

**ÇUKUROVA UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES**

PhD THESIS

Emine AVŞAR AYDIN

**AN APPROXIMATE SOLUTION FOR THE PLANE WAVE DIFFRACTION
BY AN IMPEDANCE STRIP: H-POLARIZATION CASE**

DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

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We certify that the thesis titled above was reviewed and approved for the award of degree of the Doctor of Philosophy by the board of jury on 25/03/2016.

.....
Prof. Dr. Turgut İKİZ
SUPERVISOR

.....
Prof. Dr. Faruk KARADAĞ
MEMBER

.....
Prof. Dr. Ali AKDAĞLI
MEMBER

.....
Assoc. Prof. Dr. Sami ARICA
MEMBER

.....
Asst. Prof. Dr. Şule ÇOLAK
MEMBER

This PhD Thesis is written at the Department of Institute of Natural And Applied Sciences of Çukurova University.

Registration Number:

Prof. Dr. Mustafa GÖK
Director
Institute of Natural and Applied Sciences

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ABSTRACT

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AN APPROXIMATE SOLUTION FOR THE PLANE WAVE DIFFRACTION BY AN IMPEDANCE STRIP: H-POLARIZATION CASE

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: Prof. Dr. Ali AKDAĞLI
: Assoc. Prof. Dr. Sami ARICA
: Asst. Prof. Dr. Şule ÇOLAK

In this study, the diffraction of H-polarized plane wave by an infinitely long strip which has the same impedance on both faces with a width of $2a$ is investigated by using an analytical-numerical method. The diffracted field is obtained by an integral equation in terms of the electric and magnetic currents induced by the incident-field. This integral equation is reduced to two uncoupled integral equations that include only induced electric and magnetic currents separately. Both of the currents are defined as a sum of infinite series of Gegenbauer polynomials with unknown coefficients satisfying the edge conditions. The integral equations are transformed to linear algebraic equations by using analytical methods and the unknown coefficients are determined by solving numerically obtained matrix equations. Numerical examples on the RCS (radar cross section) are presented, and the far field scattering characteristics of the strip are discussed in detail. Some of the obtained results are compared with the other existing method.

Key Words: Impedance, Strip, Analytic, Numeric, Diffraction

ÖZ

DOKTORA TEZİ

BİR EMPEDANS ŞERİTTEN DÜZLEM DALGANIN SAÇILIMI İÇİN YAKLAŞIK BİR ÇÖZÜM: H-POLARİZE DURUMDA

Emine AVŞAR AYDIN

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: Prof. Dr. Ali AKDAĞLI

: Doç. Dr. Sami ARICA

: Yrd. Doç. Dr. Şule ÇOLAK

Bu çalışmada, bir analitik-nümerik metot kullanılarak her iki yüzeyi de aynı empedansa sahip $2a$ genişliğinde ve sonsuz uzunluktaki bir şerite gönderilen H-polarize düzlemsel dalgaların saçılması araştırılmıştır. Gelen alan tarafından şerit üzerinde indüklenen elektrik ve manyetik akımlar bir integral denklemle gösterilerek saçılan alan ifadesi elde edilmiştir. Bu integral denklemi birbirinden bağımsız sadece elektrik ve manyetik akımları içeren iki integral denkleme indirgenebilmiştir. Bu iki akım da kenar koşullarını sağlayacak şekilde katsayıları belirsiz Gegenbauer polinomlarının birer sonsuz serisi şeklinde tanımlanmıştır. Analitik yöntemler kullanılarak integral denklemler birer lineer cebirsel eşitlikler sistemine dönüştürülmüş ve elde edilen matris eşitlikler nümerik olarak çözülerek bilinmeyen katsayılar belirlenmiştir. RCS (radar kesit alanı) nümerik örnekler sunulmuş ve şeritin uzak alan saçılım karakteristikleri detaylı olarak tartışılmıştır. Elde edilen bazı sonuçlar mevcut bir method ile karşılaştırılmıştır.

Anahtar Kelimeler: Empedans, Şerit, Analitik, Nümerik, Saçılma

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CONTENTS	PAGE
ABSTRACT	I
ÖZ	II
ACKNOWLEDGMENTS	III
CONTENTS	IV
LIST OF FIGURES	VI
SYMBOLS AND ABBREVIATIONS	VIII
1. INTRODUCTION	1
1.1 Literature Review	2
1.2 Comparison of Numerical-Analytical Methods	8
1.3 Aims and the Scope of the Thesis	10
2. FORMULATION OF THE PROBLEM FOR H-POLARIZED CASE	13
2.1. Application of Boundary Conditions	17
3. REDUCTION OF LINEAR EQUATIONS TO SYSTEM OF LINEAR ALGEBRAIC EQUATIONS	29
3.1. General Expressions for Current Density Functions in Transform Domain	29
3.2. System of Linear Algebraic Equations for f_n^e	38
3.3. System of Linear Algebraic Equations for f_n^m	41
4. CALCULATION OF FIELD ANALYSIS	45
4.1. Scattered Far Field and Total Scattering Cross Section	45
5. NUMERICAL ANALYSIS	49
5.1. Evaluation of $D_{ln}^{E_1}$ and $D_{ln}^{E_2}$	49
5.1.1. Evaluation of $R_{ln1}^{E_2}(t, l)$	52
5.1.2. Evaluation of $R_{ln1}^{E_1}(t, l)$	56
5.1.3. Evaluation of $R_{ln2}^{E_2}(t, l)$	57
5.1.4. Evaluation of $R_{ln2}^{E_1}(t, l)$	75
5.2. Evaluation of $d_{ln}^{E_1}$ and $d_{ln}^{E_2}$	86
6. RESULTS	93

7. CONCLUSION	103
REFERENCES.....	105
CURRICULUM VITAE	111
APPENDIX.....	115
A1: Singularities of the integrand $R_{\ln 2}^{E_1}$	115
A2: Singularities of the integrand $R_{\ln 2}^{E_2}$	118



LIST OF FIGURES

PAGE

Figure 2.1. Geometry of the problem.....	14
Figure 5.1. The contour for the product of Bessel functions	58
Figure 5.2. Location of poles of the integrand of $R_{ln2}^{E_2}$	65
Figure 5.3. Location of poles of the integrand of $R_{ln2}^{E_1}$	81
Figure 6.1. Monostatic RCS [dB], $q_0 = 60^\circ$, $ka=5.0$. - - - - : $h_1 = h_2 = 0.0$; ——— : $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; - . . . - : $h_1 = h_2 = 3.0$ (Veliev et al. (1998)).....	94
Figure 6.2. Monostatic RCS [dB], $ka=5.0$ ($h = 1.5$ and $h = 3.0$)	94
Figure 6.3. Monostatic RCS [dB], $q_0 = 60^\circ$, $ka=15.0$. - - - - : $h_1 = h_2 = 0.0$; ——— : $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; - . . . - : $h_1 = h_2 = 3.0$ (Veliev et al. (1998)).....	95
Figure 6.4. Monostatic RCS [dB], $ka=15.0$ ($h = 1.5$ and $h = 3.0$)	95
Figure 6.5. Bistatic RCS [dB], $q_0 = 60^\circ$, $ka=5.0$. - - - - : $h_1 = h_2 = 0.0$; ——— : $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; - . . . - : $h_1 = h_2 = 3.0$ (Veliev et al. (1998)).....	96
Figure 6.6. Bistatic RCS [dB], $ka=5.0$ ($h = 1.5$ and $h = 3.0$)	96
Figure 6.7. Bistatic RCS [dB], $q_0 = 60^\circ$, $ka=15.0$. - - - - : $h_1 = h_2 = 0.0$; ——— : $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; - . . . - : $h_1 = h_2 = 3.0$ (Veliev et al. (1998)).....	97
Figure 6.8. Bistatic RCS [dB], $ka=15.0$ ($h = 1.5$ and $h = 3.0$)	97
Figure 6.9. Monostatic RCS [dB], $ka=5.0$ ($h = 1.5$ and $h = 1.5i$)	98
Figure 6.10. Monostatic RCS [dB], $ka=15.0$ ($h = 1.5$ and $h = 1.5i$)	98
Figure 6.11. Bistatic RCS [dB], $q_0 = 90^\circ$, $ka=5.0$ ($h = 1.5$ and $h = 3.0$)	99
Figure 6.12. Bistatic RCS [dB], $q_0 = 90^\circ$, $ka=15.0$ ($h = 1.5$ and $h = 3.0$)	99
Figure 6.13. Bistatic RCS [dB], $q_0 = 60^\circ$, $ka=15.0$ ($h = 1.5$ and $h = 1.5i$)	100

Figure 6.14. The total radiation pattern for $ka=5$	100
Figure 6.15. The total radiation pattern for $ka=15$	101



SYMBOLS AND ABBREVIATIONS

t	: The time constant
l	: The wavelength
k	: The wave number
s	: Conductivity
c	: Euler's constant
w	: The radian per second (angular frequency)
ϵ_0	: Permittivity of free space
μ_0	: Permeability of free space
Z	: The impedance
Y	: The admittance
E_z^i	: The incident electric field
E_z^s	: The scattered electric field
H_z^i	: The incident magnetic field
H_z^s	: The scattered magnetic field
I_e	: Electric current density
I_m	: Magnetic current density
$H_n^{(1)}(.)$	: Hankel function of the first kind
$J_n(.)$	: Bessel function of the first kind
$T_n(.)$	: Chebyshev polynomial of the first kind
$\Gamma(.)$	: Gamma function
$y(.)$	: Psi function
u	: Constant related to the edge condition
f_n	: Unknown coefficients of the current densities
f_n^e	: Unknown coefficients for electric current density
f_n^m	: Unknown coefficients for magnetic current density

- $F_e(\mathbf{b})$: The integral equation in spectral domain for electric current density
- $F_m(\mathbf{b})$: The integral equation in spectral domain for magnetic current density
- $f(\mathbf{j})$: The far field radiation pattern
- RCS : Radar cross section



1. INTRODUCTION

Scattering of electromagnetic waves from geometrical and physical discontinuities is one of the most essential fields in electromagnetic wave theory. However, the first considerable steps in this area of research are due to Lord Rayleigh and A.Sommerfeld. Almost a century ago, Rayleigh (1871) (from Born M. 1965) investigated the problem of scattering by a perfectly conducting sphere; and Sommerfeld (1896) overcame the problem of diffraction of plane electromagnetic waves by an absolutely conducting semi-infinite plane. Over the last few decades, the researches in the scattering of electromagnetic waves by several objects have been developed as a result of its direct applicability to civilian applications, including military ones, such as remote sensing, non-invasive diagnostics in medicine and non-destructive testing.

The received signal, which is spread by an object, can be used to resolve some of the geometrical and physical properties of the scatterer. Since the received scattered signal power is directly corresponds to the scattered field or radar cross section (RCS) of the object. Therefore, in military applications, in order to avoid the echo signal, the RCS of targets like air crafts must be cut down to a minimum level. Additionally, in order to decrease the inference caused by obstacle as buildings for instance, the RCS of such obstacles that are close to radars should also be reduced. Correspondingly, groups of engineers and scientists are also researching to extract as sufficient information as possible from low RCS values for both military and civilian applications.

Several methods exist, which are applicable to reduce the RCS of some obstacles. For instance, if the shape of the target is to be modified, scattered energy somehow may be directed toward some desired regions. But some other situations exist where shape modification is confined by the aerodynamic structure of that target, and another technique such as absorbing layer is applied for the same purpose. The absorber may consist of dielectric or magnetic materials, partially dielectric and/or magnetic materials or a number of layers of such materials. Even though such mixtures of shapes and materials propose more degrees of freedom in terms of design within

controlling the RCS of the target, similarly, the complexity of the solution procedure is also increased.

Solutions for recognized problems including half-plane, cylinder or sphere are apparently essential regarding the diffraction theory, and strip is considered to be one of the most important familiar structures due to its geometry, strips are usually accustomed to investigate the multiple diffraction phenomenon. Furthermore, a large amount of applicable issues, especially in remote sensing, modeling by conducting, impedance or resistive strips is possible. Additionally, by using the duality principle diffracting a slit in an absolute conducting plane can be reduced to a perfectly conducting strip problem.

Scattering of cracks or gaps that may occur on the obstacle's surface, which is completely or partially filled with some material, may provide a significant contribution to the overall scattering pattern. In such issues the gaps or cracks may be modeled by strips and/or slits. Accordingly, due to its adjustment to many practical problems, strips have been broadly researched by many authors by using distinctive analytical and numerical strategies.

1.1 Literature Review

A large number of analytical techniques have been developed considering diffraction by scatterers with several shapes, sizes and constituent materials. These methods can be categorized in two groups, namely, exact and approximate methods. Using an exact method is possible, when the obstacle's geometry corresponds to a coordinate system having the wave equation separable.

Furthermore, exact solutions usually include sophisticated integral expressions, which are required to apply near-field, far-field or high-frequency asymptotic for engineering purposes. Due to these constraints not many practical problems have exact solutions, and mostly severe asymptotic expressions are acquired. Approximate analytical techniques were obtained from the extension of classical optics by geometrical Theory of Diffraction (GTD) introduced by Keller (1953) (from

Keller. J.B., 1962) and Physical Theory of Diffraction (PTD) introduced by Ufimtsev (1957) (from Bhattacharyya. A.K., 1991).

In geometrical optics zero wavelength approximation is implemented and space is divided into distinct illuminated and shadow regions. It is considered that energy spreads in tube of rays, obviously the diffraction impacts are discarded in previous estimates; where these effects were first studied by GTD. At both incidence and reflection boundaries, the diffracted field becomes infinite and also at the edge which is caustic for the diffracted field. The term of a diffraction coefficient can be acquired simply by expressing the identical solution of a recognized problem in GTD form. In order to enhance GTD and overcome some of the methods' drawbacks two fundamental techniques have been developed namely, Uniform Asymptotic Theory of Diffraction (UAT) and Uniform Theory of Diffraction (UTD) introduced by Ahluwalia et.al (1968) (from Ahluwalia, D.S.,1970) and Kouyoumjian and Pathak (1974) respectively. In UAT it is considered that the field solution that occurs in an edge diffraction problem can be extended in a specific asymptotic series including a Fresnel integral, whereas in UTD Keller's diffraction, the coefficient is multiplied by a factor involving a Fresnel integral. The multiplication factor has a feature such that the field point approaches the shadow boundary and a finite field is obtained at the shadow boundaries while approaching zero.

In physical optics, surface currents are assumed to induce only at the illuminated part of the scatterer that functions as the supply of the scattered field. The accuracy of the technique can be raised by enhancing the assumed current distributions, however, it may not be able to account utterly for the presence of discontinuities on the obstacle. By PTD, Ufimtsev introduced the fringe current concept as a result of the physical or geometrical discontinuities. It was assumed that the entire current on a conducting surface includes a fringe current (nonuniform part) along with physical optics current. In distinction to GTD, PTD yields the finite fields everywhere including the shadow boundaries at the caustics.

Since its geometry is simple, the strip issue was researched by many scientists. Many techniques were proposed, but none yielded an efficient solution. So far, a severe solution for the problem of diffraction by a strip doesn't exist. However explaining the

attempts to solve this problem which had yielded significant solutions, is worth it. Some of the considerations made will be given in a sequential order as follows.

Morse and Rubenstein (1938) used a Mathieu function expansion for electromagnetic fields to obtain the exact series representation for the solution of the diffraction by strips. According to the rapid convergence of the Mathieu series, their solution yields improvement for low frequencies. However, when it comes to high frequency range their proposed method is not accurate since there is a necessity to include large number of terms while computing the infinite series.

Grinberg (1960) proposed shadow current concept for diffraction issues regarding electromagnetic waves by a conducting slit by implementing the integral equation method. However, the same method could be applied to solve the problem of diffraction by a conducting strip, which is achieved by using the duality principle. The problem with this method is reduced to the solution of a second kind Fredholm equation. According to this formulation, the asymptotic form of the solution for the high frequency case can be obtained easily. It should be taken into account that the application of integral equation method have been used by other researches before Grinberg, in previous studies, e.g. Copson (1946), Levine and Schwinger (1948) and Miles (1949). Integral equation based formulation was firstly implemented by Copson (1946) (from Senior, T.B.A., 1952) for a specific case of the Sommerfield half-plane which was an equation of the Wiener-Hopf type.

Up to 1950s, approximately all researches were focused on the diffraction by perfectly conducting structures. Senior (1952) proposed the first study for the issue of finite conductivity, by using a semi-infinite metallic sheet as the diffracting structure. The stipulation used in this study was the standard impedance condition. For E-polarization, Senior defined two induced currents namely electric and magnetic, and by using Green's theory Senior had acquired the total field at any point off the plate in terms of these induced currents and incident field. The edge conditions for current components were being given as $O(x^{-1/2})$ and $O(x^{1/2})$ for $x \rightarrow 0$, for the tangential and normal components, respectively, of the electric and magnetic currents. The asymptotic behavior introduced by Senior (1952) for the currents correspond to asymptotic behavior of the field components proposed by Meixner (1949) (from

Meixner. J., 1972). This is considered as one of the most essential information for our method. Accordingly, the information behind this behavior of currents at the edges give us the possibility to explain the currents in a series of some special functions with some unknowns.

It may be noted that the strip is used as a basic element in the formation of gratings. Diffraction by an impedance strip was researched by Faulkner (1965) (from Bowman, J.J., 1967) via a Wiener-Hopf primarily based method, where the width of the strip was considered large compared to both the wavelength and the magnitude of the surface impedance such that it was being restricted to limited values. Maliuzhinets(1960) (from Bowman. J.J.,1967) thought-out the wide strip drawback, furthermore, where the mutual interaction between the edges was neglected: then, the two edges were assumed as independent semi-infinite half planes, within the absence of the other being excited by an equivalent field alone.

Bowman (1967) has also researched the high frequency back-scattering from an absorbing wide strip with random face impedances by using an approximation based on the known half plane solutions obtained by Maliuzhinets (1958).

Multiple diffraction or the interaction among the diffracting edges of a strip is of great applicable importance additionally, diffraction problems which involve both slit and or strip geometry represent a three-part mixed boundary-value issue which strictly may be represented via triple integral equation approach that generally yields correspondingly to a modified Wiener-Hopf equation. Possibly, the solution for multiple diffractions are obtained by a repetitive procedure presented by Jones (1964) (from Serbest A.II., 1993). Kobayashi (1993) had researched numerous types of modified Wiener-Hopf geometries and conjointly conferred the solution for H and E polarization cases. Another approximate method is presented by

Tiberio and Kouyoumjian(1979) and is called extended spectral ray method which includes interpreting the incident field to the second edge as a sum of inhomogeneous plane waves that can be treated separately. By using this method, Herman and Volakis (1987) had researched several diffraction by a conductive, resistive and impedance strips where the asymptotic solutions presented for these cases also involved the impacts of surface waves. The spectral iteration technique introduced

by Büyükaksoy et. al. (1989) is considered an alternative spectral domain method furthermore, assuming that the edge is illuminated by the field diffracted from the other edge, it is possible to obtain the doubly diffracted field additionally, constructing another Wiener-Hopf problem for this configuration. A detailed discussion of the mentioned methods are given by Serbest and Büyükaksoy (1993).

The development in numerical techniques for the solution of the scattering problems has always been parallel to the evolution in computer technology. Although numerical methods may be considered as more obvious compared to analytical methods, since the matrix inversion procedure for the analysis, computer capacity limits the size of the issue that can be dealt with. Generally, for the barriers having a maximum dimension of a few wavelengths, numerical methods are able to maintain precise solutions. The Method of Moments (MOM) and Galerkin's Method (Harrington. K.F.1968) are the popularly used numerical techniques. In MOM both the basic functions and the weighting functions are distinct however, they are considered similar functions in Galerkin's Method.

In numerical methods, the scattering drawbacks are presented as integral equations of both unknown field quantity or unknown induced surface current density by applying Green's theorem:

$$\vec{E}(\mathbf{r}) = -\frac{1}{4\pi} \int_s \left\{ i\omega \mu [\vec{n} \times \vec{H}] G(\mathbf{r}, \mathbf{r}') + [\vec{n} \times \vec{E}] \times \nabla G(\mathbf{r}, \mathbf{r}') + (\vec{n} \cdot \vec{E}) \nabla G(\mathbf{r}, \mathbf{r}') \right\} ds \quad (1.1)$$

or in terms of surface current densities.

$$E(\mathbf{r}) = E^i - \frac{i}{4} \int_s \left\{ -iJ(\mathbf{r}') + M(\mathbf{r}') \frac{\partial}{\partial n} \right\} G(\mathbf{r}, \mathbf{r}') ds \quad (1.2)$$

The fundamental functions are broadly used for representing the currents on a given surface. However, in most studies the integral equation is achieved directly by applying the well-known methods such as Moment Method, Galerkin's Method or Finite Element Method.

Wandzura (1992) investigated the current fundamental functions for curved surfaces in general. Volakis (1996) researched the scattering by a narrow groove in an impedance plane, and solved the integral equations by presenting a single-basis study of the equivalent current on the narrow impedance insert. Based on these studies, the obtained integral equations are solved directly by applying either one of the above-mentioned numerical methods.

The GTD was used by Senior (1979) to obtain the expression of the scattered field by a resistive strip where the results meet the ones obtained by the numerical solution of the integral equation including strips a narrow as a sixth of a wavelength. Senior also applied MOM to inspect the contribution of front and rear- edges to the far fields. it has been represented that, for strips with a width greater than about a half wavelength, the contribution regarding the front edge is similar to a half plane one, having similar resistance and the contribution of the rear edge is proportional to the square of the current at that point on the half plane corresponding to the rear edge of the strip.

Generally, the integral equations in Eq. (1.1) and Eq. (1.2) are usually solved by numerical methods. However, they can also be reformed to a set of algebraic equations by applying some analytical techniques. Additionally, it is possible to solve the matrix equation obtained, by using standard matrix inversion algorithms. The time needed for the solution of this matrix equation is proportional to the size of the developed matrix. In the case of large bodies, the time required can be enormously large particularly for RCS estimation: Accordingly, the size of the matrix must be maintained as small as possible.

Emets and Rogowski (2015) solved a two-dimensional problem of electromagnetic wave diffraction by a plane strip with different boundary conditions on its surfaces. The diffracted field was expressed by an integral in terms of the induced electric and magnetic current densities. The related boundary-value problem in the domain of the short-wavelength was occurred as a matrix Wiener-Hopf equation which was solved through the Khrapkov method. This analytical solution is a powerful tool to study the effects of a single strip under the incidence of plane waves. It can be used to validate the numerical methods.

Apaydın and Sevgi (2015) studied on methods of moments (MoM) for modeling and simulation of scattered fields around infinite strip with one face soft and the other hard boundary conditions. Scattering cross-section was calculated numerically and compared with high-frequency asymptotics (HFA) models such as physical theory of diffraction (PTD) and theory of edge diffraction (TED). According to authors' knowledge, this method is the first application of MoM on the strip with one face soft and the other hard.

The H-polarized plane wave diffraction by a thin material strip has been solved using the Wiener-Hopf technique and approximate boundary conditions by Nagasaka and Kobayashi (2015). Employing a rigorous asymptotics, a high frequency solution for large strip width has been obtained. The numerical examples have been done on both the radar cross-section (RCS) and the far field scattering characteristics of the strip. Some of these examples have been compared with existing method and results agree reasonably well with existing method.

1.2 Comparison of Numerical-Analytical Methods

Commonly, the electrical size of the body restricts the tractability of numerical techniques, however, the geometrical complexity of the object restricts the applicability of the analytical methods. Furthermore, hybrid methods are used along with techniques based upon the extension of classical optics in order to obtain an asymptotic solution for problems regarding high region frequencies. Hybrid methods including both numerical and high frequency asymptotic techniques may have the ability to extend the class of electromagnetic scattering problems that can be handled.

The hybrid approach can be reformed as a field-based analysis where the GTD solution for the field associated with edge or surface diffraction are used as the initial point. These solutions function as the ansatz to the MoM formulation and represent the elements of a scatterer not meeting the requirements of a recognized geometry which is not cooperative with a GTD solution itself. (Burnside et.al. (1975) and Sahalov and Thiele (1981)). Alternatively, current-based formulation is possible in a situation such that the analysis proceeds from ansatz solutions for the currents obtained from physical

optics and PTD (Thiele and Newhouse (1975), Ekelman and Thiele (1980) (from Medgyesi- Mitschang, L.N., 1989)) and Medgyesi-Mitschang and Wang (1983) (from Medgyesi- Mitschang, L.N., 1989)). The option of a mixed field-current based approach also exists.

Generally, rigorous scattering theory, relies on the theory of boundary-value problems in which the progress is closely dependent, to mathematical branches. The main approach for the solution behind wave scattering is to reform the problem in terms of the integral equations satisfied by the unknown functions along with the current induced on the scatterer surface. Numerical techniques are utilized to shrink these integral equations to a system of linear algebraic equations. The presence of edges on the integration contour produce errors in current density calculations. In order to enhance the accuracy of the solution, some approaches accounting for the Meixner's edge condition in explicit form have been suggested by Shafai (1973) (from Veliev. I.E. 1993).

