ASYMPTOTIC INTEGRATION OF THE 2D EQUATIONS OF MOTION FOR AN INHOMOGENEOUS THIN ELASTIC SHELL

Dissertation

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Eskişehir 2023

ASYMPTOTIC INTEGRATION OF THE 2D EQUATIONS OF MOTION FOR AN INHOMOGENEOUS THIN ELASTIC SHELL

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DISSERTATION

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Eskişehir

Eskişehir Technical University

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January 2023

FINAL APPROVAL FOR THESIS

This thesis titled ASYMPTOTIC INTEGRATION OF THE 2D EQUATIONS OF MOTION FOR AN INHOMOGENEOUS THIN ELASTIC SHELL has been prepared and submitted by Noorullah NOORI in partial fulfillment of the requirements in "Eskişehir Technical University Directive on Graduate Education and Examination" for the Degree of Doctor of Philosophy (PhD) in Mathematics Department has been examined and approved on 23/01/2023.

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Supervisor

Prof. Dr. Nihal EGE

ABSTRACT

ASYMPTOTIC INTEGRATION OF THE 2D EQUATIONS OF MOTION FOR AN INHOMOGENEOUS THIN ELASTIC SHELL

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In this study, low-frequency dispersion analysis of cylindrical thin shells is carried out by asymptotic approximation using special scalings and compared with the theories in the literature. In particular, this asymptotic approach is applied to 3D general elasticity equations and 1D equations are obtained at the mid-surface for the problems considered in the thesis. For each problem, new results are obtained that are consistent with and contribute to the literature.

In the analysis of the first problem, the 2D annulus, it is shown that the even part of the circumferential stress is not constant with respect to the thickness and is polynomial in form, which is contrary to the characteristic of the mechanical behavior related to the almost inextensible mid-plane effect. As a second problem, an approximate 1D fourth-order equation for a 3D thin shell cylinder is obtained based on the assumption that the ratio of the shell thickness to the radius is of the same order as the ratio of the radius to a typical wavelength, and it is found that most of the coefficients in the equation coincide with their counterparts in Sanders-Koiter shell theory. Finally, the low-frequency vibrations of a thin-shell functionally graded (FGM) cylinder are considered in the plane strain framework. Similarly, a consistent approximate equation of motion at the mid-surface is derived using special scaling and dynamic relations in elasticity. The original formulas for the variation of natural frequencies and all stress components arising from the derived equation are obtained.

Keywords: Low frequency, Vibration, Asymptotic, Elasticity, Cut-off, Thin-walled.

ÖZET

HOMOJEN OLMAYAN İNCE ELASTİK KABUK İÇİN 2 BOYUTLU HAREKET DENKLEMLERİNİN ASİMPTOTİK INTEGRASYONU

Noorullah NOORI

Matematik Anabilim Dalı

Eskişehir Teknik Üniversitesi, Lisansüstü Eğitim Enstitüsü, Ocak 2023

Danışman: Prof. Dr. Nihal EGE

(Ikinci Danışman: Prof. Dr. Barış ERBAŞ)

Bu çalışmada, silindirik yapıdaki ince kabukların, özel ölçeklendirmeler kullanarak asimptotik yaklaşım ile düşük frekansta dispersiyon analizi yapılmış ve literatürde yer alan teoriler ile karşılaştırılmıştır. Bu asimptotik yaklaşım özellikle 3-Boyutlu genel elastisite denklemlere uygulanmış ve tezde ele alınan problemler için orta yüzeyde bir boyutlu denklemler elde edilmiştir. Her bir ele alınan problem için literatüre uyumlu ve katkı sağlayan yeni sonuçlar elde edilmiştir.

İlk problem olan 2-Boyutlu halkanın analizinde, neredeyse uzatılamaz orta düzlem etkisi ile ilgili mekanik davranışın özelliğine ters düşen çevresel gerilmenin çift kısmının kalınlığa göre sabit olmadığı ve polinom biçiminde olduğu ortaya konmuştur. İkinci bir problem olarak 3-Boyutlu ince kabuklu bir silindir için kabuk kalınlığının yarıçapına oranının, yarıçapın tipik bir dalga boyuna oranı ile aynı mertebede olduğu varsayımına dayandırılarak yaklaşık 1-Boyutlu dördüncü mertebeden denklem elde edilmiş ve denklemdeki çoğu katsayının Sanders- Koiter kabuk teorisindeki karşılıklarıyla örtüştüğü görülmüştür. Son olarak ise ince kabuklu fonksiyonel derecelendirilmiş (FGM) bir silindirin düşük frekanslı titreşimleri düzlem şekil değiştirme çerçevesinde ele alınmıştır. Yine benzer olarak özel ölçeklendirmeler ve elastikiyetteki dinamik ilişkiler kullanılarak orta yüzeyde tutarlı bir yaklaşık hareket denklemi türetilmiştir. Türetilen denklemden kaynaklanan doğal frekansların ve tüm gerilme bileşenlerinin değişimi için orijinal formüller elde edilmiş.

Anahtar Sözcükler: Düşük frekans, Titreşim, Asimptotik, Elastisite, İnce duvarlı.

ACKNOWLEDGMENTS

Foremost, I would like to express my deep, sincere gratitude and appreciation to my supervisors, Prof. Dr. Nihal EGE and Prof. Dr. Barış ERBAŞ for their encouragement, expert advice and financial and spiritual support, they always treated me as if I were a member of the family. I would also like to extend my thanks to Prof. Dr. Julius KAPLUNOV; without his guidance, this would not have been possible. Besides my supervisors, I would like to thank the members of my thesis committee for their encouragement and useful comments. It would be inappropriate if I omitted to mention the name of my dear friend Mohammad Qasam Hashimi, who provided technical support for drawing solid shapes for the thesis.

I am very grateful for awarding me the scholarship from YTB (Yurtdışı Türkler ve Akraba Topluluklar Başkanlığı) and the lovely people of Turkey.

Furthermore, I would like to extend my appreciation and gratitude to my family members, especially my parents, whose love made many life-changing opportunities possible for me. Without your support and guidance, I would not have been able to complete this research. So, to all of you who helped me during this research, I would like to say thank you very much.

Noorullah NOORI

January 2023

23/01/2023

STATEMENT OF COMPLIANCE WITH ETHICAL PRINCIPLES AND RULES

I hereby truthfully declare that this thesis is an original work prepared by me; that I have behaved in accordance with the scientific ethical principles and rules throughout the stages of preparation, data collection, analysis and presentation of my work; that I have cited the sources of all the data and information that could be obtained within the scope of this study, and included these sources in the references section; and that this study has been scanned for plagiarism with "scientific plagiarism detection program" used by Eskişehir Technical University, and that "it does not have any plagiarism" whatsoever. I also declare that, if a case contrary to my declaration is detected in my work at any time, I hereby express my consent to all the ethical and legal consequences that are involved.

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LIST OF ABBREVIATIONS

$\mathbb{R}^n$	:	n-dimensional Euclidean space
σ_{ij}	:	Components of stress
ϵ_{ij}	:	Components strain
ν	:	Poisson's ratio
E	:	Young's modulus
ρ	:	Mass density
c_1	:	Longitudinal wave speed
c_2	:	Transverse wave speed
O	:	Big-O notation
κ	:	Extensional rigidity
G	:	Shear modulus
D	:	Bending rigidity
Δ	:	Dilation
λ, μ	:	Lame constants
A_1, A_2	:	Lame's parameters
R_1, R_2	:	Minimum and Maximum radii of curvature

1. INTRODUCTION

The theory of elasticity deals with the relationship between forces applied to an object and the resulting deformations. The first consideration of elasticity was carried out by Galileo in his work on the nature of the resistance of solids to rupture. Later on, many scientists and mathematicians contributed to developing the theory of elasticity [53].

One of the fields of elasticity theory is shell theory. Looking around on our earth, we can see various natural shell structures as: eggs in our fridges, snails on beaches, tortoises dwelling on the land, crustaceans, and the skulls of humans and animals. At the microscopic level, every cell in our body is a thin shell. On a larger scale, every earth quake rings like a bell around the world. In this thesis, we shall be concerned with the mathematical modeling and the derivation of relevant equations of motion for man-made shell structures, which are widely used as structural elements in modern construction engineering, architecture and building, power and chemical engineering, vehicle body structures, and in other fields of technology: aircraft construction, ship buildings, rocket construction, automobiles, turbines, vessels, submarine hulls, roofing structures, etc.

In the theory of elasticity, shell is a three-dimensional body or solid material that is enclosed by two curvilinear surfaces. The surface, that divides it in half throughout its entire thickness is called the mid-surface, or the locus of points that lie between these surfaces is called the mid-surface. The distance between the surfaces, measured along the normal to the mid-surface, is the thickness of the shell at that point. A plate is a special case of the shell having no curvature.

Elastic shells can be classified into thick and thin shells. A shell is called thin if its one dimension (thickness) is small compared with the other two dimensions of the shell, where shear deformation and rotary inertia are negligible; otherwise, it is thick. Shells are also classified according to the shape of their surfaces as spherical, cylindrical, hyperbolic, paraboloidal, or other shapes.

The basic approaches which are employed in the derivations of linear shell theory can be summarized as: Direct approach, vibrational method, Method of parametric expansion and asymptotic integration, and Method of Taylor series expansion.

According to William (1965) "In the method of parametric expansion and asymptotic integration, appropriate quantities in three- dimensional elasticity are expanded into power series of some small thickness-parameter, then the shell equations are derived by asymptotic integrations."(p.3).

While deriving the stress-strain relations, strain-displacement relations, and equilibrium equations, various assumptions are made which ensure various thin shell theories. Most common of these are the Love, Naghdi, Reissner, Flugge, and Sanders shell theories. For comparison of these and other thin shell theories, Leissa's 1973 monograph [52] is a good choice. All the theories, including the literature about vibration of shells are explained there and a more recent publication [3] gives a comprehensive review of vibration of shells. The theories in [52] were first derived for isotropic shells and then expanded for laminated composite shells by the help of integration through lamina, and stress-strain relations.

The analysis of shell structures has a very long history, starting from the simplest practical shell theory that is a membrane theory [29], where bending and twisting moments are assumed to be negligibly small and then the bending theories of several shell theories. The book written by Flugge and Kraus deals with both the statics and dynamics of shells [83]. An excellent survey of the static and dynamic behaviour of laminated composite shells and homogeneous shells can be found in [67, 68, 70], a vast part of 2D analysis deals with the dynamic behaviour of circular cylindrical shells, e.g. [52]. In [86] and references therein, a brief review of 3D dynamic analysis of cylindrical shells within the framework of linear elasticity can be found.

Thin cylindrical shells as structural elements are broadly used and plenty of theories have been developed. Aron was the first who studied the cylindrical shell problem and Love provided a mathematical framework for a thin shell theory [83]. Love's mathematical framework, also known as Love's first approximation theory, included four principal assumptions which opened a door for the development of many thin shell theories. These four assumptions are also known as the Kirchhoff-Love hypothesis.

The method which is both applicable to general shell problems and simplifies the mathematics involved in the solution is called the approximate method, and this method provides a solution technique that can be applied to general shell problems without solving the complete formulation [17]. It is worth mentioning that an important concept (for singular asymptotics) of an edge effect appeared in the works of Lamb and Basset in 1890, while the boundary layer concept did not appear until 1904 in Fluid Mechanics [4]. The classical papers of Vishik and Lyusternik are a generalization of some previous results obtained by Gold'denveizer [19].

Much of the the activities in the last decades of the last century in the theory of shells has been concerned with the derivation of approximate theories; different approximate formulations based on appropriate physical assumptions or originate from asymptotic expansions are presented in [8, 19, 41, 51] and references therein. All possible shortened forms of the classical 2D dynamic equations have already been derived by Goldenveizer. Asymptotic integration of dynamic problems of shells developed and described in [20, 21, 22, 39, 40] and references therein. The equations of motion are derived in [44] using a straightforward asymptotic procedure in the framework of the Sanders-Koiter version of 2D shell theory. The results in [44] were extended to an anisotropic cylindrical shell in [45]. The technique based on a deferred approach to a limit, for analyzing the dispersion relation for long waves in a curved waveguide is presented in [9] within the framework of plane strain. The plane strain problem in 3 D elasticity for a homogeneous thin-walled elastic cylinder has been recently studied in [14].

The theory of shells outlines a general methodology for the transition from 3D to 2D (or 1D) models. For sake of simplicity, we limit ourselves to shells, which are made up of homogeneous and inhomogeneous, isotropic, and perfectly elastic material.

The main objective of this research is to suggest an efficient approach to finding 1D equations of motion through asymptotic integration for both 2D homogeneous and inhomogeneous cylinders as well as 3D homogeneous cylinders. In this thesis, mechanical parameters such as Young's modulus, Poisson's ratio, and mass density are assumed to be constant in a homogeneous shell and varying along the thickness direction in a functionally graded material (FGM) shell. The governing 1D refined equations of motion are obtained based on the principles of asymptotic analysis.

The thesis is organized as follows.

After the brief Introduction organized in Chapter 1, a concise account of some basic definitions and concepts that are frequently used throughout the thesis; equations of elasticity, the theory of surfaces and construction of shell theory are outlined in separate sections of Chapter 2.

Chapter 3 starts with a brief account of 3D equations of motion. The equations are rewritten in a simplified form, which will be the basis for our main research. In this chapter, 2D analysis for a thin elastic annulus starts with a statement of the problem followed by asymptotic scaling. Unlike Chapters 4 and 5 the displacement and stress components are separated into even and odd parts with respect to the thickness variable. With the help of iterative asymptotic integration, a 1D refined long-wave low-frequency model for a thin elastic annulus is derived from plane strain equations.

In Chapter 4, a homogeneous thin elastic cylindrical shell is subjected for study. Unlike chapters 3 and 5, the 1D equations of motion are reduced from full 3D equations of motion, and the scaled quantities are assumed in harmonic form.

In Chapter 5, FGM annulus is taken into account where the mechanical parameters are functions of thickness coordinate, the procedure of finding the result is also much common with that of homogeneous case studied in chapter 3, here a new asymptotically consistent equation of motion on the mid-surface is derived. The validity of each derived result is checked by comparison of other articles and outlined in discussions.

Finally, in Chapter 6, conclusions are given and the main results of the thesis are discussed with suggestions for future work.

2. FUNDAMENTAL EQUATIONS OF THIN SHELL THEORY

In this chapter, some basic definitions and concepts of elasticity and solid mechanics are discussed, followed by the elements of the theory of surfaces. Following the explanation of Kraus [49], the image of the geometry of the shell is clarified by the determination of its mid-surface and thickness at all points. Finally, the chapter ends with a brief introduction to shell theory and some basic postulates, concepts, and appropriate relations that will be developed for a thin elastic shell.

2.1. Equations of Elasticity

A solid material is said to be elastic if it returns to its initial shape and size after removing the external forces, and this property of materials is called elasticity. The theory of elasticity generally deals with the elasticity of deformable bodies. An elastic body is said to be homogeneous if its physical properties are the same at every point in the body. Inhomogeneous materials, unlike homogeneous materials, change their properties from point to point. Also, an elastic body is said to be isotropic if its elastic properties are the same in all directions. In the opposite case, it is considered anisotropic.

To understand the behavior of solid materials, the following concepts and definitions are necessary to be discussed.

2.1.1. Stresses

Let us consider a deformable body, to know the behavior of internal forces that are produced within the body after applying loads at a specific point O, we divide the body by a plane through the point of interest. Consider an element with a small area ΔA on the cut surface and let the resultant force ΔF act on this small area, then the stress is defined as the limiting value of the ratio $\frac{\Delta F}{\Delta A}$ at any point across the area ΔA, i.e.,

$$Stress = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}, \tag{2.1}$$

where ΔF is an internal force, if the origin of coordinates is placed at point O, with x_1 normal and x_2, x_3 tangent to ΔA, and decomposing ΔF into components parallel to x_1, x_2, and x_3, then we have

$$\sigma_1 = \lim_{\Delta A \to 0} \frac{\Delta F_{x_1}}{\Delta A},$$
$$\sigma_{12} = \lim_{\Delta A \to 0} \frac{\Delta F_{x_2}}{\Delta A}, \quad (2.2)$$
$$\sigma_{13} = \lim_{\Delta A \to 0} \frac{\Delta F_{x_3}}{\Delta A},$$

where the normal and shear stresses are denoted by σ_1 and σ_{1i}, $i = 2, 3$, respectively.

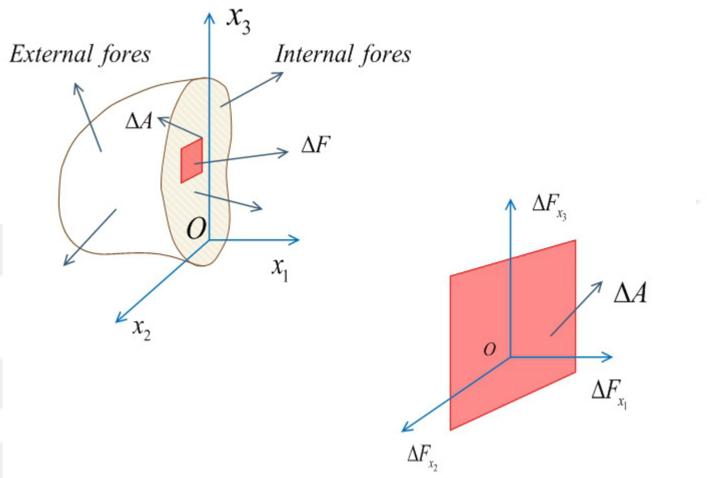

Figure 2.1. *Free body with external and internal forces, enlarged area ΔA with components of the force ΔF [93]*

Now, let us consider an infinitesimal cubic element in the cartesian coordinate system surrounding the point O, whose visible faces have outward normals along the positive x_1, x_2, and x_3 axes, respectively. Nine quantities act on the faces of this cubic element and are known as the stress components. The stress components in a 3×3 matrix notation can be written as

$$\begin{pmatrix} \sigma_1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3 \end{pmatrix}. \quad (2.3)$$

This matrix notation is known as the Cauchy stress tensor, where σ_i is the simplified form of σ_{ii} with $i = 1, 2, 3$. These nine components completely define the state of stress at a point inside the cubic element in its deformed state.

The first subscript indicates the direction of a normal to the plane on which the stress components act, and the second subscript refers to the direction of stress components on this plane.

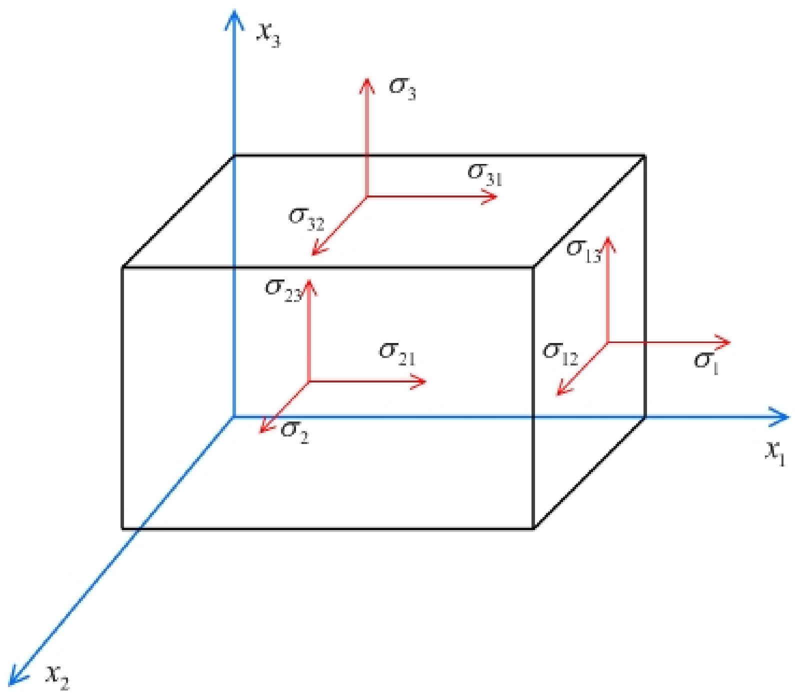

Figure 2.2. *Components of stress*

2.1.2. Strains

The deformation occurs when an external force is applied to a body, so the strain is defined as the ratio of the change in dimension of the body to the original dimension of the body and is denoted as ε. The strain of a material line element is expressed as

$$\varepsilon = \frac{\Delta l}{l}, \tag{2.4}$$

where Δl is the change in the length and l is the original length of the line element. Unlike stress, strain is a dimensionless quantity.

Strain may be classified into three types; normal strain, shear strain, and volumetric strain. The normal strain is the relative change in length of the body and is denoted by ε_{ii} $(i = 1, 2, 3)$. So ε_{11} is the relative change (elongation or contraction) of the material along the x_1 axis. Shear strain measures changes in angle with respect to two specific directions, it is denoted by ε_{ij} $(i \neq j)$. As an example, ε_{12} gives the angular change between the x_1 and x_2 axes. The volumetric strain is the relative change in the volume of the body.

2.1.3. Hooke's law

The linear relationship between stress and strain in one dimension can be written as

$$\sigma_{11} = E\varepsilon_{11}, \tag{2.5}$$

where E is a proportionality constant for the material known as the modulus of elasticity or Young's modulus after Thomas Young (1773-1829), equation (2.5) is known as Hooke's law, named for Robert Hooke (1635-1703), which simply states that the applied stress is directly proportional to strain in a solid body within the elastic limit of that solid. The generalized Hooke's law in the form of a fourth-order tensor can be written as

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}, \tag{2.6}$$

where the 81 coefficients E_{ijkl} are called the elastic constants. Hooke's law for homogeneous and isotropic bodies in terms of Lame constants is

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \tag{2.7}$$

where δ_{ij} is Kronecker delta and λ, μ are known as Lame constants.

Making use of equation (2.5) into equation (2.7), see [28], we obtain

$$\sigma_{11} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\varepsilon_{11}. \tag{2.8}$$

Latter gives us a modulus of elasticity in terms of the Lame constants as

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{2.9}$$

The Poisson's ratio, denoted by ν, is defined as the ratio of transverse strain and longitudinal strain and can be written as

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{2.10}$$

Similarly, shear modulus or modulus of rigidity, denoted by G, is defined as the ratio of shear stress to the shear strain, which can be written in terms of Young's modulus and Poisson's ratio as

$$G = \frac{E}{2(1+\nu)}. \tag{2.11}$$

Strain displacement relations can be written with the indicial notation as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (i, j = 1, 2, 3). \tag{2.12}$$

2.1.4. Equations of motion

Consider the normal and tangential components of stresses acting on the surface element of an infinitesimal rectangular parallelepiped. In order to derive the force acting on each face, we take the value of stress at the center of each face times the area of the face. With the help of Newton's second law and the body forces, the resultant forces acting on the x_1, x_2, and x_3 directions, see [57],

$$\begin{aligned}
\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} &= \rho\frac{\partial^2 u}{\partial t^2},\\
\frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} &= \rho\frac{\partial^2 v}{\partial t^2},\\
\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} &= \rho\frac{\partial^2 w}{\partial t^2},
\end{aligned} \tag{2.13}$$

these equations can be written in the form of indicial notation as

$$\frac{\partial\sigma_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2} \quad (i, j = 1, 2, 3), \tag{2.14}$$

where ρ is mass density, from Hooke's law for isotropic bodies we know that

$$\begin{aligned}
\sigma_{ii} &= \lambda\Delta + 2\mu\varepsilon_{ii},\\
\sigma_{ij} &= 2\mu\varepsilon_{ij} \quad (i, j = 1, 2, 3),
\end{aligned} \tag{2.15}$$

where Δ is the dilatation, i.e., $\Delta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$.

