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APPLICATIONS OF THE FINITE FOURIER SINE TRANSFORM TO TWO-DIMENSIONAL ELASTIC BEAM PROBLEMS

MASTER OF SCIENCE

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OMRAN ABDULQADIR A.ALRAHMAN ALZAHAWI

ABSTRACT

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OMRAN ABDULQADIR A.ALRAHMAN ALZAHAWI
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We use the finite Fourier sine transform method in order to determine the stress and displacement distributions in the two-dimensional elastic beams. We study on the two plane-stress problems of elastic beams in Cartesian coordinates in which the Saint-Venant approximation can be considered. In the first problem, a simply supported rectangular beam under transverse sinusoidal load on its top side and in the second problem, a long rectangular beam subjected uniform tension at each lateral side are considered. These problems are formulated using Airy stress function method as the fourth-order biharmonic equations. The prescribed boundary conditions for each problem are also written in terms of Airy stress function. These biharmonic equations are reduced to the fourth-order, linear, constant coefficient ordinary differential equations by applying the finite Fourier sine transform.The computed solutions of these ordinary differential equations are inverted to find the solutions of the biharmonic equations and then, the solutions of the orginal problems are obtained. These solutions coincide with the results in [20]. In order to understand the nature of the solutions, better and to clarify the results, the computed solutions are plotted by using Matlab software.

KEYWORDS: Stress and displacement distributions, two-dimensional elastic beam, Finite Fourier sine transform, plane-stress problem, Saint-Venant approximation, transverse sinusoidal load, uniform tension, Airy stress function method, biharmonic equation, fourth-order linear constant coefficient ordinary differential equation, Matlab sowftware.

ÖZET

İKİ-BOYUTLU ELASTİK KİRİŞ PROBLEMLERİNE SONLU FOURIER SİNÜS DÖNÜŞÜMÜNÜN UYGULAMALARI
YÜKSEK LİSANS TEZİ
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İki boyutlu kirişlerde, stres ve yer değişimi dağılımlarını belirlemek için sonlu Fourier sinüs dönüşümü yöntemini kullandık. Saint-Venant yaklaşımının dikkate alındığı, elastik kirişlerin iki düzlem-stres problemini Kartezyen koordinatlarda çalıştık. Birinci problemde, üst tarafında transvers sinüzoidal yük olan basitçe desteklenmiş dikdörtgensel bir kiriş ve ikinci problemde, yanal kenarlarından yeknesak gerilime maruz kalan uzun dikdörtgensel bir kiriş göz önüne alındı. Bu problemler, Airy stres fonksiyonu yöntemi kullanılarak formüle edildi. Her bir problem için saptanmış sınır koşulları da Airy stress fonksiyonu cinsinden yazıldı. Bu biharmonik denklemler sonlu Fourier sinüs dönüşümünün uygulanmasıyla, dördüncü mertebeden lineer sabit katsayılı adi diferansiyel denklemlere indirgendi. Biharmonik denklemlerin çözümlerini bulmak için, bu adi diferansiyel denklemlerin hesaplanan çözümleri evrildi ve sonra, orijinal denklemlerin çözümleri elde edildi. Bu çözümler, [20] deki sonuçlara denk geldi. Çözümlerin doğasını daha iyi anlamak ve sonuçlara açıklık getirmek için Matlab yazılımı kullanılarak, hesaplanan çözümlerin grafikleri çizildi.

ANAHTAR KELİMELER: Stres ve yer değişim dağılımları, iki boyutlu elastik kiriş, sonlu Fourier sinüs dönüşümü, düzlem-stres problemi, Saint-Venant ilkesi, transvers sinüzoidal yük, yeknesak gerilim, Airy stres fonksiyonu yöntemi, biharmonik denklem, dördüncü mertebe lineer sabit katsayılı adi diferansiyel denklem, Matlab yazılımı.

TABLE OF CONTENTS

LIST OF FIGURES

Page

LIST OF ABBREVIATIONS AND SYMBOLS

ϕ	: Airy stress function
BE	: Biharmonic equation
∇^4	: Biharmonic operator
BCs	: Boundary conditions
BVPs	:Boundary value problems
u	: Displacement vector
u, v, w	: Displacement components in the x, y and z directions respectively
E	: Young's modulus
FFST	: Finite Fourier sine transform
IFFST	:Inverse finite Fourier sine transform
λ	: Lame's constant
$\sigma_x, \sigma_y, \sigma_y$	: Normal stresses in x,y and z directions respectively
ODEs	: Ordinary differential equations
PDEs	: Partial differential equations
ν	: Poisson's ratio
x,y,z	: Rectangular coordinates
σ	: Stress tensor
e	: Strain tensor
μ	: Shear modulus
$\tau_{xy}, \tau_{yz}, \tau_{zy}$	: Shearing stresses in xy,yz and zx-planes respectively
e_x, e_y, e_y	: Strain components in x, y and z directions
2D	: Two-dimensional
3D	: Three-dimensional
∴	: Therefore

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1. INTRODUCTION

Elasticity is a measure of the body's deformation under external forces. In the elastic behavior there is no permanent deformation in the body. In other words, application of the forces not exceeding a certain limit to the body causes the deformation of the body, however, with the removal of these forces such deformation dissappears.

In elasticity studies, the main issue is the determination of the stress, strain and displacement distibution in an elastic body under the influence of external forces. A mathematical model can be constructed by the formulation in the elasticity theory under the assumptions of linear, small deformation and the problems arising from many scientific and engineering fields can be solved using this mathematical model.

The development of elasticity depends on the Newton's laws of motion, however, it has earlier roots. To understand and control the fracture of solids, Leonardo da Vinci sketched a possible test of the tensile strength of a wire in his notebook, [1]. Later, Galileo had considered the nature of the resistance of solids to rupture. Since any law connecting the displacements produced with forces or at least, any physical hypothesis yielding such a law was not known, he treated solids as inelastic. He set up a beam whose one end was built into a wall and tried to determine the resistance of it, when the tendency to break it arises from its own or an applied weight. He arrived at the following result; the beam tends to turn about an axis perpendicular to its length, and in the plane of the wall. This is known as Galileo's problem.

In the history of elasticity, two cornerstones are the discovery of Hooke's law in 1678, [2] which provided the necessary experimental foundation for the theory and the formulation of the general equations of equilibrium by Navier in 1821, [2]. These works followed by Cauchy [2-4]. He studied the basic elasticity equations and described the internal stresses and strains in a solid body. Development of the theory was continued by several eminent mathematicians and scientists, e.g., Bernoullis, Lord Kelvin, Poisson, Lamé, Green, Saint-Venant, Betti, Airy, Kirchoff, Rayleigh, Love, Timoshenko, Kolosoff, Muskhelishvilli, and others.

During two decades after World War II, a large number of analytical solutions to specific problems of elasticity in scientific and engineering interests were obtained. Using finite and boundary element theory, numerical solutions of the elasticity problems were produced in 1970s and 1980s. Anisotropic materials for composites' applications were also studied in the same period. Later, the elasticity has been used to model the materials with internal microstructures or heterogeneity and inhomogeneous graded materials.

A review of foundations of elasticity were led by the rebirth of continuum mechanics in the 1960s. A rational place for the theory within the general framework were also established at that time. Historical details can be found in the books by Todhunter and Pearson, [2-4], Love, [5] and Timoshenko [6].

Because the elasticity theory establishes mathematical models for the deformation problems, it is formulated in terms of scalars, vectors and tensor fields. All these require use of tensor notation along with tensor algebra and calculus, that is use of basic principles of continuum mechanics. The established theory by this way comprises a system of differential and algebraic relations among the stresses, strains and displacements that express particular physics at all points within the body under investigation. Combinations of field equations with BCs result the fundamental BVPs of the theory. With the use of continuum hypothesis, a displacement field at all points within the elastic body can be established. A good mathematical knowledge is necessary in order to understand formulation of problems and solution procedures. Due to the difficulty of the computations of the solutions of BVPs of elasticity, several different solution strategies have been developed, such as power series method [7], Fourier methods [8-9], integral transforms [10-12], complex variables [13], variational calculus [14-15], potential theory [16], finite differences [17], finite elements[18], and so forth.

Since the elasticity field equations have complexity, to find analytical closed-form solutions to fully 3D problems is extremely difficult. Consequently, the most of the solutions are developed for reduced problems. The reduced problems typically include axisymmetry or two-dimensionality for simplification of the specific aspects of the formulation and solution. Since in fact, all real elastic structure are 3D the theories set forth are approximate models. The nature and accuracy of the approximation bases upon the problem and the shape of loading. In general, the fundamental plane problem in elasticity are represented by two basic theories of plane-stress and strain. These theories can be applied to significantly different types of 2D elastic bodies. However, their formulations yield very similar field equations. These theories can be reduced to one governing equation in terms of a single unknown stress function. Then this reduction allows many solutions to be generated to problems [19, 20].

The Airy stress function approach [20] is a well-known method that can be conveniently handled in each of plane-stress and strain problems. With this scheme, the field equations are reduced to a single PDE. In the absence of body forces, it is the BE. This means that, the plane elasticity problem can be reduced to finding the solution to the BE in a particular domain of interest. The solution of the BE must also satisfy the given boundary conditions associated with the problem under study. There are several solution techniques for such problems, including the use of power series or polynomials, Fourier methods, integral transforms and complex variable theory. An extensive review of 2D biharmonic solutions has been given by Meleshko [21].

In this thesis we study on the applicability of finite Fourier sine transform to the 2D elastic beam problems subjected to different loads. For this aim, we consider two plane-stress problems worked out by different methods in Sadd's book, [20, Chap.8]. We find the same closed-form of the analytical solutions for the problems by using finite Fourier sine transform.

The thesis is organized as follows: in Section 2, theoretical background for the linear elasticity is provided. Two examples of the applicability of the finite Fourier sine transform technique are worked out in Section 3. The results and discussion are given in Sections 4 and 5, respectively. In Section 6, the conlusions of the work are drawn and future recommendation are presented.

2. THEORETICAL BACKGROUND

The equilibrium equations, kinematic equations and the constitutive equations are the basic items in the mathematical theory of elasticity. These equations are related to the stresses, strains and displacements. Application of the external forces to a body causes displacement and deformation of the body and developes internal forces. The intensity of internal forces is called as “stress” and the measure of deformation is defined as “strain”.

In a deformable body, the strains, stresses and displacements are related with each other and all depend on the geometry and material of the body, external forces and supports. So, for estimating the external forces that the body gains the desired shape, the stresses, strains and displacements in the body must be determined. The strain-displacement, stress-strain relations and equations of motion are involved in the determination procedures of them.

In an elasticity problem for the evaluation of stresses, derivation of a series of basic equations together with BCs is necessary. Since, all influential factors are considered during derivation process of these equations, the obtained results will be very complicated. Hence they practically, cannot be solved. Consequently, for arriving at possible solutions, some basic assumptions have to make about the properties of the body e.g., the body is continuous, perfectly elastic, homogeneous and isotropic, and the displacements and strains are small. It is important to remark that these assumptions do not lead to significant errors so long as the dimensions of body are very large in comparison with those of the particles and with the distances between neighbouring particles.

2.1 Deformation: Stress, Strain and Displacement

One of the significant aspect of the analysis and design of structures relates the deformation caused by the applied loads on the body. To determine stresses, strains and displacement and consequently forces on the body we can use the analysis of deformation. The stress and strain are specified at every point in the body by the stress tensor $\boldsymbol{\sigma}$ and strain tensor $\boldsymbol{e}$. These are the second order tensors.

In a 3D Cartesian coordinate system, at a point $P(x,\ y,\ z)$ thre stress tensor denoted by

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{2.1}$$

where σ_x is the normal stress at P across a plane perpendicular to the x-axis, and τ_{xy} is the shearing stress across this plane in the direction of y-axis. Other components can be defined similarly.

Similarly, the strain tensor is

$$\boldsymbol{e} = [\boldsymbol{e}] = \begin{bmatrix} e_x & e_{yx} & e_{zx} \\ e_{xy} & e_y & e_{zy} \\ e_{xz} & e_{yz} & e_z \end{bmatrix} \tag{2.2}$$

where $\boldsymbol{e_x}$ is the normal strain at P across a plane perpendicular to the x-axis, and $\boldsymbol{e_{xy}}$ is the shearing strain across this plane in the direction of y-axis. The definition of other components can be given similarly.

Associated with the stress and strain tensors, the displacement vector is

$$\mathbf{u} = (u, v, w)$$

where u, v and w are the components the displacement vector in the x, y and z-direcctions, respectively.

2.1.1 Kinematics of Material Deformation

When a load applied, elastic body deforms and changes shape. In order to quantify these deformations we have to know the displacement of material points in the body. We can use the continuum hypotehsis to establish displacement field at each point within the body and to construct particular measures of deformation, approximate geometry can be used. Hence, we get the strain tensor development. Since the components of strain and the displacement field are related we need the establishment of the relations between strain components and displacement components and investigation of the requirements for ensuring single-valued, continuous displacement fields. Such kinematic results are developed under the conditions of small deformation theory as the approximation for the linear elasticity. Development leads to the fundamental sets of field equations; the strain-displacement relations and compatibility equations and also further field equations including internal force and stress distribution, and elastic constitutive behavior.

In component form, the strain-displacement relations are

$$e_x = \frac{\partial u}{\partial x}, e_y = \frac{\partial v}{\partial y}, e_z = \frac{\partial w}{\partial z}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \qquad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{2.3}$$

where $\frac{\partial u}{\partial x}$ is the unit alongation in x-direction and $\frac{1}{2}\,(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$ is a measure of the angular distortion about z-axis of a plane perpendicular to z-axis. Other components can be defined similarly.

Eliminating the displacements from the strain-displacement relations (2.3) we get Saint-Venant compatibility equations

$$\begin{aligned}
&\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\,\frac{\partial^2 e_{xy}}{\partial x \partial y} \\
&\frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = 2\,\frac{\partial^2 e_{yz}}{\partial y \partial z} \\
&\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} = 2\,\frac{\partial^2 e_{zx}}{\partial z \partial x} \\
&\frac{\partial^2 e_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right) \\
&\frac{\partial^2 e_y}{\partial z \partial x} = \frac{\partial}{\partial y}(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}) \\
&\frac{\partial^2 e_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right)
\end{aligned} \tag{2.4}$$

These six equations are not all independent. However, they can be reduced to three independent fourth order relations,

$$\begin{aligned}
&\frac{\partial^4 e_x}{\partial y^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right) \\
&\frac{\partial^4 e_y}{\partial z^2 \partial x^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right) \\
&\frac{\partial^4 e_y}{\partial x^2 \partial y^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right)
\end{aligned} \tag{2.5}$$

Saint-Venant's principle, named after a French elasticity theorist, Saint-Venant was originally published in 1855. It may be expressed as:

The stress, strain and displacement fields caused by two different statically equivalent force distributions on parts of the body far away from the loading points are approximately the same .

If our solutions are restricted to points away from the boundary loading, Saint-Venant's principle allows us to change a given pointewise BC to a simpler "statically equivalent loading" that will produce essential changes in the stress locally, but has an insignificant effect on the stresses at distances which are large in comparison with the linear dimensions of the surface on which the loadings are changed. In other words, such a redistribution will not effect the resulting solution. Sometimes this is refered to as substituting a strong BC with a simpler “weak” form.

It will be important to remark that if the dimensions of a rectangular beam are of the same order then, the Saint-Venant principle cannot be used to develop an approximate solution valid away from a particular boundary.

2.1.2 Stress and Equilibrium

In the foregoing discussion, we investigated the kinematics of deformation, without regarding the force or shear distribution within the body. By using traction vector and stress tensor, a quantitative method is provided to describe boundary and internal force distributions within a continuous body. The primary application of elasticity theory can be used for determining the distribution of stress within a given body due to the fact that the maximum stress is a main contributing factor for failure of the material. The concept of equilibrium is related to these force distribution issues. The force distribution at each point of a deformable body must be balanced. The summation of forces on an infinitesimal element is required to be zero in the static case. When a body is subjected to applied external loadings, internal forces are induced inside the body. These internal forces are distributed continuously within the body from the philosophy of continuum mechanics.

2.1.3 Equations of Equilibrium

In an elastic body, the field of stress is continuously distributed within the body and determined uniquely from the applied loadings. The applied loadings satisfy the equations of static equilibrium since we primarily deal with bodies in equilibrium. Hence, the summation of forces and moments is zero. When a body is entirely in equilibrium its all parts must also be in equilibrium. Therefore, the equilibrium equations can be developed that express the vanishing of the resultant force and moment at a continuum point in the material and the resulting equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0 \qquad (2.6)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0$$

All elasticity stress fields must satisfy these relations in order to be in static equilibrium.

2.1.4. Hooke's Law

Hooke's law states that the stress is proportional to the strain within the elastic limits when an elastic body is subjected to the pure tension. This law leads to the assumption in the mathematical theory that, in general ,the components of stress are linear functions of the components of strain .

The constitutive equations are the relations characterizing the physical properties of materials. Some principles for the systematic development of constitutive equations have been established by using continuum mechanics theory. However, many constitutive laws have been developed through emprical relations based on experimental evidence.

The "generalized Hooke's law" for a linear elastic body is defined as

$$\sigma_x = \lambda(e_x + e_y + e_z) + 2\mu e_x$$

$$\sigma_y = \lambda(e_x + e_y + e_z) + 2\mu e_y \qquad (2.7)$$

$$\sigma_z = \lambda(e_x + e_y + e_z) + 2\mu e_z$$

$$\tau_{xy} = 2\mu e_{xy}, \tau_{yz} = 2\mu e_{yz}, \tau_{zx} = 2\mu e_{zx}$$

where λ is Lame's constant and μ is shear modulus (or modulus of rigidity).

These relations are named after Robert Hooke, who in 1678 first proposed. They state that the deformation of an elastic structure is proportional to the applied force. As it is seen from the relations only two independent elastic constants are needed to describe the behavior of isotropic materials, λ and μ.

From Eqs (2.7) the inverse generalized Hooke's law can be obtained as

$$e_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)],\ e_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)],$$

$$e_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)],$$

$$e_{xy} = \frac{1+\nu}{E}\tau_{xy} = \frac{1}{2\mu}\tau_{xy}, \qquad (2.8)$$

$$e_{yz} = \frac{1+\nu}{E}\tau_{yz} = \frac{1}{2\mu}\tau_{yz},$$

$$e_{zx} = \frac{1+\nu}{E}\tau_{zx} = \frac{1}{2\mu}\tau_{zx},$$

where $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the modulus of elasticity or Young's modulus and

$\nu = \frac{E}{2(\nu+1)}$ is the Poisson's ratio.

2.2. Plane Elasticity

From a point of mathematical view, the elastic problems consist in systems of PDEs that must be solved in 3D space. However, in certain situations, the problems are reduced to 2D in space and can be solved more readily than the general 3D problem. The geometry of body and the nature of the loadings on the boundaries allow identifying an irrelevant direction that is associated with a direction of the problem such that solutions is independent of this dimension and can be posed a priori for this elastic problem. In our considerations, such irrelevant direction coincides with the z-direction. Then the analysis is reduced to the xy-plane and, hence such problems can be referred as the plane elasticity problems.

