DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOME METHODS OF CATEGORY THEORY IN THE REPRESENTATION THEORY OF ARTIN ALGEBRAS

by
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İZMİR

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We have read the thesis entitled "**SOME METHODS OF CATEGORY THEORY IN THE REPRESENTATION THEORY OF ARTIN ALGEBRAS**" completed by **VÍCTOR BLASCO JIMÉNEZ** under supervision of **PROF. DR. NOYAN FEVZI ER** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

During the last decades there has been a significant development in the representation theory of Artin algebras due to different techniques which have been proved to be fruitful while dealing with problems of that nature. The main aim of this Master Thesis is to discuss some of them and see how they can be applied to a particular open problem of our own choice. Such a problem is the conjecture established in the end of the book ”Representation theory of Artin algebras” by Auslander, Reiten and Smalo which asserts that an Artin algebra with the property that its finitely generated indecomposable modules are completely determined by its composition factors is of finite representation type. Examples of rings with this property are the semisimple artinian rings and the rings of the form $\mathbb{Z}_n$. We will be able to prove the conjecture for some special cases which include the commutative, the hereditary and the $J^2 = 0$ case. Indeed, in the first two cases we will show that the converse also holds. We will also give a generic example of an Artin algebra with $J^2 = 0$ and of finite representation type but whose finitely generated indecomposable modules are not completely determined by their composition factors.

Keywords: Representation theory, Artin algebra, composition series, finite representation type

ARTIN CEBİRLERİNİN TEMSİL TEORİSİNDEKİ BAZI KATEGORİ TEORİ METOTLARI

ÖZ

Son yıllarda, artin cebirlerinin temsil kuramının doğasında var olan problemlerin üstesinden gelmede verimli olduğu kanıtlanmış yeni teknikler sayesinde bu alanda önemli gelişmeler olmuştur. Bu yüksek lisans tezinin temel amacı, bu tekniklerin bazılarını tartışmak ve kendi seçeceğimiz belirli bir açık probleme nasıl uygulanabileceğini görmektir. Böyle bir problem, Auslander, Reiten ve Smalo'nun "Artin cebirlerinin Temsil kuramı" kitabında verilmiş olan ve sonlu üretilmiş ayrıştırılamaz modülleri, kendi bileşke faktörleri tarafından tamamen belirlenebilen bir Artin modülünün sonlu temsil tiplii olduğunu ileri süren bir önsavdır. Bu özelliği sağlayan halka örnekleri olarak yarıbasit halkalar ve $\mathbb{Z}_n$ şeklinde gösterilen halkalar verilebilir. Değişmeli, kalıtsal ve $J^2 = 0$ durumlarını da içeren özel durumlar için konjektürü kanıtlayabiliyor olacağız. Dahası, ilk iki durumda tersinin de doğru olduğunu göstereceğiz. Ayrıca, sonlu temsil tipli fakat sonlu üreteçli ayrıştırılamaz modüllerinin tamamıyla kendisinin bileşke çarpanları tarafından belirlenmediği $J^2 = 0$ özelliğini sağlayan genel bir Artin cebiri örnek vereceğiz.

Anahtar kelimeler: Temsil kuramı, Artin cebir, bileşke serisi, sonlu temsil tipi

CONTENTS

LIST OF ABBREVIATIONS

f.g. : Finitely generated

PIR : Principal ideal rings

dim : Dimension

LIST OF SYMBOLS

mod$-R$	: Category of f.g. right modules over the ring R
$R-$mod	: Category of f.g. left modules over the ring R
J	: Jacobson radical
Λ	: Artin algebra
$rep_K Q$	: Category of finite-dimensional representations of Q
$m_S(M)$	: multiplicty of S as a composition factor of M
$\mathfrak{S}(M)$	: Vector with $m_{S_i}(M)$ in coordinate i
$\mathfrak{X}$	: Indecomposable modules are completely determined by composition factors

CHAPTER ONE
INTRODUCTION

During the last fifty years there has been an important development in the represention theory of Artin algebras. The new techniques have helped not only to generalize many results about finite-dimensional algebras to Artin algebras, but also to give a new insight into the structure of Artin algebras itself and, as a consequence, into the one of finite-dimensional algebras. Most part of these techniques have been developed in one way or another with the help of category theory, but two main different approaches have been taken. On the one hand, the approach that has lead, for example, the theory of quiver representations is to try to reduce a question about algebras and modules to a question about vector spaces and linear maps, and use after it some categorical argument such as an equivalence of categories to see that both questions are indeed equivalent. In this way, a relatively abstract question can be reduced to a question about linear algebra, which is normally easier to deal with. On the other hand, the approach taken in the general Auslander-Reiten theory is to systematically use homological algebra and develop categorical notions such as dualities to be able to express and attack problems concerning finite-dimensional algebras in the more general setting of Artin algebras. While the approach of quiver representations can help to understand algebras and modules by interpreting them in more concrete terms, the more abstract approach of the Auslander-Reiten theory can help to clarify and understand general properties of those.

In any case, these are just two examples of the different methods that have been recently developed. One of the aims of this Master Thesis is to illustrate some of the main categorical methods that have been developed in the last decades in a clear and pedagogical way. Even if we will be necessarily dealing with category theory, we will try to be as concrete and elementary as possible. In order to do that, whenever is possible, we will avoid abstract categorical arguments and express the main ideas by using basic ring and module theory which we believe can be generally more adequate in our particular context to understand what is going on since, in the end, we are dealing with ring and module theoretical questions. For us category theory will just be a tool, a powerful tool indeed, but not a language itself to express our ideas.

The second and more important aim of this thesis is to see how to apply those methods. We will see how those methods have been applied before in order to prove some well-known theorems but at the same time we will apply those methods ourselves to a particular open problem of our own choice. In this way, not only this thesis is an exposition of the principal categorical methods that have been developed recently in the representation theory of Artin algebras, but also a research effort carried by the author within those methods.

When one starts studying ring theory, one realizes that there are some rings that appear everywhere and, for that reason, the knowledge we have about them is bigger that for other classes of rings. Such an example is the integers and, more generally, the principal ideal domains. In those cases the extended version of the fundamental theorem for abelian groups tells us precisely how finitely generated modules over principal ideal domains decompose into indecomposable pieces, and how these indecomposable pieces look. It is the perfect situation. If R is such a principal ideal domain then the finitely generated indecomposable modules over R are R itself and the ones of the form $R/(p^r)$, where p is any non-zero prime element of R and r is some natural number. Every such $R/(p^r)$ is what is called a finite length module. That is, a module with a composition series, i.e. with a finite increasing filtration by submodules such that each quotient is a simple module. The simple modules are all of the form $R/(p)$ with $0 \neq p$ prime element of R and a composition series for $R/(p^r)$ has every composition factor isomorphic to $R/(p)$, which appears r times. We say in that case that $R/(p)$ is a composition factor of $R/(p^r)$ with multiplicity r. It can be deduced that, in the case of principal ideal domains, we cannot have two non-isomorphic finite length indecomposable modules with the same composition factors and the same multiplicities. Recall that Jordan-Hölder Theorem guarantees that, over any ring, any two composition series for a finite length modules have the same (up to isomorphism) composition factors, so it makes sense to state the property above. One can go further now and ask the question: which rings satisfy that its indecomposable right (resp. left) finite length modules are completely determined by its composition factors, in the sense that we cannot have two non-isomorphic finite length indecomposable modules with the same composition factors and the same multiplicities? For shortened, we will denoted this property by $\mathfrak{X}$ on the right (resp. left) or just by $\mathfrak{X}$ if the context is clear.

Principal ideal domains, and more generally integral domains, are non-artinian rings. In that case, the class of finite length modules is much "smaller" than the one of finitely generated ones, but for artinian rings both classes coincide. Probably the most fundamental examples of artinian rings are the full matrix rings over fields, i.e. rings of the form $M_n(K)$ with K field and $n \in \mathbb{N}$; and quotient rings of $\mathbb{Z}$, i.e. rings of the form $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$. The rings of the form $M_n(K)$ are the main examples of what are called semisimple artinian rings. Those rings are characterized by the fact that they are artinian and every indecomposable right (resp. left) module is simple. Hence, they trivially satisfy $\mathfrak{X}$ on both sides. On the other hand, if $n = p_1^{r_1} p_2^{r_2} ... p_s^{r_s}$ where the $p_i's$ are different prime numbers, then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times ... \times \mathbb{Z}_{p_s^{r_s}}$ as rings, and every such $R_i = \mathbb{Z}_{p_i^{r_i}}$ is a commutative uniserial ring. In that case, the ideals of R_i form a chain and every finitely generated indecomposable module is of the form R_i/I for some ideal I. Hence, since ideals form a chain there is at most one ideal for every length, and the length of such an ideal is bounded by the length of R_i, which is finite because it is an artinian ring. Therefore, each R_i must satisfy $\mathfrak{X}$ (and indeed $\mathbb{Z}_n$ too) because two indecomposable modules with the same composition factors with the same multiplicities have the same length in particular.

Throughout this thesis we will mainly deal with Artin R-algebras. Let R be a commutative artinian ring. A ring Λ is called an Artin R-algebra if there exists a ring homomorphism $R \to Z(\Lambda)$, where $Z(\Lambda)$ denotes the center of Λ, and which makes Λ a finitely generated module over R. If we let R be a field then we get a finite-dimensional algebra, so finite-dimensional algebras are examples of Artin algebras. Another examples are commutative artinian rings themselves. An artinian ring is said to be of finite representation type if, up to isomorphism, there is only a finite number of pairwise non-isomorphic indecomposable modules. Being of finite representation type is a left-right symmetric property, so having only finitely many indecomposable left modules implies having finitely many right modules and viceversa. Examples of classes of rings of finite representation type are $M_n(K)$ and $\mathbb{Z}/n\mathbb{Z}$. The motivation problem for this thesis is the conjecture appearing on (Auslander et al. (1997) p. 409) which can be rephrased as follows: If Λ is an Artin algebra satisfying $\mathfrak{X}$, then Λ is of finite representation type. Recall that since Artin algebras admit a duality from left to right modules and it can be shown that such a duality preserves the property $\mathfrak{X}$, we do not need to specify if we are dealing with left or right modules.

All the discussion in this thesis will be around this conjecture. We will not be able to give a complete answer to the conjecture, but we will show that it is true in three main different cases:

- The first case is when Λ is commutative. In that case we will show that the converse holds indeed, that is, a commutative artinian ring R satisfies $\mathfrak{X}$ if and only if it is of finite representation type. There is a known characterization of commutative artinian rings of finite representation type. Those are called PIR (principal ideal rings) and they are characterized by being commutative artinian rings with every ideal generated by a single element.
- The second case is when Λ is hereditary, i.e. submodules of projective modules are projective. In that case the converse also holds due to (Auslander et al. (1997) Ch. 8 Cor 2.4). That is, we will get that a hereditary Artin algebra Λ is of finite representation if and only if it satisfies $\mathfrak{X}$.
- The last case is when Λ has $J^2 = 0$, where J denotes the Jacobson radical. We will also show that the converse is not true.

The structure of the thesis is the following:

We will start the first chapter by a discussion on principal ideal domains where we will show that the extended version of the fundamental theorem for abelian groups implies that principal ideal domains satisfy $\mathfrak{X}$. After that, we will also see that a more general class of integral domains, namely Dedekind Domains, satisfy $\mathfrak{X}$. In the second part of the first chapter we will deal with commutative artinian rings. We will show first that principal ideal rings satisfy $\mathfrak{X}$, and prove the converse afterwards. The main step towards the proof will be to prove that commutative artinian rings satisfying $\mathfrak{X}$ are self-injective, for which we will use properties of the duality $D = Hom_R(-, E)$, where E denotes the injective hull of $S_1 \oplus ... \oplus S_n$ with $S_1, ..., S_n$ a complete list of simple R-modules. We will also show a more elementary alternative proof for the fact that commutative artinian rings satisfying $\mathfrak{X}$ are PIR, which will not rely directly on properties of D.

The second chapter will begin with a brief introduction to quiver representations. Since Jordan-Hölder theorem holds over any abelian category, it makes sense to define property $\mathfrak{X}$ in the general context of abelian categories. In particular, we will define it for the category of finite-dimensional quiver representations over a quiver Q. In the second section we will introduce the path algebras associated to quivers and see that the category of finite-dimensional representations of a quiver Q over a field K is equivalent to mod-KQ, where KQ is the path algebra associated Q. This will help to translate the discussion about property $\mathfrak{X}$ on KQ to a discussion which just involves vector spaces and linear maps. In the third section we will introduce the basic theory of algebraic bimodules developed in Ringel (1976). We will see under which conditions algebraic bimodules satisfy $\mathfrak{X}$. Also, we will briefly discuss about the second Brauer-Thrall conjecture, which implies in particular that any finite-dimensional algebra over an algebraically closed field satisfying $\mathfrak{X}$ must be of finite representation type, and we will show a counterexample due to Ringel for the general second Brauer-Thrall conjecture for arbitrary artinian rings. In the last section of the chapter we will show Gabriel's theorem and its consequences which imply, due to the fact that any hereditary finite-dimensional algebra over an algebraically closed field is Morita equivalent to a path algebra, that a hereditary finite-dimensional algebra over an algebraically closed field satisfies $\mathfrak{X}$ if and only if it is of finite representation type.

In the last chapter our aim will be to extend that result to arbitrary hereditary Artin algebras. The first thing to do will be to show the proof appearing in Auslander et al. (1997) that hereditary Artin algebras of finite representation type satisfy $\mathfrak{X}$. In order to prove the converse, the first reduction will be to prove that any indecomposable hereditary Artin algebra is indeed a finite-dimensional K-algebra for some field K. After this reduction we will be able to prove the theorem thanks to some exact embeddings described in Dlab & Ringel (1978) and the theory of K-species developed in Ringel (1976),Dlab & Ringel (1975),Dlab & Ringel (1974). The last part of the chapter will be devoted to prove the conjecture for Artin algebras with J^2. For, we will introduce the stable equivalence of Artin algebras appearing in Auslander et al. (1997); and we will see how, due to the result on hereditary Artin algebras, a particular stable equivalence construction helps to prove the conjecture for the $J^2 = 0$ case. Finally, we will construct a generic family of finite-dimensional algebras s.t. $J^2 = 0$ which are of finite representation type but do not satisfy $\mathfrak{X}$.

The two main resources of ideas for writing this thesis are Auslander et al. (1997) and Ringel (1976). The proof of the commutative case relies on the properties of the duality D described in Auslander et al. (1997). The proof of the hereditary case will be built on the theory of algebraic bimodules developed in Ringel (1976). As well, supplementary articles such as Dlab & Ringel (1975),Dlab & Ringel (1974) and Dlab & Ringel (1978) have helped for proving the hereditary case, specially the full and exact embeddings described in Dlab & Ringel (1978) and the characterization of the connected K-species with no oriented cycles which are of tame type appearing in Dlab & Ringel (1974). The main technique used to prove the result on algebras with $J^2 = 0$ is the stable equivalence of algebras described in Auslander et al. (1997).

CHAPTER TWO
COMMUTATIVE RINGS

2.1 Principal ideal domains

Given a f.g. abelian group A, the fundamental theorem of abelian groups tells us how A decomposes into a direct sum of indecomposable abelian groups, and it is stated as follows:

Theorem 2.1.1. *Let A be a f.g. abelian group, then $A \cong \mathbb{Z}_{p_1^{r_1}} \oplus ... \oplus \mathbb{Z}_{p_n^{r_n}} \oplus \mathbb{Z}^m$ for unique $p_1 < ... < p_n$ prime numbers, $ri, m \in \mathbb{N}$.*

It follows that any f.g. indecomposable abelian group is either isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_{p^r}$ for some prime number p and some natural number r. Furthermore, for such indecomposable $\mathbb{Z}_{p^r}$

$$0 \subseteq p^{r-1}\mathbb{Z}_{p^r} \subseteq ... \subseteq p\mathbb{Z}_{p^r} \subseteq \mathbb{Z}_{p^r} \tag{2.1}$$

is a composition series with every composition factor isomorphic to $\mathbb{Z}_p$. Thus, its length as a $\mathbb{Z}$-module, denoted by $l(\mathbb{Z}_{p^r})$, is $l(\mathbb{Z}_{p^r}) = r$ and its composition factors (up to iso morphism) are $r{*}\mathbb{Z}_p$, meaning with the symbol "$*$" that $\mathbb{Z}_p$ appears r times as a compo sition factor . Hence, two arbitrary $\mathbb{Z}_{p^r}$, $\mathbb{Z}_{q^s}$ have composition factors $r * \mathbb{Z}_p$ and $s * \mathbb{Z}_q$ respectively, and so they have the same composition factors, meaning that every sim ple $\mathbb{Z}$-module appears as a composition factor in $\mathbb{Z}_{p^r}$ as many times as in $\mathbb{Z}_{q^s}$, iff $r = s$ and $p = q$. It follows, since $\mathbb{Z}$ is not finite length (it is not artinian), that finite len gth indecomposable abelian groups are determined by their composition factors in the sense above. This motivates the following definition:

Definition. *Let R be a ring and M a finite length indecomposable right (resp. left) R-module with composition series*

$$0 \subseteq M_1 \subseteq M_2 \subseteq ... \subseteq M_n = M \tag{2.2}$$

We say that a simple right (resp. left) R-module S is a composition factor of M if there exists some j s.t. $S \cong M_{j+1}/M_j$. We denote by $m_S(M)$ the multiplicity of S as a composition factor of M, meaning by that the number of different j's s.t. $S \cong M_{j+1}/M_j$. Then we say that the right (resp. left) finite length indecomposable modules over R are determined by composition factors if, for any two finite length indecomposable modules M, N, $m_S(M) = m_S(N)$ for all simple right (resp. left) R-module $\Rightarrow M \cong N$. In that case we say that R satisfies the property $\mathfrak{X}$ on the right (resp. left). If it satisfies it in both sides (for example in the commutative case) or if it is clear that we are working only with right or left modules, we just say that R satisfies $\mathfrak{X}$.

Remark. *Recall that Jordan-Hölder theorem holds over any ring, which guarantees that any two composition series for a finite length modulele over R have the same composition factors (up to isomorphism), and so $m_S(M)$ is well-defined.*

Being a non-artinian ring is a common property of integral domains which are not fields:

Lemma 2.1.2. *Any integral domain R which is not a field is non-artinian.*

Proof. Since R is not a field, there exists some non-invertible $0 \neq a \in R$, and the ideal generated by a is properly contained in R. Consider now the descending chain

$$aR \supseteq a^2R \ldots \tag{2.3}$$

and assume there exists $n \in \mathbb{N}$ s.t. $a^nR = a^{n+1}R$. This means in particular that there is some b in R s.t. $a^n = a^{n+1}b$ and so

$$a^n = a^{n+1}b \iff a^n(1-ab) = 0 \iff a^n = 0 \text{ or } ab = 1 \tag{2.4}$$

as R is an integral domain. Since a is non-zero, the only possibility is $ab = 1$, but this contradicts a being non-invertible. Hence, we conclude that it is a strictly descending chain and so R is non-artinian. $\square$

The fundamental theorem of abelian groups is important not only because it gives a full description of the f.g. indecomposable $\mathbb{Z}$-modules but also because it extends naturally to principal ideal domains (PID) in the following way:

Theorem 2.1.3. *Let R be a PID and M be a f.g.module over R. Then there exist unique (up to order) prime elements $p_1, ..., p_n$ in R and unique natural numbers $r_1, ..., r_n, m$ s.t.*

$$M \cong R/(p_1^{r_1}) \oplus ... \oplus R/(p_n^{r_n}) \oplus R^m \tag{2.5}$$

where $(p_i^{r_i})$ denotes the ideal generated by $p_i^{r_i}$ in R.

Hence, it follows from the previous lemma that the only possible finite length modules over a PID are necessarily of the form $R/(p^r)$ for $p \in R$ prime and $r \in \mathbb{N}$. Indeed, such amodule is of finite length and have composition series of the form

$$0 \subseteq p^{r-1}R/(p^r) \subseteq ... \subseteq pR/(p^r) \subseteq R/(p^r) \tag{2.6}$$

with every composition factor isomorphic to $R/(p)$, which is simple since (p) is maximal as p is prime, and so irreducible. Since p is uniquely determined, it follows that $R/(p) \cong R/(q)$ implies $p = q$, and so the same argument as for abelian groups holds in terms of the property $\mathfrak{X}$. That is,

Proposition 2.1.4. *Every PID satisfies $\mathfrak{X}$*

Classical examples of PID's are polynomial rings of the form $K[X]$, where K is a field. Thus, we have found a property that holds on its indecomposable finite length modules. For that, the classical fundamental theorem of abelian groups has given us all the required information we needed, and so we wonder if a similar structure theorem holds in a more general class of integral domains containing PID's. Let us first notice that a proper ideal in an integral domain is never artinian, and so it is not of finite length. For, since it is proper it contains a non-invertible element and so the same argument than in the previous lemma holds. Thus, over such rings, it is only required to study indecomposable modules which are not ideals. Having this in mind we can get the result for what are called Dedekind domains, which is a class of integral domains containing PID's, and which are defined as follows.

Definition. *A Dedekind domain is an integral domain for which every non-zero ideal is projective, that is, a hereditary domain.*

A Dedekind domain may not be a PID, but one can show that any f.g. torsion module (i.e. every element is annihilated by some non-zero divisor $0 \neq r \in R$) which is indecomposable is of the form R/A, where A is an ideal of R (Jacobson (2012) Thm. 10.14 p. 646) . For the torsion-free part we can use the following well-known theorem:

Theorem 2.1.5. *(Jacobson (2012) Thm. 10.14 p. 646) Any f.g. torsion-free module over a Dedekind domain is isomorphic to a finite direct sum of ideals.*

We can write any f.g.module over a Dedekind domain as a direct sum of a torsion module and a torsion-free module. By using these results together with chinese reminder theorem we get the following:

Theorem 2.1.6. *Let M be a f.g.module over a Dedekind domain R. Then there exist prime ideals $P_1, ..., P_n$, natural numbers $r_1, ..., r_n$ and some ideals $I_1, ..., I_r$ s.t.*

$$M \cong R/P_1^{r_1} \oplus ... \oplus R/P_n^{r_n} \oplus I_1 \oplus ... \oplus I_r \tag{2.7}$$

Thus, the only possible finite length indecomposables are of the form R/P^r and since non-zero prime ideals are maximal over Dedekind domains (Jacobson (2012) Prop. 10.5 p. 623) and $P^2 \subseteq P \neq R$ for any prime P, then any simple module is of the form R/P where P is prime. Also, we have the following structural theorem on the ideals over a Dedekind domain:

Theorem 2.1.7. *(Jacobson (2012) Thm. 10.1 p. 623 and paragraph preceeding Prop. 10.4) Let $I \subseteq J \neq R$ be ideals over a Dedekind domain. Then there exists an ideal K*

s.t. $I = JK$*. Furthermore, every proper ideal can be written uniquely as product of prime ideals.*

Then, we can prove the following:

Proposition 2.1.8. *Let* P *be a prime ideal in a Dedekind domain* R *and let* $r \in \mathbb{N}$*. Then,* $0 \subseteq P^{r-1}/P^r \subseteq ... \subseteq P/P^r \subseteq R/P^r$ *is a composition series for* R/P^r *with every composition factor isomorphic to* R/P*.*

Proof. Let us prove first that all the proper submodules of R/P^r are of the form P^s/P^r with $1 \leq s \leq r-1$. For, we know that every submodule of R/P^r is of the form I/P^r where I is an ideal containing P^r. By the previous theorem this means that there exists some ideal J over R s.t. $P^r = IJ$. Then, since we know that P is prime and every proper ideal, in particular P^r, can be written uniquely as a product of prime ideals, it follows that $I = P^s$ for some $1 \leq s \leq r$, as desired. Hence, all the submodules of R/P^r are of the form P^s/P^r and theyform a chain

$$P^{r-1}/P^r \subseteq ... \subseteq P/P^r \subseteq R/P^r \tag{2.8}$$

Thus, every quotient $\frac{P^i/P^r}{P^{i+1}/P^r} \cong P^i/P^{i+1}$ must be simple, and so isomorphic to some R/Q with Q prime since every simple R-module is isomorphic to one of thatform. For any R-module M we denote by $ann_R(M) := \{a \in R : am = 0 \,\forall m \in M\}$ the annihilator of M in R. One can show easily that isomorphicmodules have the same annihilators.