Making use of equations (2.12) and (2.15) into equation (2.14), we get

$$
\begin{aligned}
(\lambda + \mu)\frac{\partial \Delta}{\partial x_1} + \mu \nabla^2 u = \rho \frac{\partial^2 u}{\partial t^2}, \\
(\lambda + \mu)\frac{\partial \Delta}{\partial x_2} + \mu \nabla^2 v = \rho \frac{\partial^2 v}{\partial t^2}, \\
(\lambda + \mu)\frac{\partial \Delta}{\partial x_3} + \mu \nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2},
\end{aligned}
\tag{2.16}
$$

where the operator ∇^2 is defined as

$$
\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \tag{2.17}
$$

relations of (2.16) are equations of motion of an isotropic elastic solid in which the body forces (gravitation and moments) are neglected.

If displacements in equations of motion are replaced by strains and the equations are grouped together after taking the derivative of equation (2.16) with respect to x_1, x_2, and x_3 respectively, we obtain

$$
\frac{\partial^2 \Delta}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2 \Delta, \tag{2.18}
$$

this equation is known as the longitudinal (dilatational) wave and travels at a velocity

$$
V_{long} = c_1 = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}. \tag{2.19}
$$

Also, by assuming the dilatation zero in the equations of motion (2.16), we get

$$
\begin{aligned}
\mu \nabla^2 u = \rho \frac{\partial^2 u}{\partial t^2}, \\
\mu \nabla^2 v = \rho \frac{\partial^2 v}{\partial t^2}, \\
\mu \nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2}.
\end{aligned}
\tag{2.20}
$$

Thus, transverse (distortional) waves travel at a velocity

$$V_s = c_2 = \left(\frac{\mu}{\rho}\right)^{1/2} = \sqrt{\frac{E}{2(1+\nu)\rho}}, \tag{2.21}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{2.22}$$

2.2. The Theory of Surfaces

2.2.1. The first fundamental form

Every surface Ω in rectangular coordinate system may be written as a single-valued and continuous functions of two parameters α and β as follows:

$$x_1 = x_1(\alpha, \beta), \qquad x_2 = x_2(\alpha, \beta), \qquad x_3 = x_3(\alpha, \beta),$$

thus, the last relations are *parametric equations of a surface*, where α and β are curvilinear coordinates of the surface. By fixing, in turn, one parameter and varying the other, we obtain a family of curves called parametric curves of the surface.

If α and β are eliminated in the parametric equations of a surface, then we obtain the familiar form of the equation of a surface as

$$F(x_1, x_2, x_3) = 0.$$

The position vector of any point $M(\alpha, \beta)$ on the surface Ω can be given by following equation:

$$\vec{r} = \vec{r}(\alpha, \beta) = x_1(\alpha, \beta)\,\hat{e_1} + x_2(\alpha, \beta)\,\hat{e_2} + x_3(\alpha, \beta)\,\hat{e_3}. \tag{2.23}$$

A differential change in the position vector $\vec{r}$, as we move from point M on the surface to another point M_1 on the surface, where both points are infinitesimally close to each other is given by $d\vec{r}$

$$d\vec{r} = \vec{r}_{,1}d\alpha + \vec{r}_{,2}d\beta, \tag{2.24}$$

where for partial derivatives of vectors, we introduce the notation

$$\vec{r}_{,1} = \frac{\partial \vec{r}}{\partial \alpha}, \qquad \vec{r}_{,2} = \frac{\partial \vec{r}}{\partial \beta}. \tag{2.25}$$

Since $ds \approx |d\vec{r}|$, then, by help of equation (2.24), we can write

$$\begin{aligned}(ds)^2 &= d\vec{r} \cdot d\vec{r} \\ &= \vec{r}_{,1} \cdot \vec{r}_{,1}(d\alpha)^2 + 2\vec{r}_{,1} \cdot \vec{r}_{,2}(d\alpha)(d\beta) + \vec{r}_{,2} \cdot \vec{r}_{,2}(d\beta)^2.\end{aligned}$$

Let us consider the following relations

$$E = \vec{r}_{,1} \cdot \vec{r}_{,1}, \qquad F = \vec{r}_{,1} \cdot \vec{r}_{,2}, \qquad G = \vec{r}_{,2} \cdot \vec{r}_{,2},$$

then

$$(ds)^2 = d\vec{r} \cdot d\vec{r} = E(d\alpha)^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2, \tag{2.26}$$

equation (2.26) is known as the *first fundamental form* of the surface Ω defined by vector $\vec{r}(\alpha, \beta)$, it allows us to determine the infinitesimal lengths, the angle between the curve, and the area of the surface. E, F, and G are called the first fundamental magnitudes. Along the parametric curves themselves, differential length of arcs take the form

$$\begin{aligned} ds_1 &= \sqrt{E} d\alpha \qquad \text{along a curve of constant } \beta \\ ds_2 &= \sqrt{G} d\beta \qquad \text{along a curve of constant } \alpha \end{aligned} \tag{2.27}$$

For an *orthogonal* net, since $\vec{r}_{,1}$ and $\vec{r}_{,2}$ are tangent to curves of constant α and β respectively, it follows that $F = 0$, then equation (2.26) becomes

$$(ds)^2 = A_1^2(d\alpha)^2 + A_2^2(d\beta)^2, \tag{2.28}$$

where $A_1^2 = E$ and $A_2^2 = G$, the quantities A_1 and A_2 are termed as Lame's parameters. Lame parameters are quantities which relate a change in arc length on the surface to the corresponding change in curvilinear coordinate.

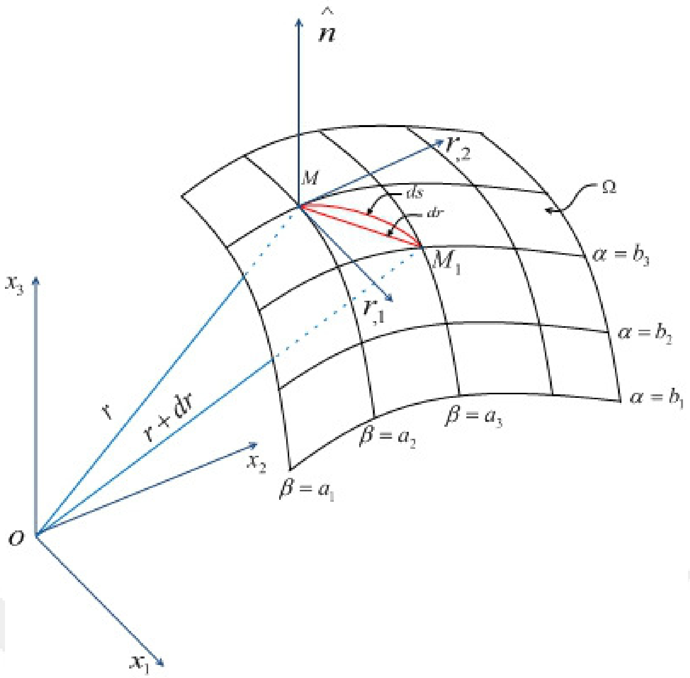

Figure 2.3. *Mid-Surface coordinates [49]*

2.2.2. The second fundamental form

At every point M on the surface Ω, there exists unit normal vector $\hat{n}(\alpha, \beta)$ which is perpendicular to the tangent plane containing $\vec{r}_{,1}$ and $\vec{r}_{,2}$. i.e.,

$$\hat{n}(\alpha, \beta) = \frac{\vec{r}_{,1} \times \vec{r}_{,2}}{|\vec{r}_{,1} \times \vec{r}_{,2}|}. \tag{2.29}$$

Note that

$$|\vec{r}_{,1} \times \vec{r}_{,2}| = |\vec{r}_{,1}|\,|\vec{r}_{,2}|\, sin\theta = \sqrt{EG} sin\theta,$$
$$\vec{r}_{,1} \cdot \vec{r}_{,2} = |\vec{r}_{,1}|\,|\vec{r}_{,2}|\, cos\theta = \sqrt{EG} cos\theta = F,$$

where θ is angle between $\vec{r}_{,1}$ and $\vec{r}_{,2}$.

So that

$$cos\theta = \frac{\vec{r}_{,1} \cdot \vec{r}_{,2}}{|\vec{r}_{,1}|\,|\vec{r}_{,2}|} = \frac{\vec{r}_{,1} \cdot \vec{r}_{,2}}{\sqrt{(\vec{r}_{,1} \cdot \vec{r}_{,1})} \cdot \sqrt{(\vec{r}_{,2} \cdot \vec{r}_{,2})}} = \frac{F}{\sqrt{EG}},$$
$$sin\theta = \sqrt{(1 - cos^2\theta)} = \sqrt{(1 - \frac{F^2}{EG})} = \sqrt{(\frac{EG - F^2}{EG})},$$

and thus the final expression for the *unit normal vector* $\hat{n}$ is

$$\hat{n}(\alpha,\beta) = \frac{\vec{r}_{,1}\times\vec{r}_{,2}}{\left|\vec{r}_{,1}\times\vec{r}_{,2}\right|} = \frac{\vec{r}_{,1}\times\vec{r}_{,2}}{\sqrt{EG}sin\theta} = \frac{\vec{r}_{,1}\times\vec{r}_{,2}}{\sqrt{EG}}\sqrt{\frac{EG}{EG-F^2}},$$

$$\hat{n}(\alpha,\beta) = \frac{\vec{r}_{,1}\times\vec{r}_{,2}}{H}, \qquad H \equiv \sqrt{(EG-F^2)}, \tag{2.30}$$

equation (2.30) requires that H does not vanish.

Now let us consider a curve on the surface Ω, see [49] for further detail. The curvature vector $\vec{K}$ of this curve is described by

$$\hat{t'} = \vec{K} = K\hat{N}, \tag{2.31}$$

where $\hat{t} = \frac{d\vec{r}}{ds}$ is the unit tangent vector to the curve; hence it lies in the tangent plane to the surface. Here $\hat{t'} = \frac{d}{ds}(\hat{t})$ and s is the variable of the arc length along the curve.
Now $\vec{K}$ can be resolved into components $\vec{K}_n$ and $\vec{K}_t$

$$\vec{K} = \vec{K}_n + \vec{K}_t, \tag{2.32}$$

where $\vec{K}_n$ is called normal curvature vector. It is the curvature of the curve projected onto the plane containing the curve's tangent $\hat{t}$ and surface normal $\hat{n}$. This plane is normal to the tangent plane to the surface. $\vec{K}_t$ is tangential curvature vector or geodesic curvature vector, it is the curvature of the curve projected onto the tangent plane to the surface.
Now, $\vec{K}_n$ is along $\hat{n}$, so it is proportional to $\hat{n}$ and can be expressed as

$$\vec{K}_n = -K_n\hat{n}, \tag{2.33}$$

where K_n is called normal curvature, the minus sign takes into account the fact that the direction of the $\vec{K}_n$ is opposite to that of the $\hat{n}$. Note that the normal curvature is a curvature of a normal section of the surface at a point, since $\hat{t}$ lies in the tangent plane and $\hat{n}$ is normal to the tangent plane.

Hence, $\hat{n} \cdot \hat{t} = 0$ and differentiating it with respect to s, we get

$$\hat{n'} \cdot \hat{t} = -\hat{t'} \cdot \hat{n}. \tag{2.34}$$

From equations (2.31) and (2.32), we have

$$\vec{K} \cdot \hat{n} = \hat{t'} \cdot \hat{n} = \vec{K}_n \cdot \hat{n}, \tag{2.35}$$

where

$$\vec{K}_t \cdot \hat{n} = 0,$$

from scalar product of equation (2.33), we have

$$K_n = -\vec{K}_n \cdot \hat{n}. \tag{2.36}$$

Making use of equations (2.34) and (2.35) into equation (2.36), we have

$$K_n = \hat{n'} \cdot \hat{t}, \tag{2.37}$$

hence

$$K_n = \hat{n'} \cdot \hat{t} = \frac{d\hat{n}}{ds} \cdot \frac{d\vec{r}}{ds} = \frac{d\hat{n} \cdot d\vec{r}}{(ds)^2} = \frac{d\hat{n} \cdot d\vec{r}}{d\vec{r} \cdot d\vec{r}}. \tag{2.38}$$

Then, since

$$d\hat{n} = \hat{n}_{,1} d\alpha + \hat{n}_{,2} d\beta,$$

$$d\vec{r} = \vec{r}_{,1} d\alpha + \vec{r}_{,2} d\beta,$$

it follows that

$$d\hat{n} \cdot d\vec{r} = L(d\alpha)^2 + 2M(d\alpha)(d\beta) + N(d\beta)^2, \tag{2.39}$$

equation (2.39) is known as the *second fundamental form* of the surface Ω defined by vector $\vec{r}(\alpha, \beta)$, where the second fundamental magnitudes are

$$L = \vec{r}_{,1} \cdot \hat{n}_{,1}, \qquad N = \vec{r}_{,2} \cdot \hat{n}_{,2}, \qquad 2M = \vec{r}_{,1} \cdot \hat{n}_{,2} + \vec{r}_{,2} \cdot \hat{n}_{,1}. \tag{2.40}$$

Second fundamental form can be used to compute the curvature of the curves obtained by intersecting the surface with normal planes, i.e., normal sections of the surface.

2.2.3. Principal curvatures

We know that

$$K_n = \frac{d\hat{n} \cdot d\vec{r}}{d\vec{r} \cdot d\vec{r}} = \frac{L(d\alpha))^2 + 2M(d\alpha)(d\beta) + N(d\beta)^2}{E(d\alpha))^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2}.$$

The simplified form of latter is

$$K_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}, \tag{2.41}$$

where , $\lambda = \frac{d\beta}{d\alpha}$, for the values of λ for which the normal curvature K_n is maximum or minimum, $\frac{dK_n}{d\lambda} = 0$. After some manipulation we find that the extreme value of $K_n(\lambda)$ is given by

$$K_n = \frac{M + N\lambda}{F + G\lambda} = \frac{L + M\lambda}{E + F\lambda}, \tag{2.42}$$

where λ is a root of

$$(MG - NF)\lambda^2 + (LG - NE)\lambda + (LF - ME) = 0. \tag{2.43}$$

These two roots will give two extremum values of normal curvature K_1 and K_2, one of them would be the maximum curvature (say K_1) and the other would be the minimum curvature (say K_2).

Since the normal curvature is a curvature of a normal section of the surface at a point, therefore, K_1 and K_2 represent the principal curvatures of the surface and the corresponding curves are called principal curves of the surface. Radii R_1 and R_2 are called principal radii of curvature. The principal radii of curvatures are the maximum and minimum radii of curvatures out of all possible radii of curvatures.

$$R_1 = \frac{1}{K_1} = \frac{E}{L} \qquad \text{(minimum radii of curvature)}$$

$$R_2 = \frac{1}{K_2} = \frac{G}{N} \qquad \text{(maximum radii of curvature)}$$

for obtaining relations above one can see [49].

Since principal curves are orthogonal to each other, then $F = M = 0.$ Hence, for the principal curves, the principal curvature would be given by

$$K_n = \frac{L + N\lambda^2}{E + G\lambda^2} = \frac{L(d\alpha)^2 + N(d\beta)^2}{E(d\alpha)^2 + G(d\beta)^2}. \tag{2.44}$$

2.2.4. Gauss-Codazzi conditions

The relations

$$\hat{n}_{,1} = \frac{A_1}{R_1}\hat{t}_1, \qquad \hat{n}_{,2} = \frac{A_2}{R_2}\hat{t}_2, \tag{2.45}$$

and

$$\begin{aligned}
\hat{t}_{1,1} &= -\frac{A_1}{R_1}\hat{n} - \frac{A_{1,2}}{A_2}\hat{t}_2,\\
\hat{t}_{1,2} &= \frac{A_{2,1}}{A_1}\hat{t}_2,\\
\hat{t}_{2,1} &= \frac{A_{1,2}}{A_2}\hat{t}_1,\\
\hat{t}_{2,2} &= -\frac{A_2}{R_2}\hat{n} - \frac{A_{2,1}}{A_1}\hat{t}_1,
\end{aligned} \tag{2.46}$$

are known as theorem of Rodriques and Weingarten formula respectively, one can find the proof of each in textbooks of differential geometry and $\hat{t}_1$, $\hat{t}_2$ and $\hat{n}$ are units vectors.

Now we derive three differential equations which are known as Gauss-Codazzi conditions, which relate the quantities A_1, A_2, R_1 and R_2 of a given surface, these are found from the equality of the mixed second derivatives of the unit vectors, these vectors have continuous second derivatives,

$$\hat{n}_{,12} = \hat{n}_{,21}.$$

With help of Rodriques theorem and rules of differentiation, the latter equation can be written as

$$\left(\frac{A_1}{R_1}\hat{t}_1\right)_{,2} = \left(\frac{A_2}{R_2}\hat{t}_2\right)_{,1},$$
$$\left[\left(\frac{A_1}{R_1}\right)_{,2}\hat{t}_1 + \frac{A_1}{R_1}\hat{t}_{1,2}\right] = \left[\left(\frac{A_2}{R_2}\right)_{,1}\hat{t}_2 + \frac{A_2}{R_2}\hat{t}_{2,1}\right].$$

Next, using Weingarten formulae, we obtain

$$\left[\left(\frac{A_1}{R_1}\right)_{,2}\hat{t}_1 + \frac{A_1}{R_1}\cdot\frac{A_{2,1}}{A_1}\hat{t}_2\right] = \left[\left(\frac{A_2}{R_2}\right)_{,1}\hat{t}_2 + \frac{A_2}{R_2}\cdot\frac{A_{1,2}}{A_2}\hat{t}_1\right],$$

hence

$$\hat{t}_1\left[\left(\frac{A_1}{R_1}\right)_{,2} - \frac{A_{1,2}}{R_2}\right] = \hat{t}_2\left[\left(\frac{A_2}{R_2}\right)_{,1} - \frac{A_{2,1}}{R_1}\right],$$

$\hat{t}_1$ can not be along $\hat{t}_2$ as they both are perpendicular, the contents in square brackets must be zero

$$\left(\frac{A_1}{R_1}\right)_{,2} = \frac{A_{1,2}}{R_2}, \qquad \left(\frac{A_2}{R_2}\right)_{,1} = \frac{A_{2,1}}{R_1}, \tag{2.47}$$

these two equations are called *Codazzi conditions*.

Now, again we take equations

$$\hat{t}_{1,12} = \hat{t}_{1,21},$$
$$\left(\hat{t}_{1,1}\right)_{,2} = \left(\hat{t}_{1,2}\right)_{,1}.$$

Using Weingarten formula

$$\left(-\frac{A_1}{R_1}\hat{n} - \frac{A_{1,2}}{A_2}\hat{t}_2\right)_{,2} = \left(\frac{A_{2,1}}{A_1}\hat{t}_2\right)_{,1},$$

after some manipulation we get

$$\left(\frac{1}{A_1}A_{2,1}\right)_{,1} + \left(\frac{1}{A_2}A_{1,2}\right)_{,2} = -\frac{A_1A_2}{R_1R_2}, \tag{2.48}$$

this equation is called *Gauss condition*.

2.3. The Construction of a Shell Theory

2.3.1. Love's first approximation

The first approximate shell theory based on classical elasticity was introduced by Love. Love's first approximation, which is also known as the Kirchhoff-Love hypothesis, depends on the following assumptions:

1. The shell is assumed to be thin,
2. The shell deflections are assumed to be small,
3. The transverse normal stress is small compared to other components of stress, and may be neglected,
4. Normals to the undeformed reference surface remain normal to the deformed reference surface.

The first assumption is basic to the entire theory, which states that the ratio of shell thickness (h) to one of the radii of curvature (R_i) shall be small compared to unity, i.e., $\frac{h}{R_i} \ll 1$. The second one ensure that

$$\varepsilon_3 = \varepsilon_{13} = \varepsilon_{23} = 0, \tag{2.49}$$

while in some references, instead of above γ_{i3} $(i = 1, 2, 3.)$ is used, where

$$\gamma_{i3} = 2\varepsilon_{i3} \qquad (i = 1, 2, 3).$$

The third one states that σ_3, is much smaller than σ_1, σ_2, i.e., $\sigma_3 \ll \sigma_1, \sigma_2$.

