires a fluid that is barotropic, inviscid and irrotational. In other words, the fluid should be easily controllable and isolated from external effects, because even a very small disturbance can significantly complicate the process.

As we mentioned above Hawking temperature is a very subtle concept. For the analog models such as water Hawking radiation gives a thermal flux at a temperature in the region of $\sim \mu K$. Comparing the Hawking temperature to the temperature in which water behaves as a liquid, it is obvious that due to gap, effect will not be detectable. Considering the low value of the Hawking temperature, one should choose a fluid that behaves inside the analogy restrictions at extremely low temperatures and preserves its liquid form. Also the difference between the Hawking temperature and the fluid temperature should not be too large for the radiation to be detectable. Otherwise the background fluctuations would mask the quantum effect.

While the conditions to observe analog Hawking radiation are nearly impossible to achieve at the present time, advancing technology and increasing researches on superfluids and quantum fluids may overcome such difficulties, hence that is why the area is at the center of current research.

One of the most easily controlled superfluid is a Bose-Einstein Condensate (BEC), that is, a dilute gas system in which every atom occupies a single quantum state. The microscopic theory of BEC is well understood and the quantum states are relatively easy to describe [23]. This is an additional benefit of working with BEC comparing to other superfluids such as superfluid Helium. Since the Hawking radiation relies on quantum effects, BEC gives a great advantage to study.

One other advantage of BEC is that it is a system that exists in liquid state at temperatures in nano-Kelvin scales. So that, the differences between Hawking temperature and the temperature of BEC will be much smaller than the water analogy or any other classical analogs. Estimated Hawking temperature for BEC for the best scenario is $T_H \sim 70nK$, while the temperature of the condensate will be $T_{BEC} \sim 90nK$. That is a very small difference compared to other models available.

As mentioned before, to get the particle creation effect, commutation relations should match with those of a quantum scalar field in curved spacetime. Unruh and Schutzhold [24] showed that for a BEC, this is in fact the case. This provides substantial advantage and strong motivation to investigate Bose-Einstein Condensates for analog systems.

Finally, the trans-Planckian problem for Hawking radiation can be addressed in the analog models. For sonic systems, trans-Planckian problem takes the form of sound modes that are created near the horizon with a characteristic wavelength smaller than the healing length so that hydrodynamic limit is no longer valid. Here the Planck scale for the analogy corresponds to the inter-atomic distance. While the problem in gravity can not be addressed, the atomic theory of physics is known in most of the cases. Therefore analogy offers a way to address the trans-Planckian problem as well. The microscopic theory of BEC is also well known and gives an opportunity to investigate the problem.

Chapter 4

METRIC CALCULATIONS AND SUPER-RESONANCE

In this chapter, we derive the partial differential equation that corresponds to the massless scalar (Klein-Gordon) wave equation from the line equation (3.5) using differential geometric techniques and Hodge duals [25]. The main advantage of such geometric techniques is that they allow a coordinate-free representation. By structuring the second order PDE for a specific fluid defined in section 3, we will be able to solve it in the frequency domain by letting $\psi = e^{-i\omega t}e^{im\phi}e^{ikz}P(r)$ by the Frobenius method. Then the indicial equation will be calculated and the condition for super-radiance derived. The analyses of the asymptotic behavior of the PDE in the limits $+\infty$ and $-\infty$, and the solutions in asymptotic regions determine the reflection and transmission coefficients of an incident wave through a fluid, and provide information for calculating the condition for super-resonance effect for the analog acoustic black hole.

4.1 *From the Acoustic Metric to the Klein-Gordon Equation*

Unruh [2] considered the propagation of sound waves in a fluid that is barotropic, inviscid and with an irrotational flow determined by the solutions fo the Klein-Gordon equation $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi) = 0$. In our case, linear perturbations of the velocity potential ψ in the Lorentzian acoustic geometry background are determined by the equation (3.5). Let us consider the (2+1)-dimensional line element (z dimension is ignored for simplicity at the beginning, since it does not affect the calculations),

$$ds^2 = -\left(\frac{c^2r^2 - a^2c^2 - a^4\Omega^2}{r^2}\right)dt^2 + \frac{2ca}{r}dtdr - 2\Omega a^2 dtd\phi + dr^2 + r^2 d\phi^2. \qquad (4.1)$$