By definition, a plane body is a region of uniform thickness bounded by two parallel planes and any closed lateral surface. The thickness of the body must be uniform, however it need not be limited. It may be very thick or very thin. Together with the restrictions on geometry of the body, we also assume that the body forces cannot vary through the thickness of the region. In other words, they vanish in the z-direction and the surface tractions or the loads on the lateral boundary must be uniformly distributed across the thickness, i.e., constant in the z-direction and must be in the plane of the model. There are two large groups associated with two families of simplifying hypotheses, plane strain problems and plane stress problems.

Once the geometry and loading have been defined, stresses can be determined by using either the plane-strain or plane-stress approach. Usually the plane-strain approach is used when the body is very thick relative to its lateral dimensions like dams. The plane-stress approach is employed when the body is relatively thin in-relation to its lateral dimensions like beams.

2.2.1 The Plane-Strain Approach

By assuming the strains in the body are plane, that is, the strains in the x and y-directions are functions of x and y alone, and the strains in the z-direction are equal to zero, we can simplify the strain-displacement relation (2.3) as

$$e_x = \frac{\partial u}{\partial x}, \qquad e_y = \frac{\partial v}{\partial y}, \qquad e_z = \frac{\partial w}{\partial z} = 0$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \; e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \; e_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \tag{2.9}$$

Similarly, from Eqs. (2.7) the stress-strain relations are obtained as:

$$\sigma_x = \lambda(e_x + e_y) + 2\mu e_x$$

$$\sigma_y = \lambda(e_x + e_y) + 2\mu e_y \tag{2.10}$$

$$\sigma_z = \lambda(e_x + e_y), \qquad \tau_{xy} = \mu e_{xy}, \; \tau_{xz} = \tau_{yz} = 0$$

In addition**,** the stress equations of equilibrium (2.6) reduces to

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0, \; \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0 \tag{2.11}$$

Any solution of a plane-strain problem must satisfy Eqs.(2.9)-(2.11) in addition to the BCs on the lateral surface and the bounding planes. The BCs can be expressed in terms of the stresses yielding x, y and z-components of stress. Therefore, the components of the stresses applied to the body on the lateral surface must be functions of x and y, but, not of z. Moreover, there must be no applied tractions on the two parallel bounding planes. As a result, τ_{yz}, τ_{zx},σ_z must be zero on these surfaces.

It is clear from Eqs.(2.10) that σ_z will be equal to zero, only when the dilation, $e_x + e_y$ is equal to zero. In most problems, it will not be equal to zero. Therefore, the solution will not be exact due to violation of the BCs on the parallel planes. In many problems this violation of the BCs can be cleared by superimposing an equal and opposite distribution of σ_z (residual solution) onto the original solution.

Only when σ_z is a linear function of x and y, we can obtain an exact solution to the residual problem. When σ_z is nonlinear, an approximate solution based on Saint-Venant's principle is often utilized.

2.2.2 The Plane Stress Approach

In the foregoing discussion, we mentioned that the plane-strain method is limited to very long or thick bodies. When the body thickness is small relative to its lateral dimensions, it will be advantageous to assume that

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \tag{2.12}$$

throughout the thickness of the plate. With this assumption the stress equations of equilibrium again reduce to

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0\,,\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0 \tag{2.13}$$

and the stress-strain relations (2.7) become

$$\sigma_x = \lambda(e_x + e_y) + 2\mu e_x, \sigma_y = \lambda(e_x + e_y) + 2\mu e_y$$

$$\sigma_z = \lambda(e_x + e_y) = 0, \tag{2.14}$$

$$\tau_{xy} = \mu e_{xy},\ \ \tau_{yz} = \mu e_{yz} = 0, \qquad \tau_{zx} = \mu e_{zx} = 0$$

Since after deformation, x and y components of the displacement vector are independent of z and its z-component is zero, using the strain-displacement relation (2.3) we also write

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x},\ \sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_z = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \qquad \tau_{xy} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tau_{xz} = \tau_{yz} = 0 \tag{2.15}$$

From the inverse generalized Hooke's law (2.8) adding the first two equations in (2.14) and noting $\nu = \frac{\lambda}{2(\lambda+\mu)}$ we obtain

$$e_z = \frac{-\lambda}{\lambda + 2\mu}(e_x + e_y) \tag{a}$$

With the value of e_z the first strain invariant $e_x + e_y$ (dilation) becomes

$$e_x + e_y \rightarrow \frac{2\mu}{\lambda+2\mu}(e_x + e_y) \tag{b}$$

Substituting this value of dilation in Eq. (b) into Eqs. (2.14) yields

$$\sigma_x = \frac{2\lambda\mu}{\lambda + 2\mu}(e_x + e_y) + 2\mu e_x, \sigma_y = \frac{2\lambda\mu}{\lambda + 2\mu}(e_x + e_y) + 2\mu e_y,$$

$$\tau_{xy} = \mu e_{xy},\ \sigma_z = \ \tau_{yz} = \tau_{zx} = 0 \tag{2.16}$$

Unfortunately, in the general case σ_x, σ_y and τ_{xy} are not independent of z and thus the BCs imposed on the lateral surface cannot be rigorously satisfied. To overcome this difficulty, average displacements and stresses over the thickness are commonly used.

Since all equations for the plane-stress and plane-strain solutions, except for the comparison between Eqs. (2.10) and (2.14), are identical, the results from plane-strain can be transformed into plane-stress by letting

$$\lambda \to \frac{2\mu}{\lambda+2\mu} \text{ or } \frac{\nu}{1-\nu} \to \nu \tag{2.17}$$

In a similar manner, a plane-stress solution can be transformed into a plane-strain solution by letting

$$\frac{2\lambda\mu}{\lambda+2\mu} \to \lambda \text{ or } \nu \to \frac{\nu}{1-\nu} \tag{2.18}$$

In the plane-stress approach it is generally assumed that

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0$$

and the unknown stresses σ_x, σ_x and τ_{xy} will have z dependence. However, the z dependence causes the violation of the BCs on the lateral surface of the body. This difficulty can be eliminated and an approximate solution to the problem can be obtained by using average values for the stresses and displacements.

In this thesis, we consider the plane stress formulations of elastic beam problems under the application of two different loads. Hence, as a solution approach for such problems we mention on the Airy's stress function method.

2.2.3 Airy Stress Function Approach

In the plane problems, we must determine three unkonwns σ_x, σ_y and τ_{xy} which satisfy the required field equations and BCs. Several solutions to such problems can be obtained by using a particular stress functions technique. One of the method employs the Airy stress function [20]. It reduces the general formulation to a single governing equation in terms of a single unknown function. Then the resulting governing equation is solvable by several methods of applied mathematics. Consequently, many analytical solutions to problems of interest can be found. The stress function formulation is based on the following general idea: We develop a representation for the stress field that satisfies equilibrium equations and yields a single governing equation from the compatibility statement. The use of two equations of equilibrium and one stress equation of compatibility form the most suitable sets of fiels equations in this determination.

The equilibrium equations in 2D are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \tag{2.19a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0 \tag{2.19b}$$

The stress compatibility equation for the case of plane strain is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = -\frac{2(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \tag{2.19C}$$

We now suppose that $\Gamma(x,y)$ defines the body force field so that the body forces intensities are given by

$$F_x = -\frac{\partial \Gamma}{\partial x}, F_y = -\frac{\partial \Gamma}{\partial y} \tag{2.20}$$

Then by substituting Eqs. (2.20) into Eqs. (2.19) and noting that $\frac{2(\lambda+\mu)}{\lambda+2\mu} = \frac{1}{1-\nu}$. we have

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \frac{\partial \Gamma}{\partial x}, \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = \frac{\partial \Gamma}{\partial y}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y - \frac{\Gamma}{1-\nu}\right) = 0 \tag{2.21}$$

Eqs. (2.21) represent the three field equations which σ_x, σ_y and τ_{xy} must satisfy.

We assume that the stresses can be represented by a stress function $\phi(x,y)$ such that

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + \Gamma, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} + \Gamma, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{2.22}$$

Substitution of Eqs. (2.22) into Eqs. (2.21) gives two equations of equilibrium are exactly satisfied and the last of Eqs. (2.21) yields

$$\nabla^4 \phi = -(1-2\nu)\nabla^2 \Gamma \,/(1-\nu) \tag{2.23}$$

Thus, if ϕ satisfies Eq. (2.23) then equilibrium and compatibility relations are immediately satisfied. The expression ϕ is known as Airy stress function. If Eq. (2.23) is solved for ϕ an expression containing x and y and a number of constants will be obtained. The constants are evaluated from the BCs given in Eqs. (2.19), and the stresses are computed from ϕ according to Eqs. (2.22). Obviously, evaluation of ϕ from Eq. (2.23) produces stresses for the plane-strain case. Stresses for the plane-stress can be obtained by letting $\frac{\nu}{1-\nu} \to \nu$, as indicated in Eq. (2.17). This substitution leads to

$$\nabla^4 \phi = -(1-\nu)\nabla^2 \Gamma \tag{2.24}$$

which is valid for plane-stress problems.

When the body force are zero or constant, then $\nabla^2 \Gamma = 0$ and Eqs.(2.23) and (2.24) both becomes

$$\nabla^4\phi = 0 \tag{2.25a}$$

This is a BE which can also be written in the form

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = \nabla^4\phi = 0 \tag{2.25b}$$

This equation shows that ϕ and thus σ_x, σ_y and τ_{xy} are not dependent of the elastic constants. This is a very important consideration in 2D elasticity.

2.3. Methods of Solution

There are many analytical solutions techniques to find the closed form solutions of the problems discussed in Section 2.2. One of well-known technique is the Fourier integral transform.

The Fourier transform is suited for BVPs of elasticity associated with the linear PDEs with constant coefficients on infinite domains while the Fourier cosine and sine transforms are well suited for solving certain problems on semi-infinite domains where the governing differential equations involves only even-order derivatives. Sometimes the domain may be finite, in these cases we use finite Fourier transforms e.g., the finite Fourier sine and cosine transforms that are obtained from the Fourier sine and cosine series.

In our work, we use the finite Fourier transform for the governing equations and the BCs to get the analytical solution for the first fundamental problems of elasticity.

2.3.1 Finite Fourier Sine Transform (FFST)

This technique was first introduced by Doetsch [22] then, developed and generalized by the other mathematicians, e.g., Kneitz [23], Koschmieder [24], Roettinger [25], and Brown [26].

The FFST of a function $f(x)$ which is continuous or piecewise continuous on a finite interval $0 < x < a$, is defined by [27]

$$\mathcal{F}_s\{f(x)\} = f_s(n) = \int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx, \ \ (n = 1,2,3,\dots) \tag{2.26}$$

In view of formula (2.27) the inverse finite Fourier sine transform is given by

$$\mathcal{F}_s^{-1}\{f_s(n)\} = f(x) = \frac{2}{a}\sum_{n=0}^{\infty} f_s(n)\sin\left(\frac{n\pi x}{a}\right) \tag{2.27}$$

This result is obtained from the fact that Fourier sine series for $f(x)$ in the interval $0 < x < a$, $f(x) = \frac{2}{a}\sum_{n=0}^{\infty} f_s(n)\sin\left(\frac{n\pi x}{a}\right)$ converges to the value $f(x)$ at each continuity point in the interval (a,b) and to the value $\frac{1}{2}[f(x+0)+f(x-0)]$ at each finite discontinuity point in (a,b).

Both $\mathcal{F}_s$ and $\mathcal{F}_s^{-1}$ are linear transformations.

3. APPLICATIONS OF THE FFST TO THE 2D PLANE ELASTICITY PROBLEMS

In order to see the applicability of the FFST for solving 2D elastic beams, we consider two problems of elastic beams that are formulated by plane-stress conditions and the Saint-Venant principle can be applied to them. These problems are solved analytically by other methods in Sadd's book [20]. The first problem is the rectangular elastic beam that is simply supported and subjected to sinusoidal loading transversally on top, (see Figure 3.1) and the second problem is the uniaxial tension of the rectangular elastic beam (see Figure 3.14). In both problems, by introducing Airy stress function $\phi(x, y)$, Eqs.(2.22) without body forces, we transform the problems together with prescribed BCs to the BVPs of the BEs in terms of Airy stress function $\phi(x, y)$. Applying FFST to BEs and also BCs we transform them into the BVPs of the fourth-order linear ODEs with constant coefficients (parameters) and compute the solution of each transform equation. Then, by inverting these solutions we obtain the formal solutions in closed-form for the BEs as Airy stress functions. Using Eqs. (2.22) assuming no body forces, we determine the complete stress field for each problem. With the use of the inverse generalized Hooke's law (2.8), we calculate the strain fields, first and then, using strain-displacement relations (2.3) we determine the horizontal and vertical components of the displacement. The obtained solutions coincide with the results given in [20].

For clarification of the results we plot the graphs of computed functions; namely Airy stress functions, the stress components and the displacement components for two different values of the applied loads**.**

For both problems, we take the dimensions of the beam as $a = 5$ and $c = 2$ and the elastic constants, Poisson's ratio and Young's modulus are chosen as $\nu = 0.3$ and $E = 5$, respectively (see Figures 3.2-3.7 and 3.15-3.20). Moreover, in order to verify the Saint-Venant approximation, graphs of the mentioned functions above are sketched by taking the dimensions of the beam as $a = 30$ and $c = 2$, (see Figures 3.8-3.13 and Figures 3.21-3.26).

3.1. Beam Under the Transverse Sinusoidal Loading

As the first application of FFST method, we consider simply supported beam carrying a sinusoidal loading that is applied transversally on top side as shown in Figure.3.1

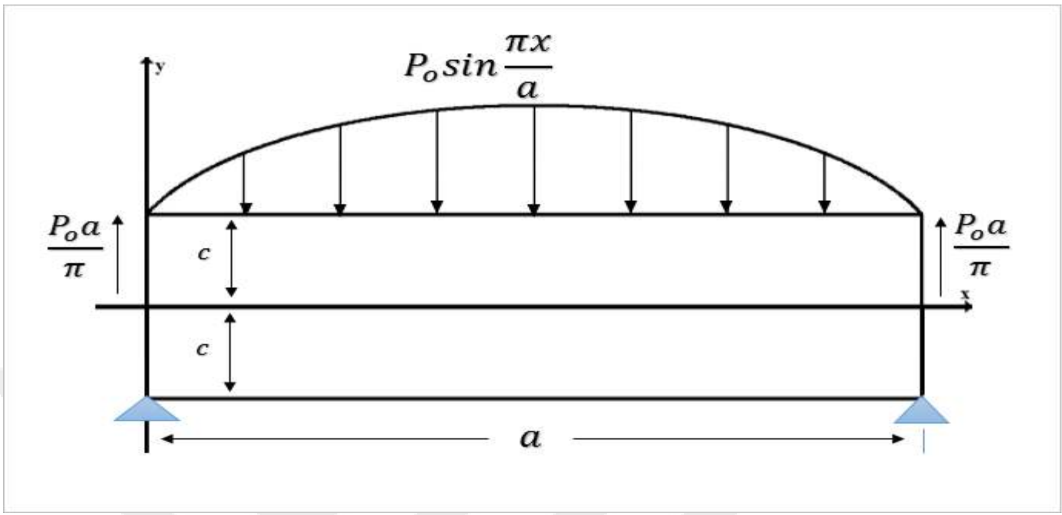

Figure3.1.A rectangular beam under transverse sinusoidal load.

The BCs are prescribed as

$$\sigma_x(0,y) = \sigma_x(a,y) = 0 \tag{3.1$_1$}$$

$$\tau_{xy}(x,\mp c) = 0 \tag{3.1$_2$}$$

$$\sigma_y(x,-c) = 0 \tag{3.1$_3$}$$

$$\sigma_y(x,c) = -P_{o} sin(\frac{\pi x}{a}) \tag{3.1$_4$}$$

$$\int_{-c}^{c} \tau_{xy}(0,y)dy = -\frac{P_o a}{\pi} \tag{3.1$_5$}$$

$$\int_{-c}^{c} \tau_{xy}(a,y)dy = \frac{P_o a}{\pi} \tag{3.1$_6$}$$

We can consider this problem as the Saint-Venant approximation to the more general case since the given BCs do not determine explicitely the pointwise distribution of shear stress on the ends of the beam, instead they rather guarantee the resultant condition based on overall problem of equilibrium. Therefore, we can obtain a solution that is valid away from the ends and it would be most useful for the case where $a \gg c$.

We now introduce the Airy stress function so that

$$\sigma_x(x,y)=\frac{\partial^2\phi}{\partial y^2}\,,\sigma_y(x,y)=\frac{\partial^2\phi}{\partial x^2}\,,\tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \tag{3.2}$$

3.1.1 The Solution of the BE

Substituting Eq. (3.2) into the equilibrium equations (2.13) by assumption of no body forces, we obtain a BE

$$\nabla^4\phi=\frac{\partial^4\phi}{\partial \mathrm{x}^4}+2\frac{\partial^4\phi}{\partial \mathrm{x}^2\,\partial \mathrm{y}^2}+\frac{\partial^4\phi}{\partial \mathrm{y}^4}=0 \tag{3.3}$$

The BCs (3.1) are also transformed in to

$$\frac{\partial^2\phi}{\partial y^2}(0,y)=\frac{\partial^2\phi}{\partial y^2}(a,y)=0 \qquad (3.4)_1$$

$$\frac{\partial^2\phi}{\partial x\partial y}(x,\mp c)=0 \qquad (3.4)_2$$

$$\frac{\partial^2\phi}{\partial x^2}(x,-c)=0 \qquad (3.4)_3$$

$$\frac{\partial^2\phi}{\partial x^2}(x,c)=-P_o\sin\left(\frac{\pi}{a}x\right) \qquad (3.4)_4$$

$$\int_{-c}^{c}\frac{\partial^2\phi}{\partial x\partial y}(0,y))dy=\frac{P_o a}{\pi} \qquad (3.4)_5$$

$$\int_{-c}^{c}\frac{\partial^2\phi}{\partial x\partial y}(a,y))dy=-\frac{P_o a}{\pi} \qquad (3.4)_6$$

Applying the FFST to the BE (3.3) transform it to the fourth-order ODE using the prescribed BCs

$$\int_0^a\left(\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\right)sin(\frac{n\pi x}{a})dx=0$$

Letting $\emptyset_n(y)=\int_0^a\phi(x,y)sin(n\beta x)dx$, (3.5)

where $\beta=\frac{\pi}{a}$

$$\int_0^a\frac{\partial^4\phi}{\partial x^4}sin(n\beta x)dx+2\int_0^a\frac{\partial^4\phi}{\partial x^2\partial y^2}sin(n\beta x)dx+\int_0^a\frac{\partial^4\phi}{\partial y^4}sin(n\beta x)dx=0 \tag{3.6}$$

Applying integration by parts twice and using Eq. (3.5) we write

$$\int_0^a\frac{\partial^2\phi}{\partial x^2}sin(n\beta x)dx=\left[\frac{\partial\phi}{\partial x}sin(n\beta x)\right]_0^a-n\beta\int_0^a\frac{\partial\phi}{\partial x}cos(n\beta x)dx$$

$$= 0 - n\beta \left\{ [\phi(x,y)\,cos(n\beta x)]_0^a + n\beta \int_0^a \phi(x,y) sin(n\beta x) dx \right\}$$

$$= -n\beta[\phi(x,y)\,cos(n\beta x)]_0^a - (n\beta)^2 \int_0^a \phi(x,y) sin(n\beta x) dx$$

$$= -n\beta[\phi(x,y)\,cos(n\beta x)]_0^a - (n\beta)^2 \phi_n(y)$$

$$= -n\beta[\phi(a,y)(-1)^n - \phi(0,y)(1)] - (n\beta)^2 \phi_n(y)$$

applying the BCs,

$$\phi(0,y) = 0, \quad \phi(a,y) = 0\,, \quad \frac{\partial^2}{\partial x^2}\phi(0,y) = 0, \quad \frac{\partial^2}{\partial x^2}\phi(a,y) = 0 \tag{3.7}$$

we get

$$\int_0^a \frac{\partial^2 \phi}{\partial x^2} sin(n\beta x) dx = -(n\beta)^2 \phi_n(y) \tag{3.8}$$