Hence, since $ann_R(R/Q) = Q$ and $ann_R(P^i/P^{i+1}) = P = ann_R(R/P)$, it follows that $Q = P$. □

Proposition 2.1.9. *every Dedekind domain satisfies* $\mathfrak{X}$

Proof. Assume we have two non-zero indecomposables M and N with the same composition factors. By the theorem above we can assume that $M = R/P^r$ and

$N = R/Q^s$. Then we know that the composition factors of M are $r * R/P$ and the ones of N are $s * R/Q$. It follows that $s = r$ and $R/Q \cong R/P$. It just remains to show that this implies $P = Q$.

But $R/Q \cong R/P \Rightarrow P = ann_R(R/P) = ann_R(R/Q) = Q$, and so $M \cong N$. $\square$

Remark. *A very nice property of Dedekind domains is the one stated in Thm 1.7, which said that every non-zero proper ideal factors into a unique product of prime ideals. By using this one could also show that the prime ideals occuring in the decomposition theorem for the torsion-part of a f.g.module are uniquely determined, and so property $\mathfrak{X}$ follows without need of the previous proposition. This property of the non-zero proper ideals plays a fundamental role in algebraic number theory since, for example, rings of integers of finite field extensions $K/\mathbb{Q}$ (i.e. those elements in K which are roots of some monic polyunomial with coefficients in $\mathbb{Z}$) are Dedekind domains, and so even if the unique factorization may not hold at the level of elements we know that it holds at the level of ideals. Historically, this helped Kummer to prove Fermat's last theorem for what are called regular primes.*

Once the property $\mathfrak{X}$ has be proven to hold on Dedekind domains, one wonders about which other classes of integral domains satisfy it, or even if all integral domains satisfy it or not. For example, one could ask if the property $\mathfrak{X}$ holds for rings such as $\mathbb{Z}[X]$ or, more generally, over unique factorization domains. The answer to that does not seem to be clear for the author.

2.2 Commutative artinian rings

We have seen that integral domains are not artinian. In this section, we will study the property $\mathfrak{X}$ in the context of commutative artinian rings and we will be able to give a complete characterization of such rings.

Firstly, recall that since R is artinian then $R/rad(R)$ is an artinian ring with Jacobson radical equal zero, so it is a semisimple ring. Thus, R/J has (up to isomorphism) only finitely many pairwise non-isomorphic simple modules $S_1, ..., S_n$. But J annihilates every simple R-module so every such simple is in fact a simple over R/J and so it must be isomorphic to one of the S_i's, and so R has also finitely many simples up to isomorphism, namely $S_1, ..., S_n$. Now, over artinian rings the class of finite length modules coincides with the one of the finitely generated ones, so given a f.g. R-module M we can associate to it the n-tuple

$$\mathfrak{S}(M) := (m_{S_1}(M), m_{S_2}(M), ..., m_{S_n}(M)) \tag{2.9}$$

where $m_{S_i}(M)$ denotes the multiplicity of S_i as a composition factor of M. In this way we can say, for any two f.g. indecomposable R-modules M, N, that $m_S(M) = m_S(N)$ $\forall S$ simple R-module $\iff \mathfrak{S}(M) = \mathfrak{S}(N)$.

And so R satisfies $\mathfrak{X} \iff \mathfrak{S}(M) = \mathfrak{S}(N)$ implies that $M \cong N$ whenever M and N are f.g. indecomposable R-modules.

We have seen previously that $\mathbb{Z}$ and, more generally every integral domain which is not a field, is not artinian. When we look at the quotient rings of $\mathbb{Z}$ they are all of the form $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. If n is prime then $\mathbb{Z}_n$ is a field, whereas if n is not prime, then $\mathbb{Z}_n$ is never an integral domain but its structure is interesting in the following sense: Writing $n = p_1^{r_1} p_2^{r_2} ... p_s^{r_s}$ we have that

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times ... \times \mathbb{Z}_{p_s^{r_s}} \tag{2.10}$$

as rings, and every $\mathbb{Z}_{p_1^{r_1}}$ is an uniserial ring, that is, all the idealsform a chain (they are of the form $p_i^t \mathbb{Z}_{p_i^{r_i}}$ with $0 \leq t \leq r_i$). That is, $\mathbb{Z}_n$ factors as a product of commutative artinian uniserial rings. Furthermore, it is well-known that all the indecomposable modules over a commutative artinian uniserial ring R are just all the cyclic $R-$modules, and so all of them are of the form R/I for some ideal I of R. Therefore, since R has at most one ideal for every length because of uniseriality, we have that

$I \neq I' \Rightarrow l(R/I) \neq l(R/I')$, and so we deduce that there is at most one indecomposable for every length and hence we get that commutative artinian uniserial rings satisfy $\mathfrak{X}$.

Furthermore, $\mathfrak{X}$ is preserved under products:

Proposition 2.2.1. *Let $\{R_i\}_{i \in I}$ be rings satisfying $\mathfrak{X}$. Then $R = \Pi_i R_i$ satisfies $\mathfrak{X}$.*

Proof. Any indecomposable module over $R = \Pi_i R_i$ is of the form $M = \Pi_i M_i$ with M_t indecomposable module over R_t for some t and $M_i = 0$ for any $i \neq t$; and as R-modules $M = \Pi_i M_i$ and $N = \Pi_i N_i$ are isomorphic iff $M_i \cong N_i$ for every i. In particular, if S and T are simple modules over R_i and over R_j respectively with $i \neq j$, then $\Pi_r S_r$ with $S_i = S$ and $S_r = 0 \, \forall r \neq i$, and $\Pi_r T_r$ with $T_j = T$ and $T_r = 0 \, \forall r \neq j$ are non-isomorphic simple R-modules. This just means that two indecomposable R-modules $M = \Pi_i M_i$ and $N = \Pi_i N_i$, with $M_i = 0$ for any $i \neq r$ and with $N_i = 0$ for any $i \neq s$ can satisfy $\mathfrak{S}(M) = \mathfrak{S}(N)$ only if $r = s$ and $\mathfrak{S}(M_s) = \mathfrak{S}(N_s)$ as modules over R_s. But in that case M_s and N_s must be indecomposable and so since R_s satisfies $\mathfrak{X}$ we must have $M_s \cong N_s$ and so $M \cong N$. Hence, R also satisfies $\mathfrak{X}$. □

Therefore, thanks to $\mathfrak{X}$ being preserved under products, we get the following:

Theorem 2.2.2. *Every ring of the form $\mathbb{Z}_n$, and more generally, any ring which is a product of commutative artinian uniserial rings satisfies $\mathfrak{X}$.*

Commutative rings which are finite products of commutative artinian uniserial rings are called commutative artinian serial rings and they have been extensively studied in the literature. Besides semisimple rings they are probably the most well-understood rings. If we look at the structure of the rings of the form $\mathbb{Z}_n$ we can see easily that every ideal is principal over them. Actually, this is not a specific feature of these rings, it holds over any commutative artinian serial ring and, furthermore, it characterizes them:

Proposition 2.2.3. *(Faith (1976) Prop. 25.4.6.B) A commutative artinian ring is serial iff every ideal is principal.*

Remark. *Because of this, commutative artinian serial rings are frequently called principal ideal rings (PIR), and among all the commutative artinian rings these are precisely the ones of finite representation type, i.e. those with (up to isomorphism) only finitely many indecomposable modules.*

Corollary 2.2.4. *Every commutative artinian ring of finite representation type satisfies* $\mathfrak{X}$.

Thus, we have found a whole class of commutative artinian rings satisfying $\mathfrak{X}$, and they are characterized by having only finitely many indecomposables up to isomorphism. A natural question that arises now is: Is every commutative artinian ring satisfying $\mathfrak{X}$ of finite representation type?

In order to answer this question we need to be able to use the the condition $\mathfrak{X}$ in a way that gives us useful information about the ring and/or its modules. Firstly, recall that every module can be embedded into an injective one, so calling E_i the injective hull of S_i and defining $E := E_1 \oplus ... \oplus E_n$ we have that $Hom_R(-, E) : mod - R \to mod - R$ is an exact functor since E is injective, and furthermore it is a duality.

Definition. *Let R be a ring. A duality D : mod$-R \longrightarrow R-$mod is a contravariant exact functor s.t. $\exists$ another contravariant exact functor D' : $R-$mod$\longrightarrow$ mod$-R$ s.t. $D' \circ D$ is naturally equivalent to the identity functor on mod$-R$ and $D \circ D'$ is naturally equivalent to the identity functor on $R-$mod.*

Remark. *If D is a duality then D' defined as above is also called a duality. A duality takes projective modules to injective modules, injectives to projectives, non-projectives to non-injectives and indecomposables to indecomposable ones.*

On the other hand, for every $i \neq j$, $Hom_R(S_i, S_j) = 0$ and, since R is commutative, for every i we have $S_i \cong R/M_i$ for some maximal ideal M_i and we have:

$$Hom_R(S_i, S_i) \cong Hom_R(R/M_i, S_i) \cong Hom_R(R, S_i) \cong S_i \tag{2.11}$$

As a consequence of these observations we get the following theorem that appears in (Auslander et al. (1997) Thm.3.1 p. 37):

Theorem 2.2.5. *Let M be a f.g.module over a commutative artinian ring. Then the multiplicity of every S_i as a composition factor of M is the same than the multiplicity as a composition factor of $Hom_R(M, E)$.*

Proof. We will proceed by induction on $l(M)$. For $l(M) = 0$ it is trivial, so let usassume then that $l(M) = 1$. This means that M is simple so we canassume WLOG that $M = S_i$ for some i. Then, since $Hom_R(S_i, -)$ commutes with finite direct sums we get

$$Hom_R(S_i, E) \cong Hom_R(S_i, E_1 \oplus ... \oplus E_n) \cong Hom_R(S_i, E_1) \oplus ... \oplus Hom_R(S_i, E_n) \tag{2.12}$$

Now, any non-zero $f_j : S_i \to E_j$ is a monomorphism because S_i is simple. Also, it has simple image in E_j and so we must have $Imf = S_j$ (this holds because S_j is essential in E_j by the definition of injective hull, that is, S_j intersects non-trivially with every non-zero submodule of E_j, and we know that $S_i \cap S_j = (0)$ for $i \neq j$). Thus, for $i \neq j$ we have $Hom_R(S_i, E_j) = 0$ and for $i = j$ we get by (1.1) that $Hom_R(S_i, E_i) \cong S_i$. That is, $Hom_R(S_i, E) \cong S_i$. So we get the same composition factors with the same multiplicities trivially.

Assume now that the result is true for f.g.modules with length $\leq n$ and let M be s.t. $l(M) = n + 1$. Let now $0 \subseteq M_1 \subseteq M_2 \subseteq ... \subseteq M_n \subseteq M$ be a composition series for M. Then $M/M_n \cong S_j$ for some j. Hence, we can consider the following exact sequence given by the inclusion of M_n into M and the projection onto M_n

$$0 \to M_n \to M \to S_j \to 0 \tag{2.13}$$

Applying $Hom_R(-, E)$ to this short exact sequence we get a exact sequence equivalent to

$$0 \to S_j \to Hom_R(M, E) \to Hom_R(M_n, E) \to 0 \tag{2.14}$$

since $Hom_R(S_j, E) \cong S_j$.

Thus, we can apply induction argument since $l(M_n) < l(M)$. The result follows since the composition factors of $Hom_R(M, E)$ are the the union of S_j and the composition factors of $Hom_R(M_n, E)$. □

Using this theorem we will show first that any commutative artinian ring satisfying $\mathfrak{X}$ is self-injective (i.e. R is injective as amodule over itself).

Theorem 2.2.6. *Let R be a commutative artinian ring satisfying $\mathfrak{X}$. Then R is self-injective.*

Proof. For, let us recall first that we can write $R = \oplus_{i=1}^n P_i$, where each P_i is a projective indecomposable module. Then, applying $Hom_R(-, E)$ to R we get

$$E \cong Hom_R(R, E) = Hom_R(\oplus_{i=1}^n P_i, E) \cong \oplus_{i=1}^n Hom_R(P_i, E) \tag{2.15}$$

And so each $Hom_R(P_i, E)$ is indecomposable because $Hom_R(-, E)$ is a duality, and is injective as it is a direct summand of E, which is injective. Then, by the theorem above it has the same composition factors as P_i with the same respective multiplicities. But this just means, by property $\mathfrak{X}$, that $P_i \cong Hom_R(P_i, E)$, and hence each P_i is injective. We conclude, since a finite direct sum of injective modules is always injective, that R is self-injective. □

One nice and important aspect of property $\mathfrak{X}$ is that it is preserved under quotients:

Proposition 2.2.7. *Let R be any unital ring satisfying $\mathfrak{X}$ on the right (resp.left), and let I be any proper ideal of R. Then R/I satisfies $\mathfrak{X}$ on the right (resp.left) too.*

Proof. We willassume that R satisfies $\mathfrak{X}$ on the right, the proof of the left case follows by a left-right symmetric argument. First of all, let us show that if M is an indecomposable R/Imodule then so is as an R-module:assume $M = N \oplus T$ as R-modules.

Then we know that $MI = 0$ and so $0 = MI = NI \oplus TI \Rightarrow 0 = TI, NI$. Thus, also N and T are R/I-modules, and hence since M is indecomposable then either $N = 0$ or $T = 0$, as desired.

Now,assume that there exists M, N indecomposable R/I-modules with composition series

$$0 \subseteq M_1 \subseteq ... \subseteq M_r = M \tag{2.16}$$

$$0 \subseteq N_1 \subseteq ... \subseteq N_r = N \tag{2.17}$$

s.t. $\mathfrak{S}(M) = \mathfrak{S}(N)$ as R/I-modules.

Claim: If S is a simple R/I-module then so is as an R-module. For, we have $SI = 0$, so assume $\exists T \subsetneq S$. Then we also have $TI = 0$, and so T is an R/I-submodule of S so it must be 0 and so S is a simple R-module too.

It follows then that the composition series above are also composition series as R-modules and thus, since R satisfies $\mathfrak{X}$, we must havs $M \cong N$ as R-modules, and so also as R/I-modules.

□

Corollary 2.2.8. *If R is a commutative artinian ring satisfying $\mathfrak{X}$ and I is any proper ideal of R, then R/I is self-injective.*

Proof. It follows by Thm. 2.2.6 and Prop. 2.2.7. □

It is remarkable, as it appears in (Faith (1976) Prop. 25.4.6.B) that artinian PIR's are characterized by the property above (i.e. every proper quotient ring is self-injective) and hence by Thm. 2.2.6 and Cor. 2.2.8 it follows that commutative artinian rings satisfying $\mathfrak{X}$ are PIR, and so we get, together with the remark after Prop. 2.2.3, the following characterization:

Theorem 2.2.9. *Let R be a commutative artinian ring, then TFAE:*

a) R is PIR

b) R/I is self-injective for every proper ideal I of R

c) R is of finite representation type

d) R satisfies $\mathfrak{X}$

We want to finish this section by showing a more elementary proof of the fact that commutative artinian rings satisfying $\mathfrak{X}$ are PIR. For that, we need the following lemmas first.

Lemma 2.2.10. *Let R be any artinian ring, then $J = rad(R)$ is nilpotent, that is, there exists some $n \in \mathbb{N}$ s.t. $J^n = 0$.*

Proof. Consider the descending sequence of ideals $J \supseteq J^2 \supseteq$ ... Since R is artinian it must stop, so there exists $n \in \mathbb{N}$ s.t. $J^n = J^{n+1}$. Then since R is artinian it is also Noetherian so every ideal is finitely generated, in particular J^n is. Then the result follows by applying Nakayama's Lemma: If M finitely generated and $MJ = M$, then $M = 0$; and so $J^n = (0)$ as desired. □

Lemma 2.2.11. *Every commutative artinian ring decomposes into a product of artinian local rings*

Proof. We know that J is nilpotent by the previous Lemma, so let $n \in \mathbb{N}$ s.t. $J^n = (0)$. Then we know that J is the intersection of all maximal ideals $\{M_i\}$ and so, since R is commutative we have $(0) = J^n = \cap_i M_i^n$ and hence by chinese reminder theorem we get

$$R = R/(0) = R/J^n = R/\cap_i M_i^n \cong \Pi_i R/M_i^n \tag{2.18}$$

Thus, it is enough to prove that each R/M_i^n is a local ring. For, every maximal ideal is in particular prime and the prime ideals of R/M_i^n are precisely those prime ideals of R containing M_i^n. Therefore, if P is prime then $M_i^n \subseteq P \Rightarrow M_i \subseteq P$, but M_i is maximal so $M_i = P$. Hence, M_i is the only maximal ideal of R/M_i^n and so we conclude that R/M_i^n is a local ring. □

Lemma 2.2.12. *If R is commutative and $R/I \cong R/J$, where I and J are ideals of R, then $I = J$.*

Proof. Viewing R/I and R/J as an R-modules, we have that the annihilators are $ann(R/I) = I$ and $ann(R/J) = J$. Now let $\phi : R/I \to R/J$ be such isomorphism. Then we have $0 = \phi(0) = (I \cdot R/I) = I \cdot \phi(R/I) = I \cdot R/J$, so $I \subseteq ann(R/J) = J$. In the same way, we get $J \subseteq I$ and hence $I = J$. □

Remark. *Notice that the last lemma clarifies the argument used in the proof of Prop. 1.1.9. Also, the last lemma shows that R has finitely many maximal ideals. For, we know that because of artinianity R has finitely many simple modules $S_1, ..., S_n$ (up to isomorphism) and that each S_i is isomorphic to R/M_i for some maximal ideal M_i. Also, for any maximal ideal N, R/N is a simple module. But if $R/M_i \cong R/N$, then $M = N$ by the last lemma, so $M_1, ..., M_n$ is a complete set of maximal ideals (up to iso morphism) of R. This also implies that product of local rings appearing in the second lemma is a finite product.*

We can now prove the theorem:

Theorem 2.2.13. *Let R be a commutative artinian ring satisfying $\mathfrak{X}$. Then R is a PIR.*

Proof. By the second lemma R decomposes into a finite product of local rings, and hence it is enough to prove that each local piece is uniserial: Recall that the property $\mathfrak{X}$ is preserved under quotients and so each local piece satisfies $\mathfrak{X}$ too since , if $R \cong R_1 \times ... \times R_n$, then for example $I = R_1 \times ... \times R_{n-1} \times (0)$ is an ideal (as the multiplication in R is defined component-wise) and so $R/I \cong R_n$ as rings.

Thus, it is enough to show that a local commutative artinian ring R satisfying $\mathfrak{X}$ is uniserial.assume it is not, then there exist ideals A and B not containing each other. Then, by artinianity, $A/A\cap B$ and $B/A\cap B$ are finite length and so there exist $A\cap B \subseteq X \subseteq A$, $A\cap B \subseteq Y \subseteq B$ s.t. both $X/A\cap B$ and $Y/A\cap B$ are simple. This just means that $R/X \cong \frac{R/A\cap B}{X/A\cap B}$ and $R/Y \cong \frac{R/A\cap B}{Y/A\cap B}$ are modules with the same length, and hence with the same composition factors set since a local ring has a unique simple module.

Furthermore, both R/X and R/Y are modules with only one maximal submodule (i.e. T/X and T/Y, where T is the unique maximal ideal of R) and hence they must be indecomposable: If $R/X \cong M \oplus N$ with $(0) \neq M, N$ then since R/X is cyclic it is finitely generated and so is any direct summand of it. Thus we can find, applying Zorn's lemma, maximal submodules $M' \subseteq M$ and $N' \subseteq N$. But then $H_1 = M \oplus N'$ and $H_2 = M' \oplus N$ are both maximal submodules of R/X, which is a contradiction.

Thus, both R/X and R/Y are indecomposable modules with the same composition factors, so $\mathfrak{X}$ applies and hence $R/X \cong R/Y$. This implies by previous lemma that $X = Y$ and so $X \subseteq B$ too, but then $A \cap B \subsetneq X \subseteq A \cap B$, which is a contradiction.

Thus we conclude that each local commutative artinian ring R satisfying $\mathfrak{X}$ is uniserial, and hence the statement follows.

$\square$

CHAPTER THREE
FINITE-DIMENSIONAL HEREDITARY ALGEBRAS

In the previous chapter we characterized the commutative artinian rings that satisfy $\mathfrak{X}$ and it turned out that those rings are precisely the ones of finite representation type. In this chapter, our focus will be on finite-dimensional algebras over fields (which are examples of artinian rings), more concretely on hereditary algebras. Recall that a ring is called left (resp. right) hereditary if every left (resp. right) ideal is projective, and just hereditary if it is both left and right. Over finite-dimensional algebras, and more generelly over artinian rings, the notions of left and right hereditary coincide so even if we are working with right or left modules we will call those rings just hereditary. We saw in the previous chapter that $\mathbb{Z}$ and more generally every Dedekind domain satisfies $\mathfrak{X}$. They are examples of hereditary rings. Furthermore, an integral domain is a hereditary ring iff it is Dedekind domain. It turns out that over an algebraically closed field every hereditary finite-dimensional algebra is Morita equivalent to a path algebra, that is, an algebra whose generators and relations are structured by a quiver; so path algebras form a huge and interesting class of hereditary rings.

After introducing the required terminology we will study the property $\mathfrak{X}$ for path algebras. The quiver structure of these algebras provides a way to study their modules in terms of vector spaces and linear maps. This will help us to translate module theoretic properties to linear algebra and so we will be able to study the property $\mathfrak{X}$ in a more concrete way.

3.1 Representations of quivers

We will start with the definition of quiver, following Schiffler (2014):

Definition. *A quiver is a tuple $Q = (Q_0, Q_1, s, t)$ where:*

- Q_0 *is a set of vertices*

- *Q_1 is a set of arrows*
- *$s : Q_1 \to Q_0$ is a map from arrows to vertices, mapping an arrow to its starting point*
- *$t : Q_1 \to Q_0$ is a map from arrows to vertices, mapping an arrow to its terminal point*

An arrow $\alpha \in Q_1$ from vertex i to vertex j will be denoted by $\alpha : i \to j$.

Example 1. *Consider the following quiver:*

$$1 \xrightarrow{\alpha} 2$$

It is denoted by A_2. In general if we have n vertices connected by a line like

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

we say that the quiver is of type A_n; but we allow the arrows inside to have different directions as in

$$1 \longrightarrow 2 \longleftarrow 3$$

which is an example of a quiver of type A_3.