First we re-organize it so that

$$ds^2 = -c^2dt^2 + \left(dr + \frac{ca}{r}dt\right)^2 + \left(rd\phi - \frac{\Omega a^2}{r}dt\right)^2, \qquad (4.2)$$

In cylindrical polar coordinates $x^\mu : (x^0 = ct,\ x^1 = r,\ x^2 = \phi)$, the natural basis 1-forms are given by

$$dx^0 = cdt, \quad dx^1 = dr, \quad dx^2 = d\phi. \qquad (4.3)$$

By inspection, we define an orthonormal co-frame $\{e^0, e^1, e^2\} \in T^*\mathbb{R}^3$ given by

$$e^0 = cdt, \quad e^1 = dr + \tfrac{ca}{r}dt, \quad e^2 = rd\phi - \tfrac{\Omega a^2}{r}dt. \qquad (4.4a)$$

So that,

$$dt = \tfrac{e^0}{c}, \quad dr = e^1 - \tfrac{a}{r}e^0, \quad d\phi = \tfrac{e^2}{r} + \tfrac{\Omega a^2}{r^2 c}e^0. \qquad (4.4b)$$

We define the Hodge duals as follows;

$${}^*e^0 = -e^1 \wedge e^2, \quad {}^*e^1 = -e^0 \wedge e^2, \quad {}^*e^2 = e^0 \wedge e^1,$$

where the volume 3-form

$${}^*1 = e^0 \wedge e^1 \wedge e^2,$$

fixes the orientation of space.

Therefore exterior derivative of a function $f(t, r, \phi)$ can be written using the new basis as,

$$df = dr\frac{\partial f}{\partial r} + dt\frac{\partial f}{\partial t} + d\phi\frac{\partial f}{\partial \phi}$$

$$= \left(e^1 - \frac{a}{r}e^0\right)\frac{\partial f}{\partial r} + \frac{e^0}{c}\frac{\partial f}{\partial t} + \left(\frac{e^2}{r} + \frac{\Omega a^2}{r^2 c}e^0\right)\frac{\partial f}{\partial \phi}. \tag{4.5}$$

We take the Hodge duals;

$$-e^1 \wedge e^2 = -\left(dr + \frac{ca}{r}dt\right)\wedge\left(rd\phi - \frac{\Omega a^2}{r}dt\right) = -rdr\wedge d\phi + \frac{\Omega a^2}{r}dr\wedge dt - cadt\wedge d\phi = {}^*e^0 \tag{4.6}$$

$$-e^0 \wedge e^2 = -cdt \wedge \left(rd\phi - \frac{\Omega a^2}{r}dt\right) = -rcdt \wedge d\phi = {}^*e^1 \tag{4.7}$$

$$e^0 \wedge e^1 = -cdt \wedge -\left(dr + \frac{ca}{r}dt\right) = cdt \wedge dr = {}^*e^2 \tag{4.8}$$

and using them, now we can write

$${}^*df = -(e^1 \wedge e^2)\frac{1}{c}\frac{\partial f}{\partial t} - \left(e^0 \wedge e^2\right)\frac{\partial f}{\partial r} + \left(e^1 \wedge e^2\right)\frac{a}{r}\frac{\partial f}{\partial r}$$

$$+ \left(e^0 \wedge e^1\right)\frac{1}{r}\frac{\partial f}{\partial \phi} - \left(e^1 \wedge e^2\right)\frac{\Omega a^2}{r^2 c}\frac{\partial f}{\partial \phi}. \tag{4.9}$$

It reads, going back to the coordinate basis

$${}^*df = \left(-\frac{r}{c}\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial r} - \frac{\Omega a^2}{cr}\frac{\partial f}{\partial \phi}\right) dr \wedge d\phi + \left(-a\frac{\partial f}{\partial t} - rc\frac{\partial f}{\partial r} + \frac{ca^2}{r}\frac{\partial f}{\partial r} - \frac{\Omega a^3}{r^2}\frac{\partial f}{\partial \phi}\right) dt \wedge d\phi$$

$$+ \left(\frac{\Omega a^2}{rc}\frac{\partial f}{\partial t} - \frac{\Omega a^3}{r^2}\frac{\partial f}{\partial r} - \frac{c}{r}\frac{\partial f}{\partial \phi} + \frac{\Omega^2 a^4}{r^3 c}\frac{\partial f}{\partial \phi}\right) dr \wedge dt. \tag{4.10}$$