A similar technique was used by Butler (1985) in which the analytical solution of the integral equation was obtained for the current induced on a confined conducting strip. In this solution the current mass was expressed as a product of a Chebyshev polynomial and a weighting operator which fulfill the edge condition and after specifying the Fourier type coefficients, the expression of far fields were obtained. So an integral equation which contains the current density and excitation was obtained as follows,

$$E_z^i(x) = -j \frac{kh}{2p} \left\{ \int_{-\infty}^{\infty} J_z(x') \ln|x - x'| dx' + (c + \ln \frac{k}{2}) \int_{-\infty}^{\infty} J_z(x') dx' \right\} \quad |x| \leq w \quad (1.3)$$

Where J_z is the unknown current density. $h = \sqrt{m/e}$ and $c = g + j(p/2)$, ($c=0.5772...$ is the Euler's constant). The solution for the current density was expressed as,

$$J_z(x) = [1 - (\frac{x}{W})^2]^{-1/2} \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n T_n(\frac{x}{W}) \quad (1.4)$$

where $T(\cdot)$ denotes the Chebyshev polynomials of the first kind and f_n 's are the unknown coefficients. By using the orthogonality properties of the polynomials, the unknown coefficients and so the expression of the current density were obtained for both E and H-polarization cases.

A substitute technique was proposed by Veliev et al. (1989, 1993) where the solution encompass any preassigned accuracy. The distributed field was represented using the Fourier transform of the corresponding surface current density which offers a number of enhancements to overcome the problem. A hybrid technique based on the semi-inversion procedure for equation operators and the method of moments was used to yield the optimal solution. The essentials of the solution and its application to the wave scattering by polygonal cylinders and flat conducting strip structures are proposed by Veliev and Veremey (1993). The previous versions of this method were suggested by (Nomura and Katsura (1957), Hongo (1972) and Otsuki (1976) (from Veliev. E.I. 1993)) for completely conducting strip and slit configurations. This analytical-numerical method, which utilizes spectral approach, somehow solves the problem to a system of linear algebraic equations for the unknown Fourier coefficients of the current density function. Appropriate truncation of the infinite system of equations can obtain the results with any desired accuracy.

It should be realized that the applicability of the truncation method cannot always be modified, additionally, the matrix elements correlated with the system of linear algebraic equations usually collapse slowly with an increase of their index.

1.3 Aims and the Scope of the Thesis

Despite there are various powerful analytical techniques, the main influence of numerical techniques is that they may be applied to a scatterer of random shape and are generally only restricted by the size of the scatterer. But this limitation is a realistic problem. Apparently, a set of linear equations which denote the scattering issue can be generated but the obtained set may be too large to be solved. Fortunately, the

developments in computer technology make the solution of many electromagnetic problems possible for a required degree of accuracy. In contrast, the asymptotic techniques work best when compared to the wavelength the scatterer size is large. However, the difficulty of the problem increases when the complex shaped bodies are of interest.

The use of analytical methods along with numerical methods may bypass these restrictions. The numerical methods are limited to the bodies having a maximum dimension of less than a few wavelengths, while the analytical methods produce accurate outcomes for the scatterer much larger than those of one wavelength. Thereby, these methods may be combined to overcome the scattering problems involving scatterers of intermediate size and size in the resonance region. Additionally by using analytical-numerical methods the computation time required may be decreased to an acceptable level.

In this study, diffraction by an impedance strip is researched by using the analytical-numerical technique proposed by Veliev and Veremey (1989). In Chapter 2, the formulation of the problem for H-polarized wave is given. By expressing the electric and magnetic currents as infinite series in terms of Gegenbauer polynomials, two integral equations in spectral domain for electric and magnetic currents are derived. In Chapter 3, the integral equations are reduced to a system of linear algebraic equations for both currents with some unknown coefficients. In Chapter 4 some physical quantities are represented in terms of the unknown coefficients which will be denoted by solving the system of linear algebraic equations. In Chapter 5 the solution of branch-cut integrals are obtained and finally in Chapter 5, the curves for both currents, far field and RCS are represented. The results are reviewed and compared with some previously obtained results.

The aims of the current research are:

- (1) to employ a new analytical-numerical technique to solve a recognized diffraction problem that still does not have a proper solution,
- (2) to obtain an accurate solution for the impedance strip which can be used to simulate many practical obstacles,
- (3) to retrieve a solution which may work in a wide frequency range,

- (4) to set a base solution and write computer codes which may be used to analyze some complex structures that could be treated as a combination of multiple strips.



2. FORMULATION OF THE PROBLEM FOR H-POLARIZED CASE

Diffraction by strips is a significant and classical subject both in electromagnetic and acoustic wave theory. First of all, due to its geometry, strips are frequently used to investigate the multiple diffraction phenomenon. Furthermore, such as remote sensing and transmission lines, a large number of practical problems can be simulated by conducting, resistive or impedance strips. In specific, scattering from resistive, conductive and impedance strips have been considered by Herman et al. (1987), while, Senior (1979) has also made an attempt to solve a problem related to resistive strip configuration. Many analytical, numerical or approximate analytical methods have been used to study a single or multiple diffraction patterns from a strip. To name a few for example, geometrical theory of diffraction (from Keller J.B. 1962), Kobayashi's potential method (from Imran et al. 2007), (from Imran et al. 2009) spectral iteration technique (SIT) (from Serbest A.H. and Büyükaksoy A. 1993), method of successive approximations (from Chakrabarti A. 1977) and the Wiener-Hopf technique (Noble, B., 1958) have positively been utilized. Some recent advances in the literature are also found on Bessel's potential spaces (from Lawrie J.B. and Abrahams I.D. 2007) and Maliuzhinetz-Sommerfeld integral representation (from Castro L.P. and Kapanadze D. 2008).

Scattering from gaps or cracks that may exist on the surface of an obstacle, which is entirely or partially filled with some material can provide a significant contribution to the overall scattering pattern. In such problems the gaps or cracks may be simulated by strips/or slits. Therefore, due to its conformity to many practical problems, strips have been extensively investigated by many authors by using different analytical and numerical methods.

The scatterer of the diffraction problem that will be formulated in this section is a strip of width $2a$ where the same impedance is assumed to be imposed on both sides. The geometry of the problem is illustrated in Figure 2.1, and η denotes the normalized impedance of the strip.

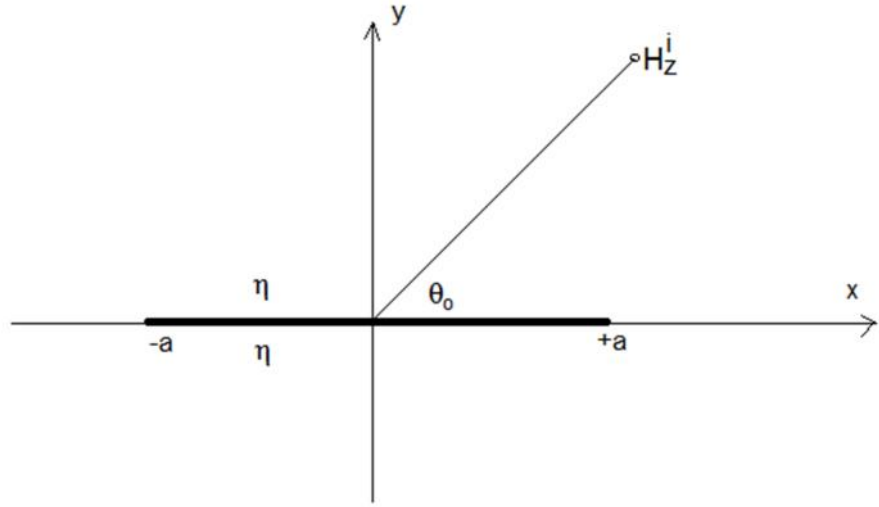


Figure 2.1. Geometry of the problem

Since the strip is uniform along the z -axis, the problem can be reduced to a two dimensional problem. The time dependence of the fields are assumed to be $\exp(-i\omega t)$ and suppressed throughout the analysis. The incident magnetic field is given as a linearly polarized plane wave

$$H_z^i(x, y) = e^{-ik(xa_0 + y\sqrt{1-a_0^2})} \quad (2.1)$$

where

$$a_0 = \cos\theta_0. \quad (2.2)$$

The total field is

$$H_z(x, y) = H_z^i(x, y) + H_z^s(x, y) \quad (2.3)$$

where H_z^s is the scattered field.

On the strip, the total field must satisfy the Leontovich boundary condition which is frequently called as impedance boundary condition, given by,

$$\left\{ \frac{\partial H_z(x, y)}{\partial y} \pm i k h H_z(x, y) \right\} \bigg|_{y=\pm 0} = 0. \quad (2.4)$$

So the total field can be expressed as (Senior, 1952)

$$H_z(x, y) = H_z^i(x, y) + \frac{1}{4} \int_{-a}^a \left\{ k Y I_m(x') - i I_e(x') \frac{\partial}{\partial y} \right\} H_0^{(1)} \left(k \sqrt{(x-x')^2 + y^2} \right) dx \quad (2.5)$$

where I_m and I_e are the equivalent magnetic and electric current densities respectively. Expression of magnetic and electric current densities can be defined as follows respectively,

$$\overline{I}_e = \hat{a}_y \times [\overline{H}]_{-}^{+} \quad (\overline{H} = H \hat{a}_z) \quad (2.6)$$

$$\overline{I}_e = \hat{a}_y \times [H_z(x, +0) - H_z(x, -0)] \Rightarrow \quad (2.7)$$

$$[H_z(x, +0) - H_z(x, -0)] \hat{a}_x = f_2(x) \hat{a}_x$$

and

$$\overline{I}_m = \hat{a}_y \times [\overline{E}]_{-}^{+} \quad (2.8)$$

$$i \omega \epsilon_0 \overline{E} = \nabla \times \overline{H} \quad \overline{E} = \frac{1}{i \omega \epsilon_0} \nabla \times \overline{H} \quad (2.9)$$

$$\overline{E} = \frac{1}{i\omega e_0} \begin{pmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z \end{pmatrix} \quad (2.10)$$

$$\overline{E} = \frac{1}{i\omega e_0} \left\{ \frac{\partial H_z}{\partial y} \hat{a}_x - \frac{\partial H_z}{\partial x} \hat{a}_y \right\} \quad (2.11)$$

$$\overline{I}_m = \frac{1}{i\omega e_0} \left\{ \frac{\partial H_z}{\partial y} \hat{a}_x - \frac{\partial H_z}{\partial x} \hat{a}_y \right\} \times \hat{a}_y \quad (2.12)$$

$$\overline{I}_m = \frac{1}{i\omega e_0} \left\{ \frac{\partial H(x, +0)}{\partial y} - \frac{\partial H(x, -0)}{\partial y} \right\} \hat{a}_z \quad (2.13)$$

$$kY = \omega \sqrt{e_0 m_0} \cdot \frac{1}{Z} = \frac{\omega \sqrt{e_0 m_0}}{\sqrt{\frac{m_0}{e_0}}} = \omega e_0 \quad (2.14)$$

$$\begin{aligned} kY \overline{I}_m &= \cancel{\omega e_0} \left(\frac{1}{\cancel{i \omega e_0}} \left\{ \frac{\partial H(x, +0)}{\partial y} - \frac{\partial H(x, -0)}{\partial y} \right\} \right) \hat{a}_z \\ &= -i \left\{ \frac{\partial H(x, +0)}{\partial y} - \frac{\partial H(x, -0)}{\partial y} \right\} = -if_1(x) \hat{a}_z \end{aligned} \quad (2.15)$$

So Eq. (2.5) can be rearranged as

$$H_z(x, y) = H_z^i(x, y) - \frac{i}{4} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} \quad (2.16)$$

$$H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) dx'$$

2.1. Application of Boundary Conditions

If we rewrite the Leontovich boundary condition for $y = +0$ and $y = -0$, and if we subtract and add these two equations we can obtain the following expressions:

$$\frac{\partial H_z(x, +0)}{\partial y} + ikh H_z(x, +0) = 0 \quad (2.17)$$

and

$$\frac{\partial H_z(x, -0)}{\partial y} - ikh H_z(x, -0) = 0 \quad (2.18)$$

The difference of Eq. (2.17) and Eq. (2.18) is

$$\frac{\partial H_z(x, +0)}{\partial y} - \frac{\partial H_z(x, -0)}{\partial y} + ikh \{H_z(x, +0) + H_z(x, -0)\} = 0 \quad (2.19)$$

And the sum is

$$\frac{\partial H_z(x, +0)}{\partial y} + \frac{\partial H_z(x, -0)}{\partial y} + ikh \{H_z(x, +0) - H_z(x, -0)\} = 0. \quad (2.20)$$

Using the expressions electric and magnetic current densities f_2 and f_1 in Eqs. (2.19) and (2.20)

$$f_1(x) + ikh \{H_z(x, +0) + H_z(x, -0)\} = 0 \quad (2.21)$$

$$f_2(x) + \frac{1}{ikh} \left\{ \frac{\partial H_z(x, +0)}{\partial y} + \frac{\partial H_z(x, -0)}{\partial y} \right\} = 0 \quad (2.22)$$

are derived.

The expressions of total field at $y = \pm 0$ can be obtained from Eqs. (2.3) and (2.16) as

$$H_z(x, +0) = H_z^i(x, +0) - \lim_{y \rightarrow +0} \frac{i}{4} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} H_0^1(k|x-x'|) dx' \quad (2.23)$$

and

$$H_z(x, -0) = H_z^i(x, -0) - \lim_{y \rightarrow -0} \frac{i}{4} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} H_0^1(k|x-x'|) dx'. \quad (2.24)$$

In Eq.(2.21) we require the sum of these two expressions as

$$\begin{aligned} H_z(x, +0) + H_z(x, -0) &= H_z^i(x, +0) + H_z^i(x, -0) \\ &- \lim_{y \rightarrow +0} \frac{i}{4} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} H_0^1(k|x-x'|) dx' \\ &- \lim_{y \rightarrow -0} \frac{i}{4} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} H_0^1(k|x-x'|) dx'. \end{aligned} \quad (2.25)$$

For the incident field we have that

$$H_z^i(x, 0) = H_z^i(x, +0) = H_z^i(x, -0) = e^{-ikx a_0} \quad (2.26)$$

and from (Senior, 1952) we have that

$$\left(\lim_{y \rightarrow +0} + \lim_{y \rightarrow -0} \right) \int_{-a}^{+a} \left\{ f_2(x') \frac{\partial}{\partial y} \right\} H_0^{(1)}(k|x-x'|) dx' = 0 \quad (2.27)$$

and

$$\begin{aligned} \lim_{y \rightarrow +0} \int_{-a}^{+a} f_1(x') H_0^{(1)}(k|x-x'|) dx' &= \lim_{y \rightarrow -0} \int_{-a}^{+a} f_1(x') H_0^{(1)}(k|x-x'|) dx' \\ &= \int_{-a}^{+a} f_1(x') H_0^{(1)}(k|x-x'|) dx'. \end{aligned} \quad (2.28)$$

So by substituting Eq. (2.26) on Eq. (2.27) and Eq.(2.28) into Eq. (2.25) one can get that;

$$-\frac{1}{ikh} f_1(x) = 2 H_z^i(x, 0) - \frac{i}{2} \int_{-a}^{+a} f_1(x') H_0^{(1)}(k|x-x'|) dx' \quad (2.29)$$

or

$$-\frac{1}{kh} f_1(x) = 2ie^{-ikx a_0} + \frac{1}{2} \int_{-a}^{+a} f_1(x') H_0^{(1)}(k|x-x'|) dx' \quad (2.30)$$

Eq. (2.30) is the integral equation for the magnetic current $f_1(x)$.

At this stage we will derive the integral equation for magnetic current in spectral domain using the integral representation of Hankel function given as;

$$H_0^{(1)}(k|x-x'|) = \frac{1}{p} \int_{-\infty}^{+\infty} \frac{e^{ik(x-x')t}}{\sqrt{1-t^2}} dt. \quad (2.31)$$

Using Eq. (2.31) in Eq. (2.30)

$$-\frac{1}{kh} f_1(x) = 2ie^{-ikxa_0} + \frac{1}{2p} \int_{-a}^{+a} f_1(x') \int_{-\infty}^{+\infty} \frac{e^{ik(x-x')t}}{\sqrt{1-t^2}} dt dx' \quad (2.32)$$

and changing the order of integration

$$-\frac{1}{kh} f_1(x) = 2ie^{-ikxa_0} + \frac{1}{2p} \int_{-\infty}^{+\infty} \left\{ \int_{-a}^{+a} f_1(x') e^{-ikx't} dx' \right\} \frac{e^{ikxt}}{\sqrt{1-t^2}} dt \quad (2.33)$$

is obtained. Where

$$F_1(t) = \int_{-a}^{+a} f_1(x') e^{-ikx't} dx' \quad (2.34)$$

is the Fourier transform of the magnetic current density $f_1(x)$. So

$$-\frac{1}{kh} f_1(x) = 2ie^{-ikxa_0} + \frac{1}{2p} \int_{-\infty}^{+\infty} F_m(t) \frac{e^{ikxt}}{\sqrt{1-t^2}} dt \quad (2.35)$$

is derived.

The following variable changes are required to be able to express the current density functions in terms of Gegenbauer polynomials.

Since $-a \leq x \leq a$, If we use the variable change

$$x = az, \quad x' = az', \quad x = ka$$

$$F_1(t) = \int_{-1}^{+1} f_1(az') e^{-i(x/a)t(az')} (adz') \quad (2.36)$$

$$F_1(t) = \int_{-1}^{+1} af_1(az') e^{-ixtz'} dz' \quad (2.36)$$

and let

$$\tilde{f}_1(z') = af_1(az'). \quad (2.37)$$

So,

$$F_1(t) = \int_{-1}^{+1} \tilde{f}_1(z') e^{-ixtz'} dz' \quad (2.38)$$

is obtained. Means that Eq. (2.35) can be arranged as;

$$-\frac{1}{\left(\frac{x}{a}\right)h} f_1(x) = 2ie^{-i\left(\frac{x}{a}\right)a_0(az)} + \frac{1}{2p} \int_{-\infty}^{+\infty} F_m(t) \frac{e^{i\left(\frac{x}{a}\right)(az)t}}{\sqrt{1-t^2}} dt \quad (2.39)$$

or

$$-\frac{1}{xh} af_1(x) = 2ie^{-ixa_0z} + \frac{1}{2p} \int_{-\infty}^{+\infty} F_m(t) \frac{e^{ixtz}}{\sqrt{1-t^2}} dt \quad (2.40)$$

and

$$-\frac{1}{xh} \tilde{f}_1(z) = 2ie^{-ixa_0z} + \frac{1}{2p} \int_{-\infty}^{+\infty} F_m(t) \frac{e^{ixtz}}{\sqrt{1-t^2}} dt. \quad (2.41)$$

If we multiply both side of the equation by e^{-ixbz} and integrate between -1 and +1

$$-\frac{1}{xh} \int_{-1}^1 \tilde{f}_1(z) e^{-ixbz} dz = 2i \int_{-1}^1 e^{-ixa_0 z} e^{-ixbz} dz + \frac{1}{2p} \int_{-1}^1 e^{-ixbz} \int_{-\infty}^{+\infty} F_m(t) e^{ixtz} \frac{dt}{\sqrt{1-t^2}} dz \quad (2.42)$$

is derived. From Eq. (2.34) it is obvious that the form on the left hand side can be expressed as

$$F_m(b) = \int_{-1}^{+1} \tilde{f}_1(z) e^{-ixbz} dz. \quad (2.43)$$

The first term on the right hand side is

$$\int_{-1}^{+1} e^{-ixa_0 z} e^{-ixbz} dz = 2 \frac{\sin x(b+a_0)}{x(b+a_0)} \quad (2.44)$$

and the second term on the right hand side by changing the order of integration;

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{F_m(t)}{\sqrt{1-t^2}} \int_{-1}^1 e^{ix(t-b)z} dz dt &= \int_{-\infty}^{+\infty} \frac{F_m(t)}{\sqrt{1-t^2}} \left(\frac{e^{ix(t-b)z}}{i(t-b)x} \right) \bigg|_{-1}^1 dt \\ &= \frac{2}{x} \int_{-\infty}^{+\infty} \frac{F_m(t)}{\sqrt{1-t^2}} \frac{\sin x(t-b)}{(t-b)} dt \end{aligned} \quad (2.45)$$

is obtained. So Eq. (2.42) can be rearranged as

$$\begin{aligned}
-\frac{1}{xh} F_m(b) &= 4i \frac{\sin x(b + a_0)}{x(b + a_0)} \\
&+ \frac{1}{xp} \int_{-\infty}^{+\infty} \frac{F_m(t)}{\sqrt{1-t^2}} \frac{\sin x(t-b)}{t-b} dt.
\end{aligned}
\tag{2.46}$$

This equation is the integral equation for the magnetic current density in spectral domain.

Now we must obtain the integral equation in spectral domain for electric current density f_2 . Since Eq. (2.22) is given as

$$f_2(x) + \frac{1}{ikh} \left\{ \frac{\partial H_z(x, +0)}{\partial y} + \frac{\partial H_z(x, -0)}{\partial y} \right\} = 0.$$

Using Eq.(2.16)

$$\begin{aligned}
\frac{1}{ikh} \left\{ \frac{\partial H_z(x, +0)}{\partial y} + \frac{\partial H_z(x, -0)}{\partial y} \right\} &= \frac{2}{ikh} \frac{\partial H_z^i(x, 0)}{\partial y} \\
&- \frac{1}{4kh} \lim_{y \rightarrow +0} \frac{\partial}{\partial y} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} H_0^1(k\sqrt{(x-x')^2 + y^2}) dx' \\
&- \frac{1}{4kh} \lim_{y \rightarrow -0} \frac{\partial}{\partial y} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} H_0^1(k\sqrt{(x-x')^2 + y^2}) dx'
\end{aligned}
\tag{2.47}$$

is obtained. So,

$$\begin{aligned}
f_2(x) = & -\frac{2}{ikh} \frac{\partial H_z^i(x,0)}{\partial y} \\
& + \frac{1}{4kh} (\lim_{y \rightarrow +0} + \lim_{y \rightarrow -0}) \frac{\partial}{\partial y} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} H_0^1(k\sqrt{(x-x')^2 + y^2}) dx'.
\end{aligned} \tag{2.48}$$

Since

$$\frac{\partial H_z^i(x,0)}{\partial y} = \frac{\partial}{\partial y} \left\{ e^{-ik(xa_0 + y\sqrt{1-a_0^2})} \right\} = -ik\sqrt{1-a_0^2} e^{-ikxa_0} \tag{2.49}$$

and from Senior (Senior, 1952),

$$\lim_{y \rightarrow \pm 0} \frac{\partial}{\partial y} \int_{-a}^{+a} f_1(x') H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) dx' = \pm 2if_1(x). \tag{2.50}$$

Then using Eq.(2.49) and Eq.(2.50) in Eq. (2.48)

$$\begin{aligned}
f_2(x) = & \frac{2}{h} \sqrt{1-a_0^2} e^{-ikxa_0} \\
& + \frac{1}{4kh} (\lim_{y \rightarrow +0} + \lim_{y \rightarrow -0}) \frac{\partial}{\partial y} \int_{-a}^{+a} \left\{ f_2(x') \frac{\partial}{\partial y} \right\} H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) dx'
\end{aligned} \tag{2.51}$$

is derived; again using the integral expression of Hankel function Eq. (2.31)

$$\frac{\partial}{\partial y} H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) = \frac{\partial}{\partial y} \frac{1}{p} \int_{-\infty}^{+\infty} e^{ik\{(x-x')t + \sqrt{1-t^2}|y|\}} \frac{dt}{\sqrt{1-t^2}} \tag{2.52}$$

and

$$\frac{\partial}{\partial y} H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) = \pm \frac{ik}{p} \int_{-\infty}^{+\infty} e^{ik\{(x-x')t + \sqrt{1-t^2}|y|\}} dt \quad (2.53)$$

where $\pm$ corresponds to $y > 0$ and $y < 0$. Then Eq. (2.51) can be rearranged as;

$$\begin{aligned} f_2(x) &= \frac{2}{h} \sqrt{1-a_0^2} e^{-ikx a_0} \\ &+ \left(\pm \frac{ik}{4kph} \right) \left(\lim_{y \rightarrow +0} + \lim_{y \rightarrow -0} \right) \frac{\partial}{\partial y} \int_{-a}^{+a} f_2(x') \int_{-\infty}^{+\infty} e^{ik\{(x-x')t + \sqrt{1-t^2}|y|\}} dt dx' \end{aligned} \quad (2.54)$$

or,

$$\begin{aligned} f_2(x) &= \frac{2}{h} \sqrt{1-a_0^2} e^{-ikx a_0} \\ &+ \left(\pm \frac{i}{4ph} \right) \left(\lim_{y \rightarrow +0} + \lim_{y \rightarrow -0} \right) \frac{\partial}{\partial y} \int_{-\infty}^{+\infty} e^{ik\{xt + \sqrt{1-t^2}|y|\}} \left\{ \int_{-a}^{+a} f_2(x') e^{-ikx't} dx' \right\} dt. \end{aligned} \quad (2.55)$$