We write the second integral in Eq. (3.6) as the form

$$\int_0^a \frac{\partial^4 \phi}{\partial x^2 \partial y^2} sin(n\beta x) dx = \frac{d^2}{dy^2} \int_0^a \frac{\partial^2 \phi}{\partial x^2} sin(n\beta x) dx$$

$$= \frac{d^2}{dy^2}[-(n\beta)^2 \phi_n(y)]$$

$$= -(n\beta x)^2 \frac{d^2}{dy^2} \phi_n(y) \tag{3.9}$$

Also, the third integral in Eq. (3.6)

$$\int_0^a \frac{\partial^4 \phi}{\partial y^4} sin(n\beta x) dx = \frac{d^4}{dy^4} \int_0^a \phi(x,y) sin(n\beta x) dx$$

$$= \frac{d^4}{dy^4} \phi_n(y) \tag{3.10}$$

Finally, using integration by parts three time we write the first integral in Eq. (3.6)

$$\int_0^a \frac{\partial^4 \phi}{\partial x^4} sin(n\beta x) dx = \left[\left[\frac{\partial^3 \phi}{\partial x^3} sin(n\beta x) \right]_0^a - n\beta \int_0^a \frac{\partial^3 \phi}{\partial x^3} cos(n\beta x) dx \right]$$

$$= -n\beta\left[\int_0^a \frac{\partial^3\phi}{\partial x^3} cos(n\beta x)dx\right]$$

$$= -n\beta\left\{\left[\frac{\partial^2\phi}{\partial \mathrm{x}^2}\cos(n\beta x)\right]_0^{\mathrm{a}} + n\beta \int_0^{\mathrm{a}} \frac{\partial^2\phi}{\partial \mathrm{x}^2}\sin(n\beta x)\mathrm{dx}\right\}$$

$$=$$

$$-n\beta\left\{\left[\frac{\partial^2}{\partial x^2}\phi(a,y)(-1)^n - \frac{\partial^2}{\partial x^2}\phi(0,y)(1)\right] + \right.$$

$$\left. n\beta \int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx\right\}$$

$$= -n\beta\left[n\beta \int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx\right]$$

$$= -(n\beta)^2[-(n\beta)^2\phi_n(y)]$$

$$= (n\beta)^4\phi_n(y) \qquad (3.11)$$

Substitution of Eqs. (3.9), (3.10) and (3.11) into Eq. (3.6) yields fourth-order linear homogenous ODE

$$\int_0^a \frac{\partial^4\phi}{\partial x^4} sin(n\beta x)dx + 2\int_0^a \frac{\partial^4\phi}{\partial x^2\partial y^2} sin(n\beta x)dx + \int_0^a \frac{\partial^4\phi}{\partial y^4} sin(n\beta x)dx = 0$$

$$\frac{d^4}{dy^4}\phi_n(y) - 2(n\beta)^2\frac{d^2}{dy^2}\phi_n(y) + (n\beta)^4\phi_n(y) = 0 \qquad (3.12)$$

3.1.2 The General Solution

The auxiliary equation of (3.12)

$[m^4 - 2(n\beta)^2m^2 + (n\beta)^4] = 0$ or $(\mathrm{m}^2 - (n\beta)^2)^2 = 0$

has roots

$m = \mp n\beta \qquad or \quad m = \mp n\beta$

Therefore the general solution of (3.12)

$$\phi_n(y) = (A_n + C_n y)\, cosh(n\beta y) + (B_n + D_n y)\, sinh(n\beta y) \qquad (3.13)$$

For convenience setting $C_n y = C_n\beta y$ and $D_n y = D_n\beta y$ in Eq. (3.13), we write

$$\phi_n(y) = (A_n + C_n\beta y)\, cosh(n\beta y) + (B_n + D_n\beta y)\, sin(n\beta y)$$

The IFFST for the general solution

$$\phi(x,y) = \mathfrak{F}_s^{-1}\{\phi_n(y)\} = \frac{2}{a}\sum_{n=1}^{\infty} \phi_n(y)\, sin(n\beta x) \tag{3.14}$$

Substitute equation (3.13) in Eq. (3.14) to get

$$\phi(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} [(A_n + C_n\beta y)\, cosh(n\beta y) + (B_n + D_n\beta y)\, sinh(n\beta y)]\, sin(n\beta x) \tag{3.15}$$

differentiating Eq. (3.15) with respect to x twice we obtain

$$\frac{\partial\phi}{\partial x} = \frac{2}{a}\beta\sum_{n=1}^{\infty} n[(A_n + C_n\beta y)\, cosh(n\beta y) + (B_n + D_n\beta y)\, sinh(n\beta y)]\, cos(n\beta x) \tag{3.16}$$

and

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{2}{a}\beta^2\, sin(n\beta x)\sum_{n=1}^{\infty} n^2\big((A_n + C_n\beta y)\, cosh(n\beta y) + (B_n + D_n\beta y) sin\, h(n\beta y)\big) \tag{3.17}$$

Since $\sigma_y(x,y) = \frac{\partial^2\phi}{\partial x^2}$

Substituting Eq(3.4)$_4$, $\sigma_y(x,c) = -P_o\, sin(\beta x)$ where $\beta = \frac{\pi}{a}$ we get

$-\frac{2}{a}\beta^2 \sum_{n=1}^{\infty} n^2[(A_n + C_n\beta c)\, cosh(n\beta c) + (B_n + D_n\beta c)\, sinh(n\beta c)]\, sin(n\beta x) = -P_o\, sin(\,\beta x)$

$1^2[(A_1 + C_1\beta c)\, cosh(1\beta c) + (B_1 + D_1\beta c)\, sinh(1\beta c)]\, sin(1\beta x) + 2^2[(A_2 + C_2\beta c)\, cosh(2\beta c) + (B_2 + D_2\beta c)\, sinh(2\beta c)]\, sin(2\beta x) + 3^2[(A_3 + C_3\beta c)\, cosh(3\beta c) + (B_3 + D_3\beta c)\, sinh(3\beta c)]\, sin(3\beta x) + \cdots + n^2[(A_n + C_n\beta c)\, cosh(n\beta c) + (B_n + D_n\beta c)\, sinh(n\beta c)]\, sin(n\beta x) + \cdots = \frac{-aP_o}{2\beta} sin(\,\beta x)$

that is

$$[(A_1 + C_1\beta c)\, cosh(\beta c) + (B_1 + D_1\beta c)\, sinh(\beta c)]\, sin(\beta x) = \frac{-aP_o}{2\beta} sin(\,\beta x)$$

This equation implies that only A_1 , B_1 , C_1 , D_1 are not zero and remaining coefficients A_k, B_k, C_k, D_k are zero when $k \geq 2$

By letting $A_1 = A$, $B_1 = B$, $C_1 = C$, $D_1 = D$ the general solution $\phi(x, y)$ reduces to

$$\phi(x, y) = [(A + C\beta y)\, cosh(\beta y) + (B + D\beta y)\, sinh(\beta y)]\, sin(\beta x) \tag{3.18}$$

Now by using Eq. (3.18) we find $\frac{\partial^2\phi}{\partial x^2}$, $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial x\partial y}$ as

$$\frac{\partial\phi}{\partial x} = \beta[(A + C\beta y)\, cosh(\beta y) + (B + D\beta y)\, sinh(\beta y)]\, cos(\beta x) \tag{3.19}$$

$$\frac{\partial^2\phi}{\partial x^2} = -\beta^2[(A + C\beta y)\, cosh(\beta y) + (B + D\beta y)\, sinh(\beta y)]\, sin(\beta x) \tag{3.20}$$

$$\therefore\ \sigma_y(x, y) = -\beta^2[(A + C\beta y)\, cosh(\beta y) + (B + D\beta y)\, sinh(\beta y)]\, sin(\beta x) \tag{3.21}$$

Differentiating (3.19) with respect to y we get

$$\frac{\partial^2\phi}{\partial y\partial x} = \beta^2\, cos(\beta x)\, [(C\ (cosh(\beta y) + \beta y\, sinh(\beta y)) + Bcosh(\beta y) +$$
$$A\, sinh(\beta y) + D(sinh(\beta y) + \beta ycosh(\beta y)] \tag{3.22}$$

By $\tau_{xy} = -\frac{\partial^2\phi}{\partial y\partial x}$

$$\tau_{xy} = -\beta^2\, cos(\beta x)\, [(C\ (cosh(\beta y) + \beta y\, sinh(\beta y)) + Bcosh(\beta y) +$$
$$A\, sinh(\beta y) + D(sinh(\beta y) + \beta ycosh(\beta y)] \tag{3.23}$$

Also we find $\frac{\partial^2\phi}{\partial y^2}$ by using Eq. (3.15)

$$\frac{\partial\phi}{\partial y} = \beta[(C\ (cosh(\beta y) + \beta y\, sinh(\beta y)) + Bcosh(\beta y) + A\, sinh(\beta y) +$$
$$D(sinh(\beta y) + \beta ycosh(\beta y)]\, sin(\beta x) \tag{3.24}$$

and

$$\frac{\partial^2\phi}{\partial y^2} = \beta^2[C(2sinh(\beta y) + \beta y\, cosh(\beta y)) + A\, cosh(\beta y) + B\, sinh(\beta y) + D(2cosh(\beta y) + \beta y\, sinh(\beta y))]\, sin(\beta x) \tag{3.25}$$

Since $\sigma_x(x,y) = \frac{\partial^2\phi}{\partial y^2}$ we have

$$\sigma_x(x,y) = \beta^2[C(2sinh(\beta y) + \beta y\, cosh(\beta y)) + A\, cosh(\beta y) + B\, sinh(\beta y) + D(2cosh(\beta y) + \beta y\, sinh(\beta y))]\, sin(\beta x) \tag{3.26}$$

Now, by applying the BCs $(3.1)_1$-$(3.1)_4$ we shall find unknown coefficient A, B, C and D. first, we use the BCs $(3.1)_2$, $\tau_{xy}(x, \mp c) = 0$ in Eq. (3.23).

Application of $\tau_{xy}(x, +c) = 0$ yields

$$\tau_{xy}(x,+c) = -\beta^2[(C\ (cosh(\beta c) + c\beta\, sinh(\beta c)) + Bcosh(\beta c) + A\, sinh(\beta c) + D(sinh(\beta c) + c\beta\, cosh(\beta c)]\, cos(\beta x) = 0$$

that is

$$C\ (cosh(\beta c) + \beta c\, sinh(\beta c)) + Bcosh(\beta c) + A\, sinh(\beta c) + D(sinh(\beta c) + \beta c\, cosh(\beta c)) = 0 \tag{3.27}$$

then $\tau_{xy}(x, -c) = 0$ gives

$$\tau_{xy}(x,-c) = -\beta^2\left(\left(C\ (cosh\left(\beta(-c)\right) + \beta(-c)\, sinh(\beta(-c))\right) + Bcosh(\beta(-c)) + A\, sinh(\beta(-c)) + D(sinh(\beta(-c)) + \beta(-c)\, cosh(\beta(-c))\right) cos(\beta x) = 0$$

that is

$$C(cosh(\beta c) + \beta c\, sinh(\beta c)) + Bcosh(\beta c) - A\, sinh(\beta c) - D(sinh(\beta c) + \beta c\, cosh(\beta c)) = 0 \tag{3.28}$$

Now adding Eqs. (3.27) and (3.28) we get

$$2C(cosh(\beta c) + \beta c\, sinh(\beta c) + 2Bcosh(\beta c) = 0$$

Solving this equation for B we obtain

$$B = -C[1 + \beta c\, tanh(\beta c)] \tag{3.29}$$

By subtracting Eq. (3.28) from Eq. (3.27) we find

$$2A\,sinh(\beta c) + 2D(sinh(\beta c) + \beta c\,cosh(\beta c)) = 0$$

solving this equation for A

$$A = -D[1 + \beta c\,coth(\beta c)] \tag{3.30}$$

Substitution Eqs. (3.29)and (3.30) into Eq. (3.21) yields

$$\sigma_y(x,y) = -\beta^2\left\{-D\left[cosh(\beta y) + \beta c\frac{cosh(\beta c)}{sinh(\beta c)}cosh(\beta y) - \beta y\,sin\,h(\beta y)\right] - C\left[sinh(\beta y) + \beta c\frac{sinh(\beta c)}{cosh(\beta c)}sinh(\beta y) - \beta y\ \ cosh(\beta y)\right]\right\}sin(\beta x) \tag{3.31}$$

Now we apply BC $(3.1)_3$ on equation (3.31) we find the coefficient C

$$\sigma_y(x,-c) = -\beta^2\left\{-D\left[cosh(\beta(-c)) + \beta c\frac{cosh(\beta c)}{sinh(\beta c)}cosh(\beta(-c)) - \beta(-c)\,sinh(\beta(-c))\right] - C\left[sinh(\beta(-c)) + \beta c\frac{sinh(\beta c)}{cosh(\beta c)}sinh(\beta(-c)) - \beta(-c)\,cosh(\beta(-c))\right]\right\}sin(\beta x) = 0$$

so

$$C = -\,coth(\beta c)\frac{[\beta c + sin\,h(\beta c)\,cosh(\,\beta c)]}{[\beta c - sin\,h(\beta c)\,cosh(\beta c)]}D \tag{3.32}$$

We apply the BC $(3.1)_4$ $\sigma_y(x,c) = -P_o\,sin\left(\frac{\pi}{a}x\right)$ to Eq. (3.31)

by replacing $\beta = \frac{\pi}{a}$, to get (3.33)

hence

$$\sigma_y(x,c) = -\beta^2\left\{-D\left(cosh(\beta c) + \beta c\frac{cosh(\beta c)}{sinh(\beta c)}cosh(\beta c) - \beta c\,sin\,h(\beta c)\right) - C\left(sinh(\beta c) + \beta c\frac{sinh(\beta c)}{cosh(\beta c)}sinh(\beta c) - \beta c\ \ cosh(\beta c)\right)\right\}sin(\beta x) = -P_o\,sin(\beta x)$$

$$\left\{-D\left(cosh(\beta c) + \beta c\left(\frac{cos^2h(\beta c) - sin^2h(\beta c)}{sinh(\beta c)}\right)\right) - C\left(sinh(\beta c) - \beta c\left(\frac{sin^2h(\beta c) - cos^2\,h(\beta c)}{cosh(\beta c)}\right)\right)\right\} = \frac{P_o}{\beta^2}$$

$$\frac{1}{cosh(\beta c)}C(\beta c - sinh(\beta c)\, cosh(\beta c)) - \frac{1}{sinh(\beta c)}D(\beta c - sinh(\beta c)\, cosh(\beta c)) = \frac{P_o}{\beta^2} \quad (3.33)$$

Substitution Eq. (3.32) in Eq. (3.33) gives the coefficient D

$$D = \frac{-\,P_o\, sin\, h(\beta c)}{2\,\beta^2[\beta c + sinh(\beta c)\, cosh(\,\beta c)]} \quad (3.34)$$

We substitute Eq. (3.34) in Eq. (3.32) and find the coefficient C as

$$C = \frac{P_o\;\; cosh(\beta c)}{2\,\beta^2[\beta c - sin\, h(\beta c)\, cosh(\beta c)]} \quad (3.35)$$

Substitution Eq. (3.34) into Eq. (3.30) yields the coefficient A as

$$A = \frac{P_o[sin\, h(\beta c) + \beta c\; cosh(\beta c)}{2\,\beta^2[\beta c + sinh(\beta c)\, cosh(\,\beta c)]} \quad (3.36)$$

and finally substitution Eq. (3.35) into Eq. (3.29) gives the coefficient B

$$B = -\frac{P_o\,[cosh(\beta c) + \beta c\; sinh(\beta c)]}{2\,\beta^2[\beta c - sin\, h(\beta c)\, cosh(\beta c)]} \quad (3.37)$$

By substituting Eqs. (3.34), (3.35), (3.36) and (3.37) into Eq. (3.18) we obtain the solution of the BE

$$\phi(x,y) = \left\{\left(\frac{P_o[sin\, h(\beta c) + \beta c\; cosh(\beta c)}{2\,\beta^2[\beta c + sinh(\beta c)\, cosh(\,\beta c)]} + \frac{P_o\;\; cosh(\beta c)}{2\,\beta^2[\beta c - sinh(\beta c)\, cosh(\beta c)]}\beta y\right) cosh(\beta y) + \left(-\frac{P_o\,[cosh(\beta c) + \beta c\; sinh(\beta c)]}{2\,\beta^2[\beta c - sinh(\beta c)\, cosh(\beta c)]} + \frac{-\,P_o\, sin\, h(\beta c)}{2\,\beta^2[\beta c + sinh(\beta c)\, cosh(\,\beta c)]}\beta y\right) sin\, h(\beta y)\right\}\;\; sin(\beta x)$$

$$\phi(x,y) = \frac{P_o}{2\,\beta^2}\left(\left(\frac{[sinh(\beta c) + \beta c\; cosh(\beta c)}{[\beta c + sinh(\beta c)\, cosh(\,\beta c)]} + \frac{cosh(\beta c)}{[\beta c - sinh(\beta c)\, cosh(\beta c)]}\beta y\right) cosh(\beta y) - \left(\frac{[cosh(\beta c) + \beta c\; sinh(\beta c)]}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} + \frac{sin\, h(\beta c)}{[\beta c + sinh(\beta c)\, cosh(\,\beta c)]}\beta y\right) sinh(\beta y)\right)\;\; sin(\beta x) \quad (3.38)$$

Figure 3.2 displays the behavior of the Airy stress function, $\phi(x,y)$, (a) for $P_o = 5$ and (b) $P_o = 10$.