By considering the above assumptions, the stress-strain relations for an isotropic, homogeneous elastic solid can be written as

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E}\left(\sigma_1 - \nu\sigma_2\right), \\ \varepsilon_2 &= \frac{1}{E}\left(\sigma_2 - \nu\sigma_1\right). \end{aligned} \tag{2.50}$$

The latter gives another convenient form as

$$\sigma_1 = \frac{E}{1-\nu^2}\left(\varepsilon_1 + \nu\varepsilon_2\right),$$
$$\sigma_2 = \frac{E}{1-\nu^2}\left(\varepsilon_2 + \nu\varepsilon_1\right), \tag{2.51}$$

$$\sigma_{12} = 2G\varepsilon_{12} = G\gamma_{12}, \qquad \sigma_{13} = \sigma_{23} = 0. \tag{2.52}$$

2.3.2. Shell coordinates

The position vector of an arbitrary point (α_1, α_2, z) in the space occupied by a thin shell, see [12] is defined as

$$\vec{R}(\alpha_1, \alpha_2, z) = \vec{r}(\alpha_1, \alpha_2) + z\hat{n}(\alpha_1, \alpha_2), \tag{2.53}$$

where z is the position of the point measured from the mid (reference) surface along with $\hat{n}$, $\hat{n}$ is the unit normal vector from the mid-surface to the point and $\vec{r}$ is the position vector of a corresponding point on the mid-surface.
The differential line element on a surface away from the mid-surface is

$$d\vec{R} = d\vec{r} + zd\hat{n} + \hat{n}dz. \tag{2.54}$$

Thus, by using the equation (2.54), the magnitude of an arbitrary differential element of length in space is given as

$$(ds)^2 = dr \cdot dr + 2zdr \cdot dn + 2dz(dr \cdot n) + z^2(dn \cdot dn)$$
$$+2zdz(n \cdot dn) + (dz)^2(n \cdot n). \tag{2.55}$$

From previous notes of section (2.2), especially equations (2.28), (2.41), principal radii of curvatures and theorem of Rodrigues we get

$$
\begin{aligned}
dr \cdot dr &= A_1^2\,(d\alpha_1)^2 + A_2^2\,(d\alpha_2)^2\,,\\
dr \cdot dr &= \frac{A_1^2}{R_1}\,(d\alpha_1)^2 + \frac{A_2^2}{R_2}\,(d\alpha_2)^2\,,\\
dr \cdot n &= 0,\\
dn \cdot dn &= (\frac{A_1}{R_1})^2\,(d\alpha_1)^2 + (\frac{A_2}{R_2})^2\,(d\alpha_2)^2\,,\\
dn \cdot dn &= 0,\\
n \cdot n &= 1.
\end{aligned}
\tag{2.56}
$$

Substituting the values of last equation into equation (2.55), one obtain

$$
(ds)^2 = A_1^2\left(1+\frac{z}{R_1}\right)^2 (d\alpha_1)^2 + A_2^2\left(1+\frac{z}{R_2}\right)^2 (d\alpha_2)^2 + (dz)^2, \tag{2.57}
$$

the first two terms on the right-hand side of this equation is the first fundamental form of a surface at a distance z from the mid-surface and should satisfy the Codazzi conditions. The last equation can be written in convenient form as

$$
(ds)^2 = g_1(d\alpha_1)^2 + g_2(d\alpha_2)^2 + g_3(dz)^2, \tag{2.58}
$$

where the Lame parameters are

$$
\begin{aligned}
g_1 &= A_1^2\left(1+\frac{z}{R_1}\right)^2,\\
g_2 &= A_2^2\left(1+\frac{z}{R_2}\right)^2,\\
g_3 &= 1.
\end{aligned}
\tag{2.59}
$$

The lengths of the edges of the fundamental element are given by

$$
\begin{aligned}
ds_1(z) &= A_1\left(1+\frac{z}{R_1}\right)d\alpha_1,\\
ds_2(z) &= A_2\left(1+\frac{z}{R_2}\right)d\alpha_2.
\end{aligned}
\tag{2.60}
$$

The corresponding area elements of the faces are

$$d\Sigma_1(z) = A_1 \left(1 + \frac{z}{R_1}\right)(d\alpha_1)(dz),$$
$$d\Sigma_2(z) = A_2 \left(1 + \frac{z}{R_2}\right)(d\alpha_2)(dz). \tag{2.61}$$

The volume of the fundamental element can be written as

$$dv(z) = A_1 A_2 \left(1 + \frac{z}{R_1}\right)\left(1 + \frac{z}{R_2}\right)(d\alpha_1)(d\alpha_2)(dz). \tag{2.62}$$

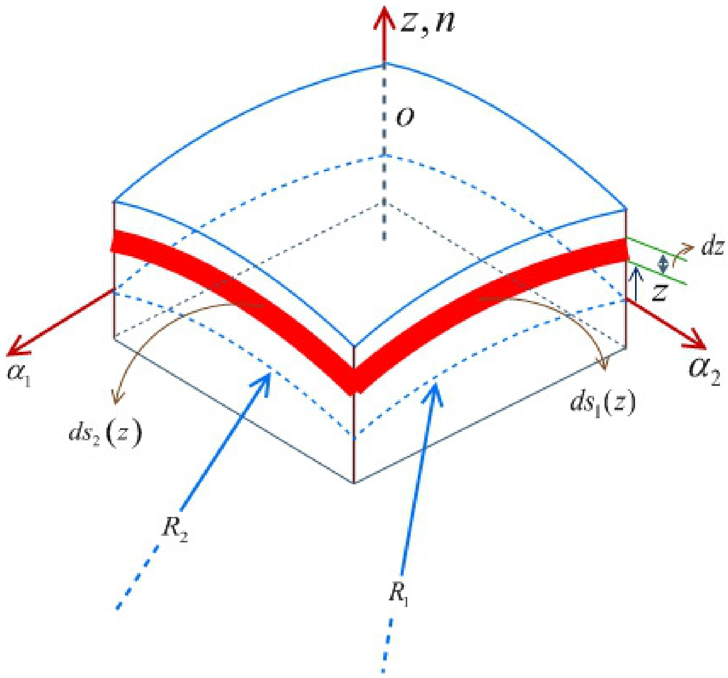

Figure 2.4. *Differential element of a shell [49]*

2.3.3. Strain-displacements relations

The normal and shear strain components in an orthogonal curvilinear system are

$$\varepsilon_i = \frac{\partial}{\partial \alpha_i}\left(\frac{U_i}{\sqrt{g_i}}\right) + \frac{1}{2g_i}\sum_{k=1}^{3}\frac{\partial g_i}{\partial \alpha_k}\frac{U_k}{\sqrt{g_k}}, \qquad i = 1,2,3$$
$$\gamma_{ij} = \frac{1}{\sqrt{g_i g_j}}\left[g_i \frac{\partial}{\partial \alpha_j}\left(\frac{U_i}{\sqrt{g_i}}\right) + g_j \frac{\partial}{\partial \alpha_i}\left(\frac{U_j}{\sqrt{g_j}}\right)\right], \quad i,j = 1,2,3, \tag{2.63}$$

where $\varepsilon_i, \gamma_{ij}$, and U_i are normal strains, shear strains, and displacement components, respectively, at an arbitrary point.
Let

$$\begin{aligned} &\alpha_1 = \alpha_1, \qquad \alpha_2 = \alpha_2, \qquad \alpha_3 = z, \\ &U_1 = U, \qquad U_2 = V, \qquad U_3 = W. \end{aligned} \tag{2.64}$$

Using equations (2.59), and (2.64) into equation (2.63), we obtain

$$\begin{aligned} \varepsilon_1 &= \frac{1}{A_1(1+\frac{z}{R_1})}\left(\frac{\partial U}{\partial \alpha_1} + \frac{V}{A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{A_1 W}{R_1}\right), \\ \varepsilon_2 &= \frac{1}{A_2(1+\frac{z}{R_2})}\left(\frac{\partial V}{\partial \alpha_2} + \frac{U}{A_1}\frac{\partial A_2}{\partial \alpha_1} + \frac{A_2 W}{R_2}\right), \\ \varepsilon_3 &= \frac{\partial W}{\partial z}, \\ \gamma_{12} &= \frac{A_1(1+\frac{z}{R_1})}{A_2(1+\frac{z}{R_2})}\frac{\partial}{\partial \alpha_2}\left(\frac{U}{A_1(1+\frac{z}{R_1})}\right) + \frac{A_2(1+\frac{z}{R_2})}{A_1(1+\frac{z}{R_1})}\frac{\partial}{\partial \alpha_1}\left(\frac{V}{A_2(1+\frac{z}{R_2})}\right), \\ \gamma_{13} &= \frac{1}{A_1(1+\frac{z}{R_1})}\frac{\partial W}{\partial \alpha_1} + A_1(1+\frac{z}{R_1})\frac{\partial}{\partial z}\left(\frac{U}{A_1(1+\frac{z}{R_1})}\right), \\ \gamma_{23} &= \frac{1}{A_2(1+\frac{z}{R_2})}\frac{\partial W}{\partial \alpha_2} + A_2(1+\frac{z}{R_2})\frac{\partial}{\partial z}\left(\frac{V}{A_2(1+\frac{z}{R_2})}\right), \end{aligned} \tag{2.65}$$

these relations are exact, i.e., Love's assumptions are not introduced in these. Now to satisfy the Kirchhoff hypothesis, the following linear relationships of displacements are taken.

$$U = u + z\beta_1, \qquad V = v + z\beta_2, \qquad W = w, \tag{2.66}$$

where β_1, β_2 are rotations that express changes in the slope of the normal to the reference surface and u, v, w are displacements of mid-surface that correspond to the direction α_1, α_2 and normal n respectively.
With help of equation (2.66), Codazzi-conditions and second assumption, the exact relations (2.65) can be written as

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{1+\frac{z}{R_1}}\left(\varepsilon_{10}+z\kappa_1\right), \\
\varepsilon_2 &= \frac{1}{1+\frac{z}{R_2}}\left(\varepsilon_{20}+z\kappa_2\right), \\
\varepsilon_3 &= \frac{\partial w}{\partial z} = 0, \\
\gamma_{12} &= \frac{1}{1+\frac{z}{R_1}}\left(\gamma_{10}+z\delta_1\right) + \frac{1}{1+\frac{z}{R_2}}\left(\gamma_{20}+z\delta_2\right), \\
\gamma_{13} &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1} + \beta_1\right) = 0, \\
\gamma_{13} &= \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} + \beta_2\right) = 0,
\end{aligned}
\tag{2.67}
$$

from third equation of (2.67), it is clear that deflection w is independent of the z coordinate. Also, from last two relations of equation (2.67), the rotations β_1, β_2 can be expressed in terms of displacement u, v, and w of the reference surface as

$$
\begin{aligned}
\beta_1 &= \frac{u}{R_1} - \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}, \\
\beta_2 &= \frac{v}{R_2} - \frac{1}{A_2}\frac{\partial w}{\partial \alpha_2},
\end{aligned}
\tag{2.68}
$$

where

$$
\begin{aligned}
\varepsilon_{10} &= \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{v}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1}, \\
\varepsilon_{20} &= \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{u}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2},
\end{aligned}
$$

$$
\begin{aligned}
\kappa_1 &= \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}, \\
\kappa_2 &= \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1},
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_{10} &= \frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{u}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2},\\
\gamma_{20} &= \frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{v}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1},\\
\delta_1 &= \frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2},\\
\delta_2 &= \frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2} - \frac{\beta_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1},
\end{aligned}
$$

$\varepsilon_{10}, \varepsilon_{20}$ are extensional strains at reference surface and κ_1, κ_2 are changes in the curvature of the reference surface during deformations.

From assumption (1), $\frac{z}{R_i}$ are small compared with unity and can be neglected, hence, the relations in (2.67) reduce to

$$
\begin{aligned}
\varepsilon_1 &= \varepsilon_{10} + z\kappa_1,\\
\varepsilon_2 &= \varepsilon_{20} + z\kappa_2,\\
\gamma_{120} &= \gamma_{12} + z\kappa_{12}.
\end{aligned}
\tag{2.69}
$$

Shear strain γ_{120} of the mid-surface is given by

$$
\gamma_{120} = \gamma_{10} + \gamma_{20}. \tag{2.70}
$$

Also, the twisting direction k_{12} is defined by

$$
\kappa_{12} = \delta_1 + \delta_2, \tag{2.71}
$$

for further discussion see [52].

2.3.4. Stress resultants and stress couples

Stress resultants and stress couples acting per unit length of the mid-surface of the shell, represent the system of internal forces and moments (bending and twisting moments) respectively acting on the differential shell element.

The stress resultants are defined as

$$T_i = \int_{-h}^{h} (1 + \frac{\alpha_3}{R_j})\sigma_{ii}\, d\alpha_3, \qquad S_{ij} = \int_{-h}^{h} (1 + \frac{\alpha_3}{R_j})\sigma_{ij}\, d\alpha_3, \qquad (i \neq j = 1, 2)\,. \tag{2.72}$$

Similarly, stress couples are defined as

$$G_i = \int_{-h}^{h} (1 + \frac{\alpha_3}{R_j})\sigma_{ii}\alpha_3\, d\alpha_3, \qquad H_{ij} = \int_{-h}^{h} (1 + \frac{\alpha_3}{R_j})\sigma_{ij}\alpha_3\, d\alpha_3, \qquad (i \neq j = 1, 2)\,. \tag{2.73}$$

Finally, the shear force resultants are defined as

$$N_i = \int_{-h}^{h} (1 + \frac{\alpha_3}{R_j})\sigma_{i3}\, d\alpha_3, \qquad (i \neq j = 1, 2)\,, \tag{2.74}$$

where h and α_3 are half thickness of mid-surface of shell and transverse coordinate respectively. Also T_i are the longitudinal stress resultants, S_{ij} are the membrane shear forces, G_i are the bending stress couples, H_{ij} are the twisting stress couples, N_i are the transverse shear forces.

Now, applying assumption (1) on the equations (2.72), (2.73), and (2.74), we will get the simplified form of these equations. In this case,

$$S_{ij} = S_{ji}, \qquad H_{ij} = H_{ji}.$$

Substituting equation (2.69) into equations (2.51) and (2.52), then inserting the result into a simplified form of equations (2.72) and (2.73) will give the constitutive equations, i.e.,

$$\begin{aligned} T_i &= k\left(\varepsilon_{i0} + \nu\varepsilon_{j0}\right), \\ S_{ij} &= S_{ji} = Gh\gamma_{ij0}, \\ G_i &= D\left(\kappa_i + \nu\kappa_j\right), \\ H_{ij} &= H_{ji} = \frac{1-\nu}{2}D\kappa_{ij}, \end{aligned} \tag{2.75}$$

where

$$k = \frac{Eh}{1-\nu^2}, \qquad G = \frac{E}{2(1+\nu)}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}. \tag{2.76}$$

k is called the extensional rigidity, D is bending (flexural) rigidity or stiffness, and G is the shear modulus of the shell. The system of equations (2.75) is also known as the stress-strain relationship.

2.3.5. Force-displacement equations

If we substitute values of $\varepsilon_{10}, \varepsilon_{20}, \gamma_{120}, \kappa_1, \kappa_2$, and κ_{12} from previous notes into equation (2.75), we get the force-displacement relationships.

2.3.6. Equilibrium equations

Equations of motion are derived by three different methods [52]. For the sake of simplicity, the equations of motion are derived in the static case, yielding equations which govern the equilibrium of a shell element.

$$\begin{aligned}
&\frac{\partial T_i A_j}{\partial \alpha_i} + \frac{\partial S_{ji} A_i}{\partial \alpha_j} + S_{ij}\frac{\partial A_i}{\partial \alpha_j} - T_j\frac{\partial A_j}{\partial \alpha_i} + A_i A_j \frac{N_i}{R_i} = 0,\\
&\frac{\partial N_1 A_2}{\partial \alpha_1} + \frac{\partial N_2 A_1}{\partial \alpha_2} - \left(\frac{T_1}{R_1} + \frac{T_2}{R_2}\right) A_1 A_2 = 0,\\
&\frac{\partial G_i A_j}{\partial \alpha_i} + \frac{\partial H_{ji} A_i}{\partial \alpha_j} + H_{ij}\frac{\partial A_i}{\partial \alpha_j} - G_j\frac{\partial A_j}{\partial \alpha_i} - A_i A_j N_i = 0,
\end{aligned} \tag{2.77}$$

The sixth equilibrium equation given below can also be derived by some appropriate method,

$$\frac{H_{12}}{R_1} - \frac{H_{21}}{R_2} + S_{12} - S_{21} = 0. \tag{2.78}$$

Effective external force terms have been omitted from the above equations of equilibrium for simplicity. Also, for dynamic equations of motion, see section (8.1) of the cited reference [49].

By eliminating shear force resultants N_i from last equation of (2.77), substituting the resulting equations for N_i into the first two equations of equation (2.77), then the equilibrium equations will be reduced to system of three equations.

2.4. A Brief Account of 3D Equations of Motion

Let us consider a linearly elastic isotropic body of uniform thickness. The mechanical parameters can be written as : E is Young's modulus, ν is Poisson's ratio, ρ is the mass density of the body and $2h$ is the thickness of the body .

In this thesis we use the orthogonal coordinate system specified by the vector equality

$$P\left(\alpha_1, \alpha_2, \alpha_3\right) = M\left(\alpha_1, \alpha_2\right) + \alpha_3 n, \tag{2.79}$$

where α_1, α_2 are parameters of the lines of curvature on the mid-surface of the body $\Gamma\left(\alpha_3 = 0\right)$; n is the unit mid-surface normal; α_3 is the distance from Γ along the normal (see figure (2.5)).

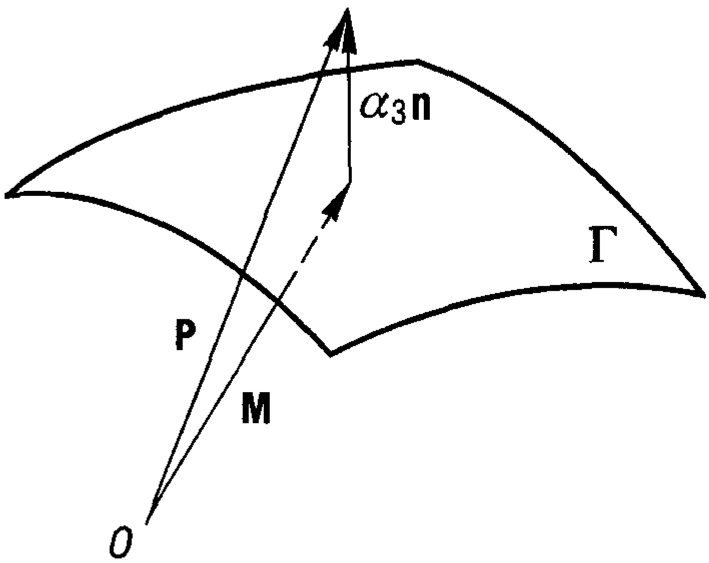

Figure 2.5. *Mid-Surface of an arbitrary shell [41]*

In the coordinates α_1, α_2, α_3 the 3D dynamic equations of elasticity can be written as, see [41],

$$\frac{1}{H_i}\frac{\partial \sigma_{ii}}{\partial \alpha_i} + \frac{1}{H_j}\frac{\partial \sigma_{ji}}{\partial \alpha_j} + \frac{\partial \sigma_{3i}}{\partial \alpha_3} + \frac{1}{H_i H_j}\frac{\partial H_j}{\partial \alpha_i}\left(\sigma_{ii} - \sigma_{jj}\right) + \frac{1}{H_i H_j}\frac{\partial H_i}{\partial \alpha_j}\left(\sigma_{ij} + \sigma_{ji}\right)$$
$$+ \frac{1}{H_i H_j}\frac{\partial H_i H_j}{\partial \alpha_3}\sigma_{3i} + \frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3}\sigma_{i3} - \rho\frac{\partial^2 v_i}{\partial t^2} = 0, \tag{2.80}$$

$$
\frac{1}{H_i}\frac{\partial \sigma_{i3}}{\partial \alpha_i}+\frac{1}{H_j}\frac{\partial \sigma_{j3}}{\partial \alpha_j}+\frac{\partial \sigma_{33}}{\partial \alpha_3}-\frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3}\sigma_{ii}-\frac{1}{H_j}\frac{\partial H_j}{\partial \alpha_3}\sigma_{jj}+\frac{1}{H_iH_j}\frac{\partial H_iH_j}{\partial \alpha_3}\sigma_{33}
+\frac{1}{H_iH_j}\frac{\partial H_j}{\partial \alpha_i}\sigma_{i3}+\frac{1}{H_iH_j}\frac{\partial H_i}{\partial \alpha_j}\sigma_{j3}-\rho\frac{\partial^2 v_3}{\partial t^2}=0 \quad (2.81)
$$

and

$$
\sigma_{ii}=\frac{E}{2(1+\nu)\chi^2}\left[\frac{\nu}{1-\nu}\left(\frac{\partial v_3}{\partial \alpha_3}+\frac{1}{H_j}\frac{\partial v_j}{\partial \alpha_j}+\frac{1}{H_iH_j}\frac{\partial H_j}{\partial \alpha_i}v_i+\frac{1}{H_j}\frac{\partial H_j}{\partial \alpha_3}v_3\right)
+\frac{1}{H_i}\frac{\partial v_i}{\partial \alpha_i}+\frac{1}{H_iH_j}\frac{\partial H_i}{\partial \alpha_j}v_j+\frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3}v_3\right], \quad (2.82)
$$

$$
\sigma_{33}=\frac{E}{2(1+\nu)\chi^2}\left[\frac{\nu}{1-\nu}\left(\frac{1}{H_i}\frac{\partial v_i}{\partial \alpha_i}+\frac{1}{H_j}\frac{\partial v_j}{\partial \alpha_j}+\frac{1}{H_iH_j}\frac{\partial H_i}{\partial \alpha_j}v_j
+\frac{1}{H_iH_j}\frac{\partial H_j}{\partial \alpha_i}v_i+\frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3}v_3+\frac{1}{H_j}\frac{\partial H_j}{\partial \alpha_3}v_3\right)+\frac{\partial v_3}{\partial \alpha_3}\right], \quad (2.83)
$$

$$
\sigma_{3i}=\frac{E}{2(1+\nu)}\left(\frac{1}{H_i}\frac{\partial v_3}{\partial \alpha_i}+\frac{\partial v_i}{\partial \alpha_3}-\frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3}v_i\right), \quad (2.84)
$$

$$
\sigma_{ij}=\frac{E}{2(1+\nu)}\left(\frac{1}{H_j}\frac{\partial v_i}{\partial \alpha_j}+\frac{1}{H_i}\frac{\partial v_j}{\partial \alpha_i}-\frac{1}{H_iH_j}\frac{\partial H_i}{\partial \alpha_j}v_i-\frac{1}{H_iH_j}\frac{\partial H_j}{\partial \alpha_i}v_j\right), \quad (2.85)
$$

where $i \neq j = 1, 2$ and

$$
\chi=\frac{c_2}{c_1}=\sqrt{\frac{1-2\nu}{2(1-\nu)}}, \quad (2.86)
$$

with

$$
c_1=\sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}, \qquad c_2=\sqrt{\frac{E}{2(1+\nu)\rho}}. \quad (2.87)
$$

Here σ_{kl} $(k, l = 1, 2, 3)$ are the stresses, v_m $(m = 1, 2, 3)$ are the displacements, t is time, H_i are Lame's coefficients, c_1 is the dilatation wave speed, c_2 is the distortion wave speed.

In the 3D dynamic equations of elasticity H_i are the Lame's coefficients which are defined by the formula

$$
H_i = A_i\left(1+\frac{\alpha_3}{R_i}\right), \quad (2.88)
$$

where A_i are the coefficients of the first quadratic form $A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2$ of the mid-surface, R_i are the principal curvature radii of the mid-surface.

The derivatives of Lame's coefficients by Minardi-Codazzi relations (2.47) can be written as

$$\frac{\partial H_i}{\partial \alpha_j} = \frac{\partial A_i}{\partial \alpha_j}\left(1 + \frac{\alpha_3}{R_j}\right), \qquad \frac{\partial H_i}{\partial \alpha_3} = \frac{A_i}{R_i}, \qquad (i \neq j = 1, 2)\,. \tag{2.89}$$

The traction - free boundary conditions along the faces $\zeta = \pm 1$ are considered as

$$\sigma_{3i} = 0. \tag{2.90}$$

Now, we obtain after straight forward calculations of (2.82)

$$\sigma_{33} = A\left(\frac{\partial v_3}{\partial \alpha_3} + \frac{\nu}{1-\nu}\left(a_1 + a_2\right)\right), \tag{2.91}$$

where

$$a_1 = \frac{1}{H_i}\frac{\partial v_i}{\partial \alpha_i} + \frac{1}{H_i H_j}\frac{\partial H_i}{\partial \alpha_j} v_j + \frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3} v_3,$$
$$a_2 = \frac{1}{H_j}\frac{\partial v_j}{\partial \alpha_j} + \frac{1}{H_i H_j}\frac{\partial H_j}{\partial \alpha_i} v_i + \frac{1}{H_j}\frac{\partial H_j}{\partial \alpha_3} v_3$$

and

$$A = \frac{E}{2(1+\nu)\chi^2}.$$

It is obvious from equation (2.91) that

$$\frac{\partial v_3}{\partial \alpha_3} = \frac{\sigma_{33}}{A} - \frac{\nu}{1-\nu}\left(a_1 + a_2\right). \tag{2.92}$$

Next, inserting the latter into equation (2.82), we have

$$\begin{aligned}\sigma_{ii} = \frac{E}{1-\nu^2}\Bigg[&\left(\frac{1}{H_i}\frac{\partial v_i}{\partial \alpha_i} + \frac{1}{H_i H_j}\frac{\partial H_i}{\partial \alpha_j} v_j + \frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3} v_3\right)\\ &+ \nu\left(\frac{1}{H_j}\frac{\partial v_j}{\partial \alpha_j} + \frac{1}{H_i H_j}\frac{\partial H_j}{\partial \alpha_i} v_i + \frac{1}{H_j}\frac{\partial H_j}{\partial \alpha_3} v_3\right)\Bigg] + \frac{\nu}{1-\nu}\sigma_{33}.\end{aligned} \tag{2.93}$$

Interchanging $i \longleftrightarrow j$ in equation (2.82), we have

$$\begin{aligned}\sigma_{jj} = \frac{E}{1-\nu^2}\bigg[& \left(\frac{1}{H_j}\frac{\partial v_j}{\partial \alpha_j} + \frac{1}{H_iH_j}\frac{\partial H_j}{\partial \alpha_i}v_i + \frac{1}{H_j}\frac{\partial H_j}{\partial \alpha_3}v_3\right) \\ & + \nu\left(\frac{1}{H_i}\frac{\partial v_i}{\partial \alpha_i} + \frac{1}{H_iH_j}\frac{\partial H_i}{\partial \alpha_j}v_j + \frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3}v_3\right)\bigg] + \frac{\nu}{1-\nu}\sigma_{33}.\end{aligned} \tag{2.94}$$

Similarly, from equations (2.91) (2.93) and (2.94), we get

$$E\frac{\partial v_3}{\partial \alpha_3} = \sigma_{33} - \nu\left(\sigma_{ii} + \sigma_{jj}\right), \tag{2.95}$$

equations (2.93) and (2.95) are the simplified form of equations (2.82) and (2.83) respectively.