Let us use the variable change in Eq. (2.36) again;

$$x = az, \quad x' = az', \quad x = ka$$

$$\int_{-a}^{+a} f_2(x') e^{-ikx't} dx' = \int_{-1}^1 f_2(az') e^{-ikaz't} az' dz' \quad (2.56)$$

$$= a \int_{-1}^1 f_2(az') e^{-ikaz't} dz' = aF_e(t)$$

where

$$F_e(t) = \int_{-1}^{+1} \tilde{f}_2(z) e^{-ikaz't} dz \quad (2.57)$$

and

$$\tilde{f}_2(z) = f_2(az). \quad (2.58)$$

Then Eq. (2.55) can be expressed as,

$$\begin{aligned} f_2(x) = & \frac{2}{h} \sqrt{1-a_0^2} e^{-ikx a_0} \\ & + \left(\pm \frac{i}{4ph} \right) \left(\frac{ka}{k} \right) \left(\lim_{y \rightarrow +0} + \lim_{y \rightarrow -0} \right) \int_{-\infty}^{+\infty} \frac{\partial}{\partial y} \left\{ e^{ik\{xt + \sqrt{1-t^2}|y|\}} \right\} F_e(t) dt \end{aligned} \quad (2.59)$$

or

$$\begin{aligned} f_2(x) = & \frac{2}{h} \sqrt{1-a_0^2} e^{-ikx a_0} \\ & - \left(\pm \frac{i}{4ph} \right) \left(\frac{x}{k} \right) (\pm ik) \left(\lim_{y \rightarrow +0} + \lim_{y \rightarrow -0} \right) \int_{-\infty}^{\infty} e^{ik\{xt + \sqrt{1-t^2}|y|\}} F_e(t) \sqrt{1-t^2} dt \end{aligned} \quad (2.60)$$

and finally,

$$f_2(x) = + \frac{2}{h} \sqrt{1-a_0^2} e^{-ikx a_0} - \frac{x}{2ph} \int_{-\infty}^{\infty} e^{ikxt} F_e(t) \sqrt{1-t^2} dt \quad (2.61)$$

Eq. (2.61) can be represented in terms of $\tilde{f}_2(z)$ as follows

$$\tilde{f}_2(z) = + \frac{2}{h} \sqrt{1-a_0^2} e^{-ixza_0} - \frac{x}{2ph} \int_{-\infty}^{\infty} e^{ixzt} F_e(t) \sqrt{1-t^2} dt \quad (2.62)$$

Multiply both side of Eq.(2.62) by e^{-ixzb} and integrate from -1 to +1 with respect to z

$$\int_{-1}^1 \tilde{f}_2(z) e^{-ixzb} dz = \frac{2}{h} \sqrt{1-a_0^2} \int_{-1}^1 e^{-ix(b+a_0)z} dz$$

$$- \frac{x}{2ph} \int_{-1}^1 e^{-ixzb} \int_{-\infty}^{\infty} F_e(t) \sqrt{1-t^2} e^{ixtz} dt dz. \quad (2.63)$$

From Eq. (2.61)

$$F_e(b) = \int_{-1}^1 \tilde{f}_2(z) e^{-ixbz} dz \quad (2.64)$$

and

$$\int_{-1}^1 e^{-ix(b+a_0)z} dz = 2 \frac{\sin x(b+a_0)}{x(b+a_0)} \quad (2.65)$$

and

$$\int_{-1}^1 e^{-ixzb} \int_{-\infty}^{\infty} F_e(t) \sqrt{1-t^2} e^{ixtz} dt dz = 2 \int_{-\infty}^{\infty} F_e(t) \sqrt{1-t^2} \frac{\sin x(t-b)}{x(t-b)} dt. \quad (2.66)$$

Then Eq. (2.63) can be expressed as

$$F_e(b) = \frac{4}{h} \sqrt{1-a_0^2} \frac{\sin x(b+a_0)}{x(b+a_0)} - \frac{1}{ph} \int_{-\infty}^{\infty} F_e(t) \sqrt{1-t^2} \frac{\sin x(t-b)}{t-b} dt \quad (2.67)$$

This is the integral equation in spectral domain for electric current density $f_2(x)$.



3. REDUCTION OF LINEAR EQUATIONS TO SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

In this section, the solution of the integral for electric $F_e(\mathbf{b})$ and magnetic $F_m(\mathbf{b})$ current densities will be reduced to the solution of two uncoupled system of linear algebraic equations. The first step of the reduction process is to express the current densities in Fourier transform domain and obtain a general spectral expression for currents. Then by using the constraints implied by edge conditions, electric and magnetic current density expressions will be obtained in transform domain. Finally, they will be written in the form of infinite system of linear algebraic equations involving Gegenbauer polynomial coefficients as unknowns.

3.1. General Expressions for Current Density Functions in Transform Domain

Since it is necessary to express the current functions in spectral domain, the Fourier transform $F(\mathbf{b})$ of the current density function $\tilde{f}(\mathbf{z})$ must be found as

$$F(\mathbf{b}) = \int_{-1}^{+1} \tilde{f}(\mathbf{z}) e^{-i\mathbf{x}\mathbf{b}\mathbf{z}} d\mathbf{z}. \quad (3.1)$$

The current density function $\tilde{f}(\mathbf{z})$ is defined for $|\mathbf{z}| \leq 1$ and it is zero elsewhere. Let $\tilde{f}(\mathbf{z})$ be represented by a uniformly convergent series, such as,

$$\tilde{f}(\mathbf{z}) = (1 - \mathbf{z}^2)^u \sum_{n=0}^{\infty} f_n C_n^{u+\frac{1}{2}}(\mathbf{z}) \quad (3.2)$$

where $C_n^{u+\frac{1}{2}}(\mathbf{z})$ denote the Gegenbauer polynomials and u is a constant related to the edge condition. The value of u in Eq. (3.2) will be determined by enforcing the

functions such as to satisfy the edge conditions for electric and magnetic current densities separately.

For the electric current density function $\tilde{f}_e(z)$, and the magnetic current density function $\tilde{f}_m(z)$, from Meixner's (1972) edge conditions, v can be determined for $z \rightarrow 0$ as,

$$\tilde{f}_e(z) = O(z^{-1/2}) \text{ and } \tilde{f}_m(z) = O(z^{1/2}). \quad (3.3)$$

The order relations given above for electric and magnetic current densities can be obtained respectively as $u = -1/2$ and $u = 1/2$ by considering the asymptotic behavior of Gegenbauer polynomials together with Eq. (3.2). The integration in Eq. (3.1) is divided into two parts, as follows

$$F(b) = \int_{-1}^0 \tilde{f}(z) e^{-ixbz} dz + \int_0^{+1} \tilde{f}(z) e^{-ixbz} dz \quad (3.4)$$

By replacing z with $-z$ and by changing the order of upper and lower limits in the first term on the right-hand-side, Eq. (3.4) will be written as:

$$F(b) = \int_0^1 \tilde{f}(-z) e^{-ixbz} dz + \int_0^1 \tilde{f}(z) e^{-ixbz} dz. \quad (3.5)$$

Now to determine $\tilde{f}(-z)$ the following formula (Prodnikov, A. P., 1983, p. 732) will be used

$$C_n^l(-z) = (-1)^n C_n^l(z) \quad (3.6)$$

and, $\tilde{f}(-z)$ can be expressed as:

$$\tilde{f}(-z) = (1-z^2)^u \sum_{n=0}^{\infty} (-1)^n f_n C_n^{u+\frac{1}{2}}(z). \quad (3.7)$$

By substituting Eqs. (3.7) and (3.2) into Eq. (3.5),

$$F(b) = \int_0^{+1} (1-z^2)^u \sum_{n=0}^{\infty} f_n C_n^{u+\frac{1}{2}}(z) \{e^{-ixbz} + (-1)^n e^{ixbz}\} dz \quad (3.8)$$

is obtained. If n is even, i.e. $n=2p$, then Eq. (3.8) takes the form of,

$$F_{ev}(b) = 2 \int_0^{+1} (1-z^2)^u \sum_{p=0}^{\infty} f_{2p} C_{2p}^{u+\frac{1}{2}}(z) \left\{ \frac{e^{-ixbz} + e^{ixbz}}{2} \right\} dz \quad (3.9)$$

or

$$F_{ev}(b) = 2 \int_0^{+1} (1-z^2)^u \sum_{p=0}^{\infty} f_{2p} C_{2p}^{u+\frac{1}{2}}(z) \cos(xbz) dz. \quad (3.10)$$

By changing the order of integration and summation

$$F_{ev}(b) = 2 \sum_{p=0}^{\infty} f_{2p} K_{2p} \quad (3.11)$$

is written with

$$K_{2p} = \int_0^{+1} (1-z^2)^u C_{2p}^{u+\frac{1}{2}}(z) \cos(xbz) dz. \quad (3.12)$$

On the other hand, if n is odd, i.e. $n=2p+1$ then Eq. (3.8) takes the form of,

$$F_{od}(b) = \int_0^{+1} (1-z^2)^u \sum_{p=0}^{\infty} f_{2p+1} C_{2p+1}^{u+\frac{1}{2}}(z) 2i \left\{ \frac{e^{-ixbz} - e^{ixbz}}{2i} \right\} dz \quad (3.13)$$

or

$$F_{od}(b) = -2i \int_0^{+1} (1-z^2)^u \sum_{p=0}^{\infty} f_{2p+1} C_{2p+1}^{u+\frac{1}{2}}(z) \sin(xbz) dz. \quad (3.14)$$

Again by changing the order of integration and summation, it yields

$$F_{od}(b) = -2i \sum_{p=0}^{\infty} f_{2p+1} K_{2p+1} \quad (3.15)$$

with

$$K_{2p+1} = \int_0^{+1} (1-z^2)^u C_{2p+1}^{u+\frac{1}{2}}(z) \sin(xbz) dz. \quad (3.16)$$

It can easily be seen from Eq. (3.8) that,

$$F(b) = F_{ev}(b) + F_{od}(b) \quad (3.17)$$

Now, in order to express K_{2p} and K_{2p+1} in a form convenient for numerical calculations, the following expressions (Prudnikov, A.P., 1983, p. 534) are being used:

$$\int_0^1 (1-x^2)^{z-\frac{1}{2}} C_{2n+1}^z(x) \sin(ax) dx = (-1)^n p \frac{\Gamma(2n+2z+1) J_{2n+z+1}(a)}{(2n+1)! \Gamma(z) (2a)^z} \quad (3.18)$$

$$\int_0^1 (1-x^2)^{z-\frac{1}{2}} C_{2n}^z(x) \cos(ax) dx = (-1)^n p \frac{\Gamma(2n+2z) J_{2n+z}(a)}{(2n)! \Gamma(z) (2a)^z}. \quad (3.19)$$

As usual, $\Gamma(\cdot)$ denotes the Gamma functions and $J_n(\cdot)$ denotes the well-known Bessel function of the first kind. In Eqs. (3.18) and (3.19), by setting $x = z$, $z = u + \frac{1}{2}$ and $a = xb$, the followings can be written

$$K_{2p} = \int_0^1 (1-z^2)^u C_{2p}^{u+\frac{1}{2}}(z) \cos(xbz) dz = (-1)^p \frac{\Gamma(2p+2u+1) J_{2p+u+\frac{1}{2}}(xb)}{(2p)! \Gamma(u+\frac{1}{2}) (2xb)^{u+\frac{1}{2}}} \quad (3.20)$$

and

$$K_{2p+1} = \int_0^1 (1-z^2)^u C_{2p+1}^{u+\frac{1}{2}}(z) \sin(xbz) dz = (-1)^p p \frac{\Gamma(2p+2u+2) J_{2p+u+\frac{3}{2}}(xb)}{(2p+1)! \Gamma(u+\frac{1}{2}) (2xb)^{u+\frac{1}{2}}} \quad (3.21)$$

From Eq. (3.11), Eq. (3.15) and Eq. (3.17) $F(b)$ is

$$F(b) = 2 \sum_{p=0}^{\infty} f_{2p} K_{2p} - 2i \sum_{p=0}^{\infty} f_{2p+1} K_{2p+1} \quad (3.22)$$

and from Eqs. (3.20) and (3.21), it can be written as

$$F(b) = 2 \sum_{p=0}^{\infty} f_{2p} (-1)^p p \frac{\Gamma(2p+2u+1)}{\Gamma(2p+1)} \frac{J_{u+\frac{1}{2}+2p}(xb)}{\Gamma(u+\frac{1}{2}) (2xb)^{u+\frac{1}{2}}} - 2i \sum_{p=0}^{\infty} f_{2p+1} (-1)^p p \frac{\Gamma(2p+2u+2)}{\Gamma(2p+2)} \frac{J_{u+\frac{3}{2}+2p}(xb)}{\Gamma(u+\frac{1}{2}) (2xb)^{u+\frac{1}{2}}} \quad (3.23)$$

or

$$F(b) = \frac{2p}{\Gamma(u + \frac{1}{2})} \sum_{p=0}^{\infty} (-1)^p \left\{ f_{2p} \frac{\Gamma(2p + 2u + 1)}{\Gamma(2p + 1)} \frac{J_{u + \frac{1}{2} + 2p}(xb)}{(2xb)^{u + \frac{1}{2}}} \right. \\ \left. - i f_{2p+1} \frac{\Gamma(2p + 2u + 2)}{\Gamma(2p + 2)} \frac{J_{u + \frac{3}{2} + 2p}(xb)}{(2xb)^{u + \frac{1}{2}}} \right\}. \quad (3.24)$$

By rearranging the last equation, the Fourier transform of the current density function can be obtained as follows:

$$F(b) = \frac{2p}{\Gamma(u + \frac{1}{2})} \sum_{n=0}^{\infty} (-i)^n f_n \frac{\Gamma(n + 2u + 1)}{\Gamma(n + 1)} \frac{J_{n + u + \frac{1}{2}}(xb)}{(2xb)^{u + \frac{1}{2}}}. \quad (3.25)$$

This completes the calculation for the Fourier transform of a current element represented in terms of Gegenbauer polynomials given by Eq. (3.2). Since no restriction is imposed on the current series expression during the derivation, it is obvious that this representation is valid for both electric and magnetic currents.

Now considering the edge conditions separately for electric and magnetic current components, the corresponding spectral expressions in the Fourier domain can easily be obtained. First, the magnetic current density will be obtained simply by substituting $u = 1/2$ which yields

$$F_m(b) = p \sum_{n=0}^{\infty} (-i)^n (n+1) f_n^m \frac{J_{n+1}}{xb}. \quad (3.26)$$

Similarly, by inserting $u = -1/2$ in Eq. (3.25) it is possible to obtain the Fourier transform of the electric current density. But, it is obviously seen that due to

the presence of $[\Gamma(u + 1/2)]^{-1}$ term in the expression, the current density will have a singularity in this case. Therefore, in order to remove this singularity the series expansion given by Eq. (3.2) will be used for $u = -1/2$:

$$\tilde{f}_e(z) = (1 - z^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} f_n^e C_n^0(z). \quad (3.27)$$

In (Prudnikov, A.P., 1983, p. 734) it is given that,

$$C_n^0(z) = \frac{2}{n} T_n(z) \quad (3.28)$$

where $T_n(\cdot)$ denotes the well-known Chebyshev polynomials which is valid for $n \neq 0$; therefore, Eq. (3.27) can be written as,

$$\tilde{f}_e(z) = (1 - z^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} f_n^e \frac{2}{n} T_n(z). \quad (3.29)$$

From Eq. (3.1), the Fourier transform of $\tilde{f}(z)$ for $u = -\frac{1}{2}$ is,

$$F_e(b) = \int_{-1}^{+1} (1 - z^2)^{-\frac{1}{2}} \sum_{n=1}^{\infty} f_n^e \frac{2}{n} T_n(z) e^{-ixbz} dz \quad (3.30)$$

or

$$F_e(b) = \sum_{n=1}^{\infty} \frac{2}{n} f_n^e \int_{-1}^{+1} (1 - z^2)^{-\frac{1}{2}} T_n(z) e^{-ixbz} dz. \quad (3.31)$$

The following equality is given for $n \neq 0$ in (Prudnikov, A.P., 1983, p.p. 453)

$$\int_{-a}^a \frac{1}{\sqrt{a^2 - x^2}} T_n\left(\frac{x}{a}\right) e^{ipx} dx = i^n p J_n(ap) \quad (3.32)$$

so the integral in Eq. (3.31) can be calculated as,

$$\int_{-1}^1 (1 - z^2)^{-\frac{1}{2}} T_n(z) e^{-ixbz} dz = p (-i)^n J_n(xb) \quad (3.33)$$

and by substituting Eq. (3.33) into Eq. (3.31),

$$F_e(b) = p \sum_{n=1}^{\infty} (-i)^n \frac{2}{n} f_n^e J_n(xb) \quad (3.34)$$

is obtained.

Now, in order to calculate the contribution for $n=0$, let the analysis start with the series expansion for the Fourier transform of the current given by Eq. (3.25):

$$F(b) = \frac{2p}{\Gamma(u + \frac{1}{2})} \sum_{n=0}^{\infty} (-i)^n f_n \frac{\Gamma(n + 2u + 1)}{\Gamma(n + 1)} \frac{J_{n+u+\frac{1}{2}}(xb)}{(2xb)^{u+\frac{1}{2}}}.$$

The term $\Gamma(n + 2u + 1)$ can be written in a more convenient form:

$$\Gamma(n + 2u + 1) = \Gamma(n + 2(u + \frac{1}{2})) \quad (3.35)$$

which gives (Prudnikov, A. P., 1983, p. 720)

$$\Gamma(n + 2u + 1) = (2(u + \frac{1}{2}))_n \Gamma(2(u + \frac{1}{2})) \quad (3.36)$$

where $(.)_n$ denotes that

$$(2(u + \frac{1}{2}))_n = \frac{\Gamma(n + 2u + 1)}{\Gamma(2u + 1)}. \quad (3.37)$$

The following identify (Prudnikov, A. P., 1983, p. 721)

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{p}} \Gamma(z) \Gamma(z + \frac{1}{2}) \quad (3.38)$$

is used with Eq. (3.36) and

$$\Gamma(n + 2u + 1) = (2(u + \frac{1}{2}))_n \Gamma(u + \frac{1}{2}) \Gamma(u + 1) \frac{2^{2(u+\frac{1}{2})-1}}{\sqrt{p}} \quad (3.39)$$

is obtained. By substituting Eq. (3.39) into Eq. (3.25) and considering that $n = 0$ with $u = -1/2$ for the electric current

$$F_e(b) \Big|_{n=0} = p f_0^e J_0(xb) \quad (3.40)$$

is derived. Here, the following equality is being taken into account

$$\Gamma(n + 2u + 1) \Big|_{n=0} = \Gamma(u + \frac{1}{2}) \Gamma(u + 1) \frac{2^{2(u+\frac{1}{2})-1}}{\sqrt{p}} \quad (3.41)$$

and the analysis gives the complete expression for the electric current density

$$F_e(b) = p f_0^e J_0(xb) + 2p \sum_{n=1}^{\infty} \frac{f_n^e}{n} (-i)^n J_n(xb) \quad (3.42)$$

for any value of n .

3.2. System of Linear Algebraic Equations for f_n^e

In the previous section, the Fourier transform of $\tilde{f}_e(z)$ for $u = -\frac{1}{2}$ was obtained as

$$F_e(b) = p \sum_{n=0}^{\infty} X_n J_n(xb) \quad (3.43)$$

where

$$X_n = f_0^e \quad \text{for } n = 0 \quad (3.44)$$

and

$$X_n = 2(-i)^n \frac{f_n^e}{n} \quad \text{for } n \neq 0. \quad (3.45)$$

If Eq. (3.43) is substituted into Eq. (2.67), the problem is reduced to that of finding the unknowns f_n^e with $n = 0, 1, 2, \dots$ as follows:

$$hp \sum_{n=0}^{\infty} X_n J_n(xb) = 4\sqrt{1 - \cos^2 q_0} \frac{\sin x(b + \cos q_0)}{x(b + \cos q_0)} \quad (3.46)$$

$$- \frac{1}{p} \int_{-\infty}^{\infty} \frac{\sqrt{1-t^2} \sin x(t-b)}{t-b} \left\{ p \sum_{n=0}^{\infty} X_n J_n(xt) \right\} dt.$$

By rearranging Eq. (3.46)

$$\begin{aligned}
\text{hp } \sum_{n=0}^{\infty} X_n J_n(xb) &= +4\sqrt{1-\cos^2 q_0} \frac{\sin x(b + \cos q_0)}{x(b + \cos q_0)} \\
&- \sum_{n=0}^{\infty} X_n \int_{-\infty}^{\infty} \frac{\sqrt{1-t^2} \sin x(t-b)}{t-b} J_n(xt) dt
\end{aligned} \tag{3.47}$$

is obtained. In order to be able to express Eq. (3.47) in a more convenient form for numerical calculations, both sides of the equations will be multiplied by

$$\frac{J_{l+t}(xb)}{b^t} \quad \text{for } l = 0, 1, 2, \dots$$

and by integrating each term with respect to b from $-\infty$ to ∞

$$\begin{aligned}
\text{hp } \int_{-\infty}^{\infty} \frac{J_{l+t}(xb)}{b^t} \sum_{n=0}^{\infty} X_n J_n(xb) db &= 4 \int_{-\infty}^{\infty} \sqrt{1-\cos^2 q_0} \frac{\sin x(b + \cos q_0)}{x(b + \cos q_0)} \frac{J_{l+t}(xb)}{b^t} db \\
&- \int_{-\infty}^{\infty} \frac{J_{l+t}(xb)}{b^t} \sum_{n=0}^{\infty} X_n \int_{-\infty}^{\infty} \frac{\sqrt{1-t^2} \sin x(t-b)}{t-b} J_n(xt) dt db
\end{aligned} \tag{3.48}$$

is written. If Eq. (3.48) is rearranged by changing the order of integration and summation, it yields

$$\begin{aligned}
\text{hp } \sum_{n=0}^{\infty} X_n \int_{-\infty}^{\infty} \frac{J_{l+t}(xb) J_n(xb)}{b^t} db &= 4 \int_{-\infty}^{\infty} \frac{\sqrt{1-\cos^2 q_0}}{x} \frac{\sin x(b + \cos q_0)}{b + \cos q_0} \frac{J_{l+t}(xb)}{b^t} db \\
&- \sum_{n=0}^{\infty} X_n \int_{-\infty}^{\infty} \sqrt{1-t^2} J_n(xt) \left\{ \int_{-\infty}^{\infty} \frac{\sin x(t-b)}{t-b} \frac{J_{l+t}(xb)}{b^t} db \right\} dt.
\end{aligned} \tag{3.49}$$

The integrals in the first and second terms on the right-hand side of Eq. (3.49) can be calculated as follows from the equality given in (Prudnikov, A.P., 1983, p.197):

$$\frac{1}{p} \int_{-\infty}^{\infty} \frac{J_{l+t}(xb)}{b^t} \frac{\sin x(t-b)}{t-b} db = \frac{J_{l+t}(xt)}{t^t}. \quad (3.50)$$

This equality will give the value of the first integral on the right-hand side of Eq. (3.49) by setting $t = -\cos q_0$ which yields

$$\frac{1}{p} \int_{-\infty}^{\infty} \frac{J_{l+t}(xb)}{b^t} \frac{\sin x(b + \cos q_0)}{b + \cos q_0} db = \frac{J_{l+t}(-x \cos q_0)}{(-\cos q_0)^t}. \quad (3.51)$$

By using the series representation of Bessel functions, the negative argument functions can be expressed as follows in terms of Bessel functions with positive arguments:

$$\frac{1}{p} \int_{-\infty}^{\infty} \frac{J_{l+t}(xb)}{b^t} \frac{\sin x(b + \cos q_0)}{b + \cos q_0} db = (-1)^l \frac{J_{l+t}(x \cos q_0)}{\cos^t q_0}. \quad (3.52)$$

Now, substitution of Eq. (3.50), Eq. (3.51) into Eq. (3.49) yields that,

$$\begin{aligned} h \sum_{n=0}^{\infty} X_n \int_{-\infty}^{\infty} \frac{J_{l+t}(xb) J_n(xb)}{b^t} db &= \frac{4\sqrt{1-\cos^2 q_0}}{x} (-1)^l \frac{J_{l+t}(x \cos q_0)}{\cos^t q_0} \\ &\quad - \sum_{n=0}^{\infty} X_n \int_{-\infty}^{\infty} \frac{J_{l+t}(xt)}{t^t} J_n(xt) \sqrt{1-t^2} dt. \end{aligned} \quad (3.53)$$

The integral on the left-hand side of Eq. (3.53) is denoted as

$$d_{ln}^{E1} = \int_{-\infty}^{\infty} \frac{J_{l+t}(xb) J_n(xb)}{b^t} db \quad (3.54)$$

and the integral on the right hand side is named as

$$D_{\ln}^{E1} = \int_{-\infty}^{\infty} \frac{J_{l+t}(xt)J_n(xt)}{t^t} \sqrt{1-t^2} dt \quad (3.55)$$

So, by substituting these two equations into Eq. (3.53)

$$h \sum_{n=0}^{\infty} X_n d_{\ln}^{E1} = \lambda^{E1} - \sum_{n=0}^{\infty} X_n D_{\ln}^{E1} \quad (3.56)$$

is obtained with

$$\lambda^{E1} = -\frac{4\sqrt{1-\cos^2 q_0}}{x} (-1)^l \frac{J_{l+t}(x \cos q_0)}{\cos^t q_0}. \quad (3.57)$$

Now Eq. (3.56) can be arranged simply as

$$\lambda^{E1} = \sum_{n=0}^{\infty} X_n (h d_{\ln}^{E1} + D_{\ln}^{E1}) \quad (3.58)$$

which gives an infinite system of linear algebraic equations for f_n^e . The details of the numerical computation will be shown in the following sections which requires to manipulate analytically the integral seen above.