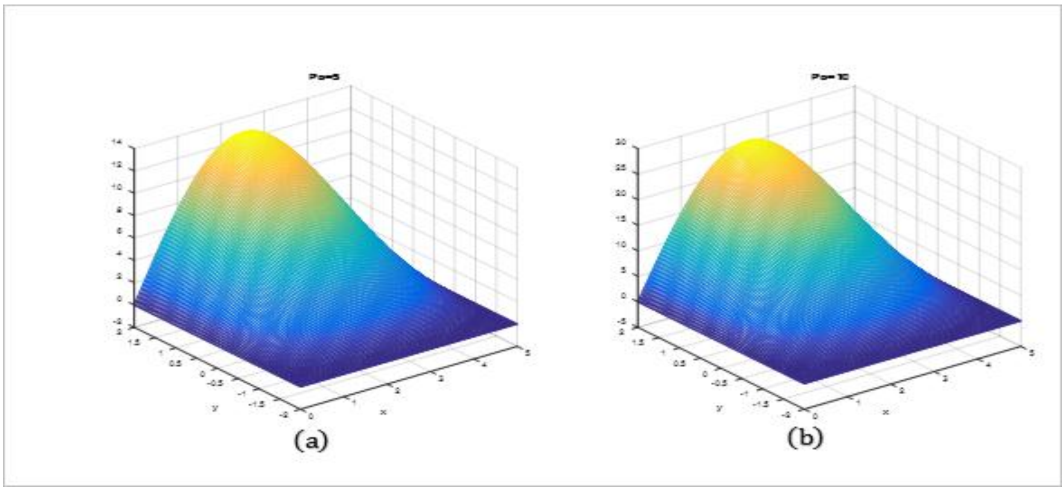

Figure 3.2. The Airy stress function, (a) for $P_o = 5$ and (b) $P_o = 10$

Substitution of Eqs. (3.34), (3.35), (3.36) and (3.37) into Eq. (3.21) yields the function

$$\sigma_y(x,y) = -\frac{P_o}{2}\left(\left(\frac{[sinh(\beta c)+\beta c\, cosh(\beta c)}{[\beta c+sinh(\beta c)\, cosh(\beta c)]}+\frac{cosh(\beta c)}{[\beta c-sinh(\beta c)\, cosh(\beta c)]}\beta y\right) cosh(\beta y) - \left(\frac{[cosh(\beta c)+\beta b\, sinh(\beta c)]}{[\beta c-sinh(\beta c)\, cosh(\beta c)]}+\frac{sinh(\beta c)}{[\beta c+sinh(\beta c)\, cosh(\beta c)]}\beta y\right) sinh(\beta y)\right) sin(\beta x) \quad (3.39)$$

The function $\sigma_y(x,y)$ is depicted in Figure 3.3 for (a) $P_o = 5$ and (b) $P_o = 10$

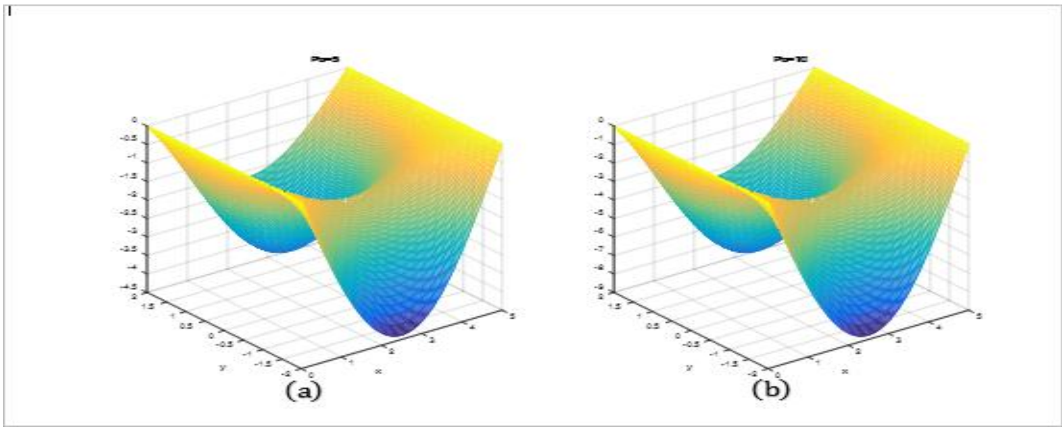

Figure 3.3. The vertical stresss component $\sigma_y(x,y)$, for (a) $P_o = 5$ and (b) $P_o = 10$

Substitution Eqs. (3.34), (3.35), (3.36) and (3.37) into Eq. (3.26) to find the function

$$\sigma_x(x,y) = \beta^2[C(2sinh(\beta y) + \beta y\, cosh(\beta y)) + A\, cosh(\beta y) + B\, sinh(\beta y) + D(2cosh(\beta y) + \beta y\, sinh(\beta y))]\, sin(\beta x)$$

$$\sigma_x(x,y) = \frac{P_o}{2} sin(\beta x) \left(\frac{cosh(\beta c) - \beta c\, sinh(\beta c)}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} sinh(\beta y) + \frac{[\beta c\, cosh(\beta c) - sinh(\beta c)}{[\beta c + sinh(\beta c)\, cosh(\beta c)]} cosh(\beta y) + \frac{[cosh(\beta c)]\beta y\, cosh(\beta y)}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} - \frac{sinh(\beta c)[\beta y\, sinh(\beta y)]}{[\beta c + sinh(\beta c)\, cosh(\beta c)]} \right) \tag{3.40}$$

The function $\sigma_x(x,y)$ is sketched in Figure 3.4 (a) for $P_o = 5$ and (b) $P_o = 10$.

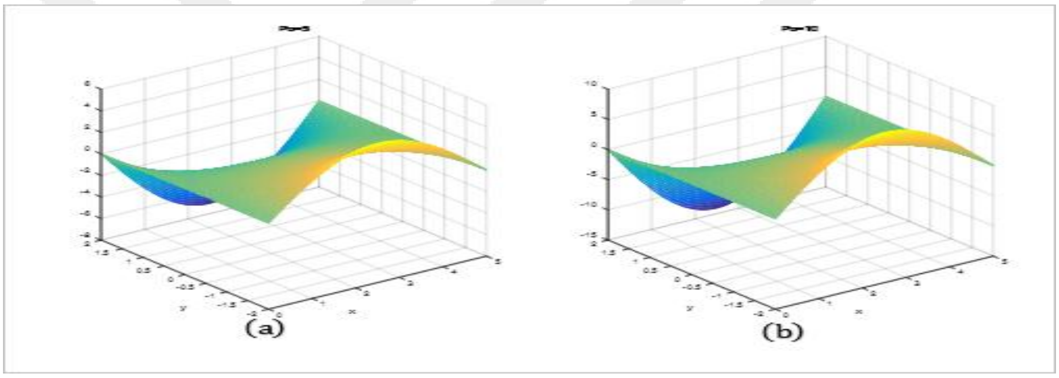

Figure 3.4. The horizontal stress component $\sigma_x(x,y)$, (a) for $P_o = 5$ and (b) $P_o = 10$

Substitution Eqs. (3.36), (3.35), (3.36) and (3.37) into Eq. (3.23) to find the function

$$\tau_{xy}(x,y) = -\beta^2[(C\,(cosh(\beta y) + \beta y\, sinh(\beta y)) + Bcosh(\beta y) + A\, sinh(\beta y) + D(sinh(\beta y) + \beta y cosh(\beta y)]\, cos(\beta x)$$

$$\tau_{xy} = -\frac{P_o}{2} cos(\beta x) \left(\frac{\beta y\, cosh(\beta c) sinh(\beta y) - \beta c\, sinh(\beta c)\, cosh(\beta y)}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} + \frac{\beta c\, cosh(\beta c) sinh(\beta y) - \beta y\, sinh(\beta c)\, cosh(\beta y)}{[\beta c + sinh(\beta c)\, cosh(\beta c)]} \right) \tag{3.41}$$

The function $\tau_{xy}(x,y)$is plotted in Figure 3.5. (a) for $P_o = 5$ and (b) $P_o = 10$.

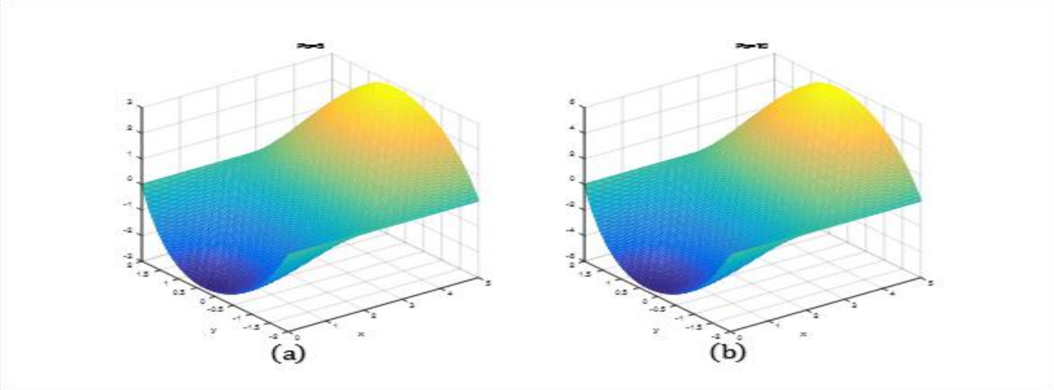

Figure 3.5. The shear stress $\tau_{xy}(x,y)$, (a) for $P_o = 5$ and (b) $P_o = 10$

Now we satisfy the BC $(3.1)_1$ in equation (3.40)

$$\sigma_x(0,y) = \frac{P_o}{2}\sin\big(\beta(0)\big)\left(\frac{cosh(\beta c)-\beta c\, sinh(\beta c)}{[\beta c-sinh(\beta c)\, cosh(\beta c)]}sinh(\beta y)+\frac{[\beta c\, cosh(\beta c)-sinh(\beta c)}{[\beta c+sinh(\beta c)\, cosh(\beta c)]}cosh(\beta y)+\frac{[cosh(\beta c)]\beta y\, cosh(\beta y)}{[\beta c-sinh(\beta c)\, cosh(\beta c)]}-\frac{sinh(\beta c)[\beta y\, sinh(\beta y)]}{[\beta c+sinh(\beta c)\, cosh(\beta c)]}\right)$$

$$\sigma_x(0,y) = 0$$

Next, we want to satisfy $\sigma_x(a,y) = 0$ when $\beta = \frac{\pi}{a}$

$$\sigma_x(a,y) = \frac{P_o}{2}\sin\left(\frac{\pi}{a}(a)\right)\left\{\frac{cosh(\beta c)-\beta c\, sinh(\beta c)}{[\beta c-sinh(\beta c)\, cosh(\beta c)]}sinh(\beta y)+\frac{[\beta c\, cosh(\beta c)-sinh(\beta c)}{[\beta c+sinh(\beta c)\, cosh(\beta c)]}cosh(\beta y)+\frac{[cosh(\beta c)]\beta y\, cosh(\beta y))}{[\beta c-sinh(\beta c)\, cosh(\beta c)]}-\frac{sinh(\beta c)[\beta y\, sinh(\beta y))]}{[\beta c+sinh(\beta c)\, cosh(\beta c)]}\right\}$$

$$\sigma_x(a,y) = 0$$

so the BC $(3.1)_1$, $\sigma_x(0,y) = \sigma_x(a,y) = 0$ are satisfied.

We satisfy the BC $(3.1)_2$ by using Eq. (3.41)

$$\tau_{xy} = -\frac{P_o}{2}cos(\beta x)\left(\frac{\beta y\, cosh(\beta c)sinh(\beta y)-\beta c\, sinh(\beta c)\, cosh(\beta y)}{[\beta c-sinh(\beta c)\, cosh(\beta c)]}+\frac{\beta c\, cosh(\beta c)sinh(\beta y)-\beta y\, sinh(\beta c)\, cosh(\beta y)}{[\beta c+sinh(\beta c)\, cosh(\beta c)]}\right)$$

First we satisfy $\tau_{xy}(x,c) = 0$

$\tau_{xy}(x,c) =$

$-\frac{P_o}{2}cos(\beta x)\left(\frac{\beta c\, cosh(\beta c) sinh(\beta c) - \beta c\, sinh(\beta c)\, cosh(\beta c)}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} + \right.$

$\left.\frac{\beta c\, cosh(\beta c) sinh(\beta c) - \beta c\, sinh(\beta c)\, cosh(\beta c)}{[\beta c + sinh(\beta c)\, cosh(\beta c)]}\right)$

$\tau_{xy}(x, c) = 0.$

Second we satisfy $\tau_{xy}(x,-c) = 0$

$\tau_{xy} =$

$-\frac{P_o}{2}cos(\beta x)\left(\frac{\beta c\, cosh(\beta c) sinh(\beta c) - \beta c\, sinh(\beta c)\, cosh(\beta c)}{[\beta c - sin\, h(\beta c)\, cosh(\beta c)]} + \right.$

$\left.\frac{-\beta c\, cosh(\beta c) sinh(\beta c) + \beta c\, sinh(\beta c)\, cosh(\beta c)}{[\beta c + sinh(\beta c)\, cosh(\beta c)]}\right)$

$\tau_{xy}(x,-c) = 0$, as desired.

hence, the BC $(3.1)_2$ are satisfied.

Now, we satisfy the BC $(3.1)_3$ by using Eq. (3.39)

$\sigma_y(x,-c) = 0$

$\sigma_y(x,-c) = -\frac{P_o}{2}sin(\beta x)\left\{\frac{[sinh(\beta c)\, cosh(\beta(-c))]}{[\beta c + sinh(\beta c)\, cosh(\beta c)]} + \frac{\beta c\, cosh(\beta c)\, cosh(\beta(-c))}{[\beta c + sinh(\beta c)\, cosh(\beta c)]} + \right.$

$\frac{\beta(-c)\ cosh(\beta c)\, cosh(\beta(-c))}{[\beta c - sin\, h(\beta c)\, cosh(\beta c)]} - \frac{[cosh(\beta c)\, sinh(\beta(-c))]}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} - \frac{\beta c\, [sinh(\beta c)\, sinh(\beta(-c))]}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} -$

$\left.\frac{\beta(-c)\ sinh(\beta c)\, sinh(\beta(-c))}{[\beta c + sinh(\beta c)\, cosh(\beta c)]}\right\}$

Arranging this expression we arrive at

$$\sigma_y(x,-c) = -\frac{P_o}{2}sin(\beta x)\left\{\frac{\beta c[sinh(\beta c)\, cosh(\beta c)] - \beta c[sinh(\beta c)\, cosh(\beta c)]}{(\beta c)^2 - sinh^2(\beta c)\, cosh^2(\beta c)]}\right\} = 0$$

so the BC $(3.1)_3$ are also satisfied

Next, we satisfy the BC $(3.1)_4$, $\sigma_y(x,c) = -P_0\, sin(\beta x)$ by using Eq. (3.39)

$\sigma_y(x,c) = -\frac{P_o}{2}sin(\beta x)\left\{\frac{[sinh(\beta c)\, cosh(\beta c)] + \beta c\, cosh^2(\beta c)}{[\beta c + sinh(\beta c)\, cosh(\beta c)]} + \frac{\beta c\, cosh^2(\beta c)}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} - \right.$

$\left.\frac{[cosh(\beta c)\, sinh(\beta c)] + \beta c\, [sinh^2(\beta c)]}{[\beta c - sinh(\beta c)\, cosh(\beta c)]} - \frac{\beta c\, [sinh^2(\beta c)]}{[\beta c + sinh(\beta c)\, cosh(\beta c)]}\right\}$

by arranging we get

$$\sigma_y(x,c) = -\,P_o\, sin(\beta x)\left\{\frac{(\beta c)^{\,2} - sinh^2(\beta c)\, cosh^2(\beta c)]}{(\beta c)^{\,2} - sinh^2(\beta c)\, cosh^2(\beta c)]}\right\} = -\,P_o\, sin(\beta x)$$

Therefore the BC $(3.1)_4$ is satisfied.

Then, to satisfy the BC $(3.1)_5$

$$\int_{-c}^{c} \tau_{xy}(x,y)dy = \int_{-c}^{c} -\frac{\partial^2\phi}{\partial x\partial y}(x,y)dy = -\frac{\partial}{\partial x}\int_{-c}^{c}\frac{\partial\phi}{\partial y}(x,y)dy = -\frac{\partial}{\partial x}[\phi(x,y)]_{-c}^{c}$$

$$= -\frac{\partial}{\partial x} sin(\beta x)\Big(\frac{P_o(sinh(\beta c)+\beta c\, cosh(\beta c)}{2\beta^2(\beta c+sinh(\beta c)\, cosh(\beta c))} cosh(\beta(c) + \frac{P_o\beta c\, cosh(\beta c) cosh(\beta c)}{2\beta^2(\beta c-sinh(\beta c)\, cosh(\beta c))} - \frac{P_o(cosh(\beta c)+\beta c\, sinh(\beta c))}{2\beta^2(\beta c-sinh(\beta c)\, cosh(\beta c))} sinh(\beta c) + \frac{P_o\beta c\, sinh(\beta c) sinh(\beta c)}{2\beta^2(\beta c-sinh(\beta c)\, cosh(\beta c))} - \frac{P_o(sinh(\beta c)+\beta c\, cosh(\beta c)}{2\beta^2(\beta c+sinh(\beta c)\, cosh(\beta c))} cosh(\beta(-c) - \frac{P_o\beta(-c) cosh(\beta c) cosh(\beta(-c)}{2\beta^2(\beta c-sinh(\beta c)\, cosh(\beta c))} + \frac{P_o(cosh(\beta c)+\beta c\, sinh(\beta c))}{2\beta^2(\beta c-sinh(\beta c)\, cosh(\beta c))} sinh\big(\beta(-c)\big) + \frac{P_o\beta(-c)\, sinh(\beta c) sinh\big(\beta(-c)\big)}{2\beta^2(\beta c+sinh(\beta c)\, cosh(\beta c))}\Big)$$

by arranging we get

$$\int_{-c}^{c} \tau_{xy}(x,y)dy = -\frac{P_o\beta\, cosh(\beta x)}{2\beta^2}\Big[\frac{\beta c - sinh(\beta c) cosh(\beta c)}{(\beta c - sinh(\beta c)\, cosh(\beta c))}\Big] = -\frac{P_o\beta\, cosh(\beta x)}{2\beta^2}$$

for $x = 0$ it becomes

$$\int_{-c}^{c} \tau_{xy}(0,y)dy = -\frac{P_o\, cosh(\beta(0))}{\beta} = -\frac{P_o}{\beta}$$

since $\beta = \frac{\pi}{a}$ we have $\int_{-c}^{c} \tau_{xy}(0,y)dy = -\frac{P_o a}{\pi}$, as desired

Finally, we satisfy the BC $(3.1)_6$.

$$\int_{-c}^{c} \tau_{xy}(x,y)dy = -\frac{P_o\, cosh(\beta x)}{\beta}\Big[\frac{\beta c - sinh(\beta c) cosh(\beta c)}{(\beta c - sinh(\beta c)\, cosh(\beta c))}\Big]$$

$$\int_{-c}^{c} \tau_{xy}(a,y)dy = -\frac{P_o\, cosh\left(\frac{\pi}{a}(a)\right)}{\beta} = -\frac{P_o(-1)}{\frac{\pi}{a}} = \frac{P_o a}{\pi}$$

So, the BC $(3.1)_6$ is satisfied.