Definition. *Let Q be a quiver. We say that there is a path of length m between $i, j \in Q_1$ if there is a sequence of arrows $\alpha_1, \alpha_2, ..., \alpha_m$ s.t. $s(\alpha_1) = i$, $t(\alpha_m) = j$ and $t(\alpha_k) = s(\alpha_{k+1})$ $\forall k = 1, ..., m-1$. In that case we say that i is the source or starting point of $\alpha = \alpha_1, \alpha_2, ..., \alpha_m$ and j is the target or terminal point of α, denoting them by $s(\alpha) = i$ and $t(\alpha) = j$ respectively.*

Definition. *Let Q be a quiver. We say that there is a walk of length m from i to j with $i, j \in Q_1$ if there is a sequence of arrows $\alpha_1, \alpha_2, ..., \alpha_m$ s.t. the following conditions are satisfied :*

- *$s(\alpha_1) = i$ or $t(\alpha_1) = i$*

- *$s(\alpha_m) = j$ or $t(\alpha_m) = j$*
- *For every $k = 1, ..., m-1$ either $t(\alpha_k) = s(\alpha_{k+1})$ or $s(\alpha_k) = t(\alpha_{k+1})$ or $s(\alpha_k) = s(\alpha_{k+1})$ or $t(\alpha_k) = t(\alpha_{k+1})$*

A quiver is called finite if both Q_0 and Q_1 are finite sets, and it is called connected if there exists at least one walk between any two $i \neq j \in Q_0$. This is equivalent to say that there is a sequence of edges connecting i and j in the underlying graph of Q. As an example, the underlying graph of

$$1 \longrightarrow 2 \longleftarrow 3$$

is

$$1 — 2 — 3$$

We will introduce now the key notion of quiver representation. For that we have to fix an arbitrary field K first.

Definition. *A representation $M = (M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q is a collection of K-vector spaces $\{M_i\}_{i \in Q_0}$ and a collection of K-linear maps $\phi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ for each arrow $\alpha \in Q_1$.*

A representation M is called finite-dimensional if each vector space is finite-dimensional. In this case, the dimension vector of M is the vector **dim**$M := (dimM_i)_{i \in Q_0}$. An element of a representation M is a tuple $(m_i)_{i \in Q_0}$, with $m_i \in M_i \; \forall i$. We will assume from now on that all our representations are finite-dimensional.

representations are related to each other via morphisms of representations which we will introduce now:

Definition. *Let $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two representations of a quiver Q. Then a morphism of representations $f : M \to N$ is a collection $(f_i)_{i \in Q_0}$ of linear* *maps $f_i : M_i \to N_i$ s.t. for each arrow $\alpha : i \to j$ in Q_1 we have (commutative diagram) $f_j \circ \phi_\alpha(m) = \psi_\alpha \circ f_i(m)$ for all $m \in M_i$.*

A morphism of representations $f : M \to N$ is called a monomorphism if every f_i is injective, epimorphism if every f_i is surjective, and it is called isomorphism if it is both a monomorphism and an epimorphism.

Definition. *Let Q be a quiver and $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two representations of Q. Then*

$$M \oplus N = (M_i \oplus N_i, \left(\begin{smallmatrix} \phi_\alpha & 0 \\ 0 & \psi_\alpha \end{smallmatrix}\right)) \tag{3.1}$$

is a representation of Q called the direct sum of M and N.

Definition. *A non-zero representation M of Q is called indecomposable if M cannot be written as a direct sum of two non-zero repesentations.*

Given a quiver Q and a field K, the finite-dimensional representations of Q together with the morphisms of representations form a category which we denote by $\mathfrak{rep}_K Q$. Besides finite direct sums and indecomposability, the notions of kernels, and cokernels apply also in here (For exact definitions see Schiffler (2014) Ch. 1). Furthermore, $\mathfrak{rep}_K Q$ is a K-linear abelian category and as in any abelian category, Jordan-Hölder theorem holds for finite length objects , meaning that if we have two composition series for the same finite length object then the multiplicity of every simple object in each composition series is the same. By a simple object in a general abelian category A we mean an object S such that whenever we have a monomorphism $f : T \to S$ in A then either $f = 0$ or f is an isomorphism. Clearly, this definition coincides with the usual one for modules over rings.

Thus, it makes sense to talk about property $\mathfrak{X}$ for general abelian categories. We say in that case that an abelian category A satisfies $\mathfrak{X}$ on its finite length objects, or just that A satisfies $\mathfrak{X}$. In particular we can talk about it in our particular context of quiver representations. But in order to understand how to construct composition series for a general finite length quiver representation it is necessary first to give the definition of subrepresentation and quotient representation.

Definition. *A representation $L = (L_i, \psi_\alpha)$ is called a subrepresentation of a representation $M = (M_i, \phi_\alpha)$ if, for every vertex i, L_i is a vector subspace of M_i and the inclusion maps $\iota_i : L_i \to M_i$ form a morphism of representations $\iota : L \to M$.*

Example 2. *Let $Q = A_2$. Then we can consider the representation*

$$K \xrightarrow{1} K$$

where 1 denotes the identity map. Then the representation

$$0 \xrightarrow{0} K$$

is a subrepresentation of it.

Given a representation $M = (M_i, \phi_\alpha)$ and a subrepresentation $L = (L_i, \psi_\alpha)$ of M , we can construct the quotient representation M/L as follows: For every vertex i write the vector space M_i/L_i and for each arrow $\alpha : i \to j$ define $\delta_\alpha : M_i/L_i \to M_j/L_j$ by the rule $\delta_\alpha(m_i + L_i) = \phi_\alpha(m_i) + L_j$, which can be checked to be well-defined.

We are ready now to study how the composition factors of a given finite length representation look. For our study it would be good to know how the simple representations (i.e. those ones with no proper subrepresentation) that appear as composition factors in a composition series of a finite length representation look. If we assume that Q has no oriented cycles (i.e. for each vertex i in the quiver there is no path starting at i and ending at i) then the possible simple representations are very easy to describe:

Definition. *Let Q have no oriented cycles and let i be a vertex of Q. Then the simple representation $S(i) = (S(i)_j, \phi_\alpha)$ is defined as the representation where $S(i)_i = K$, $S(i)_j = 0$ if $j \neq i$ and $\phi_\alpha = 0$ for every arrow $\alpha \in Q_1$.*

Remark. *Over a quiver with no oriented cycles every representation has a subrepresentation isomorphic to one of the form $S(i)$ for some i, so the $S(i)$'s are, up to isomorphism, all the simple representations of Q (Schiffler (2014) Prop. 2.9 p. 43).*

Remark. *Over a quiver with no oriented cycles the class of finite length representations coincide with the class finite-dimensional ones.*

Since every finite-dimensional vector space V is isomorphic to K^n for some n we can assume for simplicity that all vector spaces appearing in a quiver representation are of that form. We can now state the following theorem:

Theorem 3.1.1. *Let Q be a connected quiver with no oriented cycles and let $M = (K^{n_i}, \phi_\alpha)_{i \in Q_0}$ be a finite-dimensional representation. Then, any composition series for M has composition factors $\{n_i * S(i)\}_{i \in Q_0}$.*

Proof. First recall that, since Q has no oriented cycles, the finite-dimensional repre sentations are precisely the finite length ones by the remark above. Then Jordan-Hölder holds and so any composition series for M has the same composition factors. Hence, in order to prove the theorem, it is enough to construct one composition series for $M = (K^{n_i}, \phi_\alpha)$ satisfying the conditions of the statement. Recall also that if L is a subrepre sentation of M then the composition factors of M are the union of the composition factors of L and the ones of M/L.

Let us proceed by induction on $n = dimM := \Sigma_{i \in Q_0} n_i$. If $n = 1$ then one vertex has dimension one and the others has dimension zero, and since there is no cycle in the quiver, every ϕ_α must be 0, and so $M = S(i)$ for some i, and the result follows. assume then that the result is true for $n \geq 1$ and take M such that $dimM = n + 1$. Since Q has no oriented cycles and is finite we must have at least one vertex that is a source, that is, there is no arrow coming to it. WLOG we may assume that one of such vertices is the vertex 1. If $n_1 > 0$, then consider the subrepresentation $L = (L_i, \psi_\alpha)$ with vector space $L_1 = K^{n_1 - 1}$ and K^{n_i} in the others, and with linear maps $\psi_\alpha = \phi_\alpha$ if $s(\alpha) \neq 1$. For the other arrows, since Q is connected there must be one arrow

$\alpha : 1 \to j$ for some j. For every such arrow write the linear map $\psi_\alpha = \phi_\alpha|_{K^{n_1-1}}$ in there. That is, L is the representation with same vector spaces and linear maps as the ones of M except for the vertex 1 and the arrows coming from 1. Clearly $0 \neq L$ since $dimL = dimM - 1 > 0$. We need to check that L is actually a subrepresentation of M. For, since clearly every L_i is a vector subspace of M_i we can consider the map $\iota : L \to M$ given by the inclusions $\iota_i : L_i \to M_i$. If α is an arrow with $s(\alpha) \neq 1$, then $t(\alpha) \neq 1$ too since 1 is a source. Hence, in that case we have $L_{s(\alpha)} = M_{s(\alpha)}$ and $L_{t(\alpha)} = M_{t(\alpha)}$, and so trivially we have $\iota_{t(\alpha)} \circ \psi_\alpha = \phi_\alpha \circ \iota_{s(\alpha)}$. If $s(\alpha) = 1$, then for every $m \in L_1$ we have $\iota_{s(\alpha)}(m) = m$ and $\iota_{t(\alpha)}(\psi_\alpha(m)) = \psi_\alpha(m) = \phi_\alpha(m)$ and hence $\iota_{t(\alpha)} \circ \psi_\alpha = \phi_\alpha \circ \iota_{s(\alpha)}$. This proves that ι is a morphism of representations, and so L is a subrepresentation of M with $dimL = n$, and hence induction applies. On the other hand, M/L is isomoprhic to $S(1)$ so the composition factors of M are $\{S(1)\} \cup \{(n_1 - 1) * S(1), ..., n_m * S(m)\} = \{n_1 * S(1), ..., n_m * S(m)\}$, where $m = |Q_0|$. Hence, the result follows for the case $n_1 > 0$.

It is left to study the case of $n_1 = 0$. In that case, we can run the same process with any source i of Q with $n_i > 0$. If every source i of Q has corresponding $n_i = 0$, then construct the quiver Q' by removing all sources of Q and all arrrows coming out of them. Then Q' is again a finite connected quiver with no oriented cycles so it must contain some source. If for some source the corresponding vector space in M is non-zero we stop, else we continue constructing Q'' by removing the sources of Q' and all arrrows coming from them, and so on. Since M is non-zero and finite, at some point by iterating this process we must find some source r of $Q^{(t)}$ with corresponding vector space $0 \neq M_r$. In that case, for any $\alpha : s \to r$ we must have $M_s = 0$ because otherwise M_s would be a source in $Q^{(t-1)}$, which is a contradiction. Hence, in that case $\phi_\alpha = 0$, and this also happens for every arrow deleted in the process of constructing $Q^{(t)}$ from Q. Then, since we are assuming that M is not simple, we can construct again the non-zero subrepresentation $L = (L_i, \psi_\alpha)$ with $L_r = K^{n_r-1}$, $L_h = K^{n_h}$ for $h \neq r$, and the linear maps given like in the construction of the case $n_1 > 0$. We have $\phi_\alpha = 0$ for any arrow $\alpha : i \to j$ removed during the construction of $Q^{(t)}$, and so trivially $\iota_j \circ \psi_\alpha = \phi_\alpha \circ \iota_i$. For the remaining α's that appear in $Q^{(t)}$ the same argument that in the case above with $n_1 \neq 0$ holds since $\psi_\alpha = \phi_\alpha$ or $\psi_\alpha = \phi_\alpha|_{K^{n_r-1}}$ and the ι_i's are inclusions, and so the relations $\iota_{t(\alpha)} \circ \psi_\alpha = \phi_\alpha \circ \iota_{s(\alpha)}$ are satisfied. Thus, the statement follows.

□

3.2 The path algebra of Q

In this section we will introduce the path K algebra KQ associated to a finite quiver Q. Our aim is to be able to study the property $\mathfrak{X}$ in the context of path algebras. For that we will show that the f.g. modules over a path algebra KQ can be understood in terms of the finite-dimensional quiver represention of Q, and with the help of that we will be able to translate the property $\mathfrak{X}$ from modules to representations in an appropiate way.

We start with the definition of the path algebra KQ:

Definition. *Let Q be a (non-necessarily connected) finite quiver, K be any field and $e_1, ..., e_n$ be the vertices of Q. Denote by A the set of all the paths in Q. The path algebra KQ is the K-algebra with basis $B = A \cup \{e_1, ..., e_n\}$ and and with multiplication defined on two basis elements α, β by*

$$\alpha\beta = \alpha \cdot \beta \text{ if } s(\beta) = t(\alpha), \text{ and 0 otherwise} \tag{3.2}$$

We use the convection that $s(e_i) = e_i = t(e_i)$ so that the multiplication above makes sense. Also, we impose the following condition: (1) If α is in B with $s(\alpha) = i$ and $t(\alpha) = j$, then $e_i \cdot \alpha = \alpha = \alpha \cdot e_j$.

Remark. *We have that the unital element $1 \in KQ$ is given by $1 = \Sigma_{i \in Q_0} e_i$ which, by (1), makes sense since Q is a finite quiver. furthermore, also by (1) every e_i is an idempotent element, that is, $e_i^2 = e_i$ and they are orthogonal to each other (i.e. $e_i e_j = 0$ if $i \neq j$). Also, notice that if Q contains some cycle α then $\alpha, \alpha^2, ...$ are* *linearly independent elements, so KQ is an infinite-dimensional algebra. On the other hand, if no cycle appears in Q, then there is a longest path in Q, and since Q_1 is finite there are finitely many possibilities of combining all arrows (or paths of length 1), so it follows that KQ is finite-dimensional. Since we are dealing with finite-dimensional algebras, from now on we will only focus on path algebras coming from finite quivers with no oriented cycles. We will also assume the quivers involved to be connected.*

Example 3. *The path algebra associated to $Q = A_2$ is the upper triangular matrix ring* $\left(\begin{smallmatrix} K & K \\ 0 & K \end{smallmatrix}\right)$.

Example 4. *The path algebra associated to the one-loop quiver*

$$\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$$

is $K[X]$.

The class of path algebras (over finite quivers) has the common property that all of them are hereditary rings. Conversely, if the field K is algebraically closed, then any finite-dimensional hereditary K-algebra is Morita equivalent to a path algebra! Therefore, the study of path algebras provides useful information and tools to understand hereditary algebras in general.

Finite-dimensional K-algebras are examples of artinian rings and so finite length modules coincide with finitely generated ones. Let us denote by mod$-KQ$ the ca tegory of all finitely generated right modules over KQ, where K is any field and Q is a finite connected quiver with no oriented cycles. We can relate the objects in mod$-KQ$ with the finite-dimensional representations of Q in the following way. Consider the functor F : mod$-KQ \to \mathfrak{rep}_K Q$ defined as:

- Let M be in mod$-KQ$, then define $F(M)$ as the representation $(M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ where $M_i = Me_i$ is the vector space consisting of all me_i

with $m \in M$, and for any arrow $\alpha : i \to j$, the map $\phi_\alpha : M_i \to M_j$ is given by $\phi_\alpha(me_i) = m(e_i\alpha)$

- Let $f : N \to M$ be a morphism in mod$-KQ$. Then $F(f) : F(N) \to F(M)$ is the morphism of quiver representations defined on each vertex i as $f_i : N_i \to M_i$ sending me_i to $f(m)e_i$

F is a well-defined functor and it can be checked (Schiffler (2014) Thm. 5.4) that is full, faithful and dense. In other words, F induces an equivalence of categories between mod$-KQ$ and $\mathfrak{rep}_K Q$, and every equivalence of abelian categories is additive and exact. That is, F commutes with finite direct sums and takes exact sequences to exact sequences. Indeed, exactness of F could be read directly from its definition. This is fundamental for us because thanks to the exactness and to the fact, which we will show now, that F takes simple modules to simple representations we can relate the composition factors of an object $M \in$ mod$-KQ$ to its image under F as follows:

Theorem 3.2.1. *Let $0 \subseteq M_1 \subseteq M_2 \subseteq ... \subseteq M_n = M$ be a composition series for M in mod$-KQ$. Then $0 \subseteq F(M_1) \subseteq F(M_2) \subseteq ... \subseteq F(M_n) = F(M)$ is a composition series for $F(M)$ and $F(M_i/M_{i-1}) \cong F(M_i)/F(M_{i-1})$. Thus, if $T_1, ..., T_r$ are the composition factors of M, then $F(T_1), ..., F(T_r)$ are the composition factors of $F(M)$.*

Proof. Let M be a simple $A := KQ$ module. Then we know that $M \cdot J = 0$, where J denotes the Jacobson radical of KQ. It can be shown (Schiffler (2014) Ch. 4 Cor. 4.5) that J is the ideal generated by all the arrows in Q_1. This means that for every $0 \neq m \in M$ we have $m\alpha = 0$ for any $\alpha \in Q_1$, and so also $m\beta = 0$ for any path β in Q. Hence, for $x \in KQ$ we have that $mx \neq 0$ implies $x = r_1e_1 + .. + r_ne_n$ for some $r_1, .., r_n \in K$. Now, since M is simple, we have that $M \cong A/I$ for some maximal right ideal I of A and so we can assume WLOG that $M = A/I$. assume that there exist two different e_i, e_j s.t. $0 \neq e_i + I, e_j + I$. This means that I does not contain e_i and e_j. But then $I' = I + e_jA$ does not contain e_i since all paths are linearly independent in A and so either $e_i \in I$, which is impossible, or $e_i = e_jy \in e_jA$ for some $y \in A$. But in the latter case $e_i = e_jy$ implies $e_i = e_ie_i = e_i(e_jy) = (e_ie_j)y = 0$, so it is also impossible.

Hence, $e_i \notin I'$. But this contradicts the maximality of I. Thus, since $1 = \Sigma_{i \in Q_0} e_i$, we have that one and only one e_i do not belong to I. This means, since M is simple, that $e_i + I$ is a generator for A/I, so every element can be written in the form $e_i x + I$, with x a path in KQ. But we know that $M \cdot J = 0$ and $J \subseteq I$, so we can conclude that $0 \neq e_i x + I \Leftrightarrow e_i x = e_i a$ with $0 \neq a \in K$. Hence, M is a one-dimensional vector space over K and $Me_j = 0$ for any $j \neq i$. This just means, by the construction of $F(M)$, that $F(M) = S(i)$ and so it is a simple representation. Thus, F takes simple modules to simple representations.

We can proceed now by induction on the length of M. For $l(M) = 0$ it is trivial since the zero module corresponds to the zero representation under F. If $l(M) = 1$, then M is simple and so is $F(M)$. assume now that $l(M) = n+1$ and that the statement holds for $l(M) \leq n$, and let

$$0 \subseteq M_1 \subseteq M_2 \subseteq ... \subseteq M_{n+1} = M \tag{3.3}$$

be a composition series for M. Consider then the short exact sequence given by inclusion and projection

$$0 \to M_n \to M \to M/M_n \to 0 \tag{3.4}$$

Then, since F is exact, we have that

$$0 \to F(M_n) \to F(M) \to F(M/M_n) \to 0 \tag{3.5}$$

is exact. Since M/M_n is simple, then $F(M/M_n)$ is also simple and the exactness tells us that $F(M/M_n) \cong F(M)/F(M_n)$. Also, then induction hypothesis tells us, since $l(M_n) < l(M)$, that

$$0 \subseteq F(M_1) \subseteq F(M_2) \subseteq ... \subseteq F(M_n) \tag{3.6}$$

is a composition series for $F(M_n)$ and hence

$$0 \subseteq F(M_1) \subseteq F(M_2) \subseteq ... \subseteq F(M_n) \subseteq F(M) \tag{3.7}$$

is a composition series for $F(M)$. Thus, the result follows. □

Remark. *Recall that F induces an equivalence of categories, so there exists some functor $G : \mathfrak{rep}_K Q \to mod{-}KQ$ s.t. $F \circ G(B) \cong B$ for any B in $\mathfrak{rep}_K Q$ and $G \circ F(C) \cong C$ for any C in mod$-KQ$, and G induces an equivalence of categories too. Then, in particular, if T_1 and T_2 are non-isomorphic simple modules then so are $F(T_1)$ and $F(T_2)$ (and viceversa), because as an equivalence of categories, F (and G) maps isomorphisms to isomorphisms and non-isomorphisms to non-isomomorphisms. This means, since F is dense, that we have a bijection between simple modules $\{T_i\}$ and simple representations $\{F(T_i)\}$.*

Every equivalence between abelian categories preserves indecomposable objects:

Lemma 3.2.2. *Let $F : A \to B$ be a functor which induces an equivalence of abelian categories. Then M is an indecomposable object in A if and only if $F(M)$ is an indecomposable object in B.*

Proof. A functor $F : A \to B$ which induces an equivalence of abelian categories is additive, and so is its inverse. Call the inverse G. It is enough to show that if M is indecomposable then so is $F(M)$ since the other way around follows by arguing with G instead of F.

For, let M be indecomposable in A, and assume $F(M)$ is not indecomposable. Then we can write $F(M) = N \oplus T$ with $0 \neq N, T$ in B. But then, by applying G we get $M \cong G(F(M)) \cong G(N) \oplus G(T)$. But $0 \neq G(N)$ since otherwise $N \cong F(G(N)) = F(0) = 0$. Similarly, $0 \neq G(T)$. This contradicts M being indecomposable.

□

KQ is an artinian ring so, up to isomorphism, we can list its simple right modules as $S_1, ..., S_n$. Then, as in the previous chapter, for any f.g. KQ-module M we can paraphrase $\mathfrak{S}(M) := (m_{S_1}(M), m_{S_2}(M), ..., m_{S_n}(M))$, where $m_{S_i}(M)$ denotes the multiplicity of S_i as a composition factor of M.

Proposition 3.2.3. *Let M and N be f.g. right modules over KQ. Then $\mathfrak{S}(M) = \mathfrak{S}(N)$ iff* $\mathbf{dim}F(M) = \mathbf{dim}F(N)$.

Proof. It follows from Thm. 3.2.1 together with Thm. 3.1.1 of the previous section.

□

Corollary 3.2.4. *Let K be any field and let Q be any finite connected quiver with no oriented cycles. Then KQ satisfies $\mathfrak{X}$ iff for any two f.d. indecomposable representations V, W of Q with $\mathbf{dim}V = \mathbf{dim}W$ we have $V \cong W$.*

Proof. For the "if part" assume we have two indecomposable modules M and N with the same composition factors. Then $F(M)$ and $F(N)$ are indecomposable (F induces an equivalence between abelian categories) with $\mathbf{dim}F(M) = \mathbf{dim}F(N)$ by the previous proposition. Hence, by assumption, $F(M) \cong F(N)$ and hence $M \cong G \circ F(M) \cong G \circ F(N) \cong N$, as desired.

For the "only if" we know that F is an equivalence of categories and so in particular is dense, meaning that any f.d. representation is isomorphic to one of the form $F(M)$ with M in mod$-KQ$. Hence, if we have two indecomposable representations V and W s.t. $\mathbf{dim}V = \mathbf{dim}W$, we know that $V \cong F(M)$ and $W \cong F(N)$ for some indecomposable modules M, N. Then by previous proposition it follows that M and N have the same composition factors and so they must be isomorphic by assumption. Hence, also V and W are isomorphic.

□

Remark. *If Q is not connected then we can write $KQ \cong \Pi_i KQ_i$, where the Q_i's are the connected components of Q. Therefore, since any indecomposable module over*

$\Pi_i KQ_i$ is an indecomposable module over some KQ_i and by the previous chapter we know that $\mathfrak{X}$ is preserved under quotient rings and under products of rings, it follows then that KQ satisfies $\mathfrak{X}$ iff every KQ_i satisfies $\mathfrak{X}$. Hence, for our aim of classifying path algebras satisfying $\mathfrak{X}$ we only need to deal with the ones coming from connected quivers.