3.3. System of Linear Algebraic Equations for f_n^m

In the previous section, the Fourier transform of $\tilde{f}_m(z)$ for $u = \frac{1}{2}$ was obtained as,

$$F_m(b) = p \sum_{n=0}^{\infty} Y_n \frac{J_{n+1}(xb)}{xb} \quad (3.59)$$

where

$$Y_n = (-i)^n (n+1) f_n^m. \quad (3.60)$$

If this equation is substituted into Eq. (2.46)

$$\begin{aligned} -\frac{1}{xh} p \sum_{n=0}^{\infty} Y_n \frac{J_{n+1}(xb)}{xb} &= 4i \frac{\sin x(b + \cos q_0)}{x(b + \cos q_0)} \\ &+ \frac{1}{xp} \int_{-\infty}^{\infty} \frac{\sin x(t-b)}{(t-b)\sqrt{1-t^2}} \left\{ p \sum_{n=0}^{\infty} Y_n \frac{J_{n+1}(xt)}{xt} \right\} dt \end{aligned} \quad (3.61)$$

is obtained. By changing the order of integration and summation and by rearranging the equation

$$\begin{aligned} -\frac{p}{h} \sum_{n=0}^{\infty} Y_n \frac{J_{n+1}(xb)}{xb} &= 4i \frac{\sin x(b + \cos q_0)}{(b + \cos q_0)} \\ &+ \sum_{n=0}^{\infty} Y_n \left\{ \int_{-\infty}^{\infty} \frac{\sin x(t-b)}{(t-b)\sqrt{1-t^2}} \frac{J_{n+1}(xt)}{xt} dt \right\} \end{aligned} \quad (3.62)$$

is written. By multiplying both sides of Eq. (3.62) by $b^{-1} J_{l+1}(xb)$ and integrating each term with respect to b from $-\infty$ to ∞

$$\begin{aligned} -\frac{p}{h} \int_{-\infty}^{\infty} \left\{ \frac{J_{l+1}(xb)}{b} \sum_{n=0}^{\infty} Y_n \frac{J_{n+1}(xb)}{xb} \right\} db &= 4i \int_{-\infty}^{\infty} \frac{\sin x(b + \cos q_0)}{(b + \cos q_0)} \frac{J_{l+1}(xb)}{b} db \\ &+ \int_{-\infty}^{\infty} \frac{J_{l+1}(xb)}{b} \sum_{n=0}^{\infty} Y_n \left\{ \int_{-\infty}^{\infty} \frac{\sin x(t-b)}{(t-b)\sqrt{1-t^2}} \frac{J_{n+1}(xt)}{xt} dt \right\} db \end{aligned} \quad (3.63)$$

is found. Or

$$\begin{aligned}
 -\frac{p}{h} \sum_{n=0}^{\infty} Y_n \int_{-\infty}^{\infty} \frac{1}{x} \frac{1}{b^2} J_{l+1}(xb) J_{n+1}(xb) db &= 4i \int_{-\infty}^{\infty} \frac{\sin x(b + \cos q_0)}{(b + \cos q_0)} \frac{J_{l+1}(xb)}{b} db \\
 &+ \sum_{n=0}^{\infty} Y_n \int_{-\infty}^{\infty} \left\{ \frac{J_{n+1}(xt)}{xt} \frac{1}{\sqrt{1-t^2}} \int_{-\infty}^{\infty} \frac{\sin x(t-b)}{(t-b)} \frac{J_{l+1}(xb)}{b} db \right\} dt.
 \end{aligned} \tag{3.64}$$

Each term in Eq. (3.64) will be written later in a form more convenient for numerical calculations. For using similar notations with the previous section

$$d_{\ln}^{E2} = \int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db \tag{3.65}$$

is written. Now to evaluate the integral on the right-hand side of Eq. (3.64), Eq. (3.50) is used with $t = 1$, and $t = -\cos q_0$ which gives

$$\frac{1}{p} \int_{-\infty}^{\infty} \frac{J_{l+1}(xb)}{b} \frac{\sin x(b + \cos q_0)}{b + \cos q_0} db = \frac{J_{l+1}(-x \cos q_0)}{-\cos q_0}. \tag{3.66}$$

By substituting, Eq. (3.66) into Eq. (3.64)

$$-\frac{p}{hx} \sum_{n=0}^{\infty} d_{\ln}^{E2} Y_n = 4i \frac{J_{l+1}(-x \cos q_0)}{-\cos q_0} p + \sum_{n=0}^{\infty} Y_n \int_{-\infty}^{\infty} \frac{J_{n+1}(xt) J_{l+1}(xt) p}{xt \sqrt{1-t^2} t} dt \tag{3.67}$$

is obtained, or

$$-\frac{1}{h} \sum_{n=0}^{\infty} d_{\ln}^{E2} Y_n = \lambda^{E2} + \sum_{n=0}^{\infty} Y_n D_{\ln}^{E2} \tag{3.68}$$

is written, where

$$D_{\ln}^{E2} = \int_{-\infty}^{\infty} J_{n+1}(x t) J_{l+1}(x t) \frac{1}{t^2 \sqrt{1-t^2}} dt \quad (3.69)$$

$$\lambda^{E2} = 4i(-1)^l x \frac{J_{l+1}(x \cos q_0)}{\cos q_0}. \quad (3.70)$$

Finally, Eq. (3.68) can be expressed as,

$$-\lambda^{E2} = \sum_{n=0}^{\infty} Y_n \left(\frac{1}{h} d_{\ln}^{E2} + D_{\ln}^{E2} \right) \quad (3.71)$$

which is the system of linear algebraic equations for f_n^m

4. CALCULATION OF FIELD ANALYSIS

The total field expression Eq. (2.16) involving the electric and magnetic current densities as unknowns is the fundamental formula for field analysis. In the present section, first the field analysis will be presented in a formal way by assuming that the current densities are known. Then, the steps for numerical determination of the unknowns will be given.

4.1. Scattered Far Field and Total Scattering Cross Section

As stated in Eq. (2.5), the scattered field expression was given as

$$H_z(x, y) = H_z^i(x, y) + \frac{1}{4} \int_{-a}^a \left\{ k Y I_m(x') - i I_e(x') \frac{\partial}{\partial y} \right\} H_0^{(1)} \left(k \sqrt{(x - x')^2 + y^2} \right) dx.$$

In order to obtain the scattered far field, the first step is to substitute the asymptotic expression of Hankel for large argument. Which is given as [5, p.337]

$$H_n^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{2n+1}{4}\pi)}. \quad (4.1)$$

Using

$$x = r \cos j, \quad y = r \sin j \quad (4.2)$$

in the following expression which is a part of the argument of the Hankel function:

$$\begin{aligned}
 (x-x')^2 + y^2 &= r^2 \cos^2 j - 2r x' \cos j + x'^2 + r^2 \sin^2 j \\
 &= r^2 - 2r x' \cos j + x'^2
 \end{aligned}$$

or

(4.3)

$$(x-x')^2 + y^2 = r^2 \left(1 - 2 \frac{x'}{r} \cos j + \left(\frac{x'}{r}\right)^2\right),$$

is written. Then, by inserting it into the argument

$$\begin{aligned}
 k\sqrt{(x-x')^2 + y^2} &= kr \sqrt{1 - 2 \frac{x'}{r} \cos j + \left(\frac{x'}{r}\right)^2} \\
 &= kr \sqrt{1 - 2 \frac{kx'}{kr} \cos j + \left(\frac{kx'}{kr}\right)^2}
 \end{aligned}$$
(4.4)

and by using the binomial expansion formula

$$\begin{aligned}
 k\sqrt{(x-x')^2 + y^2} &= kr \left\{ 1 - \frac{kx'}{kr} \cos j + O\left(\frac{1}{(kr)^2}\right) \right\} \\
 &= kr \left(1 - \frac{x'}{r} \cos j\right) + O\left(\frac{1}{kr}\right).
 \end{aligned}$$
(4.5)

is obtained. So as $kr \rightarrow \infty$ Eq. (3.2) can be expressed as,

$$\begin{aligned}
 H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) &\approx H_0^{(1)}\left(kr\left(1 - \frac{x'}{r} \cos j\right)\right) \\
 &= \sqrt{\frac{2}{\pi kr}} \left(1 - \frac{x'}{r} \cos j\right)^{-\frac{1}{2}} e^{i\left\{kr\left(1 - \frac{x'}{r} \cos j\right) - \frac{\pi}{4}\right\}} \left\{1 + O\left(\frac{1}{kr}\right)\right\}
 \end{aligned}$$
(4.6)

and by rearranging simply

$$H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) = \sqrt{\frac{2}{ipkr}} e^{i(kr(1-\frac{x'}{r}\cos j))} \left\{ 1 + O\left(\frac{1}{kr}\right) \right\} \quad (4.7)$$

is written. If Eq. (4.7) is substituted into the equation of scattered field the following is obtained

$$H_z^s(x, y) = H_z^s(r, j) = -\frac{i}{4} \sqrt{\frac{2}{ipkr}} e^{ikr} \int_{-a}^{+a} \left\{ f_1(x') + f_2(x') \frac{\partial}{\partial y} \right\} e^{-ikx'\cos j} dx'. \quad (4.8)$$

The following of the partial derivative in polar coordinates

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial j}{\partial y} \frac{\partial}{\partial j} \quad (4.9)$$

with Eq. (4.2)

$$\frac{\partial}{\partial y} = \sin j \frac{\partial}{\partial r} + \frac{\cos j}{r} \frac{\partial}{\partial j} \quad (4.10)$$

and considering for large r , the second term on the right-hand side is obviously zero. So, Eq. (4.8) can be written as

$$H_z^s(r, j) = -\frac{i}{4} \sqrt{\frac{2}{ipkr}} e^{ikr} \int_{-1}^{+1} \left\{ \tilde{f}_1(z') + \frac{x \sin j}{4} \tilde{f}_2(z') \right\} e^{-ixz'\cos j} dz'. \quad (4.11)$$

Then, let the scattered field be expressed as

$$H_z^s(r, j) = A(kr) f(j) \quad (4.12)$$

where,

$$A(kr) = \sqrt{\frac{2}{\rho kr}} e^{i(kr - \frac{\rho}{4})} \quad (4.13)$$

and,

$$f(j) = f_e(j) + f_m(j). \quad (4.14)$$

In this case $f(j)$ represents the far field radiation pattern. The integrals in Eq. (4.11) are the Fourier transforms of the current density functions. By using Eq. (3.43) and Eq. (3.59) the $f_e(j)$ and $f_m(j)$ can be obtained as follows:

$$f_e(j) = \frac{\rho}{4} \sum_{n=0}^{\infty} X_n J_n(x \cos j) \quad (4.15)$$

and,

$$f_m(j) = \frac{x \sin j}{4} \rho \sum_{n=0}^{\infty} Y_n \frac{J_{n+1}(x \cos j)}{x \cos j}. \quad (4.16)$$

As known the total scattering cross section can be calculated as [5, p.102],

$$\frac{S_s}{4a} = -\frac{1}{x} \text{Re}\{f(q)\} \quad (4.17)$$

where q is the incident angle. It is obvious that the calculation of the RCS requires to know the values of X_n and Y_n . From the analysis accomplished in the previous sections, it should be clear that the determination of X_n and Y_n is reduced to numerical evaluation of $D_{\ln}^{E1}, D_{\ln}^{E2}, d_{\ln}^{E1}$ and $d_{\ln}^{E2}$.

5. NUMERICAL ANALYSIS

As shown in the previous section the analysis of the scattered field is reduced to the numerical evaluation of the functions $d_{ln}^{E_{1,2}}$ and $D_{ln}^{E_{1,2}}$. The integral expressions of these terms given by equations Eq. (3.54), Eq. (3.55), Eq. (3.65), and Eq. (3.69) are not convenient for numerical calculations. Therefore, these integrals must be evaluated in terms of some well-known functions which are convenient for numerical calculations.

5.1. Evaluation of $D_{ln}^{E_1}$ and $D_{ln}^{E_2}$

In this section, the infinite integrals in the expression of $D_{ln}^{E_1}$ and $D_{ln}^{E_2}$ will be evaluated. From Eq. (3.55) and Eq. (3.69) the followings are given

$$D_{ln}^{E_1} = \left\{ 1 + (-1)^{l+n} \right\} \int_0^\infty \frac{J_{l+t}(xt) J_n(xt) \sqrt{1-t^2}}{t^t} dt$$

and

$$D_{ln}^{E_2} = \left\{ 1 + (-1)^{l+n} \right\} \int_0^\infty J_{n+1}(xt) J_{l+1}(xt) \frac{1}{t^2 \sqrt{1-t^2}} dt.$$

Now, it is useful to denote the integrals in these expressions with

$$R_{ln}^{E_1}(t, l) = \int_0^\infty J_{l+t}(xt) J_{n+1}(xt) \frac{\sqrt{1-t^2}}{t^{l+t}} dt \quad (5.1)$$

and,

$$R_{ln}^{E_2}(t, l) = \int_0^\infty J_{l+t}(xt) J_{n+1}(xt) \frac{dt}{t^{l+t} \sqrt{1-t^2}} \quad (5.2)$$

which yields

$$D_{ln}^{E_1} = \left\{1 + (-1)^{l+n}\right\} R_{ln}^{E_1}(t, 0) \quad (5.3)$$

and

$$D_{ln}^{E_2} = \left\{1 + (-1)^{l+n}\right\} R_{ln}^{E_2}(1, 1). \quad (5.4)$$

At this stage, it is necessary to choose the proper branch for $\sqrt{1-t^2}$ branch is chosen and assume that it is chosen as

$$\text{Im}\left\{\sqrt{1-t^2}\right\} > 0, \quad \text{for } |t| \rightarrow \infty, \quad (5.5)$$

which gives

$$\sqrt{1-t^2} = i\sqrt{t^2-1} \quad |t| > 1. \quad (5.6)$$

If Eq. (5.1) is rewritten in the form of

$$R_{ln}^{E_1}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+1}(xt) \frac{\sqrt{1-t^2}}{t^{l+t}} dt + \int_1^\infty J_{l+t}(xt) J_{n+1}(xt) \frac{\sqrt{1-t^2}}{t^{l+t}} dt. \quad (5.7)$$

and if Eq. (5.6) is used in Eq. (5.7) we obtain that,

$$R_{ln}^{E_1}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+1}(xt) \frac{\sqrt{1-t^2}}{t^{l+t}} dt + i \int_1^\infty J_{l+t}(xt) J_{n+1}(xt) \frac{\sqrt{t^2-1}}{t^{l+t}} dt \quad (5.8)$$

is obtained. This gives

$$R_{ln}^{E_1}(t, l) = R_{ln1}^{E_1}(t, l) + i R_{ln2}^{E_1}(t, l) \quad (5.9)$$

with

$$R_{ln1}^{E_1}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+1}(xt) \frac{\sqrt{1-t^2}}{t^{l+t}} dt \quad (5.10)$$

and

$$R_{ln2}^{E_1}(t, l) = \int_1^\infty J_{l+t}(xt) J_{n+1}(xt) \frac{\sqrt{t^2-1}}{t^{l+t}} dt. \quad (5.11)$$

In the same manner let Eq. (5.2) be written in the form of

$$R_{ln}^{E_2}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+1}(xt) \frac{dt}{t^{l+t} \sqrt{1-t^2}} + \int_1^\infty J_{l+t}(xt) J_{n+1}(xt) \frac{dt}{t^{l+t} \sqrt{1-t^2}}. \quad (5.12)$$

Again using Eq. (5.6) in Eq. (5.12) yields

$$R_{ln}^{E_2}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+1}(xt) \frac{dt}{t^{l+t} \sqrt{1-t^2}} - i \int_1^\infty J_{l+t}(xt) J_{n+1}(xt) \frac{dt}{t^{l+t} \sqrt{t^2-1}} \quad (5.13)$$

with

$$R_{ln}^{E_2}(t, l) = R_{ln1}^{E_2}(t, l) - iR_{ln2}^{E_2}(t, l), \quad (5.14)$$

where the following definitions are assumed

$$R_{ln1}^{E_2}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+l}(xt) \frac{dt}{t^{l+t} \sqrt{1-t^2}}, \quad (5.15)$$

and,

$$R_{ln2}^{E_2}(t, l) = \int_1^\infty J_{l+t}(xt) J_{n+l}(xt) \frac{dt}{t^{l+t} \sqrt{t^2-1}}. \quad (5.16)$$

As seen through these steps the numerical calculations of $D_{ln}^{E_{1,2}}$ is reduced to the evaluation of $R_{ln1}^{E_1}$, $R_{ln2}^{E_1}$, $R_{ln1}^{E_2}$ and $R_{ln2}^{E_2}$ which will be given in the following sections.

5.1.1. Evaluation of $R_{ln1}^{E_2}(t, l)$

In the previous section, the function $R_{ln1}^{E_2}(t, l)$ was defined as

$$R_{ln1}^{E_2}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+l}(xt) \frac{dt}{t^{l+t} \sqrt{1-t^2}}.$$

For the product of Bessel functions, the following equality can be written from (Abramowitz, M., 1968. P.360)

$$J_\nu(z) J_\mu(z) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{2}\right)^{m+\nu+2k} \frac{\Gamma(m+\nu+2k+1)}{k! \Gamma(m+\nu+k+1) \Gamma(\nu+k+1) \Gamma(m+k+1)}, \quad (5.17)$$

and again from (Prudnikov, A.P., 1983, p.721) the recurrence formula for double argument is used.

So, the gamma function in the numerator of Eq. (5.17) can be written as,

$$\Gamma\left(2\frac{m+v+2k+1}{2}\right) = \frac{2^{v+m+2k}}{\sqrt{p}} \Gamma\left(\frac{m+v+2k+1}{2}\right) \Gamma\left(\frac{v+m+2k+1}{2} + \frac{1}{2}\right).$$

or

$$\Gamma\left(2\frac{m+v+2k+1}{2}\right) = \frac{2^{v+m+2k}}{\sqrt{p}} \Gamma\left(k + \frac{m+v+1}{2}\right) \Gamma\left(k + \frac{v+m+2}{2}\right). \quad (5.18)$$

Substitution of Eq. (5.18) into Eq. (5.17) yields that,

$$J_v(z) J_m(z) = \frac{1}{\sqrt{p}} \sum_{k=0}^{\infty} (-1)^k z^{m+v+2k} \times \frac{\Gamma\left(k + \frac{m+v+1}{2}\right) \Gamma\left(k + \frac{m+v+2}{2}\right)}{\Gamma(k+1) \Gamma(m+v+k+1) \Gamma(v+k+1) \Gamma(m+k+1)} \quad (5.19)$$

where $\Gamma(k+1)$ is used for the $k!$ function.

If the following variable changes are made $v = l + t$, $m = n + l$ and $z = xt$ we obtain that,

$$J_{l+t}(xt) J_{n+l}(xt) = \frac{1}{\sqrt{p}} \sum_{k=0}^{\infty} h_{kln}^{lt}(xt)^{l+n+l+t+2k} \quad (5.20)$$

is obtained with

$$h_{kln}^{lt} = (-1)^k \frac{\Gamma\left(k + \frac{l+n+l+t+1}{2}\right) \Gamma\left(k + \frac{l+n+l+t+2}{2}\right)}{\Gamma(k+1) \Gamma(k+l+n+l+t+1) \Gamma(k+n+l+1) \Gamma(k+l+t+1)}. \quad (5.21)$$

So by substituting Eq. (5.20) into the equation of $R_{ln1}^{E_2}(t, l)$

$$R_{ln1}^{E_2}(t, l) = \int_0^1 \frac{1}{\sqrt{p}} \sum_{k=0}^{\infty} h_{kln}^{lt} (xt)^{l+n+l+t+2k} \frac{dt}{t^{l+t} \sqrt{1-t^2}} \quad (5.22)$$

is found. By changing the order of integration and summation

$$R_{ln1}^{E_2}(t, l) = \frac{1}{\sqrt{p}} \sum_{k=0}^{\infty} h_{kln}^{lt} (x)^{l+n+l+t+2k} \int_0^1 \frac{t^{2k+l+n}}{\sqrt{1-t^2}} dt \quad (5.23)$$

is written and to evaluate the integral in Eq. (5.23), the substitution $t = \sqrt{r}$ will be used with $dt = dr / 2\sqrt{r}$ which yields

$$\int_0^1 \frac{t^{2k+l+n}}{\sqrt{1-t^2}} dt = \int_0^1 \frac{r^{k+\left(\frac{l+n}{2}\right)}}{\sqrt{1-r}} \frac{dr}{2\sqrt{r}}. \quad (5.24)$$

Note that in (Spiegel, M.R., 1968. P.103) the following definitions given

$$b(x, y) = \int_0^1 d^{x-1} (1-d)^{y-1} dd, \quad (5.25)$$

where $b(x, y)$ denotes the beta function. By making the following variable changes $y = 1/2$ and $x = k + \left[(l+n)/2\right] + 1$

$$\frac{1}{2} \int_0^1 \frac{r^{k+\left(\frac{l+n-1}{2}\right)}}{\sqrt{1-r}} dr = \frac{1}{2} b\left(k + \frac{l+n-1}{2} + 1, \frac{1}{2}\right) \quad (5.26)$$

is written. Again from (Spiegel, M.R., 1968. P.103) the following relationship between the beta function and gamma function is given

$$b(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (5.27)$$

for $\text{Re } x > 0$ and $\text{Re } y > 0$. Since $l \geq 0$ and $n \geq 0$, it is obvious that the above identity can be applied for Eq. (5.26). This yields

$$\int_0^1 \frac{t^{2k+l+n}}{\sqrt{1-t^2}} dt = \frac{1}{2} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(k + \frac{l+n+2}{2}\right)} \quad (5.28)$$

or

$$\int_0^1 \frac{t^{2k+l+n}}{\sqrt{1-t^2}} dt = \frac{\sqrt{p}}{2} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right)}{\Gamma\left(k + \frac{l+n+2}{2}\right)} \quad (5.29)$$

since $\Gamma(1/2) = \sqrt{p}$. By substituting (5.29) into (5.23) the following equation for $R_{ln1}^{E_2}(t, l)$

$$R_{ln1}^{E_2}(t, l) = \frac{x^{l+n+1+t}}{2} \sum_{k=0}^{\infty} h_{kln}^{l,t} x^{2k} \frac{\Gamma\left(k + \frac{l+n}{2} + \frac{1}{2}\right)}{\Gamma\left(k + \frac{l+n}{2} + 1\right)} \quad (5.30)$$

is obtained.

5.1.2. Evaluation of $R_{ln1}^{E_1}(t, l)$

The integral which will be calculated in this section is,

$$R_{ln1}^{E_1}(t, l) = \int_0^1 J_{l+t}(xt) J_{n+l}(xt) \frac{\sqrt{1-t^2}}{t^{l+t}} dt$$

where first Eq. (5.20) will be substituted in place of the product of Bessel functions. It gives

$$R_{ln1}^{E_1}(t, l) = \int_0^1 \frac{1}{\sqrt{\rho}} \sum_{k=0}^{\infty} h_{kln}^{t,l}(xt)^{l+n+l+t+2k} \frac{\sqrt{1-t^2}}{t^{l+t}} dt \quad (5.31)$$

or by changing the order of integration and summation it yields

$$R_{ln1}^{E_1}(t, l) = \frac{1}{\sqrt{\rho}} \sum_{k=0}^{\infty} h_{kln}^{t,l}(xt)^{l+n+l+t+2k} \int_0^1 t^{2k+l+n} \sqrt{1-t^2} dt. \quad (5.32)$$

In a similar manner by using the same substitution $t = \sqrt{r}$, the integral in Eq. (5.32) can be written as

$$\int_0^1 t^{2k+l+n} \sqrt{1-t^2} dt = \int_0^1 r^{k+\left(\frac{l+n}{2}\right)} \sqrt{1-r} \frac{dr}{2\sqrt{r}} \quad (5.33)$$

or

$$\int_0^1 t^{2k+l+n} \sqrt{1-t^2} dt = \frac{1}{2} \int_0^1 r^{k+\left(\frac{l+n-1}{2}\right)} \sqrt{1-r} dr \quad (5.34)$$

and by using Eq. (5.25) in Eq. (5.34)

$$\int_0^1 t^{2k+l+n} \sqrt{1-t^2} dt = \frac{1}{2} b \left(k + \frac{l+n+1}{2}, \frac{3}{2} \right) \quad (5.35)$$

is found. Since the condition $\text{Re} \left\{ k + (l+n+1)/2 \right\} > 0$ is satisfied, by using Eq. (5.27) in Eq. (5.35) the following equation is obtained:

$$\int_0^1 t^{2k+l+n} \sqrt{1-t^2} dt = \frac{1}{2} \frac{\Gamma \left(k + \frac{l+n+1}{2} \right) \Gamma \left(\frac{3}{2} \right)}{\Gamma \left(k + \frac{l+n}{2} + 2 \right)}. \quad (5.36)$$

Finally the desired result is found as follows,

$$R_{ln1}^{E_1}(t, l) = \frac{x^{l+n+l+t}}{4} \sum_{k=0}^{\infty} h_{kln}^{l,t} x^{2k} \frac{\Gamma \left(k + \frac{l+n}{2} + \frac{1}{2} \right)}{\Gamma \left(k + \frac{l+n}{2} + 2 \right)} \quad (5.37)$$

by substituting Eq. (5.36) into Eq. (5.32).