3.1.3 The Displacement Vector Components $\boldsymbol{u(x,y), v(x,y)}$

By using Eqs. (3.21) and (3.26)

$$\frac{\partial u}{\partial x} = e_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) = \frac{1}{E} sin(\beta x)\,(\beta^2[C(2sinh(\beta y) + \beta y\, cosh(\beta y)) + A\, cosh(\beta y) + B\, sinh(\beta y) + D(2cosh(\beta y) + \beta y\, sinh(\beta y))] - \nu[-\beta^2[(A + C\beta y)\, cosh(\beta y) + (B + D\beta y)\, sinh(\beta y)]])$$

and by arranging the last expression we get

$$\frac{\partial u}{\partial x} = \frac{\beta^2}{E} sin(\beta x)\,[A(1+\nu)\, cosh(\beta y) + B(1+\nu)\, sinh(\beta y) + C((1+\nu)\beta y\, cosh(\beta y) + 2\, sinh(\beta y)) + D\big((1+\nu)\beta y\, sinh(\beta y) + 2\, cosh(\beta y)\big] \tag{3.42}$$

Also,

$$\frac{\partial v}{\partial y} = e_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) = \frac{1}{E} sin(\beta x)\,([-\beta^2[(A + C\beta y)\, cosh(\beta y) + (B + D\beta y)\, sinh(\beta y)] - \nu(\beta^2[C(2sinh(\beta y) + \beta y\, cosh(\beta y)) + A\, cosh(\beta y) + B\, sinh(\beta y) + D(2cosh(\beta y) + \beta y\, sinh(\beta y))])$$

making required cancelations

$$\frac{\partial v}{\partial y} = -\frac{\beta^2}{E} sin(\beta x)\,(A(1+\nu)\, cosh(\beta y) + C((1+\nu)\beta y\, cosh(\beta y) + 2\nu\, sinh(\beta y)) + B(1+\nu)\, sinh(\beta y) + D((1+\nu)\beta y\, sinh(\beta y) + 2\nu\, cosh(\beta y))) \tag{3.43}$$

Integrating Eqs. (3.42) and (3.43) with respect to x and y respectively we obtain

$$u(x,y) = -\frac{\beta}{E} cos(\beta x)\,[A(1+\nu)\, cosh(\beta y) + B(1+\nu)\, sinh(\beta y) + C((1+\nu)\beta y\, cosh(\beta y) + 2\, sinh(\beta y)) + D\big((1+\nu)\beta y\, sinh(\beta y) + 2\, cosh(\beta y)\big] + h(y) \tag{3.44}$$

and

$v(x,y) = -\frac{\beta}{E} sin(\beta x)\,(A(1+\nu)\,sinh(\beta y) + C((1+\nu)\beta y\,sinh(\beta y) - (1-$
$\nu)\,cosh(\beta y)) + B(1+\nu)\,cosh(\beta y) +$
$D\big((1+\nu)\beta y\,cosh(\beta y) - (1-\nu) sinh(\beta y)\big) + g(x)$ (3.45)

Now, we apply the formula, $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{\mu}$, where $\mu = \frac{E}{2(1+\nu)}$ shear modulus

$$\frac{\tau_{xy}}{\mu} = \frac{2(1+\nu)}{E}\tau_{xy}$$
$$= -\frac{2(1+\nu)\beta^2}{E}(cos(\beta x)\,[(C+B+D\beta y)\,cosh(\beta y) + (D+A+C\beta y)\,sinh(\beta y)] \quad (3.46)$$

and

$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{\beta^2}{E} cos(\beta x)\,[A(1+\nu)\,sinh(\beta y) + B(1+\nu)\,cosh(\beta y) +$
$C((1+\nu)\beta\,cosh(\beta y) + (1+\nu)\beta y\,sinh(\beta y) + 2\,cosh(\beta y)) + D\big((1+$
$\nu)\beta\,sinh(\beta y) + (1+\nu)\beta y\,cosh(\beta y) + 2\,sinh(\beta y)\big] + h'(y) +$

$[-\frac{\beta^2}{E} cos(\beta x)\,(A(1+\nu)\,sinh(\beta y) + C((1+\nu)\beta y\,sinh(\beta y) - (1-$
$\nu)\,cosh(\beta y)) + B(1+\nu)\,cosh(\beta y) +$
$D\big((1+\nu)\beta y\,cosh(\beta y) - (1-\nu) sinh(\beta y)\big)] + g'(x)$

$=$
$-\frac{\beta^2}{E} cos(\beta x)\,[2A(1+\nu)\,sinh(\beta y) + 2B(1+\nu)\,cosh(\beta y) + C(2\,cosh(\beta y) +$
$2(1+\nu)\beta y\,sinh(\beta y) + 2\nu\,cosh(\beta y)) +$
$D(2\,sinh(\beta y) + 2(1+\nu)\beta y\,cosh(\beta y) + 2\nu\,sinh(\beta y)] + h'(y) + g'(x)$ (3.47)

Since Eqs. (3.46) and (3.47) are equal we have

$-\frac{\beta^2}{E} cos(\beta x)\,[A(1+\nu)\,sinh(\beta y) + B(1+\nu)\,cosh(\beta y) +$
$C((1+\nu)\beta y\,sinh(\beta y) + (\nu+1)\,cosh(\beta y)) + D((\nu+1)\,sinh(\beta y) +$
$(1+\nu)\beta y\,cosh(\beta y)] + h'(y) + g'(x) =$
$\frac{2(1+\nu)}{E}(-\beta^2[(C+B+D\beta y)\,cosh(\beta y) + (D+A+C\beta y)\,sinh(\beta y)]\,cos(\beta x))$

performing cancellation we get

$$h'(y) + g'(x) = 0 \tag{3.48}$$

or

$$h'(y) = -g'(x) = constant \tag{3.49}$$

Integrating $h'(y), g'(x)$ with respect to y and x respectively we write

$$h(y) = \gamma_o y + \delta_o \tag{3.50}$$

also

$$g(x) = \gamma_o x + \varepsilon_o \tag{3.51}$$

here $\gamma_o, \delta_o, \varepsilon_o$ are arbitrary constants

We use Eq. (3.50) into Eq. (3.44)

$$\begin{aligned} u(x,y) = -\frac{\beta}{E} cos(\beta x)\, [&A(1+\nu)\, cosh(\beta y) + B(1+\nu)\, sinh(\beta y) \\ &+ C((1+\nu)\beta y\, cosh(\beta y) + 2\, sinh(\beta y)) \\ &+ D\big((1+\nu)\beta y\, sinh(\beta y) + 2\, cosh(\beta y)\big] + \gamma_o y + \delta_o \end{aligned} \tag{3.52}$$

3.1.3.1 Simply Supported Beam, We Choose Displacement BCs as

Since we have a simply supported beam, the displacement fixity conditions can be selected as

$$u(0,0) = 0 \text{ and } v(0,0) = v(a,0) = 0 \tag{3.53}$$

$u(0,0) = -\frac{\beta}{E} cos(0)\, [A(1+\nu)\, cosh(0) + B(1+\nu)\, sinh(0) + C((1+\nu)\beta(0)\, cosh(0) + 2\, sinh(0)) + D\big((1+\nu)\beta(0)\, sinh(0) + 2\, cosh(0)\big] + \gamma_o(0) + \delta_o = 0$

$$-\frac{\beta}{E}[A(1+\nu) + 2D] + \delta_o = 0$$

$$\delta_o = \frac{\beta}{E}[A(1+\nu) + 2D] \tag{3.54}$$

$v(x,y) = -\frac{\beta}{E} sin(\beta x)\,(A(1+\nu)\,sinh(\beta y) + C((1+\nu)\beta y\, sinh(\beta y) - (1 -$
$\nu)\,cosh(\beta y)) + B(1+\nu)\,cosh(\beta y) +$
$D\big((1+\nu)\beta y\, cosh(\beta y) - (1-\nu) sinh(\beta y)\big) + \gamma_o x + \varepsilon_o$ (3.55)

$v(0,0) = -\frac{\beta}{E} sin(0)\,(A(1+\nu)\,sinh(0) + C((1+\nu)\beta(0)\,sinh(0) - (1 -$
$\nu)\,cosh(0)) + B(1+\nu)\,cosh(0) +$
$D\big((1+\nu)\beta(0)\,cosh(0) - (1-\nu) sinh(0)\big) + \gamma_o(0) + \varepsilon_o = 0$

i,e

$\varepsilon_o = 0$

$v(a,0) = -\frac{\beta}{E} sin(\pi)\,(A(1+\nu)\,sinh(0) + C((1+\nu)\beta(0)\,sinh(0) - (1 -$
$\nu)\,cosh(0)) + B(1+\nu)\,cosh(0) +$
$D\big((1+\nu)\beta(0)\,cosh(0) - (1-\nu) sinh(0)\big) + \gamma_o(a) + \varepsilon_o = 0$

$\gamma_o(a) + \varepsilon_o = 0 \rightarrow \gamma_o(a) + 0 = 0 \quad \rightarrow \gamma_o = 0$

∴, The condition in Eq. (3.52) determine the rigid body terms, given the result

$\delta_o = \frac{\beta}{E}[A(1+\nu) + 2D]$ only horizontal translation

$\gamma_o = 0, \varepsilon_o = 0$ no rotation, no vertical translation

In comparision with strength of materials theory, the vertical centerline displacement is determined for $y = 0$ in Eq. (3.54) with $\gamma_o = 0, \varepsilon_o = 0$ yields

$v(x,0) = -\frac{\beta}{E} sin(\beta x)\,(A(1+\nu)\,sinh(0) + C((1+\nu)\beta(0)\,sinh(0) - (1 -$
$\nu)\,cosh(0)) + B(1+\nu)\,cosh(0) + D\big((1+\nu)\beta(0)\,cosh(0) - (1-\nu) sinh(0)\big)$

$$v(x,0) = -\frac{\beta}{E} sin(\beta x)\,(-C(1-\nu)) + B(1+\nu)$$

Since

$B = -C[1 + \beta c\, tanh(\beta c)]$

we get

$$v(x,0) = -\frac{\beta}{E} sin(\beta x)\,(-C(1-v)) - C[1 + \beta c\; tanh(\beta c)](1+v)$$

$$v(x,0) = \frac{\beta C}{E} sin(\beta x)\,[2 + \beta c(1+v) tanh(\beta c)] \tag{3.56}$$

The final form of u and v are

$$u(x,y) = -\frac{\beta}{E} cos(\beta x)\,[A(1+v)\,cosh(\beta y) + B(1+v)\,sinh(\beta y) + C((1+v)\beta y\,cosh(\beta y) + 2\,sinh(\beta y)) + D\big((1+v)\beta y\,sinh(\beta y) + 2\,cosh(\beta y)\big] + \frac{\beta}{E}[A(1+v) + 2D] \tag{3.57}$$

$$v(x,y) = -\frac{\beta}{E} sin(\beta x)\,(A(1+v)\,sinh(\beta y) + C((1+v)\beta y\,sinh(\beta y) - (1-v)\,cosh(\beta y)) + B(1+v)\,cosh(\beta y) + D\big((1+v)\beta y\,cosh(\beta y) - (1-v) sinh(\beta y)\big) \tag{3.58}$$

The variations of the functions $u(x,y)$ and $v(x,y)$ with respect to x and y are sketched in Figure 3.6 and 3.7 respectively. For each one we consider two different values of loads as $5\sin(\frac{\pi x}{5})$ and $10\sin\left(\frac{\pi x}{5}\right)$.

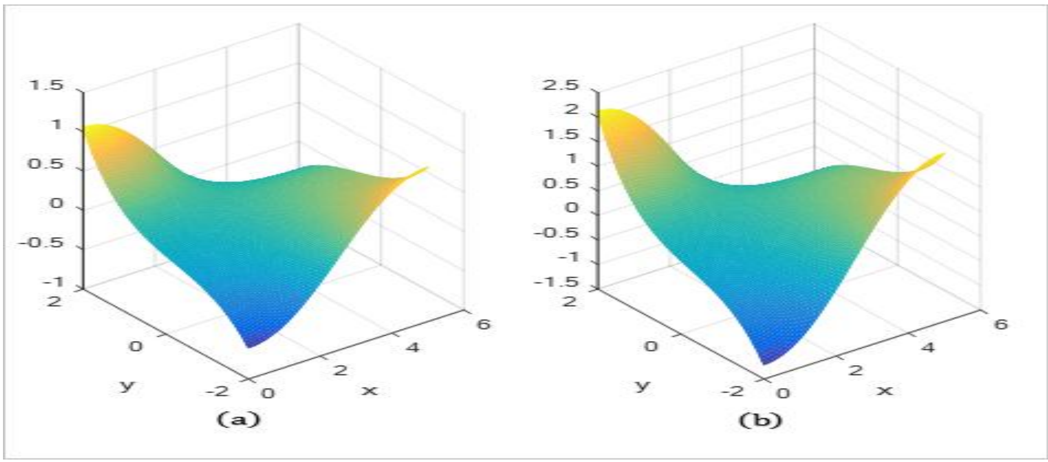

Figure 3.6. The horizontal displacement components $u(x,y)$, (a) for $P_o = 5$ and (b) $P_o = 10$

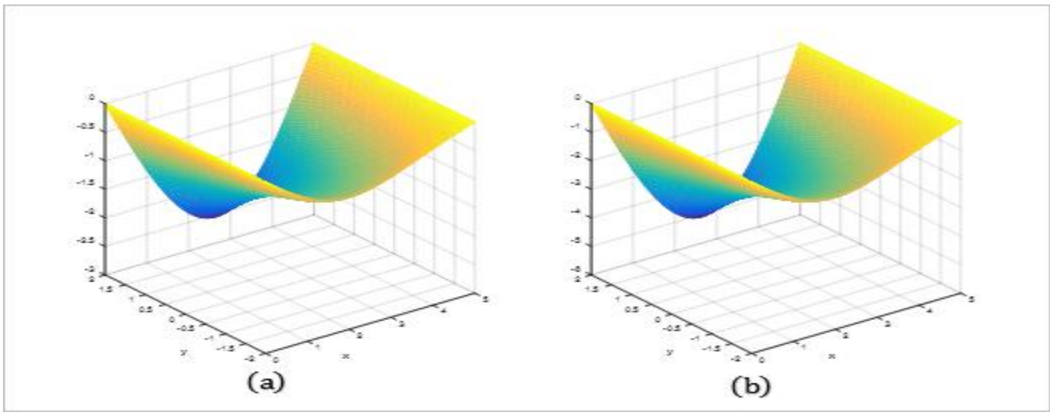

Figure 3.7. The vertical displacement components $v(x,y)$ (a) for $P_o = 5$ and (b) $P_o = 10$

For the case $a \gg c$

$$C = \frac{-3P_o a^5}{4c^3\pi^5}$$

and the relation (3.56) becomes

$$v(x,0) = \frac{-3P_o a^4}{2c^3\pi^4}\frac{1}{E}sin(\beta x)\left[1 + \frac{(1+\nu)}{2}(\frac{\pi}{a}c)tanh(\beta c)\right] \tag{3.59}$$

The corresponding deflection from strength of materials theory is given by

$$v(x,0) = \frac{-3P_o a^4}{2c^3\pi^4}\frac{1}{E}sin(\beta x) \tag{3.60}$$

In the next, as the verification of the Saint-Venant approximation for the problem under consideration, the variations of the Airy stress function, stress and displacement components are plotted in Figure 3.8-3.13 for $P_o = 5$, by taking the dimensions of the beam as $a = 30$ and $c = 2$.

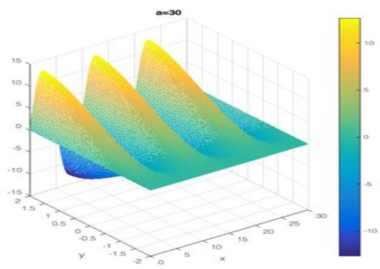

Figure 3.8. The Airy stress function for $P_o = 5$

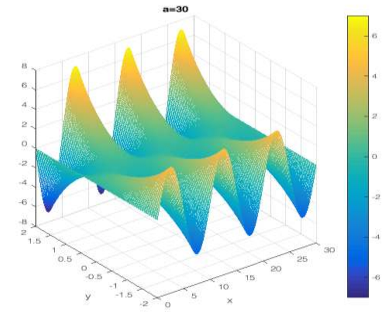

Figure 3.9. The horizontal stress component $\sigma_x(x, y)$ for $P_o = 5$

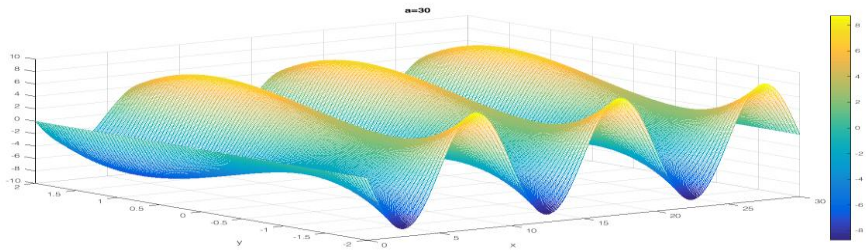

Figure 3.10. The vertical stress component $\sigma_y(x, y)$ for $P_o = 5$

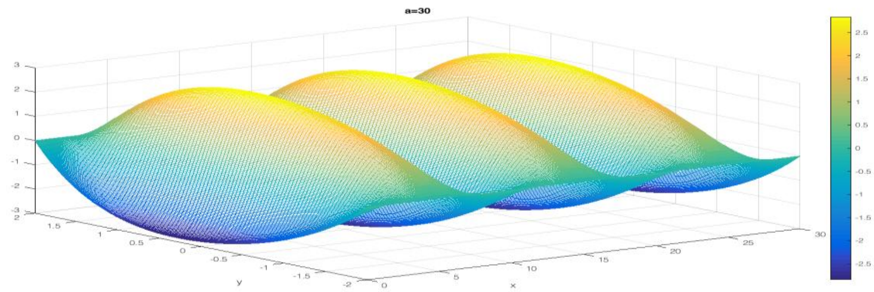

Figure 3.11. The shear stress $\tau_{xy}(x, y)$ for $P_o = 5$

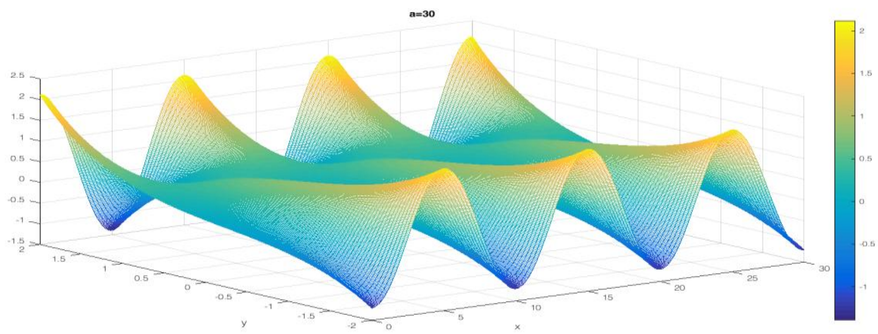

Figure 3.12. The horizontal displacement component $u(x, y)$ for $P_o = 5$

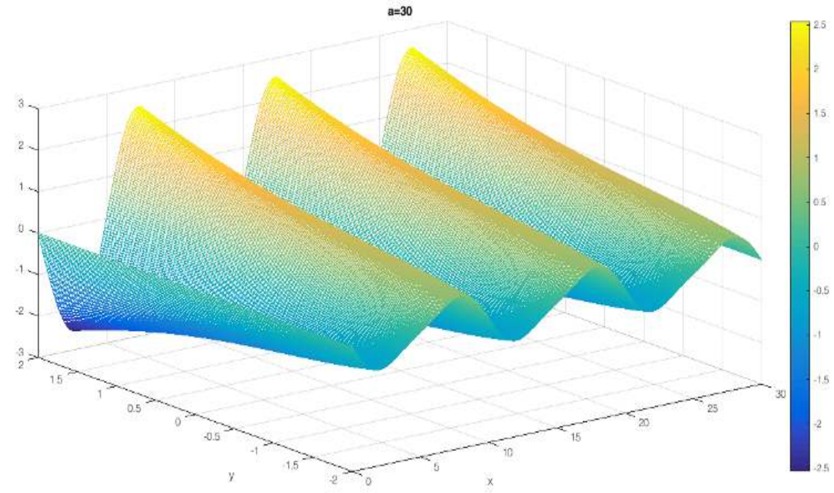

Figure 3.13. The vertical displacement component $v(x, y)$ for $P_o = 5$

3.2. Beam Subjected to the Uniform Tension

As the second example for the applications of FFST method, we study uniaxial tension of a beam. Again we consider the 2D plane-stress case of a rectangular beam under uniform tension P at each end, as shown in Figure 3.14.

