ALGEBRAIC SOLUTION OF THE HAMILTONIANS OF THE TWO-LEVEL QUANTUM OPTICAL SYSTEMS

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Abstract

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In this thesis an algebraic method is developed for solution of two-level quantum optical problems based on $su(2)$ and $su(1,1)$ Lie algebras. The method provides a general framework within which many related problems can similarly be solved. We concentrate our attention to the solution of the Jaynes-Cummings Hamiltonian, as well as other quantum optical Hamiltonians. It is shown that the solution of quantum optical problems can be reduced to a homogenous ordinary differential equation. Our systematic approach reproduces a number of earlier results and also leads to some novelties. Most of the quantum optical Hamiltonians possess hidden group structure based on $su(2)$ or $su(1,1)$ algebra and they can be treated in the framework of the quasi-exactly solvable problems.

Furthermore construction of the quantum optical Hamiltonians using finite-group invariance is discussed. Generalization of the method to the various quantum physics problems is indicated.

Key words: Exactly solvable systems, Lie algebraic methods, Quantum optical problems.

Öz

İki Seviyeli Kuantum Optik Sistem Hamiltonyan İşlemcilerinin Cebirsel Çözümü

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Bu tezde $su(1,1)$ ve $su(2)$ Lie cebirlerine bağlı olarak iki seviyeli kuantum optik problemlerinin çözümü için bir metod geliştirilmiştir. Bu metod buna benzer birçok problemin çözümüne genel bir bakış açısı sağlar. Çalışmamızda Jaynes-Cummings Hamiltonyanın çözümüne ve diğer kuantum optik Hamiltonyanlara odaklanıldı. Kuantum optik problemlerin çözümünün homojen adi bir diferansiyel denkleme indirgendiği gösterilmiştir. Sistematik yaklaşımımız literatürdeki mevcut birçok sonucu yeniden üretir ve bazı yeniliklerede öncülük eder. Birçok kuantum optik sistemler $su(1,1)$ ve $su(2)$ cebirlerine dayalı gizli grup simetrisine sahiptir ve yarım-çözülebilir problemler doğrultusunda incelenebilir.

Ayrıca kuantum optik Hamiltonyanların yapısı sonlu grup değişmezliği kulanılarak incelenmiştir. Metodumuzun çeşitli kuantum fizik problemlerine genellemesi gösterilmiştir.

Anahtar kelimeler: Kesin çözülebilir sistemler, Lie cebiri metodlar, Kuantum optiksel problemler.

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a : Bosonic annihilation operator

a^{+} : Bosonic creation operator

b : Bosonic annihilation operator

b^{+} : Bosonic creation operator

$\sigma_0, \sigma_+, \sigma_-$: Pauli matrices

ω : Effective frequency

g : Gyromagnetic ratio

μ : Bohr magneton

ω_c : Cyclotron frequency of electron

m^* : Effective mass of electron

κ : Coupling constant

$J_+; J_-; J_0$: $Su(2)$ algebra generators

N : $Su(2)$ algebra number operator

K_+, K_-, K_0 : $Su(1,1)$ algebra operator

M : $Su(1,1)$ algebra number operator

Ψ : Wavefunction

E : Eigenvalue

λ : Spectral parameter

Chapter 1

Introduction

During the last decade a great deal of attention has been paid to examine different quantum optical models. Recently some algebraic techniques which improve both analytical and numerical solutions of the problems, have been suggested and developed for some quantum optical systems [1, 9]. In general, the study of two-level systems, in a one and two-dimensional geometry, coupled to bosonic modes has been the subject of intense attention because of its extensive applicability in the various fields of physics [10, 17]. Jaynes-Cummings Model (JC) model is one of the major paradigms describing atom-field interactions in quantum optics today. It was introduced as a highly simplified model to explain, qualitatively, the remarkable features of the interaction of matter with a quantizied radiation field in a cavity. Despite the fact that it considers matter in the highly simplified form of a single-mode field, it demonstrates quite a few uniquely quantum mechanical features. Its importance has been enchanced recently by experimental realization in Rydberg atom masers [18, 19]. Various generalizations of the basic JC model have been considered in the literature[20, 21] multiphoton interactions, nonlinear media, deformed radiation fields.

There are many approaches to solve quantum optical models, in particular for JC model. Recently some algebraic techniques which improve both analytical and numerical solutions of the problems, have been suggested and developed for some quantum optical systems. It is well-known that the Lie algebraic techniques [22, 25] are very powerfull in describing many physical problems while improving both analytical and numerical solutions as well as understanding the nature of physical structures.

In our work we deal with the algebraic solution of the Hamiltonians of the two-level quantum optical systems especially we focus on JC model.A general Hamiltonian is described some parameters, from where depending on the choices of parameters various models can be obtained. We concentrate our attention to the solution of JC models with and without rotating-wave approximation

(RWA) by constructing the proper realization of $su(2)$ and $su(1,1)$ Lie algebras. It is also seen that the algebras $su(2)$ and $su(1,1)$ describing symmetry of our general Hamiltonian can be imbedded into a larger algebra that contains both [26, 27]. This algebra is $Sp(4,R)$ that provides a unified treatment of the various approaches to the solution of the such systems.

1.1 Analysis of Some JC Models

JC model has been studied by many authors and along many generalized forms, has two apparent advantages. First, the irreducible invariant subspace of the Hilbert space is finite, and it is mathematically solvable. Second this model exhibits many fascinating quantum effects which can be tested by experiments, such as the quantum collapse and revival of atomic inversion, and optical Schrödinger-cat states. The remarkable advance in cavity quantum electrodynamics (QED) experiments involving single atoms (usually Rydberg atoms) within single-mode cavities (the micromaser) and the possibility of finding solutions (often exact) to fundamental models of the quantum theory of interacting field and atoms have excited many efforts to exploit and extend this model. As a result, various modifications and many generalized forms of JC models have been proposed.

The fundamental JC models are easily described with RWA [28, 29]. It is easy to find the solutions using RWA. There are some modifications on JC model. Some studies on this popular model is related with dipole squeezing investigated by Dun-huan Liu and others [29]. Also there are some works about its dependency of intensity and time [30]. Two photon JC model interactions are found in literature [31, 32].

Some theories are improved on JC model without using RWA for example Eduard Tur [33, 34] suggets solutions without RWA. One of the interesting aspect is algebraic structure and analytic solutions of generalized three-level JC model [35].

1.2 Fundamental JC Models

In addition to its exact solvability within the RWA, the most interesting aspect of its dynamics is the entanglement between the atom and the field. Much attention has been focused on the entanglement of the field and the atom. Recently, entanglement as a physical source has been used in quantum information science such as quantum teleportation, superdense coding and quantum cryptography. The details of the entanglement between two subsystem in the presence of such an environment is worth to study. JC model is employed in the dispersive approximation in a dissipative cavity at zero temperature to study the entanglement between the atom and the field as well as the decoherence induced by the cavity [31]. It is demonstrated that the coherence property of the field is affected by cavity not only in qualitatively but also in quantitavely when two-photon process and Stark shift is involved, the influence of dissipation on entanglement make the amplitude of each state supressed, the larger the intensity of the field is the weaker the entanglement of subsytem and the larger maxima degree of mixture state.

One of the possible generalizations to the JC model is the intensity-dependent JC model [36] which is qualitatively different from that of the usual one. As in the original JC model, the intensity-dependent JC model is analytically solvable due to the neglect of the so-called counter-rotating terms. But in the presence of the counter rotating terms some differences occur so effects of the intensity-dependent JC model in the intensity-dependent JC model are investigated. In order to solve the intensity-dependent JC model in which the counter-rotating terms included, it is need to resort to numerical diagonalization method because analytical solutions are not available. In the presence of some unitary transformation which decouples the spin degree of freedom from, and thus the numerical calculations are considerably simplified.

The modified JC model can be used to describe a two-level atom placed in the common domain of two cavities, where there exists a "free-exchange" of excitations between the modes. The modified JC model is studied by a dynamical algebraic method. With the help of $su(2)$ algebraic structure, it is easy to obtain the eigenvalues, eigenstates, time evolution operator and atomic inversion operators for the system [37].

1.3 Two-Photon JC Model

JC model has been studied by many authors, numerous non-classical effects have been predicted. If the levels of an isolated atom are degenerate in the projection of the total electronic angular momenta on the quantization axis, the original JC model becomes, in general, invalid. Recently, Reshetov [38] considered the degeneracy of atomic levels and generalized the JC model to the case of atomic level degenerate in the projections of the angular momenta within RWA. Atom population inversion is the population difference between the upper and lower levels is important to study the degeneracy of atomic levels. Comparing generalized JC model with RWA with original JCM by means of atom population inversion it is seen that the revival period of generalized JC model becomes longer than that of the original JC model and the maximum amplitude of atomic inversion decreases. It is concluded that the 'collapse and revival' is related to the angular momenta of two atomic levels. Both non-RWA JC model and generalized non-RWA JC model show quantum chaos but the chaos of generalized JC model are much weaker than that of the original JC model. The degeneracy of angular momentum can restrain the phenomenon of quantum chaos.

Squeezing of a quantized electromagnetic field could be produced both therotically and experimentally through various physical processes, such as degenerate parametric processes, phase conjugation or four-wave mixing, harmonic generation, resonance fluorescence, the free-electron laser. Also, the squeezing of quantum fluctuations in one of the atomic dipole variables has been extensively studied, and the available studies have shown that atomic dipole squeezing (ADS) could be generated in numerous physical situations, such as the cooperative Dicke system, the Rydberg atom maser, resonance fluorescence. Using Heisenberg uncertainty relation (HUR) and Schrödiger uncertainty relation (SUR) a comparative study on the squeezing properties of quantum fluctuations in atomic dipole variables in the two-photon JC model can be done [32]. There are differences between the cases; the influence of initial atomic coherence or thermal photons on ADS and the possibility of realization of ADS in two-photon micromaser experiments with Rydberg atoms in a high-Q cavity.

Histrocally, the Heisenberg lower limit is used as the reference level for the definition of squeezing in observables. However, in the measurement of observables corresponding to noncanonical operators, it may not work [39]. Then, one should employ the SUR for further studies, where the Schrödinger lower limit sets a higher bound on quantum fluctuations as the Heisenberg did. In the absence of thermal photons; in both cases (HUR and SUR) the squeeze period is determined by the atom-field interaction, and the initial atomic coherence only influences the squeeze duration and the height of squeeze peak, in some special cases SUR case turns to HUR. In the precence of thermal photons; for sufficiently small photon number, the influence of thermal photons on ADS is very little, then with the increase of photon number, the squeeze duration where ADS could appear is shortened, the height of squeeze peak decreases gradually and the position of squeeze peaks changes slightly. These conclusions apply to both cases, but ADS in the HUR case disappears more quickly than that in the SUR case with the increase of thermal photons.