5.1.3. Evaluation of $R_{ln2}^{E_2}(t, l)$

The expression of $R_{ln2}^{E_2}(t, l)$ was given as

$$R_{ln2}^{E_2}(t, l) = \int_1^{\infty} J_{l+t}(xt) J_{n+1}(xt) \frac{dt}{t^{l+t} \sqrt{t^2-1}}$$

and the first step will be to apply the contour integral representation of the product of the Bessel functions (Veliev, 1993, p.502)

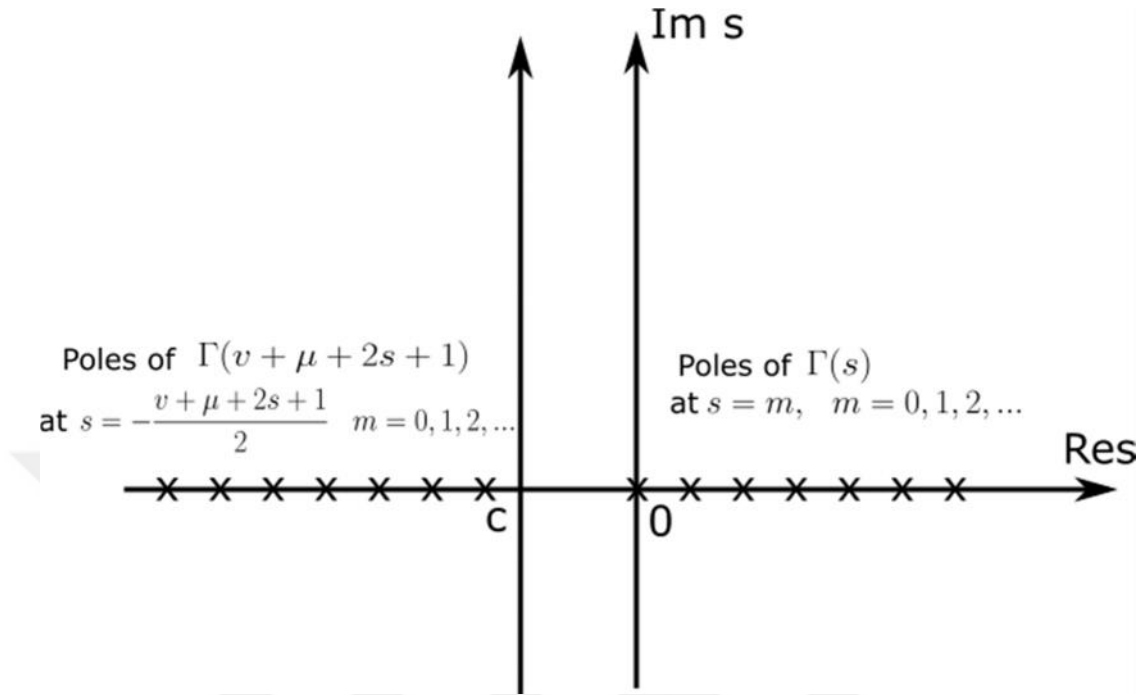


Figure 5.1. The contour for the product of Bessel functions

$$J_m(z) J_v(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{z}{2}\right)^{m+v+2s} \frac{\Gamma(-s)\Gamma(m+v+2s+1)}{\Gamma(m+s+1)\Gamma(v+s+1)\Gamma(m+v+s+1)} ds. \quad (5.38)$$

As known the poles of Gamma functions are at negative integers. Since m and v are assumed to be positive real, the poles of $\Gamma(-s)$ are on the right-hand side of the contour while the poles of $\Gamma(m+v+2s+1)$ are on the left-hand side of the contour. The poles of $\Gamma(-s)$ are simple poles and they are located at

$$s = m, \quad (m = 0, 1, 2, \dots), \quad (5.39)$$

and the poles of $\Gamma(m+v+2s+1)$ are also simple poles located at

$$m + v + 2s + 1 = -m, \quad (m = 0, 1, 2, \dots),$$

yielding

$$s = -\frac{m + v + m + 1}{2}. \quad (5.40)$$

So the contour must be chosen such that the following condition is satisfied:

$$-\operatorname{Re} \frac{m + v + 1}{2} < c < 0. \quad (5.41)$$

Let the substituting $m = l + t$, $v = n + l$, $z = xt$ be made and rewrite Eq. (5.38) to obtain that

$$J_{l+t}(xt) J_{n+l}(xt) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)}{\Gamma(l+t+s+1)\Gamma(n+l+s+1)} \times \frac{\Gamma(l+n+l+t+2s+1)}{\Gamma(l+n+t+l+s+1)} \times \left(\frac{xt}{2}\right)^{l+n+t+l+2s} ds. \quad (5.42)$$

In this case the contour must satisfy that,

$$-\operatorname{Re} \frac{l+n+t+l+1}{2} < c < 0 \quad (5.43)$$

As we used previously the double argument formula from (Prudnikov, A.P., 1983, p.721) yields

$$\Gamma(l+n+l+t+2s+1) = \frac{2^{l+n+l+t+2s}}{\sqrt{p}} \quad (5.44)$$

$$\times \Gamma\left(\frac{l+n+l+t+2s+1}{2}\right) \Gamma\left(\frac{l+n+l+t+2s+2}{2}\right)$$

or

$$\Gamma(l+n+l+t+2s+1) = \frac{2^{l+n+l+t+2s}}{\sqrt{p}} \quad (5.45)$$

$$\times \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+2}{2}\right).$$

By substituting Eq. (5.45) into Eq. (5.42)

$$J_{l+t}(xt) J_{n+l}(xt) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)}{\sqrt{p}} \frac{\Gamma\left(s + \frac{l+n+l+t+1}{2}\right)}{\Gamma(s+n+l+1)} \quad (5.46)$$

$$\times \frac{\Gamma\left(s + \frac{l+n+l+t+2}{2}\right) (xt)^{2s+l+n+l+t}}{\Gamma(s+l+t+1) \Gamma(s+l+n+l+t+1)} ds$$

is obtained. Let Eq. (5.46) be substituted into the main integral equation

$$R_{ln2}^{E_2}(t, l) = \int_1^\infty \frac{1}{t^{l+t} \sqrt{t^2-1}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)}{\sqrt{p}} \quad (5.47)$$

$$\times \frac{\Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+2}{2}\right) (xt)^{2s+l+n+l+t}}{\Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1)} ds dt$$

is found or simply by changing the order of integration

$$\begin{aligned}
R_{ln2}^{E_2}(t, l) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)}{\sqrt{p}} \frac{\Gamma\left(s + \frac{l+n+l+t+1}{2}\right)}{\Gamma(s+l+t+1)} \\
&\times \frac{\Gamma\left(s + \frac{l+n+l+t+2}{2}\right)}{\Gamma(s+n+l+1)\Gamma(s+l+n+l+t+1)} (x)^{2s+l+n+l+t} \int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt ds
\end{aligned} \tag{5.48}$$

is written. Now, to calculate the second integral on the right-hand side of Eq. (5.48), the following variable change will be made:

$$t = r^{\frac{1}{2}}, \quad \text{and} \quad dt = -\frac{1}{2} r^{-\frac{3}{2}} dr,$$

which yields

$$\int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt = -\frac{1}{2} \int_1^0 \frac{r^{-s-\frac{(l+n)}{2}}}{\sqrt{r^{-1}-1}} r^{-\frac{3}{2}} dr = \frac{1}{2} \int_0^1 \frac{r^{-s-\frac{(l+n)}{2}}}{\sqrt{1-r}} r^{-\frac{3}{2}+\frac{1}{2}} dr$$

or

$$\int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt = \frac{1}{2} \int_0^1 r^{-s-\frac{(l+n)}{2}-1} (1-r)^{-\frac{1}{2}} dr. \tag{5.49}$$

From Eq. (5.25)

$$\frac{1}{2} b(a, b) = \frac{1}{2} \int_0^1 r^{a-1} (1-r)^{b-1} dr.$$

If $\text{Re } a > 0$ and $\text{Re } b > 0$, where $b(a, b)$ is the previously defined beta function which gives

$$\int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt = \frac{1}{2} b\left(-s - \frac{l+n}{2}, \frac{1}{2}\right). \quad (5.50)$$

To be able to use Eq. (5.27) in Eq. (5.50), the condition $\operatorname{Re}\left(-s - (l+n)/2\right) > 0$ or $\operatorname{Re}\left(s + (l+n)/2\right) < 0$ must be satisfied. It is known that s denotes the integration variable and lies on the contour: therefore, $\operatorname{Re}\{s\}$ is taken to be equal to a constant c and so $c + (l+n)/2 < 0$ or $c < -(l+n)/2$ is written. From Eq. (5.43) it can be stated that for any t and $l=0$ the contour must satisfy the condition $-(l+n+t+1)/2 < c < -(l+n)/2$. Under this condition by using Eq. (5.27) equation Eq. (5.50) can be written in the form of

$$\int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt = \frac{1}{2} \frac{\Gamma\left(-s - \frac{l+n}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-s - \frac{l+n}{2} + \frac{1}{2}\right)}$$

or

$$\int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt = \frac{\sqrt{p}}{2} \frac{\Gamma\left(-s - \frac{l+n}{2}\right)}{\Gamma\left(-s - \frac{l+n-1}{2}\right)}. \quad (5.51)$$

Now, the following identity (Prudnikov, A.P., 1983, p.721) for gamma function will be introduced

$$\Gamma(z) = \frac{p^z}{\Gamma(1-z) \sin p z} \quad (5.52)$$

and applied to the both Gamma functions in Eq. (5.51), which yields

$$\int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt = \frac{\sqrt{p}}{2} \frac{p}{\Gamma\left(1+s+\frac{l+n}{2}\right) \sin p \left(-s-\frac{l+n}{2}\right)} \times \frac{\Gamma\left(1+s+\frac{l+n-1}{2}\right) \sin p \left(-s-\frac{l+n-1}{2}\right)}{p} \quad (5.53)$$

If the following well-known trigonometric identities $\sin(z-p/2) = -\cos(z)$ and $\sin(-z) = -\sin(z)$ are used, Eq. (5.53) reduced to

$$\int_1^\infty \frac{t^{2s+l+n}}{\sqrt{t^2-1}} dt = -\frac{\sqrt{p}}{2} \frac{\Gamma\left(s+\frac{l+n+1}{2}\right) \cos p \left(s+\frac{l+n}{2}\right)}{\Gamma\left(s+\frac{l+n+1}{2}\right) \sin p \left(s+\frac{l+n}{2}\right)}. \quad (5.54)$$

Now, by substituting Eq. (5.52) for negative argument and Eq. (5.54) into Eq. (5.48)

$$R_{ln2}^{E_2}(t, l) = \frac{-1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\Gamma(s+1) \sin p s} \frac{\Gamma\left(s+\frac{l+n+l+t+1}{2}\right)}{\Gamma(s+l+t+1)} \times \frac{\Gamma\left(s+\frac{l+n+l+t+2}{2}\right) (x)^{2s+l+n+l+t}}{\Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1)} \times \left[-\frac{1}{2} \frac{\Gamma\left(s+\frac{l+n+1}{2}\right) \cos p \left(s+\frac{l+n}{2}\right)}{\Gamma\left(s+\frac{l+n+2}{2}\right) \sin p \left(s+\frac{l+n}{2}\right)} \right] \quad (5.55)$$

is obtained or by rearranging Eq. (5.55)

$$\begin{aligned}
R_{ln2}^{E_2}(t, l) &= \frac{1}{2p} \int_{c-i\infty}^{c+i\infty} \frac{p \cos p \left(s + \frac{l+n}{2} \right)}{2 \sin p s \sin p \left(s + \frac{l+n}{2} \right)} \\
&\times \frac{\Gamma \left(s + \frac{l+n+l+t+1}{2} \right) \Gamma \left(s + \frac{l+n+l+t+2}{2} \right)}{\Gamma(s+n+l+1) \Gamma(s+1+n+l+t+1) \Gamma(s+1)} \\
&\times \frac{\Gamma \left(s + \frac{l+n+1}{2} \right) (x)^{2s+l+n+l+t}}{\Gamma(s+l+1+t) \Gamma \left(s + \frac{l+n+2}{2} \right)} ds
\end{aligned} \tag{5.56}$$

is found. By introducing $W_1(s)$ and $W_2(s)$ as follows

$$\begin{aligned}
W_1(s) &= \Gamma \left(s + \frac{l+n+l+t+1}{2} \right) \Gamma \left(s + \frac{l+n+l+t+2}{2} \right) \\
&\times \Gamma \left(s + \frac{l+n+1}{2} \right) (x)^{2s+l+n+l+t}
\end{aligned} \tag{5.57}$$

and

$$W_2(s) = \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) \Gamma(s+l+1+t) \Gamma \left(s + \frac{l+n+2}{2} \right), \tag{5.58}$$

Eq. (5.56) is reduced to

$$R_{ln2}^{E_2}(t, l) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p \cos p \left(s + \frac{l+n}{2} \right)}{2 \sin p s \sin p \left(s + \frac{l+n}{2} \right)} \frac{W_1(s)}{W_2(s)} ds. \quad (5.59)$$

To calculate the integral in Eq. (5.59), the integration contour will be deformed into the right-half plane (Figure 5.2.). So by determining the singularities of the integrand and using the residue theorem Eq. (5.59) can be evaluated. It is known that for $\text{Re } l \geq 0$, $\text{Re } t \geq 0$, $l > 0$. c can be chosen such that $-(l+n+1)/2 < c < -(l+n)/2$. As explained in Appendix the only poles on the right-half plane of the integration contour are double poles which are located at $s = k$ ($k = 0, 1, 2, 3, \dots$) and the simple poles which are located at $s = k - (l+n)/2$ ($k = 0, 1, 2, 3, \dots, (l+n)/2 - 1$).

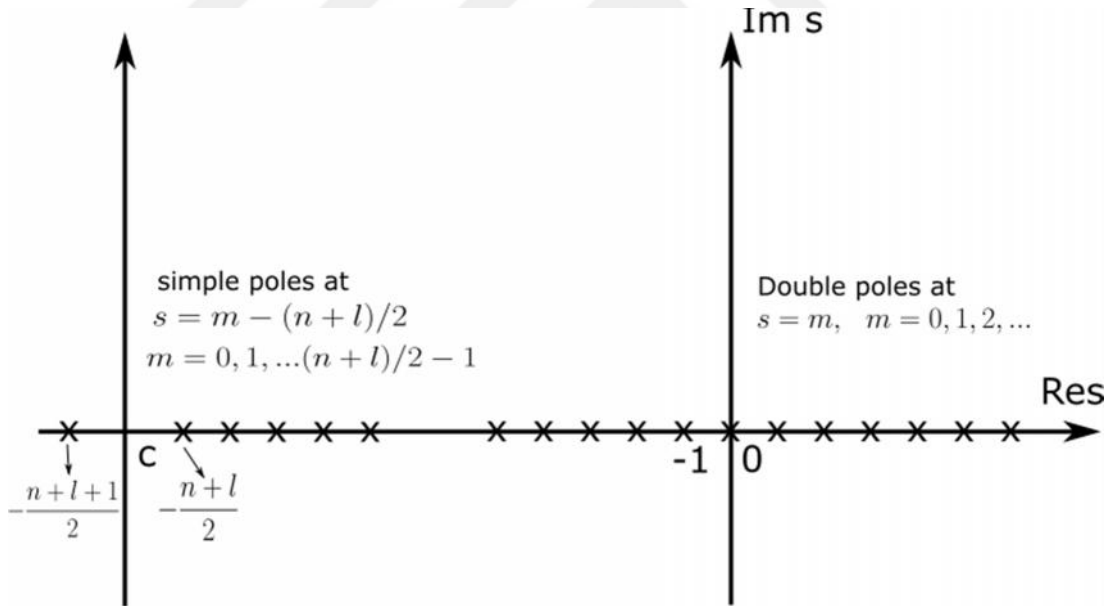


Figure 5.2. Location of poles of the integrand of $R_{ln2}^{E_2}$

Now, the contributions of these poles to the integral must be calculated by using the residue theorem as follows:

$$\begin{aligned}
R_{ln2}^{E_2}(t, l) = & - \sum_{k=0}^{\infty} \text{Resudues of} \left\{ \frac{p \csc p \left(s + \frac{l+n}{2} \right) W_1(s)}{2 \sin p s \sin p \left(s + \frac{l+n}{2} \right) W_2(s)} \right\} \Bigg|_{s=k} \\
& - \sum_{k=0}^{\frac{l+n}{2}-1} \text{Resudues of} \left\{ \frac{p \csc p \left(s + \frac{l+n}{2} \right) W_1(s)}{2 \sin p s \sin p \left(s + \frac{l+n}{2} \right) W_2(s)} \right\} \Bigg|_{s=k-\frac{l+n}{2}}
\end{aligned} \quad (5.60)$$

As known, if $f(z)$ has a pole of order k at $z = a$, then the residue a_1 is calculated as

$$a_1 = \lim_{z \rightarrow a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left\{ (z-a)^k f(z) \right\}. \quad (5.61)$$

First, let the residue (a_1) be calculated at $s = k - (l+n)/2$ as

$$\begin{aligned}
a_1 = & \lim_{s \rightarrow k - \frac{l+n}{2}} \left\{ \left(s - k + \frac{l+n}{2} \right) \frac{p \csc p \left(s + \frac{l+n}{2} \right) x^{2s+l+n+l+t}}{2 \sin p s \sin p \left(s + \frac{l+n}{2} \right) \Gamma(s+1)} \right. \\
& \times \frac{\Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+2}{2}\right)}{\Gamma\left(s + \frac{l+n+2}{2}\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1)} \Bigg\}
\end{aligned} \quad (5.62)$$

and by rearranging Eq. (5.62)

$$\begin{aligned}
a_1 = & \lim_{s \rightarrow k - \frac{l+n}{2}} \left\{ \left(s - k + \frac{l+n}{2} \right) \frac{\mathfrak{p} \cos \mathfrak{p} \left(s + \frac{l+n}{2} \right)}{2 \sin \mathfrak{p} \left(s + \frac{l+n}{2} \right)} \right\} \\
& \times \lim_{s \rightarrow k - \frac{l+n}{2}} \left\{ \frac{x^{2s+l+n+l+t}}{\sin \mathfrak{p} s \Gamma(s+1) \Gamma \left(s + \frac{l+n+2}{2} \right)} \right\} \\
& \times \lim_{s \rightarrow k - \frac{l+n}{2}} \left\{ \frac{\Gamma \left(s + \frac{l+n+1}{2} \right) \Gamma \left(s + \frac{l+n+l+t+1}{2} \right) \Gamma \left(s + \frac{l+n+l+t+2}{2} \right)}{\Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+t+l+1)} \right\}.
\end{aligned} \tag{5.63}$$

is obtained. The last term will be evaluated simply by taking the limit for $s \rightarrow k - (l+n)/2$ and for the second limit on the right-hand side of Eq. (5.63) first recurrence relation given by Eq. (5.52) will be used

$$\Gamma(s+1) \Gamma(l-s-1) = \frac{\mathfrak{p}}{\sin \mathfrak{p}(s+1)} \tag{5.64}$$

and

$$\Gamma(s+1) = \frac{\mathfrak{p}}{\Gamma(-s) \sin \mathfrak{p}(s+1)} \tag{5.65}$$

will be written. Then the second limit will be equal to

$$\begin{aligned}
\lim_{s \rightarrow k - \frac{l+n}{2}} \frac{x^{2s+l+n+l+t}}{\sin p s \Gamma(s+1)} &= \lim_{s \rightarrow k - \frac{l+n}{2}} \frac{\Gamma(-s) \sin p (s+1) x^{2s+l+n+l+t}}{p \sin p s} \\
&= - \frac{\Gamma\left(-k + \frac{l+n}{2}\right)}{p} x^{2k+l+t}.
\end{aligned}
\tag{5.66}$$

To calculate the first limit the following formula (Prudnikov, A.P., 1983, p.721)

$$\Gamma(1+z)\Gamma(1-z) = \frac{p z}{\sin p z}, \tag{5.67}$$

is used to express sine function in terms of the Gamma function and by simplifications

$$\lim_{s \rightarrow k - \frac{l+n}{2}} \frac{\left(s - k + \frac{l+n}{2}\right) p \cos p \left(s + \frac{l+n}{2}\right)}{2 \sin p \left(s + \frac{l+n}{2}\right)} = \frac{1}{2} \tag{5.68}$$

is obtained. If we substitute Eq. (5.66) and Eq. (5.68) into Eq. (5.63) we obtain that,

$$\begin{aligned}
a_1 &= - \frac{1}{2p} \frac{\Gamma\left(-k + \frac{l+n}{2}\right) \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k + \frac{l+t+1}{2}\right)}{\Gamma(k+1) \Gamma\left(k - \frac{l+n}{2} + l+t+1\right) \Gamma\left(k - \frac{l+n}{2} + n+l+1\right)} \\
&\quad \times \frac{\Gamma\left(k + \frac{l+t}{2} + 1\right) e^{2k+l+t}}{\Gamma\left(k + \frac{l+n}{2} + l+t+1\right)}.
\end{aligned}
\tag{5.69}$$

Now, the residue b_1 at $s = k$ will be calculated and according to Eq. (5.61) the residue at $s = k$ is given by

$$b_1 = \lim_{s \rightarrow k} \frac{d}{ds} \left\{ (s-k)^2 \frac{p \csc p s W_1(s)}{2 \sin^2 p s W_2(s)} \right\} \quad (5.70)$$

where

$$\frac{\csc p \left(s + \frac{l+n}{2} \right)}{\sin p \left(s + \frac{l+n}{2} \right)} = \frac{(-1)^{\frac{l+n}{2}} \csc p s}{(-1)^{\frac{l+n}{2}} \sin p s} \quad (5.71)$$

is used. The differentiation in Eq. (5.70) simply gives

$$\begin{aligned} \frac{d}{ds} \left\{ (s-k)^2 \frac{p \csc p s W_1(s)}{2 \sin^2 p s W_2(s)} \right\} &= (-1)^k \frac{d}{ds} \left\{ (s-k)^2 \frac{p \csc p (s-k)}{2 \sin^2 p (s-k)} \right\} \frac{W_1(s)}{W_2(s)} \\ &+ (s-k)^2 \frac{p \csc p s}{2 \sin^2 p (s-k)} \frac{dW_1(s)}{ds W_2(s)}. \end{aligned} \quad (5.72)$$

The first differentiation in Eq. (5.72) can be evaluated as follows:

$$\begin{aligned} \frac{d}{ds} \left\{ (s-k)^2 \frac{p \csc p (s-k)}{2 \sin^2 p (s-k)} \right\} &= 2(s-k) \frac{p \csc p (s-k)}{2 \sin^2 p (s-k)} \\ &+ (s-k)^2 \frac{1}{4 \sin^4 p (s-k)} \times \{ -p^2 \sin p (s-k) 2 \sin^2 p (s-k) \\ &- p \csc p (s-k) 4 \sin p (s-k) p \csc p (s-k) \}, \end{aligned} \quad (5.73)$$

or

$$\begin{aligned} \frac{d}{ds} \left\{ (s-k)^2 \frac{p \cos p(s-k)}{2 \sin^2 p(s-k)} \right\} &= (s-k) \frac{p \cos p(s-k)}{\sin^2 p(s-k)} - \frac{p^2 (s-k)^2}{2 \sin^4 p(s-k)} \\ &\times \{ \sin^3 p(s-k) + 2 \sin p(s-k) \cos^2 p(s-k) \}. \end{aligned} \quad (5.74)$$

If Eq.(5.74) is rearranged, it will be able to take the limit by using $\lim_{x \rightarrow 0} [(\sin x)/x] = 1$. So the limit of Eq. (5.74) will be equal to

$$\frac{p}{2}(s-k) = 0. \quad (5.75)$$

Then from Eq. (5.72) the residue b_1 is

$$b_1 = \lim_{s \rightarrow k} \left\{ (s-k)^2 \frac{p \cos p s}{2 \sin^2 p(s-k)} \frac{dW_1(s)}{ds W_2(s)} \right\}, \quad (5.76)$$

and,

$$b_1 = (-1)^k \frac{1}{2p} \lim_{s \rightarrow k} \left\{ \frac{W_1(s)}{W_2(s)} \right\}. \quad (5.77)$$