Thus, we have stated the property $\mathfrak{X}$ for path algebras in terms of their representations. This will be the key to classify, over algebraically closed fields and over finite quivers with no oriented cycles, the path algebras satisfying $\mathfrak{X}$.

3.3 The second Brauer-Thrall conjecture

Let R be an artinian ring satisfying $\mathfrak{X}$. Since it is artinian it has, up to isomorphism, only finitely many simple modules $S_1, ..., S_r$, and since it satisfies $\mathfrak{X}$ we have that if M is an indecomposable module of length m, then M is completely determined by its composition factors $n_1 * S_1, ..., n_r * S_r$, with $m = n_1 + ... + n_r$. In particular this means that, for a fixed length m, the number of indecomposable modules of length m is at most r! times the number of ways of decomposing m as a sum of r non-negative integers, and thus finite. It follows that, over an artinian ring satisfying $\mathfrak{X}$, the number of indecomposables of any given length m is always finite.

A particular setting where this property is important is the one of finite-dimensional algebras over algebraically closed fields. Finite-dimensional algebras are examples of artinian rings, so the above property holds for the ones satisfying $\mathfrak{X}$. The classical second Brauer-Thrall conjecture asserts that a finite-dimensional algebra over an infinite field is either of finite type or for an infinite number of distinct integers n_i there is an infinite number of pairwise non-isomorphic indecomposable modules of length n_i. An affirmative answer to this would imply that finite-dimensional algebras over infinite fields satisfying $\mathfrak{X}$ are of finite type. It is astonishing that we have an affirmative answer for it if we assume that the field is algebraically closed (see the survey Ringel (2005) and the article Bongartz (2016) for accounting on the history of the conjecture and its solutions). Thus, over algebraically closed fields any finite-dimensional algebra satisfying $\mathfrak{X}$ must be of finite representation type.

In order to prove the second Brauer-Thrall conjecture, one could try to argue by induction on the dimensions of the algebras as vector spaces as follows: If we want to prove the second Brauer-Thrall conjecture for R we can assume in particular that every factor algebra R/I of R with $0 \neq I$ is of finite representation type. For seeing this, just recall that the dimension of every such R/I is lower than the one of R and so applying induction argument they must satisfy the second Brauer-Thrall conjecture, that is, either they are of finite representation type or else we can find infinitely many integers n_i with infinitely many pairwise non-isomorphic indecomposable modules of length n_i. But in the last case the category of f.g. modules over R/I is a full subcatagory of mod$-R$ and so every indecomposable module over R/I is indecomposable over R. Hence, also R satisfies the second Brauer-Thrall conjecture hypothesis. Thus, in this approach one could assume that R is of infinite type and that every R/I is of finite type.

This idea was used in the approach made by Nazarova and Rojter to the conjecture (see the report Ringel (1980) about Nazarova and Rojter's approach). Another idea behind this approach was to find, for any finite-dimensional K-algebra A of infinite representation type over K algebraically closed field , a "good" embedding of categories mod$-R_M \to$ mod$-A$, where R_M is some ring of the form $\left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$, with F and G are division rings and M is an $F, G-$ bimodule. Over this kind of rings, indecomposable modules are in some sense easier to understand than for arbitrary finite-dimensional algebras. If such a "good" embedding is possible then every indecomposable module over such R_M can be seen as an indecomposable over A and the embedding takes non-isomorphic indecomposable modules over R_M to non-isomorphic indecomposable modules over A. So, if R_M satisfies the conditions of the second Brauer-Thrall conjecture then so must A do. Therefore, in that case the study of the second Brauer-Thrall conjecture over A could be reduced to the study over rings of the form R_M.

Eventhough Nazarova and Rojter could not give a satisfactory proof for the conjecture (they claimed to have one such proof but it contained some gaps that could not be solved within their methods; for more details about this see Ringel (2005)), the idea of finding such an embedding for a finite-dimensional algebra of infinite representation type is a very valid idea, which we will indeed use repetidly in this thesis.

This section is devoted to study some classes of formal 2×2 upper triangular matrix rings in which the second Brauer-Thrall conjecture holds. Also, we will show an example given by Ringel of an artinian ring, which is a formal 2×2 upper triangular matrix ring of infinite representation type not satisfying the second Brauer-Thrall conjecture. We will follow the article Ringel (1976).

3.3.1 Algebraic bimodules

We start with a discussion on algebraic bimodules. Let F and G be two division rings, and let M be a $F,G-$ bimodule (left module over F and right over G) which is finite-dimensional over both division rings. In this case, the two by two upper triangular matrix ring $R_M = \left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ is a finite-dimensional algebra over some field K if and only if there is a field K such that both F and G are finite-dimensional (division) algebras over K and K operates centrally on M. That is, F and G are finite-dimensional vector spaces over K with ring homomorphisms $K \to Z(F)$, $K \to Z(G)$ and $am = ma \;\; \forall a \in K, \; m \in M$. If the above conditions are satisfied, M is called an algebraic bimodule. Recall that, in this case, the dimension of M over K is also finite .

It can be shown (see for example Auslander et al. (1997) Prop. 2.2 p. 74) that the category of f.g. right modules over R_M is equivalent to the following category $\mathfrak{L}(M)$:

- The objects are the tuples (V, W, ϕ) where V is a f.d. right vector space over F, W is a f.d. right vector space over G and $\phi : V \otimes_F M \to W$ is a G-linear map.

- A morphism $\alpha : (V, W, \phi) \to (V', W', \phi')$ is given by an F-linear map

  $\alpha_F : V \to V'$ and a G-linear map $\alpha_G : W \to W'$ s.t. $\alpha_G \circ \phi = \phi' \circ (\alpha_F \otimes 1)$

The tuple $X = (V, W, \phi)$ will be called a representation of M and we will call $dimX = (dim_F V, dimW_G)$ its dimension vector.

Also, if we call F the functor from $\mathfrak{L}(M)$ to mod-R_M that induces the equivalence, we have the following (see for example Auslander et al. (1997) Prop. 2.3 p. 75 and Prop. 2.5 p. 77).

Proposition 3.3.1. *Let R_M and $\mathfrak{L}(M)$ be defined as above. Then we have:*

- *The only simple representations of R_M are $(F, 0, 0)$ and $(0, G, 0)$*
- *$F(X)$ is simple in mod-R_M if and only if X is simple in $\mathfrak{L}(M)$*
- *$0 \to X \to Y \to Z \to 0$ is exact in $\mathfrak{L}(M)$ if and only if*

  $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact in mod-R_M

Thus, in view of Thm. 3.2.1, since F takes simples to simples and exact sequences to exact sequences, then the length of X as an object in $\mathfrak{L}(M)$ is the same than the length of $F(X)$ in mod$-R_M$, and if $X_1, ..., X_n$ are the composition factors of X then $F(X_1), ..., F(X_n)$ are the composition factors of $F(X)$. Therefore, there is no danger for us in working with the category $\mathfrak{L}(M)$ instead of mod$-R_M$ and identifying modules with representations in the same way as for path algebras. Indeed, the notions of algebraic bimodules and more generally K-species which we will see later, generalize the previous one on path algebras. For example the Kronecker algebra, that is the path algebra associated to the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

is the 2×2 upper triangular matrix ring $\left(\begin{smallmatrix} F & F\oplus F \\ 0 & F \end{smallmatrix}\right)$, and so in that case $K = F = G$ and $M = F \oplus F$.

Associated to an algebraic bimodule M we have the following quadratic form $q(X,Y)$ defined on $\mathbb{Q}^2$ as follows:

$$q(X,Y) = f_1X^2 + f_2Y^2 - mXY \tag{3.8}$$

where $f_1 = dim_K F$, $f_2 = dim_K G$ and $m = dim_K M$

It was shown by Dlab and Ringel (see Dlab & Ringel (1975), Dlab & Ringel (1974)) that R_M is of finite-representation type (i.e. it has only finitely many isoclasses of f.g. indecomposable modules) iff q is positive definite (i.e. $q(X,Y) > 0\ \forall (X,Y)$). It would be nice if we could interpret this result just in terms of $a = dim_G M$ and $b = dim_F M$, that is without refering to the central field K. Actually, as Ringel showed in Ringel (1976) we can do that by defining the (usually non-symmetric) bilinear form $\tilde{q}$ on $\mathbb{Q}^2$ by

$$\tilde{q}((X_1,Y_1),(X_2,Y_2)) = f_1X_1X_2 + f_2Y_1Y_2 - mX_1Y_2 \tag{3.9}$$

whose corresponding quadratic form is just q. In turns out Ringel (1976) that we can interpret $\tilde{q}$ as follows: For any two representations Z, T of M, we have $\tilde{q}(dimZ, dimT) = dim_K Hom(Z,T) - dim_K Ext^1(Z,T)$, and from the equality

$$dim_K G \cdot dim_G M = dim_K M = dim_K F \cdot dim_F M \tag{3.10}$$

we deduce that $\tilde{q}$ is given (up to scalar multiplication) by the matrix $\left(\begin{smallmatrix} a & -ab \\ 0 & b \end{smallmatrix}\right)$ which only depends on $a = dim_G M$ and $b = dim_F M$.

Then we have:

Theorem 3.3.2. *(Ringel (1976) p. 281-282) R_M is of finite type iff $ab \leq 3$.*

If R_M is of infinite representation type (that is, $ab \geq 4$) , then for every $x \in \mathbb{Z}^2_{\geq 0}$ s.t. $q(x) = 0$ there exists at least one indecomposable representation X of M with $dim(X) = x$. Furthermore, for such M we can always find some $x \in \mathbb{Z}^2_{\geq 0}$ of the form $x = (1, s)$ s.t. $q(x) = 0$. Then, for every multiple of nx with $n \in \mathbb{N}$ we have again $q(nx) = 0$ and so we can construct indecomposable representations of arbitrary large dimension! But this is not all, we have the following lemma:

Lemma 3.3.3. *(Ringel (1976) p.282) In case K is infinite, together with $ab \geq 4$, for each $x = (1, s)$ s.t. $q(x) = 0$ we can find an infinite family $\{X_i\}$ of pairwise non-isomorphic indecomposable representations of M with $dimX_i = x$. If K is finite, we can find at least two of them.*

Hence, provided that K is infinite, given a vector of the form $x = (1, s) \in \mathbb{Z}^2_{\geq 0}$ s.t. $q(x) = 0$ we can find infinitely many indecomposable with that dimension. Indeed the the same is true for any vector of the form $x_n = (n, ns)$(Ringel (1976) p. 283). Thus, we have infinitely many vectors x_n with infinitely many indecomposable repre sentations with that dimension. This reminds us of the second Brauer-Thrall conjecture. Indeed, this just means that the finite-dimensional algebras of the form R_M satisfy the second Brauer-Thrall conjecture. But for being truly convinced we should show first that the length, after identifying modules with representations, of two representations with the same dimension vector is the same. In order to do that, let us first recall the definitions of subrepresentation and quotient representation, which are similar to the ones in the quiver case:

Definition. *We say that (V_1, W_1, ψ) is a subrepresentation of (V, W, ϕ) if V_1 is a subvector space of V, W_1 is a subvector space of W and the inclusions maps $\iota : V_1 \to V$, $W_1 \to W$ form a morphism of representations. That is to say, in a sense, that $\psi = \phi|_{V_1 \otimes M}$ upon identifying image in $V \otimes M$.*

Definition. *Let $(V_1, W_1, \psi = \phi|_{V_1 \otimes M})$ be a subrepresentation of (V, W, ϕ). Then the quotient representation $(V/V_1, W/W_1, \delta)$ is the one given by $\delta : V/V_1 \otimes M \to W/W_1$, with $\delta((a + V_1) \otimes m) := \phi(a \otimes m) + W_1$*

Lemma 3.3.4. *Let $X = (V, W, \phi)$ be a representation of M with $dim_F V = n$ and $dim W_G = m$. Then in any compositin series for X the simple representation $(F, 0, 0)$ appears n times as a composition factor and the simple representation $(0, G, 0)$ appears m times. That is, $l(X) = dim_F V + dim W_G$.*

Proof. We know by Prop. 3.3.1 that the only simple representations of M are $(F, 0, 0)$ and $(0, G, 0)$. If $n = 0$ then $X = (0, W, 0) \cong (0, G, 0)^m$ and the result follows. otherwise, consider the representation $X_1 = (V_1, W, \psi)$ where V_1 is a subspace of V with dim $n - 1$ and $\psi = \phi|_{V_1 \otimes M}$, where $n = dim_F V$. Then clearly X_1 is a subrepresentation of X and we have $X/X_1 \cong (F, 0, 0)$. If $n - 1 > 0$ repeat the process with X_1 and construct X_2 similarly. Continue this process until reaching $X_n = (0, W, \phi|0) = (0, W, 0)$. Then clearly if $m = dim W_G$ then $(0, W, 0) \cong (0, G, 0)^m$. Thus, we can construct a composition series of the form

$$0 \subseteq (0, G, 0) \subseteq (0, G \oplus G, 0) \subseteq ... \subseteq X_n \subseteq X_{n-1} \subseteq ... \subseteq X_1 \subseteq X \tag{3.11}$$

and so X has length $n + m$, which does not depend on the particular choices of V, W and ϕ. □

Remark. *The last lemma tells us in particular that two indecomposable representations have the same dimension vector if and only if they have the same composition factors because the only simple representations are $(F, 0, 0)$ and $(0, G, 0)$, which appear as many times as composition factors as the dimensions of V over F and W over G respectively. This just means, by our previous discussion and Lemma 3.3.3, that if $ab \geq 4$ then we can find at least two non-isomorphic indecomposable representations with the same dimension vector and hence with the same composition factors; and so R_M does not satisfy $\mathfrak{X}$. On the other hand, it can be shown (see Ringel (1976) Thm. 3) that for finite type ones (i.e. $ab \leq 3$) we can find at most one indecomposable representation for a given dimension vector. Hence, R_M satisfies $\mathfrak{X}$ iff $ab \leq 3$ iff R_M is of finite representation type.*

The properties of the two by two upper triangular rings of the form $\left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ that we have just studied imply in particular that the rings of the form $\left(\begin{smallmatrix} K & K^n \\ 0 & K \end{smallmatrix}\right)$ are of finite representation type iff $n = 1$. These rings appear as the path algebras of the quivers given by two vertices and n arrows between then (with the same direction). In case $n = 1$ this is precisely the path algebra associated to A_2 and now we can say that its path algebra, that is, $\left(\begin{smallmatrix} K & K \\ 0 & K \end{smallmatrix}\right)$ satisfies $\mathfrak{X}$ and it is of finite representation type. It will turn out that every A_m also satisfies $\mathfrak{X}$ regardless of the directions of the arrows involved. For $n = 2$ we have the Knonecker algebra, which by our dicussion above has infinite representation type for any field K. It is the probably the most classical, basic and important example of a finite-dimensional algebra of infinite representation type. In the next chapter the idea of finding, for a given finite-dimensional algebra A of infinite representation type, a "good" embedding mod$-R_M \to$ mod$-A$ for some ring of the form $R_M = \left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ will play a fundamental role for proving that hereditary Artin algebras of infinite representation type do not satisfy $\mathfrak{X}$.

But, what do we understand by a "good" embedding? By that we mean a full and exact embedding of abelian categories. These notions are defined as follows:

Definition. *Let A, B and C be abelian categories, with B subcategory of A, and let $F : C \to A$ be a functor Then:*

- *B is a full subcategory of A if for any two objects X, Y in B we have $Hom_B(X, Y) = Hom_A(X, Y)$*
- *B is an exact subcategory of A if the inclusion functor from B into A preserves exact sequences (i.e. it is an exact functor).*

  In other words, if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in B, then it is also an exact sequence in A
- *F is called a full exact embedding if it induces an equivalence of (abelian) categories $C \cong F(C)$, and $F(C)$ is a full and exact subcategory of A*

Notice that such a functor may not preserve the length of the objects since it may not take simple objects to simple objects. Anyway, what is important for us is the following:

Proposition 3.3.5. *Let A and B be abelian categories in which every object has finite length. assume that B does not satisfy $\mathfrak{X}$ and that there exists $F : B \to A$ full exact embedding of abelian categories. Then A does not satisfies $\mathfrak{X}$.*

Proof. Let us first show that the equivalence of abelian categories $B \to F(B)$ preserves no-$\mathfrak{X}$. We know by Lemma 3.2.2. that M is indecomposable in B if and only if $F(M)$ is indecomposable in $F(B)$. Let

$$0 = M_0 \subseteq M_1 \subseteq ... \subseteq M_n = M \tag{3.12}$$

be a composition series for M and let $M_{i+1}/M_i = S_i$ be the corresponding simple objects $\forall i = 0, .., n-1$. Consider now, for every i, the short exact sequence

$$0 \to M_i \to M_{i+1} \to S_i \to 0 \tag{3.13}$$

Then, since F induces an equivalence of categories $B \to F(B)$ and every such functor is exact, we have that

$$0 \to F(M_i) \to F(M_{i+1}) \to F(S_i) \to 0 \tag{3.14}$$

is a short exact sequence in $F(B)$.

This just means that, for any i, $F(S_i) \cong F(M_{i+1})/F(M_i)$ and that

$$0 \subseteq F(M_1) \subseteq ... \subseteq F(M_n) = F(M) \tag{3.15}$$

is a series for $F(M)$. It may not be a composition series since the $F(S_i)$'s may not be simple, but since every object in A, and so in $F(B)$, has finite length, it can be refined to a composition series by Jordan-Hölder; and so at least we know that the composition factors of $F(M)$ are the union of the composition factors of all the $F(S_i)'s$.

This means in particular that if M and N have the same composition factors with the same multiplicities, then so does $F(M)$ and $F(N)$. Hence, if we assume that B does not satisfy $\mathfrak{X}$, then there must exist two non-isomorphic indecomposable objects X and Y with the same composition factors with the same multiplicities. But then $F(M)$ and $F(N)$ are non-isomorphic indecomposable (F is an equivalence), finite length objects, and have the same composition factors with the same multiplicities. Thus, $F(B)$ does not satisfy $\mathfrak{X}$.

Now that this is established, it remains to show that indeed A does not satisfy $\mathfrak{X}$. For that, recall that a simple object in $F(B)$ may not be simple in A, but at least we can find a composition series in A since every object is a finite length object. This means again that if M and N have the same composition factors with the same multiplicities in B then so does $F(M)$ and $F(N)$ in A. Hence, we just need to show that $F(M)$ is an indecomposable object in A if M is indecomposable in B. For, we know that $F(M)$ is indecomposable in $F(B)$ and $F(B)$ is a full and exact subcategory of A. Assume that $F(M) = T \oplus H$, with $0 \neq T, H$ in A, and let π be the projection of $F(M)$ onto H, viewing it as an endomorphism of $F(M)$. Then $\pi^2 = \pi$ is a non-trivial idempotent of $\mathrm{End}_A(F(M))$. But $F(B)$ is a full subcategory of A and $F(M)$ is an object in $F(B)$, so $\mathrm{End}_B(F(M))$ $=\mathrm{End}_A(F(M))$. This means that π is a morphisms in $F(B)$ and so it is a non-trivial idempotent of $\mathrm{End}_B(F(M))$. But this just means that $F(M)$ is not indecomposable in $F(B)$, which is a contradiction.

□

Corollary 3.3.6. *Let R, S be artinian rings such that there is a full and exact embedding F : mod$-S \to$ mod$-R$. If S does not satisfy $\mathfrak{X}$, then R does not satisfy $\mathfrak{X}$.*

Proof. It follows by the above proposition since over artinian rings any finitely generated module has finite length. □

Remark. *The main argument used in Prop. 3.3.5 is that a full and exact embedding $F : C \to D$ between abelian categories where every object is of finite length "preserve" composition factors in the sense that if A and B are finite length objects* *in C with the same composition factors and the same multiplicities then $F(A)$ and $F(B)$ are finite length objects in D with the same composition factors and the same multiplicities. This means, since F takes indecomposable objects to indecomposable ones and C and $F(C)$ are equivalent as abelian categories, that C satisfies $\mathfrak{X}$ if and only if $F(C)$ satisfies $\mathfrak{X}$ also. Hence, in particular if F induces an equivalence of categories then C satisfies $\mathfrak{X}$ if and only if D satisfies $\mathfrak{X}$. An important application of this fact is that $\mathfrak{X}$ is a Morita invariant, i.e. a property preserved under equivalences of module categories.*

3.3.2 Ringel's counterexample

We would like to finish this section by giving an example of an artinian ring with infinite center which is of infinite type where for every length there are at most two indecomposables. This just means that the second-Brauer Thrall conjecture cannot be generalized to arbitrary artinian rings, say with infinite center. Indeed, the ring Ringel constructed is an "almost" counterexample to the assertion that artinian rings satisfying $\mathfrak{X}$ must be of finite type.

Let us recall first that the second Brauer-Thrall conjecture does not hold over finite fields. Indeed, just consider for example, over an arbitrary finite field K, the Kronecker K-algebra, which is of infinite representation type by the discussion above. We know that any finite-dimensional representation is given by two finite-dimensional K-vector spaces V, W and two K-linear maps $\phi, \psi : V \to W$ [see final comment on section 3.3.1], and that two representations have the same length iff the sum of the dimensions of the vector spaces involved is the same. Hence, for a fixed n the representations with length n are precisely those tuples (V, W, ϕ, ψ) with $dimV + dimW = n$. Therefore, the possible dimensions of the vector spaces appearing is finite. It is just at most two times the number of ways of writing n as a sum of two non-negative integers. Then, assuming that $V = K^r$, $W = K^s$ with $r = s$ the possible linear maps between V and W are given by $s \times r$ matrices over a finite field, and thus there are only finitely many possible linear maps between V and W. Then, since a representation is given by two vector spaces and two linear maps, the possible number of representations with length n is always finite. Thus, the Kronecker algebra is a counterexample. Indeed, the same holds for any path algebra KQ of infinite representation type, where Q is a finite quiver with no oriented cycles, that is, they are finite-dimensional algebras of infinite representation type not satisfying the second Brauer-Thrall conjecture. Indeed, that is why second Brauer-Thrall conjecture is stated over infinite fields. Hence, the natural generalization of the second Brauer-Thrall conjecture to arbitrary artinian rings is the one that forces the centers of the rings to be infinite since any K-algebra contains a copy of K in its center, and otherwise examples over finite fields such that the Kronecker algebra, which is an artinian ring, would automatically be counterexamples for the conjecture itself.

The first part of this section was mainly devoted to the study of algebraic bimo dules. In this one we will focus on the non-algebraic case, that is, $\left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ is a not a finite-dimensional algebra over any field . In this case, the dimension of $\left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ over its center is infinite. Anyway, $\left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ is still artinian (Auslander et al. (1997) Ch. 3 Prop. 2.1 p. 72) if we impose that $dim_F M$ and $dim_G M$ are finite. Indeed, the one we will construct will satisfy that $dim_F M = 2$ and $dim_G M = 2$.

Let us construct it. For, let F be any field, and let $\delta : F \to F$ be a non-zero derivation of F. That is, δ is a map which satisfies

$$\delta(x + y) = \delta(x) + \delta(y) \ , \ \ \delta(xy) = x\delta(y) + \delta(x)y \tag{3.16}$$

Then, we can construct the F, F- bimodule $M_F(\delta)$, where the underlying vector space is $F \oplus F$, and with F-multiplication on the left the canonical one (i.e. $a*(x, y) = (ax, ay)$) but with right multiplication given by

$$(x, y) * a = (xa + y\delta(a), ya) \tag{3.17}$$

Then $M_F(\delta)$ has both left and right dimension over F equal to 2.