Finally taking one more exterior derivative, we get

$$d^*df = -\frac{\partial}{\partial t}\left(\frac{r}{c}\frac{\partial f}{\partial t} - a\frac{\partial f}{\partial r} + \frac{\Omega a^2}{cr}\frac{\partial f}{\partial \phi}\right) dt \wedge dr \wedge d\phi$$

$$-\frac{\partial}{\partial r}\left(a\frac{\partial f}{\partial t} + rc\frac{\partial f}{\partial r} - \frac{ca^2}{r}\frac{\partial f}{\partial r} + \frac{\Omega a^3}{r^2}\frac{\partial f}{\partial \phi}\right) dr \wedge dt \wedge d\phi$$

$$-\frac{\partial}{\partial \phi}\left(-\frac{\Omega a^2}{rc}\frac{\partial f}{\partial t} + \frac{\Omega a^3}{r^2}\frac{\partial f}{\partial r} + \frac{c}{r}\frac{\partial f}{\partial \phi} - \frac{\Omega^2 a^4}{r^3 c}\frac{\partial f}{\partial \phi}\right) d\phi \wedge dr \wedge dt.$$

After some long and careful calculations,

$$^*(d^*df) = -\frac{r}{c}\frac{\partial^2 f}{\partial t^2} + a\frac{\partial^2 f}{\partial t \partial r} - \frac{\Omega a^2}{c}\frac{\partial^2 f}{\partial t \partial \phi} + \cdots$$

we finally arrive at the massless Klein-Gordon equation relative to the metric (3.5). Since the linear perturbations of the velocity potential is a solution to the massless Klein-Gordon equation, the PDE to be solved,

$$\vec{\nabla}^\mu \vec{\nabla}_\mu \psi = 0, \tag{4.11}$$

is formed in explicit form as (putting in back the z-dependence by hand):

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2a}{cr}\frac{\partial^2}{\partial t \partial r} - \frac{2a^2\Omega}{c^2 r^2}\frac{\partial^2}{\partial t \partial \phi} + \left(1 - \frac{a^2}{r^2}\right)\frac{\partial^2}{\partial t^2} + \frac{2a^3\Omega}{cr^3}\frac{\partial^2}{\partial r \partial \phi}\right.$$

$$\left. + \frac{c^2 r^2 - a^4\Omega^2}{c^2 r^4}\frac{\partial^2}{\partial \phi^2} + \frac{r^2 + a^2}{r^3}\frac{\partial}{\partial r} - \frac{2a^3\Omega}{cr^4}\frac{\partial}{\partial \phi} + \frac{\partial^2}{\partial z^2}\right]\psi = 0. \tag{4.12}$$

4.2 Frobenius Method

The equation (4.12) is strongly hyperbolic, but it is completely separable [17]. We can analyze the equation in the frequency domain by separation of variables

$$\psi = e^{-i\omega t} e^{im\phi} e^{ikz} P(r), \tag{4.13}$$

where k and m are the axial and azimuthal wave numbers, respectively. To avoid polydromy problems [17], that is to make ψ single valued, m should be taken as an integer and k to be a real number defined by the boundary conditions along the z axis.

By inserting (4.13) into (4.12), we obtain a second order ODE for the radial part:

$$\left(1 - \frac{a^2}{r^2}\right)\frac{d^2P}{dr^2} + \left(-\frac{2ia\omega}{cr} + \frac{2ia^3\Omega m}{cr^3} + \frac{r^2+a^2}{r^3}\right)\frac{dP}{dr}$$
$$+ \left(\frac{\omega^2}{c^2} - k^2 - \frac{2a^2\Omega\omega m}{c^2r^2} - \frac{c^2r^2m^2 + a^4\Omega^2m^2}{c^2r^4} - \frac{2ima^3\Omega}{cr^4}\right)P = 0. \tag{4.14}$$

From this point on the Frobenius method can be applied to (4.14). This method is used to find solutions for linear second order, homogeneous ODEs, provided that the point of expansion is a regular singular point [26]. That is to say, the Frobenius method tells us a power series solution of the form $P = \sum\limits_{k=0}^{\infty} a_k (r-a)^{k+s}$ could be formed.