Since the following is easily written

$$\frac{d}{ds} \left\{ \frac{W_1(s)}{W_2(s)} \right\} = \frac{W_1'(s)W_2(s) - W_1(s)W_2'(s)}{W_2^2(s)}. \quad (5.78)$$

It is necessary to calculate $W_1'(s)$, $W_2'(s)$ and $W_2^2(s)$. Then

$$\begin{aligned}
W_1'(s) &= 2 \ln x x^{2s+l+n+l+t} \\
&\times \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \\
&+ x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \\
&+ x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \\
&+ x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right)
\end{aligned} \tag{5.79}$$

is found and

$$\begin{aligned}
W_2'(s) &= \Gamma'(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) \\
&+ \Gamma(s+1) \Gamma'\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) \\
&+ \Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma'(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) \\
&+ \Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma'(s+n+l+1) \Gamma(s+l+n+l+t+1) \\
&+ \Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma'(s+l+n+l+t+1)
\end{aligned} \tag{5.80}$$

is obtained. On the other hand $W_2^2(s)$,

$$\begin{aligned}
W_2^2(s) &= \Gamma^2(s+1) \Gamma^2\left(s + \frac{l+n}{2} + 1\right) \Gamma^2(s+l+t+1) \\
&\times \Gamma^2(s+n+l+1) \Gamma^2(s+l+n+l+t+1).
\end{aligned} \tag{5.81}$$

If Eq. (5.79), Eq. (5.80), Eq. (5.81) are substituted into Eq. (5.78)

$$\begin{aligned}
\frac{d}{ds} \left\{ \frac{W_1(s)}{W_2(s)} \right\} = & \left\{ \left\{ 2 \ln x x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \times \right. \right. \\
& \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) + \\
& + x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \\
& + x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \\
& \left. + x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \right\} \times \\
& \left\{ \Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) \right\} - \\
& \left\{ x x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \right\} \times \\
& \left\{ \Gamma'(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) + \right. \\
& \Gamma(s+1) \Gamma'\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) + \\
& \Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma'(s+l+t+1) \Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) + \\
& \Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma'(s+n+l+1) \Gamma(s+l+n+l+t+1) + \\
& \left. \Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1) \Gamma'(s+l+n+l+t+1) \right\} \times \\
& \frac{1}{\Gamma^2(s+1) \Gamma^2\left(s + \frac{l+n}{2} + 1\right) \Gamma^2(s+l+t+1)} \times \frac{1}{\Gamma^2(s+n+l+1) \Gamma^2(s+l+n+l+t+1)}
\end{aligned}
\tag{5.82}$$

are obtained. In (Prudnikov, A.P., 1983, p722) the Psi function $y(z)$ is defined as

$$y(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$
(5.83)

If Eq. (5.83) is used in Eq. (5.82)

$$\begin{aligned} \frac{d}{ds} \left\{ \frac{W_1(s)}{W_2(s)} \right\} &= \frac{x^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+1}{2}\right)}{\Gamma(s+1) \Gamma\left(s + \frac{l+n}{2} + 1\right) \Gamma(s+l+t+1) \Gamma(s+n+l+1)} \\ &\quad \times \frac{\Gamma\left(s + \frac{l+n+l+t}{2} + 1\right)}{\Gamma(s+l+n+l+t)} \times \left\{ 2 \ln x + y\left(s + \frac{l+n+1}{2}\right) \right. \\ &\quad + y\left(s + \frac{l+n+l+t+1}{2}\right) + y\left(s + \frac{l+n+l+t}{2} + 1\right) \\ &\quad - y(s+1) - y\left(s + \frac{l+n}{2} + 1\right) \\ &\quad \left. - y(s+l+t+1) - y(s+n+l+1) - y(s+l+n+l+t+1) \right\} \end{aligned} \quad (5.84)$$

can be obtained, or it can arranged as:

$$\begin{aligned} \frac{d}{ds} \left\{ \frac{W_1(s)}{W_2(s)} \right\} &= \frac{\Gamma\left(s + \frac{l+n+1}{2}\right) h_{ln}^{l,t} x^{2s+l+n+l+t}}{\Gamma\left(s + \frac{l+n}{2} + 1\right)} \times \left\{ 2 \ln x + y\left(s + \frac{l+n+1}{2}\right) \right. \\ &\quad + y\left(s + \frac{l+n+l+t+1}{2}\right) + y\left(s + \frac{l+n+l+t}{2} + 1\right) - y(s+1) \\ &\quad \left. - y\left(s + \frac{l+n}{2} + 1\right) - y(s+l+t+1) - y(s+n+l+1) - y(s+l+n+l+t+1) \right\} \end{aligned} \quad (5.85)$$

where,

$$h_{ln}^{l,t} = \frac{\Gamma\left(k + \frac{l+n+l+t+1}{2}\right) \Gamma\left(k + \frac{l+n+l+t+2}{2}\right)}{\Gamma(k+1) \Gamma(k+l+n+l+t+1) \Gamma(k+n+l+1) \Gamma(k+l+t+1)}. \quad (5.86)$$

According to (5.21) for $h_{kln}^{l,t}$

$$h_{kln}^{l,t} = (-1)^k h_{ln}^{l,t}(k) \quad (5.87)$$

can be written and by substituting (5.86) into (5.77)

$$\begin{aligned} b_1 = & (-1)^k \frac{1}{2p} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right)}{\Gamma\left(k + \frac{l+n}{2} + 1\right)} h_{ln}^{l,t} x^{2k+l+n+l+t} \times \left\{ 2 \ln x + y \left(k + \frac{l+n+1}{2} \right) \right. \\ & + y \left(k + \frac{l+n+l+t+1}{2} \right) + y \left(k + \frac{l+n+l+t}{2} + 1 \right) - y(k+1) \\ & \left. - y \left(k + \frac{l+n}{2} + 1 \right) - y(k+l+t+1) - y(k+n+l+1) - y(k+l+n+l+t+1) \right\} \end{aligned} \quad (5.88)$$

can be obtained. Then by using Eq. (5.87),

$$\begin{aligned} b_1 = & \frac{1}{2p} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right)}{\Gamma\left(k + \frac{l+n}{2} + 1\right)} h_{kln}^{l,t} x^{2k+l+n+l+t} \times \left\{ 2 \ln x + y \left(k + \frac{l+n+1}{2} \right) \right. \\ & + y \left(k + \frac{l+n+l+t+1}{2} \right) + y \left(k + \frac{l+n+l+t}{2} + 1 \right) - y(k+1) \\ & \left. - y \left(k + \frac{l+n}{2} + 1 \right) - y(k+l+t+1) - y(k+n+l+1) - y(k+l+n+l+t+1) \right\}. \end{aligned} \quad (5.89)$$

is found. Now let Eq. (5.89) and Eq. (5.69) be substituted into Eq. (5.60), to obtain the desired solution such as,

$$\begin{aligned}
R_{ln2}^{E_2}(t, l) = & \sum_{k=0}^{\frac{l+n}{2}-1} \frac{1}{2p} \frac{\Gamma\left(-k + \frac{l+n}{2}\right) \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k + \frac{l+t+1}{2}\right)}{\Gamma(k+1) \Gamma\left(k - \frac{l+n}{2} + n + l + 1\right)} \\
& \times \frac{\Gamma\left(k + \frac{l+t}{2} + 1\right)}{\Gamma\left(k + \frac{l+n}{2} + l + t + 1\right) \Gamma\left(k - \frac{l+n}{2} + l + t + 1\right)} e^{2k+l+t} W_n \\
& - \sum_{k=0}^{\infty} \frac{1}{2p} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right)}{\Gamma\left(k + \frac{l+n}{2} + 1\right)} h_{kln}^{l,t} e^{2k+l+n+l+t} \times \left\{ 2 \ln x + y \left(k + \frac{l+n+1}{2}\right) \right. \\
& + y \left(k + \frac{l+n+l+t+1}{2}\right) + y \left(k + \frac{l+n+l+t}{2} + 1\right) - y(k+1) \\
& \left. - y \left(k + \frac{l+n}{2} + 1\right) - y(k+l+t+1) - y(k+n+l+1) - y(k+l+n+l+t+1) \right\}
\end{aligned} \tag{5.90}$$

where,

$$\begin{aligned}
W_n &= 0, & l+n &= 0 \\
W_n &= 1, & l+n &\neq 0.
\end{aligned} \tag{5.91}$$

5.1.4. Evaluation of $R_{ln2}^{E_1}(t, l)$

The expression of $R_{ln2}^{E_1}(t, l)$ is,

$$R_{ln2}^{E_1}(t, l) = \int_1^\infty J_{l+t}(xt) J_{n+l}(xt) \frac{\sqrt{t^2-1}}{t^{l+t}} dt.$$

It is obvious that $l+n = \text{even}$ in this case also. Again we will use the contour integral representation of product of the Bessel functions Eq. (5.38). Again the contour must be chosen such that, (see Figure 5.3.)

$$-\operatorname{Re} \frac{l+n+l+t+1}{2} < c < 0.$$

For $\operatorname{Re} l \geq 1$ and $\operatorname{Re} t \geq 1$.

$$-\operatorname{Re} \frac{l+n+3}{2} < c < 0. \quad (5.92)$$

Hence in this case $l=t=1$. By substituting Eq. (5.38) into the equation of $R_{ln2}^{E_2}(t, l)$

$$\begin{aligned} R_{ln2}^{E_1}(t, l) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)}{\sqrt{p}} \frac{\Gamma\left(s + \frac{l+n+l+t+1}{2}\right)}{\Gamma(s+l+t+1)} \\ &\times \frac{\Gamma\left(s + \frac{l+n+l+t+2}{2}\right)(xt)^{2s+l+n+l+t}}{\Gamma(s+n+l+1)\Gamma(s+l+n+l+t+1)} \int_1^\infty t^{2s+l+n} \sqrt{t^2-1} dt ds \end{aligned} \quad (5.93)$$

is obtained. To solve the second integral in above equation the following assumption is used

$$t = r^{\frac{1}{2}}, \quad dt = -\frac{1}{2} r^{-\frac{3}{2}} dr. \quad (5.94)$$

So,

$$\int_1^\infty t^{2s+l+n} \sqrt{t^2-1} dt = \frac{1}{2} \int_0^1 r^{-s-\frac{(l+n)}{2}-2} \sqrt{1-r} dr. \quad (5.95)$$

The integral on the right-side of equation given above can be evaluated in terms of Gamma function if the following conditions are again satisfied.

$$\operatorname{Re}\left(-s - \frac{l+n}{2} - 1\right) > 0, \quad \operatorname{Re}\left(s + \frac{l+n}{2} + 1\right) < 0. \quad (5.96)$$

Since,

$$b(x, y) = \int_0^1 r^{x-1} (1-r)^{y-1} dr. \quad (5.97)$$

If $\operatorname{Re}\{x\} > 0$ and $\operatorname{Re}\{y\} > 0$. Since s lies on the contour so,

$$\operatorname{Re}(s) = c \quad (5.98)$$

from Eq. (5.96),

$$\begin{aligned} c + \frac{l+n}{2} + 1 &< 0, \\ c &< -\frac{l+n}{2} - 1. \end{aligned} \quad (5.99)$$

To satisfy both Eq. (5.99) and Eq. (5.92), the contour must be chosen as follows

$$-\frac{l+n+3}{2} < c < -\frac{l+n+2}{2}. \quad (5.100)$$

So Eq. (5.95) can be expressed as

$$\int_1^\infty t^{2s+l+n} \sqrt{t^2-1} dt = \frac{1}{2} b\left(-s - \frac{l+n}{2} - 1, \frac{3}{2}\right) \quad (5.101)$$

and by using Eq. (5.27) in Eq. (5.101)

$$\int_1^\infty t^{2s+l+n} \sqrt{t^2-1} dt = \frac{\sqrt{p}}{4} \frac{\Gamma\left(-s-\frac{(l+n)}{2}-1\right)}{\Gamma\left(-s-\frac{l+n}{2}+\frac{1}{2}\right)} \quad (5.102)$$

is obtained. From Eq. (5.52) it can be found that

$$\frac{1}{\Gamma\left(-s-\frac{l+n}{2}+\frac{1}{2}\right)} = \frac{\Gamma\left(s+\frac{l+n}{2}+\frac{1}{2}\right) \sin p \left(s-\frac{l+n}{2}+\frac{1}{2}\right)}{p} \quad (5.103)$$

and,

$$\Gamma\left(-s-\frac{l+n}{2}-1\right) = \frac{p}{\Gamma\left(s+\frac{l+n}{2}+2\right) \sin p \left(-s-\frac{l+n}{2}-1\right)}. \quad (5.104)$$

If Eq. (5.103) and Eq. (5.104) are substituted into Eq. (5.102)

$$\begin{aligned} \int_1^\infty t^{2s+l+n} \sqrt{t^2-1} dt &= \frac{\sqrt{p}}{4} \frac{p}{\Gamma\left(s+\frac{l+n}{2}+2\right) \sin p \left(-s-\frac{l+n}{2}-1\right)} \\ &\quad \times \frac{\Gamma\left(s+\frac{l+n}{2}+\frac{1}{2}\right) \sin p \left(s-\frac{l+n}{2}+\frac{1}{2}\right)}{p} \end{aligned} \quad (5.105)$$

is derived. From trigonometric identities it is obvious that $\sin(A-p/2) = -\cos A$ and $\sin(A+p) = -\sin A$. By using these identities in Eq. (5.105) it can be obtained that,

$$\int_1^\infty t^{2s+l+n} \sqrt{t^2-1} dt = \frac{\sqrt{p}}{4} \frac{\Gamma\left(s + \frac{l+n}{2} + \frac{1}{2}\right) \cos p \left(s + \frac{l+n}{2}\right)}{\Gamma\left(s + \frac{l+n}{2} + 2\right) \sin p \left(s + \frac{l+n}{2}\right)}. \quad (5.106)$$

Substitute Eq. (5.106) into Eq. (5.93) to get,

$$\begin{aligned} R_{ln2}^{E_1}(t, l) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)}{\sqrt{p}} \frac{\Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+2}{2}\right)}{\Gamma(s+l+t+1) \Gamma(s+n+l+1)} \\ &\quad \times \frac{\sqrt{p}}{4} \frac{(xt)^{2s+l+n+l+t} \Gamma\left(s + \frac{l+n+1}{2}\right) \cos p \left(s + \frac{l+n}{2}\right)}{\Gamma(s+l+n+l+t+1) \Gamma\left(s + \frac{l+n}{2} + 2\right) \sin p \left(s + \frac{l+n}{2}\right)} ds. \end{aligned} \quad (5.107)$$

Furthermore, for negative arguments if Eq. (5.52) is used in Eq. (5.107)

$$\begin{aligned} R_{ln2}^{E_1}(t, l) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p \cos p \left(s + \frac{l+n}{2}\right)}{4 \sin p s \sin p \left(s + \frac{l+n}{2}\right)} \frac{\Gamma\left(s + \frac{l+n+l+t+1}{2}\right)}{\Gamma(s+1) \Gamma(s+l+t+1)} \\ &\quad \times \frac{\Gamma\left(s + \frac{l+n+l+t+2}{2}\right) \Gamma\left(s + \frac{l+n+1}{2}\right) (x)^{2s+l+n+l+t}}{\Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) \Gamma\left(s + \frac{l+n}{2} + 2\right)} \end{aligned} \quad (5.108)$$

is obtained. By introducing the following functions

$$\begin{aligned} W_1(s) &= \Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t}{2} + 1\right) \\ &\quad \times \Gamma\left(s + \frac{l+n+1}{2}\right) (xt)^{2s+l+n+l+t} \end{aligned} \quad (5.109)$$

and,

$$W_3(s) = \Gamma(s+n+l+1)\Gamma(s+l+n+l+t+1)\Gamma(s+1) \\ \times \Gamma(s+l+1+t)\Gamma\left(s+\frac{l+n}{2}+2\right). \quad (5.110)$$

Eq. (5.108) can be written in terms of $W_1(s)$ and $W_3(s)$ as follows:

$$R_{ln2}^{E_1}(t, l) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p \cosh\left(s+\frac{l+n}{2}\right)}{4 \sinh p s \sinh\left(s+\frac{l+n}{2}\right)} \frac{W_1(s)}{W_3(s)} ds \quad (5.111)$$

From trigonometric identities it can be written as

$$\cosh\left(s+\frac{l+n}{2}\right) = \cosh p s \cosh\left(\frac{l+n}{2}\right) = (-1)^{\frac{l+n}{2}} \cosh p s \quad (5.112)$$

and

$$\sinh\left(s+\frac{l+n}{2}\right) = \sinh p s \cosh\left(\frac{l+n}{2}\right) = (-1)^{\frac{l+n}{2}} \sinh p s. \quad (5.113)$$

By substituting Eq. (5.112) and Eq. (5.113) into Eq. (5.111) it can be derived that

$$R_{ln2}^{E_1}(t, l) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p \cosh p s}{4 \sinh^2 p s} \frac{W_1(s)}{W_3(s)} ds. \quad (5.114)$$

The pole of the integrand in Eq. (5.114) are, (see appendix)

double poles at $s = k, \quad (k = 0, 1, 2, \dots)$

simple poles at $s = k - \frac{l+n}{2} \quad (k = -1, 0, 1, 2, \dots)$.

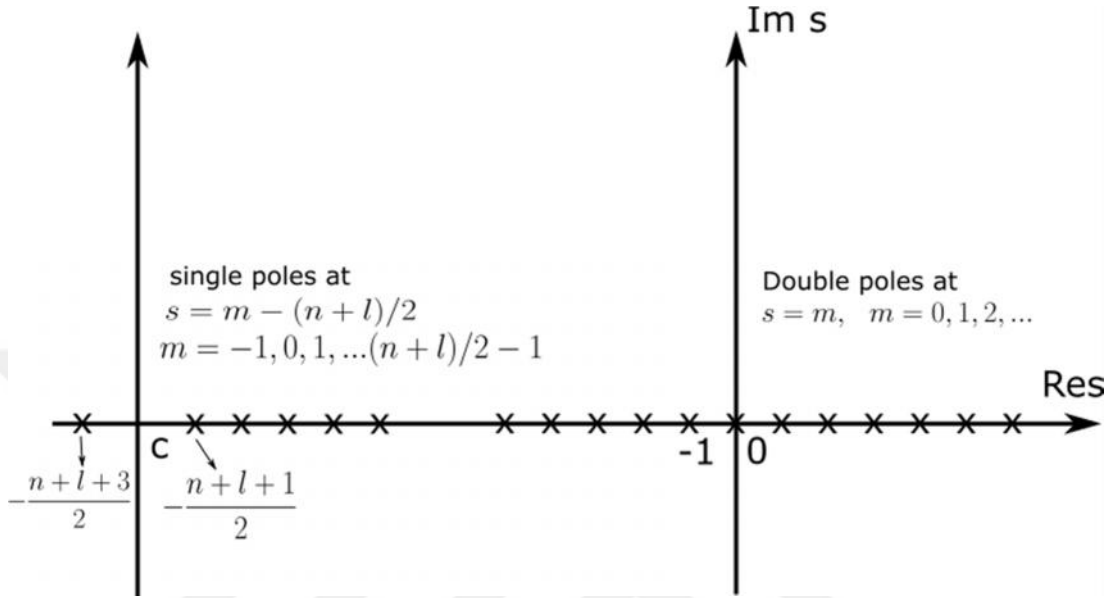


Figure 5.3. Location of poles of the integrand of $R_{ln2}^{E_1}$

So from residue theorem

$$R_{ln2}^{E_1}(t, l) = - \sum_{k=-1}^{\frac{l+n}{2}-1} \text{Residues of } \left\{ \frac{p \cos p s W_1(s)}{4 \sin^2 p s W_3(s)} \right\} \Big|_{s=k-\frac{l+n}{2}} \quad (5.115)$$

$$- \sum_{k=0}^{\infty} \text{Residues of } \left\{ \frac{p \cos p s W_1(s)}{4 \sin^2 p s W_3(s)} \right\} \Big|_{s=k}.$$

Firstly the residues (a_1) at $s = k - \frac{l+n}{2}$ will be calculated as follows.

$$a_1 = \lim_{s \rightarrow k - \frac{l+n}{2}} \left\{ \left(s - k + \frac{l+n}{2} \right) \frac{p \cos p s W_1(s)}{4 \sin^2 p s W_3(s)} \right\} \quad (5.116)$$

or,

$$\begin{aligned}
a_1 &= \lim_{s \rightarrow k - \frac{l+n}{2}} \left(s - k + \frac{l+n}{2} \right) \frac{\rho \csc \rho \left(s + \frac{l+n}{2} \right)}{4 \sin \rho \left(s + \frac{l+n}{2} \right)} \\
&\times \lim_{s \rightarrow k - \frac{l+n}{2}} \frac{1}{\sin \rho s \Gamma(s+1)} \\
&\times \frac{\Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k + \frac{l+t+1}{2}\right)}{\Gamma(k+2) \Gamma\left(k - \frac{l+n}{2} + l+t+1\right) \Gamma\left(k - \frac{l+n}{2} + n+l+1\right)} \\
&\times \frac{\Gamma\left(k + \frac{l+t}{2} + 1\right)}{\Gamma\left(k + \frac{l+n}{2} + l+t+1\right)}.
\end{aligned} \tag{5.117}$$

From Eq. (5.69) a_1 can be evaluated as,

$$\begin{aligned}
a_1 &= -\frac{1}{4\rho} \frac{\Gamma\left(-k + \frac{l+n}{2}\right) \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k + \frac{l+t+1}{2}\right)}{\Gamma(k+2) \Gamma\left(k - \frac{l+n}{2} + l+t+1\right) \Gamma\left(k - \frac{l+n}{2} + n+l+1\right)} \\
&\times \frac{\Gamma\left(k + \frac{l+t}{2} + 1\right) e^{2k+l+t}}{\Gamma\left(k + \frac{l+n}{2} + l+t+1\right)}.
\end{aligned} \tag{5.118}$$

Now the residue b_1 at $s = k$ will be calculated as

$$b_1 = \lim_{s \rightarrow k} \frac{d}{ds} \left\{ (s-k)^2 \frac{\rho \csc \rho s}{4 \sin^2 \rho s} \frac{W_1(s)}{W_3(s)} \right\}. \tag{5.119}$$

The differentiation in above equation will be

$$\begin{aligned} \frac{d}{ds} \left\{ (s-k)^2 \frac{\mathbf{p} \cos \mathbf{p} s}{4 \sin^2 \mathbf{p} s} \frac{W_1(s)}{W_3(s)} \right\} &= (-1)^k \frac{d}{ds} \left\{ (s-k)^2 \frac{\mathbf{p} \cos \mathbf{p} (s-k)}{4 \sin^2 \mathbf{p} (s-k)} \right\} \frac{W_1(s)}{W_3(s)} \\ &+ (s-k)^2 \frac{\mathbf{p} \cos \mathbf{p} s}{4 \sin^2 \mathbf{p} (s-k)} \frac{d}{ds} \frac{W_1(s)}{W_3(s)}. \end{aligned} \quad (5.120)$$

From the previous section the first differentiation will be

$$\lim_{s \rightarrow k} \frac{\mathbf{p}}{2} (s-k) = 0. \quad (5.121)$$

Then the residue b_1 is

$$b_1 = \frac{1}{4} (-1)^k \frac{d}{ds} \left\{ \frac{W_1(s)}{W_3(s)} \right\} \Big|_{s=k}. \quad (5.122)$$

The expression $\frac{d}{ds} \left\{ \frac{W_1(s)}{W_3(s)} \right\}$ can be evaluated by using Eq. (5.78). The only difference between $W_2(s)$ and $W_3(s)$ is the term $\Gamma\left(s + \frac{l+n}{2} + 1\right)$ which appeared in $W_2(s)$ and $\Gamma\left(s + \frac{l+n}{2} + 2\right)$ which appeared in $W_3(s)$. Therefore by replacing $y\left(s + \frac{l+n}{2} + 1\right)$ in the solution of Eq. (5.78) by $y\left(s + \frac{l+n}{2} + 2\right)$ it can be obtain that,

$$\begin{aligned}
\frac{d}{ds} \left\{ \frac{W_1(s)}{W_3(s)} \right\} &= \frac{\Gamma\left(s + \frac{l+n+1}{2}\right) h_{ln}^{l,t} x^{2s+l+n+l+t}}{\Gamma\left(s + \frac{l+n}{2} + 2\right)} \times \left\{ 2 \ln x + y\left(s + \frac{m+n+1}{2}\right) \right. \\
&\quad + y\left(s + \frac{l+n+l+t+1}{2}\right) + y\left(s + \frac{l+n+l+t}{2} + 1\right) - y(s+1) \\
&\quad \left. - y\left(s + \frac{l+n}{2} + 2\right) - y(s+l+t+1) - y(s+n+l+1) - y(s+l+n+l+t+1) \right\}
\end{aligned} \tag{5.123}$$

where,

$$h_{kln}^{l,t} = \frac{\Gamma\left(k + \frac{l+n+l+t+1}{2}\right) \Gamma\left(k + \frac{l+n+l+t+2}{2}\right)}{\Gamma(k+1) \Gamma(k+l+n+l+t+1) \Gamma(k+n+l+1) \Gamma(k+l+t+1)}. \tag{5.124}$$

According to Eq. (5.21)

$$h_{kln}^{l,t} = (-1)^k h_{ln}^{l,t}(k) \tag{5.125}$$

and so

$$\begin{aligned}
b_1 &= (-1)^k \frac{1}{4p} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right)}{\Gamma\left(k + \frac{l+n}{2} + 2\right)} h_{ln}^{l,t} x^{2k+l+n+l+t} \times \left\{ 2 \ln x \right. \\
&\quad + y\left(s + \frac{l+n+1}{2}\right) + y\left(s + \frac{l+n+l+t+1}{2}\right) + y\left(s + \frac{l+n+l+t}{2} + 1\right) - y(s+1) \\
&\quad \left. - y\left(s + \frac{l+n}{2} + 2\right) - y(s+l+t+1) - y(s+n+l+1) - y(s+l+n+l+t+1) \right\}
\end{aligned} \tag{5.126}$$

or by using Eq. (5.125),

$$\begin{aligned}
b_1 = & \frac{1}{2p} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right)}{\Gamma\left(k + \frac{l+n}{2} + 2\right)} h_{ln}^{l,t} x^{2k+l+n+l+t} \times \left\{ 2 \ln x \right. \\
& + y \left(s + \frac{l+n+1}{2} \right) + y \left(s + \frac{l+n+l+t+1}{2} \right) + y \left(s + \frac{l+n+l+t}{2} + 1 \right) - y(s+1) \\
& \left. - y \left(s + \frac{l+n}{2} + 2 \right) - y(s+l+t+1) - y(s+n+l+1) - y(s+l+n+l+t+1) \right\}.
\end{aligned}
\tag{5.127}$$

Finally by substituting a_1 and b_1 into Eq. (5.115)

$$\begin{aligned}
R_{ln2}^{E_1}(t, l) = & \sum_{k=-1}^{\frac{l+n}{2}-1} \frac{1}{4p} \frac{\Gamma\left(-k + \frac{l+n}{2}\right) \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k + \frac{l+t+1}{2}\right)}{\Gamma(k+2) \Gamma\left(k - \frac{l+n}{2} + l+t+1\right)} \\
& \times \frac{\Gamma\left(k + \frac{l+t}{2} + 1\right)}{\Gamma\left(k + \frac{l+n}{2} + l+t+1\right) \Gamma\left(k - \frac{l+n}{2} + n+l+1\right)} e^{2k+l+t} \\
& - \sum_{k=0}^{\infty} \frac{1}{4p} \frac{\Gamma\left(k + \frac{l+n+1}{2}\right)}{\Gamma\left(k + \frac{l+n}{2} + 2\right)} h_{kln}^{l,t} e^{2k+l+n+l+t} \times \left\{ 2 \ln x + y \left(k + \frac{l+n+1}{2} \right) \right. \\
& + y \left(k + \frac{l+n+l+t+1}{2} \right) + y \left(k + \frac{l+n+l+t}{2} + 1 \right) - y(k+1) \\
& \left. - y \left(k + \frac{l+n}{2} + 2 \right) - y(k+l+t+1) - y(k+n+l+1) - y(k+l+n+l+t+1) \right\}
\end{aligned}
\tag{5.128}$$

is obtained as the desired solution.