3. ASYMPTOTIC ANALYSIS OF A 2D HOMOGENEOUS ANNULUS

In this chapter, we investigate the asymptotic integration of 2D dynamic equations of motion for a thin elastic annulus, which is assumed to be a cross-section of a thin homogenous cylindrical shell. Low-frequency vibrations are considered for a thin elastic annulus. The 2D dynamic equations starting from the 3D equations are subjected to asymptotic analysis for leading and next-order (second-order) approximation. The displacement and stress components are first separated into their even and odd parts, then normalized under appropriate assumptions. A 1D equation that will govern the vibrations of a thin elastic cylindrical shell at the lowest cut-off frequencies is derived. For validity and accuracy of these approximate solutions, the result is compared with other relevant results.

3.1. Statement of the Problem

Consider a thin elastic cylindrical shell of thickness $2h$ for which the geometrical parameter $\eta = \frac{h}{R}$ is small compared to unity, where R is the radius of curvature of the mid-surface of the shell. The mechanical parameters can be written as : E is Young's modulus, ν is Poisson's ratio, and ρ is the mass density of the shell.

We define the curvilinear coordinates α_2 and α_3 as

$$\alpha_2 = R\theta, \qquad \alpha_3 = h\zeta, \tag{3.1}$$

where $\theta(0 \leq \theta < 2\pi)$ is the dimensionless circumferential coordinate and $\zeta(-1 \leq \zeta \leq 1)$ is the dimensionless transverse coordinate , while the longitudinal coordinate α_1 is not shown in figure (3.1).

Also, A_i and R_i of (2.88) can be considered throughout the thesis as

$$\begin{aligned} R_1 &= \infty, \qquad A_1 = 1, \\ R_2 &= R, \qquad A_2 = 1. \end{aligned} \tag{3.2}$$

With the help of above given relations we can proceed our study, in first attempt we deal with the thin elastic homogeneous shell, which will be base for our further study on an inhomogeneous case. In an inhomogeneous shell, the above mentioned mechanical parameters will be assumed as functions of thickness variable α_3.

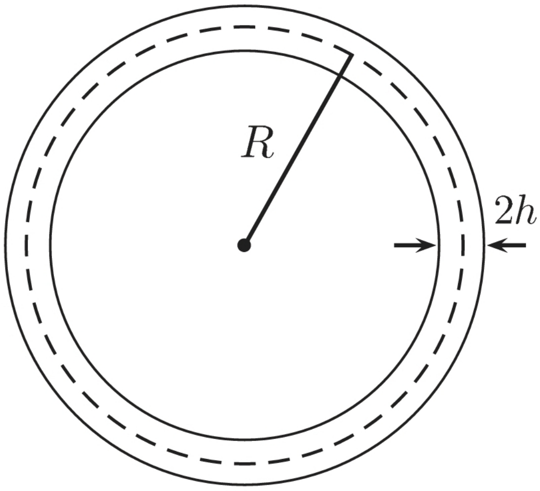

Figure 3.1. *A thin elastic annulus.*

The 3D equations of motion can be reduced to 2D plane strain equations of motion as, see section (2.4),

$$\frac{R}{R+\alpha_3}\frac{\partial \sigma_{22}}{\partial \alpha_2}+\frac{\partial \sigma_{32}}{\partial \alpha_3}+\frac{2}{R+\alpha_3}\sigma_{32}-\rho\frac{\partial^2 v_2}{\partial t^2}=0, \tag{3.3}$$

$$\frac{R}{R+\alpha_3}\frac{\partial \sigma_{32}}{\partial \alpha_2}+\frac{\partial \sigma_{33}}{\partial \alpha_3}-\frac{1}{R+\alpha_3}\sigma_{22}+\frac{1}{R+\alpha_3}\sigma_{33}-\rho\frac{\partial^2 v_3}{\partial t^2}=0, \tag{3.4}$$

with

$$\sigma_{22}=\frac{E}{1-\nu^2}\left(\frac{R}{R+\alpha_3}\frac{\partial v_2}{\partial \alpha_2}+\frac{1}{R+\alpha_3}v_3\right)+\frac{\nu}{1-\nu}\sigma_{33}, \tag{3.5}$$

$$E\frac{\partial v_3}{\partial \alpha_3}=(1-\nu^2)\sigma_{33}-\nu(1+\nu)\sigma_{22}, \tag{3.6}$$

$$\sigma_{32}=\frac{E}{2(1+\nu)}\left(\frac{R}{R+\alpha_3}\frac{\partial v_3}{\partial \alpha_2}+\frac{\partial v_2}{\partial \alpha_3}-\frac{1}{R+\alpha_3}v_2\right), \tag{3.7}$$

where v_i, $i=2,3$ are displacements and σ_{2i}, $i=2,3$ are the stress components.

For the sake of simplicity, the traction-free boundary conditions along the faces $\zeta=\pm1$, is considered as

$$\sigma_{3i}=0. \tag{3.8}$$

For separating symmetric and antisymmetric motion, let us first separate the displacement and stress components into their even and odd parts with respect to the thickness variable α_3, as

$$\sigma_{ij} = \sigma_{ij,e} + \sigma_{ij,o}, \qquad v_i = v_{i,e} + v_{i,o}, \qquad i, j = 2, 3, \tag{3.9}$$

where the quantities with suffixes e and o, are even and odd quantities, respectively. Thus, equations (3.3)-(3.7) become,

$$\frac{R^2}{R^2-\alpha_3^2}\frac{\partial\sigma_{22,e}}{\partial\alpha_2} - \frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial\sigma_{22,o}}{\partial\alpha_2} + \frac{\partial\sigma_{32,o}}{\partial\alpha_3} + \frac{2R}{R^2-\alpha_3^2}\sigma_{32,e} - \frac{2\alpha_3}{R^2-\alpha_3^2}\sigma_{32,o} - \rho\frac{\partial^2 v_{2,e}}{\partial t^2} = 0, \tag{3.10}$$

$$-\frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial\sigma_{22,e}}{\partial\alpha_2} + \frac{R^2}{R^2-\alpha_3^2}\frac{\partial\sigma_{22,o}}{\partial\alpha_2} + \frac{\partial\sigma_{32,e}}{\partial\alpha_3} + \frac{2R}{R^2-\alpha_3^2}\sigma_{32,o} - \frac{2\alpha_3}{R^2-\alpha_3^2}\sigma_{32,e} - \rho\frac{\partial^2 v_{2,o}}{\partial t^2} = 0, \tag{3.11}$$

$$\frac{R^2}{R^2-\alpha_3^2}\frac{\partial\sigma_{32,e}}{\partial\alpha_2} - \frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial\sigma_{32,o}}{\partial\alpha_2} + \frac{\partial\sigma_{33,o}}{\partial\alpha_3} - \frac{R}{R^2-\alpha_3^2}\sigma_{22,e} + \frac{\alpha_3}{R^2-\alpha_3^2}\sigma_{22,o} + \frac{R}{R^2-\alpha_3^2}\sigma_{33,e} - \frac{\alpha_3}{R^2-\alpha_3^2}\sigma_{33,o} - \rho\frac{\partial^2 v_{3,e}}{\partial t^2} = 0, \tag{3.12}$$

$$\frac{R^2}{R^2-\alpha_3^2}\frac{\partial\sigma_{32,o}}{\partial\alpha_2} - \frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial\sigma_{32,e}}{\partial\alpha_2} + \frac{\partial\sigma_{33,e}}{\partial\alpha_3} - \frac{R}{R^2-\alpha_3^2}\sigma_{22,o} + \frac{\alpha_3}{R^2-\alpha_3^2}\sigma_{22,e} + \frac{R}{R^2-\alpha_3^2}\sigma_{33,o} - \frac{\alpha_3}{R^2-\alpha_3^2}\sigma_{33,e} - \rho\frac{\partial^2 v_{3,o}}{\partial t^2} = 0, \tag{3.13}$$

with

$$\sigma_{22,e} = \frac{E}{1-\nu^2}\left(\frac{R^2}{R^2-\alpha_3^2}\frac{\partial v_{2,e}}{\partial \alpha_2} - \frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial v_{2,o}}{\partial \alpha_2} + \frac{R}{R^2-\alpha_3^2}v_{3,e} \right.$$
$$\left. -\frac{\alpha_3}{R^2-\alpha_3^2}v_{3,o}\right) + \frac{\nu}{1-\nu}\sigma_{33,e}, \quad (3.14)$$

$$\sigma_{22,o} = \frac{E}{1-\nu^2}\left(\frac{R^2}{R^2-\alpha_3^2}\frac{\partial v_{2,o}}{\partial \alpha_2} - \frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial v_{2,e}}{\partial \alpha_2} + \frac{R}{R^2-\alpha_3^2}v_{3,o} \right.$$
$$\left. -\frac{\alpha_3}{R^2-\alpha_3^2}v_{3,e}\right) + \frac{\nu}{1-\nu}\sigma_{33,o}, \quad (3.15)$$

$$E\frac{\partial v_{3,o}}{\partial \alpha_3} = (1-\nu^2)\sigma_{33,e} - \nu(1+\nu)\sigma_{22,e}, \quad (3.16)$$
$$E\frac{\partial v_{3,e}}{\partial \alpha_3} = (1-\nu^2)\sigma_{33,o} - \nu(1+\nu)\sigma_{22,o}, \quad (3.17)$$

$$\sigma_{32,e} = \frac{E}{2(1+\nu)}\left(\frac{R^2}{R^2-\alpha_3^2}\frac{\partial v_{3,e}}{\partial \alpha_2} - \frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial v_{3,o}}{\partial \alpha_2} + \frac{\partial v_{2,o}}{\partial \alpha_3} \right.$$
$$\left. -\frac{R}{R^2-\alpha_3^2}v_{2,e} + \frac{\alpha_3}{R^2-\alpha_3^2}v_{2,o}\right), \quad (3.18)$$

$$\sigma_{32,o} = \frac{E}{2(1+\nu)}\left(\frac{R^2}{R^2-\alpha_3^2}\frac{\partial v_{3,o}}{\partial \alpha_2} - \frac{R\alpha_3}{R^2-\alpha_3^2}\frac{\partial v_{3,e}}{\partial \alpha_2} + \frac{\partial v_{2,e}}{\partial \alpha_3} \right.$$
$$\left. -\frac{R}{R^2-\alpha_3^2}v_{2,o} + \frac{\alpha_3}{R^2-\alpha_3^2}v_{2,e}\right). \quad (3.19)$$

Also, the even and odd quantities of stresses must satisfy the boundary conditions at $\alpha_3 = \pm h$ as

$$\sigma_{3i,e} = \sigma_{3i,o} = o, \qquad i = 2, 3. \quad (3.20)$$

3.2. Asymptotic Scaling

Keeping in mind that asymptotic order neither changes with differentiation with respect to the dimensionless variables θ and ζ, see equation (3.1), nor changes with dimensionless time τ, given by

$$\tau = \frac{\eta c_2}{R} t. \tag{3.21}$$

Let us define the dimensionless frequency as

$$\Omega = \frac{\omega R}{\eta c_2}, \qquad c_2 = \sqrt{\frac{E}{2(1+\nu)\rho}}, \tag{3.22}$$

where ω is the original frequency.

The scaled stress and displacement components are as follows

$$\begin{aligned} &\sigma_{22,e} = E\eta^2\sigma^*_{22,e}, \qquad \sigma_{22,o} = E\eta\sigma^*_{22,o}, \qquad \sigma_{33,e} = E\eta^2\sigma^*_{33,e}, \\ &\sigma_{33,o} = E\eta^3\sigma^*_{33,o}, \qquad \sigma_{32,e} = E\eta^2\sigma^*_{32,e}, \qquad \sigma_{32,o} = E\eta^3\sigma^*_{32,o} \end{aligned} \tag{3.23}$$

and

$$v_{2,e} = Rv^*_{2,e}, \quad v_{2,o} = R\eta v^*_{2,o}, \quad v_{3,e} = Rv^*_{3,e}, \quad v_{3,o} = R\eta^3 v^*_{3,o}, \tag{3.24}$$

in above scaling the starred quantities have the same asymptotic order.

In addition, we suppose

$$\frac{\partial v^*_{2,e}}{\partial\theta} + v^*_{3,e} = \eta^2 e_*, \tag{3.25}$$

meaning that the annulus is nearly inextensible along the circumferential direction. The proposed scaling is motivated by previous asymptotic considerations in the framework of the Kirchhoff-Love theory for cylindrical shells, see [14].

On substituting the dimensionless variables together with equations (3.23) – (3.25) into equations (3.10) – (3.19) and the boundary conditions (3.20), we obtain

$$\frac{1}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{22,e}}{\partial\theta}-\frac{\zeta}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{22,o}}{\partial\theta}+\frac{\partial\sigma^*_{32,o}}{\partial\zeta}+\frac{2}{1-\eta^2\zeta^2}\sigma^*_{23,e}$$
$$-\frac{2\eta^2\zeta}{1-\eta^2\zeta^2}\sigma^*_{23,o}+\frac{\Omega^2}{2(1+\nu)}v^*_{2,e}=0, \quad (3.26)$$

$$\frac{1}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{22,o}}{\partial\theta}-\frac{\eta^2\zeta}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{22,e}}{\partial\theta}+\frac{\partial\sigma^*_{32,e}}{\partial\zeta}+\frac{2\eta^2}{1-\eta^2\zeta^2}\sigma^*_{23,0}$$
$$-\frac{2\eta^2\zeta}{1-\eta^2\zeta^2}\sigma^*_{23,e}+\frac{\eta^2\Omega^2}{2(1+\nu)}v^*_{2,o}=0, \quad (3.27)$$

$$\frac{1}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{23,e}}{\partial\theta}-\frac{\eta^2\zeta}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{23,o}}{\partial\theta}+\frac{\partial\sigma^*_{33,o}}{\partial\zeta}-\frac{1}{1-\eta^2\zeta^2}\sigma^*_{22,e}$$
$$+\frac{\zeta}{1-\eta^2\zeta^2}\sigma^*_{22,o}+\frac{1}{1-\eta^2\zeta^2}\sigma^*_{33,e}-\frac{\eta^2\zeta}{1-\eta^2\zeta^2}\sigma^*_{33,o}+\frac{\Omega^2}{2(1+\nu)}v^*_{3,e}=0, \quad (3.28)$$

$$\frac{\eta^2}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{23,o}}{\partial\theta}-\frac{\eta^2\zeta}{1-\eta^2\zeta^2}\frac{\partial\sigma^*_{23,e}}{\partial\theta}+\frac{\partial\sigma^*_{33,e}}{\partial\zeta}-\frac{1}{1-\eta^2\zeta^2}\sigma^*_{22,o}$$
$$+\frac{\eta^2\zeta}{1-\eta^2\zeta^2}\sigma^*_{22,e}+\frac{\eta^2}{1-\eta^2\zeta^2}\sigma^*_{33,o}-\frac{\eta^2\zeta}{1-\eta^2\zeta^2}\sigma^*_{33,e}+\frac{\eta^4\Omega^2}{2(1+\nu)}v^*_{3,o}=0. \quad (3.29)$$

In similar way, we arrive at equations of the form

$$\sigma^*_{22,e}=\frac{1}{1-\nu^2}\left(\frac{e_*}{1-\eta^2\zeta^2}-\frac{\zeta}{1-\eta^2\zeta^2}\frac{\partial v^*_{2,o}}{\partial\theta}-\frac{\eta^2\zeta}{1-\eta^2\zeta^2}v^*_{3,o}\right)+\frac{\nu}{1-\nu}\sigma^*_{33,e}, \quad (3.30)$$

$$\sigma^*_{22,o}=\frac{1}{1-\nu^2}\left(-\frac{\eta^2\zeta}{1-\eta^2\zeta^2}e_*+\frac{1}{1-\eta^2\zeta^2}\frac{\partial v^*_{2,o}}{\partial\theta}+\frac{\eta^2}{1-\eta^2\zeta^2}v^*_{3,o}\right)+\frac{\nu}{1-\nu}\eta^2\sigma^*_{33,o}, \quad (3.31)$$

$$\frac{\partial v^*_{3,0}}{\partial\zeta}=(1-\nu^2)\sigma^*_{33,e}-\nu(1+\nu)\sigma^*_{22,e}, \quad (3.32)$$

$$\frac{\partial v^*_{3,e}}{\partial\zeta}=\eta^4(1-\nu^2)\sigma^*_{33,o}-\eta^2\nu(1+\nu)\sigma^*_{22,o}, \quad (3.33)$$

$$\eta^2\sigma^*_{32,e} = \frac{1}{2(1+\nu)}\left(\frac{1}{1-\eta^2\zeta^2}\frac{\partial v^*_{3,e}}{\partial\theta} - \frac{\eta^4\zeta}{1-\eta^2\zeta^2}\frac{\partial v^*_{3,o}}{\partial\theta} + \frac{\partial v^*_{2,o}}{\partial\zeta} - \frac{1}{1-\eta^2\zeta^2}v^*_{2,e} + \frac{\eta^2\zeta}{1-\eta^2\zeta^2}v^*_{2,o}\right), \quad (3.34)$$

$$\eta^4\sigma^*_{32,o} = \frac{1}{2(1+\nu)}\left(\frac{\eta^4}{1-\eta^2\zeta^2}\frac{\partial v^*_{3,o}}{\partial\theta} - \frac{\eta^2\zeta}{1-\eta^2\zeta^2}\frac{\partial v^*_{3,e}}{\partial\theta} + \frac{\partial v^*_{2,e}}{\partial\zeta} - \frac{\eta^2}{1-\eta^2\zeta^2}v^*_{2,o} + \frac{\eta^2\zeta}{1-\eta^2\zeta^2}v^*_{2,e}\right), \quad (3.35)$$

where

$$\sigma^*_{3i,e} = \sigma^*_{3i,o} = 0, \qquad i = 2, 3, \quad (3.36)$$

at $\zeta = \pm 1$.

The solutions to equations (3.26) to (3.35) are sought in the form of an asymptotic series

$$\sigma^*_{i2,m} = \sigma^{(0)}_{i2,m} + \eta^2\sigma^{(1)}_{i2,m}\cdots \qquad v^*_{i,m} = v^{(0)}_{i,m} + \eta^2 v^{(1)}_{i,m}\cdots \qquad e_* = e_0 + \eta^2 e_1\cdots \quad (3.37)$$

where $i = 2, 3$ and $m = e, o$.

3.3. The Leading-Order Approximation

Let us substitute equation (3.37) into equations (3.26)–(3.35) and discarding $O(\eta^2)$ terms, to yield

$$\frac{\partial\sigma^{(0)}_{22,e}}{\partial\theta} - \zeta\frac{\partial\sigma^{(0)}_{22,o}}{\partial\theta} + \frac{\partial\sigma^{(0)}_{32,o}}{\partial\zeta} + 2\sigma^{(0)}_{32,e} + \frac{\Omega_0^2}{2(1+\nu)}v^{(0)}_{2,e} = 0, \quad (3.38)$$

$$\frac{\partial\sigma^{(0)}_{22,o}}{\partial\theta} + \frac{\partial^{(0)}_{32,e}}{\partial\zeta} = 0, \quad (3.39)$$

$$\frac{\partial\sigma^{(0)}_{32,e}}{\partial\theta} + \frac{\partial\sigma^{(0)}_{33,o}}{\partial\zeta} - \sigma^{(0)}_{22,e} + \zeta\sigma^{(0)}_{22,o} + \sigma^{(0)}_{33,e} + \frac{\Omega_0^2}{2(1+\nu)}v^{(0)}_{3,e} = 0, \quad (3.40)$$

$$\frac{\partial\sigma^{(0)}_{33,e}}{\partial\zeta} - \sigma^{(0)}_{22,o} = 0, \quad (3.41)$$

$$\sigma_{22,e}^{(0)} = \frac{1}{1-\nu^2}\left(e_0 - \zeta\frac{\partial v_{2,o}^{(0)}}{\partial\theta}\right) + \frac{\nu}{1-\nu}\sigma_{33,e}^{(0)}, \tag{3.42}$$

$$\sigma_{22,o}^{(0)} = \frac{1}{1-\nu^2}\frac{\partial v_{2,o}^{(0)}}{\partial\theta}, \tag{3.43}$$

$$\frac{\partial v_{3,o}^{(0)}}{\partial\zeta} = (1-\nu^2)\sigma_{33,e}^{(0)} - \nu(1+\nu)\sigma_{22,e}^{(0)}, \tag{3.44}$$

$$\frac{\partial v_{3,e}^{(0)}}{\partial\zeta} = 0, \tag{3.45}$$

$$\frac{\partial v_{3,e}^{(0)}}{\partial\theta} + \frac{\partial v_{2,o}^{(0)}}{\partial\zeta} - v_{2,e}^{(0)} = 0, \tag{3.46}$$

$$\frac{\partial v_{2,e}^{(0)}}{\partial\zeta} = 0, \tag{3.47}$$

with the boundary conditions at $\zeta = \pm 1$

$$\sigma_{3i,e}^{(0)} = \sigma_{3i,o}^{(0)} = 0, \qquad i = 2, 3. \tag{3.48}$$

Integrating these equations with respect to thickness variable ζ, it is evident from equations (3.47) and (3.45) that

$$v_{2,e}^{(0)} = V_2^{(0)}(\theta,\tau), \qquad v_{3,e}^{(0)} = V_3^{(0)}(\theta,\tau), \tag{3.49}$$

where $V_2^{(0)}(\theta,\tau)$ and $V_3^{(0)}(\theta,\tau)$ are arbitrary functions of θ and τ.

Also taking into account the latter relation, the leading terms of inextensible conditions (3.25), gives

$$V_3^{(0)} = -\frac{\partial V_2^{(0)}}{\partial\theta}. \tag{3.50}$$

Then, we have from equations (3.46), (3.43), (3.41) and (3.39), that

$$v_{2,o}^{(0)} = \zeta P_0, \tag{3.51}$$

$$\sigma_{22,o}^{(0)} = \frac{\zeta}{1-\nu^2} P_1, \tag{3.52}$$

$$\sigma_{33,e}^{(0)} = -\frac{1-\zeta^2}{2(1-\nu^2)} P_1, \tag{3.53}$$

$$\sigma_{32,e}^{(0)} = \frac{1-\zeta^2}{2(1-\nu^2)} P_2. \tag{3.54}$$

The boundary conditions (3.48) are taken into account while deriving equations (3.53) and (3.54).