1.4 Some Different JC Models

A few authors treated the time-dependent version of JC model [40, 41]. Using the underlying $su(2)$ algebra of the JC model a time-dependent interaction term that allows analytical solution for even off-resonance conditions was also constructed in [25] where exact solutions for the time evolution of any state were found. The effect of detuning on the Rabi oscillations and the collapse and revival of inversion was indicated. It was also shown that at resonance, the time-dependent JC model is analytically solvable for an arbitrary interaction term.

In some other studies[42, 43], an attempt was made to go beyond the framework of the RWA. In these works, continual integration methods and perturbation theory were used. The continual integration methods, which are based on variation principles, are unable to give an estimate of the solution error, and the series of the perturbation theory converge only for a certain relationship between parameters of a problem. Despite the fact that the convergence rate may be different for different cases, a special method converges for any parameters of the model. The method is based on the possibility of representing a matrix element of the resolvent of a self-adjoint operator in the form of a continued fraction.

The JC model in the general nonresonant case without RWA has been considered [34]. The analysis has been carried out using the resolvent formalism. It has been shown that one, given the matrix of Hamiltonian resolvent, can easily find all basic physical quantities corresponding to the given model. The matrix of the resolvent of the total Hamiltonian for the given model has been found. In terms of continued fractions matrix elements of the resolvent have been expressed. It was also shown that these fractions uniformly converge to meromorphic functions, which corresponds to a purely point spectrum of the total Hamiltonian. Time evolution in the case of exact resonance for different coupling constants was numerically studied. It has been obtained that the RWA is not satisfactory for large coupling constants even in the case of exact resonance. In this case, probabilities of multiphoton transitions increase with increasing coupling constant.

In quantum optics and laser physics JC model has exhibited a lot of interesting non classical features of atom field interactions. One of them is the squeezing of the radiation field, which has been extensively investigated due to its potential applications in optical communications, and weak signal corerections for the past ten years. It has been shown theoritically and experimentally that the squeezed field can be produced by various approaches. Meanwhile increased attention has also been devoted to the squeezing of the fluctuations of atomic dipole variables, moreover some authors have shown that the squeezed atoms can radiate a squeezed field. By virtue of the numerical method of Graham and Höhnerbach [44], the quantum fluctuations of the atomic dipole variables in the multi-photon JC model with and without RWA when it is restricted to the following initial condition is investigated; the atom in its superposition state and the field in vacuum state. It is found that even under the conditions in which the RWA is considered to be valid there are significant effects of virtual-photon field on dipole squeezing predicted in the RWA, and dipole squeezing turns to disappear for sufficiently strong coupling [29]. In order to analyze the squeezing properties of atom Hermitian conjugate operators are used.

Recently, the JC model has been generalized to the case of multilevel atoms interacting with one or more discrete field modes, or to the case involving a multiphoton transition through the intermediary of virtual levels. Specifically, attention has been given to the case in which an atom undergoes a non-resonant two-photon process (known as the two-photon JC model) and the study of collapse and revival phenomenon of Rabi oscillation is reported. Further, both one- and two-photon JC models have been extended to include the effects of a Kerr-like medium [44] and some significant results for the dynamical evolution of the atom, the quasi probability distribution of the cavity field and the emission spectrum are reported. The quantum phase properties of the JC model with a Kerr medium is described by applying the Hermitian phase operator formalism introduced by Pegg and Barnett [46]. The results show that phase properties get considerably modified due to the presence of Kerr-like nonlinearity [45]. In particular, for both one- as well as two-photon JC model regular/symmetric (in the case of the one-photon JC model) or periodic (in the case of the two-photon JC model) behaviour of phase properties become quite irregular as the coupling srength of the Kerr medium increases. This happens because of the competing interaction of the field between the atom and the Kerr-like medium. Since the interest in Kerr-like media is quite relevant in relation to the so-called quantum non-demolition measurements [45].

It has been shown that all types of two-level JC model have an $su(2)$ structure and can be treated as a linear function of the generators of an $su(2)$ group. Thus any two-level JC model can be treated as a spin $\frac{1}{2}$ system in an external magnetic field. From this algebraic structure, it is easy to obtain evolution matrix, as well as the eigenvalues and eigenstates of the Hamiltonian which are simply algebraic expressions. Based on the existence of conservation of excitation and the project property of fermion annihilation operators, a unified $su(n)$ algebraic structure of a generalized N-level JC model can be constructed.

It is possible to write the N-level JC model of different types of interactions, Ξ, Λ,and $\vee$ types [35]. For all three types of interaction, there are only N-1 ways of coupling is restricted between N atomic levels: for $\Xi - type$ only adjacent levels are coupled, for $\vee - type$ the coupling is restricted between the lowest level and other levels, and for $\Lambda - type$ only the highest level is coupled with other levels [35].

When a Hamiltonian is expressed as a linear function of Lie group's generators, there are many algebraic methods to obtain solutions of the equations of motion. One of the methods that can deal with general Lie algebraic structure is the algebraic dynamics. An important procedure of algebraic dynamics to obtain solutions of a linear system is to find a gauge transformation that transforms the time-dependent Hamiltonian into a linear function of the Cartan operators of Lie algebra. Then the exact solutions are obtained by the inverse gauge transformation. In the case when the considered Hamiltonian is autonomous (it is not dependent on time explicitly), only algebraic equations are needed to solve to obtain the solutions of the solutions of the equations of motion, as well as the eigenvalues and eigenstates of the Hamiltonian, if the gauge transformation is let to be independent of time. As a result the solutions for the N-level JC model are algebraic expressions when N ≤ 4 [35].

According to standart Lie algebraic theory, if a Hermitian operator is a linear function of the generators of a compact semisimple Lie group, it can be transformed into a linear combination of the Cartan operators of the corresponding Lie algebra and applying some procedure [35] the solutions of a linear autonomous system can be obtained. This thesis starts from finite group invariance of JC Hamiltonian and Jahn-Teller Systems. In chapter 3 a general method is developed to solve quantum optical systems. Solutions of JC models and some other physical systems are given in chapter 4. In chapter 5 a different type of solution is discussed. Finally two-photon Rabi Hamiltonian is investigated.

Chapter 2

Finite Group Invariance of JC Model and Jahn-Teller Systems

Main purpose of this chapter is to study the $E \times e$ JT and JC models under the finite group invariance . A method is presented to obtain finite group invariance of the $E \times e$ systems.

2.1 Finite Subgroups of $su(2)$ and JC Model

In this section we investigate the symmetry properties of the JC model. It is well known that the finite subgroups of $su(2)$ have been focus of interest from the both physical and mathematical point of view, and it can be generated by discrete rotations around the properly chosen two axes. For example the two dimensional matrix generators of dihedral groups can be generated as

$$A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \qquad B = \begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{2.1}$$

where $\phi = \frac{2\pi}{n}$, and the generation relations of the A and B are given by

$$A^n = B^2 = (AB)^2 = 1 \tag{2.2}$$

Dihedral groups include one and two-dimensional irreducible repsentations. The matrix realizations of the two-dimensional irreducible representations can be obtained from (2.1). It is obvious that the characters of the one-dimensional irreducible representations are ∓ 1. When a physical system possesses the symmetry of a finite group, the coupling between energy levels can be determined by the direct product of the corresponding representation. For dihedral groups the decomposition of the symmetric direct product of two-dimensional representation E can be expressed as

$$[E \times E] = e \qquad [E \times E] = b_1 \qquad [E \times E] = b_1 + b_2 \tag{2.3}$$

where e is two-dimensional representation and b_1 and b_2 are the one-dimensional representations, with the characters ± 1. Equation (2.3)describes the symmetric E state interactions for different dihedral groups.In this study we deal with the $[E \times E] = b_1$ interaction. Our task is now to construct the physical Hamiltonian which is invariant under the dihedral groups. Let us consider the following polynomial function

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_{ij}x_ix_jq^0 + \sum_{i=1}^{2}\sum_{i=1}^{2}\beta_{ij}x_ix_jq^1 + H.C. \tag{2.4}$$

where x_i represents the coordinates of the field and q_i is the coordinate of the atom. When atom interacts with field then the coupling type may be classified according to (2.3) and the coordinates x_i transform with the two-dimensional representation of the group while q_i transform with the one-dimensional representations. In general we can write

$$P(x_i, q) = P(Ax_i, g_1 q) = P(Bx_i, g_2 q) \tag{2.5}$$

where g_1 and g_2 are the characters of the b_1. The invariant matrix potential can be obtained from the relation

$$V = \frac{\partial P}{\partial x_i x_j} \tag{2.6}$$

It is interesting that the procedure presented above leads to the construction of the JC Hamiltonian. From the relations (2.1), (2.4) and (2.5) one can obtain the following expressions

$$\begin{aligned} P(Ax_i, g_1 q) = \{&\frac{1}{2}\left[\alpha_{11} + \alpha_{22} + (\alpha_{11} - \alpha_{22})\cos\left(\frac{4\pi}{n}\right) - \alpha_{12}\sin\left(\frac{4\pi}{n}\right)\right]x_1^2 + \\ &\frac{1}{2}\left[\alpha_{11} + \alpha_{22} - (\alpha_{11} - \alpha_{22})\cos\left(\frac{4\pi}{n}\right) - \alpha_{12}\sin\left(\frac{4\pi}{n}\right)\right]x_2^2 + 7a \\ &\left[(\alpha_{11} - \alpha_{22})\sin\left(\frac{4\pi}{n}\right) - \alpha_{12}\cos\left(\frac{4\pi}{n}\right)\right]x_1 x_2\}(g_1 q) \end{aligned} \tag{2.7}$$

$$\begin{aligned} P(Bx_i, g_2 q) = \{&\frac{1}{2}\left[\alpha_{11} + \alpha_{22} + (\alpha_{11} - \alpha_{22})\cos\left(\frac{4\pi}{n}\right) - \alpha_{12}\sin\left(\frac{4\pi}{n}\right)\right]x_1^2 + \\ &\frac{1}{2}\left[\alpha_{11} + \alpha_{22} - (\alpha_{11} - \alpha_{22})\cos\left(\frac{4\pi}{n}\right) + \alpha_{12}\sin\left(\frac{4\pi}{n}\right)\right]x_2^2 - 7b \\ &\left[(\alpha_{11} - \alpha_{22})\sin\left(\frac{4\pi}{n}\right) - \alpha_{12}\cos\left(\frac{4\pi}{n}\right)\right]x_1 x_2\}(g_2 q) \end{aligned} \tag{2.8}$$

The solutions of the (2.7) and (2.8) for integer values of n gives us the following relation for the coefficients of the polynomial function $P(x_i, q)$.