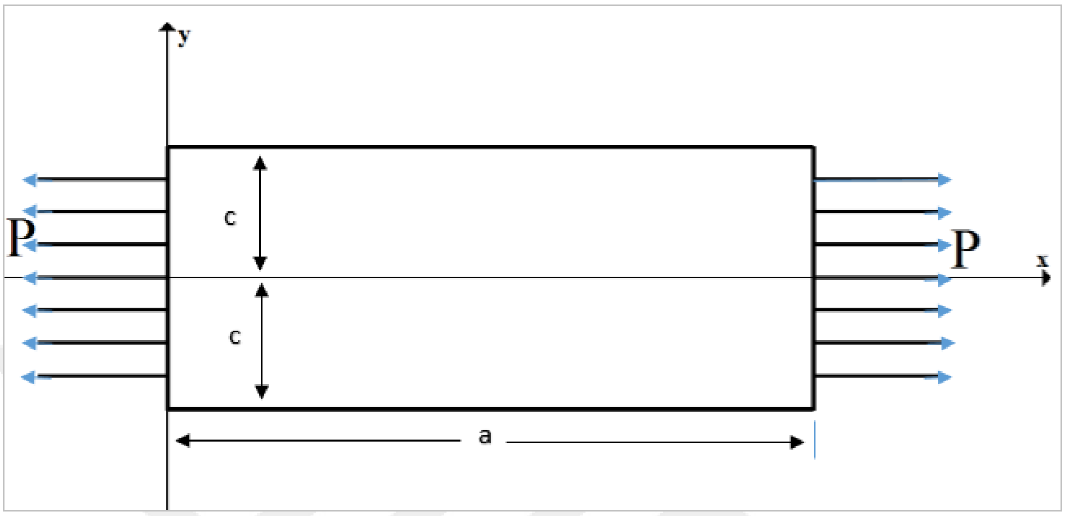

Figure 3.14. Beam under uniform tension.

We can also consider this problem as the Saint-Venant approximation to the more general case since we have nonuniformly distributed tensile forces at the ends $x = 0$ and $x = a$. Hence, the prescribed BCs are replaced by the statically equivalent uniform distribution, and the solution to be developed will be valid at points away from these ends, that is, at the points where $a \gg c$.

The BCs are prescribed as

$$\sigma_x(0, y) = P, \sigma_x(a, y) = P \qquad (3.61)_1$$

$$\tau_{xy}(0, y) = 0\,, \tau_{xy}(a, y) = 0 \qquad (3.61)_2$$

$$\sigma_y(x, -c) = 0, \sigma_y(x, c) = 0 \qquad (3.61)_3$$

Let $\phi = \phi(x, y)$ be the Airy's stress function so that the stresses are defined as

$$\sigma_x(x, y) = \frac{\partial^2\phi}{\partial y^2}\,, \sigma_y(x, y) = \frac{\partial^2\phi}{\partial x^2}\,, \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} \qquad (3.62)$$

Substitution these into the equilibrium equations, (2.13) without body forces yields a BE

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \qquad (3.63)$$

Using the procedure given in [28] we compute

$$\frac{\partial^2\phi}{\partial x^2}|_1 = 0 \ , at\ y = -c \tag{3.64$_1$}$$

$$\frac{\partial^2\phi}{\partial y^2}|_2 = P\ , at\ x = a \tag{3.64$_2$}$$

$$\frac{\partial^2\phi}{\partial x^2}|_3 = 0\ , at\ y{=}c \tag{3.64$_3$}$$

$$\frac{\partial^2\phi}{\partial y^2}|_4 = P\ , at\ x{=}0 \tag{3.64$_4$}$$

and

$$\phi|_1 = \frac{P}{2}c^2\ \ , at\ \ y = -c \tag{3.65$_1$}$$

$$\phi|_2 = \frac{P}{2}y^2 + Cy\quad , at\ \ x = a \tag{3.65$_2$}$$

$$\phi|_3 = \frac{P}{2}c^2 - \tilde{C}x\ \ ,\ at\ \ y = c\ (\tilde{C} = \frac{c}{a}C) \tag{3.65$_3$}$$

$$\phi|_4 = \frac{P}{2}y^2,\ at\ x = 0 \tag{3.65$_4$}$$

and

$$\frac{\partial^2\phi}{\partial y^2}|_1 = P,\ at\ y = -c \tag{3.66$_1$}$$

$$\frac{\partial^2\phi}{\partial x^2}|_2 = 0,\ at\ x = a \tag{3.66$_2$}$$

$$\frac{\partial^2\phi}{\partial y^2}|_3 = P,\ at\ y{=}c \tag{3.66$_3$}$$

$$\frac{\partial^2\phi}{\partial x^2}|_4 = 0\ ,\qquad at\ x = 0 \tag{3.66$_4$}$$

3.2.1 The Method of Solution

By taking the FFST of the BE (3.63)

$$\int_0^a \left(\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}\right) sin(\frac{n\pi x}{a})dx = 0$$

Using the linearity of integration

$$\int_0^a \frac{\partial^4\phi}{\partial x^4} sin(\frac{n\pi x}{a})dx + 2\int_0^a \frac{\partial^4\phi}{\partial x^2\partial y^2} sin(\frac{n\pi x}{a})dx + \int_0^a \frac{\partial^4\phi}{\partial y^4} sin(\frac{n\pi x}{a})dx = 0 \quad (3.67)$$

Letting $\phi_n(y) = \int_0^a \phi(x,y) sin(n\beta x)d$, where $\beta = \frac{\pi}{a}$ and integrating by parts

$$\int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx = -n\beta \left\{ [\phi(x,y)\,cos(n\beta x)]_0^a + n\beta \int_0^a \phi(x,y) sin(n\beta x)dx \right\}$$

$$= -n\beta[\phi(x,y)\,cos(n\beta x)]_0^a - (n\beta)^2 \int_0^a \phi(x,y) sin(n\beta x$$

$$= -n\beta[\phi(x,y)\,cos(n\beta x)]_0^a - (n\beta)^2\phi_n(y)$$

$$= -n\beta[\phi(a,y)(-1)^n - \phi(0,y)(1)] - (n\beta)^2\phi_n(y)$$

Appling the first of the BCs in $(3.65)_2$, $(3.66)_2$ and $(3.65)_4$, $(3.66)_4$

$$\int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx = -n\beta([\frac{P}{2}y^2 + Cy](-1)^n - \frac{P}{2}y^2(1)) - (n\beta)^2\phi_n(y)$$

$$= n\beta\frac{P}{2}y^2(1-(-1)^n) - n\beta\, Cy(-1)^n - (n\beta)^2\phi_n(y) \quad (3.68)$$

Write the third term of equation (3.67) as the form

$$\int_0^a \frac{\partial^4\phi}{\partial y^4} sin(n\beta x)dx = \frac{d^4}{dy^4}\int_0^a \phi(x,y) sin(n\beta x)d = \frac{d^4}{dy^4}\phi_n(y) \quad (3.69)$$

Also, write the second term of Eq. (3.67) as the form

$$\int_0^a \frac{\partial^4\phi}{\partial x^2\partial y^2} sin(n\beta x)dx = \frac{d^2}{dy^2}\int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx$$

$$= \frac{d^2}{dy^2}[n\beta\frac{P}{2}y^2(1-(-1)^n) - n\beta\, Cy(-1)^n - (n\beta)^2\phi_n(y)]$$

$$= n\beta P(1-(-1)^n) - \frac{d^2}{dy^2}(n\beta)^2\phi_n(y) \quad (3.70)$$

Also, write the first term of Eq. (3.67) as below form and integrating by part for the following

$$\int_0^a \frac{\partial^4\phi}{\partial x^4} sin(n\beta x)dx = \left[\left[\frac{\partial^3\phi}{\partial x^3} sin(n\beta x)\right]_0^a - n\beta \int_0^a \frac{\partial^3\phi}{\partial x^3} cos(n\beta x)dx\right]$$

$$= -n\beta \left[\int_0^a \frac{\partial^3\phi}{\partial x^3} cos(n\beta x)dx\right]$$

$$= -n\beta \left[\left[\frac{\partial^2\phi}{\partial x^2} cos(n\beta x)\right]_0^a + n\beta \int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx\right]$$

$$= -n\beta \left[[\frac{\partial^2}{\partial x^2}\phi(a,y)(-1)^n - \frac{\partial^2}{\partial x^2}\phi(0,y)(1)] + n\beta \int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx\right]$$

$$= -n\beta[0-0] - n\beta\left[n\beta \int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx\right] = -(n\beta)^2 \int_0^a \frac{\partial^2\phi}{\partial x^2} sin(n\beta x)dx$$

$$= -(n\beta)^2 \left[n\beta \frac{P}{2} y^2(1-(-1)^n) - n\beta\, Cy(-1)^n - (n\beta)^2\phi_n(y)\right]$$

$$= -(n\beta)^3 \frac{P}{2} y^2[1-(-1)^n] + (n\beta)^3 Cy(-1)^n + (n\beta)^4\phi_n(y) \tag{3.71}$$

Substituting Eqs. (3.69), (3.70) and (3.71) in Eq. (3.67) to get the fourth-order ODE

$$\frac{d^4}{dy^4}\phi_n(y) - 2(n\beta)^2 \frac{d^2}{dy^2}\phi_n(y) + (n\beta)^4\phi_n(y) = F_n(y) \tag{3.72}$$

where

$$\mathrm{F_n(y)} = (n\beta)^3 \frac{P}{2} y^2[1-(-1)^n] - (n\beta)^3 Cy(-1)^n - 2n\beta P(1-(-1)^n) \tag{3.73}$$

So, we get a nonhomogeneous linear ODE of the fourth-order, the solution of which is

$$\phi_n(y) = \phi_n^h(y) + \phi_n^p(y) \tag{3.74}$$

where $\phi_n^h(y)$ is the homogeneous solution and $\phi_n^p(y)$ is the particular solution

3.2.2 The Homogeneous Solution

$$\frac{d^4}{dy^4}\phi_n(y) - 2(n\beta)^2 \frac{d^2}{dy^2}\phi_n(y) + (n\beta)^4\phi_n(y) = 0$$

The auxiliary equation is

$$[m^4 - 2(n\beta)^2 m^2 + (n\beta)^4] = 0$$

or

$$(m^2 - (n\beta)^2)^2 = 0$$

$$m = \mp n\beta \qquad or \quad m = \mp n\beta$$

Therefore, the homogeneous solution is:

$$\phi_n^h(y) = (A_n + C_n(y-c))\frac{sinh(n\beta(y+c))}{sinh(2n\beta c)} + (B_n + D_n(y+c))\frac{sinh(n\beta(y-c))}{sinh(2n\beta c)} \quad (3.75)$$

3.2.3 The Particular Solution

$$D = \frac{1}{f(y)} \rightarrow (D^2 - (n\beta)^2)^2 . \phi_{\mathrm{n}}^p(\mathrm{y}) = f_n(y) \rightarrow \phi_{\mathrm{n}}^p(\mathrm{y}) = \frac{1}{(D^2 - (n\beta)^2)^2} f_n(y)$$

$$\phi_n^p(y) = \frac{1}{(n\beta)^4}\left(1 + 2\frac{1}{\boldsymbol{n^2\beta^2}} D^2 + \cdots\right)\left(n^3\beta^3 \frac{P}{2} y^2[1-(-1)^n] - n^3\beta^3 C y(-1)^n - 2n\beta P(1-(-1)^n)\right)$$

$$\phi_n^p(y) = \frac{1}{n\beta}\frac{P}{2} y^2(1-(-1)^n) - \frac{1}{n\beta} C y(-1)^n \quad (3.76)$$

The general solution is

$$\phi_n(y) = (A_n + C_n(y-c))\frac{sinh(n\beta(y+c))}{sinh(2n\beta)} + (B_n + D_n(y+c))\frac{sinh(n\beta(y-c))}{sinh(2n\beta)} + \frac{1}{\boldsymbol{n\beta}}\frac{P}{2} y^2(1-(-1)^n) - \frac{1}{n\beta} C y(-1)^n \quad (3.77)$$

We now compute the coefficients A_n, B_n, C_n and D_n in the solution (3.77) using FFST of $(3.65)_1$

At $y = -c$

$$\phi_n(-c) = \frac{P}{2} c^2 \frac{(1-(-1)^n)}{n\beta} + \frac{c}{n\beta} C(-1)^n \quad (3.78)$$

$$\phi_n(-c) = (A_n + C_n(-c-c))\frac{sinh(n\beta(-c+c))}{sinh(2n\beta)} + (B_n + D_n(-c+c))\frac{sinh(n\beta(-c-c))}{sinh(2n\beta)} + \frac{P}{2n\beta} c^2(1-(-1)^n) + \frac{c}{n\beta} C(-1)^n = \frac{P}{2n\beta} c^2(1-(-1)^n) + \frac{c}{n\beta} C(-1)^n$$

We get

$$B_n = 0 \quad (3.79)$$

At $y = c$ using FFST of(3.65)$_3$

$$\phi_n(c) = \frac{P}{2}c^2\left[\frac{1-(-1)^n}{n\beta}\right] - \frac{c}{n\beta}C(-1)^n \tag{3.80}$$

$$\phi_n(c) = (A_n + C_n(c-c))\frac{sinh(n\beta(c+c))}{sinh(2n\beta)} + (B_n + D_n(c+c))\frac{sinh(n\beta(c-c))}{sinh(2n\beta)} + \frac{P}{2n\beta}c^2(1-(-1)^n) - \frac{c}{n\beta}C(-1)^n = \frac{P}{2n\beta}c^2(1-(-1)^n) - \frac{c}{n\beta}C(-1)^n$$

We get

$$A_n = 0 \tag{3.81}$$

To compute C_n and D_n we differentiate Eq. (3.77) with respect to y twice

$$\frac{d^2\phi_n}{dy^2} = 2n\beta C_n\frac{cosh(n\beta(y+c))}{sinh(2n\beta)} + 2n\beta D_n\frac{cosh(n\beta(y-c))}{sinh(2n\beta)} + (A_n + C_n(y-c))\frac{sinh(n\beta(y+c))}{sinh(2n\beta)} + (B_n + D_n(y+c))\frac{sinh(n\beta(y-c))}{sinh(2n\beta)} + \frac{P}{n\beta}(1-(-1)^n) \tag{3.82}$$

In addition, substituting $A_n = B_n = 0$ in Eq. (3.82) and using the BCs (3.66)$_1$ and (3.66)$_3$ we get

At $y = -c$ since $\frac{d^2\phi_n(-c)}{dy^2} = \frac{P}{n\beta}(1-(-1)^n)$ we write

$$2n\beta C_n\frac{cosh(n\beta(-c+c))}{sinh(2n\beta)} + 2n\beta D_n\frac{cosh(n\beta(-c-c))}{sinh(2n\beta)} + (A_n + C_n(-c-c))\frac{sinh(n\beta(-c+c))}{sinh(2n\beta)} + (B_n + D_n(-c+c))\frac{sinh(n\beta(-c-c))}{sinh(2n\beta)} + \frac{P}{n\beta}(1-(-1)^n) = \frac{P}{n\beta}(1-(-1)^n)$$

after making cancellations

$$2n\beta C_n\frac{1}{sinh(2n\beta)} + 2n\beta D_n\frac{cosh(2n\beta)}{sinh(2n\beta)} = 0 \text{ or } C_n + D_n cosh(2n\beta) = 0$$

So

$$C_n = -D_n cosh(2n\beta) \tag{3.83}$$

At $y = c$ since $\frac{d^2\phi_n(c)}{dy^2} = \frac{P}{n\beta}(1-(-1)^n)$ we have

$$2n\beta C_n\frac{cosh(n\beta(c+c))}{sinh(2n\beta)} + 2n\beta D_n\frac{cosh(n\beta(c-c))}{sinh(2n\beta)} + (A_n + C_n(c-c))\frac{sinh(n\beta(c+c))}{sinh(2n\beta)} + (B_n + D_n(c+c))\frac{sinh(n\beta(c-c))}{sinh(2n\beta)} + \frac{P}{n\beta}(1-(-1)^n) = \frac{P}{n\beta}(1-(-1)^n)$$

$$2n\beta C_n\frac{cosh(2n\beta)}{sinh(2n\beta)} + 2n\beta D_n\frac{1}{sinh(2n\beta)} += 0 \text{ , or } C_n.\cosh(2n\beta c) + D_n = 0 \tag{3.84}$$

Substituting Eq. (3.83) into Eq. (3.84) we obtain

$\left(-D_n.cosh(2n\beta)\right)cosh(2n\beta) + D_n = 0$, that is

$D_n(1 - \cosh^2(2n\beta)) = 0$ therefore

$$D_n = 0 \tag{3.85}$$

Also, substituting Eq. (3.85) into Eq. (3.83) we get

$$C_n = 0 \tag{3.86}$$

Therefore, the general solution is

$$\phi_n(y) = \frac{P}{2n\beta} y^2[1-(-1)^n] - \frac{1}{n\beta} Cy(-1)^n \tag{3.87}$$

The IFFST of Eq. (3.87) is computed

$$\phi(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \phi_n(y)\, sin(n\beta x)$$

i.e.

$$\phi(x,y) = \frac{P}{a\beta}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}y^2\right) sin(n\beta x) - \frac{2C}{a\beta}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}y^2\right) sin(n\beta x) \tag{3.88}$$

By using the tables in [29, p.46] we write

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\beta x) = -\frac{1}{2}\beta x \quad , \quad 0 < x < a \tag{3.89}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\beta x) = \frac{\pi}{2} - \frac{x}{2}\beta \quad , \quad 0 < x < a \tag{3.90}$$

By rearranging Eq. (3.88) and using Eqs. (3.89) and (3.90) we obtain

$$\phi(x,y) = \frac{P}{a\beta}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}sin(n\beta x)\right)y^2 - \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}sin(n\beta x)\right)y^2\right)$$

$$-\frac{2C}{a\beta}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}sin(n\beta x)\right)y$$

$$= \frac{P}{\pi}\left\{\left(\frac{\pi}{2} - \frac{1}{2}\beta x\right)y^2 - \left(-\frac{1}{2}\beta x\right)y^2\right\} - \frac{2C}{\pi}\left(-\frac{1}{2}\beta x\right)y$$

$$= \frac{P}{2}y^2 - \frac{P}{2a}xy^2 + \frac{P}{2a}xy^2 + \frac{C}{a}xy = \frac{P}{2}y^2 + \frac{C}{a}xy \tag{3.91}$$

Since in our problem the BCs at $x = 0$ and $x = a$ are given as

$\tau_{xy}(0,y) = \tau_{xy}(a,y) = 0$, we have

$$-\frac{\partial^2\phi}{\partial x\partial y}(0,y) = 0 \text{ and } -\frac{\partial^2\phi}{\partial x\partial y}(a,y) = 0 \tag{3.92}$$

From Eq. (3.91)

$\frac{\partial\phi}{\partial x}(x,y) = \frac{C}{a}y$ and $\frac{\partial^2\phi}{\partial x\partial y}(x,y) = \frac{C}{a}$

Since $\frac{C}{a} = 0$ at $x = 0$ and $x = a$, we have

$$C = 0 \tag{3.93}$$

∴The solution of the BE $\nabla^4\phi = 0$ with the specified BCs are

$$\phi(x,y) = \frac{P}{2}y^2 \tag{3.94}$$

which is plotted in Figure 3.15 for (a) $P = 5$ and (b) $P = 10$

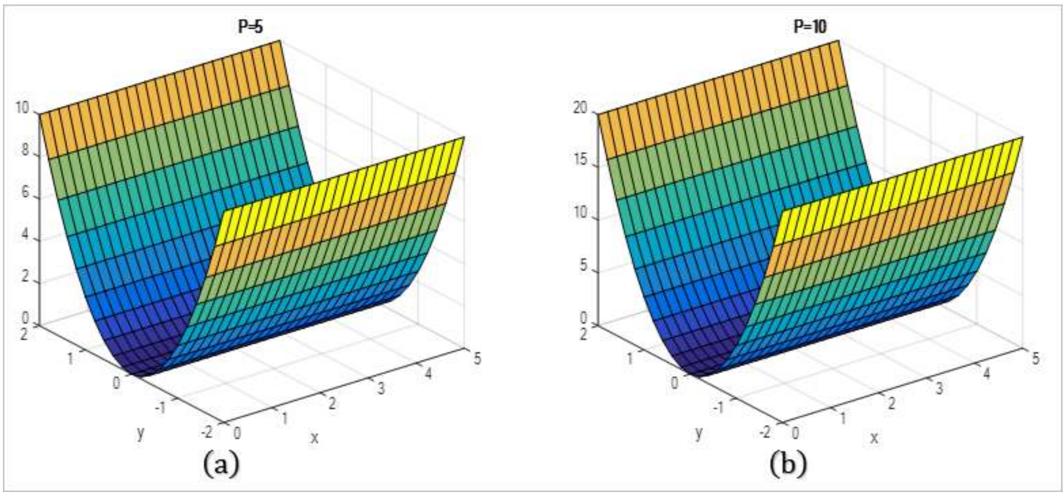

Figure 3.15. The Airy stress function, (a) for $P = 5$and (b) $P = 10$

Hence the stress components are

$$\sigma_x(x,y) = \frac{\partial^2\phi}{\partial y^2}(x,y) = P$$

$$\sigma_y(x,y) = \frac{\partial^2\phi}{\partial x^2}(x,y) = 0$$

and

$$\tau_{xy}(x,y) = -\frac{\partial^2\phi}{\partial x \partial y}(x,y) = 0$$

These three functions are plotted in Figures 3.16, 3.17 and 3.18 for $P = 5$ and $P = 10$ respectively.