The notation

$$\begin{aligned} P_i &= \frac{\partial^i}{\partial\theta^i}\left(V_2^{(0)} - \frac{\partial V_3^{(0)}}{\partial\theta}\right), \qquad i = 0, 1, 2, \cdots, \\ P_i'(\theta) &= P_{i+1}(\theta), \end{aligned} \tag{3.55}$$

is adapted throughout thesis for 2D problems.

Now, we assume general quadratic form

$$\sigma_{22,e}^{(0)} = a_1 + a_2\zeta^2. \tag{3.56}$$

where $a_i, i = 1, 2$ are 1D differential operators. The last formula, in fact, contradicts the traditional uniform distribution of the circumferential stress σ_{22} across the thickness. The semi-membrane approximation is usually derived using such an assumption.

Next, integrating (3.40) with respect to ζ and considering the boundary conditions (3.48), we obtain

$$\sigma_{33,o}^{(0)} = -\frac{\zeta(3-\zeta^2)}{6(1-\nu^2)} P_3 + \frac{\zeta(1-\zeta^2)}{2(1-\nu^2)} P_1 + a_1\zeta + \frac{1}{3}a_2\zeta^3 - \frac{\zeta}{2(1+\nu)}\Omega_0^2 V_3^{(0)}, \tag{3.57}$$

with

$$a_1 = \frac{1}{3(1-\nu^2)} P_3 - \frac{1}{3}a_2 + \frac{1}{2(1+\nu)}\Omega_0^2 V_3^{(0)}. \tag{3.58}$$

It is worth nothing that the expressions for each of operators a_1 and a_2 can be determined only at the next-order approximation. Also, as it follows from equations (3.56) and (3.58),

$$\int_{-1}^{1} \sigma_{22,e}^{(0)}\, d\zeta = \frac{2}{3(1-\nu^2)} P_3 + \frac{1}{(1+\nu)} \Omega_0^2 V_3^{(0)},$$

i.e., the formulation does not depend on the unknown operators a_1 and a_2. This is why the aforementioned semi-membrane formulation based on engineering hypotheses and operating with membrane forces (stress resultants) but not stresses, is valid at the leading order.

Now, substituting equations (3.51), (3.53) and (3.56) into equation (3.42), yields

$$e_0 = \left(1-\nu^2\right)\left(a_1 + a_2\zeta^2\right) + \frac{\nu + (2-3\nu)\zeta^2}{2(1-\nu)} P_1. \tag{3.59}$$

Integrating (3.38) at leading order, it follows that

$$\sigma_{32,o}^{(0)} = -\frac{\zeta(3-2\zeta^2)}{3(1-\nu^2)} P_2 - \zeta \frac{\partial a_1}{\partial \theta} - \frac{\zeta^3}{3}\frac{\partial a_2}{\partial \theta} - \frac{\zeta}{2(1+\nu)} \Omega_0^2 V_2^{(0)}. \tag{3.60}$$

On substituting the latter into the boundary conditions (3.48) and taking into account equation (3.58), one obtains

$$\frac{\Omega_0^2}{2(1+\nu)}\left(V_2^{(0)} + \frac{\partial V_3^{(0)}}{\partial \theta}\right) = -\frac{1}{3(1-\nu^2)}\left(P_4 + P_2\right). \tag{3.61}$$

Using the adapted notation in equation (3.61), we arrive at the sought for leading order equation in displacements

$$\frac{\Omega_0^2}{2(1+\nu)}\left(V_2^{(0)} - \frac{\partial^2 V_2^{(0)}}{\partial \theta^2}\right) = -\frac{1}{3(1-\nu^2)}\left(\frac{\partial^6 V_2^{(0)}}{\partial \theta^6} + 2\frac{\partial^4 V_2^{(0)}}{\partial \theta^4} + \frac{\partial^2 V_2^{(0)}}{\partial \theta^2}\right). \tag{3.62}$$

Where the relation between $V_2^{(0)}$ and $V_3^{(0)}$ is employed, see (3.50).

Integrating equation (3.44) with respective thickness variable ζ, and taking into account (3.53) and (3.56), we get

$$v_{3,0}^{(0)} = -\frac{\zeta(3-\zeta^2)}{6}P_1 - \nu(1+\nu)\left(\zeta a_1 + \frac{\zeta^3}{3}a_2\right). \tag{3.63}$$

Thus, we expressed all the given quantities in terms of 1D functions $V_2^{(0)}$ and $V_3^{(0)}$. In current section some quantities are still dependent on the operators a_1 and a_2 related to each other by equation (3.58). It is worth mentioning that these operators will be determined in next-order approximation.

3.4. Next-Order Approximation

At next order, let us substitute (3.37) into (3.26)–(3.35) with (3.25) and keeping $O(\eta^2)$ terms, yields

$$\frac{\partial \sigma_{22,e}^{(1)}}{\partial \theta} + \zeta^2 \frac{\partial \sigma_{22,e}^{(0)}}{\partial \theta} - \zeta \frac{\partial \sigma_{22,o}^{(1)}}{\partial \theta} - \zeta^3 \frac{\partial \sigma_{22,o}^{(0)}}{\partial \theta} + \frac{\partial \sigma_{32,o}^{(1)}}{\partial \zeta} + 2\sigma_{32,e}^{(1)} + 2\zeta^2 \sigma_{32,e}^{(0)} - 2\zeta \sigma_{32,o}^{(0)} + \frac{\Omega_0^2}{2(1+\nu)} v_{2,e}^{(1)} + \frac{\Omega_1^2}{2(1+\nu)} v_{2,e}^{(0)} = 0, \tag{3.64}$$

$$\frac{\partial \sigma_{22,o}^{(1)}}{\partial \theta} + \zeta^2 \frac{\partial \sigma_{22,o}^{(0)}}{\partial \theta} - \zeta \frac{\partial \sigma_{22,e}^{(o)}}{\partial \theta} + \frac{\partial \sigma_{32,e}^{(1)}}{\partial \zeta} + 2\sigma_{32,o}^{(0)} - 2\zeta \sigma_{32,e}^{(0)} + \frac{\Omega_0^2}{2(1+\nu)} v_{2,o}^{(0)} = 0, \tag{3.65}$$

$$\zeta^2 \frac{\partial \sigma_{32,e}^{(0)}}{\partial \theta} + \frac{\partial \sigma_{32,e}^{(1)}}{\partial \theta} - \zeta \frac{\partial \sigma_{32,o}^{(0)}}{\partial \theta} + \frac{\partial \sigma_{33,o}^{(1)}}{\partial \zeta} - \sigma_{22,e}^{(1)} - \zeta^2 \sigma_{22,e}^{(0)} + \zeta \sigma_{22,o}^{(1)} + \zeta^3 \sigma_{22,o}^{(0)} + \sigma_{33,e}^{(1)} + \zeta^2 \sigma_{33,e}^{(0)} - \zeta \sigma_{33,o}^{(0)} + \frac{\Omega_0^2}{2(1+\nu)} v_{3,e}^{(1)} + \frac{\Omega_1^2}{2(1+\nu)} v_{3,e}^{(0)} = 0, \tag{3.66}$$

$$\frac{\partial \sigma_{32,o}^{(0)}}{\partial \theta} - \zeta \frac{\partial \sigma_{32,e}^{(0)}}{\partial \theta} + \frac{\partial \sigma_{33,e}^{(1)}}{\partial \zeta} - \sigma_{22,o}^{(1)} - \zeta^2 \sigma_{22,o}^{(0)} + \zeta \sigma_{22,e}^{(0)} + \sigma_{33,o}^{(0)} - \zeta \sigma_{33,e}^{(0)} = 0, \tag{3.67}$$

$$\sigma_{22,e}^{(1)} = \frac{1}{1-\nu^2}\left(\zeta^2 e_0 + e_1 - \zeta \frac{\partial v_{2,o}^{(1)}}{\partial \theta} - \zeta^3 \frac{\partial v_{2,o}^{(0)}}{\partial \theta} - \zeta v_{3,o}^{(0)}\right) + \frac{\nu}{1-\nu}\sigma_{33,e}^{(1)}, \tag{3.68}$$

$$\sigma_{22,o}^{(1)} = \frac{1}{1-\nu^2}\left(-\zeta e_0 + \frac{\partial v_{2,o}^{(1)}}{\partial \theta} + \zeta^2 \frac{\partial v_{2,o}^{(0)}}{\partial \theta} + v_{3,o}^{(0)}\right) + \frac{\nu}{1-\nu}\sigma_{33,o}^{(0)}, \tag{3.69}$$

$$\frac{\partial v_{3,o}^{(1)}}{\partial \zeta} = (1-\nu^2)\sigma_{33,e}^{(1)} - \nu(1+\nu)\sigma_{22,e}^{(1)}, \tag{3.70}$$

$$\frac{\partial v_{3,e}^{(1)}}{\partial \zeta} = -\nu(1+\nu)\sigma_{22,o}^{(0)}, \tag{3.71}$$

$$\sigma_{32,e}^{(0)} = \frac{1}{2(1+\nu)}\left(\frac{\partial v_{3,e}^{(1)}}{\partial \theta} + \zeta^2 \frac{\partial v_{3,e}^{(0)}}{\partial \theta} + \frac{\partial v_{2,o}^{(1)}}{\partial \zeta} - v_{2,e}^{(1)} - \zeta^2 v_{2,e}^{(0)} + \zeta v_{2,o}^{(0)}\right), \tag{3.72}$$

$$-\zeta \frac{\partial v_{3,e}^{(0)}}{\partial \theta} + \frac{\partial v_{2,e}^{(1)}}{\partial \zeta} - v_{2,o}^{(0)} + \zeta v_{2,e}^{(0)} = 0, \tag{3.73}$$

$$e_0 = \frac{\partial v_{2,e}^{(1)}}{\partial \theta} + v_{3,e}^{(1)}, \tag{3.74}$$

with the boundary conditions at $\zeta = \pm 1$

$$\sigma_{3i,e}^{(1)} = \sigma_{3i,o}^{(1)} = 0, \qquad i = 2, 3. \tag{3.75}$$

The adopted notation in leading-order is also valid for next-order approximation, for simplicity at next order we introduce the notation

$$\begin{aligned} Q_i &= \frac{\partial^i}{\partial \theta^i}\left(V_2^{(1)} - \frac{\partial V_3^{(1)}}{\partial \theta}\right), \qquad i = 0, 1, 2, \cdots, \\ Q_i^{'}(\theta) &= Q_{i+1}(\theta), \end{aligned} \tag{3.76}$$

Again we begin by integrating these equations with respect to the thickness variable ζ, from equations (3.73) and (3.71), we have

$$v_{2,e}^{(1)} = V_2^{(1)}(\theta, \tau), \qquad v_{3,e}^{(1)} = -\frac{\nu \zeta^2}{2(1-\nu)} P_1 + V_3^{(1)}(\theta, \tau). \tag{3.77}$$

Next, comparison of equation (3.59) and (3.74) gives

$$V_3^{(1)} + \frac{\partial V_2^{(1)}}{\partial \theta} = \frac{\nu + 2(1-\nu)\zeta^2}{2(1-\nu)} P_1 + (1-\nu^2)(a_1 + a_2 \zeta^2). \tag{3.78}$$

Since the left-hand side of the latter is independent of ζ, then the coefficients of ζ^2 must equal to zero, hence we get

$$a_2 = -\frac{1}{1-\nu^2} P_1. \tag{3.79}$$

Now, inserting value of a_2 from latter into equation (3.58), we have

$$a_1 = \frac{1}{3(1-\nu^2)} P_3 + \frac{1}{3(1-\nu^2)} P_1 + \frac{1}{2(1+\nu)} \Omega_0^2 V_3^{(0)}. \tag{3.80}$$

Also, (3.78) gives

$$V_3^{(1)} + \frac{\partial V_2^{(1)}}{\partial \theta} = \frac{1}{3} P_3 + \frac{2+\nu}{6(1-\nu)} P_1 + \frac{1}{2}(1-\nu)\Omega_0^2 V_3^{(0)}. \tag{3.81}$$

Thus, all quantities that were dependent on the operators a_1 and a_2 are now fully defined. It is also worth nothing that the well-expected assumption on leading-order uniform variation of the normal stress across the thickness for which $a_2 \equiv 0$ in equation (3.56) would lead to a contradiction in equation (3.78). In fact, in the latter case, we would have

$$\frac{(1+2\nu)\zeta^2}{1-\nu} P_1 = 0, \tag{3.82}$$

which is generally not the case.

From equation (3.72) and (3.69) respectively, we obtain

$$v_{2,o}^{(1)} = \frac{6\zeta - (2-\nu)\zeta^3}{6(1-\nu)} P_2 + \zeta Q_0 \tag{3.83}$$

and

$$\sigma_{22,o}^{(1)} = \frac{(4-\nu)\zeta - 2(1-\nu)\zeta^3}{6(1-\nu)(1-\nu^2)} P_3 - \frac{\zeta(5-7\zeta^2)}{6(1-\nu^2)} P_1 + \frac{\zeta}{1-\nu^2} Q_1 - \frac{\zeta}{2(1-\nu^2)} \Omega_0^2 V_3^{(0)}. \tag{3.84}$$

Inserting the previous derived equations related to stress in equation (3.67), then integrating with respect to thickness variable ζ, we arrive at

$$\sigma_{33,e}^{(1)} = \frac{(4-3\nu)\zeta^2 - 2(1-\nu)\zeta^4}{4(1-\nu)(1-\nu^2)}P_3 - \frac{\zeta^2(10-9\zeta^2)}{8(1-\nu^2)}P_1 + \frac{\zeta^2}{2(1-\nu^2)}Q_1 - \frac{(2-\nu)\zeta^2}{4(1-\nu^2)}\Omega_0^2 V_3^{(0)} + C.$$

Applying the boundary conditions (3.75) in latter, we have

$$\sigma_{33,e}^{(1)} = -\frac{(2-\nu)-(4-3\nu)\zeta^2 + 2(1-\nu)\zeta^4}{4(1-\nu)(1-\nu^2)}P_3 + \frac{1-10\zeta^2+9\zeta^4}{8(1-\nu^2)}P_1 - \frac{1-\zeta^2}{2(1-\nu^2)}Q_1 + \frac{(2-\nu)-(2-\nu)\zeta^2}{4(1-\nu^2)}\Omega_0^2 V_3^{(0)}. \quad (3.85)$$

Integrating (3.65) with respect to thickness varibale ζ, we arrive at

$$\sigma_{32,e}^{(1)} = -\frac{(2+\nu)\zeta^2 - (1-\nu)\zeta^4}{12(1-\nu)(1-\nu^2)}P_4 + \frac{(50-37\zeta^2)\zeta^2}{24(1-\nu^2)}P_2 - \frac{\zeta^2}{2(1-\nu^2)}Q_2 + \frac{(2-\nu)\zeta^2}{4(1-\nu^2)}\Omega_0^2\frac{\partial V_3^{(0)}}{\partial\theta} - \frac{\zeta^2}{4(1+\nu)}\Omega_0^2 P_0 + C,$$

and, as before, applying the boundary conditions (3.75), we get

$$\sigma_{32,e}^{(1)} = \frac{(1+2\nu)-(2+\nu)\zeta^2 + (1-\nu)\zeta^4}{12(1-\nu)(1-\nu^2)}P_4 - \frac{13-50\zeta^2+37\zeta^4}{24(1-\nu^2)}P_2 + \frac{1-\zeta^2}{2(1-\nu^2)}Q_2 - \frac{(2-\nu)(1-\zeta^2)}{4(1-\nu^2)}\Omega_0^2\frac{\partial V_3^{(0)}}{\partial\theta} + \frac{1-\zeta^2}{4(1+\nu)}\Omega_0^2 P_0. \quad (3.86)$$

Inserting the previous derived equations related to stress and displacement in equation (3.66), we have

$$\begin{aligned}\frac{\partial\sigma_{33,o}^{(1)}}{\partial\zeta} = &-\frac{(1+2\nu)-(2+\nu)\zeta^2+(1-\nu)\zeta^4}{12(1-\nu)(1-\nu^2)}P_5 \\ &+ \frac{(25-19\nu)-(122-104\nu)\zeta^2+97(1-\nu)\zeta^4}{24(1-\nu)(1-\nu^2)}P_3 - \frac{1-30\zeta^2+45\zeta^4}{8(1-\nu^2)}P_1 \\ &- \frac{1-\zeta^2}{2(1-\nu^2)}Q_3 + \frac{1-3\zeta^2}{2(1-\nu^2)}Q_1 - \frac{1}{2(1+\nu)}\Omega_0^2 V_3^{(1)} - \frac{1}{2(1+\nu)}\Omega_1^2 V_3^{(0)} \\ &- \frac{(2-\nu)(1-3\zeta^2)}{4(1-\nu^2)}\Omega_0^2 V_3^{(0)} + \frac{(2-\nu)-(2-\nu)\zeta^2}{4(1-\nu^2)}\Omega_0^2\frac{\partial^2 V_3^{(0)}}{\partial\theta^2} - \\ &- \frac{(1-\nu)-\zeta^2}{4(1-\nu^2)}\Omega_0^2 P_1 + b_1 + b_2\zeta^2 + b_3\zeta^4,\end{aligned}$$

with

$$\sigma_{22,e}^{(1)} = b_1 + b_2\zeta^2 + b_3\zeta^4, \tag{3.87}$$

where $b_i, i = 1, 2, 3$ are 1D differential operators.

Integrating the latter relation with respect to ζ, we arrive at

$$\begin{aligned}\sigma_{33,o}^{(1)} = &-\frac{15(1+2\nu)\zeta - 5(2+\nu)\zeta^3 + 3(1-\nu)\zeta^5}{180(1-\nu)(1-\nu^2)}P_5\\ &+\frac{15(25-19\nu)\zeta - 10(61-52\nu)\zeta^3 + 291(1-\nu)\zeta^5}{360(1-\nu)(1-\nu^2)}P_3\\ &-\frac{\zeta - 10\zeta^3 + 9\zeta^5}{8(1-\nu^2)}P_1 - \frac{\zeta(3-\zeta^2)}{6(1-\nu^2)}Q_3 + \frac{\zeta(1-\zeta^2)}{2(1-\nu^2)}Q_1\\ &-\frac{\zeta}{2(1+\nu)}\Omega_0^2V_3^{(1)} - \frac{\zeta}{2(1+\nu)}\Omega_1^2V_3^{(0)} - \frac{(2-\nu)(\zeta-\zeta^3)}{4(1-\nu^2)}\Omega_0^2V_3^{(0)}\\ &-\frac{3(1-\nu)\zeta - \zeta^3}{12(1-\nu^2)}\Omega_0^2P_1 + \frac{(2-\nu)(3\zeta-\zeta^3)}{12(1-\nu^2)}\Omega_0^2\frac{\partial^2V_3^{(0)}}{\partial\theta^2}\\ &+b_1\zeta + \frac{1}{3}b_2\zeta^3 + \frac{1}{5}b_3\zeta^5.\end{aligned} \tag{3.88}$$

Then, using the boundary conditions (3.75), we get

$$\begin{aligned}b_1 =&\frac{4+11\nu}{90(1-\nu)(1-\nu^2)}P_5 - \frac{7}{45(1-\nu^2)}P_3 + \frac{1}{3(1-\nu^2)}Q_3 + \frac{1}{2(1+\nu)}\Omega_0^2V_3^{(1)}\\ &+\frac{1}{2(1+\nu)}\Omega_1^2V_3^{(0)} + \frac{2-3\nu}{12(1-\nu^2)}\Omega_0^2P_1 - \frac{2-\nu}{6(1-\nu^2)}\Omega_0^2\frac{\partial^2V_3^{(0)}}{\partial\theta^2} - \frac{1}{3}b_2 - \frac{1}{5}b_3.\end{aligned} \tag{3.89}$$

Now, inserting b_1 from equation (3.89) into equation (3.87), we have

$$\begin{aligned}\sigma_{22,e}^{(1)} = &\frac{4+11\nu}{90(1-\nu)(1-\nu^2)}P_5 - \frac{7}{45(1-\nu^2)}P_3 + \frac{1}{3(1-\nu^2)}Q_3 + \frac{1}{2(1+\nu)}\Omega_0^2V_3^{(1)}\\ &+\frac{1}{2(1+\nu)}\Omega_1^2V_3^{(0)} + \frac{2-3\nu}{12(1-\nu^2)}\Omega_0^2P_1 - \frac{2-\nu}{6(1-\nu^2)}\Omega_0^2\frac{\partial^2V_3^{(0)}}{\partial\theta^2} - \frac{1}{3}(1-3\zeta^2)b_2\\ &-\frac{1}{5}(1-5\zeta^4)b_3.\end{aligned} \tag{3.90}$$

Next, taking into account equations (3.49), (3.52) (3.56), (3.60), (3.72), (3.84) and (3.86) in equation (3.64), we arrive at

$$
\begin{aligned}
\frac{\partial \sigma_{32,o}^{(1)}}{\partial \zeta} = &- \frac{4+11\nu}{90(1-\nu)(1-\nu^2)}P_6 - \frac{(1+44\nu)-30(2+\nu)\zeta^2+45(1-\nu)\zeta^4}{90(1-\nu)(1-\nu^2)}P_4 \\
&+ \frac{[13-(2-3\nu)\Omega_0^2]-100\zeta^2+111\zeta^4}{12(1-\nu^2)}P_2 - \frac{1}{3(1-\nu^2)}Q_4 \\
&- \frac{1-2\zeta^2}{1-\nu^2}Q_2 + \frac{(2-\nu)}{6(1-\nu^2)}\Omega_0^2\frac{\partial^3 V_3^{(0)}}{\partial \theta^3} + \frac{(2-\nu)(1-2\zeta^2)}{2(1-\nu^2)}\Omega_0^2\frac{\partial V_3^{(0)}}{\partial \theta} \\
&- \frac{\Omega_0^2}{2(1+\nu)}(V_2^{(1)} + \frac{\partial V_3^{(1)}}{\partial \theta}) - \frac{\Omega_1^2}{2(1+\nu)}(V_2^{(0)} + \frac{\partial V_3^{(0)}}{\partial \theta}) + \frac{1-3\zeta^2}{3}\frac{\partial b_2}{\partial \theta} \\
&+ \frac{1-5\zeta^4}{5}\frac{\partial b_3}{\partial \theta} - \frac{1-\zeta^2}{2(1+\nu)}\Omega_0^2 P_0,
\end{aligned}
$$

integrating latter relation with respect to thickness variable at $\zeta = 1$, we have

$$
\begin{aligned}
\sigma_{32,o}^{(1)} = &- \frac{(4+11\nu)\zeta}{90(1-\nu)(1-\nu^2)}P_6 - \frac{(1+44\nu)\zeta-10(2+\nu)\zeta^3+9(1-\nu)\zeta^3}{90(1-\nu)(1-\nu^2)}P_4 \\
&+ \frac{15[13-(2-3\nu)\Omega_0^2]\zeta-500\zeta^3+333\zeta^5}{180(1-\nu^2)}P_2 - \frac{\zeta}{3(1-\nu^2)}Q_4 \\
&- \frac{\zeta(3-2\zeta^2)}{3(1-\nu^2)}Q_2 + \frac{(2-\nu)\zeta}{6(1-\nu^2)}\Omega_0^2\frac{\partial^3 V_3^{(0)}}{\partial \theta^3} + \frac{(2-\nu)(3\zeta-2\zeta^3)}{6(1-\nu^2)}\Omega_0^2\frac{\partial V_3^{(0)}}{\partial \theta} \\
&- \frac{\zeta}{2(1+\nu)}\Omega_0^2(V_2^{(1)} + \frac{\partial V_3^{(1)}}{\partial \theta}) - \frac{\zeta}{2(1+\nu)}\Omega_1^2(V_2^{(0)} + \frac{\partial V_3^{(0)}}{\partial \theta}) + \frac{\zeta(1-\zeta^2)}{3}\frac{\partial b_2}{\partial \theta} \\
&+ \frac{\zeta(1-\zeta^4)}{5}\frac{\partial b_3}{\partial \theta} - \frac{\zeta(3-\zeta^2)}{6(1+\nu)}\Omega_0^2 P_0,
\end{aligned}
\tag{3.91}
$$

and applying the boundary condition at $\zeta = -1$, we have

$$
\begin{aligned}
\frac{\Omega_1^2}{2(1+\nu)}(V_2^{(0)} + \frac{\partial V_3^{(0)}}{\partial \theta}) = &-\frac{4+11\nu}{90(1-\nu)(1-\nu^2)}P_6 + \frac{2-5\nu}{18(1-\nu)(1-\nu^2)}P_4 \\
&+ \frac{7}{45(1-\nu^2)}P_2 - \frac{1}{3(1-\nu^2)}\left(Q_4+Q_2\right) + \Bigg[-\frac{2-3\nu}{12(1-\nu^2)}P_2 - \frac{1}{3(1+\nu)}P_0 \\
&+ \frac{2-\nu}{6(1-\nu^2)}\frac{\partial^3 V_3^{(0)}}{\partial \theta^3} + \frac{2-\nu}{6(1-\nu^2)}\frac{\partial V_3^{(0)}}{\partial \theta} - \frac{1}{2(1+\nu)}\left(V_2^{(1)} + \frac{\partial V_3^{(1)}}{\partial \theta}\right)\Bigg]\Omega_0^2.
\end{aligned}
\tag{3.92}
$$

Paying attention to the latter, it is clearly seen that the operators b_2 and b_3 are eliminated after applying boundary conditions.