For the values of n except that n = 2 and 4, the generators g_1 and g_2 take the value 1 and $\alpha_{11} = \alpha_{22}$, $\alpha_{12} = 0$. Let us turn our attention to the construction

n	g_1	g_2	α_{11}	α_{22}	α_{12}
2	1	1	α_{11}	α_{22}	0
	1	-1	0	0	α_{12}
4	1	1	α_{11}	α_{11}	0
	1	-1	0	0	α_{12}
	-1	-1	α_{11}	$-\alpha_{11}$	α_{12}

Table 2.1: Coefficients of the polynomial $P(x_i, q)$ for dihedral groups

of the invariant Hamiltonian. We assume that the decomposition of $E \times E$ is in the form

$$E \times E = a_0 + b_1 \tag{2.9}$$

For $n = 2$. The Hamiltonian can be written as

$$H = H_0 + V(a_0) + V(b_1) \tag{2.10}$$

where

$$V(a_0) + H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{1}{2}m\omega^2 q^2 \tag{2.11}$$

and $V(a_0)$ and $V(b_1)$ can be obtained by using the invariance of the Hamiltonian. By using the relation (2.6) and Table 1.1 we can obtain

$$V(b_1) = \alpha_{12} q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_{11} q \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.12}$$

By using the differential realizations of the bosonic operators

$$a^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dq} + q\right) \qquad ; \qquad a = \frac{1}{\sqrt{2}}\left(\frac{d}{dq} + q\right) \tag{2.13}$$

The Hamiltonian (2.10) can be expressed as

$$H = a^+ a + \alpha_1 \sigma_0 + \alpha_{12}(a^+ + a)(\sigma_+ + \sigma_-) \tag{2.14}$$

For the values of $n = 4$, we obtain two invariant Hamiltonians which can be transformed onto each others

$$H = a^+ a + \alpha_{12}(a^+ + a)(\sigma_+ + \sigma_-) \quad or \quad H = a^+ a + \alpha_{11}(a^+ + a)\sigma_0 \tag{2.15}$$

In this case the energy splitting term $\frac{1}{2}\mu\sigma$ is missing.

$E \times e$ JT Hamiltonian is a system with doubly degenerate electronic state and doubly degenerate JT active vibrational state. The JT effect describes the interaction of degenerate electronic states through non-totally symmetric, usually non-degenerate, nuclear modes. This effect plays an important role in explaining the structure and dynamics of the solids and molecules in degenerate electronic states. For $[E \times e]$ JT Hamiltonian equation (2.12) can be written as

$$V(b_1) = \alpha_{12} q_1 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_{11} q_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.16}$$

and JT Hamiltonian can be expressed as

$$H = a_1^+ a_1 + a_2^+ a_2 + \alpha_{11}\sigma_0 + \alpha_{12}[(a_1 + a_2^+)\sigma_+ + (a_1^+ + a_2)\sigma_-] \tag{2.17}$$

Therefore we have constructed JC model and JT Hamiltonian by using group invariance. Clearly, JC Hamiltonian is invariant under D_2 group.

One can also investigate the symmetry properties of the modified JC Hamiltonian. In this case we consider the interaction $E \otimes \varepsilon$. The polynomial function can be written as

$$P(x_i, q) = \sum_{i=1}^{2}\sum_{j=1}^{2} \alpha_{ij} x_i x_j + \sum_{i=1}^{2}\sum_{i=1}^{2} \beta_{ijk} x_i x_j q_k + H.C. \tag{2.18}$$

We assume that the same generators

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.19}$$

transform q_i ,q_1 and x_i, x_1. In this case the invariant polynomial takes the form

$$P(x, q) = \alpha_{112} x_1^2 + \alpha_{222} x_2^2 + \beta_{121}(x_1 x_2 q_1) \tag{2.20}$$

when the generators transform the x_2 and x_1 we obtain

$$P(x, q) = \alpha_{112} x_1^2 + \alpha_{222} x_2^2 + \beta_{222}(x_1^2 q_2 - x_2^2 q_2) \tag{2.21}$$

Using (2.6) for modified JC model we get

$$V(b_1) = \begin{pmatrix} \alpha_{112} & 0 \\ 0 & 2\alpha_{222} \end{pmatrix} + \beta_{222} q_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.22}$$

One can obtain other possible forms of the polynomial function that is invariant under the dihedral groups. In this section we have presented a procedure to investigate the symmetry of the JC model and JT Hamiltonian.

Chapter 3

An Algebraic Development For Matrix Hamiltonians

A quantum model is exactly solvable if all of its energy levels and corresponding wavefunctions having convenient (explicit) expressions can be obtained, or, more precisely, if the spectral problem for this model can be reduced to a problem of algebra (in the case of quantum mechanics) or classical analysis (in the field theoretical case).

It is well-known that exactly solvable models play an extremely important role in many fields of quantum physics.

First of all, they may be interesting in themselves as models of actual physical situations. For example, the behavior of many quantum systems near their equilibrium can be described by the harmonic oscillator, the spectrum of the hydrogen atom can be found from the Coulomb problem, which is also exactly solvable, and one-dimensional completely integrable non relativistic field theoretical models have proved to be good approximations for the so-called quasi-one-dimensional systems.

Second, exactly solvable models can be successfully used as a training ground for elaborating various approximate and qualitative methods of studying exactly non-solvable systems, and for testing theoretical hypotheses of a general nature. Sometimes the analysis of exact solutions allows one to reveal certain properties of the system which do not change even after it has undergone considerable deformation. The study of such properties is very helpful for a better understanding of the general structure of quantum models. By analogy with the cases of the simple harmonic oscillator and the Coulomb problem, the study of which led to comprehension of many fundamental principles of quantum mechanics, the analysis of such completely integrable systems as magnetic chains made a valuable contribution to the understanding of physical nature of excitations in systems with many degrees of freedom.

Third, exactly solvable models can be used as zeroth-order approximations by constructing various perturbative schemes. For example, the rapid progress of quantum field theory in the fifties was attained in the framework of perturbation theory: the role of the unperturbed problem was played in this case by the free field or, in other words, by the infinite-dimensional harmonic oscillator. At present, there are some preconditions for constructing perturbation theory near the completely integrable one-dimensional field theoretical models and magnetic chains.

Finally, exactly solvable models are also interesting from a purely mathematical point of view, since the phenomenon of exact solvability can often be explained as being a consequence of certain hidden symmetry present in the model under consideration. It is remarkable that the group describing this symmetry appears not only necessary attribute of exact solvability, but as a unique language in which this phenomenon has simple and transparent mathematical sense. As an example, it is sufficient to recall the simple harmonic oscillator and the Heisenberg algebra associated with it. In the same sense, the various completely integrable systems obtained by means of the inverse scattering method turn out to be connected with various finite- and infinite-dimensional Lie algebras.

The last fifteen years have been marked by the discovery of a number of remarkable methods of building and solving exactly solvable models. They are: the quantum inverse scattering method, the Leznov-Savelyev approach (Takhtajan and Faddeev 1979),the projection method (Olshanetsky and Perelomov 1983), the method of Gelfand-Levitan-Marchenko equations (Zakhariev and Suzko 1985), methods based on the use of Witten's supersymmetric quantum mechanics and the Darboux theorem (Infeld and Hull 1951, Gendenshtein 1983, Adrianov et al 1984, Gendenshtein and Krive 1985),and so on.Brute-force attempts to find new exactly solvable models and methods of their construction encounter serious difficulties, connected probably with the fact that the usual requirement of exact solvability, understood as the possibility of writing down the entire spectrum of the Hamiltonian in more or less closed form, is too strict.

A possible way out of this impasse might involve only an essentially new approach to the problem, based on some constructive expansion of the concept of exact solvability in quantum physics. One such approach was proposed several years ago. It was based on the idea of relaxing the standard requirements of exact solvability and seeking models for which the spectral problem could be solved exactly only for certain limited parts of the spectrum but not for the whole spectrum. This idea has gradually crystallized in the last decade. The critical impetus has been given in the works of Zaslavsky and Ulyanov (1984), Bagrov and Vshivtsev (1986), Turbiner and Ushveridze (1987). Following the terminology introduced by Turbiner and Ushveridze, we shall call these models "quasi-exactly solvable" referring to their "order" as the number of states for which exact results can be obtained [47].

In this chapter a general Hamiltonian of a two-level system is constructed also the present chapter emphasizes a general method to solve physical systems.

3.1 A Model Hamiltonian for Two Level Quantum Optical Systems

The most general form of the Hamiltonian of a two level system in two dimensional geometry can be expressed as [48]

$$H = H_0 + \beta\sigma_0 + (\kappa_1 a + \kappa_2 a^+ + \kappa_3 b + \kappa_4 b^+)\sigma_+ + (\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+)\sigma_- \quad (3.1)$$

where

$$H_0 = \hbar\omega_1 a^+ a + \hbar\omega_2 b^+ b \quad (3.2)$$

a, b and a^+, b^+ are bosonic annihilation and creation operators, respectively and $\omega_{i,}\beta, \kappa_i$ and γ_1 are physical constants and $\sigma_0, \sigma_+, \sigma_-$ are usual Pauli matrices and they are given by,

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.3)$$

The Hamiltonian (3.1) includes various physical Hamiltonians depending on the choice of the parameters. If

a) $\omega_1 = \omega_2 = \omega$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$ and $\kappa_1 = \kappa_4 = \gamma_2 = \gamma_3 = \kappa$, the Hamiltonian, H, reduces to the $E \otimes \varepsilon$ Jahn-Teller (JT) Hamiltonian [11]

b) $\omega_1 = \omega_2 = \omega$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$ and $\kappa_1 = -\kappa_4 = \gamma_2 = -\gamma_3 = \kappa$, the Hamiltonian,H, becomes the Hamiltonians of quantum dots including spin-orbit coupling [17]

c) $\kappa_2 = \kappa_3 = \kappa_4 = \gamma_1 = \gamma_3 = \gamma_4 = 0, \kappa_1 = \gamma_2$ and $\beta = \frac{\omega_0}{2}$ the Hamiltonian,H, becomes JC Hamiltonian with RWA

d) $\omega_2 = \kappa_3 = \kappa_4 = \gamma_3 = \gamma_4 = 0$, $\kappa_1 = \kappa_2 = \gamma_1 = \gamma_2 = \kappa, \beta = \frac{\hbar\omega_0}{2}$ and $\omega_1 = \omega$ then we get JC Hamiltonian without RWA

e) $\kappa_2 = \kappa_4 = \gamma_1 = \gamma_3 = 0$, $\kappa_1 = \gamma_2 = \lambda_1$, $\kappa_3 = \gamma_4 = \lambda_2$, and $\beta = \hbar\omega_0$ the Hamiltonian,H,takes the form of Modified JC (MJC) Hamiltonian.