We shall need the following general lemma in order to show that $M_F(\delta)$ is a non-algebraic bimodule:

Lemma 3.3.7. *Let F/E be a finite field extension and let $\delta : F \to F$ be a derivation s.t. $\delta(E) = 0$. Then $\delta = 0$.*

Proof. Let $\alpha \in F$, and let $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in E[X]$ be its minimum polynomial over E. This means that $f(\alpha) = 0$ and that $g(x) \neq 0$ for every polynomial in $E[X]$ of degree less than n. We must show that $\delta(\alpha) = 0$.

For, we have

$$0 = \delta(0) = \delta(\alpha^n+a_{n-1}\alpha^{n-1}+...+a_1\alpha+a_0) = \delta(\alpha^n)+\delta(a_{n-1}\alpha^{n-1})+...+\delta(a_1\alpha)+\delta(a_0) \tag{3.18}$$

Now, $\delta(a_0) = 0$ since $a_0 \in E$. Also we have, for every $1 \leq r \leq n$, $\delta(\alpha^r) = r\alpha^{r-1}\delta(\alpha)$ and (writing $a_n = 1$)

$$\delta(a_r\alpha^r) = a_r\delta(\alpha^r) + \delta(a_r)\alpha^r = ra_r\alpha^{r-1}\delta(\alpha) \tag{3.19}$$

since every $\delta(a_r) = 0$ as $a_r \in E$.

Therefore, we get

$$0 = \delta(0) = \delta(\alpha^n+a_{n-1}\alpha^{n-1}+...+a_1\alpha+a_0) = n\alpha^{n-1}\delta(\alpha)+a_{n-1}\alpha^{n-2}\delta(\alpha)+...+a_1\delta(\alpha) \tag{3.20}$$

$$= (a_n\alpha^{n-1} + a_{n-1}\alpha^{n-2} + ... + a_1)\delta(\alpha) \tag{3.21}$$

Finally, since $g(x) = a_nx^{n-1} + a_{n-1}x^{n-2} + ... + a_1$ is a polynomial over E of degree lower than n we must have $g(\alpha) \neq 0$, and so $\delta(\alpha) = 0$.

$\square$

Corollary 3.3.8. *$M_F(\delta)$ is a non-algebraic bimodule.*

Proof. If it were algebraic we should be able to find a subfield E of F s.t. F is a finite field extension of E and such that $a * (x, y) = (ax, ay) = (xa + y\delta(a), ya) = (x, y) * a$ for every $a \in E$ and $x, y \in F$. But that would imply $\delta(a) = 0$ for every $a \in E$ and so also, since F is a finite field extension of E, it should be 0 on F too by Lemma 3.7, and hence $\delta = 0$. But this would contradict the assumption of δ being non-zero. $\square$

Previously, we have shown that for algebraic bimodules M we can understand the modules over $\left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ as tuples (V, W, ϕ) where V is a f.d. right vector space over F, W is a f.d. right vector space over G and $\phi : V \otimes M \to W$ is a G-linear map. Indeed, exactly by the same arguments given in (Auslander et al. (1997) Prop. 2.2. p. 74) we have an equivalence of categories between mod$-R_M$, where $R_M = \left(\begin{smallmatrix} F & M_F(\delta) \\ 0 & F \end{smallmatrix}\right)$, and $\mathfrak{L}(M_F(\delta))$ whose objects are the tuples (V, W, ϕ) where V and W are f.d. right vector spaces over F, and $\phi : V \otimes M_F(\delta) \to W$ is a F-linear map.

The arguments used in (Auslander et al. (1997) Prop. 2.3 p. 75 Prop. 2.5 p. 77) of only depends on the construction of the equivalence functor and the artinianity of the upper triangular ring involved, and so those theorems also apply for non-algebraic bimodules. We also saw in Lemma 3.4 that, for algebraic modules, the length of a module corresponds to the sum of the dimensions $dim_F V + dim_G W$, similarly as for path algebras, and that the composition factors of (V, W, ϕ) are $\{dim_F V * (F, 0, 0), dim_G W * (0, G, 0)\}$. In that proof we did not use in any moment that M was algebraic. Therefore, we can also identify modules with representations in the non-algebraic case and study the property $\mathfrak{X}$ in terms of representations.

A field F equipped with a non-zero derivation $\delta : F \to F$ is called a differential field, and a differential field is called differentially closed if every finite system of differential equations which has a solution in some differential field extension of F has also a solution in F. Cozzens showed in Cozzens (1970) that if F is differentially closed then $R_\delta^F = F[T; \delta] := \{\Sigma_{i=0}^n a_i T^i : a_i \in F\}$ and with right multiplication defined by

$Ta = aT + \delta(a)$ has only one simple module S which is injective (and thus example of V-ring). This means in particular that R_δ^F contains only one indecomposable of finite length, namely S, because any finite length module contains a simple submodule, and so contains S, but S is injective so it splits in it. Hence, R_δ^F trivially satisfies $\mathfrak{X}$. Anyway, what is important for us are the following theorem and corollary:

Theorem 3.3.9. *(Ringel (1976) Thm. 4) Let M be a non-simple non-algebraic bimodule with $dim_F M = dim\ M_G = 2$. In $\mathfrak{L}(M)$ there is precisely one indecomposable representation with dimension vector $(n, n+1)$ and one with $(n+1, n)$ for every $n \in \mathbb{N} \cup \{0\}$. All the other indecomposable representation have dimension vector (x, x) and their direct sums form a full, exact abelian subcategory of mod$-R_M$, which is denoted as $\mathfrak{r}(M)$. The category $\mathfrak{r}(M)$ is equivalent, as abelian categories, to the product category $\mathfrak{M}_{R^F_{\delta,\epsilon}} \times \mathfrak{U}$, where $\mathfrak{M}_{R^F_{\delta,\epsilon}}$ is the category of finite length modules over the corresponding $R^F_{\delta,\epsilon}$, and $\mathfrak{U}$ is a category with only one simple object U and such that for every $n \in \mathbb{N}$ there is exactly one indecomposable object with length n. $R^F_{\delta,\epsilon}$ is a skew polynomial ring in one variable for some automorphism ϵ of F and some $(1, e)$-derivation δ of F.*

Conversely, given a division ring F, an automorphism ϵ of F and a $(1, e)$-derivation δ of F, there exists a bimodule M over F with $\mathfrak{r}(M) = \mathfrak{M}_{R^F_{\delta,\epsilon}} \times \mathfrak{U}$.

Corollary 3.3.10. *(Ringel (1976) Cor. 4.2) Let $R_M = \left(\begin{smallmatrix} F & M_F(\delta) \\ 0 & F \end{smallmatrix}\right)$, where F is a differentially closed field and $0 \neq \delta : F \to F$ a derivation . Then, for every $n \in \mathbb{N} \cup \{0\}$ there is precisely one indecomposable representation with dimension vector $(n, n+1)$ and one with $(n+1, n)$. All the other indecomposable representation have dimension vector (x, x) and their direct sums form a full, exact abelian subcategory of mod$-R_M$ which is equivalent, as abelian categories, to the product category $\mathfrak{M}_{R^F_\delta} \times \mathfrak{U}$, where $\mathfrak{M}_{R^F_\delta}$ is the category of finite length modules over the corresponding R^F_δ, and $\mathfrak{U}$ is a category with only one simple object U, and such that for every $n \in \mathbb{N}$ there is exactly one indecomposable object with length n over U. In that case R^F_δ has only one simple module S, which is injective.*

Let us understand what two results are saying. Firstly, recall that $M_F(\delta)$ is not a simple bimodule since $F \oplus 0$ is a proper submodule of it. Therefore by these two results it follows that in $R_M = \left(\begin{smallmatrix} F & M_F(\delta) \\ 0 & F \end{smallmatrix}\right)$ for any $n \in \mathbb{N}$ we have exactly two indecomposable modules with length $2n+1$, and the rest of the indecomposable modules of finite length have even length. Furthermore, by our previous discussion we know that $\mathfrak{M}_{R_\delta^F}$ has exactly one indecomposable module of finite length , which is simple injective, and so denoting it by S, we know that the indecomposable objects in $\mathfrak{M}_{R_\delta^F} \times \mathfrak{U}$ are S and $\{U_n\}_{n\in\mathbb{N}}$, with U_n indecomposable object in $\mathfrak{U}$ with length n. Since $\mathfrak{U}$ contains only one simple object, namely U, a composition series for U_n in $\mathfrak{U}$ has necessarily the form

$$0 \subseteq U = U_1 \subseteq ... \subseteq U_{n-1} \subseteq U_n \tag{3.22}$$

with $U_{i+1}/U_i \cong U \ \ \forall i.$

We need to be careful about something here: S and U are simple objects in $\mathfrak{M}_{R_\delta^F} \times \mathfrak{U}$ but they are not in mod$-R_M$ (after identifying $\mathfrak{M}_{R_\delta^F} \times \mathfrak{U}$ with the full exact abelian subcategory of mod$-R_M$). Indeed, the only simple objects in mod$-R_M$ are (up to isomorphism) $(F, 0, 0)$ and $(0, F, 0)$, and hence since both S and U have even length, they cannot be simple. Actually, an object can be indecomposable in a subcategory and not be indecomposable in the category itself, but in our case the indecomposability of S and the U_n's is guaranteed by the final paragaraph of Prop. 3.3.5 since $\mathfrak{M}_{R_\delta^F} \times \mathfrak{U}$ is equivalent to a full exact abelian subcategory of mod$-R_M$.

After identifying modules with representations we can say that every indecomposable object in mod$-R_M$ corresponding to one in $\mathfrak{M}_{R_\delta^F} \times \mathfrak{U}$ has dimension vector (n, n), and so one would expect the simple objects S and U in $\mathfrak{M}_{R_\delta^F} \times \mathfrak{U}$ to have dimension vector $(1, 1)$. Indeed, that is the case (Ringel (1976) Thm. 7.4). This means that the length of S and U is two in mod$-R_M$, and that every U_n has length $2n$. Hence, it follows that in mod$-R_M$ we have for every odd length exactly two indecomposable modules, two with length two, and one for every other even length. Thus, R_M is of infinite representation type but does not satisfy the hypothesis of the second Brauer-Thrall conjecture.

In relation to the property $\mathfrak{X}$ we saw in the previous subsection that, for algebraic bimodules, the dimension vector determines completely the composition factors, but that proof again did not use that the bimodules appearing were algebraic.

This means, since for R_M there is only one indecomposable object with dimension vector $(n, n+1)$, one with $(n+1, n)$, one with (n, n) for $n > 1$ and two with $(1, 1)$, that the ring constructed by Ringel does not satisfy $\mathfrak{X}$ but it "almost" does: There are only two indecomposable non-isomorphic modules with the same composition factors (i.e. S and U). It is possible then that a further investigation on non-algebraic bimodules could lead to the discovery of an artinian ring of infinite type satisfying $\mathfrak{X}$.

3.4 Gabriel's theorem

From our discussion in the previous section about the Brauer-Thrall conjecture we deduced that any finite-dimensional algebra over an algebraically closed field satisfying $\mathfrak{X}$ must be of finite representation type. In particular, this holds over hereditary algebras. Furthermore, since every hereditary finite-dimensional algebra over an algebraically closed field K is Morita equivalent to a path algebra KQ where Q is a finite connected quiver, and we know that $\mathfrak{X}$ is a Morita invariant, then in order to classify, over algebraically closed fields, the hereditary algebras satisfying $\mathfrak{X}$ is enough to focus on the ones of the form KQ which are of finite representation type.

During this section we will fix an arbitrary field K and, if we do not state the contrary, all our quivers Q will be assume to be finite connected with no oriented cycles. We will follow Krause (2008). For any quiver, we will denote by Γ_Q its underlying graph, that is, in Γ_Q we do not have directions in the arrows, and we call them edges. Now, let $|Q_0| = n$ and label the vertices as $\{1, ..., n\}$. Denote by $d_{ij} = d_{ji}$ the number of edges between i and j in Γ_Q. Then Γ_Q induces a quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}^n$ given by

$$q(x) = q((x_1, ..., x_n)) = \sum_{i=1}^{n} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j \tag{3.23}$$

Definition.

(1) *q is called positive definite if* $q(x) > 0 \;\; \forall 0 \neq x \in \mathbb{Z}^n$

(2) *q is called positive semi-definite if* $q(x) \geq 0 \;\; \forall 0 \neq x \in \mathbb{Z}^n$

We can charaterize which Γ_Q have corresponding quadratic form positive definite or semidefinite in terms of the classical Dynkin diagrams A_n, D_n, E_6, E_7, E_8, which appear in the classification theory of simple Lie algebras, and the Euclidean or Extended Dynkin diagrams $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$:

Theorem 3.4.1. *(Krause (2008) Thm. 4.2.1) Let Γ be a connected graph and q its corresponding quadratic form. Then:*

(1) *q is positive definite iff Γ is a Dynkin diagram of the form A_n, D_n, E_6, E_7, E_8*

(2) *q is positive semi-definite but not positive definite iff Γ is an Euclidean diagram of the form $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$*

Definition. *Let Γ be a graph which is Dynkin or Euclidean.*

We define $\Delta = \{x \in \mathbb{Z}^n : \; q(x) \leq 1\}$. A non-zero element $x = (x_1, ..., x_n)$ in Δ is called a root, and a root is called positive if $x_i \geq 0 \; \forall i$.

Proposition 3.4.2. *(Krause (2008) Prop. 4.3.1) If Γ is Dynkin then Δ is a finite set.*

We are able now to state the classical Gabriel's theorem:

Theorem 3.4.3. *(Krause (2008) Thm. 5.1.1) Let Q be a quiver s.t. Γ_Q is one of the form A_n, D_n, E_6, E_7, E_8. Then the assignment $V \to \mathbf{dim}(V)$ induces a bijection between the isomorphism classes of indecomposable representations of Q and the positive roots corresponding to the quadratic form q associated to Γ_Q. In particular, there are finitely many isomorphism classes of indecomposable representations.*

Remark. *Due to the equivalence of categories between $\mathfrak{rep}_K Q$ and mod$-KQ$ we deduce that KQ is of finite representation type if Γ_Q is one of the form A_n, D_n, E_6, E_7, E_8. Also, in those cases Gabriel's theorem is saying in particular that the indecomposable representations are completely determined by its dimension vectors. In follows from Cor. 2.4 that, if Γ_Q is one of the form A_n, D_n, E_6, E_7, E_8, then KQ satisfies $\mathfrak{X}$; and this does not depend on the field K neither on the orientation of the arrows. In fact, a change of orientation does not change the number of indecomposable representations.*

Remark. *In case Γ_Q is of the form $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ then, for any $i \in Q_0$ we can construct infinitely many indecomposable representations of Q of the form $\{C^r I(i)\}_{r\in\mathbb{N}}$ and infinitely many of the form $\{C^{-r} P(i)\}_{r\in\mathbb{N}}$, where each $P(i)$ is the projective cover of $S(i)$ and $I(i)$ is the injective hull of $S(i)$. Both of them always exist for any vertex i (see Schiffler (2014) Ch. 2) and they are indecomposable. Here $C : \mathfrak{rep}_K Q \to \mathfrak{rep}_K Q$ and $C^- : \mathfrak{rep}_K Q \to \mathfrak{rep}_K Q$ are the Coxeter functors associated to Γ_Q, which were introduced in Bernstein et al. (1973) and have proven to be a fundamental tool for constructing indecomposable representation of arbitrary length under good circunstances like in these cases. Therefore, we conclude that KQ, where Γ_Q is of the form $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ is always of infinite representation type, and hence if the field is algebraically closed then it does not satisfy $\mathfrak{X}$ by the second Brauer-Thrall conjecture.*

Remark. *If the quadratic form q associated to Γ_Q is indefinite (i.e. $\exists x \in \mathbb{Z}^n$ s.t. $q(x) < 0$) then it can be shown that KQ is always of infinite representation type since in that case we can find a subquiver Q' of Q of Euclidean type, and so KQ does not satisfy $\mathfrak{X}$ neither. In the indefinite case we say that KQ is of wild representation type.*

If KQ is not of finite type, neither wild, we say that it is of tame type (i.e. Γ_Q is of the form $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$). These definitions extend to finite-dimensional algebras over algebraically closed fields due to Drozd's theorem Drozd (2006).

If Q is a finite quiver with no oriented cycles then we can write $KQ \cong \Pi_i KQ_i$, where each Q_i is a connected component of Q. Since Q is finite, the number of connected components is finite. Hence, if KQ is of finite representation type, then every KQ_i must be of finite representation type, and vicerversa; and if KQ is of infinite representation type then at least one KQ_i must be of infinite representation type, and viceversa. Thus, it follows:

Corollary 3.4.4. *Let Q be a finite quiver with no oriented cycles and let K be any field. Call $Q_1, ..., Q_n$ the different connected components of Q. Then KQ is of finite representation type if and only if every Γ_{Q_i} is a Dynkin diagram of the form A_n, D_n, E_6, E_7, E_8.*

From these results together with the fact that over an algebraically closed field K any hereditary finite-dimensional algebra is Morita equivalent to KQ, where Q is a finite quiver with no oriented cycles, we deduce the following:

Theorem 3.4.5. *A hereditary finite-dimensional algebra A over an algebraically closed field satisfies $\mathfrak{X}$ if and only if A is of finite representation type.*

Our main aim in the next chapter will be to extend this result to arbitrary fields, and more generally to hereditary Artin algebras. For, we will deal with K-species and the Auslander-Reiten Theory.

CHAPTER FOUR
ARTIN ALGEBRAS AND K-SPECIES

This chapter is devoted to study, on the one hand, the basic representation theory of Artin algebras and, on the other the basic facts of the representation theory of K-species. The former helps in many cases to generalize discussions on finite-dimensional algebras over fields to algebras which are finitely generated over commutative artinian rings, while the latter is a generalized version of the representation theory of quivers with applications to the general representation theory of finite-dimensional algebras.

We have seen previously that any finite-dimensional hereditary algebra over an algebraically closed field is of finite representation type if and only if it satisfies $\mathfrak{X}$ [Ch. 3 Thm. 3.4.5], and also that over an arbitrary field those (hereditary) algebras induced by quivers of Dynkin type are of finite type and satisfy $\mathfrak{X}$ [Ch. 3 Thm. 3.4.3 and the first remark after it]. We will show indeed that a hereditary Artin algebra of finite representation satisfies $\mathfrak{X}$. The proof of such a fact appears in the book Auslander et al. (1997).

On the other side of the picture, the theory of K-species, and in particular the one of algebraic bimodules we have seen previously, will help us to show that finite-dimensional hereditary algebras of infinite representation type do not satisfy $\mathfrak{X}$, and hence with the theorem in Auslander et al. (1997) we will get the whole picture: A finite-dimensional hereditary algebra over an arbitrary field is of finite representation type if and only if it satisfies $\mathfrak{X}$. We will also indicate that this result extends naturally to arbitrary hereditary Artin algebras due to the fact that the center of any indecomposable hereditary Artin algebra is a field.

We will also show that a categorical tool such as the stable equivalence of algebras can sometimes help us to translate results about hereditary Artin algebras to Artin algebras with $J^2 = 0$, where J denotes the Jacobson radical. In particular, we will show that any Artin algebra with $J^2 = 0$ satisfying $\mathfrak{X}$ must be of finite type.

4.1 Hereditary Artin algebras of finite representation type satisfy $\mathfrak{X}$

4.1.1 The Auslander-Reiten translation

Through this section R will always denote a commutative artinian ring. Recall that a ring Λ is an R-algebra if there exists a ring homomorphism $R \to Z(\Lambda)$, where $Z(\Lambda)$ denotes the center of Λ. We say that Λ is an Artin R-algebra if it is finitely generated as a module over R. Clearly, an Artin R-algebra is a left and right artinian ring, and also an Artin R-algebra Λ is finitely generated over its center, which is also a commutative artinian ring, and hence it is a Artin $Z(\Lambda)$-algebra.

Let us first recall that over an artinian ring S finite length modules coincide with finitely generated ones and that projective covers exist for any finitely generated mo dule. Indeed, for a f.g. module M we can always find a projective cover P which is also f.g., and any two such f.g. projective covers are isomorphic (Auslander et al. (1997) Ch. 2), so we can say that such P is the projective cover of M without any loss of generality. Furthermore, if $S_1, ..., S_n$ are (up to isomorphism) all the simple S-modules, then denoting by P_i the projective cover of S_i we have that P_i is indecomposable (Auslander et al. (1997) Ch. 2) and $P_1, ..., P_n$ gives a complete list (up to isomorphism) of all f.g. projective indecomposable S-modules. This means that we have a decomposition $S \cong \oplus_i P_i^{n_i}$ for some non-negative integers n_i, $1 \leq i \leq n$.

Recall also that if N is a submodule of M with M of finite length, then N is also of finite length and $l(M) = l(N) + l(M/N)$. Hence, over an artinian ring submodules of finitely generated modules are finitely generated (indeed this is true over any noetherian ring). Now, let M be a f.g. module over S and let $f_0 : P_0 \to M$ be its projective cover. Then, $Ker f_0$ is f.g. since P_0 is also, and so we can consider again a projective cover $f_1 : P_1 \to Ker f_0$. We say then that the exact sequence $P_1 \to P_0 \to M \to 0$ is a minimal projective resolution of M, and it is unique (up to isomorphism of sequences) since projective covers are unique.

Consider now the functor $Hom_\Lambda(-,\Lambda) : \Lambda-mod\longrightarrow$ mod$-\Lambda$, where Λ is an Artin R-algebra. It can be checked (Auslander et al. (1997) Ch. 2 Thm. 3.3) that it induces a duality between the category of f.g. projective left Λ-modules and the one of right f.g. projective ones. Let now $P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of M and apply the functor $Hom_\Lambda(-,\Lambda)$ to f_1. Then we get a morphism

$$f_1* : Hom_\Lambda(P_0,\Lambda) \to Hom_\Lambda(P_1,\Lambda) \tag{4.1}$$

We call Cokerf_1* the transpose of M, and we denote it by TrM, which is a right f.g. module. The association $M \to TrM$ does not necessarily induce a functor between $\Lambda-$mod and mod$-\Lambda$, but it has the following useful properties:

Proposition 4.1.1. *Let Tr be defined as above. Then:*

(a) Tr commutes with finite direct sums

(b) $TrM = 0$ iff M is projective

(c) If M and N are finitely generated modules with no projective direct summands, then $TrM \cong TrN \Leftrightarrow M \cong N$

(d) If M is indecomposable non-projective then so is $Tr(M)$

(e) $Tr : \Lambda-$mod$\to$ mod$-\Lambda$ induces a bijection between the left indecomposable f.g. modules with no projective direct summands and the right indecomposable f.g. modules with no projective direct summands

In our proof of the conjecture for the commutative case we were using the fact that, for any commutative artinian ring R, $Hom_R(-,E)$ induces a duality in mod$-R$, where $E = \oplus_i E_i$ with each E_i being the injective hull (which is indecomposable) of the simple S_i . Indeed, such a duality also holds over Artin algebras. Namely, if Λ is an Artin R-algebra with left simple Λ-modules $S_1, ..., S_n$ and $E = \oplus_i E_i$, where each E_i is the injective hull of S_i, then $Hom_R(-,E) : \Lambda-$mod$\longrightarrow$mod$-\Lambda$ is a duality. We will denote it by D. The same duality could have been define from right modules to left modules, so if the context is clear we will denote also the duality from right to left as D.