We rewrite (4.14) in a form that will help us to apply the Frobenius method:

$$(r-a)^2 \ddot{P} + (r-a)\, A(r)\dot{P} + B(r)(r-a)P = 0, \tag{4.15}$$

where

$$A(r) = \frac{r^2}{r+a}\left(-\frac{2ia\omega}{cr} + \frac{2ia^3\Omega m}{cr^3} + \frac{r^2+a^2}{r^3}\right), \tag{4.16}$$

and

$$B(r) = \frac{r^2}{r+a}\left(\frac{\omega^2}{c^2} - k^2 - \frac{2a^2\Omega\omega m}{c^2r^2} - \frac{c^2r^2m^2 + a^4\Omega^2m^2}{c^2r^4} - \frac{2ima^3\Omega}{cr^4}\right). \tag{4.17}$$

The radial equation (4.15) has two regular singular points at $r = \pm a$. Applying the method at the singular point r =a, which is also our horizon as shown before, the indicial equation is calculated. One of the roots of the indicial equation for (4.14), gives a condition for an important phenomena called super-resonance.

To start, suppose P(r) and its derivatives at the singular point r=a can be expanded as power series as follows:

$$\begin{aligned} P &= \sum_{k=0}^{\infty} a_k (r-a)^{k+s}, \\ \dot{P} &= \sum_{k=0}^{\infty} a_k (k+s)(r-a)^{k+s-1}, \\ \ddot{P} &= \sum_{k=0}^{\infty} a_k (k+s)(k+s-1)(r-a)^{k+s-2}. \end{aligned} \tag{4.18}$$

Then putting all the derivatives in (4.18) into (4.15), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} a_k (k+s)(k+s-1)(r-a)^{k+s} + \sum_{k=0}^{\infty} A(r) a_k (k+s)(r-a)^{k+s} \\ + \sum_{k=0}^{\infty} B(r) a_k (r-a)^{k+s+1} = 0, \end{aligned} \tag{4.19}$$

$$\sum_{k=0}^{\infty} a_k \left(k+s\right)\left(r+a\right)^{k+s} \left(\left(k+s-1\right) + \mathrm{A}(r)\right) + \sum_{k=1}^{\infty} \mathrm{a}_{k-1} \mathrm{B}(r)(r-a)^{k+s} = 0, \tag{4.20}$$

$$a_0(r-a)^s s\left(s-1+A(r)\right)+\sum_{k=0}^{\infty}(r-a)^{k+s}\left(a_k(k+s)(k+s-1+A(r))\right)+$$
$$\sum_{k=0}^{\infty}a_{k-1}B(r)=0. \tag{4.21}$$

The indicial equation, by definition, is obtained by taking the coefficient of the lowest order term $(r-a)^{k+s}$ (corresponding to k=0) equal to zero. So that, the first part of (4.21) is the indicial equation or the characteristic equation with the coefficient

$$a_0 s\left(s-1+A(r)\right)=0. \tag{4.22}$$

Setting $r=a$ for the indicial equation gives two roots for s; $s_1=0$ and $s_2=1-\mathrm{A}(a)$. For the non-zero root, it follows from (4.16) that

$$s_2=1-\mathrm{A}(a)=\frac{ia}{c}(\omega-m\Omega). \tag{4.23}$$

Thus the solution for the nonzero root of the indicial equation for the radial part of the Klein-Gordon equation at the horizon gives us $\frac{ia}{c}(\omega-m\Omega)$. As it was shown in section 4.3, this is the equation which the super-resonance condition relies on.

Note that, we used the Frobenius method only to find indicial equation, not to solve the ODE. The radial equation (4.14) contains complex functions A and B, and Frobenius method is not preferred as a method to solve for complex function [27].