Satisfying the appropriate boundary condition in equation (3.75) and inserting values of P and Q, we obtain

$$
\begin{aligned}
&\frac{1}{2(1+\nu)}\left(V_2^{(1)}-\frac{\partial^2 V_2^{(1)}}{\partial\theta^2}\right)\Omega_0^2+\frac{1}{2(1+\nu)}\left(V_2^{(0)}-\frac{\partial^2 V_2^{(0)}}{\partial\theta^2}\right)\Omega_1^2\\
&=\frac{1}{45(1-\nu^2)}\left(5\frac{\partial^{10}V_2^{(0)}}{\partial\theta^{10}}+13\frac{\partial^8 V_2^{(0)}}{\partial\theta^8}+18\frac{\partial^6 V_2^{(0)}}{\partial\theta^6}+17\frac{\partial^4 V_2^{(0)}}{\partial\theta^4}+7\frac{\partial^2 V_2^{(0)}}{\partial\theta^2}\right)\\
&-\frac{1}{3(1-\nu^2)}\left(\frac{\partial^6 V_2^{(1)}}{\partial\theta^6}+2\frac{\partial^4 V_2^{(1)}}{\partial\theta^4}+\frac{\partial^2 V_2^{(1)}}{\partial\theta^2}\right)+\frac{1-\nu}{4(1+\nu)}\Omega_0^4\frac{\partial^2 V_2^{(0)}}{\partial\theta^2}\\
&+\frac{1}{3(1-\nu^2)}\left((1-\nu)\frac{\partial^6 V_2^{(0)}}{\partial\theta^6}+(3-2\nu)\frac{\partial^4 V_2^{(0)}}{\partial\theta^4}+(3-2\nu)\frac{\partial^2 V_2^{(0)}}{\partial\theta^2}+(1-\nu)V_2^{(0)}\right)\Omega_0^2.
\end{aligned}
\tag{3.93}
$$

3.5. Discussion and Refined 1D Equations

We know that

$$v_{2,e}^{*}=v_{2,e}^{(0)}+\eta^2 v_{2,e}^{(1)},\qquad \Omega^2=\Omega_0^2+\eta^2\Omega_1^2, \tag{3.94}$$

Taking into account the latter, the sum of equation (3.62) and (3.93) expressed in dimensionless form, can be written as

$$
\begin{aligned}
\frac{1}{2(1+\nu)}\left(v_{2,e}^{*}-\frac{\partial^2 v_{2,e}^{*}}{\partial\theta^2}\right)\Omega^2=&-\frac{1}{3(1-\nu^2)}\left(\frac{\partial^6 v_{2,e}^{*}}{\partial\theta^6}+2\frac{\partial^4 v_{2,e}^{*}}{\partial\theta^4}+\frac{\partial^2 v_{2,e}^{*}}{\partial\theta^2}\right)\\
&+\eta^2\frac{1}{45(1-\nu^2)}\left(5\frac{\partial^{10}v_{2,e}^{*}}{\partial\theta^{10}}+13\frac{\partial^8 v_{2,e}^{*}}{\partial\theta^8}+18\frac{\partial^6 v_{2,e}^{*}}{\partial\theta^6}+17\frac{\partial^4 v_{2,e}^{*}}{\partial\theta^4}+7\frac{\partial^2 v_{2,e}^{*}}{\partial\theta^2}\right)\\
&-\eta^2\frac{1}{3(1-\nu^2)}\left((1-\nu)\frac{\partial^6 v_{2,e}^{*}}{\partial\theta^6}+(3-2\nu)\frac{\partial^4 v_{2,e}^{*}}{\partial\theta^4}+(3-2\nu)\frac{\partial^2 v_{2,e}^{*}}{\partial\theta^2}\right.\\
&\left.+(1-\nu)v_{2,e}^{*}\right)\Omega^2+\eta^2\Omega^4\frac{1-\nu}{4(1+\nu)}\frac{\partial^2 v_{2,e}^{*}}{\partial\theta^2}.
\end{aligned}
\tag{3.95}
$$

The derived equation (3.95) in terms of original variables, gives

$$
\begin{aligned}
&-\frac{R^2}{h^2}\frac{\rho}{E}\left(1-R^2\frac{\partial^2}{\partial\alpha_2^2}\right)\frac{\partial^2 v_{2,e}}{\partial t^2}=-\frac{1}{3(1-\nu^2)}\left(R^4\frac{\partial^6}{\partial\alpha_2^6}+2R^2\frac{\partial^4}{\partial\alpha_2^4}+\frac{\partial^2}{\partial\alpha_2^2}\right)v_{2,e}\\
&\quad+\frac{1}{45(1-\nu^2)}\frac{h^2}{R^2}\left(5R^8\frac{\partial^{10}}{\partial\alpha_2^{10}}+13R^6\frac{\partial^8}{\partial\alpha_2^8}+18R^4\frac{\partial^6}{\partial\alpha_2^6}+17R^2\frac{\partial^4}{\partial\alpha_2^4}\right.\\
&\quad\left.+7\frac{\partial^2}{\partial\alpha_2^2}\right)v_{2,e}+\frac{2\rho}{3(1-\nu)E}\left[(1-\nu)R^6\frac{\partial^6}{\partial\alpha_2^6}+(3-2\nu)R^4\frac{\partial^4}{\partial\alpha_2^4}\right.\\
&\quad\left.+(3-2\nu)R^2\frac{\partial^2}{\partial\alpha_2^2}+(1-\nu)\right]\frac{\partial^2 v_{2,e}}{\partial t^2}+\frac{(1-\nu^2)\rho^2}{E^2}\frac{R^6}{h^2}\left[\left(\frac{\partial^2}{\partial\alpha_2^2}\right)\frac{\partial^4}{\partial t^4}\right]v_{2,e}.
\end{aligned}
\tag{3.96}
$$

Now, multiplying both side of latter by R^2h^2, we arrive to our desired equation, i.e.,

$$
L_1^{(0)}\frac{\partial^2 v_{2,e}}{\partial t^2}+h^2L_2^{(0)}v_{2,e}+h^2L_1^{(1)}\frac{\partial^2 v_{2,e}}{\partial t^2}+L_2^{(1)}\frac{\partial^4 v_{2,e}}{\partial t^4}+h^4L_3^{(1)}v_{2,e}=0, \tag{3.97}
$$

with operators

$$
\begin{aligned}
L_1^{(0)}&=\frac{\rho}{E}R^4\left(1-R^2\frac{\partial^2}{\partial\alpha_2^2}\right),\\
L_2^{(0)}&=-\frac{R^2}{3(1-\nu^2)}\left(R^4\frac{\partial^6}{\partial\alpha_2^6}+2R^2\frac{\partial^4}{\partial\alpha_2^4}+\frac{\partial^2}{\partial\alpha_2^2}\right),\\
L_1^{(1)}&=\frac{2\rho}{3E}R^2\left[(1-\nu)R^6\frac{\partial^6}{\partial\alpha_2^6}+(3-2\nu)R^4\frac{\partial^4}{\partial\alpha_2^4}+(3-2\nu)R^2\frac{\partial^2}{\partial\alpha_2^2}+(1-\nu)\right],
\end{aligned}
$$

$$
\begin{aligned}
L_2^{(1)}&=\frac{(1-\nu^2)\rho^2}{E^2}R^8\frac{\partial^2}{\partial\alpha_2^2},\\
L_3^{(1)}&=\frac{1}{45(1-\nu^2)}\left(5R^8\frac{\partial^{10}}{\partial\alpha_2^{10}}+13R^6\frac{\partial^8}{\partial\alpha_2^8}+18R^4\frac{\partial^6}{\partial\alpha_2^6}+17R^2\frac{\partial^4}{\partial\alpha_2^4}+7\frac{\partial^2}{\partial\alpha_2^2}\right),
\end{aligned}
\tag{3.98}
$$

where

$$
v_{2,e}=R\left(V_2^{(0)}+\frac{h^2}{R^2}V_2^{(1)}\right). \tag{3.99}
$$

Keeping only the first two terms in the latter, we arrive at its leading-order counterpart,

$$
L_1^{(0)}\frac{\partial^2 v_{2,e}}{\partial t^2}+h^2L_2^{(0)}v_{2,e}=0, \tag{3.100}
$$

the latter equation is identical to (3.62). We also write down the formulae for the associated refined 2D displacement

$$v_2 = v_{2,e} + \frac{\alpha_3}{R} v_{2,e} + \alpha_3 R \frac{\partial^2 v_{2,e}}{\partial \alpha_2^2} \tag{3.101}$$

and

$$v_3 = -\frac{\partial v_{2,e}}{\partial \theta} - \frac{\alpha_3^2}{R^2} \frac{\nu}{2(1-\nu)} \left(R \frac{\partial v_{2,e}}{\partial \alpha_2} + R^3 \frac{\partial^3 v_{2,e}}{\partial \alpha_2^3} \right). \tag{3.102}$$

By making use of e_0 from equation (3.59), together with the values of a_1 and a_2 from equations (3.80) and (3.79) respectively, we have on mid-surface

$$R \frac{\partial v_{2,e}}{\partial \alpha_2} + v_{3,e} = -\frac{h^2}{3R^2} \left(R^2 \frac{\partial^2 v_{3,e}}{\partial \alpha_2^2} + R^4 \frac{\partial^4 v_{3,e}}{\partial \alpha_2^4} \right) + \frac{h^2}{R^2} \frac{2+\nu}{6(1-\nu)} \left(v_{3,e} + R^2 \frac{\partial^2 v_{3,e}}{\partial \alpha_2^2} \right)$$
$$- R^2 \frac{(1-\nu^2)\rho}{E} \frac{\partial v_{3,e}}{\partial t^2}, \tag{3.103}$$

where

$$v_{3,e} = R \left(V_3^{(0)} - \frac{\alpha_3^2}{R^2} \frac{\nu}{2(1-\nu)} P_1 - \frac{h^2}{R^2} V_3^{(1)} \right). \tag{3.104}$$

This formula assures all η^2 corrections to the assumption of circumferential inextensibility. As an example, the natural frequencies can be derived from equation (3.97), on substituting $v_{2,e} = \exp\left(i(n\alpha_2/R - \omega t)\right)$, we get

$$\frac{1}{2(1+\nu)} \left(1+n^2\right) \Omega^2 = \frac{n^2(1-n^2)^2}{3(1-\nu^2)} - \frac{\eta^2}{45(1-\nu^2)} \left(5n^{10} - 13n^8 + 18n^6\right.$$
$$\left. -17n^4 + 7n^2\right) - \frac{\eta^2}{3(1-\nu^2)} \big[-(1-\nu)n^6 + (3-2\nu)n^4 - (3-2\nu)n^2 \tag{3.105}$$
$$+ (1-\nu) \big] \Omega^2 - \eta^2 \frac{(1-\nu)n^2}{4(1+\nu)} \Omega^4,$$

where $\Omega = \eta^{-1}\omega R/c_2$ and n is the wavenumber.

Inserting the two-term expansion $\Omega^2 = \Omega_0^2 + \eta^2\Omega_1^2 + \cdots$ into this relation, we obtain

$$\Omega_0^2 = \frac{2}{3(1-\nu)}\frac{n^2(1-n^2)^2}{1+n^2}, \tag{3.106}$$

$$\Omega_1^2 = -\frac{2n^2(1-n^2)^2\left[17(1-\nu)-(9+\nu)n^2+11(1-\nu)n^4+(17-7\nu)n^6\right]}{45(1-\nu)^2(1+n^2)^3}. \tag{3.107}$$

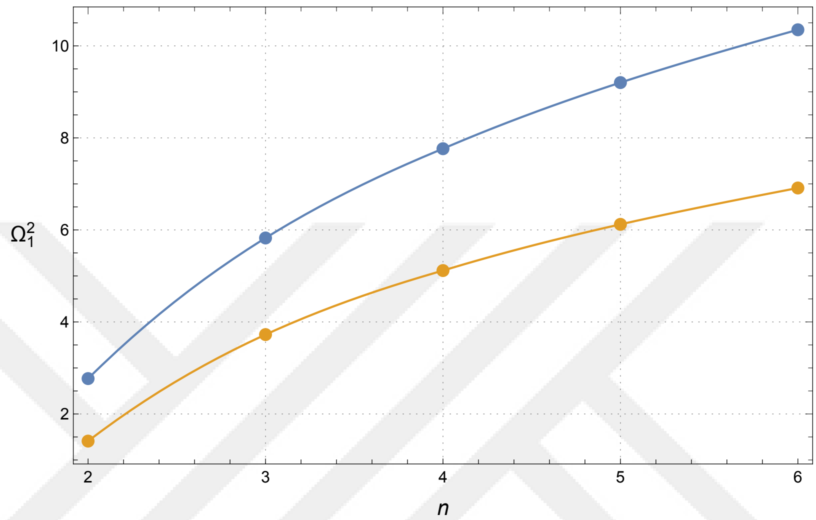

Figure 3.2. *Comparison of* (3.107) *(blue line) and* (3.108) *(orange line) at* $\nu = 0.3$

Let us compare the frequency obtained above with the results of [44]. To this end, the convenient form of equation of $(3.6)_2$ of [44] which is the counterpart of the considered frequency in terms of notations introduced in section (3.2) can be written as

$$\Omega_1^2 = -\frac{8}{9(1-\nu)}\frac{n^6(1-n^2)^2}{(1+n^2)^3}. \tag{3.108}$$

It is clearly seen that (3.106) agrees with $(3.6)_1$, but Ω_1^2 in $(3.6)_2$ of [44] does not coincide with formula (3.107). However, equation (3.107) coincides with the result of the asymptotic analysis of the exact dispersion relation in plane elasticity of cited paper [44] within higher-order terms.

If we substitute $v_{2,e} = V\exp\left(i(n\alpha_2/R-\omega t)\right)$, $v_{3,e} = iW\exp\left(i(n\alpha_2/R-\omega t)\right)$ into equation (3.103), we obtain

$$V + \frac{1}{n}W = \frac{h^2}{R^2}(1-n^2)\frac{6+3\nu+6(1+\nu)n^2}{18(1+\nu)n}W + \frac{2R^2(1+\nu)\rho}{3E}\frac{n(1-n^2)^2}{1+n^2}W, \tag{3.109}$$

which also does not agree with equation (5.1) of [44].

4. ASYMPTOTIC DERIVATION OF 3D EQUATIONS OF MOTION FOR HOMOGENEOUS ELASTIC SHELL

In this chapter, a refined 1D consistent equation is obtained from full 3D dynamic equations of motion for a thin elastic cylindrical shell, and low-frequency vibration is considered for the problem. For the first time, the quantities are normalized by a novel special scaling, and unlike the previous chapter, the stress and displacement components are not separated into even and odd parts. Except for the first-order frequency Ω^2 in the frequency expansion, all of the coefficients in the derived fourth-order equation correspond to their counterparts in the Sanders-Koiter shell theory.

4.1. Statement of the Problem

Consider a thin elastic cylindrical shell of thickness $2h$ for which the geometrical parameter $\eta = \frac{h}{R}$ is small compared to unity, where R is the radius of curvature of the mid-surface of the shell, see figure (4.1) .

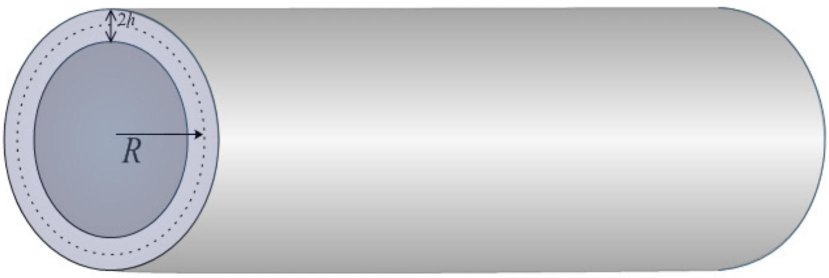

Figure 4.1. *A thin elastic cylindrical shell*

The traditional three orthogonal coordinates along the mid-surface α_i, $i = 1, 2, 3$, are taken in the form

$$\alpha_1 = \frac{R^2}{h}\xi, \qquad \alpha_2 = R\theta, \qquad \alpha_3 = h\zeta, \tag{4.1}$$

where ξ is the longitudinal coordinate $(-\infty < \xi < \infty)$, θ is the circumferential coordinate (angular coordinate) $(0 \le \theta < 2\pi)$, and ζ is the transverse coordinate $(-1 \le \zeta \le 1)$. Also, define the dimensionless frequency by

$$\Omega = \frac{\omega R}{\eta c_2}, \qquad c_2 = \sqrt{\frac{E}{2(1+\nu)\rho}}, \tag{4.2}$$

where ω is the original frequency, E is the Young's modulus, ρ is the mass density, and ν is the Poisson's ratio; in what follows the factor $e^{i\omega t}$, where t is time, is omitted. The first formula (4.1) defines a typical wavelength L of order $\frac{R^2}{h}$ which is much greater not only then the shell thickness ($h/L \sim \eta^2 \ll 1$) but also then the shell radius ($R/L \sim \eta \ll 1$). Assuming, in addition, $\Omega \sim 1$ we restrict ourselves to the lowest vibration modes investigate in [44] within the framework of the 2D shell theory.

Let us introduce the 3D dimensionless displacement and stress components as

$$v_1 = u_1 \cos n\theta, \qquad v_2 = u_2 \sin n\theta, \qquad v_3 = u_3 \cos n\theta \tag{4.3}$$

and

$$\begin{aligned} \sigma_{11} &= s_{11} \cos n\theta, & \sigma_{22} &= s_{22} \cos n\theta, & \sigma_{21} &= s_{21} \sin n\theta, \\ \sigma_{31} &= s_{31} \cos n\theta, & \sigma_{32} &= s_{32} \sin n\theta, & \sigma_{33} &= s_{33} \cos n\theta, \end{aligned} \tag{4.4}$$

where stresses and displacements denote 2D unknown quantities, n is the circumferential wave number. As is clear, the circumferential coordinate is separated in the displacement and stress components.

The 3D equations of motion can be reduced to simplified form written below, see section (2.4),

$$\eta^2 \frac{\partial s_{1i}}{\partial \xi} + (-1)^{i-1} \frac{n\eta}{1+\eta\zeta} s_{2i} + \frac{\partial s_{3i}}{\partial \zeta} + \frac{2^{i-1}\eta}{1+\eta\zeta} s_{3i} + \eta^3 \frac{E}{2(1+\nu)R} \Omega^2 u_i = 0, \quad i = 1,2 \tag{4.5}$$

$$\eta^2 \frac{\partial s_{31}}{\partial \xi} + \frac{n\eta}{1+\eta\zeta} s_{32} + \frac{\partial s_{33}}{\partial \zeta} - \frac{\eta}{1+\eta\zeta} s_{22} + \frac{\eta}{1+\eta\zeta} s_{33} + \eta^3 \frac{E}{2(1+\nu)R} \Omega^2 u_3 = 0, \tag{4.6}$$

together with

$$s_{11} = \frac{E}{(1-\nu^2)R} \left(\eta \frac{\partial u_1}{\partial \xi} + \frac{\nu e}{1+\eta\zeta} \right) + \frac{\nu}{1-\nu} s_{33}, \tag{4.7}$$

$$s_{22} = \frac{E}{(1-\nu^2)R} \left(\eta\nu \frac{\partial u_1}{\partial \xi} + \frac{e}{1+\eta\zeta} \right) + \frac{\nu}{1-\nu} s_{33}, \tag{4.8}$$

$$\frac{E}{h}\frac{\partial u_3}{\partial \zeta} = s_{33} - \nu\left(s_{11} + s_{22}\right), \tag{4.9}$$

$$s_{31} = \frac{E}{2(1+\nu)h}\left(\eta^2\frac{\partial u_3}{\partial \xi} + \frac{\partial u_1}{\partial \zeta}\right), \tag{4.10}$$

$$s_{32} = \frac{E}{2(1+\nu)R}\left(\frac{1}{\eta}\frac{\partial u_2}{\partial \zeta} - \frac{1}{1+\eta\zeta}(u_2 + nu_3)\right), \tag{4.11}$$

$$s_{21} = \frac{E}{2(1+\nu)R}\frac{\eta}{1+\eta\zeta}\left(\eta\zeta\frac{\partial u_2}{\partial \xi} + s\right). \tag{4.12}$$

where

$$e = nu_2 + u_3, \qquad s = \frac{\partial u_2}{\partial \xi} - \frac{n}{\eta}u_1. \tag{4.13}$$

s and e are the deformations corresponding to mid-surface shear and circumferential extension respectively. We also impose homogeneous boundary conditions on traction free faces, i.e.,

$$s_{3i} = 0, \quad \zeta = \pm 1 \quad i = 1, 2, 3. \tag{4.14}$$

The main purpose of this chapter is to reduce the 3D equations to a 1D form over the long-wave, low-frequency domain defined in this section.