Now, starting with a non-projective indecomposable f.g. left module M we know that TrM is an indecomposable non-projective right module, and so $DTrM$ is an indecomposable non-injective left module. Indeed, the composition $DTr : \Lambda-\text{mod}\longrightarrow\text{mod}-\Lambda$ induces a bijection between the indecomposable non-projective f.g. left modules and the indecomposable non-injective left ones, even if it not necessarily a functor. This is a fundamental point. Also, starting in the reverse order with a non-injective indecomposable module N, then $TrDN$ is an indecomposable non-projective module. $DTr(TrDN) \cong N$ if N is an inde composable non-injective and $TrD(DTrM) \cong M$ if M is an indecomposable non-projective. DTr is called the Auslander-Reiten translation, and TrD is called its inverse. The Auslander-Reiten translation has proven to be a very powerful tool in the representation theory of Artin algebras, and we will see now how to apply it to show that hereditary Artin algebras of finite representation type satisfy $\mathfrak{X}$.

4.1.2 Hereditary Artin algebras of finite representation type

Our aim now is to show, by using the Auslander-Reiten translation, that hereditary Artin algebras of finite representation type satisfy X. In order to do that, we first need to give the definitions of preprojective and preinjective module associated to a hereditary Artin R-algebra Λ. Let M be indecomposable non-projective module over Λ. Then we know that $0 \neq DTrM$ since DTr only kills projective modules. If now $DTrM$ is non-projective then we can apply again DTr and so on. We say that M is a preprojective module if there exists some non-negative integer n such that $(DTr)^n M$ is projective. By convention, $DTr^0 M = M$, so indeed projective modules are preprojective too. In the same fashion we say that M is preinjective if $(TrD)^n M$ is injective for some non-negative integer n.

Hereditary Artin algebras of finite representation type are characterized by the fact that every f.g. module is preinjective and preprojective:

Theorem 4.1.2. *(Auslander et al. (1997) Ch. VI Prop. 1.13) Let Λ be a hereditary Artin algebra over R. Then Λ is of finite representation type if and only if every f.g. indecomposable module is preprojective if and only if every f.g. indecomposable is preinjective.*

Let us define now the Grothendieck group $K_0(\Lambda)$ associated to an Artin algebra, or more generally to an artinian ring Λ: Let $F_0(\Lambda)$ denote the free abelian group generated by all the isoclasses of f.g. left Λ-modules. For each M in $\Lambda-$mod, denote by $[M]$ its isoclass. Now, let $R_0(\Lambda)$ be the subgroup generated by all the expressions of the form $[A] + [C] - [B]$ whenever there is a short exact sequence $0 \to A \to B \to C \to 0$. Then $K_0(\Lambda)$ is defined as $F_0(\Lambda)/R_0(\Lambda)$.

We have the following useful fact about $K_0(\Lambda)$:

Lemma 4.1.3. *Let M be a module with $\mathfrak{S}(M) = (m_1, ..., m_n)$, and call N the semisimple module $S_1^{m_1} \oplus ... \oplus S_n^{m_n}$. Then $[M] = [N]$ in $K_0(\Lambda)$.*

Proof. Let $0 = M_0 \subseteq M_1 \subseteq ... \subseteq M_r = M$ be a composition series for M. Then for $i = 1, ..., r$ we have the short exact sequences

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0 \tag{4.2}$$

Therefore, in $K_0(\Lambda)$ we have

$$[M] = [M_{r-1}]+[M/M_{r-1}] = [M_{r-2}]+[M_{r-1}/M_{r-2}]+[M/M_{r-1}] = ... = \Sigma_{i=1}^{i=r}[M_i/M_{i-1}] \tag{4.3}$$

which is equal to $[S_1]^{m_1} \oplus ... \oplus [S_n]^{m_n} = [N]$, and so $[M] = [N]$. □

Denoting by $S_1, ..., S_n$ the simple Λ-modules (up to isomorphism) we have that $[S_1], ..., [S_n]$ is indeed a $\mathbb{Z}-$basis for $K_0(\Lambda)$ (Auslander et al. (1997) Ch.1 Thm. 1.7).

This means in particular that $[S_1^{m_1} \oplus ... \oplus S_r^{n_r}] = [S_1^{m_1} \oplus ... \oplus S_r^{m_s}]$ in $K_0(\Lambda)$ if and only if $r = s$ and $m_i = n_i$ for all i. It follows then from the Lemma above that $[M] = [N]$ in $K_0(\Lambda)$ iff M and N have the same composition factors with the same multiplicities.

Indeed, over a hereditary Artin algebra, or more generally over an Artin algebra of finite global dimension (see Auslander et al. (1997) for definition and basic properties), the set $\{[P_1], ..., [P_n]\}$, where each P_i is the projective cover of S_i, is also a basis; and also $\{[E_1], ..., [E_n]\}$ is, where each E_i is the injective hull of S_i (see Auslander et al. (1997) Ch. 8 Lemma 2.1).

Define now $\mathfrak{c} : K_0(\Lambda) \to K_0(\Lambda)$ given by $\mathfrak{c}([P_i]) = -[E_i]$, which is clearly a group isomorphism by the discussion above. It is called the Coxeter transformation associated to a hereditary Artin algebra Λ. What is fundamental for us is the following theorem (Auslander et al. (1997) Ch. 8 Prop 2.2):

Theorem 4.1.4. *Let Λ be a hereditary Artin algebra and let $\mathfrak{c}$ be its Coxeter transformation. We then have the following:*

- *If M is a f.g. indecomposable non-projective module, then $\mathfrak{c}([M]) = [DTrM]$*
- *If M is a f.g. indecomposable non-injective module, then $\mathfrak{c}^{-1}([M]) = [TrDM]$*

We say that a non-zero element X in $K_0(\Lambda)$ is positive (resp. negative) if all its coordinates with respect to the basis $[S_1], ..., [S_n]$ are non-negative (resp. non-positive). We have the following:

Proposition 4.1.5. *(Auslander et al. (1997) Ch. 8 Prop 2.2) Let Λ be a hereditary Artin algebra, and let M be a f.g. indecomposable module. Then:*

- *M is projective iff $\mathfrak{c}([M])$ is negative*
- *M is injective iff $\mathfrak{c}^{-1}([M])$ is negative*

Since any $[M]$ in $K_0(\Lambda)$ contains precisely modules with the same composition factors and the same multiplicities than M, in order to prove that hereditary Artin al gebras of finite representation type satisfy $\mathfrak{X}$ it is enough to show then that if M and N are indecomposable modules such that $[M] = [N]$ in $K_0(\Lambda)$, then $M \cong N$.

Theorem 4.1.6. *(Auslander et al. (1997) Ch. 8 Cor. 2.3) Let M and N be inde composable modules over a hereditary Artin algebra s.t. $[M] = [N]$, and let $\mathfrak{c}$ be its Coxeter transformation. Then we have:*

a) If M is projective then $M \cong N$

b) If M is preprojective then $M \cong N$

c) If M is injective then $M \cong N$

d) If M is preinjective then $M \cong N$

Proof. We only prove part $a)$ and $b)$ since $c)$ and $d)$ are the duals of them.

It is clear that if M is projective then also N is projective by the previous proposition since $[M] = [N] \Rightarrow \mathfrak{c}([M]) = \mathfrak{c}([N])$. Let now M' be a maximal submodule of M. Then, $M/M' \cong S_j$ for some j and, if

$$0 \subseteq N_1 \subseteq ... \subseteq N_r = N \tag{4.4}$$

is a composition series for N there must exists some i such that $N_{i+1}/N_i \cong S_j$ since $[M] = [N]$. Consider now the projections $\pi_1 : N_{i+1} \to S_j$ and $\pi_2 : M \to S_j$. Then since M is projective there must exist a non-zero homomorphism $f : M \to N$ s.t. $\pi_1 \circ f = \pi_2$. Then, since Λ is hereditary, submodules of projective modules are projective and so Imf is projective. This means that $\tilde{f} : M \to Imf$ splits, and so $M \cong Kerf \oplus Imf$. But M is indecomposable and f is non-zero, so we must have $Kerf = 0$, and so f is a monomorphism. Therefore, since $[M] = [N]$, they have in particular the same length, and hence since $l(M) = l(N) = l(Imf) + l(N/Imf)$ we conclude that f is also onto and hence f is an isomorphism. This proves part $a)$.

For part $b)$ since M is preprojective we have that, for some non-negative integer $n+1$, $\mathfrak{c}^{n+1}([M])$ is negative. Let $n+1$ be the least such number. Then $[M]=[N] \Rightarrow \mathfrak{c}^{n+1}([M]) = \mathfrak{c}^{n+1}([N])$ and so also N is preprojective. This means in particular, since $\mathfrak{c}^{n}([M]) = [(DTr)^n M]$, that $[(DTr)^n M] = [(DTr)^n N]$, and so both $(DTr)^n M$ and $(DTr)^n N$ are projective indecomposable. Hence, by $a)$ we have $(DTr)^n M \cong (DTr)^n N$ and so

$$M \cong (TrD)^n((DTr)^n M) \cong (TrD)^n((DTr)^n N) \cong N \tag{4.5}$$

as desired.

□

Corollary 4.1.7. *(Auslander et al. (1997) Ch. 8 Cor. 2.4) Let Λ be an hereditary Artin algebra of finite representation type. Then Λ satisfies $\mathfrak{X}$.*

Proof. It follows since over hereditary Artin algebras of finite representation type every f.g. indecomposable module is preprojective by Thm. 4.1.2. □

Corollary 4.1.8. *Any finite-dimensional hereditary K-algebra of finite representation type satisfies $\mathfrak{X}$.*

4.2 Hereditary Artin algebras of infinite representation type do not satisfy $\mathfrak{X}$

4.2.1 Hereditary Artin algebras of infinite representation type

We have seen in the last section that hereditary Artin algebras of finite representation type satisfy $\mathfrak{X}$. Our aim in this section is to prove that this fact indeed chracterizes them, meaning that a hereditary Artin algebra satisfies $\mathfrak{X}$ if and only if it is of finite representation type. In order to do that we will prove that hereditary Artin algebras of infinite representation type do not satisfy $\mathfrak{X}$. For, the theory of algebraic bimodules studied in the Chapter 2 will play a fundamental role.

Let us start by reminding that in Ch. 2 Prop. 2.2.1 we showed that if a ring R is a finite product of rings R_i and each R_i satisfies $\mathfrak{X}$, then also R satisfies $\mathfrak{X}$. Indeed, the converse is true since if $R = \prod\limits_{1 \le i \le n} R_i$ then any indecomposable module over R is an indecomposable module over some R_i. Hence, we get:

Lemma 4.2.1. *Let R and R_i's be rings $R = \prod\limits_{1 \le i \le n} R_i$. Then R satisfies $\mathfrak{X}$ iff every R_i satisfies $\mathfrak{X}$.*

Remark. *Assuming that R is an artinian ring, we know that any decomposition of R into a product of indecomposable rings will be finite. So, if it is of infinite representation type then at least one of the indecomposable direct factors must be of infinite representation type, and so if we want to prove that R does not satisfy $\mathfrak{X}$, it is enough to show that such an indecomposable direct factor does not. Hence, if we want to prove that an Artin algebra of infinite representation type does not satisfy $\mathfrak{X}$ it is enough to show that an indecomposable Artin algebra of infinite representation type does not satisfy it. Thus, we may assume that all the Artin algebras in the sequel are indecomposable.*

Definition. *An Artin algebra Λ is called basic if we can write $\Lambda = P_1 \oplus ... \oplus P_n$ with P_i's pairwise non-isomorphic projective indecomposable modules. This is equivalent to saying that Λ/J is a product of division rings.*

Suppose now that Λ is an arbitrary Artin algebra and $\Lambda = P_1^{n_1} \oplus ... \oplus P_m^{n_m}$ with $n_i > 0$ for any i, and the P_i's pairwise non-isomorphic indecomposable projective modules. Let now $P = P_1 \oplus ... \oplus P_m$ and let $\Gamma = End_\Lambda(P)^{op}$ (the opposite ring of $End_\Lambda(P)$). It can be checked that Γ is a basic Artin algebra Morita equivalent to Λ (see Auslander et al. (1997) p. 35-36). Hence, since $\mathfrak{X}$ is a Morita invariant, by the remark above we can assume that all the Artin algebras that we are dealing with are indecomposable and basic.

The following theorem, which appears in (Dlab & Ringel (1978) Section 4 Lemma 1), is the first result towards proving our aim:

Theorem 4.2.2. *Let R be a indecomposable basic semiprimary ring (i.e. J is nilpotent and R/J is artinian), and assume there exist primitive orthogonal idempotents $g \neq f$ such that $F = fRf$ and $G = gRg$ are division rings. Then there is a full and exact embedding of the category of finite length modules over the upper triangular matrix ring $\left(\begin{smallmatrix} F & fJg \\ 0 & G \end{smallmatrix}\right)$ into mod$-R$.*

Since every artinian ring is semiprimary this result applies in our context. Assume now that R is furthermore hereditary artinian, and let $e_1, ..., e_n$ be a complete set of orthogonal primitive idempotents of R. Then the following proposition tells us that every e_iRe_i is a division ring.

Proposition 4.2.3. *Let R be a hereditary artinian ring and let e be a primitive idempotent of R. Then $eRe \cong End_R(Re)$ is a division ring.*

Proof. Let us first show that $End_R(Re)$ is a division ring. For, let $0 \neq f : Re \to Re$. Then Imf is a submodule of Re, and Re is projective. Therefore, since R is hereditary, submodules of projective modules are projective and so Imf is projective. This just means that $\tilde{f} : Re \to Imf$ splits, but Re is indecomposable, so $Kerf = 0$ and so f is a monomorphism. Re is cyclic so its length as a module over R is bounded by the length of R which is finite since R is artinian. Thus, Re is a finite length module and so since a monomorphism between two finite length modules with the same length must be an isomorphism we conclude that f is an isomorphism and so $End_R(Re)$ is a division ring.

Now, every $g : Re \to Re$ is given by $g(e) = xe$ for some $x \in R$, and so given by $g(e) = g(e^2) = eg(e) = exe$, and every such g is well-defined since

$$ae = 0 \Rightarrow aee = 0 \Rightarrow 0 = aexe = g(ae) \tag{4.6}$$

Hence, the result follows. □

Corollary 4.2.4. *Let R be a hereditary local artinian ring. Then R is a division ring.*

Proof. It follows from Prop. 4.2.3 since over a local rings the only idempotents are 0 and 1, and so in particular $e = 1$ is a primitive idempotent. $\square$

Thus, since division rings trivially satisfy $\mathfrak{X}$ we may assume from now on also that our hereditary artinian rings are not local. In those cases we can always find primitive orthogonal idempotents $1 \neq f, g$. Hence, from Thm. 4.2.2, Prop. 4.2.3 and the fact that the notions of finitely generated and finite length module coincide over artinian rings we get:

Corollary 4.2.5. *Let R be a basic indecomposable hereditary artinian ring, and let $\{e_1, ..., e_n\}$ be a complete set of orthogonal primitive idempotents of R. Then $F_i = e_iRe_i$ is a division ring for any i, and for any $i \neq j$ we can find a full exact embedding of the category of finitely generated modules over the upper triangular matrix ring $\left(\begin{smallmatrix} F_i & e_iJe_j \\ 0 & F_j \end{smallmatrix}\right)$ into mod$-R$.*

Hence, in view of this corollary and Prop. 3.3.5 of Ch. 3, if we want to prove that a basic indecomposable hereditary Artin algebra Λ of infinite representation type do not satisfy $\mathfrak{X}$ it could be a good strategy to prove that one of the rings of the form $\left(\begin{smallmatrix} e_i\Lambda e_i & e_iJe_j \\ 0 & e_j\Lambda e_j \end{smallmatrix}\right)$ does not satisfy $\mathfrak{X}$. Indeed, whenever such a ring is a finite-dimensional algebra over some field K, the theory of algebraic bimodules developed in the previous chapter tells us precisely under which conditions they do not satisfy $\mathfrak{X}$, that is, precisely whenever $dim_{e_i\Lambda e_i} e_iJe_j \cdot dim\ e_iJe_{j_{e_j\Lambda e_j}} \geq 4$, since e_iJe_j is a left module over $e_i\Lambda e_i$ and a right module over $e_j\Lambda e_j$.

In fact, every indecomposable hereditary Artin algebra Λ is a finite-dimensional al gebra over some field K, namely $Z(\Lambda)$, as the following theorem shows:

Theorem 4.2.6. *(Robson & Small (1974) Thm. 4) Let Λ be an indecomposable hereditary Artin algebra. Then the center of Λ is a field.*

Proof. Call Z its center. Let us recall first that if Z contains a non-trivial idempotent e then by (Auslander et al. (1997) Ch. 2 Prop. 5.1) we can write $\Lambda \cong e\Lambda \times (1-e)\Lambda$ as rings. But Λ is indecomposable, so Z does not contain a non-trivial idempotent, and so it is a local ring.

We are going to show now that Z is an integral-domain. For, assume $ab = 0$ for some $0 \neq a, b \in Z$. This means that $0 \neq ann_\Lambda(b)$. Since $\Lambda b \cong \Lambda / ann_\Lambda(b)$ we have the following short exact sequence

$$0 \to ann_\Lambda(b) \to \Lambda \to \Lambda b \to 0 \tag{4.7}$$

which must split since Λ is hereditary and Λb is a submodule of Λ, which is projective. Hence, we get $\Lambda \cong b\Lambda \oplus ann_\Lambda(b)$. Then since any direct summand of Λ is generated by an idempotent, we can say $ann_\Lambda(b) = e\Lambda$ with e idempotent. Since b is in the center we have $ann_\Lambda(b) = ann(b)_\Lambda$, and since an artinian ring is right hereditary if and only if is left hereditary we can repeat the same argument on the right and get $e\Lambda = ann_\Lambda(b) = ann(b)_\Lambda = \Lambda f$ for some idempotent f.

By the equality above there exists $s, t \in R$ s.t. $f = es$ and $e = tf$. Then we have $f = es = e^2 s = ef$ and $ef = tff = tf = e$, and so we get $e = f$. Let now $r \in \Lambda$, then there exist $x, y \in \Lambda$ s.t. $er = xe$ and $re = ey$. So we have $ey = eey = ere = xee = xe$ and hence $re = er$. This just means that e is a central idempotent. But this contradicts the fact that Z does not contain a non-trivial idempotent. Hence, Z is an integral domain.

Now, from Ch. 2 Lemma 2.1.2 we know that an integral domain which is not a field is never artinian. Thus, since Z is artinian as Λ is an Artin algebra we conclude that it must be a field. $\square$

Corollary 4.2.7. *Let Λ be an indecomposable hereditary Artin algebra and let e and f be primitive orthogonal idempotents in Λ. Then $\left(\begin{smallmatrix} e\Lambda e & eJf \\ 0 & f\Lambda f \end{smallmatrix}\right)$ is a finite-dimensional al gebra over $Z(\Lambda)$.*

Proof. By the theorem above we know Λ is a finite-dimensional algebra over its center

$Z(\Lambda)$. Let $E = eZ(\Lambda)e = Z(\Lambda)e$. Then the homomorphism $\phi : Z(\Lambda) \to Z(\Lambda)e$ given by $\phi(a) = ae$ is an isomorphism: It is clearly well-defined and onto, and if $ae = 0$, with $0 \neq a$, then $0 = ae = a^{-1}ae = e$ contradicting e being a primitive idempotent; so it is mono. This means that $e\Lambda e$ is a $Z(\Lambda)$-algebra since $E \subseteq Z(e\Lambda e)$. Also, since $e\Lambda e$ is a subring of Λ and Λ is a finite-dimensional algebra over $Z(\Lambda)$, then also $e\Lambda e$ must be a finite-dimensional algebra over $Z(\Lambda)$ because the dimension of $e\Lambda e$ as a vector space over $Z(\Lambda)$ is lower than or equal to the one of Λ, which is finite. Similar argument holds for $f\Lambda f$, so they are finite-dimensional algebras over $Z(\Lambda)$. Also, by Prop. 4.2.3 both $e\Lambda e$ and $f\Lambda f$ are division rings. Hence, they are finite-dimensional division algebras over $Z(\Lambda)$. On the other hand, $Z(\Lambda)$ acts centrally on $eJf \subseteq \Lambda$ and eJf is a finite-dimensional vector space over $Z(\Lambda)$. Therefore by the characterization of algebraic bimodules given at the begining of Section 3.1 Ch. 3 we conclude that $\left(\begin{smallmatrix} e\Lambda e & eJf \\ 0 & f\Lambda f \end{smallmatrix}\right)$ is a finite-dimensional algebra over $Z(\Lambda)$.

□

Thus, it follows:

Theorem 4.2.8. *Let Λ be a basic indecomposable hereditary Artin algebra and let $\{e_1, ..., e_n\}$ be a complete set of primitive orthogonal idempotents for Λ. Denote $F_i = e_i\Lambda e_i$ and ${}_iJ_j = e_iJe_j$ for every $i, j = 1, ..., n$. If there exist $i \neq j$ s.t. $dim_{F_i}{}_iJ_j \cdot dim\ {}_iJ_j{}_{F_j} \geq 4$, then Λ does not satisfy $\mathfrak{X}$.*

Proof. By the corollary above we know that for $i \neq j$ every $\left(\begin{smallmatrix} F_i & {}_iJ_j \\ 0 & F_j \end{smallmatrix}\right)$ is a finite-dimensional $Z(\Lambda)$-algebra. This means, by the introductory paragraph to Section 3.1 of Ch. 3, that e_iJe_j is an algebraic F_i, F_j-bimodule. Also, Cor. 4.2.5 tells us that we have a full exact embedding of the category of finitely generated modules over the upper triangular matrix ring $\left(\begin{smallmatrix} F_i & {}_iJ_j \\ 0 & F_j \end{smallmatrix}\right)$ into mod$-\Lambda$. Furthermore, the remark after Ch. 3 Lemma 3.3.4 implies that $\left(\begin{smallmatrix} F_i & {}_iJ_j \\ 0 & F_j \end{smallmatrix}\right)$ does not satisfy $\mathfrak{X}$ iff $dim_{F_i}{}_iJ_j \cdot dim\ {}_iJ_j{}_{F_j} \geq 4$ since e_iJe_j is an algebraic F_i, F_j-bimodule. Finally, from

Prop. 3.3.5 of Ch. 3 it follows that if $\left(\begin{smallmatrix} F_i & {}_iJ_j \\ 0 & F_j \end{smallmatrix}\right)$ does not satisfy $\mathfrak{X}$ then Λ does not satisfy $\mathfrak{X}$ neither. $\square$

So, in view of the above theorem, we deduce in particular that basic indecomposable hereditary Artin algebras of infinite representation type s.t. $dim_{F_i i}J_j \cdot dim\ {}_iJ_{j_{F_j}} \geq 4$ for some $i \neq j$ do not satisfy $\mathfrak{X}$. Therefore, we have reduced our question to the study of basic indecomposable hereditary Artin algebras of infinite representation type s.t. $dim_{F_i i}J_j \cdot dim\ {}_iJ_{j_{F_j}} \leq 3\ \forall i \neq j$, and so in order to prove our aim it is just left to show that such an algebra cannot satisfy $\mathfrak{X}$.