The corresponding Hamiltonians are illustrated in the first column and their algebras and appropriate realizations are illustrated in the last column of Table 3.1. The parameters $\kappa' = \sqrt{\frac{m\omega}{4\hbar}}\kappa, \quad \kappa'' = 2ic\sqrt{m\omega\hbar}, \quad \lambda' = \sqrt{\frac{m\omega}{4\hbar}}\lambda_{R.}$

It can be checked easily that for some certain values of κ_i, γ_i and β the equation can be solved in the framework of $su(1,1)$ or $su(2)$ Lie algebra technique

Related	Parameters											Symmetry
Hamiltonian	κ_1	κ_2	κ_3	κ_4	γ_1	γ_2	γ_3	γ_4	ω_1	ω_2	β	Group
H_{JT}	κ'	0	0	κ'	0	κ'	κ'	0	ω	ω	$\frac{\mu}{2}$	$su(1,1)$
H_{dot}	λ'	0	0	$-\lambda'$	0	λ'	$-\lambda'$	0	ω	ω	$\frac{\mu}{2}gB$	$su(1,1)$
H_{JC}	κ	κ	0	0	κ	κ	0	0	ω	0	$\frac{\hbar\omega_0}{2}$	$su(1,1)$
H_{JC}^{RWA}	κ	0	0	0	0	κ	0	0	ω	0	$\frac{\hbar\omega_0}{2}$	$su(1,1)$
H_{MJC}	λ_1	0	λ_2	0	0	λ_1	0	λ_2	ω	ω	$\hbar\omega_0$	$su(2)$

Table 3.1: List of the Hamiltonians depending on the choices of the parameters of Hamiltonian 3.1

and the Hamiltonian can be related to the various physical Hamiltonians of interest. The resulting Hamiltonians are summarized in the Table 3.1.

The results of our investigation are that, for the Hamiltonians whose list given in Table 3.1, are demonstrated in two categories; exactly [27] [49, 50] and quasi-exactly solvable Hamiltonians [51, 54]. It will be shown that, the rotating wave approximated JC Hamiltonian, and modified JC Hamiltonian, as they expected, exactly solvable, while JT, JC, quantum dot and Rabi Hamiltonians are quasi-exactly solvable.

One way to relate a Hamiltonian with an appropiate Lie algebra is to construct its bosonic and fermionic representation. We are inserted in the two-level system in a one and two-dimensional geometry, whose Hamiltonians are given in terms of bosons-fermions or matrix-differential equations. Therefore, it is worth to express a suitable differential realizations of the bosons. By the use of the differential realization of the operators one can easily find the connection between boson-fermion and matrix differential equation formalism of the Hamiltonians. To this end, let us start by introducing the following differential realizations of the boson operators:

$$\begin{aligned} a^+ &= \frac{\ell}{2}(x+iy) - \frac{1}{2\ell}(\partial_x + i\partial_y) \\ a &= \frac{\ell}{2}(x-iy) + \frac{1}{2\ell}(\partial_x - i\partial_y) \\ b^+ &= \frac{\ell}{2}(x-iy) - \frac{1}{2\ell}(\partial_x - i\partial_y) \\ b &= \frac{\ell}{2}(x+iy) + \frac{1}{2\ell}(\partial_x + i\partial_y) \end{aligned} \tag{3.4}$$

where $\ell = \sqrt{\frac{m\omega}{\hbar}}$ is the length parameter and the boson operators obey the usual commutation relation

$$[a, a^+] = [b, b^+] = 1; \qquad [a, b^+] = [b, a^+] = [a, b] = [a^+, b^+] = 0 \tag{3.5}$$

In principle, if a Hamiltonian is expressed by boson operators, one could rely directly on the known formulae of the action of boson operators on a state with a defined number of particles without solving differential equations. Apart from the mentioned method, sometimes the Hamiltonians can be put in a simple form by using the transformation properties of the bosons.

3.2 Method

In our method the bosonised Hamiltonians are connected with $su(1,1)$ and $su(2)$, as well as $Sp(4,R)$ Lie algebras. This connection opens the way to an algebraic treatment of a large class of physical Hamiltonians of practical interest. In this section we present a general procedure to solve the Hamiltonian (3.1). Consider eigenvalue equation

$$H\Psi = E\Psi \tag{3.6}$$

where Ψ is two component wavefunction and E is eigenvalues of the Hamiltonian H. Consequently the Hamiltonian (3.1) can be written as

$$(H_0 - E - \beta)\Psi_1 + (\kappa_1 a + \kappa_2 a^+ + \kappa_3 b + \kappa_4 b^+)\Psi_2 = 0 \tag{3.7}$$

$$(H_0 - E + \beta)\Psi_2 + (\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+)\Psi_1 = 0 \tag{3.8}$$

These coupled equations may be solved by using various techniques. In here we follow new strategies. In the first step we eliminate Ψ_2(or Ψ_1) between the above equation

$$\Psi_2 = -(H_0 - E + \beta)^{-1}(\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+)\Psi_1 \tag{3.9}$$

substituting (3.9) into (3.7) we obtain following equation

$$(H_0 - E - \beta)\Psi_1 -$$

$$(\kappa_1 a + \kappa_2 a^+ + \kappa_3 b + \kappa_4 b^+)(H_0 - E + \beta)^{-1}(\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+)\Psi_1 = 0 \tag{3.10}$$

The last equation can be solved by performing a suitable realizations of the $su(1,1)$ and $su(2)$ algebra. In our formalism, in order to obtain an adequate form of the (3.10), in the next step, we use the relations

$$\begin{aligned} a(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta - \omega_1)^{-1} a \\ a^+(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta + \omega_1)^{-1} a^+ \\ b(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta - \omega_2)^{-1} b \\ b^+(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta + \omega_2)^{-1} b^+ \end{aligned} \tag{3.11}$$

by setting $\omega_1 = \omega_2 = \omega$, we obtain the following general expression

$$(H_0 - E + \beta + \hbar\omega)(H_0 - E + \beta - \hbar\omega)(H_0 - E - \beta)\Psi_1 =$$

$$\kappa_1(H_0 - E + \beta + \hbar\omega)(\gamma_1 a^2 + \gamma_2 aa^+ + \gamma_3 ab + \gamma_4 ab^+)\Psi_1 +$$
$$\kappa_2(H_0 - E + \beta - \hbar\omega)(\gamma_1 a^+ a + \gamma_2 a^{+^2} + \gamma_3 a^+ b + \gamma_4 a^+ b^+)\Psi_1 +$$
$$\kappa_3(H_0 - E + \beta + \hbar\omega)(\gamma_1 ba + \gamma_2 ba^+ + \gamma_3 b^2 + \gamma_4 bb^+)\Psi_1 +$$
$$\kappa_2(H_0 - E + \beta - \hbar\omega)(\gamma_1 b^+ a + \gamma_2 b^+ a^+ + \gamma_3 b^+ b + \gamma_4 b^{+^2})\Psi_1. \quad (3.12)$$

This equation will lead us to find the exact and quasi-exact solution of several kinds of physical systems.

3.3 Realizations of $su(2)$ by Boson Operators

It is well known that if the Hamiltonians characterized by single or double boson operators, then the simplest way to find the symmetry algebra of the corresponding Hamiltonian is that to construct the single or double boson realizations of the algebras. In this section we introduce some basic boson realizations of $su(2)$ which we need to solve the Hamiltonians given in previous section.The $su(2)$ algebra can be constructed by two mode bosons in two dimensions by introducing the following three generators

$$J_+ = a^+ b; \qquad J_- = b^+ a; \qquad J_0 = \frac{1}{2}(a^+ a - b^+ b) \quad (3.13)$$

satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm; \qquad [J_+, J_-] = 2J_0 \quad (3.14)$$

The number operator which commutes with the generators of the $su(2)$ algebra is given by

$$N = a^+ a + b^+ b \quad (3.15)$$

Casimir invariant of $su(2)$ can be related to N by

$$C = \frac{1}{4}N(N+2). \quad (3.16)$$

The eigenvalues of C are given by

$$\langle C \rangle = j(j+1) \quad (3.17)$$

It is obvious that the irreducible representations of $su(2)$ can be characterized by the total boson number $N = 2j$. The application of the realization (J_+, J_-) on a set of $2j+1$ states, leads to the $(2j+1)$-dimensional unitary irreducible representation for each $j = 0, 1/2, 1, ...$If the basis states $|j, m\rangle$ $(m = j, j-1, ... - j)$, then the action of the operators on the basis is given by

$$J_0 \, |j, m\rangle = m \, |j, m\rangle$$

$$J_{\pm}|j,m\rangle = \sqrt{(j\mp m)(j\mp m+1)}\,|j,m\pm 1\rangle$$
$$C|j,m\rangle = j(j+1)\,|j,m\rangle \tag{3.18}$$

Furthermore, the single boson realizations of $su(2)$ algebra can be constructed by defining three operators

$$J'_{+} = \sqrt{2j-N'a}; \qquad J'_{-} = a^{+}\sqrt{2j-N'}; \qquad J'_{0} = j-N'; \qquad N' = a^{+}a \tag{3.19}$$

also satisfy the commutation relations of the $su(2)$ algebra.

3.4 Realizations of $su(1,1)$ by Boson Operators

The Lie algebra $su(1,1)$ possesses interesting realizations by bosons and more appropriate to solve the many physical problems. Using the set of boson operators (3.4) we introduce the three operators

$$K_{+} = a^{+}b^{+}; \qquad K_{-} = ab; \qquad K_{0} = \frac{1}{2}(a^{+}a+b^{+}b+1) \tag{3.20}$$

The reader can easily check that the K$'s$ satisfy the $su(1,1)$ commutation relations

$$[K_0, K_{\pm}] = \pm K_{\pm}; \qquad [K_{+}, K_{-}] = -2K_0. \tag{3.21}$$

While in the $su(2)$ case the number operator was the sum, N , of the boson number operators, in the present case it is the operator

$$M = a^{+}a - b^{+}b, \tag{3.22}$$

which is the difference of the number operators. The casimir invariant of the $su(1,1)$ is related to M by

$$C = \frac{1}{4}(1+M)(1-M). \tag{3.23}$$

Therefore if the eigenvalues of the operator C is $k(1-k)$ we find that $M = (1-2k)$.Consequently, the action of the realization (3.8) on the states $|k,n\rangle$, $(n = 0, 1, 2, ...)$, leads to infinite dimensional unitary irreducible representation, so called positive representation $D^{+}(k)$, corresponds to any $k = 1/2, 1, 3/2, ...$

$$K_0|k,n\rangle = (k+n)\,|k,n\rangle$$

$$K_{+}|k,n\rangle = \sqrt{(2k+n)(n+1)}\,|k,n+1\rangle$$
$$K_{-}|k,n\rangle = \sqrt{(2k+n-1)n}\,|k,n-1\rangle$$

$$C\,|k,n\rangle = k(1-k)\,|k,n\rangle \tag{3.24}$$

We will end this section by introducing two different single boson realizations of the $su(1,1)$ algebra. One of them can be constructed by the operators:

$$L_+ = \frac{1}{2}a^{+^2}; \qquad L_- = \frac{1}{2}a^2; \qquad L_0 = \frac{1}{2}(a^+a + \frac{1}{2}); \qquad M' = a^+a; \tag{3.25}$$

For the single-mode bosonic realization of $su(1,1)$ that we require here, the Bargmann index k is equal to either 1/4 or 3/4 which split the Hilbert space of the boson space into two independent subspace. The other realization is

$$S_+ = a^+\sqrt{M' + 2k}; \qquad S_- = \sqrt{M' + 2k}a; \qquad S_0 = M' + k; \quad k\rangle 0. \tag{3.26}$$

It will be shown that in particular the realizations (3.20) and (3.25) play dominant role on the solution of the Hamiltonian (3.1).