4.3 Super-resonance

In Chapter 2, the ergoregion is defined as the negative energy region where objects have to co-rotate with the black hole. Presence of both the ergoregion and horizon can generate an important effect called the Penrose process that was predicted in 1969 [3]. In fact, the Penrose process is basically a mechanism to extract rotational energy from the ergoregion of a black hole. Since the rotational energy of a black hole is found in the ergoregion which lies outside the event horizon, energy extraction would be possible. Penrose's idea of extraction of energy from a Kerr black hole relies on pair production, considering a particle that splits into a pair of particles inside the ergoregion; one with negative energy, the other with positive energy. The negative energy partner relative to an observer at infinity will be captured by the black hole while the other with positive energy escapes to infinity. However, in order to escape to infinity the particle with positive energy should have relatively high speeds that is quite unlikely. Therefore, to understand Penrose process alternative ways have been presented [28], [29]. In particular, as shown by Christodoulou [30], $\sim$ %29 of the rotational energy of a Kerr black hole is available in principle for extraction.

The wave counterpart of the Penrose process is called super-radiance, which was first suggested by Zel'dovich in 1971 [4]. Zel'dovich pointed out that a cylinder made of absorbing material, rotating around its axis with angular velocity Ω can amplify incident waves of angular velocity ω, provided the condition

$$\omega < m\Omega \tag{4.24}$$

is satisfied, where m is azimuthal quantum number with respect to the rotation axis. Suppose an incident wave at infinity with a unit amplitude has frequency ω and azimuthal quantum number m. When such wave encounters a black hole, it will be partly absorbed and partly reflected back if the condition $\omega < m\Omega$ is satisfied. Hence provided the reflection coefficient $|R|^2 > 1$, super-radiance amplification will be accomplished. Since we are talking about acoustic waves now, we may refer to super-radiance as super-resonance in our analogy.

This process is purely kinematic; depending on the rotation of objects and it is taking place in the background spacetime of a rotating black hole. Thus, it is possible to reproduce a similar effect in rotating solutions describing acoustic black holes. The possibility to observe rotational super-resonance in analogue models was first considered by Schutzold and Unruh in the Draining Bathtub Model [22].

For the specific acoustic metric, we considered, the radial part of the Klein-Gordon equation (4.14) is already calculated. To carry out the calculation further, we define a new radial coordinate called the Regge-Wheeler tortoise coordinate, r_*, which will map $r \in [r_H, \infty]$ to $r_* \in [-\infty, +\infty]$ according to

$$r_* = \int \frac{r^2}{r^2 - a^2} dr. \tag{4.25}$$

By examining the asymptotic behavior of the wave at $r = a$ and $r = \infty$ in terms of the tortoise coordinate, one can calculate the reflection coefficient of the incident wave and see under what conditions it is amplified.

First we start with the behavior of the incident wave on the horizon at $r = a$. In order to make the calculations simpler, we substitute $P = R(r)H(r_*)$ and define a function $f = 1 - a^2/r^2$, so that the radial differential equation (4.14) transforms into;

$$fP'' + AP' + BP = 0, \tag{4.26}$$

$$\frac{d}{dr}\left(\frac{dR}{dr}H + \frac{dH}{dr_*}\frac{dr_*}{dr}R\right) + A\left(\frac{dR}{dr}H + \frac{dH}{dr_*}\frac{dr_*}{dr}R\right) + BRH = 0, \tag{4.27}$$

$$R\frac{d^2H}{{dr_*}^2} + \left(2\frac{1}{f}\frac{dR}{dr} + \frac{1}{f^2}\frac{df}{dr}R + \frac{AR}{f^2}\right)\frac{dH}{dr_*} + \left(\frac{1}{f^2}\frac{d^2R}{dr^2} + \frac{A}{f^3}\frac{dR}{dr} + \frac{BR}{f^3}\right)H = 0, \tag{4.28}$$

where

$$A(r) = -\frac{2ia\omega}{cr} + \frac{2ia^3\Omega m}{cr^3} + \frac{r^2+a^2}{r^3}, \tag{4.29}$$

$$B(r) = \frac{\omega^2}{c^2} - k^2 - \frac{2a^2\Omega\omega m}{c^2r^2} - \frac{c^2r^2m^2 + a^4\Omega^2m^2}{c^2r^4} - \frac{2ima^3\Omega}{cr^4}, \tag{4.30}$$

$$\frac{df}{dr} = \frac{2a^2}{r^3}. \tag{4.31}$$

The coefficient of $\frac{dH}{dr_*}$ is taken as zero, so the ingoing mode of the wave for the tortoise coordinate at $r_* = -\infty$ (that is $r = r_H$) is eliminated to give us a first order ODE for R [12]:

$$\frac{dR}{dr} + R\left(\frac{1}{2f}\frac{df}{dr} + \frac{A}{2f}\right) = 0. \tag{4.32}$$