It turns out that such a ring has the structure of what is called a tensor algebra, as we will show now. But firstly, the following lemmas are needed:

Lemma 4.2.9. *Let R be a basic hereditary ring, and let $e_1, ..., e_n$ be a complete set of primitive orthogonal idempotents of R. If for some $i \neq j$ we have $0 \neq e_iRe_j$, then $0 = e_jRe_i$.*

Proof. One can show first that, as abelian groups, $e_iRe_j \cong Hom_R(Re_i, Re_j)$ by a similar argument as in Prop. 4.2.3. Assume now that $\exists 0 \neq f : Re_i \to Re_j$. Then, since R is hereditary, Imf is projective and so $\tilde{f} : Re_i \to Imf$ must split. But Re_i is indecomposable because e_i is primitive, and so we must have $Kerf = 0$. This just means that f is a monomorphism, and so $l(Re_i) \leq l(Re_j)$. Now, since R is basic, principal indecomposables are pairwise non-isomorphic, that is, $Re_i \cong Re_j \Leftrightarrow i = j$. But, if $0 \neq e_jRe_i$ then there must exist a non-zero $g : Re_j \to Re_i$ which must be mono, and so $l(Re_j) \leq l(Re_i)$. But then, $l(Re_i) = l(Re_j)$ and f is an isomorphism, which is a contradiction. Hence, $0 = e_jRe_i$. $\square$

Lemma 4.2.10. *(Eilenberg andT (1955) Thm. 1) Let R be a semiprimary ring and let $I \subseteq J^2$ be an ideal such that R/I is hereditary. Then $I = 0$.*

Remark. *In the proof of Lemma 4.2.9 we have shown in particular that $0 \neq Hom_R(Re_i, Re_j)$ implies that any $0 \neq f \in Hom_R(Re_i, Re_j)$ is a* *monomorphism, and so $l(Re_i) < l(Re_j)$ since $Re_i \cong Re_j \Leftrightarrow i = j$ as R is basic. By the same reason, if $l(Re_i) = l(Re_j)$ with $i \neq j$, then $0 = Hom_R(Re_i, Re_j)$. Therefore, after reordering the principal indecomposable modules Re_i's in such a way that $l(Re_j) < l(Re_i) \Rightarrow j < i$, we can assume that $e_iRe_j = 0$ for every $j > i$ Also, $J \neq R$ for any unital ring R, and so $e_iJe_i \subsetneq e_iRe_i$, but e_iRe_i is a division ring and e_iJe_i is a proper submodule of it, so we must have $0 = e_iJe_i$. Hence, we can assume from now on that, if $e_1, ..., e_n$ are primitive orthogonal idempotents of R, then $0 = e_iJe_j$ for $j \geq i$.*

Coming back to the specific case that we are studying, we first notice that we can write $J = \bigoplus\limits_{j<i} e_iJe_j$, and so J has a natural bimodule structure over $F = \Pi_i e_i\Lambda e_i = \Pi_i F_i$. Now, the condition $dim_{F_i} e_iJe_j \cdot dim\ e_iJe_j{}_{F_j} \leq 3$ means that either $e_iJe_j = 0$ or $dim_{F_i} e_iJe_j = 1$ or $dim\ e_iJe_j{}_{F_j} = 1$. In any case, e_iJe_j must be either 0 or a simple F_i, F_j-bimodule, and so J is a semisimple bimodule over F. This means in particular that there exists a bimodule M over F s.t. $J = J^2 \oplus M$ as bimodules over F.

On the other hand, since Λ is basic, we can write $J = \bigoplus\limits_{j<i} e_i\Lambda e_j$ and so $F + J = \Lambda$. Furthermore, clearly $F \cap J = 0$ because $\sum\limits_{i,j} e_i\Lambda e_j = \sum\limits_{i} e_i\Lambda e_i \oplus \sum\limits_{i\neq j} e_i\Lambda e_j$.

We can construct now the tensor ring T associated to F and J:

Write $T = \sum\limits_{n=0}^{\infty} T^{(n)}$, where $T^{(0)} = F$, $T^{(1)} = M$ and for every $n > 1$ write the n-fold tensor product $T^{(n)} = T^{(1)} \otimes_F T^{(1)} \otimes_F ... \otimes_F T^{(1)}$ (n times). Recall also that $J = J^2 \oplus M$ where M is a bimodule over F. The multiplication on T is defined through the canonical isomorphism $T^{(i)} \otimes_F T^{(j)} \to T^{(i+j)}$ and extended distributively.

Proposition 4.2.11. *With the notations above, let T be the tensor ring associated to F and J. Then there exists $m \in \mathbb{N}$ s.t. $T^{(m)} = 0$. In that case T is a semiprimary ring.*

Proof. By the first paragraph of (Dlab & Ringel (1975) p. 388) there exists $m \in \mathbb{N}$ s.t. $T^{(m)} = 0$ if and only if T is semiprimary if and only if every sequence of vertices

$i_1, ..., i_n$ with $0 \neq e_{i_k} M e_{i_{k+1}}$ for $1 \leq k \leq n-1$ is of bounded length. We know by the remark after Lemma 4.2.10 that $e_i M e_j \subseteq e_i J e_j = 0 \ \ \forall j \geq i$. This means that $0 \neq e_{i_k} M e_{i_{k+1}}$ implies $i_{k+1} < i_k$, and so $i_1 > i_2 > ... > i_{n-1} > i_n$. Since the number of e_i's is finite, every such sequence is bounded precisely by the number of e_i's. Thus T is semiprimary and so there exists $m \in \mathbb{N}$ s.t. $T^{(m)} = 0$. □

Hence, T is a semiprimary ring. By (Dlab & Ringel (1975) p. 388), $rad\, T = \sum_{n=1}^{\infty} T^{(n)}$ and so $rad^2 T = \sum_{n=2}^{\infty} T^{(n)}$.

Proposition 4.2.12. *(Dlab & Ringel (1975) p. 389) With the notations above, $\Lambda \cong T$ as rings.*

Proof. Define $f : T \to \Lambda$ as the identity on F and M and extend it distributively. We claim that M generates J as a ring: For, J is nilpotent, so $0 = J^m$ for some m, and

$$J = J^2 \oplus M \Rightarrow J^{m-1} = MJ^{m-2} \Rightarrow \tag{4.8}$$

$$\Rightarrow J^{m-2} = MJ^{m-2} + MJ^{m-3} \Rightarrow ... \Rightarrow J = MJ^{m-2} + MJ^{m-3} + ... + MJ^2 + MJ + M \tag{4.9}$$

We have $MJ^{m-1} = MJ^{m-2}J = J^{m-1}J = 0$ and so we get:

$$MJ^{m-2} = M(J^2 + M)J^{m-3} = MJ^{m-1} + M^2 J^{m-3} = M^2 J^{m-3}.$$

In case of $m = 3$ (with the assumption $J^0 = 1$) we get:

$$J = MJ + M = M^2 + M \tag{4.10}$$

as desired. Otherwise, if $m \geq 4$ we get :

$$MJ^{m-2} = M^2 J^{m-3} = M^2(J^2 + M)J^{m-4} = M^2 J^{m-2} + M^3 J^{m-4} = M(MJ^{m-2}) + M^3 J^{m-4} = \tag{4.11}$$

$$= MJ^{m-1} + M^3J^{m-4} = M^3J^{m-4} \tag{4.12}$$

and

$$MJ^{m-3} = M(J^2 + M)J^{m-4} = MJ^{m-2} + M^2J^{m-4} = M^3J^{m-4} + M^2J^{m-4} \tag{4.13}$$

Thus, if $m = 4$ we can write

$$J = MJ^{m-2}+MJ^{m-3}+...+MJ^2+MJ+M = M^3J^{m-4}+M^2J^{m-4}+MJ^{m-4} = M^3+M^2+M \tag{4.14}$$

If $m > 4$ we can make a similar argument with MJ^{m-4}, and so on. The claim follows inductively.

Thus, M generates J as a ring and so, due to the fact that $F + J = \Lambda$, we conclude that f is onto. Hence, $\Lambda \cong T/I$ for $I = Kerf$. But now, since f is the identity on F and M and T is semiprimary, we must have $I \subseteq \sum_{n=2}^{\infty} T^{(n)} = rad^2T$. Then, by Lemma 4.2.10, since T is semiprimary and $I \subseteq rad^2T$ we conclude that $I = 0$ and so f is an isomorphism, as desired.

□

Thus, the ring we are studying is a tensor ring, and indeed a finite-dimensional al gebra over $K = Z(\Lambda)$ since we are assuming that Λ is indecomposable and hereditary in particular. The interesting fact is that the representation theory of such a tensor K-algebra can be understood in simple terms thanks to the theory of K-species that we are going to illustrate now.

4.2.2 K-species

Let K be a field, and let $F_1, ..., F_n$ be finite-dimensional division algebras over K. Consider now F_i, F_j-bimodules ${}_iM_j$ s.t. K operates centrally on every ${}_iM_j$ and every ${}_iM_j$ is a finite dimensional vector space over K. Then $S = (F_i, {}_iM_j)_{1\leq i,j\leq n}$ is called a K-species.

From S we can derive a quiver $Q(S)$ by writing an edge $i \to j$ whenever $0 \neq {}_iM_j$. We call S connected if the underlying graph of $Q(S)$ is connected, and we say that S has no oriented cycle if the derived oriented graph $Q(S)$ does not have an oriented cycle. The latter condition implies in particular that ${}_iM_i = 0$ for every i and that, if $i \neq j$, $0 \neq {}_iM_j \Rightarrow 0 = {}_jM_i$. We will assume throughout that S is a connected K-species without oriented cycles.

A representation (V_i, ϕ_{ij}) of S is given by a collection of right vector speces over F_i and F_j-linear maps $\phi_{ij} : V_i \otimes {}_iM_j \to V_j$. Such representation is called finite-dimensional provided that all V_i are finite-dimensional vector spaces. A homomorphism between two representations (V_i, ϕ_{ij}) and (W_i, ψ_{ij}) is given by a collection of F_i-linear maps $\alpha_i : V_i \to W_i$ such that $\alpha_j\phi_{ij} = \psi_{ij}(\alpha_i \otimes 1)$.

We denote by $\mathfrak{L}(S)$ the category of all finite-dimensional representations of S together with all the homomorphisms between them, which is indeed an abelian category. We say that S is of finite representation type if there are only finitely many isomorphism classes of indecomposable representations in $\mathfrak{L}(S)$. Otherwise, it is said to be of infinite representation type and in that case, as well as for quiver representations, the notions of tame and wild representation type apply by considering a quadratic form defined on $Q(S)$ which coincides with the usual one for quiver representations (see Ringel (1976) p. 1).

We want to note that in case all $F_i = K$ and all ${}_iM_j \in \{0, K\}$ a representation of S is just a representation of the quiver $Q(S)$. So, the representaion theory of K-species contains in some sense the representation theory of quivers. Also notice that the representations of a K-species of the form $S = (F_1, F_2, {}_1M_2)$ are just the representations of the algebraic bimodule ${}_1M_2$ so indeed the theory of algebraic bimodules we developed in the previous chapter is one part of the general representation theory of K-species.

What is interesting for us is that we can associate a tensor K-algebra with a connected K-species as follows:

Definition. *Let $S = (F_i, {}_iM_j)_{1\leq i,j\leq n}$ be a connected K-species. Write $F = \Pi_i F_i$ and $M = \underset{i,j}{\oplus_i} M_j$, which is a bimodule over F. Then $T(S) = F \oplus M \oplus M^{(2)} \oplus ... = \sum_{n=0}^{\infty} M^{(n)}$ with $M^{(0)} = F$, $M^{(1)} = M$ and $M^{(n)} = M \otimes_F ... \otimes_F M$ (n times) is the tensor al gebra associated with S.*

Remark. *By (Dlab & Ringel (1975) p. 388) $T(S)$ is semiprimary iff $M^{(m)} = 0$ for some m iff every sequence $i_1, ..., i_n$ with $0 \neq {}_{i_k}M_{i_{k+1}}$, for $1 \leq k \leq n-1$ is of bounded length. This means in particular that if $T(S)$ is semiprimary then S cannot contain any oriented cycle, since otherwise we can find an infinite path in S.*

It turns out that the category of finite dimensional representations of a connected K-species S without oriented cycles is equivalent, as an abelian category, to the category of f.g. right modules over $T(S)$, in a similar manner to $\mathfrak{rep}_K Q$ being equivalent to mod$-KQ$.

Proposition 4.2.13. *(Dlab & Ringel (1975) Prop. 10.1 p. 386) Let S be a connected K-species without oriented cycles. Then there is an equivalence of abelian categories between $\mathfrak{L}(S)$ and mod$-T(S)$.*

Remark. *Since we are assuming that S is connected with no oriented cycles, similar arguments to the ones used in the second chapter [Thm 2.1.1 and Lemma 2.3.4] tell us that $\mathfrak{L}(S)$ satisfies $\mathfrak{X}$ if and only if whenever we have two finite-dimensional indecomposable representations $V = (V_i, \phi_{ij})$ and $W = (W_i, \psi_{ij})$ of S s.t. $dim_{F_i} V_i = dim_{F_i} W_i$ $\forall i$, then $V \cong W$.*

We can do the reverse process, meaning, start with our specific indecomposable basic hereditary Artin algebra Λ of infinite representation type s.t. $dim_{F_i i}J_j \cdot dim\ {}_iJ_{j F_j} \leq 3\ \ \forall i \neq j$, where ${}_iJ_j = e_iJe_j$ and $F_i = e_i\Lambda e_i\ \ \forall i$, which turned out to be a tensor $Z(\Lambda)-$algebra [Prop 4.2.12]. We can write $M = \underset{i,j}{\oplus} e_iMe_j = \underset{i,j}{\oplus} {}_iM_j$, where M was such that $J = M \oplus J^2$ as $F = \prod\limits_i F_i-$modules. Associate to it now the K-species $S(\Lambda) = (F_i, {}_iM_j)_{1\leq i,j\leq n}$. Then $T(S(\Lambda)) \cong \Lambda$ by the definition of the tensor algebra associated to $S(\Lambda)$ since we know the tensor algebra structure of Λ by Prop. 4.2.12. Hence, in particular $T(S(\Lambda))$ is artinian and so semiprimary; and so by the remark above $S(\Lambda)$ does not contain any oriented cycles. Also, it is connected since Λ was assumed to be indecomposable.

Therefore, by Prop. 4.2.13 and the remark after Ch. 3 Cor. 3.3.6 we have:

Theorem 4.2.14. *There is an equivalence of abelian categories between* $\mathfrak{L}(S(\Lambda))$ *and mod*$-\Lambda$*, and so* Λ *satisfies* $\mathfrak{X}$ *if and only if* $\mathfrak{L}(S(\Lambda))$ *satisfies* $\mathfrak{X}$*.*

Thus, in view of Corollary 4.1.8 and Theorem 4.2.8, our discussion has been reduced to study the indecomposable representations over a connected K-species $S = (F_i, {}_iM_j)_{1\leq i,j\leq n}$ without oriented cycles s.t. $dim_{F_i i}M_j \cdot dim\ {}_iM_{j F_j} \leq 3\ \ \forall i \neq j$ (since $dim_{F_i i}M_j \leq dim_{F_i i}J_j\ \forall i, j$ because M is an $F-$ direct summand of J).

Before continuing we want to recall that, if S is a connected K-species of wild re presentation type, then we have a full exact embedding mod$-F[X, Y] \to \mathfrak{L}(S)$ for some field F (see Ringel (1976) Thm. 2). In that case, $F[X, Y]$ never satisfies $\mathfrak{X}$. A possible way of seeing this is by considering the quotient ring $A = F[X, Y]/(X^2, Y^2)$. A is a commutative artinian ring but it is not PIR since $0 \neq I = (X, Y)$ is a proper ideal of A. Then, by [Ch. 2 Thm. 2.2.9] it follows that A does not satisfy $\mathfrak{X}$, and hence $F[X, Y]$ does not satisfy $\mathfrak{X}$ neither. Thus, wild K-species do not satisfy $\mathfrak{X}$, and so we can assume that our K-species of infinite representation type have all tame type.

Luckily for us, a complete characterization of the connected K-species with no oriented cycles which are of tame type has been done Dlab & Ringel (1974), and a complete list of all the possible cases appears in there (see p. 3 of the article). So an inspection case by case of the possible K-species that can satisfy our condition could be done in order to see that they cannot satisfy $\mathfrak{X}$. However, a more general theorem applies to all connected K-species with no oriented cycles of tame representation type. Before stating the theorem we need the following definitions:

Definition. *Let M be an algebraic bimodule over F and G, where F and G are division rings. Then M is said to have type $\tilde{A}_{ab}$ if $dim_F M = a$ and $dim\ M_G = b$.*

Definition. *(Ringel (1976) 7.1) Let M be an algebraic bimodule s.t. $ab = 4$, $a \leq b$. Given an indecomposable representation $V = (V_1, V_2, \phi)$ its defect σV is defined as $\sigma V = dim_F V_1 - dim V_{2G}$ in case $2 = a = b$ and by $\sigma V = 2 \cdot dim_F V_1 - dim V_{2G}$ in case $a = 1$ and $b = 4$. A representation is called regular if it is the direct sum of indecomposable representations with zero defect. We denote by $\mathfrak{r}(M)$ the full subcategory of all regular (finite-dimensional) representations of M , which is indeed an abelian exact subcategory of $\mathfrak{L}(M)$ closed under extensions.*

Remark. *The definition in (Ringel (1976) 7.1) is stated for what are called affine bimo dules, but an algebraic bimodule M s.t. $ab = 4$ with $a \leq b$ is affine since it is a finite-dimensional K-algebra, where F and G are finite-dimensional algebras over a field K which acts centrally on M, so the affine conditions in (Ringel (1976) 6.1) are satisfied.*

We are ready now to state the theorem:

Theorem 4.2.15. *(Dlab & Ringel (1974) Thm. 5.1. p. 33) Let S be a connected K-species with no oriented cycles and of tame representation type. Then there exists an algebraic bimodule M of type $\tilde{A}_{14}$ or $\tilde{A}_{22}$ and a full exact embedding $\mathfrak{F} : \mathfrak{r}(M) \to \mathfrak{L}(S)$.*

Corollary 4.2.16. *If S be a connected K-species with no oriented cycles and of tame representation type. Then $\mathfrak{L}(S)$ does not satisfy $\mathfrak{X}$.*

Proof. For the case $2 = a = b$, the billinear form $\tilde{q}$ associated to M defined in [Ch. 3 Section 3] has the form (up to scalar multiplication) $\left(\begin{smallmatrix} 2 & -4 \\ 0 & 2 \end{smallmatrix}\right)$. In that case for

$x = (1,1)$ we have $\tilde{q}((1,1)) = 0$, and so there is an indecomposable representation V with dimension vector $(1,1)$, which satisfies $\sigma V = 0$, so it is a regular representation. Furthermore, $(1,1)$ is a fundamental vector so we know by Ch. 3 Lemma 3.3.3 that there are at least two non-isomorphic indecomposable (regular) representations with dimension vector $(1,1)$.

For the case of $a = 1, b = 4$ we have $\tilde{q}((1,2)) = 0$, so the same argument holds since $(1,2)$ is a fundamental vector and every indecomposable representation with dimension vector $(1,2)$ is regular in that case.

In any case, this just means that M has two non-isomorphic indecomposable regular representations with the same dimension vector and so $\mathfrak{r}(M)$ does not satisfy $\mathfrak{X}$. Therefore, by Thm. 4.2.14 and Ch. 3 Prop. 3.3.5 we conclude that $\mathfrak{L}(S)$ cannot satisfy $\mathfrak{X}$.

$\square$

Thus, by Thm. 4.2.14 we get the following:

Proposition 4.2.17. *Let Λ be an indecomposable basic hereditary Artin algebra of infinite representation type s.t. $dim_{F_i}{}_iJ_j \cdot dim\ {}_iJ_j{}_{F_j} \leq 3\ \ \forall i \neq j$, where $F_i = e_i\Lambda e_i$ and ${}_iJ_j = e_iJe_j$. Then Λ does not satisfy $\mathfrak{X}$.*

And together with Thm. 4.2.8 we get:

Theorem 4.2.18. *Any basic indecomposable hereditary Artin algebra of infinite representation type does not satisfy $\mathfrak{X}$.*

Finally, by Corollary 4.1.7 and the remark after 4.2.1 together with the fact that every Artin algebra is Morita equivalent to a basic Artin algebra we get:

Theorem 4.2.19. *The following are equivalent for a hereditary Artin algebra Λ:*

a) Λ *is of finite representation type*

b) Λ *satisfies* $\mathfrak{X}$

4.3 Artin algebras with $J^2 = 0$

Our motivation problem for this Master Thesis was the conjecture appearing in the end of the Auslander et al. (1997) book which asserts that if an Artin algebra Λ satisfies $\mathfrak{X}$ then it must be of finite representation type. In the previous section we have proved the conjecture for the hereditary case. In that case actually the converse was true , i.e. a hereditary Artin algebra is of finite representation if and only if it satisfies $\mathfrak{X}$. In this section, we will use that result to prove the conjecture also for Artin algebras with $J^2 = 0$. Our main tool in this case will be the stable equivalence of algebras which we will now introduce. We will follow the chapter on stable equivalence in Auslander et al. (1997).

4.3.1 Stable equivalence

Let Λ an Artin algebra and let $\underline{mod}\ \Lambda$ be the following category:

- Its objects are the objects of mod$-\Lambda$
- $\underline{Hom}\ (A, B) := \frac{Hom(A,B)}{P(A,B)}$ for any A, B in mod$-\Lambda$. $P(A, B)$ denotes the homomorphisms $f : A \to B$ that factor through a projective module, i.e. $\exists Q$ projective module and $g : A \to Q$, $h : Q \to B$ s.t. $f = h \circ g$

This means in particular that, for any projective module Q, $\underline{Hom}\ (Q, Q) = 0$, and so projective modules are the zero object of this category. We can therefore think of $\underline{mod}\ \Lambda$ as mod$-\Lambda$ modulo projectives.

Definition. *We say that two Artin algebras* Λ *and* Γ *are stably equivalent if there exists an equivalence of categories* $F : \underline{mod}\, \Lambda \to \underline{mod}\, \Gamma$.

Remark. *If two Artin algebras are Morita equivalent then the same functor that induces the equivalence induces a stable equivalence between them since projective modules and maps that factor though projectives are preserved under Morita equivalence. But it is not true that two stably equivalent algebras are necessarily Morita equivalent. As a generic example, let* A *be any indecomposable Artin algebra and let* B *be any semisimple algebra. Then any module over* B *is projective so trivially* A *is stably equivalent to* $A \times B$*. But they are not Morita equivalent since one is an indecomposable algebra and the other is not.*

Though, while dealing with Artin algebras many properties of Λ can be transfered to Γ under stable equivalence. For example, the representation type. Before showing that let us recall the following: Each M in mod$-\Lambda$ can be written uniquely (up to i somorphism) by Krull-Schdmit as $M' \oplus M_P$ where M' is projective and M_P has no projective direct summands. Then since Hom commutes with finite sums we have that $\underline{Hom}\,(M, A) = \underline{Hom}\,(M_P, A)$ and $\underline{Hom}\,(A, M) = \underline{Hom}\,(A, M_P)$ for any A in $\underline{mod}\, \Lambda$. Hence, we have that $\underline{mod}\, \Lambda$ is equivalent to the category whose objects are the ones with $M = M_P$, and morphisms as in $\underline{mod}\, \Lambda$. Call this category $\underline{mod}\, \Lambda_P$. For more details see (Auslander et al. (1997) p. 104-105).

Proposition 4.3.1. *(Auslander et al. (1997) 1.1 p. 336) Let* Λ *and* Γ *be stably equivalent Artin algebras. Then* Λ *is of finite representation type if and only if* Γ *is of finite representation type.*

Proof. First of all recall that Artin algebras have only finitely many indecomposable projective modules (i.e. the projective covers of the simples). Thus, it is enough to show that Λ has only finitely many indecomposable non-projective modules up to isomorphism if and only if Γ has only finitely many indecomposable non-projective modules up to isomorphism.