Chapter 4

Solutions to the Hamiltonians

In this chapter we give the solutions of several quantum optical models in the framework of our method which is developed in chapter 3.

4.1 Quasi-Exactly Solvable Hamiltonians

4.1.1 Solution of the JC Hamiltonian

JC Hamiltonian without RWA is one of the QES problems that can be associated to the Hamiltonian (3.1). With the choices of the parameters $\omega_2 = \kappa_3 = \kappa_4 = \gamma_3 = \gamma_4 = 0$, $\kappa_1 = \kappa_2 = \gamma_1 = \gamma_2 = \kappa, \beta = \frac{\hbar\omega_0}{2}$ and $\omega_1 = \omega$ thus the general expression (3.12) takes the form

$$\left(H_0 - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right)\left(H_0 - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right)\left(H_0 - E - \frac{\hbar\omega_0}{2}\right)\Psi_1 =$$

$$\begin{aligned} &\kappa^2(H_0 - E + \frac{\hbar\omega_0}{2} + \hbar\omega)(a^2 + aa^+)\Psi_1 + \\ &\kappa^2(H_0 - E + \frac{\hbar\omega_0}{2} - \hbar\omega)(a^+a + a^{+^2})\Psi_1. \end{aligned} \tag{4.1}$$

The JC Hamiltonian can be expressed in terms of the $su(1,1)$ generators given in (3.25)

$$\left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right)\left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right)\left(\hbar\omega M' - E - \frac{\hbar\omega_0}{2}\right)\Psi_1 =$$

$$\begin{aligned} &\kappa^2\left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right)(2L_0 + M' + 1)\Psi_1 + \\ &\kappa^2\left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right)(2L_+ + M')\Psi_1 \end{aligned} \tag{4.2}$$

Here the Bargmann index k is $\frac{1}{4}$ for the even state $\Psi_1 = \left|\frac{1}{4}, 2n\right\rangle$ and $k = \frac{3}{4}$ for odd state $\Psi_1 = \left|\frac{3}{4}, 2n+1\right\rangle$.Under the action of the operators on the even state $\left|\frac{1}{4}, 2n\right\rangle$ we obtain the following recurrence relation

$$\left(2\hbar\omega n - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right)\left(2\hbar\omega n - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right)\left(2\hbar\omega n - E - \frac{\hbar\omega_0}{2}\right)|2n\rangle =$$

$$\kappa^2\left(\hbar\omega n - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right)(\sqrt{2n(2n-1)}\,|2n-2\rangle + (2n+1)\,|2n\rangle)$$

$$\kappa^2\left(\hbar\omega n - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right)(\sqrt{(2n+1)(2n+2}\,|2n+2\rangle + 2n\,|2n\rangle) \qquad (4.3)$$

The eigenvalues can be obtained from the recurrence relation and its ground state energy is given by

$$n = 0; E = \frac{1}{2}\left(-\hbar\omega \pm \sqrt{4\kappa^2 + \hbar^2(\omega - \omega_0)^2}\right). \qquad (4.4)$$

In order to obtain the excited states energies, it is enough to perform calculation in (4.4), but for higher values of n (i.e $n\rangle 5$) it requires numerical treatments.

4.1.2 Solution of the Rashba Hamiltonian

Another physically important QES Hamiltonian is the Hamiltonian of the quantum a dot including spin orbit coupling obtained from the formula (3.12), by setting the parameters: $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$, we obtain $\kappa_4 = \kappa_{-1} = \gamma_{-2} = \gamma_3 = -\sqrt{\frac{m\omega}{4\hbar}}\lambda_R$ and $\beta = \frac{1}{2}g\mu B$:

$$\left(H_0 - E + \frac{1}{2}g\mu B + \hbar\omega\right)\left(H_0 - E + \frac{1}{2}g\mu B - \hbar\omega\right)\left(H_0 - E - \frac{1}{2}g\mu B\right)\Psi_1 =$$

$$\frac{m\omega}{4\hbar}\lambda_R^2\left(H_0 - E + \frac{1}{2}g\mu B + \hbar\omega\right)(aa^+ + ab)\Psi_1 +$$

$$\frac{m\omega}{4\hbar}\lambda_R^2\left(H_0 - E + \frac{1}{2}g\mu B - \hbar\omega\right)(b^+a^+ + b^+b)\Psi_1 \qquad (4.5)$$

The appropriate Lie algebra of this system is $su(1,1)$ and it can be written in terms of the generators (3.20):

$$\left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B\right)\left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B - 2\hbar\omega\right)\left(2\hbar\omega K_0 - E - \frac{1}{2}g\mu B - \hbar\omega\right)\Psi_1 =$$

$$\frac{m\omega}{4\hbar}\lambda_R^2\left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B\right)\left(K_0 + K_- + \frac{M}{2}\right)\Psi_1 +$$

$$\frac{m\omega}{4\hbar}\lambda_R^2\left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B - 2\hbar\omega\right)\left(K_0 + K_+ - \frac{M+1}{2}\right)\Psi_1. \qquad (4.6)$$

The basis function of the (4.6) is $\Psi_1 = |k, n\rangle$, by using the action of the generators on the state, we obtain the three term recurrence relation

$$(2\hbar\omega(k+n)-E_+)(2\hbar\omega(k+n)-E_+-2\hbar\omega)(2\hbar\omega(k+n)-E_--\hbar\omega)\left|k,n\right\rangle =$$
$$\frac{m\omega}{4\hbar}\lambda_R^2(2\hbar\omega(k+n)-E_+)\left((n+\frac{1}{2})\left|k,n\right\rangle+\sqrt{(2k+n-1)n}\left|k,n-1\right\rangle\right)+$$
$$\frac{m\omega}{4\hbar}\lambda_R^2(2\hbar\omega(k+n)-E_+-2\hbar\omega)\left((2k+n-1)\left|k,n\right\rangle+\sqrt{(2k+n)(n+1)}\left|k,n+1\right\rangle\right). \tag{4.7}$$

where $E_\pm = E \mp \frac{1}{2}g\mu B$.The solution of the recurrence relation for each values of $k = 0, 1, 2, ...$and $n = 0, 1, 2, ..., k$ gives an expression for the eigenvalues of the Hamiltonian H_{dot}.

4.1.3 Solution of the $E \times \varepsilon$ JT Hamiltonian

Other example is the JT Hamiltonian which can also be treated as the QES problem. In this case the parameters take the values: $\gamma_1 = \gamma_4 = \kappa_2 = \kappa_3 = 0$, $\gamma_2 = \gamma_3 = \kappa_1 = \kappa_4 = \sqrt{\frac{m\omega}{4\hbar}}\kappa$ and $\beta = \frac{\mu}{2}$. Thus the eigenvalue equation takes the form:

$$\left(H_0 - E + \frac{\mu}{2} + \hbar\omega\right)\left(H_0 - E + \frac{\mu}{2} - \hbar\omega\right)\left(H_0 - E - \frac{\mu}{2}\right)\Psi_1 =$$
$$\frac{m\omega}{4\hbar}\kappa^2\left(H_0 - E + \frac{\mu}{2} + \hbar\omega\right)(aa^+ + ab)\Psi_1 +$$
$$\frac{m\omega}{4\hbar}\kappa^2\left(H_0 - E + \frac{\mu}{2} - \hbar\omega\right)(b^+a^+ + b^+b)\Psi_1. \tag{4.8}$$

The JT problem can also possesses $su(1,1)$ symmetry and in terms of its generators it can be written as

$$\left(2\hbar\omega K_0 - E + \frac{\mu}{2}\right)\left(2\hbar\omega K_0 - E + \frac{\mu}{2} - 2\hbar\omega\right)\left(2\hbar\omega K_0 - E - \frac{\mu}{2} - \hbar\omega\right)\Psi_1 =$$
$$\frac{m\omega}{4\hbar}\kappa^2\left(2\hbar\omega K_0 - E + \frac{\mu}{2}\right)\left(K_- + K_0 + \frac{M+1}{2}\right)\Psi_1 +$$
$$\frac{m\omega}{4\hbar}\kappa^2\left(2\hbar\omega K_0 - E + \frac{\mu}{2} - 2\hbar\omega\right)\left(K_+ + K_0 - \frac{M+1}{2}\right)\Psi_1. \tag{4.9}$$

This algebraic equation with the basis $\Psi_1 = \left|k,n\right\rangle$, leads to three term recurrence relation:

$$(2\hbar\omega(k+n)-E^+)(2\hbar\omega(k+n)-E^+-2\hbar\omega)(2\hbar\omega(k+n)-E^--\hbar\omega)\left|k,n\right\rangle =$$
$$\frac{m\omega}{4\hbar}\kappa^2(2\hbar\omega(k+n)-E^+)\left(\sqrt{(2k+n-1)n}\left|k,n-1\right\rangle+(n+1)\left|k,n\right\rangle\right)$$
$$\frac{m\omega}{4\hbar}\kappa^2(2\hbar\omega(k+n)-E^+-2\hbar\omega)\left(\sqrt{(2k+n)(n+1)}\left|k,n+1\right\rangle+(2k+n-1)\left|k,n\right\rangle\right). \tag{4.10}$$

where $E^{\pm} = E \mp \frac{\mu}{2}$.The eigenvalues of the JT Hamiltonian can be obtained by solving the recurrence relation (4.10).

As a consequence we have demonstrated that the solution of the Hamiltonian (3.1) can be treated within the method presented in the previous section for certain values of the parameters.

4.2 Exactly Solvable Hamiltonians

4.2.1 Solution of the JC Hamiltonian with RWA

The JC Hamiltonian with the RWA can be obtained by setting $\kappa_2 = \kappa_3 = \kappa_4 = \gamma_1 = \gamma_3 = \gamma_4 = 0, \kappa_1 = \gamma_2$ and $\beta = \frac{\omega_0}{2}$. In this case the formula (3.12) takes the form

$$\left(H_0 - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right)\left(H_0 - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right)\left(H_0 - E - \frac{\hbar\omega_0}{2}\right)\Psi_1 =$$

$$\kappa^2(H_0 - E + \beta + \hbar\omega)aa^+\Psi_1 \tag{4.11}$$

The algebraic form of the Hamiltonian can be obtained by combining (4.11) and the $su(1,1)$ realization given in (3.25), yields

$$\left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right)\left(\hbar\omega M' - E - \frac{\hbar\omega_0}{2}\right)\Psi_1 = \kappa^2(M' + 1)\Psi_1 \tag{4.12}$$

If the wavefunction $\Psi_1 = \left|\frac{1}{4}, n\right\rangle$ then the action of the (4.12) on the state leads to the following expression for the eigenvalues of the JC Hamiltonian

$$E = \left(n - \frac{1}{2}\right)\hbar\omega \pm \frac{1}{2}\sqrt{4\kappa^2(1+n) + \hbar^2(\omega - \omega_0)^2} \tag{4.13}$$

It is obvious that when the coupling constant κ is zero then the result is the eigenvalues of the shifted simple harmonic oscillator.