Renaming this parenthesis in (4.32) as θ, as a result, the solution of (4.32) takes the form of $R = R_0 exp(-\int \theta dr)$ where

$$\theta = \frac{2ia\left(a^2\Omega m - \omega r^2\right) + c(r^2+2a^2)}{rc\left(r^2-a^2\right)}, \tag{4.33}$$

$$-\int \theta dr = \frac{1}{2c}(-3c + 2ia(\omega - m\Omega)log[r^2-a^2] + 4(c + iam\Omega)log[r]), \tag{4.34}$$

$$R = exp\left(\frac{1}{2c}\left(-3c + 2ia(\omega - m\Omega)log[r^2-a^2] + 4(c+iam\Omega)log[r]\right)\right). \tag{4.35}$$

Now we have an explicit solution for R which we insert into (4.28) and solve for H:

$$\frac{d^2H}{d{r_*}^2} + \left(\frac{1}{Rf^2}\frac{d^2R}{dr^2} + \frac{A}{Rf^3}\frac{dR}{dr} + \frac{B}{f^3}\right)H = 0. \tag{4.36}$$

Given A and B by (4.29) and (4.30), we have the final equation

$$\frac{d^2H}{d{r_*}^2} + \left(\frac{\omega^2}{c^2} + \frac{5a^4}{4r^6} + \frac{a^2c^2\left(m^2-3/2\right) + \Omega^2a^4m^2}{c^2r^4} + \frac{1}{4r^2c^2}\left(c^2 - 4m^2 + 8\Omega a^2\omega\right)\right)H = 0. \tag{4.37}$$

Substituting $r_* \to -\infty$ and $r \to r_H$ to the final form of the equation we reach a simple ODE

$$\frac{d^2H(r_*)}{dr^{*2}} + (\omega - \Omega m)^2 H(r_*) = 0. \tag{4.38}$$

We impose the boundary condition that, on the horizon, out of the two solutions of this equation, only the ingoing one is physical. So that

$$H(r_*) = Te^{-i(\omega - \Omega m)r_*} \tag{4.39}$$

where T is the transmission coefficient.

For the second limit, when $r_* \to +\infty$, as $r \to +\infty$, we can simply use the first equation without getting into too much detail. By inserting the values into the radial ODE (4.14), we obtain

$$\frac{d^2H(r_*)}{dr^{*2}} + \omega^2 H(r_*) = 0, \tag{4.40}$$

with the solution

$$H(r_*) = e^{i\omega r_*} + \mathrm{R}e^{-i\omega r_*}. \tag{4.41}$$

Here the first term corresponds to an ingoing wave with unit amplitude and the second term to reflected wave where R is the reflection coefficient.

Wronskian of the solutions (and their complex conjugates) will give us the Wronskian at infinity

$$H^*(r_*) = e^{-i\omega r_*} + \mathrm{R}^* e^{+i\omega r_*}, \tag{4.42}$$

$$W(+\infty) = -2i\omega \left(1 - |R|^2\right). \tag{4.43}$$

Again for the first case $r = a$, $r_* = -\infty$, Wronskian of the solutions give

$$W(-\infty) = -2i(\omega - m\Omega) \left|T^2\right|. \tag{4.44}$$

Recalling the fact that two linearly independent solutions of a differential equation must lead to a constant Wronskian, we conclude

$$W(\infty) = W(-\infty), \tag{4.45}$$

so that

$$1 - |R|^2 = \left(\frac{\omega - m\Omega}{\omega}\right) \left|T^2\right|. \tag{4.46}$$

Here R and T are the amplitudes of the reflection and transmission coefficients of the scattered wave. It is obvious from the above equality that, for frequencies in the range of

$$0 < \omega < m\Omega_H, \tag{4.47}$$

the reflection coefficient has a magnitude larger than unity. This is precisely the amplification factor that emerges in our earlier analysis of super-resonance.