For, first recall that if we have a module M s.t. $End_\Lambda(M)$ is a local ring then M is indecomposable, because otherwise the projections into every direct summand gives non-trivial idempotents. Thus, calling F the functor that induces the stably equivalence, then F induces a bijective correspondence between the objects in $\underline{mod}\ \Lambda_P$ and in $\underline{mod}\ \Gamma_P$, so we just need to show that F takes indecomposable objects to indecomposable ones. In order to do that it is enough to show that if M is a non-projective module in $\underline{mod}\ \Lambda_P$ s.t. $End_\Lambda(M)$ is local then so is $End_\Gamma(F(M))$; and being local is equivalent to the radical to be a maximal submodule, and so to the quotient $End_\Lambda(M)/rad\ End_\Lambda(M)$ to be a simple artinian ring. Hence, since $\underline{mod}\ \Lambda_P$ and $\underline{mod}\ \Gamma_P$ are equivalent categories, it is enough to show that $P(M,M) \subseteq rad\ End_\Lambda(M)$ and $P(F(M), P(F(M)) \subseteq rad\ End_\Gamma(F(M))$ because in that case, due to the stable equivalence, we will have $End_\Lambda(M)/rad\ End_\Lambda(M) \cong End_\Gamma(F(M))/rad\ End_\Gamma(F(M))$. Since F takes non-projective modules to non-projective ones, we just need to show indeed that $P(M,M) \subseteq rad\ End_\Lambda(M)$.

For, we will show that every $f \in P(M,M)$ is non-invertible, and so it is in $rad\ End_\Lambda(M)$. Assume there exists such an invertible $f : M \to M$ in $P(M,M)$. This means that there exists some projective module P and some $g : M \to P$ and $h : P \to M$ s.t. $f = h \circ g$. Since f is invertible, g is a monomorphism and h is an epimorphism, and so $P = g(M) + Kerh$.

Therefore, $g(M) \cong M$ and $P/Kerh \cong M \cong g(M)$; and so $g(M) \cong P/Kerh = g(M) + Kerh/Kerh \cong g(M)/g(M) \cap Kerh$. Since $M \cong g(M)$ is a finite length module and $l(g(M)) = l(g(M)/g(M) \cap Kerh)$ we must have $g(M) \cap Kerh = 0$, and so $P = g(M) \oplus Kerh$. But then $M \cong g(M)$ is a direct summand of a projective module, so it is projective. But this is a contradiction. Thus, f cannot be invertible.

$\square$

4.3.2 Artin algebras with $J^2 = 0$

Let Λ be any Artin algebra such that $J^2 = 0$, where J denotes the Jacobson radical, and let Γ be the upper triangular matrix algebra $\left(\begin{smallmatrix} \Lambda/J & J \\ 0 & \Lambda/J \end{smallmatrix}\right)$. Our first aim is to show that Λ and Γ are stably equivalent. Once this is established we will see how to use it for proving that if Λ satisfies $\mathfrak{X}$ then it must be of finite representation type.

We need first the following result, which follows as a corollary of (Auslander et al. (1997) Ch. 3 Prop. 2.7):

Proposition 4.3.2. *Suppose T and U are semisimple Artin R-algebras and let M be a non-zero $T, U-$ bimodule where R acts centrally and M is finitely generated over R. Then $\Delta = \left(\begin{smallmatrix} T & M \\ 0 & U \end{smallmatrix}\right)$ is an hereditary Artin R-algebra.*

Corollary 4.3.3. *Let Λ be any Artin R-algebra such that $J^2 = 0$ and let Γ be the upper triangular matrix algebra $\left(\begin{smallmatrix} \Lambda/J & J \\ 0 & \Lambda/J \end{smallmatrix}\right)$. Then Γ is hereditary.*

Proof. Λ/J is semisimple since Λ is artinian, and J is f.g. over Λ because any artinian ring is also noetherian and so submodules of f.g. modules are also finitely generated. Therefore, J is also f.g. over R since Λ is an Artin R-algebra . Finally, since J is an ideal over Λ then obviously R acts centrally on J. $\square$

Remark. *From here it could also be deduced that the upper triangular matrix rings associated to algebraic bimodules that we have studied previously are hereditary rings.*

Now, by (Auslander et al. (1997) Prop. 2.2 p. 74) it follows again that we can see the modules over Γ as tuples (A, B, ϕ) where A is a left module over Λ/J, B is a left module over Λ/J and $\phi : A \otimes J \to B$ is a Λ/J-linear map. Then the same results that we stated for example in [Ch. 3 Prop. 3.3.1] applies in our context since both are direct consequences of the results in (Auslander et al. (1997) p. 75-77). This tells us in particular that any simple module over Γ is either of the form $(S, 0, 0)$ or $(0, S, 0)$, where

S is some simple module over Λ/J (and therefore over Λ, since J annihilates simples); and also that [see Ch. 3 Lemma 3.3.4] if $A = T_1 \oplus ... \oplus T_n$, and $B = H_1 \oplus ... \oplus H_m$ with H_i, T_j simple Λ/J modules then

$$0 \subseteq (0, H_1, 0) \subseteq (0, H_1 \oplus H_2, 0) \subseteq ... \subseteq (0, B, 0) \subseteq \tag{4.15}$$

$$\subseteq (T_1, B, \phi(T_1)) \subseteq (T_1 \oplus T_2, B, \phi(T_1 \oplus T_2)) \subseteq ... \subseteq (A, B, \phi) \tag{4.16}$$

is a composition series for (A, B, ϕ), and $\{(T_j, 0, 0)\} \cup \{(0, H_i, 0)\}$ are its composition factors.

We can define the functor $F : mod - \Lambda \to mod - \Gamma$ by $F(M) = (M/JM, JM, f)$ where $f : J \otimes_{\Lambda/J} M/JM \to JM$ is induced by the morphism $\mathfrak{r} : J \otimes_\Lambda M \to JM$ given by $\mathfrak{r}(x \otimes m) = xm$. In other words, $f(x \otimes \tilde{m}) = xm$. We need to check that f is well-defined: For, let $\phi : J \times M/JM \to J \otimes_\Lambda M$ be the map given by $\phi(x, \tilde{m}) = x \otimes m$. If $(x, \tilde{m}) = (x, \tilde{n})$ then $m - n = a_1 m_1 + .. + a_n m_n$ for some $a_i \in J$ and $m_i \in M$. Therefore, $x \otimes (m - n) = x \otimes a_1 m_1 + .. + x \otimes a_n m_n = x a_1 \otimes m_1 + .. + x a_n \otimes m_n = 0$ since $x a_i \in J^2 = 0$. Thus, $x \otimes m = x \otimes n$ and so ϕ is well-defined. Therefore, by the universal property of tensor products, there exists a map $\psi : J \otimes_{\Lambda/J} M/JM \to J \otimes_\Lambda M$ s.t. $\psi(x \otimes \tilde{m}) = \phi(x, \tilde{m}) = x \otimes m$. Then f is just the composition $\mathfrak{r} \circ \psi$.

For morphisms, recall that if $g : M \to N$ is a morphism in $mod - \Lambda$ then $g(JM) = Jg(M) \subseteq JN$ and so the restriction of g to JM has image in JN. Call the restriction $\tilde{g}$. Then it induces a morphism $g' : M/JM \to N/JN$. We can define $F(g) = (g', \tilde{g})$.

It follows from the comments above that $l(M) = l(F(M))$ and that, if $T_1, ..., T_n$ are the composition factors of JM and $H_1, ..., H_m$ are the composition factors of M/JM, then $(0, T_1, 0), ..., (0, T_n, 0), (H_1, 0, 0), ..., (H_m, 0, 0)$ are the composition factors of $F(M)$. This is fundamental for us in order to see that $\mathfrak{X}$ is preserved under F.

Indeed, F induces an stable equivalence with the following properties, which will be helpful for us:

Theorem 4.3.4. *(Auslander et al. (1997) p. 345-349; 2.1, 2,2, 2.3, 2.4) Let F be defined as above. Then F induces a stable equivalence* $\underline{mod} - \Lambda \to \underline{mod} - \Gamma$*, which we denote also by* F*, and it has the following properties:*

a) M *in mod* $-\Lambda$ *is indecomposable if and only if* $F(M)$ *is indecomposable..*

b) M *and* M' *are isomorphic if and only if* $F(M)$ *and* $F(M')$ *are isomorphic*

c) Let $X = (A, B, \phi)$ *in mod*$-\Gamma$*. Then* $X \cong F(M)$ *for some* M *in mod*$-\Lambda$ *if and only if* ϕ *is an epimorphism.*

d) Let $X = (A, B, \phi)$ *indecomposable in mod*$-\Gamma$ *. Then* $X \ncong F(M)$ *for any* M *in mod*$-\Lambda$ *if and only if* $X \cong (0, S, 0)$*, with* S *some simple* Λ*-module.*

We are in a good position now to establish our main theorem of this section:

Theorem 4.3.5. *Let* Λ *be an Artin algebra with* $J^2 = 0$ *that satisfies* $\mathfrak{X}$*, and let* $\Gamma = \left(\begin{smallmatrix} \Lambda/J & J \\ 0 & \Lambda/J \end{smallmatrix}\right)$*. Then* Γ *satifies* $\mathfrak{X}$*.*

Proof. We need to show that if $X = (A, B, \phi)$ and $Y = (A', B', \psi)$ are two indecomposable modules of Γ with the same composition factors, then they are i somorphic. Since Λ/J is semisimple, then both A and B are semisimple modules and so we can write $A \cong T_1 \oplus ... \oplus T_n$, and $B \cong H_1 \oplus ... \oplus H_m$ with H_i, T_j simple Λ/J-modules (and so simple Λ-modules). We know that in that case the composition factors of X are $\{(T_j, 0, 0)\} \cup \{(0, H_i, 0)\}$. Hence, since X and Y have the same composition factors and Λ/J is semisimple we must also have $A' \cong T_1 \oplus ... \oplus T_n$, and $B' \cong H_1 \oplus ... \oplus H_m$. Thus, $A \cong A'$ and $B \cong B'$ as Λ/J-modules (and so as Λ-modules).

Now, by Thm. 4.3.4 we know that the only indecomposable modules over Γ which are not of the form $F(M) = (M/JM, JM, f)$ with M indecomposable over Λ, are the ones of the form $(0, S, 0)$ with S simple module over Λ. Therefore, since those mo dules are simple they are trivially completely determined by their composition factors, and so we just need to check that the ones of the form $F(M) = (M/JM, JM, f)$ with M indecomposable over Λ are completely determined by their composition factors.

For, let M and N indecomposable modules over Λ s.t. $F(M) = (M/JM, JM, f)$ and $F(N) = (N/JN, JN, f)$ have the same composition factors. This means, by the first paragraph above, that $M/JM \cong N/JN$ and $JM \cong JN$ as Λ-modules. Hence, since the composition factors of M are the union of the composition factors of JM and M/JM, it follows that M and N have the same composition factors, and so they must be isomorphic since they are indecomposable and Λ satisfies $\mathfrak{X}$. But then, by Thm 4.3.4, $F(N) \cong F(M)$. We conclude that Γ satisfies $\mathfrak{X}$.

$\square$

Corollary 4.3.6. *Any Artin algebra Λ with $J^2 = 0$ that satisfies $\mathfrak{X}$ must be of finite re presentation type.*

Proof. By Thm. 4.3.5 we know that $\Gamma = \begin{pmatrix} \Lambda/J & J \\ 0 & \Lambda/J \end{pmatrix}$ also satisfies $\mathfrak{X}$, and Γ is hereditary by Corollary 4.3.3. Therefore, by Thm. 4.2.19, Γ must be of finite representation type. But then, by Proposition 4.3.1, Λ is of finite representation type. $\square$

4.3.3 A counterexample

We would like to finish this section by showing a counterexample for the converse of Corollary 4.3.6, that is, there exists an Artin algebra with $J^2 = 0$ of finite representation type not satisfying $\mathfrak{X}$.

Let Q be the following quiver

$$\begin{tikzcd} 1 \arrow[r, bend right, "\alpha"'] & 2 \arrow[l, bend right, "\beta"'] \end{tikzcd}$$

and let K be any field. Consider the path algebra KQ, and let I be the ideal generated by $\alpha \cdot \beta$ and $\beta \cdot \alpha$. Call A the algebra KQ/I. If we remember the equivalence of categories between mod$-KQ$ and $\mathfrak{rep}_K Q$, a similar equivalence holds between mod-KQ/I and $\mathfrak{rep}_K Q/I$, where $\mathfrak{rep}_K Q/I$ is the full subcategory of representations of Q whose objects are representations of Q

$$\begin{tikzcd} V \arrow[r, bend right, "f"'] & W \arrow[l, bend right, "g"'] \end{tikzcd}$$

s.t. $0 = fg = gf$. Such an equivalence occurs whenever I is what is called an admissible ideal of KQ, i.e. I does not contain vertices, arrows and there exists some $m \in \mathbb{Z}$ s.t. I contains every path of length greater than m (the latter condition guarantees that KQ/I is a finite-dimensional K-algebra), such as in our case. For more details see (Schiffler (2014) Section 5.2). We can therefore identify modules with representations as we did in Ch. 3.

Consider now the indecomposable representations P_1

$$\begin{tikzcd} K \arrow[r, bend right, "1"'] & K \arrow[l, bend right, "0"'] \end{tikzcd}$$

and P_2

$$\begin{tikzcd} K \arrow[r, bend right, "0"'] & K \arrow[l, bend right, "1"'] \end{tikzcd}$$

These are representations of Q/I since $0 = 1 \cdot 0 = 0 \cdot 1$. They are non-isomorphic because for being isomorphic we should have, after identifying invertible linear maps from K to K with non-zero scalars in K, non-zero elements $a, b \in K$ s.t. $0 \cdot a = b \cdot 1$ (commutative of the diagrams). But this is impossible, so they are not isomorphic.

Consider now the simple subrepresentation S_2 of P_1

$$\begin{tikzcd} 0 \arrow[r, bend right, "0"'] & K \arrow[l, bend right, "0"'] \end{tikzcd}$$

and the simple subrepresentation S_1 of P_2

$$\begin{tikzcd} K \arrow[r, bend right, "0"'] & 0 \arrow[l, bend right, "0"'] \end{tikzcd}$$

One can check that $P_1/S_2 \cong S_1$ and $P_2/S_1 \cong S_2$, and so both P_1 and P_2 have the same composition factors. But this just means that $A = KQ/I$ does not satisfy $\mathfrak{X}$!

It is not hard to check that the only indecomposable representations of Q/I are S_1, S_2, P_1, P_2 and so A is of finite representation type. Therefore, the length of any indecomposable module over A is at most 2. We can write $A = A_1 \oplus ... \oplus A_n$ with $n = |Q_0|$ and where A_i is the projective indecomposable module corresponding to P_i under the equivalence of categories , so we must have $n = 2$ and $l(A_1) = l(A_2) = 2$. In both cases, we necessarily have that $0 \subseteq JA_i \subseteq A_i$ is a composition series for A_i and so in particular $J^2A_i = 0$, since J^2A_i is properly contained in JA_i by a similar argument than in Ch. 2 Lemma 2.2.9. Hence, $J^2 = J^2A = J^2A_1 \oplus J^2A_2 = 0$. Therefore, we get:

Proposition 4.3.7. *Let K be a field . Then there exists a finite-dimensional K-algebra A with $J^2 = 0$ and of finite representation type which does not satisfy $\mathfrak{X}$.*

Remark. *The example we have seen is a particular case of a more general construction. Namely, let Q_n be an n-oriented cycle, and let I_n be the ideal generated by all the paths of length n in Q. There are exactly n projective indecomposable* *representations P_i of Q_n/I_n and each of them has length n. Each P_i has K in every vertex and every linear map is 1 except 0 in the one which terminal point is the vertex i, and so they are pairwise non-isomorphic. If we call S_i the simple representation with K in position i and 0 otherwise, then one can check that the composition factors of each P_i are $S_1, ..., S_n$; and so KQ/I does not satisfy $\mathfrak{X}$. Furthermore, ever P_i has a unique composition series, namely $0 \subseteq M_1^i \subseteq M_2^i \subseteq ... \subseteq M_{n-1}^i \subseteq P_i$ with $M_1^i = S_{i+1}$ (with the exception $M_1^n = S_1$), $P_i/M_{n-1}^i \cong S_i$, $M_r^i/M_{r-1}^i \cong S_{r+i}$ for every $r = 2, ..., n-i$, and $M_r^i/M_{r-1}^i \cong S_{r+i-n}$ for every $r = n-i+1, ..., n-1$. This means that the composition factors of the P_i's are permuted cyclily by a permutation of order n.*

The counterexample we have shown is therefore the particular case for $n = 2$ of the above general construction. All those algebras KQ_n/I_n are examples of what are called Nakayama algebras. These are Artin algebras characterized by the fact that indecomposable projectives and indecomposable injectives are uniserial.

CHAPTER FIVE
CONCLUSION

In retrospective, the main results that have been proven in this master thesis are, on the one hand, the characterization of the commutative artinian rings satisfying $\mathfrak{X}$, and, on the other hand, the analogous characterization for hereditary Artin algebras. In both cases, the property $\mathfrak{X}$ is equivalent to such a ring to be of finite representation type. In the commutative case, indeed, those rings are what are called principal ideal rings (PIR). Both characterizations answer indirectly the conjecture appearing in Auslander et al. (1997) for both the commutative artinian and the hereditary Artin algebra case. Also, the conjecture has been answered for the case of Artin algebras with $J^2 = 0$. As far as the concern of the author, these results have not appeared previously in the literature.

The first aim of this master thesis was to show, in the more possible elementary way, some categorical methods which have proven to be fruitful while dealing with problems in the representation theory of Artin algebras. During the thesis we have discussed some of them, such as dualities, Auslander-Reiten translation, stable equivalence, full exact embeddings, representations of quivers, algebraic bimodules, and K-species. The second and more important aim of this thesis was to see how those methods can apply. For, I chose a conjecture which appears in Auslander et al. (1997) that somehow motivated me, and then I tried to say what can be said with the limited knowledge I have. I believe that after months of dedication to my problem, I have been able to apply those methods that I have learned and get some results and at least some insight into the problem.

Due to the limited time one has for preparing the thesis, some topics and discussions that I would have loved to include in the thesis have not appeared or had just posted as questions. In the first part of the first chapter we proved that Dedekind domains satisfy $\mathfrak{X}$ and we asked whether every integral domain satisfy $\mathfrak{X}$ or which bigger class of integral domains containing Dedekind domains satisfies it. Due to my poor knowledge in the matter I could not take this question further.

Anyway, the problem in its general version makes more sense to me to be studied in the artinian setting because in that case finitely generated modules coincide with finite length ones. In this direction, one class of very interesing and elementary artinian rings is the one of Nakayama algebras. In that case, every left (resp.right) indecomposable module is uniserial and the $DTr-$orbits of simple modules contain only simple modules. Due to that fact, one can construct what is called the Kupisch series associated to a Nakayama algebra. I believe, since uniserial modules over Nakayama algebras are completely determined by its lenght and top and the $DTr-$orbits of a simple module can be written explicitely in terms of powers and quotients of radicals of projective indecomposable modules, that two Nakayama algebras with the same Kupisch series either both satisfy $\mathfrak{X}$ or both do not. If that can be shown to be true, then given what is called an admissible sequence we can construct a Nakayama algebra which Kupisch series is that admissible sequence. That Nakayama algebra is a quotient of a path algebra over some admissible ideal, so in that case it is relatively easy to show which admissible sequences give Nakayama algebras satisfying $\mathfrak{X}$. Therefore it would be possible to give a complete characterization of the Nakayama algebras satisfying $\mathfrak{X}$ in terms of admissible sequences if one can show that two Nakayama algebras with the same Kupisch series either both satisfy $\mathfrak{X}$ or both do not.

Another thing I would have loved to discuss about is the possible existence of a counterexample for the general conjecture in the hereditary artinian case, i.e. a hereditary artinian ring of infinite representation type satisfying $\mathfrak{X}$. I believe that such as a counterexample can be constructed in the following way: Let F be a field and let $n > 1$. Then construct the ring R of $n \times n$ lower triangular matrix rings over F writing $M_F(\delta)$ instead of F in the position $(n, 1)$, where $M_F(\delta)$ denotes the non-algebraic bimodule defined in the section of the second chapter with title ”Ringel's counterexample”. The finitely generated indecomposable modules over this ring were extensively studied in the article Dlab & Ringel (1978). The authors showed in Thm. 2 that the preprojective and preinjective indecomposable modules are completely determined by their composition factors. The direct sums of the remainder indecomposables form an abelian, exact, extension closed subcategory of mod$-R$, which is equivalent to a product of categories $U \times H$, where U is a hereditary serial category and H is the category of what are called homogeneous modules. It seems, due to Thm. 2 again, that the modules in U are completely determined by their composition factors (if one takes $n = 2$ the arguments get a bit simpler). Finally, they showed also that H is equivalent to mod-R_δ^F, which was also defined in "Ringel's counterexample". We know that, if we take F to be a differentially closed and $0 \neq \delta$, then mod$-R_\delta^F$ contains only one finite length module, which is injective indeed. Therefore, it looks that a priori, if we take $n = 2$ (for simplicity), F to be a differentially closed and $0 \neq \delta$, then indeed R gives such as a counterexample.

In this Master Thesis we have dealed with a problem concerning composition series and composition factors. Such as a problem led to the question of when is it true that the finitely generated indecomposable modules over an artinian ring R are completely determined by their composition factors. It is possible to give a necessary condition for that to happen just in terms of extensions of modules: If $S_1, ..., S_n$ are, up to isomorphism, the only simple R-modules then a necessary condition for R to satisfy $\mathfrak{X}$ is that any non-split short exact sequence of the form $0 \to S_i \to M \to S_j \to 0$ has only one possible choice for M up to isomorphism. That is the case since any such a non-split short exact sequence must have M indecomposable module (otherwise $M \cong S_i \oplus S_j$ and so the short exact sequence splits) and if we can find two such short exact sequences with middle terms $M \ncong N$, then both M and N are indecomposable modules with the same composition factors, and so R does not satisfy $\mathfrak{X}$. Furthermore, if $0 \to S_j \to M' \to S_i \to 0$ is a non-split short exact sequence, we also need $M \cong M'$ for R to satisfy $\mathfrak{X}$, so indeed the necessary condition above can be strengthen. I do not believe that this condition is a sufficient one in general since we are only dealing with the extensions of simples over simples, but it may be true that we can find an homological characterization of $\mathfrak{X}$ in terms of Ext_R^1.

Finally, I would like to mention that some methods such as almost split sequences and the Auslander-Reiten quiver, which have not been used in this Master Thesis, may help to give another perspective to the problem. Indeed, the original conjecture was stated firstly in the article Auslander & Reiten (1985) by Auslander and Reiten, and in that article the authors gave sufficient conditions for an Artin algebra to satisfy $\mathfrak{X}$, and they express them in terms of properties of the associated Auslander-Reiten quiver. It is worth to mention that, in this direction, the Thm. 2.1 of the article Simson (2002) by D. Simson, together with our results which assert that no-$\mathfrak{X}$ is preserved under full and exact embeddings and algebraic bimodules with $a \cdot b \geq 4$ do not satisfy $\mathfrak{X}$, has as a consequence that the conjecture is true for what are called loop-finite algebras, which is a class of Artin algebras containing the class of hereditary algebras in particular. Therefore, [Ch. 4 Thm. 4.2.18] could have followed as a corollary of all those results, but the proof of (Simson (2002) Thm. 2.1) is rather complicated and technical so I preferred to prove the hereditary algebra case directly.

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