4.2.2 Solution of the MJC Hamiltonian

Now we will consider the MJC model. The Hamiltonian can be obtained by choosing parameters:$\kappa_2 = \kappa_4 = \gamma_1 = \gamma_3 = 0$, $\kappa_1 = \gamma_2 = \lambda_1$, $\kappa_3 = \gamma_4 = \lambda_2$, and $\beta = \hbar\omega_0$ then (3.12) takes form:

$$(H_0 - E + \hbar\omega_0 + \hbar\omega)(H_0 - E + \hbar\omega_0 - \hbar\omega)(H_0 - E - \hbar\omega_0)\Psi_1 =$$

$$\lambda_1(H_0 - E + \hbar\omega_0 + \hbar\omega)(\lambda_1 aa^+ + \lambda_2 ab^+)\Psi_1 +$$

$$\lambda_2(H_0 - E + \hbar\omega_0 + \hbar\omega)(\lambda_1 ba^+ + \lambda_2 bb^+)\Psi_1 \tag{4.14}$$

We insert (3.13) and (3.15), the realization of $su(2)$, into (4.14) we obtain the following expressions

$$(\hbar\omega N - E + \hbar\omega_0 - \hbar\omega)(\hbar\omega N - E - \hbar\omega)\Psi_1 =$$
$$((\lambda_1^2 + \lambda_2^2)\left(1 + \frac{N}{2}\right) + (\lambda_1^2 - \lambda_2^2)J_0 + \lambda_1\lambda_2(J_+ + J_-))\Psi_1 \tag{4.15}$$

The above equation is not diagonal, but it can be diagonalized by similarity transformation, by the operator

$$O = e^{\frac{\alpha}{2}(J_+ - J_-)}. \tag{4.16}$$

The action of the operator on the generators of $su(2)$ is given by

$$\begin{aligned} O(J_+ + J_-)O^\dagger &= (J_+ + J_-)\cos\alpha + 2J_0\sin\alpha \\ OJ_0O^\dagger &= J_0\cos\alpha - \frac{J_+ + J_-}{2}\sin\alpha \\ ONO^\dagger &= N. \end{aligned} \tag{4.17}$$

The transformations of the generators of $su(2)$ by the operator O are given by

$$(\hbar\omega N - E + \hbar\omega_0 - \hbar\omega)(\hbar\omega N - E - \hbar\omega_0)\Psi_1 =$$
$$((\lambda_1^2 + \lambda_2^2)\left(1 + \frac{N}{2}\right)\Psi_1 + ((\lambda_1^2 - \lambda_2^2)\cos\alpha + 2\lambda_1\lambda_2\sin\alpha)J_0\Psi_1 +$$
$$(\lambda_1\lambda_2\cos\alpha - \frac{(\lambda_1^2 - \lambda_2^2)}{2}\sin\alpha)(J_+ + J_-)\Psi_1 \tag{4.18}$$

The Hamiltonian takes the diagonal form when the following condition hold:

$$\alpha = \cos^{-1}\left(\frac{(\lambda_1^2 - \lambda_2^2)}{(\lambda_1^2 + \lambda_2^2)}\right). \tag{4.19}$$

The final form of the diagonal Hamiltonian under the condition (4.19) is given by

$$(\hbar\omega N - E + \hbar\omega_0 - \hbar\omega)(\hbar\omega N - E - \hbar\omega_0)\Psi_1 =$$
$$(\lambda_1^2 + \lambda_2^2)\left(1 + \frac{N}{2}\right)\Psi_1 + (\lambda_1^2 + \lambda_2^2)J_0\Psi_1 \tag{4.20}$$

It is obvious that when $\Psi_1 = |j, m\rangle$, which is the eigenstate of the operators J_0, and N, we obtain the following expression for the eigenvalues of the MJC Hamiltonian:

$$E = (2j - \frac{1}{2})\hbar\omega - \hbar\omega_0 \pm \frac{1}{2}\sqrt{\hbar^2\omega^2 + 4(\lambda_1^2 + \lambda_2^2)(j + m + 1)}. \tag{4.21}$$

We have solved JC models and various physical Hamiltonians in the framework of the method given in section 3.

Chapter 5

Rabi Hamiltonian and JC Model with Kerr Nonlinearity

In previous chapter using our method solutions to physical Hamiltonians are given. This method is not unique, there are another approaches to solve physical Hamiltonians. In this chapter we express one of these methods and solve Rabi Hamiltonian and JC Model with Kerr nonlinearity.

5.1 Rabi Hamiltonian

In the cyclooctatetraene molecular ion, which is a particular case of molecules having a fourfold axis symmetry, in the resonant excitation of double molecules or dimers, a doubly degenerate state becomes coupled by a single mode. This system is known as the $E \otimes \beta$ Jahn-Teller system which helps us to understand the more complex cases of the Jahn-Teller effect. The $E \times \beta$ Jahn-Teller system coupled to a system executing harmonic oscillations whose energy eigenvalues differ by 2μ is characterized by the Rabi Hamiltonian

$$H = a^+a + \kappa\sigma_3(a^+ + a) + \mu(\sigma^+ + \sigma^-) \tag{5.1}$$

where $\sigma^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices and the parameter κ is a linear coupling constant. Hamiltonian (5.1) can be expressed as a differential equation in the Bargmann-Fock space by using the realizations of the bosonic operators

$$a^+ = z \qquad\qquad a = \frac{d}{dz} \tag{5.2}$$

In this formulation, the Schrödinger equation consists of two independent sets of linear first-order differential equations. Substituting (5.2) into (5.1) we obtain a system of two linear differential equations for the functions $\Psi_1(z)$ and $\Psi_2(z)$

$$(z+\kappa)\frac{d\Psi_1(z)}{dz} + (\kappa z - E)\Psi_1(z) + \mu\Psi_2(z) = 0$$

$$(z-\kappa)\frac{d\Psi_2(z)}{dz} - (\kappa z + E)\Psi_2(z) + \mu\Psi_1(z) = 0 \tag{5.3}$$

where E is the eigenvalue of the Rabi Hamiltonian. We eliminate $\Psi_2(x)$ between the two equations, and substituting

$$z = \kappa(2x-1) \qquad \Psi_1(x) = e^{-2\kappa^2 x}R(x) \tag{5.4}$$

we obtain a second order differential equation

$$x(1-x)\frac{d^2R(x)}{dx^2} + [\kappa^2(4x^2-2x-1) + E(2x-1) - x + 1]\frac{dR(x)}{dx}$$

$$+[\kappa^4(-4x+3) - E^2 + 2E\kappa^2(-2x+1) + \mu^2]R(x) = 0 \tag{5.5}$$

In order to understand quasi exact solvability of (5.5) the standard way is to demonstrate that the Hamiltonian can be expressed in terms of generators of the Lie algebra. Let us consider the algebra $su(1,1)$ expressed in the following form

$$J_- = \frac{d}{dx} \qquad J_0 = x\frac{d}{dx} - j \qquad J_+ = x^2\frac{d}{dx} - 2jx \tag{5.6}$$

The generators obey the commutation relation

$$[J_+, J_-] = -2J_0 \qquad [J_0, J_\pm] = \pm 2J_\pm \tag{5.7}$$

If $2j$ is a positive integer, the algebra possesses $(2j+1)$-dimensional subspace

$$R(x) = \{1, x, x^2, ..., x^{2j}\} \tag{5.8}$$

We first introduce the following linear and bilinear combination of the algebraic operators

$$T = -J_+J_- + J_-J_0 - jJ_- + 4\kappa^2 J_+ - (4\kappa^2 - 2j + 1)J_0 + \mu^2 \tag{5.9}$$

for which one can define the spectral problem

$$TR(x) = \lambda R(x) \tag{5.10}$$

where λ is a spectral parameter. The algebraic structure (5.9) is quasi exactly solvable. The insertion of (5.6) into (5.9) leads to the following differential equation

$$x(1-x)\frac{d^2R(x)}{dx^2} + [2j(2x-1) + (x-1)(4\kappa^2 x - 1)]\frac{dR(x)}{dx}$$
$$+[\mu^2 + j(1-2j) + 4\kappa^2(1-2x) - \lambda]R(x) = 0 \tag{5.11}$$

Equations (5.5) and (5.11) are identical under the conditions

$$E = 2j - \kappa^2 \qquad \lambda = j(1 + 2j - 4\kappa^2) \tag{5.12}$$

Thus we have shown that the Rabi Hamiltonian is QES. There are various techniques to solve (5.11). Here we obtain its solution with the theory of orthogonal polynomials.

5.2 Determination of Eigenvalues and Eigenfunctions of the Rabi Hamiltonian

We seek a solution of (5.11) to obtain eigenfunction and eigenvalues of the QES. Since the function $R(x) = \{1, x, x^2, ..., x^{2j}\}$ forms a basis function for $su(1,1)$ algebra, we search for a solution of (5.11) by substituting the polynomial of degree $2j$

$$R(x) = \sum_{m=0}^{2j} a_m x^m \tag{5.13}$$

The wavefunction is itself the generating function of the energy polynomials. The eigenvalues are then produced just by the roots of such polynomials. Therefore (5.13) can be written in the form

$$R(x) = \sum_{m=0}^{2j} a_m P_m(\kappa) x^m \tag{5.14}$$

Substituting (5.14) into (5.11) and carrying out a straightforward calculation, we obtain the expression

$$R(x) = 1 + \frac{4P_{2j-1}(\kappa)(\kappa x)^{2j}}{\mu^2} + \sum_{m=0}^{2j} P_m(\kappa) x^m \tag{5.15}$$

Here $P_m(\kappa)$ satisfies the recurrence relation

$$(m+1)(m+1-2j)P_{m+1}(\kappa) + [m - 2j - 4\kappa^2 - m) + \mu^2]P_m(\kappa)$$

$$+4\kappa^2(m-1-2j)P_{m-1}(\kappa) = 0 \tag{5.16}$$

with the initial conditions $P_{-1}(\kappa) = 0$ and $P_0(\kappa) = 1$. Certain properties of the polynomial $P_m(\kappa)$ can be discussed. If κ_j is a root of the polynomial $P_{m+1}(\kappa)$, the series (5.15) truncates at m $2j+1$ and κ_j belongs to the spectrum of the Rabi Hamiltonian. Therefore the solution given in (5.15) terminates at $m = 2j$ and it becomes a polynomial of degree $2j$. The first four of them are given by

$$P_1(\kappa) = \mu^2 \tag{5.17}$$

$$P_2(\kappa) = \mu^2(4\kappa^2 + \mu^2 - 1) \tag{5.18}$$

$$P_3(\kappa) = \mu^2(32\kappa^4 + 4(3\mu^2 - 8)\kappa^2 + \mu^2(\mu^2 - 5) + 4) \tag{5.19}$$

$$\begin{aligned} P_4(\kappa) = {} & \mu^2(384\kappa^6 + 16(11\mu^2 - 54)\kappa^4 + 8(3\mu^4 - 29\mu^2 + 54)\kappa^2 + \\ & \mu^2(\mu^2 - 7)^2 - 36) \end{aligned} \tag{5.20}$$

for $j = 0, 1/2, 3/2$, respectively.The components of the eigenfunctions are expressed as

$$\Psi_1(x) = N_1 e^{-\kappa^2 x} R(x) \tag{5.21}$$

$$\Psi_2(x) = N_2 e^{-\kappa^2 x}(2(j - \kappa^2 x)R(x) - xR'(x)) \tag{5.22}$$

where N_1 and N_2 are normalization constants. It is obvious that the degrees of polynomials in the expressions for $\Psi_1(x)$ and $\Psi_2(x)$ are $2j$ and $2j+1$, respectively. The polynomials given in (3.38-3.41) are exactly the same results obtained by the method of Juddian isolated exact solution. In the following section we discuss the results.