4.2. Asymptotic Scaling

Let us introduce dimensionless displacements and stresses by

$$u_1 = \eta R u_1^*, \qquad u_2 = R u_2^*, \qquad u_3 = R u_3^* \tag{4.15}$$

and

$$\begin{aligned} s_{11} &= E\eta s_{11}^*, \qquad s_{12} = E\eta^2 s_{12}^*, \qquad s_{22} = E\eta s_{22}^*, \\ s_{31} &= E\eta^3 s_{31}^*, \qquad s_{32} = E\eta^2 s_{32}^*, \qquad s_{33} = E\eta^2 s_{33}^*. \end{aligned} \tag{4.16}$$

In addition, let

$$e = \eta R e_*, \qquad s = \eta R s_*, \tag{4.17}$$

because such special arrangements are made to incorporate the generation of the deformations corresponding to mid-surface shear and circumferential extension.

On using (4.17), the constraints (4.13) take the form

$$nu_2^* + u_3^* = \eta e_*, \qquad \frac{\partial u_2^*}{\partial \xi} - nu_1^* = \eta s_*, \tag{4.18}$$

where all starred quantities are assumed to be of order one. This scaling is motivated by asymptotic consideration in [46] within the framework of Sanders-Koiter version of 2D classical shell theory.

Now, inserting the dimensionless variables together with the scaling given by formulae (4.15) and (4.16) into the equations (4.5)–(4.12), we obtain

$$\frac{\partial s_{11}^*}{\partial \xi} + \frac{n}{1+\eta\zeta} s_{21}^* + \frac{\partial s_{31}^*}{\partial \zeta} + \frac{\eta}{1+\eta\zeta} s_{31}^* + \frac{\eta}{2(1+\nu)} \Omega^2 u_1^* = 0, \tag{4.19}$$

$$\eta^2 \frac{\partial s_{21}^*}{\partial \xi} - \frac{n}{1+\eta\zeta} s_{22}^* + \frac{\partial s_{32}^*}{\partial \zeta} + \frac{2\eta}{1+\eta\zeta} s_{32}^* + \frac{\eta}{2(1+\nu)} \Omega^2 u_2^* = 0, \tag{4.20}$$

$$\eta^3 \frac{\partial s_{31}^*}{\partial \xi} + \frac{\eta n}{1+\eta\zeta} s_{32}^* + \frac{\partial s_{33}^*}{\partial \zeta} - \frac{1}{1+\eta\zeta} s_{22}^* + \frac{\eta}{1+\eta\zeta} s_{33}^* + \frac{\eta}{2(1+\nu)} \Omega^2 u_3^* = 0, \tag{4.21}$$

with

$$s_{11}^* = \frac{1}{1-\nu^2} \left(\eta \frac{\partial u_1^*}{\partial \xi} + \frac{\nu}{1+\eta\zeta} e_* \right) + \eta \frac{\nu}{1-\nu} s_{33}^*, \tag{4.22}$$

$$s_{22}^* = \frac{1}{1-\nu^2} \left(\eta\nu \frac{\partial u_1^*}{\partial \xi} + \frac{1}{1+\eta\zeta} e_* \right) + \eta \frac{\nu}{1-\nu} s_{33}^*, \tag{4.23}$$

$$\frac{\partial u_3^*}{\partial \zeta} = \eta^3 s_{33}^* - \nu\eta^2 \left(s_{11}^* + s_{22}^* \right), \tag{4.24}$$

$$\frac{\partial u_1^*}{\partial \zeta} = -\eta \frac{\partial u_3^*}{\partial \xi} + 2\eta^3 (1+\nu) s_{31}^*, \tag{4.25}$$

$$\frac{\partial u_2^*}{\partial \zeta} = \frac{\eta}{1+\eta\zeta} (nu_3^* + u_2^*) + 2\eta^3 (1+\nu) s_{32}^*, \tag{4.26}$$

$$s_{21}^* = \frac{1}{2(1+\nu)} \frac{1}{1+\eta\zeta} \left(\zeta \frac{\partial u_2^*}{\partial \xi} + s_* \right), \tag{4.27}$$

Boundary conditions (4.14) take the form

$$s_{3i}^* = 0, \quad \zeta = \pm 1 \quad i = 1, 2, 3. \tag{4.28}$$

We expand all the starred stress and displacement components in asymptotic series as

$$f^* = f^{(0)} + \eta f^{(1)} + \eta^2 f^{(2)} + \cdots \tag{4.29}$$

In this case, we also have for the dimensionless frequency

$$\Omega^2 = \Omega_0^2 + \eta\Omega_1^2 + \eta^2\Omega_2^2 + \cdots , \tag{4.30}$$

corresponding to a near cut-off expansion around the frequency value $\Omega \approx \Omega_0$, earlier adapted within the 2D set-up discussed in the previous chapter and many in others studies.

4.3. The Leading-Order Approximation

Let us substitute (4.29) and (4.30) into equations (4.19) – (4.27) and neglect $O(\eta)$ terms, yields

$$\frac{\partial s_{31}^{(0)}}{\partial \zeta} + \frac{\partial s_{11}^{(0)}}{\partial \xi} + n s_{21}^{(0)} = 0, \tag{4.31}$$

$$\frac{\partial s_{32}^{(0)}}{\partial \zeta} - n s_{22}^{(0)} = 0, \tag{4.32}$$

$$\frac{\partial s_{33}^{(0)}}{\partial \zeta} - s_{22}^{(0)} = 0, \tag{4.33}$$

$$s_{11}^{(0)} = \frac{\nu}{1-\nu^2} e^{(0)}, \tag{4.34}$$

$$s_{22}^{(0)} = \frac{1}{1-\nu^2} e^{(0)}, \tag{4.35}$$

$$\frac{\partial u_3^{(0)}}{\partial \zeta} = 0, \tag{4.36}$$

$$\frac{\partial u_1^{(0)}}{\partial \zeta} = 0, \tag{4.37}$$

$$\frac{\partial u_2^{(0)}}{\partial \zeta} = 0, \tag{4.38}$$

$$s_{21}^{(0)} = \frac{1}{2(1+\nu)} \left(s^{(0)} + \zeta \frac{\partial u_2^{(0)}}{\partial \xi} \right), \tag{4.39}$$

with the boundary conditions at $\zeta = \pm 1$

$$s_{3i}^{(0)} = 0, \quad i = 1, 2, 3. \tag{4.40}$$

First, integrating equations (4.37), (4.38) and (4.36) respectively with respect to thickness variable ζ, we obtain

$$u_1^{(0)} = U^{(0)}(\xi), \quad u_2^{(0)} = V^{(0)}(\xi), \quad u_3^{(0)} = W^{(0)}(\xi). \tag{4.41}$$

On account of (4.18) at leading order, we write

$$nu_2^{(0)} + u_3^{(0)} = 0, \quad \frac{\partial u_2^{(0)}}{\partial \zeta} - nu_1^{(0)} = 0, \tag{4.42}$$

and from (4.41), we get

$$W^{(0)} = -nV^{(0)} \tag{4.43}$$

and

$$U^{(0)} = \frac{1}{n}\frac{\partial V^{(0)}}{\partial \xi}. \tag{4.44}$$

Also, from (4.43) and (4.44), it is obvious that

$$U^{(0)} = -\frac{1}{n^2}\frac{\partial W^{(0)}}{\partial \xi}. \tag{4.45}$$

Next, integrating equation (4.32) with respect to ζ and taking into account boundary conditions (4.40) at $i = 2$, we get

$$s_{32}^{(0)} = -n \int_{\zeta}^{1} s_{22}^{(0)}(s)\, ds \tag{4.46}$$

and

$$n \int_{-1}^{1} s_{22}^{(0)}(s)\, ds = 0, \tag{4.47}$$

demonstrating that normal circumferential stress resultant disappears at leading order. As a result, the associated stress couple appears to be more significant. This observation nicely illustrates the basic idea of the semi-membrane shell theory, see [24, 27]
Similarly, integrating equation (4.33) with respect to ζ and taking into account boundary conditions (4.40) at $i = 3$, we get

$$s_{33}^{(0)} = -\int_{\zeta}^{1} s_{22}^{(0)}(s)\, ds \tag{4.48}$$

and

$$\int_{-1}^{1} s_{22}^{(0)}(s)\, ds = 0. \tag{4.49}$$

The relations above do not form a close set of equations and not allow calculating the leading order term Ω_0^2 in the asymptotic series (4.30). This motivates to proceed to the first order approximation.

4.4. First-Order Approximation

Let us substitute (4.29) and (4.30) into equations (4.19) – (4.27) and neglect $O(\eta^2)$ terms, yields

$$\frac{\partial s_{11}^{(1)}}{\partial \xi} + n s_{21}^{(1)} - n\zeta s_{21}^{(0)} + \frac{\partial s_{31}^{(1)}}{\partial \zeta} + s_{31}^{(0)} + \frac{1}{2(1+\nu)}\Omega_0^2 u_1^{(0)} = 0, \tag{4.50}$$

$$\frac{\partial s_{32}^{(1)}}{\partial \zeta} - n s_{22}^{(1)} + n\zeta s_{22}^{(0)} + 2 s_{32}^{(0)} + \frac{1}{2(1+\nu)}\Omega_0^2 u_2^{(0)} = 0, \tag{4.51}$$

$$\frac{\partial s_{33}^{(1)}}{\partial \zeta} + n s_{32}^{(0)} - s_{22}^{(1)} + \zeta s_{22}^{(0)} + s_{33}^{(0)} + \frac{1}{2(1+\nu)}\Omega_0^2 u_3^{(0)} = 0, \tag{4.52}$$

$$s_{11}^{(1)} = \frac{1}{1-\nu^2}\left(\frac{\partial u_1^{(0)}}{\partial \xi} + \nu e^{(1)} - \nu\zeta e^{(0)}\right) + \frac{\nu}{1-\nu} s_{33}^{(0)}, \tag{4.53}$$

$$s_{22}^{(1)} = \frac{1}{1-\nu^2}\left(\nu\frac{\partial u_1^{(0)}}{\partial \xi} + e^{(1)} - \zeta e^{(0)}\right) + \frac{\nu}{1-\nu} s_{33}^{(0)}, \tag{4.54}$$

$$\frac{\partial u_3^{(1)}}{\partial \zeta} = 0, \tag{4.55}$$

$$\frac{\partial u_1^{(1)}}{\partial \zeta} = -\frac{\partial u_3^{(0)}}{\partial \xi}, \tag{4.56}$$

$$\frac{\partial u_2^{(1)}}{\partial \zeta} = n u_3^{(0)} + u_2^{(0)}, \tag{4.57}$$

$$s_{21}^{(1)} = \frac{1}{2(1+\nu)}\left(\zeta \frac{\partial u_2^{(1)}}{\partial \xi} - \zeta^2 \frac{\partial u_2^{(0)}}{\partial \xi} + s^{(1)} - \zeta s^{(0)}\right), \tag{4.58}$$

with the boundary conditions at $\zeta = \pm 1$

$$s_{3i}^{(1)} = 0, \quad i = 1, 2, 3. \tag{4.59}$$

We begin by integrating equations (4.55)–(4.57) with respect to the thickness variable ζ, yields

$$u_3^{(1)} = W^{(1)}, \tag{4.60}$$

$$u_1^{(1)} = n\zeta \frac{\partial V^{(0)}}{\partial \xi} + U^{(1)}, \tag{4.61}$$

$$u_2^{(1)} = \zeta P_0 + V^{(1)}. \tag{4.62}$$

The relation

$$P_k = \begin{cases} (-1)^{\frac{k}{2}} n^k P_0 & k = 2, 4, 6, \cdots \\ (-1)^{\frac{k-1}{2}} n^k P_0 & k = 1, 3, 5, \cdots \end{cases} \tag{4.63}$$

is adapted throughout the chapter, where $P_0 = (1 - n^2)V^{(0)}$.

The first order terms of $(4.18)_1$ gives

$$e^{(0)} = \zeta P_1 + nV^{(1)} + W^{(1)}. \tag{4.64}$$

Integrating (4.35) from -1 to 1 with respect to ζ and employing (4.49), we get

$$\int_{-1}^{1} e^{(0)}\, ds = 0, \tag{4.65}$$

from latter we get

$$W^{(1)} = -nV^{(1)}. \tag{4.66}$$

Substituting (4.64) together with (4.66) in (4.35), we obtain

$$s_{22}^{(0)} = \frac{\zeta}{1-\nu^2} P_1, \tag{4.67}$$

comparison of equations (4.34) and (4.35) and taking into account (4.67), we have

$$s_{11}^{(0)} = \frac{\nu\zeta}{1-\nu^2} P_1. \tag{4.68}$$

On considering (4.67) equations (4.46) and (4.48) can be rewritten as

$$s_{32}^{(0)} = \frac{1-\zeta^2}{2(1-\nu^2)} P_2, \tag{4.69}$$

$$s_{33}^{(0)} = -\frac{1-\zeta^2}{2(1-\nu^2)} P_1. \tag{4.70}$$

Additionally, the first order terms of $(4.18)_2$ gives

$$s^{(0)} = \frac{\partial u_2^{(1)}}{\partial \zeta} - nu_2^{(1)} \tag{4.71}$$

On considering (4.61) and (4.62), we have

$$s^{(0)} = (1-2n^2)\zeta \frac{\partial V^{(0)}}{\partial \xi} + \frac{\partial V^{(1)}}{\partial \xi} - nU^{(1)}. \tag{4.72}$$

Integrating (4.31) with respect to ζ and taking into account upper boundary conditions (4.40) at $i = 1$, we obtain

$$s_{31}^{(0)} = \frac{n(1-n^2)(1-\zeta^2)}{2(1-\nu^2)} \frac{\partial V^{(0)}}{\partial \xi} + \frac{n(1-\zeta)}{2(1+\nu)} \left(\frac{\partial V^{(1)}}{\partial \xi} - nU^{(1)} \right), \tag{4.73}$$

then, taking into account boundary conditions (4.40) at $\zeta = 1$, we obtain

$$nU^{(1)} = \frac{\partial V^{(1)}}{\partial \xi}. \tag{4.74}$$

Making use of (4.44) and (4.72) in equation (4.39), we obtain

$$s_{21}^{(0)} = \frac{(1-n^2)\zeta}{1+\nu}\frac{\partial V^{(0)}}{\partial \xi}. \tag{4.75}$$

Inserting (4.74) into (4.73), we have

$$s_{31}^{(0)} = \frac{n(1-n^2)(1-\zeta^2)}{2(1-\nu^2)}\frac{\partial V^{(0)}}{\partial \xi}. \tag{4.76}$$

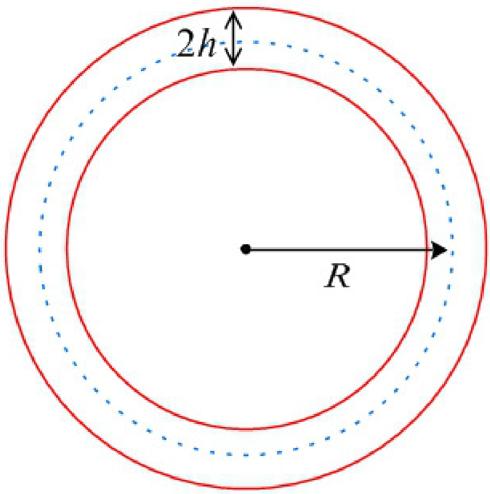

Figure 4.2. *Cross-Section of thin elastic shell*

Integrating (4.51) and (4.52) with respect to thickness variable ζ and taking into account the boundary conditions (4.59) at $i = 2$ and $i = 3$ respectively, we arrive at

$$s_{32}^{(1)} = -n\int_{\zeta}^{1} s_{22}^{(1)}(s)\,ds + \frac{1-3\zeta+2\zeta^3}{3(1-\nu^2)}P_2 + \frac{1-\zeta}{2(1+\nu)}\Omega_0^2 V^{(0)} \tag{4.77}$$

and

$$s_{33}^{(1)} = -\int_{\zeta}^{1} s_{22}^{(1)}(s)\,ds + \frac{2-3\zeta+\zeta^3}{6(1-\nu^2)}P_3 + \frac{\zeta(1-\zeta^2)}{2(1-\nu^2)}P_1 - \frac{n(1-\zeta)}{2(1+\nu)}\Omega_0^2 V^{(0)}. \tag{4.78}$$

Now, the solvability of (4.51) and taking into account the boundary conditions (4.59) at $i = 2$, we obtain

$$n \int_{-1}^{1} s_{22}^{(1)}(s)\, ds = \frac{2}{3(1-\nu^2)} P_2 + \frac{1}{1+\nu} \Omega_0^2 V^{(0)}. \tag{4.79}$$

Similarly, the solvability of (4.52) and taking into account the boundary conditions (4.59) at $i = 3$, we obtain

$$\int_{-1}^{1} s_{22}^{(1)}(s)\, ds = \frac{2}{3(1-\nu^2)} P_3 - \frac{n}{1+\nu} \Omega_0^2 V^{(0)}. \tag{4.80}$$

Comparison of (4.79) and (4.80), subject to (4.63), yields

$$\frac{1+n^2}{1+\nu} \Omega_0^2 V^{(0)} = \frac{2n^2(1-n^2)^2}{3(1-\nu^2)} V^0. \tag{4.81}$$

The latter equation can be written in simplified form as

$$\Omega_0^2 = \frac{2}{3(1-\nu)} \frac{n^2(1-n^2)^2}{1+n^2}. \tag{4.82}$$

At a given n, this is the expression of the lowest cut-off frequency within the classical 2D shell theory, e.g., see previous chapter and [42, 44]. For $n = 0$ and $n = 1$, we have $\Omega_0^2 = 0$. In this case, the assumptions underlying the asymptotic consideration above are violated. These two modes correspond to the so-called bar (or beam) type vibration of a cylindrical shell and assume a special treatment.

Although we determined the frequency, we do not yet have the sought for equation of motion. Therefore, we should proceed to higher orders.

4.5. Second-Order Approximation

Let us substitute (4.29) and (4.30) into equations (4.19) – (4.27) and neglect $O(\eta^3)$ terms, yields

$$\frac{\partial s_{11}^{(2)}}{\partial \xi} + n s_{21}^{(2)} - n\zeta s_{21}^{(1)} + n\zeta^2 s_{21}^{(0)} + \frac{\partial s_{31}^{(2)}}{\partial \zeta} + s_{31}^{(1)} - \zeta s_{31}^{(0)} + \frac{1}{2(1+\nu)} \Omega_0^2 u_1^{(1)} + \frac{1}{2(1+\nu)} \Omega_1^2 u_1^{(0)} = 0, \tag{4.83}$$

$$\frac{\partial s_{21}^{(0)}}{\partial \xi} - ns_{22}^{(2)} + n\zeta s_{22}^{(1)} - n\zeta^2 s_{22}^{(0)} + \frac{\partial s_{32}^{(2)}}{\partial \zeta} + 2s_{32}^{(1)} - 2\zeta s_{32}^{(0)} + \frac{1}{2(1+\nu)}\Omega_0^2 u_2^{(1)} + \frac{1}{2(1+\nu)}\Omega_1^2 u_2^{(0)} = 0, \tag{4.84}$$

$$ns_{32}^{(1)} - n\zeta s_{32}^{(0)} + \frac{\partial s_{33}^{(2)}}{\partial \zeta} - s_{22}^{(2)} + \zeta s_{22}^{(1)} - \zeta^2 s_{22}^{(0)} + s_{33}^{(1)} - \zeta s_{33}^{(0)} + \frac{1}{2(1+\nu)}\Omega_0^2 u_3^{(1)} + \frac{1}{2(1+\nu)}\Omega_1^2 u_3^{(0)} = 0, \tag{4.85}$$

$$s_{11}^{(2)} = \frac{1}{1-\nu^2}\left(\frac{\partial u_1^{(1)}}{\partial \xi} + \nu e^{(2)} - \nu\zeta e^{(1)} + \nu\zeta^2 e^{(0)}\right) + \frac{\nu}{1-\nu} s_{33}^{(1)}, \tag{4.86}$$

$$s_{22}^{(2)} = \frac{1}{1-\nu^2}\left(\nu\frac{\partial u_1^{(1)}}{\partial \xi} + e^{(2)} - \zeta e^{(1)} + \zeta^2 e^{(0)}\right) + \frac{\nu}{1-\nu} s_{33}^{(1)}, \tag{4.87}$$

$$\frac{\partial u_3^{(2)}}{\partial \zeta} = -\nu\left(s_{11}^{(0)} + s_{22}^{(0)}\right), \tag{4.88}$$

$$\frac{\partial u_1^{(2)}}{\partial \zeta} = -\frac{\partial u_3^{(1)}}{\partial \xi}, \tag{4.89}$$

$$\frac{\partial u_2^{(2)}}{\partial \zeta} = nu_3^{(1)} - n\zeta u_3^{(0)} + u_2^{(1)} - \zeta u_2^{(0)}, \tag{4.90}$$

$$s_{21}^{(2)} = \frac{1}{2(1+\nu)}\left(\zeta\frac{\partial u_2^{(2)}}{\partial \xi} - \zeta^2\frac{\partial u_2^{(1)}}{\partial \xi} + \zeta^3\frac{\partial u_2^{(0)}}{\partial \xi} + s^{(2)} - \zeta s^{(1)} + \zeta^2 s^{(0)}\right), \tag{4.91}$$

with the boundary conditions at $\zeta = \pm 1$

$$s_{3i}^{(2)} = 0, \quad i = 1, 2, 3. \tag{4.92}$$

We begin by integrating equations (4.88)–(4.90) with respect to the thickness variable ζ, yields

$$u_3^{(2)} = -\frac{\nu\zeta^2}{2(1-\nu)}P_1 + W^{(2)}, \tag{4.93}$$

$$u_1^{(2)} = n\zeta\frac{\partial V^{(1)}}{\partial \xi} + U^{(2)}, \tag{4.94}$$

$$u_2^{(2)} = \zeta Q_0 + V^{(2)}. \tag{4.95}$$

The relation

$$Q_k = \begin{cases} (-1)^{\frac{k}{2}} n^k Q_0 & k = 2, 4, 6, \cdots \\ (-1)^{\frac{k-1}{2}} n^k Q_0 & k = 1, 3, 5, \cdots \end{cases} \tag{4.96}$$

is also adapted throughout the chapter, where $Q_0 = (1-n^2)V^{(1)}$.