Here it is important to remember that we calculated the reflection coefficient by assuming density of fluid and speed of sound through the fluid are both constant. In the literature, derivations of super-resonance through direct calculations of reflection and transmission coefficients for a vortex geometry can be found for some particular values of density and rotation velocities [12], [31], [32].

One big advantage of comparing the acoustic black hole metric (3.5) with the Kerr metric[1]is the non-existence of an upper limit on the rotational velocity of acoustic black holes. In other words, g_{tt} component of the metric contains the rotational velocity on the numerator, unlike the Kerr metric. Therefore rotational velocity

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[1]Kerr metric reads

$$ds^2 d\tau^2 = -\left(1 - \frac{2mr}{\rho^2}\right) c^2 dt^2 + \frac{\rho^2}{\Lambda^2} dr^2 + \rho^2 d\theta^2 + \left(r^2 + \alpha^2 + \frac{2mr\alpha^2}{\rho^2}\sin^2\theta\right)\sin^2\theta \, d\phi^2 + \frac{4mr\alpha\sin^2\theta}{\rho^2} c dt d\phi$$

where

$$\rho^2 \equiv r^2 + \alpha^2\cos^2\theta, \quad \Lambda \equiv r^2 + \alpha^2 - 2mr, \quad m \equiv GM/c^2, \quad \alpha \equiv J/Mc.$$

Here M is the mass of the rotating black hole, G is the gravitational constant and J is the angular momentum of the rotating black hole [33].

can be made very large in principle to enhance the super-resonance effect in acoustic black holes making it more efficient and hopefully experimentally observable [16], [34]. However, in practice if we try to perform an actual experiment, rotation parameter will be limited from above, that is because in order to respect the hydrodynamic approximations rotational velocity can not be increased indefinitely.

A related phenomenon is the so called "black hole bomb", first proposed by Press and Teukolsky [35]. The super-resonance effect has an amplified scattered wave. If the same wave can be amplified over and over again, we may extract as much rotational energy as possible from our system. This idea maybe realized by enclosing a rotating black hole between two reflecting mirrors, amplifying the wave at each bounce, so that total extracted energy become so large that increases exponentially. Thus the cycle goes on until the radiation pressure becomes so large that the system goes unstable, the mirrors are destroyed, thus creating a black hole bomb [36].

Chapter 5

CONCLUSION

Analogue black holes prove to be an effective method for investigating the kinematics of gravitational black holes. In this thesis, I used the (3+1)-dimensional Draining Bathtub (DBT) model with a sink at the origin in analogy with spinning Kerr black holes. I started with the fundamental equations of fluid dynamics and derived the equation of motion satisfied by the velocity potential of an inviscid Newtonian fluid that is barotropic and irrotational. Then I showed the equivalence of this equation to the Klein-Gordon equation for a massless scalar field propagating in a Lorentzian spacetime.

Once the analogy is set, I analyzed the DBT model and through the effective metric, determine the corresponding event horizon and ergoregion. Using differential geometric techniques, I derived the PDE corresponding to the massless scalar Klein-Gordon wave equation from the line element of the DBT model. The DBT model is particularly applicable to the study of the Penrose process of energy extraction from black holes. I studied the Super-resonance effect which arise from the rotational kinematics of a black hole by calculating the reflection and transmission coefficients for a scattered wave. The super-resonance condition can also be derived from the asymptotic behavior of the solution. Interestingly, from a calculation of the roots of the indicial equation at the acoustic horizon (event horizon) of the model, the super-resonance condition can be determined by the Frobenius method.

This thesis work reproduces the mathematical equivalence of the governing kinematic equations of the rotating Kerr black hole to that of the DBT model, and provides an alternative method to derive the super-radiance condition. Even though astrophysical realization of the super-radiance and the Hawking radiation is yet to be seen, experimental studies in the labs conducted in recent years report the presence of event horizons, mode conversion and negative norm modes in the analog models [10], [37], which are the underlying features of the Hawking radiation and super-radiance.

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