The eigenfunction can be obtained for a given j. As an example consider the $j = 1/2$ case. The polynomial $P_m(\kappa) = 0$. The zeros of the $P_2(\kappa)$ are given by

$$\mu = 0 \qquad\qquad \mu = \pm\sqrt{1 - 4\kappa^2} \tag{5.23}$$

Under the assumption $\mu \neq 0$ we obtain the exact solution of the Rabi hamiltonian, and its normalized eigenfunctions can be written as

$$\Psi_1(x) = \frac{(8 + 4\mu^2 + \mu^4)e^{-\kappa^2 x}}{2\kappa^2\mu^4} \tag{5.24}$$

$$\Psi_2(x) = \frac{(728 - 288\mu^2 - 19\mu^4 + 6\mu^6 + \mu^8)e^{-\kappa^2 x}}{32\kappa^6}((1 + 2\kappa^2)x - 1)(\mu^2 + 4\kappa^2 x) \tag{5.25}$$

with the eigenvalues

$$E = 1 - \kappa^2 \tag{5.26}$$

The condition for the normalizability of the functions is given by

$$Re(\kappa^2) > 0 \qquad and \qquad -\frac{\pi}{4} < \arg(\kappa) < \frac{\pi}{4} \tag{5.27}$$

When $j = 1$ the roots of the polynomial $P_3(\kappa)$ can be obtained from (5.19) and they read

$$\mu = 0 \qquad \mu = \sqrt{\frac{5}{2} - 6\kappa^2 \pm \sqrt{16\kappa^4 + 8\kappa^2 + 9}} \tag{5.28}$$

The corresponding eigenfunctions are obtained by evaluating (5.24) and (5.25). The unnormalized eigenfunctions for $j = 1$ are given by

$$\Psi_1(x) = e^{-\kappa^2 x}\left(1 + (8\kappa^2 + \mu^2 - 4)\left(x + \frac{4\kappa^2}{\mu^2}x^2\right)\right)$$

$$\begin{aligned} \Psi_2(x) &= 2 + (14\kappa^2 + 2\mu^2 - 9)x - [8\kappa^2 + \mu^2 - 4] \\ &\times \left[\left(1 + 2\kappa^2 - \frac{8\kappa^2}{\mu^2}\right)x^2 - \frac{4\kappa^2(1 + 2\kappa^2)x^3}{\mu^2}\right] \end{aligned} \tag{5.29}$$

We have checked that the eigenfunctions are normalizable under the condition given in (5.27).

5.3 Eigenvalues of the JC Model with Kerr Nonlinearity

In this part we discuss the eigenvalues of effective Hamiltonian which represents the JC model surrounded by a Kerr medium.This Hamiltonian has the form;

$$H = \omega a^{+}a + \frac{1}{2}\omega_0\sigma_0 + \kappa(a^{+}\sigma_{-} + a\sigma_{+}) + \lambda a^{+}aa^{+}a \tag{5.30}$$

where κ is atom-field coupling constant , λ is nonlinearity of Kerr medium, $a^{+}$and a are creation and annihilation operators, $\sigma_{0,}\sigma_{+,}\sigma_{-}$ are Pauli matrices. Number operator of this structure is ,

$$N = a^{+}a + \frac{1}{2}\sigma_0 \tag{5.31}$$

The invariance algebra if the Hamiltonian (5.30) is generated by introducing the generators

$$J_{+} = a\sigma_{+}, \qquad J_{-} = a^{+}\sigma_{-}, \qquad J_0 = a^{+}a + \sigma_0 \tag{5.32}$$

yields the commutation relation,

$$[J_0, J_{\pm}] = \pm J_{\pm}, \qquad [J_{+}, J_{-}] = (1 + 2J_0)(J_0 - N) - \frac{1}{2} \tag{5.33}$$

The commutation relation $[J_{+}, J_{-}]$ is a polynomial in J_0. Since the deformation is quadratic in J_0, we have a quadratic algebra. The algebras of type have been considered as deformed $su(2)$ algebra. We can easily express the Hamiltonian (5.30) in terms of the generators of the deformed $su(2)$ algebra,

$$H = \omega(2N - J_0) + \omega_0(J_0 - N) + \kappa(J_{+} + J_{-}) + \lambda(2N - J_0) \tag{5.34}$$

Deformed $su(2)$ algebra generators have the forms

$$\begin{aligned}
J_{+}\left|j, m\right\rangle &= \sqrt{(j - m)(j + m + 1)}\left|j, m + 1\right\rangle \\
J_{-}\left|j, m\right\rangle &= \sqrt{(j + m)(j - m + 1)}\left|j, m - 1\right\rangle \\
J_0\left|j, m\right\rangle &= m\left|j, m\right\rangle \\
N\left|j, m\right\rangle &= 2j\left|j, m\right\rangle
\end{aligned} \tag{5.35}$$

If equation (5.35) is inserted in the Hamiltonian (5.34) and thinking about its eigenvalue $H\Psi = E\Psi$, we get

$$[\omega(4j-m)+\omega_0(m-2j)+\lambda(4j-m)^2]\left|j,m\right\rangle+\kappa\left[\sqrt{(j - m)(j + m + 1)}\left|j, m + 1\right\rangle\right]$$

$$+\kappa\left[\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle\right]-E\,|j,m\rangle=0 \tag{5.36}$$

Different eigenvalues can be found for the equation (5.36) when different j and m values are used. Starting from the condition $j=0, m=0$ it is seen that $E=0$. If we write $j=1$, m will take the values of $-1, 0$ and 1. The matrixform of this condition can be constructed as;

$$\begin{pmatrix} 5\omega+25\lambda-3\omega_0 & \sqrt{2}\kappa & 0 \\ \sqrt{2}\kappa & 4\omega+16\lambda-2\omega_0 & \sqrt{2}\kappa \\ 0 & \sqrt{2}\kappa & 3\omega+9\lambda 3\omega_0 \end{pmatrix} \tag{5.37}$$

For $j=2$, m will take the values $-2,-1,0,1,2$ this situation can be represented by the matrix,

$$\begin{pmatrix} 10\omega+100\lambda-6\omega_0 & 2\kappa & 0 & 0 & 0 \\ 0 & 9\omega+81\lambda-5\omega_0 & \sqrt{6}\kappa & 0 & 0 \\ 0 & \sqrt{6}\kappa & 8\omega+64\lambda-4\omega_0 & \sqrt{6}\kappa & 0 \\ 0 & 0 & \sqrt{6}\kappa & 7\omega+49\lambda-3\omega_0 & 0 \\ 0 & 0 & \sqrt{6}\kappa & 2\kappa & 6\omega+36\lambda-2\omega_0 \end{pmatrix} \tag{5.38}$$

For $j=N$, m will take the values of $-N,...,N$ and the general matrix has the $2N+1$ dimension and it can be represented as

$$\begin{pmatrix} (5N)\omega+(5N)^2\lambda-3N\omega_0 & ... & ... & ... \\ ... & A\omega+A^2\lambda-B\omega_0 & ... & ... \\ ... & ... & (A-1)\omega+(A-1)^2\lambda-(B-1)\omega_0 & ... \\ ... & ... & ... & ... \end{pmatrix} \tag{5.39}$$

where A=(5N-1) and B=(3N-1) equating the determinant of this matrice to zero we can find the eigenvalues as done above.

Consequently we have obtained the results for the eigenvalues of the JC Hamiltonian with Kerr nonlinearity.

Chapter 6

Two-Photon Rabi Hamiltonian

The Rabi Hamiltonian [55] is a popular and successful model for the interaction between matter and electromagnetic radiation. Some Jahn-Teller systems can also be characterized by Rabi Hamiltonian. In general this Hamiltonian has been studied within the RWA, which results in the well-known JC Hamiltonian. The full Rabi Hamiltonian is Quasi- Exactly Solvable [14].

When the atomic transitions are induced by the absorption and emission of two photons rather than one, the original Rabi Hamiltonian is extended to the two-photon Rabi Hamiltonian. Such non-linear systems have been of considerable interest with applications including two photon lasers and two-photon bistability [56, 57].

The two-photon Rabi Hamiltonian is studied in [58, 59], within the RWA. Recently, its exact isolated solution has been obtained by Emary and Bishop [57]. Without any approximation they demonstrated that there exist isolated exact solutions for the two-photon Rabi Hamiltonian.

In this chapter we discuss the quasi-exact solvability of the two-photon Rabi Hamiltonian. The Hamiltonian will be transformed into the form of a QES matrix differential equation in section 6.1. In section 6.2 we present a solution for the eigenstates and eigenvalues of the Rabi Hamiltonian in the context of $sl_2(R)$ algebra. Section 6.3 shows that there also exists a solution of the Rabi Hamiltonian without using a matrix realization of the algebra. Finally, in last part we discuss the validity of our method and suggest the possible extensions of the problem.

6.1 Solution of The Two Photon Rabi Hamiltonian

The Hamiltonian of the interaction of a two level atom with two photon can be expressed as [57];

$$h = \frac{\omega_0}{2}\sigma_0 + \omega_1 a^+ a + g((a^+)^2 + (a)^2)(\sigma_+ + \sigma_-) \tag{6.1}$$

where ω_0 is the atomic level splitting, ω_1 is the frequency of the boson mode, $\sigma_+, \sigma_-, \sigma_0$ are Pauli matrices and g is the coupling strength of the atom to the field. When the $(a^+)^2$ and a^2 replaced by a^+ and a respectively, in the last term the Hamiltonian turns out the standard Rabi Hamiltonian.