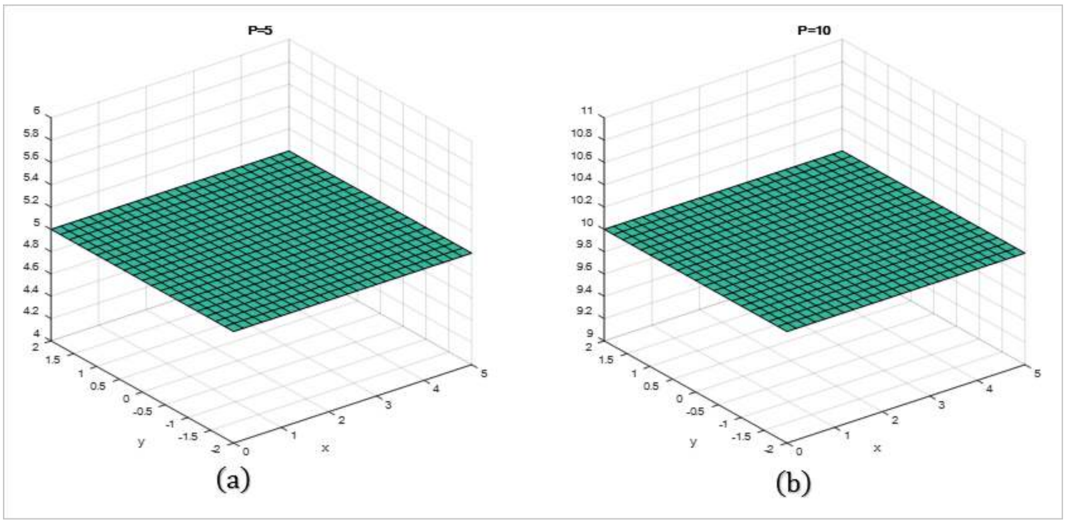

Figure 3.16. The horizontal stress components $\sigma_x(x,y)$, (a) for $P = 5$ and (b) $P = 10$

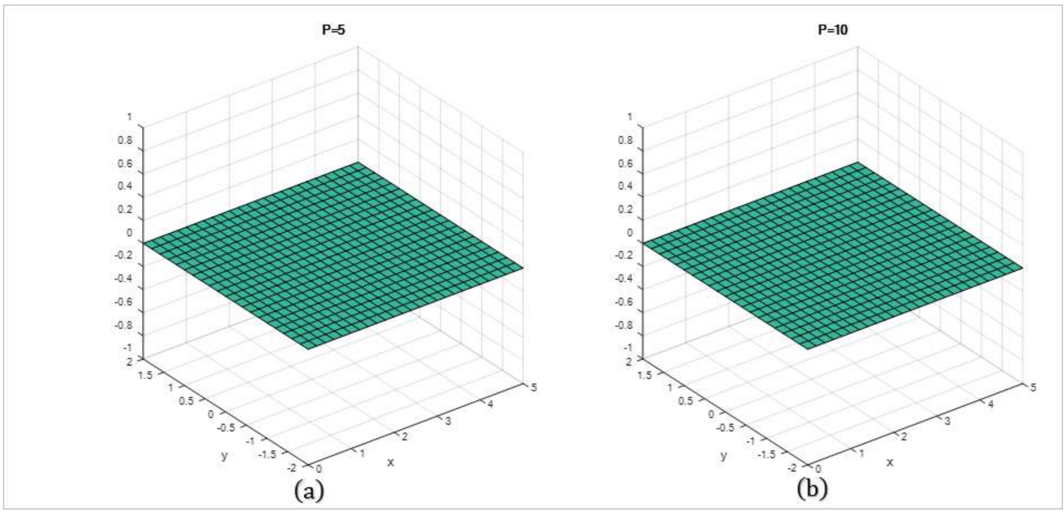

Figure 3.17. The vertical stress components $\sigma_y(x, y)$, (a) for $P = 5$ and (b) $P = 10$

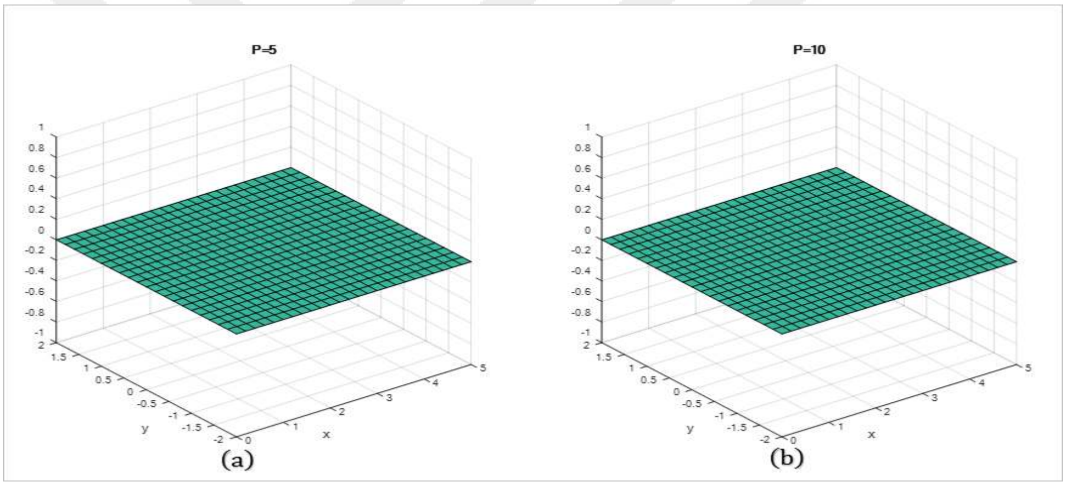

Figure 3.18. The shear stress $\tau_{xy}(x, y)$ (a) for $P = 5$ and (b) $P = 10$

3.2.4 Determination of the Components of Displacement

3.2.4.1 Solution of BE in terms of Harmonic Functions

The solution of the homogenous BE can be written in terms of harmonic functions as

$$\phi(x, y) = x.\gamma(x, y) + y.\gamma^c(x, y) + \rho(x, y) \tag{3.95}$$

where $\gamma(x, y), \rho(x, y)$ and their conjugates $\gamma^c(x, y), \rho^c(x, y)$ are harmonic functions.

Differentiating Eq. (3.95) w.r.t x and y twice respectively

$$\frac{\partial \phi}{\partial x} = 1.\gamma(x, y) + x\frac{\partial \gamma}{\partial x} + y\frac{\partial \gamma^c}{\partial x} + \frac{\partial \rho}{\partial x} \tag{3.96}$$

$$\frac{\partial^2\phi}{\partial x^2} = 2.\frac{\partial\gamma}{\partial x} + x\frac{\partial^2\gamma}{\partial x^2} + y\frac{\partial^2\gamma^c}{\partial x^2} + \frac{\partial^2\rho}{\partial x^2} \tag{3.97}$$

$$\frac{\partial\phi}{\partial y} = x\frac{\partial\gamma}{\partial y} + \gamma^c(x,y) + y\frac{\partial\gamma^c}{\partial y} + \frac{\partial\rho}{\partial y} \tag{3.98}$$

$$\frac{\partial^2\phi}{\partial y^2} = x\frac{\partial^2\gamma}{\partial y^2} + 2\frac{\partial\gamma^c}{\partial y} + y\frac{\partial^2\gamma^c}{\partial y^2} + \frac{\partial^2\rho}{\partial y^2} \tag{3.99}$$

By adding Eqs. (3.97) and (3.99) and using the Cauchy-Riemann equations we obtain

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\frac{\partial\gamma}{\partial x} + x\frac{\partial^2\gamma}{\partial x^2} + y\frac{\partial^2\gamma}{\partial x^2} + \frac{\partial^2\rho}{\partial x^2} + x\frac{\partial^2\gamma}{\partial y^2} + 2\frac{\partial\gamma^c}{\partial y} + y\frac{\partial^2\gamma}{\partial y^2} + \frac{\partial^2\rho}{\partial y^2}$$

$$\nabla^2\phi = 2\left(\frac{\partial\gamma}{\partial x} + \frac{\partial\gamma^c}{\partial y}\right) + x\left(\frac{\partial^2\gamma}{\partial x^2} + \frac{\partial^2\gamma}{\partial y^2}\right) + y\left(\frac{\partial^2\gamma}{\partial x^2} + \frac{\partial^2\gamma}{\partial y^2}\right) + \left(\frac{\partial^2\rho}{\partial x^2} + \frac{\partial^2\rho}{\partial y^2}\right)$$

$$\nabla^2\phi = 2\left(\frac{\partial\gamma}{\partial x} + \frac{\partial\gamma^c}{\partial y}\right) \tag{3.100}$$

By Cauchy-Riemann equations

$$\frac{\partial\gamma}{\partial x} = \frac{\partial\gamma^c}{\partial y}\,,\ \frac{\partial\gamma}{\partial y} = \frac{\partial\gamma^c}{\partial x} \tag{3.101}$$

$$\nabla^2\phi = 2\left(\frac{\partial\gamma}{\partial x} + \frac{\partial\gamma}{\partial x}\right)$$

$$\nabla^2\phi = 4\frac{\partial\gamma}{\partial x}$$

$$\frac{\partial\gamma}{\partial x} = \frac{1}{4}\nabla^2\phi \tag{3.102}$$

Also

$$\nabla^2\phi = 2\left(\frac{\partial\gamma}{\partial x} + \frac{\partial\gamma^c}{\partial y}\right) = 2\left(\frac{\partial\gamma^c}{\partial y} + \frac{\partial\gamma^c}{\partial y}\right) = 4\frac{\partial\gamma^c}{\partial y}$$

$$\frac{\partial\gamma^c}{\partial y} = \frac{1}{4}\nabla^2\phi \tag{3.103}$$

By integrating Eqs. (3.102) with respect to x and (3.103) with respect to y

$$\gamma(x,y) = \int_0^x \frac{1}{4}\nabla^2\phi dx + \alpha(y) \tag{3.104}$$

$$\gamma^c(x,y) = \int_0^y \frac{1}{4}\nabla^2\phi\, dy + \beta(x) \tag{3.105}$$

Differentiation of Eq. (3.102) with respect to x and Eq. (3.104) with respect to y twice, adding and using Eq. (3.94)

$$\frac{\partial^2\gamma}{\partial x^2} + \frac{\partial^2\gamma}{\partial y^2} = \frac{1}{4}\frac{\partial}{\partial x}\nabla^2\phi + \frac{1}{4}\int_0^x \frac{\partial^2}{\partial y^2}(\nabla^2\phi)dx + \alpha''(y)$$

So,

$$\alpha''(y) = 0 \tag{3.106}$$

since ϕ is a quadratic polynomial its Laplacian is a constant, i.e., $\nabla^2\phi = 2$, hence its partial derivatives with respect to x or y are always zero and γ is a harmonic function i.e., $\nabla^2\gamma = 0$.

Now, differentiating Eq. (3.103) with respect to x and Eq. (3.105) with respect to y twice, then adding and using of Eq. (3.94)

$$\frac{\partial^2\gamma^c}{\partial y^2} + \frac{\partial^2\gamma^c}{\partial x^2} = \frac{1}{4}\frac{\partial}{\partial y}\nabla^2\phi + \frac{1}{4}\int_0^y \frac{\partial^2}{\partial x^2}(\nabla^2\phi)dy + \beta''(x)$$

Thus,

$$\beta''(x) = 0 \tag{3.107}$$

since $\nabla^2\phi = 2$ and $\nabla^2\gamma^c = 0$.

Integrating Eqs. (3.106) and (3.107) twice we compute $\alpha(y)$ and $\beta(x)$ as

$$\alpha(y) = \iota_0 + \iota_1 y \tag{3.108}$$

$$\beta(x) = m_0 + m_1 x \tag{3.109}$$

Substitute Eqs. (3.108) and (3.109) into Eqs. (3.104) and (3.105) respectively we find

$$\gamma(x,y) = \frac{1}{4}\int_0^x \nabla^2\phi dx + \alpha(y)$$

$$= \tfrac{1}{4}\int_0^x \nabla^2\phi dx + \iota_0 + \iota_1 y \text{ since } \phi(x,y) = \tfrac{P}{2}y^2$$

$$= \tfrac{1}{4}\int_0^x \left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right)\left(\tfrac{P}{2}y^2\right)dx + \iota_0 + \iota_1 y$$

$$= \tfrac{1}{4}\int_0^x P dx + \iota_0 + \iota_1 y = \tfrac{1}{4}[Px|_0^x] + \iota_0 + \iota_1 y = \tfrac{1}{4}[Px - 0] + \iota_0 + \iota_1 y$$

$$\gamma(x,y) = \tfrac{P}{4}x + \iota_0 + \iota_1 y \tag{3.110}$$

Similarly

$$\gamma^c(x,y) = \frac{1}{4}\int_0^y \nabla^2\phi dy + \beta(x)$$

$$= \tfrac{1}{4}\int_0^y \nabla^2\phi dy + m_0 + m_1 x \text{ , since } \phi(x,y) = \tfrac{P}{2}y^2$$

$$= \tfrac{1}{4}\int_0^y \left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right)\left(\tfrac{P}{2}y^2\right)dy + m_0 + m_1 x$$

$$= \tfrac{1}{4}\int_0^y P dy + m_0 + m_1 x = \tfrac{1}{4}\left[Py|_0^y\right] + m_0 + m_1 x = \tfrac{1}{4}[Px - 0] + m_0 + m_1 x$$

$$\gamma^c(x,y) = \tfrac{P}{4}y + m_0 + m_1 x \tag{3.111}$$

Substituting Eqs. (3.110) and (3.111) into Eq. (3.95)

$$\frac{P}{2}y^2 = x\left(\frac{P}{4}x + \iota_0 + \iota_1 y\right) + y\left(\frac{P}{4}y + m_0 + m_1 x\right) + \rho(x, y)$$

$$\frac{P}{4}x^2 + \iota_0 x + \iota_1 xy - \frac{P}{4}y^2 + m_0 y + m_1 xy + \rho(x, y) = 0$$

by arranging we get

$$\rho(x, y) = \frac{P}{4}y^2 - \frac{P}{4}x^2 - xy(\iota_1 + m_1) - \iota_0 x - m_0 y \quad (3.112)$$

3.2.4.2 Displacement Components $u(x, y)$ and $v(x, y)$

To determine the velocity components we use the formulas

$$\frac{E}{1+\nu}u = -\frac{\partial\emptyset}{\partial x} + 4(1-\nu)\gamma(x, y) \quad (3.113)$$

$$\frac{E}{1+\nu}v = -\frac{\partial\emptyset}{\partial y} + 4(1-\nu)\gamma^c(x, y) \quad (3.114)$$

where ν, E are Poison's ratio & Young's modulus respectively

Substitution of Eqs. (3.95), (3.110) and (3.111) into Eqs. (3.113) and (3.114) gives

$$\frac{E}{1+\nu}u = 4(1-\nu)(\frac{P}{4}x + \iota_0 + \iota_1 y) \quad (3.115)$$

and

$$\frac{E}{1+\nu}v = -Py + 4(1-\nu)(\frac{P}{4}y + m_0 + m_1 x) \quad (3.116)$$

3.2.4.2.1 Elimination of the Rigid Body Translation

Applying $u(0,0) = 0$ in Eq. (3.115)

$$\frac{E}{1+\nu}u(0,0) = 4(1-\nu)(\frac{P}{4}(0) + \iota_0 + \iota_1(0)) \Longrightarrow 4(1-\nu)\iota_0 = 0$$

then

$$\iota_0 = 0 \quad (3.117)$$

Similarly

$$\frac{E}{1+\nu}v(0,0) = -P(0) + 4(1-\nu)(\frac{P}{4}(0) + m_0 + m_1(0)) \Longrightarrow 4(1-\nu)m_0 = 0$$

then

$$m_0 = 0 \quad (3.118)$$

So, we get

$$\frac{E}{1+\nu}u = 4(1-\nu)(\frac{P}{4}x + \iota_1 y) \quad (3.119)$$

$$\frac{E}{1+\nu}v = -Py + 4(1-\nu)(\frac{P}{4}y + m_1 x) \quad (3.120)$$

From Eq. (3.119) we find

$$\frac{E}{1+\nu}\frac{\partial u}{\partial y} = 4(1-\nu)\iota_1 \quad (3.121)$$

Similarly, from Eq. (3.120) we get

$$\frac{E}{1+\nu}\frac{\partial v}{\partial x} = 4(1-\nu)m_1 \tag{3.122}$$

3.2.4.2.2 Elimination of the Rigid Body Rotation

Applying the condition $\frac{\partial u(0,0)}{\partial y} - \frac{\partial v(0,0)}{\partial x} = 0$ that can be used only the first fundamental problem we write

$$\frac{E}{1+\nu}\left(\frac{\partial u(0,0)}{\partial y} - \frac{\partial v(0,0)}{\partial x}\right) = 4(1-\nu)\iota_1 - 4(1-\nu)m_1$$

$$4(1-\nu)\iota_1 - 4(1-\nu)m_1 = 0 \Longrightarrow \iota_1 - m_1 = 0$$

$$\iota_1 = m_1$$

This equality holds only when (for “This condtion is satisfied only for”)

$$\iota_1 = m_1 = 0 \tag{3.123}$$

Therefore, the displacements components are

$$\frac{E}{1+\nu}u = Px - \nu Px = (1-\nu)Px \tag{3.124}$$

and

$$\frac{E}{1+\nu}v = -Py + 4(1-\nu)\left(\frac{P}{4}y\right) = -\nu Py \tag{3.125}$$

The variations of u and v are sketched for $P = 5$ and $P = 10$ in Figures 3.19, 3.20 respectively.