5.2. Evaluation of $d_{ln}^{E_1}$ and $d_{ln}^{E_2}$

The integral on the left-hand-side of Eq. (3.54) can be divided as,

$$\begin{aligned} d_{ln}^{E_1} &= \int_{-\infty}^{\infty} \frac{J_{l+t}(xb) J_n(xb)}{b^t} db \\ &= \int_{-\infty}^0 \frac{J_{l+t}(xb) J_n(xb)}{b^t} db + \int_0^{\infty} \frac{J_{l+t}(xb) J_n(xb)}{b^t} db \end{aligned} \quad (5.129)$$

or

$$d_{ln}^{E_1} = \int_0^{\infty} \frac{J_{l+t}(-xb) J_n(-xb)}{(-b)^t} db + \int_0^{\infty} \frac{J_{l+t}(xb) J_n(xb)}{b^t} db. \quad (5.130)$$

Bessel function with a negative argument appeared in the first integral can be evaluated as follows:

$$J_l(-z) = \left(\frac{-z}{2}\right)^l \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{-z}{2}\right)^{2n}}{\Gamma(n+1)\Gamma(n+l+1)} \quad (5.131)$$

and

$$J_l(-z) = (-1)^l \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{\Gamma(n+1)\Gamma(n+l+1)}. \quad (5.132)$$

So

$$J_l(-z) = (-1)^l J_l(z) \quad (5.133)$$

and by Eq. (5.133) into Eq. (5.130)

$$d_{ln}^{E_1} = (-1)^{l+n} \int_0^\infty \frac{J_{l+t}(xb) J_n(xb)}{(b)^t} db + \int_0^\infty \frac{J_{l+t}(xb) J_n(xb)}{b^t} db \quad (5.134)$$

or

$$d_{ln}^{E_1} = [1 + (-1)^{l+n}] \int_0^\infty \frac{J_{l+t}(xb) J_n(tb)}{(b)^t} db. \quad (5.135)$$

From (Prudnikov, A.P., 1983, p.211) we have that

$$\int_0^\infty x^{a-1} J_m(c, t) J_v(c, t) dx = 2^{a-1} c^{-a} \frac{\Gamma(1-a) \Gamma\left(\frac{a+v+m}{2}\right)}{\Gamma\left(1+\frac{m-v-a}{2}\right) \Gamma\left(1+\frac{m+v-a}{2}\right) \Gamma\left(1+\frac{v-m-a}{2}\right)}. \quad (5.136)$$

By substituting $a = -t + 1$, $m = 1 + t$, $v = n$, $x = b$, and $c = x$ it can be obtained that

$$d_{ln}^{E_1} = \frac{1}{x} \left(\frac{x}{2}\right)^t \frac{[1 + (-1)^{l+n}] \Gamma(t) \Gamma\left(\frac{n+l+1}{2}\right)}{\Gamma\left(\frac{n-l+1}{2}\right) \Gamma\left(\frac{l-n+1+2t}{2}\right) \Gamma\left(\frac{n+l+2t+1}{2}\right)}. \quad (5.137)$$

To obtain the expression for $d_{ln}^{E_2}$, rewrite the first integral in Eq. (3.65) in the form of

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db \\
&= \int_{-\infty}^0 \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db + \int_0^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db.
\end{aligned} \tag{5.138}$$

And by changing the order of the limits and by changing b by $-b$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db \\
&= \int_0^{\infty} \frac{J_{n+1}(-xb) J_{l+1}(-xb)}{b^2} db + \int_0^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db.
\end{aligned} \tag{5.139}$$

By using Eq. (5.133) in Eq. (5.139) it can be obtained that:

$$\int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db = \left\{1 + (-1)^{n+l}\right\} \int_0^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db. \tag{5.140}$$

Again by using Eq. (5.136) with $a = -1$, $m = n+1$, $v = l+1$, $x = b$ and $c = x$

$$\int_0^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db = \frac{1}{4} x \frac{\Gamma(2) \Gamma\left(\frac{l+n+1}{2}\right)}{\Gamma\left(1 + \frac{n-l+1}{2}\right) \Gamma\left(1 + \frac{n+l+3}{2}\right) \Gamma\left(1 + \frac{l-n-1}{2}\right)}, \tag{5.141}$$

or

$$\int_0^\infty \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db = \frac{x}{4} \frac{\Gamma\left(\frac{l+n+1}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{n-l}{2}\right) \Gamma\left(2 + \frac{n+l+1}{2}\right) \Gamma\left(\frac{3}{2} + \frac{l-n}{2}\right)}. \quad (5.142)$$

Since,

$$\Gamma(1+z) = z\Gamma(z) \quad (5.143)$$

and

$$\begin{aligned} \Gamma\left(2 + \frac{n+l+1}{2}\right) &= \left(1 + \frac{n+l+1}{2}\right) \Gamma\left(1 + \frac{n+l+1}{2}\right) \\ &= \left(1 + \frac{n+l+1}{2}\right) \left(\frac{n+l+1}{2}\right) \Gamma\left(\frac{n+l+1}{2}\right). \end{aligned} \quad (5.144)$$

Again from (Prudnikov, A.P., 1983, p.721)

$$\Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos \pi z} \quad (5.145)$$

since,

$$\begin{aligned} \Gamma\left(\frac{3}{2} + \frac{n-l}{2}\right) \Gamma\left(\frac{3}{2} + \frac{l-n}{2}\right) &= \Gamma\left(\frac{3}{2} + \frac{n-l}{2}\right) \Gamma\left(\frac{3}{2} - \frac{n-l}{2}\right) \\ \Gamma\left(\frac{3}{2} + \frac{n-l}{2}\right) \Gamma\left(\frac{3}{2} + \frac{l-n}{2}\right) &= \Gamma\left(1 + \frac{1}{2} + \frac{n-l}{2}\right) \Gamma\left(1 + \frac{1}{2} - \frac{n-l}{2}\right) \end{aligned} \quad (5.146)$$

using Eq. (5.144) and Eq. (5.145) in Eq. (5.146) yields that,

$$\begin{aligned}
& \Gamma\left(1+\frac{1}{2}+\frac{n-l}{2}\right)\Gamma\left(1+\frac{1}{2}-\frac{n-l}{2}\right) \\
&= \left(\frac{1}{2}+\frac{n-l}{2}\right)\Gamma\left(\frac{1}{2}+\frac{n-l}{2}\right)\left(\frac{1}{2}-\frac{n-l}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n-l}{2}\right) \\
&= \left\{\frac{1}{4}-\left(\frac{n-l}{2}\right)^2\right\}\Gamma\left(\frac{1}{2}+\frac{n-l}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n-l}{2}\right) \\
&= \left\{\frac{1}{4}-\left(\frac{n-l}{2}\right)^2\right\}\frac{p}{\cosp\left(\frac{n-l}{2}\right)} = (-1)^{\frac{n-1}{2}} p \left\{\frac{1}{4}-\left(\frac{n-l}{2}\right)^2\right\}. \tag{5.147}
\end{aligned}$$

It is obvious that

$$\cosp\left(\frac{n-l}{2}\right) = \begin{cases} +1 & \frac{n-l}{2} > 0 \\ -1 & \frac{n-l}{2} < 0 \end{cases} \tag{5.148}$$

By substituting Eq. (5.146) and Eq. (5.147) into Eq. (5.142)

$$\begin{aligned}
& \int_0^\infty \frac{J_{n+1}(xb)J_{l+1}(xb)}{b^2} db = \\
& \frac{x}{4} \frac{\Gamma\left(\frac{l+n+1}{2}\right)}{\left(1+\frac{n+l+1}{2}\right)\left(\frac{n+l+1}{2}\right)\Gamma\left(\frac{n+l+1}{2}\right)\left(\frac{1-(n-l)^2}{4}\right)\cosp\left(\frac{n-l}{2}\right)} \tag{5.149}
\end{aligned}$$

or

$$\int_0^\infty \frac{J_{n+1}(xb)J_{l+1}(xb)}{b^2} db = \frac{4x \cosp\left(\frac{n-l}{2}\right)}{p(n+l+3)(n+l+1)(1-(n-l)^2)}. \tag{5.150}$$

is obtained. From Eq. (5.150) it is clear that if $l + n = \text{odd}$

$$\int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db = 0, \quad (5.151)$$

and if $l + n = \text{even}$

$$\int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db = 2 \int_0^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db. \quad (5.152)$$

By using Eq. (5.152) in Eq. (5.150)

$$\int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db = \frac{8x \cos p \left(\frac{n-l}{2} \right)}{p(n+l+3)(n+l+1) \{1 - (n-l)^2\}} \quad (5.153)$$

or

$$\int_{-\infty}^{\infty} \frac{J_{n+1}(xb) J_{l+1}(xb)}{b^2} db = \frac{8x (-1)^{\frac{n-l}{2}}}{p(n+l+3)(n+l+1) \{1 - (n-l)^2\}}. \quad (5.154)$$

can be obtained as desired solution.



6. RESULTS

The approach method used in this thesis is a hybrid method named as analytical-numerical method. In general, the aim of using hybrid methods is to eliminate the disadvantages of the analytical methods which operate well at high frequencies and of the numerical methods which operate well at low frequencies. Stated in other words its aim is, to obtain an accurate solution for a wide frequency range. By comparing the results obtained by Veliev et al. (1998) and the ones obtained by using this method it can be easily concluded that our results are much close to results obtained by Veliev et al. (1998). Figure 6.1, Figure 6.2, Figure 6.3, and Figure 6.4 illustrate the monostatic RCS as a function of incidence angle for $ka=5.0, 15.0$. In order to investigate the effect of the surface impedance on the scattered far field, two different cases have been considered as in $\eta=1.5$ and $\eta=3.0$. In view of the two RCS curves for the impedance strip, the backscattered field is not affected by the impedance of the strip surface in the shadow region. On the other hand, Figure 6.5, Figure 6.6, Figure 6.7, and Figure 6.8 illustrate the bistatic RCS as a function of observation angle for $\theta=60^\circ$ and $ka=5.0, 15.0$. It is seen from the figures that our RCS results agree quite well with the results of Veliev et al. (1998).

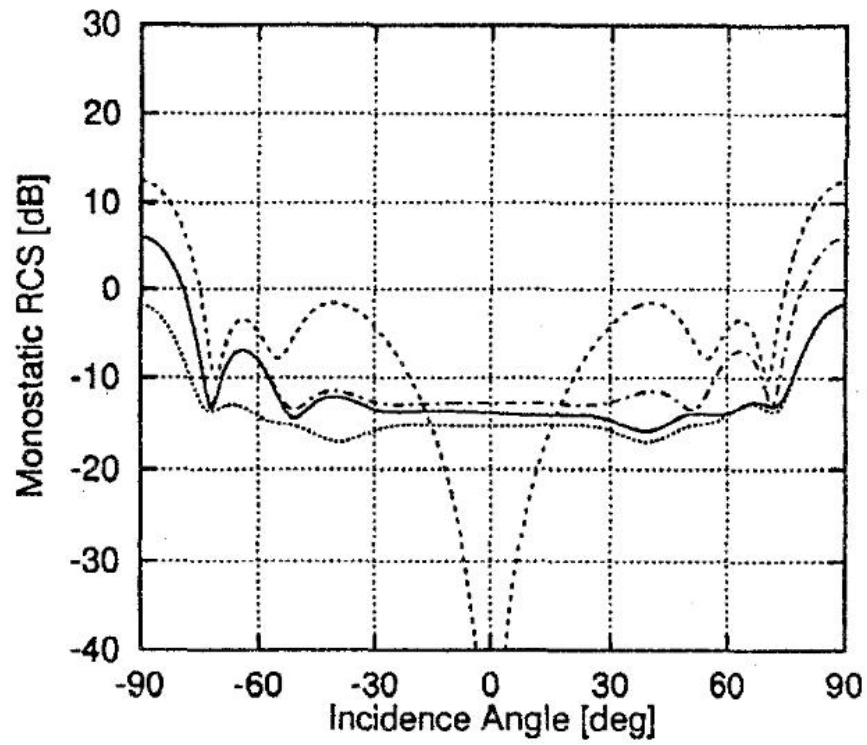


Figure 6.1. Monostatic RCS [dB], $q_0 = 60^\circ$, $ka=5.0$. ----: $h_1 = h_2 = 0.0$; —: $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; -.-.- : $h_1 = h_2 = 3.0$ (Veliev et al. (1998))

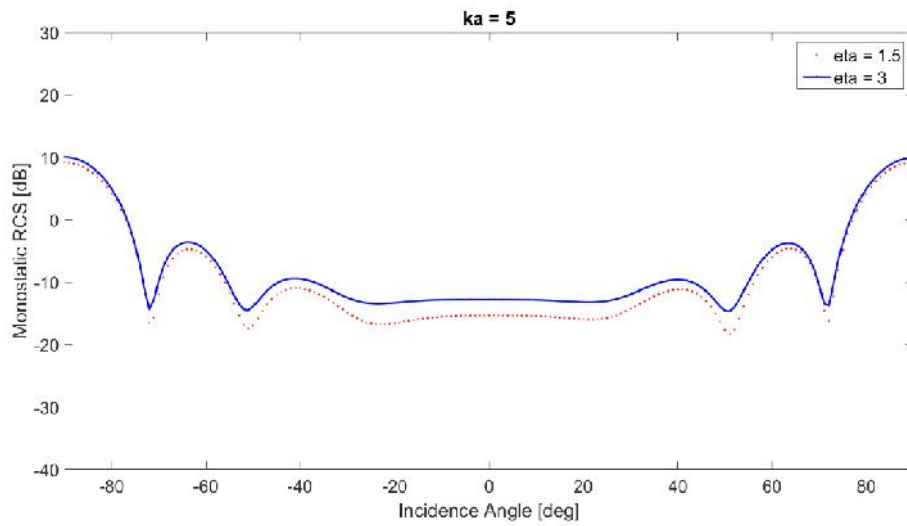


Figure 6.2. Monostatic RCS [dB], $ka=5.0$ ($h = 1.5$ and $h = 3.0$)

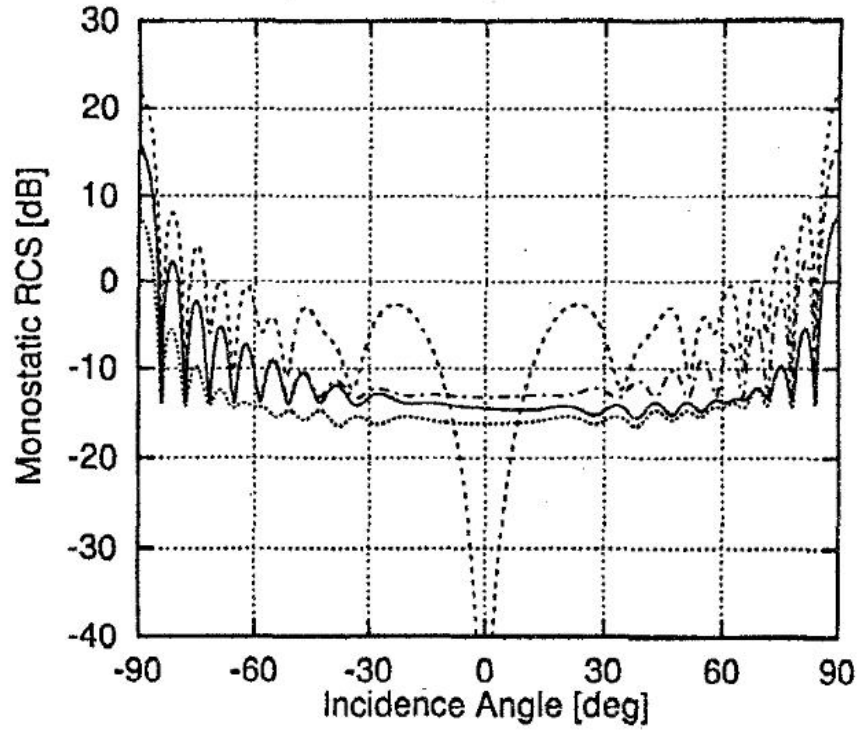


Figure 6.3. Monostatic RCS [dB], $q_0 = 60^\circ$, $ka = 15.0$. ----: $h_1 = h_2 = 0.0$; —: $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; -.-.- : $h_1 = h_2 = 3.0$ (Veliev et al. (1998))

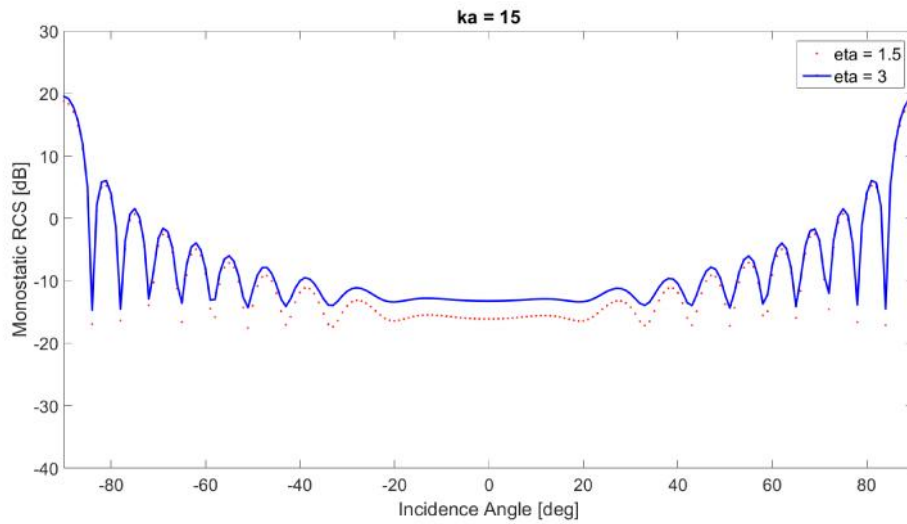


Figure 6.4. Monostatic RCS [dB], $ka = 15.0$ ($h = 1.5$ and $h = 3.0$)

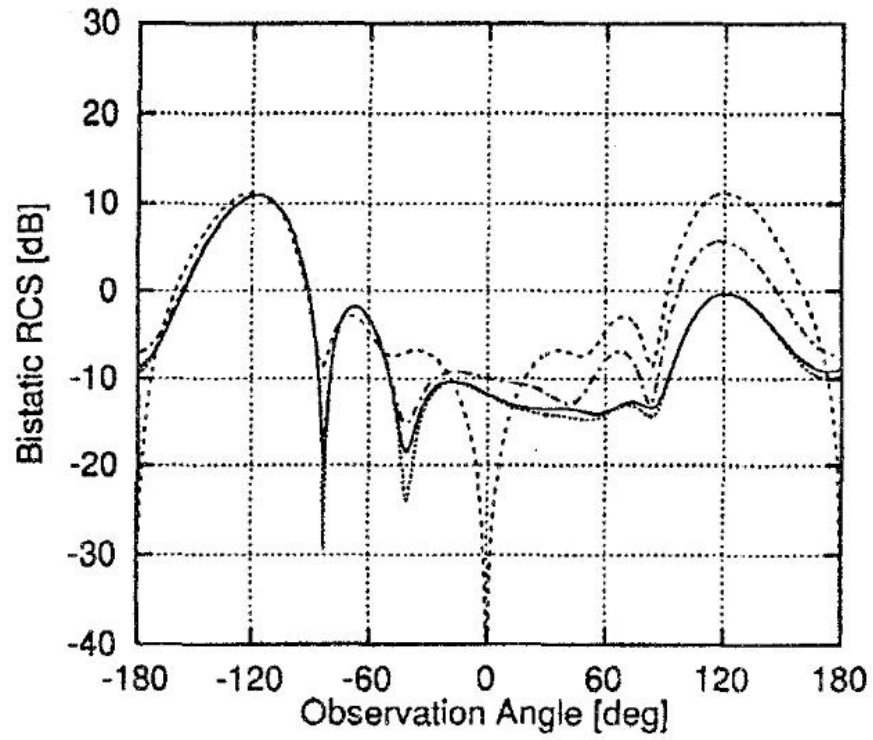


Figure 6.5. Bistatic RCS [dB], $q_0 = 60^\circ$, $ka = 5.0$. ----: $h_1 = h_2 = 0.0$; —: $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; -.-.- : $h_1 = h_2 = 3.0$ (Veliev et al. (1998))

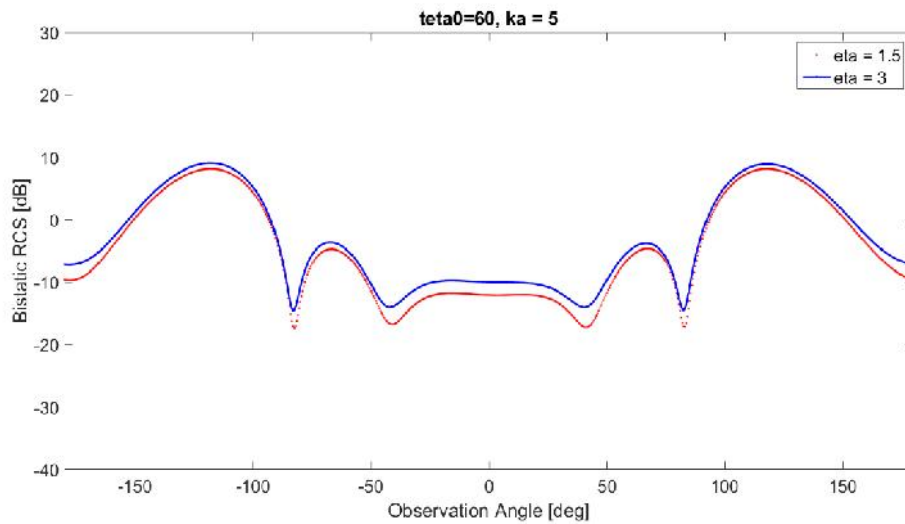


Figure 6.6. Bistatic RCS [dB], $ka = 5.0$ ($h = 1.5$ and $h = 3.0$)