For the sake of simplicity it is convenient to rescale the Hamiltonian as $h = \omega_1 H$, where

$$H = a^+ a + \kappa((a^+)^2 + (a)^2)(\sigma_+ + \sigma_-) + \mu\sigma_0 \tag{6.2}$$

with $\mu = \frac{\omega_0}{2\omega_1}$ and $\kappa = \frac{2g}{\omega_1}$. Since our aim is to obtain the quasi-exact solution of (6.2) , it is worth to express (6.2) as linear and bilinear combinations of the generators of the $sl_2(R)$ algebra. In the first we simply use similarity transformation by introducing the operator

$$U = \frac{1}{\sqrt{2}}(\sigma_0 + \sigma_+ + \sigma_-)S; \quad U^\dagger = \frac{1}{\sqrt{2}}S^\dagger(\sigma_0 + \sigma_+ + \sigma_-) \tag{6.3}$$

where S is given by

$$S = \exp\left[\frac{-\kappa}{2\epsilon}a^2\right]\exp\left[\frac{1-\epsilon}{4\kappa}a^{+2}\right]; \quad \epsilon = \sqrt{1-4\kappa^2}. \tag{6.4}$$

The transformation of (6.2) is given by

$$\begin{aligned}\tilde{H} = UHU^\dagger = \frac{1}{\epsilon}\left(1 + \left(\epsilon^2 - 1\right)\sigma_0\right)a^+ a + \frac{1-\epsilon}{2\epsilon}\left(1 - (1+\epsilon)\sigma_0\right) - \\ \frac{\epsilon - 1}{4\epsilon^2\sqrt{1-\epsilon^2}}\left(2\epsilon^2 a^{+2} + (1+\epsilon)^2 a^2\right)(1-\sigma_0) + \mu(\sigma_+ + \sigma_-).\end{aligned} \tag{6.5}$$

Hamiltonian (6.5) can be expressed as a differential equation in the Bargmann-Fock space by using the realization of the bosonic operators

$$a^+ = z, a = \frac{d}{dz} \tag{6.6}$$

by introducing a new variable x

$$z^2 = \frac{\kappa}{\epsilon}x \tag{6.7}$$

and inserting (6.6) and (6.7) in (6.5), we obtain the following matrix Hamiltonian;

$$\tilde{H} = \begin{pmatrix} 2\epsilon x\frac{d}{dx} + \frac{1}{2}(\epsilon - 1) & \mu \\ \mu & -\frac{4(\epsilon+1)}{\epsilon}x\frac{d^2}{dx^2} - \frac{2(\epsilon+1+(\epsilon^2-2)x)}{\epsilon}\frac{d}{dx} - \frac{(1-\epsilon)(2+\epsilon-2x)}{2\epsilon} \end{pmatrix}. \tag{6.8}$$

Then the eigenvalue problem can be written as

$$\tilde{H}\psi(x) = E\psi(x) \tag{6.9}$$

where E is eigenvalue of the two-photon Rabi Hamiltonian and $\psi(x)$ is a two-component spinor

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.10}$$

Here ψ_1 and ψ_2 are function of x. The equation (6.9) reads explicitly

$$\left(2\epsilon x\frac{d}{dx} + \frac{\epsilon - 1}{2} - E\right)\psi_1 + \mu\psi_2 = 0 \tag{6.11}$$

$$\left(-\frac{4(\epsilon+1)}{\epsilon}x\frac{d^2}{dx^2} - \frac{2(\epsilon+1+(\epsilon^2-2)x)}{\epsilon}\frac{d}{dx} - \frac{(1-\epsilon)(2+\epsilon-2x)}{2\epsilon} - E\right)\psi_2 + \mu\psi_1 = 0 \tag{6.12}$$

In the next section we discuss the algebraic properties of these coupled differential equations.

6.2 Algebraic Analysis of The Rabi Hamiltonian

The standard Rabi Hamiltonian has been expressed as linear and bilinear combinations of the $sl_2(R)$ algebra [14]. In this case we can also express the two-photon Rabi Hamiltonian by $sl_2(R)$ algebra. Let us consider the following generators of the algebra:

$$J_- = x\frac{d^2}{dx^2} + (1-k)\frac{d}{dx}$$

$$J_+ = x$$

$$J_0 = x\frac{d}{dx} + \frac{1-k}{2}. \tag{6.13}$$

These generators are known as the Gelfand-Dyson representation of $su(1,1)$ algebra. Their connection to the Schwinger representation has been obtained by a similarity transformation[27]. The generators satisfy the $su(1,1) \approx sl_2(R)$ commutation relations

$$[J_+, J_-] = -2J_0; \quad [J_0, J_\mp] = \pm 2J_\pm, \tag{6.14}$$

and the Casimir operator of the algebra is given by

$$J = J_0^2 - \frac{1}{2}\left[J_+J_- + J_-J_+\right] = \frac{1}{4}\left(k^2 - 1\right). \tag{6.15}$$

Eliminating ψ_2 in (6.11) we can write a third order differential equation which takes the form

$$8(1+\epsilon)x^2\frac{d^3\psi_1}{dx^3} + 2\left(\frac{(1+\epsilon)(11\epsilon - 2E - 1)}{\epsilon} + 2(\epsilon^2 - 1)x\right)x\frac{d^2\psi_1}{dx^2} -$$

$$\left(\frac{(1+\epsilon)(2E+1-5\epsilon)}{\epsilon} + \frac{6(2-\epsilon^2)\epsilon - 2 - 4\epsilon}{\epsilon}x + 2(\epsilon - 1)x^2\right)\frac{d\psi_1}{dx} +$$

$$\left(\mu^2 + \frac{(\epsilon - 1 - 2E)(2x - 2 + \epsilon(\epsilon + 2E + 1 - 2x)}{4\epsilon}\right)\psi_1 = 0. \tag{6.16}$$

Let us consider the following algebraic quasi exactly solvable operator: L$=4(\epsilon^2 - 2)J_-J_+ + 2(1-\epsilon)J_+J_0 + 4(1+\epsilon)J_-J_0 + (1+k)(\epsilon - 1)J_+ +$

$$4(k-2)(\epsilon+1)J_- + 2(7 + (2k-3)\epsilon^2)J_0 \tag{6.17}$$

which can be written as eigenvalue problem

$$L\psi_1 = \lambda\psi_1 \tag{6.18}$$

where λ is the eigenvalue of L. When E and λ are chosen to be

$$E = -\frac{1}{2} + \left(\frac{1}{2} + 2k\right)\epsilon$$

$$\lambda = -(1+k)(1 - (1+2k)\epsilon^2) - \mu^2 \tag{6.19}$$

and the realizations given in(6.13) are used the equations (6.16) and (6.17) become identical. Thus the two-photon Rabi Hamiltonian is QES. The values of energy E given in (6.19) are known as the energy base lines. The result (6.19) is exactly the same as those given by Emary and Bishop[57].

6.3 Results and Discussions

When applied the generators of the $sl_2(R)$ algebra on the space polynomials

$$\psi_1 = P_k = \langle x^0, x^1, ..., x^k \rangle \tag{6.20}$$

the algebraic operator L given in (6.17) leads to the following recurrence relations:

$$2(k-n)(\epsilon-1)P_{n+1} + (\mu^2 + 2(k-n)(1+4n-(1+2k+2n)\epsilon^2)P_n -$$

$$4(k-n)n(2n-1)(1+\epsilon)P_{n-1} = 0 \tag{6.21}$$

where $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, k$. The initial conditions of the polynomial is given by

$$P_0 = P_{1/2} = 1. \tag{6.22}$$

The Juddian points can be calculated by solving the recurrence relation. The wave function is itself the generating function of energy polynomials. The eigenvalues are then produced just by the roots of such polynomials. If ϵ_j is a root of the polynomial

P_{n+1} , then ϵ_j belongs to the spectrum of the Hamiltonian; and the polynomial wave function terminates at $n = k$. The first few values of them are listed below;

$$\begin{aligned}
P_1 &= 2 + \mu^2 - 6\epsilon^2 \\
P_2 &= -\mu^4 - 2\mu^2(7 - 17\epsilon^2) - 8(3 - 30\epsilon^2 + 35\epsilon^4) \\
P_3 &= \mu^6 + 2\mu^4(11 - 25\epsilon^2) + 4\mu^2(111 - 722\epsilon^2 + 807\epsilon^4) + \\
&\quad 144(5 - 105\epsilon^2 + 315\epsilon^4 - 231\epsilon^6)
\end{aligned} \tag{6.23}$$

for $k = 1, 2, 3$ and

$$\begin{aligned}
P_{3/2} &= 6 + \mu^2 - 10\epsilon^2 \\
P_{5/2} &= -\mu^4 - 2\mu^2(13 - 23\epsilon^2) - 8(15 - 70\epsilon^2 + 63\epsilon^4) \\
P_{7/2} &= \mu^6 + 4\mu^4(17 - 31\epsilon^2) + 4\mu^2(303 - 1338\epsilon^2 + 1231\epsilon^4) + \\
&\quad 144(35 - 315\epsilon^2 + 693\epsilon^4 - 429\epsilon^6)
\end{aligned} \tag{6.24}$$

for $k = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ respectively. The first two of the (6.23) and the first one of the (6.24) were also obtained in [57].

Chapter 7

Conclusion

The work presented in this thesis has been aimed to develop a method to solve various kinds of physical Hamiltonians. In the beginning we have briefly discussed some different JC models [60, 61] found in literature. These models can be solved exactly or quasi-exactly hence we demonstrate the importance and properties of exact and quasi-exact solvabilty.

A general Hamiltonian (3.1) has been defined, by changing the parameters a number of models can be obtained. Here we have systematically discussed exact and quasi-exact solvability of this Hamiltonian within the framework of $su(2)$ and $su(1,1)$ Lie algebra. Choosing suitable parameters we get JC Hamiltonians with and without RWA. We have solved this JC models using a technique which we have developed. We have benefited from the realizations of $su(2)$ and $su(1,1)$ Lie algebra by boson operators.

In this study we also investigated quasi-exact solution of Rabi Hamiltonians. The standard Rabi Hamiltonian has been expressed by using the differential realization of the generators of $sl_2(R)$ algebra which are given by

$$
\begin{aligned}
J_+ &= x^2\frac{d}{dx} - 2kx \\
J_0 &= x\frac{d}{dx} - k \\
J_- &= \frac{d}{dx}
\end{aligned}
\tag{7.1}
$$

while the two- photon Rabi Hamiltonian has been expressed by using the realization given in(6.13). It is obvious that the the realizations (6.13) and (7.1) can be mappen onto each others. Instead of the realization (6.13) should we use the realization (7.1) in (6.17) we obtain a second order differential equation. Then the equation would be transformed into the form of Schrödinger equation. In this case the corresponding potential is the Lamé type of potential.

We have considered the JC Hamiltonian under the effects of dihedral groups D_4, D_6, D_8. It has permitted us to build the matrix generators of these groups by using the generator formula(2.2). As a result of the interactions between the symmetric states and the E states Jahn-Teller matrices have been obtained.

Finally we present a method to solve the problem of the JC Hamiltonian with Kerr nonlinearity. A general matrice depending on j, m values has given. Using number operator and generators of deformed $su(2)$ algebra, eigenvalues of JC Hamiltonian with Kerr nonlinearity have been obtained.

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