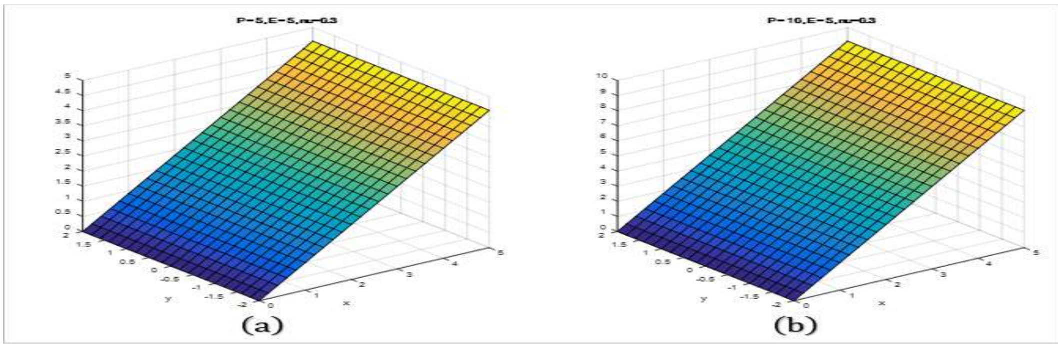

Figure 3.19. The horizontal displacement component $u(x, y)$, (a) for $P = 5$ and (b) $P = 10$

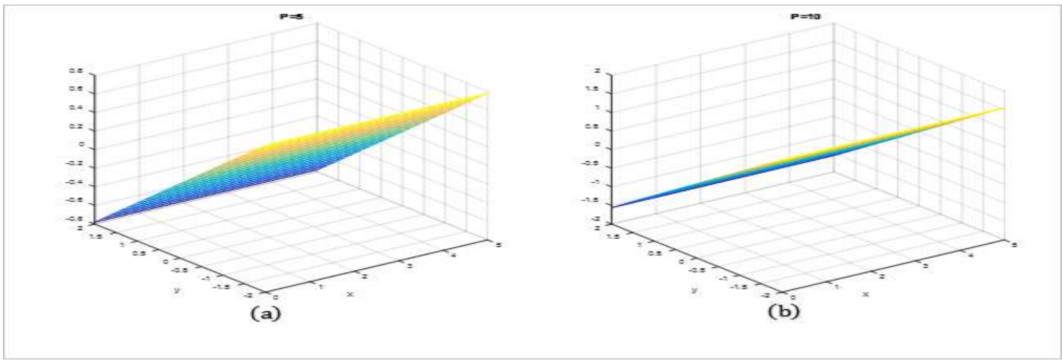

Figure 3.20. The vertical displacement component $v(x,y)$, (a) for $P = 5$ and (b) $P = 10$

Finally, for the verification of the Saint-Venant principle holds in this problem, the Airy stress function, stress and displacement components are sketched in Figures 3.21-3.26 for $P = 5$ and taking $a = 30, c = 2$

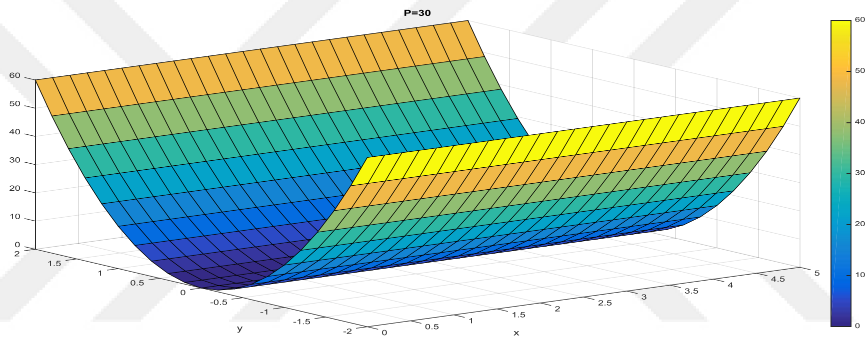

Figure 3.21. The Airy stress function $\phi(x,y)$, for $P = 5$

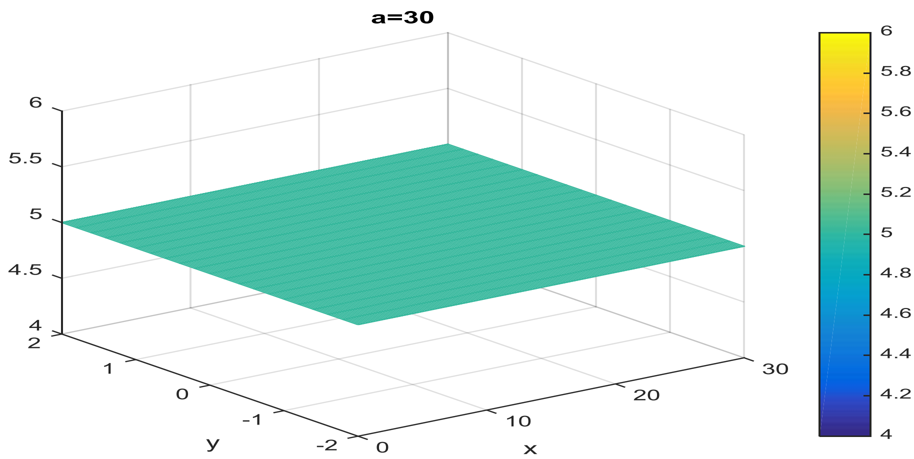

Figure 3.22. The horizontal stress component $\sigma_x(x,y)$, for $P = 5$

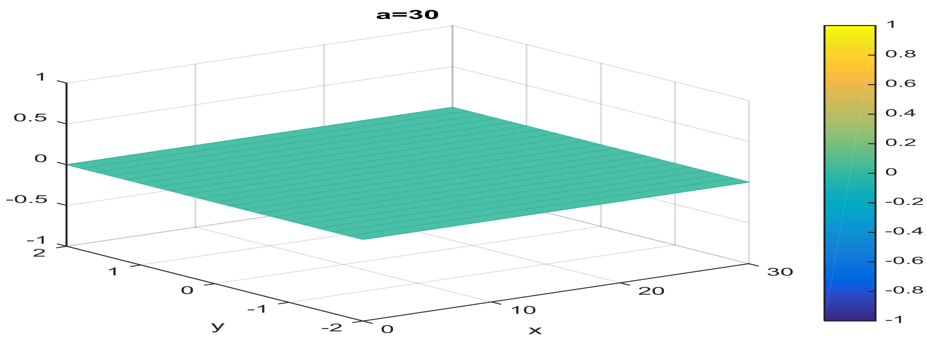

Figure 3.23. The vertical stress component $\sigma_y(x,y)$, for $P = 5$

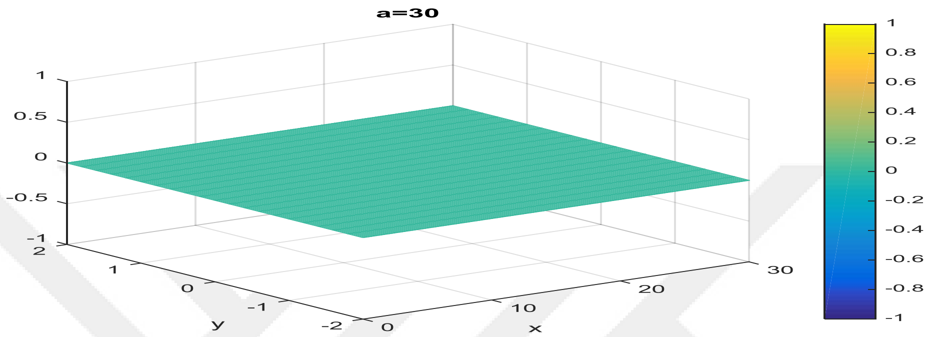

Figure 3.24. The shear stress $\tau_{xy}(x,y)$, for $P = 5$

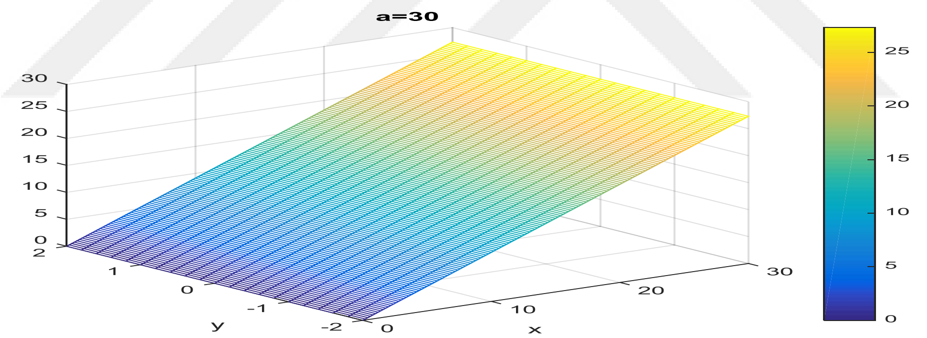

Figure 3.25. The horizontal displacement component $u(x,y)$, for $P = 5$

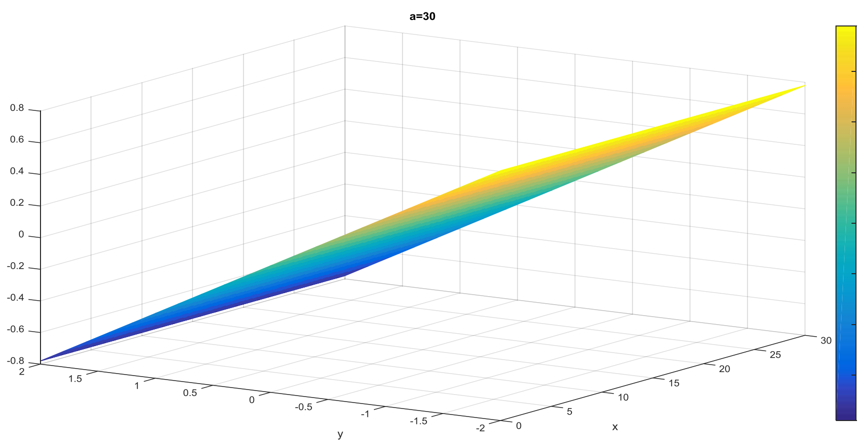

Figure 3.26. The vertical displacement component $v(x,y)$, for $P = 5$

4. RESULTS

Depending on our aim, we applied FFST method to compute analytical solutions of the 2D elastic rectangular beams of finite extent under two different loads. These problems are formulated as BEs by Airy stress function approch. The prescribed BCs are also written in terms of Airy stress function. Then by applying FFST method to the problems and then inversing the transformed solution in Fourier integral space we obtained the analytical solution of each of the problems in original coordinates. Using these computed Airy stress functions we found the horizontal, vertical and shear stresses, σ_x, σ_y and τ_{xy} at each point of the beams, respectively. The horizontal and vertical components of the displacement vectors, u and v were also computed, respectively. The behaviors of these functions under two different values of the loads are plotted in Figures 3.2-3.7 and 3.15-3.20.

For each problem, we take the dimensions of the beams as $a = 5$ and $c = 2$, and the elastic constants, Poisson ratio and Young's modulus as $\nu = 0.3$ and $E = 5$, respectively.

In the first application, we assumed that the beam subjected to transverse sinusoidal load on top. The computed solution, $\phi(x, y)$ and the resulting vertical stress σ_y, horizontal stress σ_x and shear stress τ_{xy} were plotted in Figures 3.2-3.5. The corresponding horizontal and vertical displacement components, u and v were also sketched in Figures 3.6-3.7. All sketchings are given for the values of the loads $5\, sin(\frac{\pi x}{a})$ and $10 sin\,(\frac{\pi x}{a})$. Under these loads, we see that all graphs have the same shapes except the values of the functions. The deformation of the beam is shaped by the sinusoidal load applied transversally from top side in each of these figures.

In order to verify the Saint-Venant approximation in this problem we also plotted the computed functions for a longer beam by taking $a = 30$ and $c = 2$ in Figures 3.8-3.13. Periodic behavior of all functions due to the applied periodic force can easily be seen in figures.

In the second application, the beam is supposed to be subjected to uniaxial tension, P at each lateral side. Similarly, the computed solution, $\phi(x, y)$ and the resulting vertical stress σ_y, horizontal stress σ_x and shear stress τ_{xy} were plotted in Figures 3.15-3.18. The corresponding horizontal and vertical displacement components, u and v were also sketched in Figures 3.19-3.20. To observe the effect of tension on the deformation of the beam, we take $P = 5$ and $P = 10$ and we see that the corresponding graphs, all have the same shapes, except the function values. Since applied tension is constant, the stress components are plotted as constant functions, that is $\sigma_y(x, y) = 0$, $\sigma_x(x, y) = P$ and $\tau_{xy}(x, y) = 0$. The displacement components, $u(x, y)$ and $v(x, y)$ vary linearly as seen from the graphs. As we did for the first problem, to show that this problem can also be approximated by Saint-Venant principle we sketched all graphs by taking beam's dimensions as $a = 30$ and $c = 2$. All graphs have the similar shapes as those of shorter beam as seen in figures 3.21-3.26.

5. DISCUSSION

In this thesis, the FFST has been applied successfully for finding the distributions of the normal stresses, $\sigma_x(x, y)$ and $\sigma_y(x, y)$, and shear stress $\tau_{xy}(x, y)$ in the finite 2D rectangular beams that are assumed to be linear elastic, isotropic and homogeneous. Depending on the load, we consider two problems. In the first problem, it was assumed that the beam was subjected to transverse sinusoidal loading on the top and in the second problem, uniaxial tension was applied at the lateral sides of the beam.

To simplify the plane stress elasticity problems, each problem was formulated as BEs in Airy stress function $\phi(x, y)$, subject to the BCs due to applied loads by using the Airy stress function method.So, the problems under consideration were transformed into BVPs of the fourth-order BE and worked out such BVPs by using FFST.

Application of FFST method to the biharmonic stress compatibility equations expressed in terms of the Airy stress function reduced these BEs to the fourth-order, linear ODEs with constant parameters (coefficients), found as in Eq.(3.12) and Eqs.(3.72)-(3.73) in terms of Airy stress function in Fourier transform space.While the reduced ODE was homogeneous for the first problem it was nonhomogeneous for the second problem.

Using the well-known solution techniques for linear, costant coefficients ODEs of higher-order, each ODE was solved and the general solutions were computed. Since the applications of the transformed BCs. yield a simple form for the solution of the reduced ODE, by inversing it, we found the general solution of the 2D beam problem under transverse sinusoidal load on top in the original variables. Then, using Eq.(2.22) with the assumption of no body forces we computed normal and shear stresses and the corresponding horizontal and vertical components of the displacement vector. However, for the 2D beam problem under uniaxial tension at the lateral sides, after computing all unknown coefficients (depending on the parameter β) explicitly and writing the solution explicitly in the Fourier transform space we could achieve inversing that solution and obtained the general solution to the second problem as in Eq.(3.94). Again, using Eq.(2.22) without body forces we computed stress components and displacement components, completely.

The obtained solutions for both problems coincide with the results given in Sadd's book [20].

For the illustration of the computed results, each function was graphed for two different loads. In the first problem, we assumed that two loads of amount $5sin\ (\frac{\pi x}{a})$ and $10sin\ (\frac{\pi x}{a})$ applied to the 2D rectangular elastic beam with length $a\ =\ 5$ and width $c\ =\ 2$.The behavior of the functions under these two values of loads were plotted as similar, that is, the amount of load did not affect the behavior of them, only function values change. This result is expected since our study is restricted to linear elasticity problems.

In the second problem, when we supposed that the constant tensions of amount, $P\ =\ 5$ and $P\ =\ 10$ were applied, in all figures we observed that normal stresses have constant values and horizontal and vertical displacement components vary linearly. As the first problem, beam dimensions were taken as $a\ =\ 5$, and $c\ =\ 2$. In the skecthing of the displacement components, the elastic constants, Poisson's ratio and the Young's modulus were chosen as $v\ =\ 0.3$ and $E\ =\ 5$, respectively.

Finally, we intended to verify the Saint-Venant approximation in our problems. By considering longer beams, that is by taking as $a\ =\ 30$, and $c\ =\ 2$ and by applying only one value of the load, we sketched the graphs of all computed functions. The load was taken as $5sin\ (\frac{\pi x}{a})$ in the first problem and $P\ =\ 5$ in the second problem. The effect of the periodic load applied in the first problem can be easily seen in all figures from Figure 3.8 to Figure 3.11. Since all stress components are constant and displacement components vary linearly in the second problem, we obtained the similar behavior for the longer beam as shown in Figures 3.21-3.26.

6. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, we proposed to apply FFST method for finding the closed-form of the solutions to BVPs for finite, isotropic, elastic 2D beams having rectangular shape under two different loads. In the first application, the load is supposed to be sinusoidal and applied transversally on top and in the second application, load is as assumed as uniaxial tension applied at the lateral sides. Both problems are formulated in plane-stress conditions.

To simplify the 2D finite, elastic beam problems to the problems of finding suitable biharmonic stress functions that satisfy the BCs of the given problems, we used Airy stress function method and formulated the problems under consideration as fourth-order BEs. The prescribed BCs were also written in terms of Airy stress function.

Since the Fourier transform operation is a linear operation, and greatly simplifies the solution of the biharmonic equation we apply the FFST method to our problems to reduce the biharmonic stress compatibility equations expressed in terms of the Airy's stress function into linear ODEs of the fourth order with constant coefficients (parameters), which are easier to solve than the original BEs. After solving these ODEs with use of the transformed BCs in Fourier transform space, the general solutions in the original variables were obtained by inversing them. And, hence, the components of stress and displacement were determined at each point of the beams. The solutions in Sadd's book [20] have been recovered here.

We also verify the Saint-Venant approximation in our problems. When the computed functions were plotted for the longer beam in each problem, the periodic effect of the sinusoidal loading in the first problem could be readily observed in the graphs of the stress and displacement components. The constant normal and shear stresses and linear displacement components could be clearly seen in the figures for the second problem.

The FFST technique were applied to our problems succesfully. This means that, our approach is convenient for the worked problems. One reason is that, there were no shear stresses at the boundaries in our problems, and the second one and the most important one is that the series involved in the IFFSTs are summable.

However, this may not be true for more complicated solutions. Hence these approaches may not be applicable in such problems.

In this thesis, two simple 2D elastic, finite beam problems were solved by using FFST method. As the future works we can suggest:

•to generalize this study as determination of the classes of 2D or 3D elastic beam problems can be solved by FFST or more generally by Fourier integral transform techniques, or

•to apply the other integral transform techniques to similar problems and determine which techniques can be applicable.

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7. REFERENCES

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