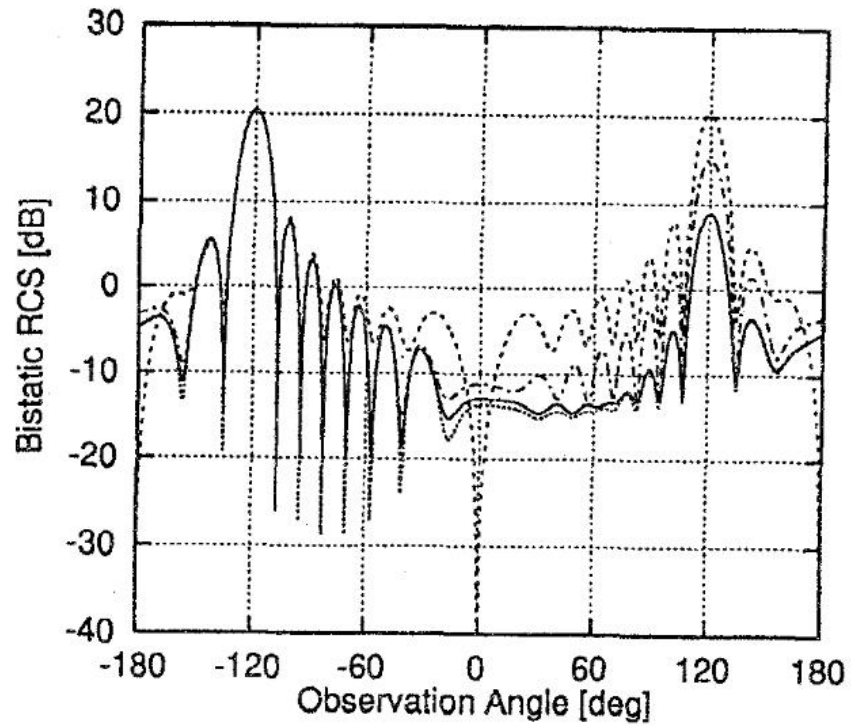


Figure 6.7. Bistatic RCS [dB], $q_0 = 60^\circ$, $ka = 15.0$. ----: $h_1 = h_2 = 0.0$; ———: $h_1 = 1.5, h_2 = 3.0$; : $h_1 = h_2 = 1.5$; -.-.- : $h_1 = h_2 = 3.0$ (Veliev et al. (1998))

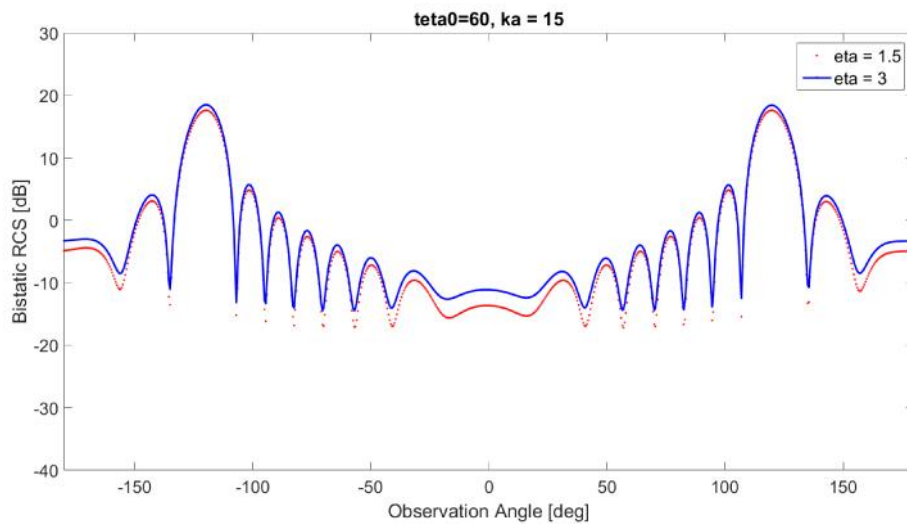


Figure 6.8. Bistatic RCS [dB], $ka = 15.0$ ($h = 1.5$ and $h = 3.0$)

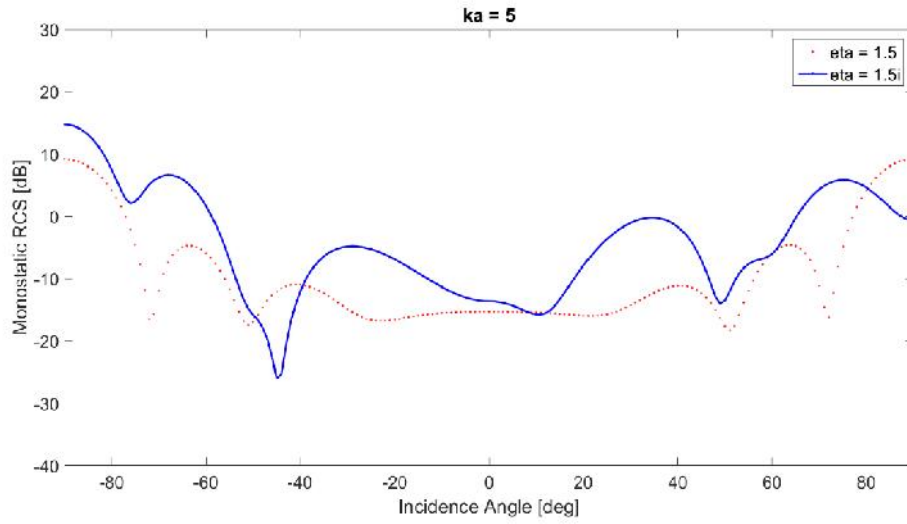


Figure 6.9. Monostatic RCS [dB], $ka=5.0$ ($h = 1.5$ and $h = 1.5i$)

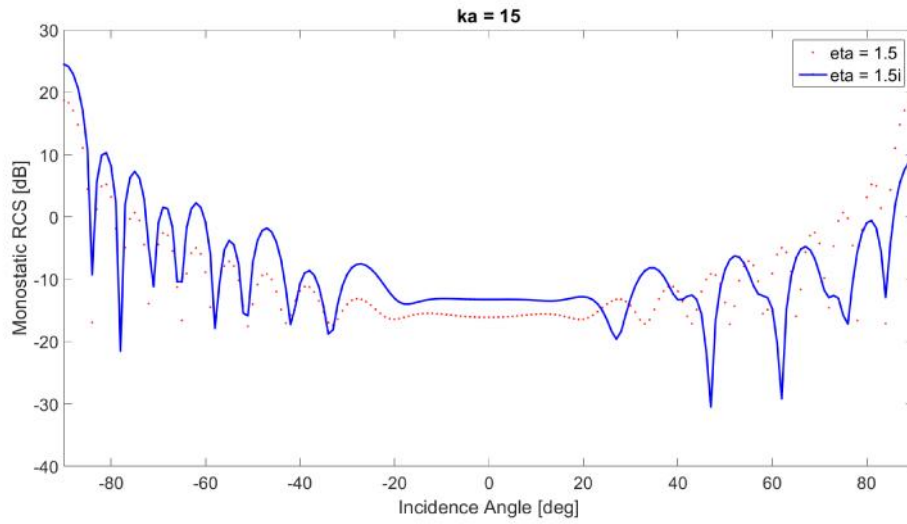


Figure 6.10. Monostatic RCS [dB], $ka=15.0$ ($h = 1.5$ and $h = 1.5i$)

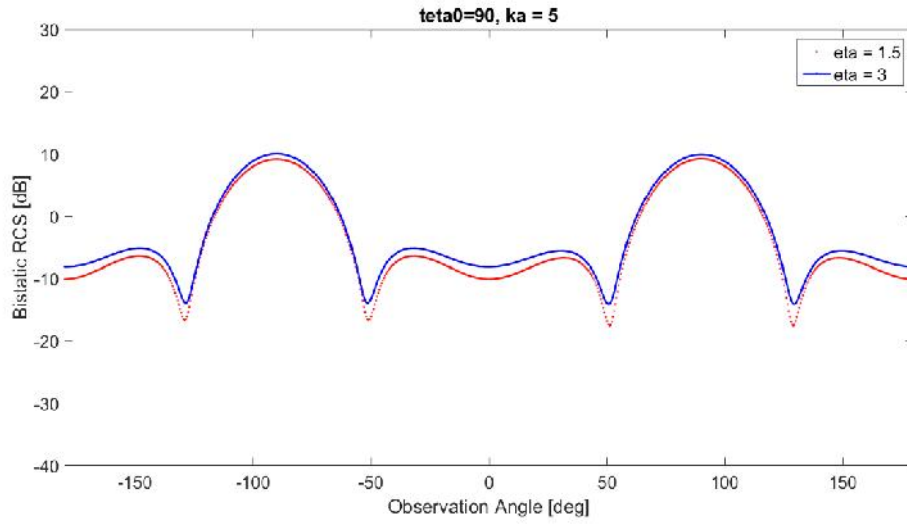


Figure 6.11. Bistatic RCS [dB], $q_0 = 90^\circ$, $ka=5.0$ ($h = 1.5$ and $h = 3.0$)

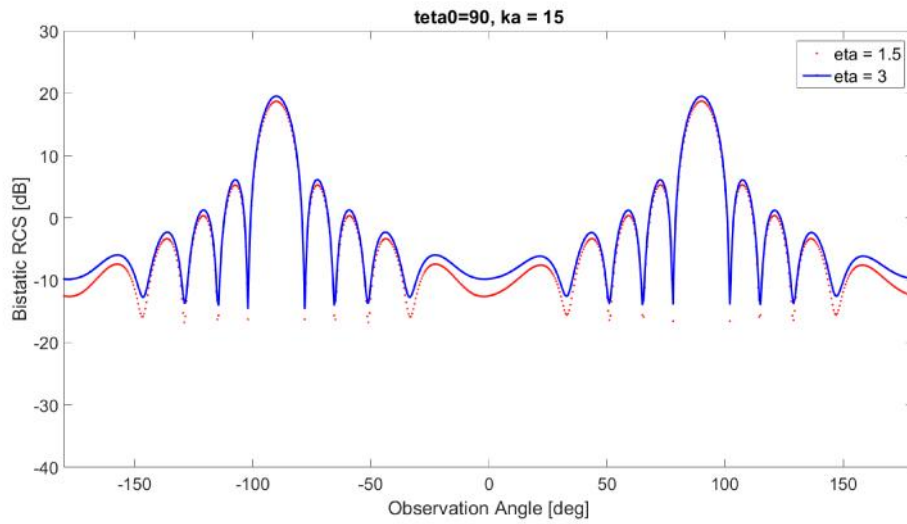


Figure 6.12. Bistatic RCS [dB], $q_0 = 90^\circ$, $ka=15.0$ ($h = 1.5$ and $h = 3.0$)

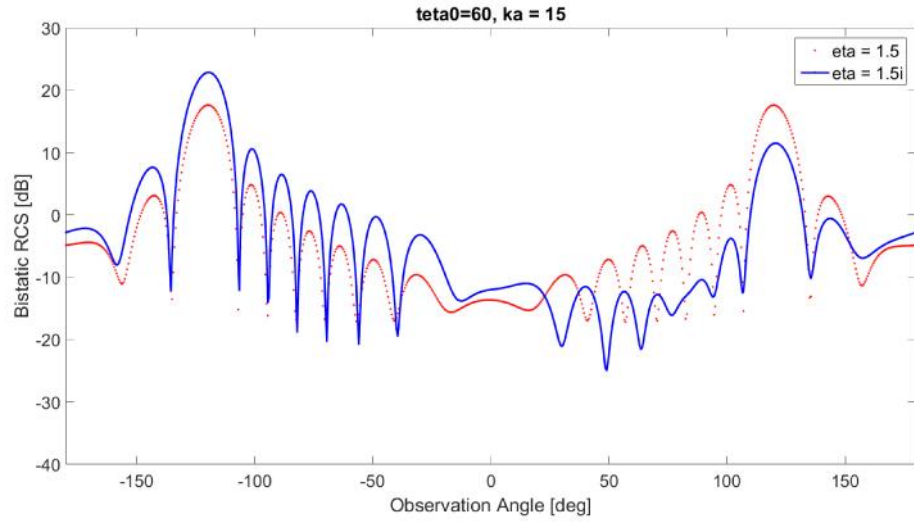


Figure 6.13. Bistatic RCS [dB], $q_0 = 60^\circ$, $ka=15.0$ ($h = 1.5$ and $h = 1.5i$)

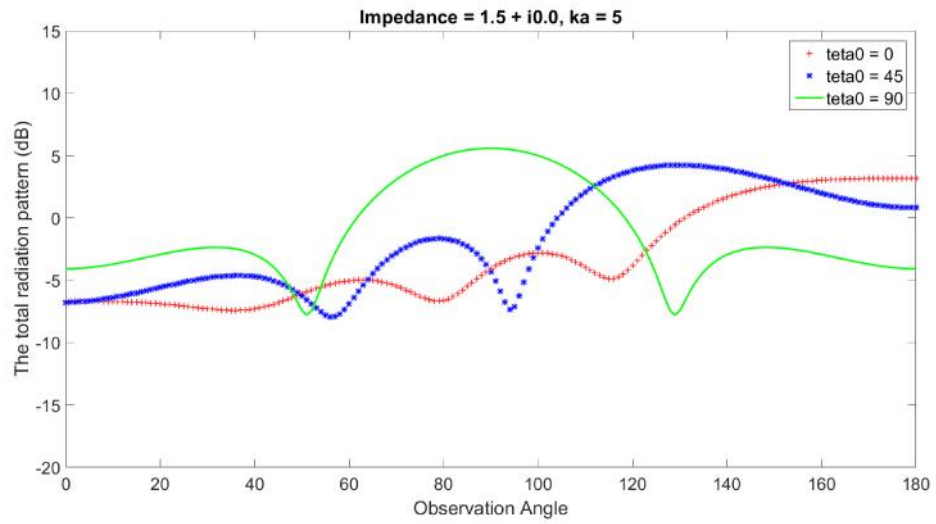


Figure 6.14. The total radiation pattern for $ka=5$

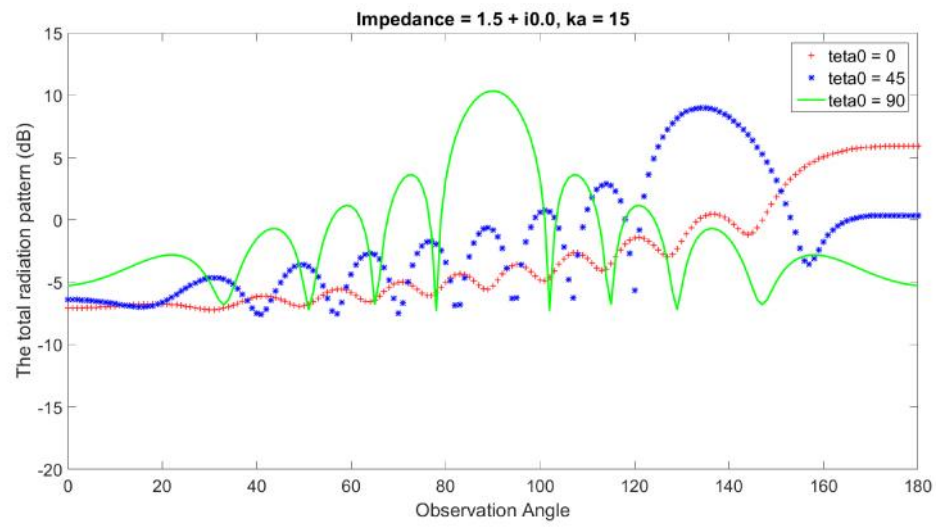


Figure 6.15. The total radiation pattern for $ka=15$



7. CONCLUSION

In this thesis, the diffraction of the H-polarized plane wave by an infinitely long strip, having the same impedance on both faces with a width of $2a$ is considered. Applying the boundary condition to an integral representation of the scattered field, the problem is formulated as simultaneous integral equations satisfied by the electric and magnetic current density functions. The integral equations are reduced to two uncoupled infinite systems of linear algebraic equations and physical quantities are obtained in terms of the solution of systems of linear algebraic equations. Numerical examples on the monostatic radar cross section (RCS), bistatic RCS, and the total scattering field radiation pattern are presented. Some obtained results are compared with the other existing results.

The system which is considered in this thesis is the simple impedance strip illuminated normally by a plane wave. The canonical strip structure is chosen in terms of its conformity to many practical problems. This method is applicable for the analysis of more complicated structures which may be considered as a combination of different strip configurations. At that rate, the mathematical sense of the solution will not change. However, there will be some extra terms in the matrix equations. Because of this, the computer codes in this thesis must be modified for different strip configurations.



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CURRICULUM VITAE

Emine was born in Adana on January 28th, 1985. She graduated from Adana Erkek High School (Foreign Language) in 2003 and entered to Electrical-Electronics Engineering Department of Mustafa Kemal University in 2004. She joined to Abant İzzet Baysal University as a research assistant on December 5th, 2008. She was charged with a duty to the Çukurova University, Electrical-Electronics Department. At 2010 she completed my MSc thesis and I started to my PhD thesis at February 2011. She has been working there as a research assistant since March 2009.





APPENDIX



APPENDIX

A1: Singularities of the integrand $R_{\ln 2}^{E_1}$

As seen from Eq.(5.56), integrals have singularities due to the poles of the integrands. Only some of them contribute to the integrals because some of the poles are eliminated by the zeros or some of the poles are located in the left hand side of the contour in complex s-plane. This appendix is devoted to explain the arguments used for the calculations of the following integrals.

$$\begin{aligned}
 R_{\ln 2}^{E_1}(t, l) = & \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p \cosh\left(s + \frac{l+n}{2}\right)}{2 \sinh p s \sinh\left(s + \frac{l+n}{2}\right)} \\
 & \times \frac{\Gamma\left(s + \frac{l+n+l+t+1}{2}\right) \Gamma\left(s + \frac{l+n+l+t+2}{2}\right)}{\Gamma(s+n+l+1) \Gamma(s+l+n+l+t+1) \Gamma(s+1)} \\
 & \times \frac{\Gamma\left(s + \frac{l+n+1}{2}\right) (x)^{2s+l+n+l+t}}{\Gamma(s+l+1+t) \Gamma\left(s + \frac{l+n+2}{2}\right)} ds
 \end{aligned}$$

The function $\Gamma[s + (l+n+l+t+1)/2]$ has simple poles located at

$$s + \frac{l+n+l+t+1}{2} = -m \text{ with } m=0,1,\dots$$

which gives the poles as:

$$s = -m - \frac{l+n+l+t+1}{2}; \quad l > 0, t > 0$$

Since $\text{Res} < 0$, all poles of this function are located on the left hand side of the integration contour which means that they have no contribution due to the direction of the integration contour.

Similarly, the function $\Gamma[s + (l + n + l + t + 1) / 2]$ has simple poles located at

$$s = -m - \frac{l + n + l + t + 2}{2}; \text{ with } m=0,1,\dots$$

which again gives the poles located on the left hand side of the integration contour; therefore, they do not contribute to the integral.

On the other hand, the function $\Gamma[s + (l + n + 1) / 2]$ has simple poles located at

$$s = -m - \frac{l + n + 1}{2}; \text{ with } m=0,1,\dots$$

and the function $\csc[s + (l + n) / 2]$ has simple zeros located at

$$s = -m - \frac{l + n}{2} - \frac{1}{2}; \text{ with } m=0,1,\dots$$

So, it can easily be seen that the poles of $\Gamma[s + (l + n + 1) / 2]$ are cancelled by the zeros of $\csc[s + (l + n) / 2]$.

The Gamma functions $\Gamma(s + l + l + t) \Gamma(s + n + l + 1) \Gamma(s + l + n + l + t + 1)$ in the denominator do have poles but they are the zeros of the integrand, therefore they do not give any singular behavior to the integral.

The functions $\Gamma(s+1)$ and $\sin \pi s$ which are in the denominator should be considered together. As is well known $\Gamma(s+1)$ has simple poles located at $s+1=-m$ or $s=-m-1$ with $m=0, 1, 2, \dots$. Some of the simple zeros of $\sin \pi s$ are at the same points. So the poles of $1/\sin \pi s$ located at $s=m$ ($m=0, 1, 2, \dots$) are cancelled by the simple

zeros of the function $1/\Gamma(s+1)$. The total contribution to the integral is due to the poles of $1/\sin \pi s$ located at

$$s = m \text{ with } m=0,1,2,\dots$$

Similarly the function $[\sin \pi (s + (l+n)/2)]^{-1}$ has simple poles located at

$$s = m - \frac{l+n}{2} \text{ with } m=0,1,2,\dots$$

and,

$$s = -m - \frac{l+n}{2} - 1 \text{ with } m=0,1,2,\dots$$

But the function $1/[\Gamma(s + (l+n)/2 + 1)]$ has simple zeros located at

$$s = -m - \frac{l+n}{2} - 1 \text{ with } m=0,1,2,\dots$$

So the only poles which contribute to the integral are located at

$$s = m - \frac{l+n}{2} \text{ with } m=0,1,2,\dots$$

Finally it is easy to conclude that the poles on the right hand side of the contour are the simple poles of $(\sin \pi s)^{-1}$ located at $s=m$, $m=0,1,2,\dots$ and the simple poles of $[\sin \pi (s + (l+n)/2)]^{-1}$ located at $s = m - (n+l)/2$. This means that if $n+l=0$ the integrand has only double poles located at $s=m$, $m=0, 1, 2,\dots$

If $n \neq 0$, $l \neq 0$ and $n+l = \text{even}$ the poles of $(\sin \pi s)^{-1}$ are at

$$s=0, 1, 2, \dots$$

And the poles of $[\sin p(s + (l + n) / 2)]^{-1}$ are at

$$s = -\frac{n+l}{2}, 1 - \frac{n+l}{2}, 2 - \frac{n+l}{2}, 0, 1, 2, \dots$$

From these two equation we conclude that the integrand has double poles at

$$s=m \quad m=0, 1, 2, \dots$$

and single poles at

$$s = m - \frac{n+l}{2} \quad m = 0, 1, 2, \dots, \frac{n+l}{2} - 1$$

As seen in Figure 5.2.

A2: Singularities of the integrand $R_{\ln 2}^{E_2}$

$$\begin{aligned} R_{\ln 2}^{E_2}(t, l) = & -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p \cos p s}{4 \sin^2 p s} \frac{\Gamma(s + \frac{l+n+l+t+1}{2})}{\Gamma(s+1)\Gamma(s+l+n+t+1)} \\ & \times \frac{\Gamma(s + \frac{l+n+l+t+2}{2})\Gamma(s + \frac{l+n+1}{2})(x)^{2s+l+n+l+t}}{\Gamma(s+n+l+1)\Gamma(s+l+n+l+t+1)\Gamma(s + \frac{l+n}{2} + 2)} \\ & \times \frac{\Gamma(s + \frac{l+n+1}{2})(x)^{2s+l+n+l+t}}{\Gamma(s+l+1+t)\Gamma(s + \frac{l+n+2}{2})} ds \end{aligned}$$

The poles of the function $\Gamma[s + (l + n + l + t + 1) / 2]$ are located at

$$s = -m - \frac{n+l+l+t+2}{2}; \text{ with } m=0,1,2,\dots$$

which are on the left hand side of the integral contour and they do not contribute to the integral.

The poles of the function $\Gamma[s+(l+n+l+t+2)/2]$ are located at

$$s = -m - \frac{l+n+1}{2}; \text{ with } m=0,1,\dots$$

They do not contribute to the integral also due to the reason explained above. The poles of the function $\Gamma[s+(l+n+1)/2]$ are located at

$$s = -m - \frac{l+n+1}{2}; \text{ with } m=0,1,2,\dots$$

But $\csc s$ has simple zeros at $s = -m - (l+n+1)/2$, since $l+n = \text{even}$. So the poles of $\Gamma[s+(l+n+1)/2]$ are cancelled by the simple zeros of $\csc s$.

The function $\sin^2 s$ has double poles at $s = \pm m, m = 0,1,2,\dots$, but the function $[\Gamma(s+1)]^{-1}$ has simple zeros at $s = -m-1, m = 0,1,2,\dots$. So the poles located between 0 and $-(n+l)/2-1$ are the simple poles. Finally the poles on the right hand side of the contour are located at

$$s = m \quad m = 0,1,2,\dots \quad \text{double poles}$$

$$s = -1, -2, -3, \dots, -\frac{n+l}{2} - 1$$

So

$$s = m - \frac{n+l}{2} \quad m = -1, 0, 1, \dots, \frac{n+l}{2} - 1$$

It is obvious that if $n+l=0$ there will be only one simple pole located at $s = -1$ as seen in Figure 5.3.