The second order terms of $(4.18)_1$ gives

$$e^{(1)} = \zeta Q_1 - \frac{\nu}{2(1-\nu)}\zeta^2 P_1 + nV^{(2)} + W^{(2)}. \tag{4.97}$$

Integrating (4.54) from -1 to 1 with respect to ζ and considering (4.64), (4.80) and (4.97), we get

$$W^{(2)} + nV^{(2)} = -\frac{\nu}{n}\frac{\partial^2 V^{(0)}}{\partial \xi^2} + \frac{1}{3}P_3 + \frac{2+\nu}{6(1-\nu)}P_1 - \frac{1}{2}n(1-\nu)\Omega_0^2 V^{(0)}, \tag{4.98}$$

inserting (4.98) back into (4.54), we get

$$s_{22}^{(1)} = \frac{\zeta}{1-\nu^2}Q_1 + \frac{1}{3(1-\nu^2)}P_3 + \frac{1-3\zeta^2}{3(1-\nu^2)}P_1 - \frac{n}{2(1+\nu)}\Omega_0^2 V^{(0)}. \tag{4.99}$$

Similarly, inserting (4.98) into (4.53), we get

$$s_{11}^{(1)} = \frac{1}{n}\frac{\partial^2 V^{(0)}}{\partial \xi^2} + \frac{\nu\zeta}{1-\nu^2}Q_1 + \frac{\nu}{3(1-\nu^2)}P_3 - \frac{\nu(1+3\zeta^2)}{6(1-\nu^2)}P_1 - \frac{n\nu}{2(1+\nu)}\Omega_0^2 V^{(0)}. \tag{4.100}$$

Substituting (4.99) into (4.77) and (4.78) respectively, we obtain

$$s_{32}^{(1)} = \frac{1-\zeta^2}{2(1-\nu^2)}Q_2 + \frac{1-\zeta}{3(1-\nu^2)}P_4 + \frac{1-4\zeta+3\zeta^3}{3(1-\nu^2)}P_2 + \frac{(1+n^2)(1-\zeta)}{2(1+\nu)}\Omega_0^2 V^{(0)} \tag{4.101}$$

and

$$s_{33}^{(1)} = -\frac{1-\zeta^2}{2(1-\nu^2)}Q_1 - \frac{\zeta(1-\zeta^2)}{6(1-\nu^2)}P_3 + \frac{5\zeta(1-\zeta^2)}{6(1-\nu^2)}P_1. \tag{4.102}$$

The second order terms of $(4.18)_2$ gives

$$s^{(1)} = (1-2n^2)\zeta\frac{\partial V^{(1)}}{\partial\xi} - nU^{(2)} + \frac{\partial V^{(2)}}{\partial\xi}. \tag{4.103}$$

Integrating (4.50) with respect to ζ and taking into account the boundary conditions (4.59) at $\zeta = 1$, we obtain

$$\begin{aligned}\sigma_{31}^{(1)} =& \frac{1-\zeta^2}{2(1-\nu^2)}\frac{\partial Q_1}{\partial\xi} + \frac{\nu(1-\zeta)}{3(1-\nu^2)}\frac{\partial P_3}{\partial\xi} - \frac{(1-\nu)+\zeta(3-\nu)-2\zeta^3(2-\nu)}{6(1-\nu^2)}\frac{\partial P_1}{\partial\xi} \\ &+ \frac{n(1-\zeta)}{2(1+\nu)}\left(\frac{\partial V^{(2)}}{\partial\xi} - nU^{(2)}\right) - \frac{1-\zeta}{2(1+\nu)}\left(n\nu\frac{\partial V^{(0)}}{\partial\xi} - U^{(0)}\right)\Omega_0^2.\end{aligned} \tag{4.104}$$

Then, substituting values of P, Q, Ω_0^2 and taking into account boundary conditions (4.59) at $\zeta = -1$, we obtain

$$\frac{1}{1+\nu}\left(nU^{(2)} - \frac{\partial V^{(2)}}{\partial\xi}\right) = \frac{2}{n^2}\frac{\partial^3 V^{(0)}}{\partial\xi^3} + \frac{1-4n^2+3n^4}{3(1-\nu)(1+n^2)}\frac{\partial V^{(0)}}{\partial\xi}. \tag{4.105}$$

Here, the relation between $U^{(2)}$ and $V^{(2)}$ are derived.

Now, making use of (4.44), (4.74) and (4.105) into (4.58), we arrived at

$$s_{21}^{(1)} = -\frac{1}{n^2}\frac{\partial^3 V^{(0)}}{\partial\xi^3} - \frac{(1-4n^2+3n^4)(1+\nu)+3(1-n^4)(1-\nu)\zeta^2}{6(1-\nu^2)(1+n^2)}\frac{\partial V^{(0)}}{\partial\xi} + \frac{(1-n^2)\zeta}{1+\nu}\frac{\partial V^{(1)}}{\partial\xi}. \tag{4.106}$$

Integrating (4.84) and (4.85) with respect to thickness variable ζ and taking into account the boundary conditions (4.92) at $i = 2$ and $i = 3$ respectively, we arrive at

$$
\begin{aligned}
s_{32}^{(2)} = -n\int_{\zeta}^{1} s_{22}^{(2)}(s)\,ds &+ \frac{(1-n^2)(1-\zeta^2)}{2(1+\nu)}\frac{\partial^2 V^{(0)}}{\partial\xi^2} + \frac{1-3\zeta+2\zeta^3}{3(1-\nu^2)}Q_2 + \frac{1-4\zeta+3\zeta^2}{6(1-\nu^2)}P_4 \\
&- \frac{1+8\zeta-24\zeta^2+15\zeta^4}{12(1-\nu^2)}P_2 + \frac{1-\zeta}{2(1+\nu)}\Omega_0^2 V^{(1)} + \frac{1-\zeta}{2(1+\nu)}\Omega_1^2 V^{(0)} \\
&+ \frac{3-4(1+n^2)\zeta+(1+4n^2)\zeta^2}{4(1+\nu)}\Omega_0^2 V^{(0)},
\end{aligned} \tag{4.107}
$$

$$
\begin{aligned}
s_{33}^{(2)} = -\int_{\zeta}^{1} s_{22}^{(2)}(s)\,ds &+ \frac{2-3\zeta+\zeta^3}{6(1-\nu^2)}Q_3 + \frac{\zeta(1-\zeta^2)}{2(1-\nu^2)}Q_1 + \frac{1-2\zeta+\zeta^2}{6(1-\nu^2)}P_5 \\
&- \frac{1+4\zeta-10\zeta^2+5\zeta^4}{12(1-\nu^2)}P_3 - \frac{5\zeta^2(1-\zeta^2)}{6(1-\nu^2)}P_1 - \frac{n(1-\zeta)}{2(1+\nu)}\Omega_0^2 V^{(1)} \\
&- \frac{n(1-\zeta)}{2(1+\nu)}\Omega_1^2 V^{(0)} + \frac{n^3-2n(1+n^2)\zeta+n(2+n^2)\zeta^2}{4(1+\nu)}\Omega_0^2 V^{(0)}.
\end{aligned} \tag{4.108}
$$

Now, the solvability of (4.84) and taking into account the boundary conditions (4.92) at $i = 2$, we obtain

$$
\begin{aligned}
n\int_{-1}^{1} s_{22}^{(2)}(s)\,ds = \frac{2}{3(1-\nu^2)}Q_2 + \frac{4}{3(1-\nu^2)}P_4 + \frac{4}{3(1-\nu^2)}P_2 + \frac{1}{1+\nu}\Omega_0^2 V^{(1)} \\
+\frac{1}{1+\nu}\Omega_1^2 V^{(0)} + \frac{2(1+n^2)}{1+\nu}\Omega_0^2 V^{(0)}.
\end{aligned} \tag{4.109}
$$

Similarly, the solvability of (4.85) and taking into account the boundary conditions (4.92) at $i = 3$, we obtain

$$
\begin{aligned}
\int_{-1}^{1} s_{22}^{(2)}(s)\,ds = \frac{2}{3(1-\nu^2)}Q_3 + \frac{2}{3(1-\nu^2)}P_5 + \frac{2}{3(1-\nu^2)}P_3 - \frac{n}{1+\nu}\Omega_0^2 V^{(1)} \\
-\frac{n}{1+\nu}\Omega_1^2 V^{(0)} + \frac{n(1+n^2)}{1+\nu}\Omega_0^2 V^{(0)}.
\end{aligned} \tag{4.110}
$$

Comparison of (4.109) and (4.110), and taking into account (4.63) and (4.82), we have

$$
\begin{aligned}
\frac{1+n^2}{1+\nu}\Omega_1^2 V^{(0)} &= -\frac{2}{3(1-\nu^2)}P_6 - \frac{2}{1-\nu^2}P_4 - \frac{4}{3(1-\nu^2)}P_2 - \frac{(1+n^2)(2-n^2)}{1+\nu}\Omega_0^2 V^{(0)} \\
&= \frac{2n^6(1-n^2)}{3(1-\nu^2)}V^{(0)} - \frac{2n^4(1-n^2)}{1-\nu^2}V^{(0)} + \frac{4n^2(1-n^2)}{3(1-\nu^2)}V^{(0)} \\
&+ \frac{2n^4(1-n^2)^2}{3(1-\nu^2)}V^{(0)} - \frac{4n^2(1-n^2)^2}{3(1-\nu^2)}V^{(0)},
\end{aligned} \tag{4.111}
$$

hence

$$\Omega_1^2 = 0. \tag{4.112}$$

i.e., there is no first order correction to the lowest cut-off frequency which is in agreement with [44]. Thus, we need to proceed to the next asymptotic order.

4.6. Third-Order Approximation

Let us substitute (4.29) and (4.30) into equations (4.19) – (4.27) and keeping $O(\eta^3)$ terms, yields

$$\begin{aligned}\frac{\partial s_{11}^{(3)}}{\partial \xi} + n s_{21}^{(3)} - n\zeta s_{21}^{(2)} + n\zeta^2 s_{21}^{(1)} - n\zeta^3 s_{21}^{(0)} + \frac{\partial s_{31}^{(3)}}{\partial \zeta} + s_{31}^{(2)} - \zeta s_{31}^{(1)} + \zeta^2 s_{31}^{(0)} \\ + \frac{1}{2(1+\nu)}\Omega_0^2 u_1^{(2)} + \frac{1}{2(1+\nu)}\Omega_1^2 u_1^{(1)} + \frac{1}{2(1+\nu)}\Omega_2^2 u_1^{(0)} = 0,\end{aligned} \tag{4.113}$$

$$\begin{aligned}\frac{\partial s_{21}^{(1)}}{\partial \xi} - n s_{22}^{(3)} + n\zeta s_{22}^{(2)} - n\zeta^2 s_{22}^{(1)} + n\zeta^3 s_{22}^{(0)} + \frac{\partial s_{32}^{(3)}}{\partial \zeta} + 2 s_{32}^{(2)} - 2\zeta s_{32}^{(1)} + 2\zeta^2 s_{32}^{(0)} \\ + \frac{1}{2(1+\nu)}\Omega_0^2 u_2^{(2)} + \frac{1}{2(1+\nu)}\Omega_1^2 u_2^{(1)} + \frac{1}{2(1+\nu)}\Omega_2^2 u_2^{(0)} = 0,\end{aligned} \tag{4.114}$$

$$\begin{aligned}\frac{\partial s_{31}^{(0)}}{\partial \xi} + n s_{32}^{(2)} - n\zeta s_{32}^{(1)} + n\zeta^2 s_{32}^{(0)} + \frac{\partial s_{33}^{(3)}}{\partial \zeta} - s_{22}^{(3)} + \zeta s_{22}^{(2)} - \zeta^2 s_{22}^{(1)} + \zeta^3 s_{22}^{(0)} \\ + s_{33}^{(2)} - \zeta s_{33}^{(1)} + \zeta^2 s_{33}^{(0)} + \frac{1}{2(1+\nu)}\Omega_0^2 u_3^{(2)} + \frac{1}{2(1+\nu)}\Omega_1^2 u_3^{(1)} \\ + \frac{1}{2(1+\nu)}\Omega_2^2 u_3^{(0)} = 0,\end{aligned} \tag{4.115}$$

$$s_{11}^{(3)} = \frac{1}{1-\nu^2}\left(\frac{\partial u_1^{(2)}}{\partial \xi} + \nu e^{(3)} - \nu\zeta e^{(2)} + \nu\zeta^2 e^{(1)} - \nu\zeta^3 e^{(0)}\right) + \frac{\nu}{1-\nu} s_{33}^{(2)}, \tag{4.116}$$

$$s_{22}^{(3)} = \frac{1}{1-\nu^2}\left(\nu\frac{\partial u_1^{(2)}}{\partial \xi} + e^{(3)} - \zeta e^{(2)} + \zeta^2 e^{(1)} - \zeta^3 e^{(0)}\right) + \frac{\nu}{1-\nu} s_{33}^{(2)}, \tag{4.117}$$

$$\frac{\partial u_3^{(3)}}{\partial \zeta} = s_{33}^{(0)} - \nu\left(s_{11}^{(1)} + s_{22}^{(1)}\right), \tag{4.118}$$

$$\frac{\partial u_1^{(3)}}{\partial \zeta} = -\frac{\partial u_3^{(2)}}{\partial \xi} + 2(1+\nu) s_{31}^{(0)}, \tag{4.119}$$

$$\frac{\partial u_2^{(3)}}{\partial \zeta} = n u_3^{(2)} - n\zeta u_3^{(1)} + n\zeta^2 u_3^{(0)} + u_2^{(2)} - \zeta u_2^{(1)} + \zeta^2 u_2^{(0)} + 2(1+\nu) s_{32}^{(0)}, \tag{4.120}$$

$$s_{21}^{(3)} = \frac{1}{2(1+\nu)}\left(\zeta\frac{\partial u_2^{(3)}}{\partial \xi} - \zeta^2\frac{\partial u_2^{(2)}}{\partial \xi} + \zeta^3\frac{\partial u_2^{(1)}}{\partial \xi} - \zeta^4\frac{\partial u_2^{(0)}}{\partial \xi} + s^{(3)} - \zeta s^{(2)} + \zeta^2 s^{(1)} - \zeta^3 s^{(0)}\right), \tag{4.121}$$

with the boundary conditions at $\zeta = \pm 1$

$$s_{3i}^{(3)} = 0, \quad i = 1, 2, 3. \tag{4.122}$$

We begin by integrating (4.118)–(4.120) with respect to thickness variable ζ respectively, we get

$$\begin{aligned} u_3^{(3)} = &-\frac{\nu\zeta}{n}\frac{\partial^2 V^{(0)}}{\partial \xi^2} - \frac{\nu\zeta^2}{2(1-\nu)}Q_1 - \frac{\nu\zeta}{3(1-\nu)}P_3 - \frac{(3-\nu)\zeta - (1+\nu)\zeta^3}{6(1-\nu)}P_1 \\ &+\frac{1}{2}n\nu\zeta\Omega_0^2 V^{(0)} + W^{(3)}, \end{aligned} \tag{4.123}$$

$$u_1^{(3)} = \frac{6n(1-n^2)\zeta - n(1-n^2)(2-\nu)\zeta^3}{6(1-\nu)}\frac{\partial V^{(0)}}{\partial \xi} - \zeta\frac{\partial W^{(2)}}{\partial \xi}, \tag{4.124}$$

$$\begin{aligned} u_2^{(3)} = &-\frac{6n^2(1-n^2)\zeta - n^2(1-n^2)(2-\nu)\zeta^3}{6(1-\nu)}V^{(0)} + n\zeta W^{(2)} \\ &+\zeta V^{(2)} + V^{(3)}. \end{aligned} \tag{4.125}$$

Also, the third order terms of (4.18) gives

$$\begin{aligned} e^{(2)} = &-\frac{2\nu}{n}\zeta\frac{\partial^2 V^{(0)}}{\partial \xi^2} \\ &-\frac{n(1-n^2)\left(1-2\nu-(1+\nu)\zeta^2+n^2(11-10\nu-3\zeta^2)+n^4(6-(2-\nu)\zeta^2)\right)}{6(1-\nu)(1+n^2)}\zeta V^{(0)} \\ &-\frac{\nu n(1-n^2)}{2(1-\nu)}\zeta^2 V^{(1)} - (1-n^2)\zeta W^{(2)} + nV^{(3)} + W^{(3)} \end{aligned} \tag{4.126}$$

and

$$\begin{aligned} s^{(2)} = &-\frac{\nu}{n^2}\zeta\frac{\partial^3 V^{(0)}}{\partial \xi^3} \\ &+\frac{(1-n^2)\left(2+\nu-n^2(14+12n^2-5\nu-2(1+n^2)(2-\nu)\zeta^2)\right)}{6(1-\nu)(1+n^2)}\zeta\frac{\partial V^{(0)}}{\partial \xi} \\ &+\frac{2n^2-1}{n}\zeta\frac{\partial W^{(2)}}{\partial \xi} + \frac{\partial V^{(3)}}{\partial \xi}. \end{aligned} \tag{4.127}$$

Inserting (4.60), (4.102) and first, second, third order terms of $(4.18)_1$ into equation (4.87), we arrive at

$$\begin{aligned} s_{22}^{(2)} =& -\frac{(1-n^2)\nu\zeta}{n(1-\nu^2)}\frac{\partial^2 V^{(0)}}{\partial\xi^2} + \frac{\nu}{n(1-\nu^2)}\frac{\partial V^{(1)}}{\partial\xi} - \frac{\nu+2(1-\nu)\zeta^2}{2(1-\nu)(1-\nu^2)}Q_1 \\ &+\frac{3(2-\nu)\zeta-2(1-\nu)\zeta^3}{6(1-\nu)(1-\nu^2)}P_3 - \frac{3(1-2\nu)\zeta-7(1-\nu)\zeta^3}{6(1-\nu)(1-\nu^2)}P_1 \\ &-\frac{(1-n^2)\zeta}{1-\nu^2}W^{(2)} + \frac{1}{1-\nu^2}\left(W^{(3)}+nV^{(3)}\right) + \frac{n\nu\zeta}{2(1-\nu^2)}\Omega_0^2 V^{(0)}. \end{aligned} \tag{4.128}$$

Integrating the latter equation from -1 to 1 with respect to ζ and comparing it with (4.110), we get

$$\begin{aligned} \frac{1}{1-\nu^2}\left(W^{(3)}+nV^{(3)}\right) =& -\frac{\nu}{n(1-\nu^2)}\frac{\partial^2 V^{(1)}}{\partial\xi^2} + \frac{1}{3(1-\nu^2)}Q_3 + \frac{2+\nu}{6(1-\nu)(1-\nu^2)}Q_1 \\ &+\frac{1}{3(1-\nu^2)}P_5 + \frac{1}{3(1-\nu^2)}P_3 - \frac{n}{2(1+\nu)}\Omega_0^2 V^{(1)} + \frac{n(1+n^2)}{2(1+\nu)}\Omega_0^2 V^{(0)}. \end{aligned} \tag{4.129}$$

Now, insert the latter back into (4.128), we obtain

$$\begin{aligned} s_{22}^{(2)} =& -\frac{(1-n^2)\nu\zeta}{n(1-\nu^2)}\frac{\partial^2 V^{(0)}}{\partial\xi^2} + \frac{1}{3(1-\nu^2)}Q_3 + \frac{1-3\zeta^2}{3(1-\nu^2)}Q_1 + \frac{1}{3(1-\nu^2)}P_5 \\ &+ \frac{2(1-\nu)+3(2-\nu)\zeta-2(1-\nu)\zeta^3}{6(1-\nu)(1-\nu^2)}P_3 - \frac{3(1-2\nu)\zeta-7(1-\nu)\zeta^3}{6(1-\nu)(1-\nu^2)}P_1 \\ &- \frac{(1-n^2)\zeta}{1-\nu^2}W^{(2)} - \frac{n}{2(1+\nu)}\Omega_0^2 V^{(1)} + \frac{n(1+n^2)(1-\nu)+n\nu\zeta}{2(1-\nu^2)}\Omega_0^2 V^{(0)}. \end{aligned} \tag{4.130}$$

Inserting (4.130) into (4.107) and (4.108) respectively, we get

$$\begin{aligned} s_{32}^{(2)} =& \frac{(1-n^2)(1-\zeta^2)}{2(1-\nu^2)}\frac{\partial^2 V^{(0)}}{\partial\xi^2} + \frac{1-\zeta}{3(1-\nu^2)}Q_4 + \frac{1-4\zeta+3\zeta^3}{3(1-\nu^2)}Q_2 \\ &+ \frac{1-\zeta}{3(1-\nu^2)}P_6 + \frac{(11-8\nu)-12(1-\nu)\zeta-3\nu\zeta^2+(1-\nu)\zeta^4}{12(1-\nu)(1-\nu^2)}P_4 \\ &- \frac{(1-7\nu)+16(1-\nu)\zeta-(54-60\nu)\zeta^2+37(1-\nu)\zeta^4}{24(1-\nu)(1-\nu^2)}P_2 \\ &+ \frac{n(1-n^2)(1-\zeta^2)}{2(1-\nu^2)}W^{(2)} + \frac{(1+n^2)(1-\zeta)}{2(1+\nu)}\Omega_0^2 V^{(1)} \\ &- \frac{n^2(2-\nu)-(1-\nu)(3-2n^4)}{4(1-\nu^2)}\Omega_0^2 V^{(0)} \\ &- \frac{2(1-\nu)(2+n^2-n^4)\zeta-((1-\nu)+n^2(4-3\nu))\zeta^2}{4(1-\nu^2)}\Omega_0^2 V^{(0)} \end{aligned} \tag{4.131}$$

and

$$s_{33}^{(2)}=\frac{(1-n^2)\nu(1-\zeta^2)}{2n(1-\nu^2)}\frac{\partial^2V^{(0)}}{\partial\xi^2}-\frac{\zeta(1-\zeta^2)}{6(1-\nu^2)}Q_3+\frac{5\zeta(1-\zeta^2)}{6(1-\nu^2)}Q_1-\frac{1-\zeta^2}{6(1-\nu^2)}P_5$$
$$-\frac{(10-7\nu)-(16-13\nu)\zeta^2+6(1-\nu)\zeta^4}{12(1-\nu)(1-\nu^2)}P_3$$
$$-\frac{(1+5\nu)+(26-32\nu)\zeta^2-27(1-\nu)\zeta^4}{24(1-\nu)(1-\nu^2)}P_1+\frac{(1-n^2)(1-\zeta^2)}{2(1-\nu^2)}W^{(2)}$$
$$-\frac{n((2-\nu)+n^2(1-\nu))(1-\zeta^2)}{4(1-\nu^2)}\Omega_0^2V^{(0)}. \tag{4.132}$$

Now, the solvability of (4.114) and taking into account the boundary conditions (4.122) at $i=2$, we obtain

$$n\int_{-1}^{1}s_{22}^{(3)}(s)\,ds=-\frac{2}{n^2}\frac{\partial^4V^{(0)}}{\partial\xi^4}+\frac{2(1-n^2)(1-\nu+n^2(3+\nu))}{3(1-\nu^2)(1+n^2)}\frac{\partial^2V^{(0)}}{\partial\xi^2}$$
$$+\frac{4}{3(1-\nu^2)}Q_4+\frac{4}{3(1-\nu^2)}Q_2+\frac{4}{3(1-\nu^2)}P_6$$
$$+\frac{58-53\nu}{15(1-\nu)(1-\nu^2)}P_4+\frac{32-27\nu}{15(1-\nu)(1-\nu^2)}P_2+\frac{2n(1-n^2)}{3(1-\nu^2)}W^{(2)}$$
$$+\frac{1}{1+\nu}\Omega_0^2V^{(2)}+\frac{2(1+n^2)}{1+\nu}\Omega_0^2V^{(1)}+\frac{1}{1+\nu}\Omega_2^2V^{(0)}$$
$$+\frac{n^2(1-2\nu)+6(1-\nu)(2-n^4)}{3(1-\nu^2)}\Omega_0^2V^{(0)}. \tag{4.133}$$

Similarly, the solvability of (4.115) and taking into account the boundary conditions (4.122) at $i=3$, we obtain

$$\int_{-1}^{1}s_{22}^{(3)}(s)\,ds=\frac{4n(1-n^2)}{3(1-\nu^2)}\frac{\partial^2V^{(0)}}{\partial\xi^2}+\frac{2}{3(1-\nu^2)}Q_5+\frac{2}{3(1-\nu^2)}Q_3+\frac{2}{3(1-\nu^2)}P_7$$
$$+\frac{28-23\nu}{15(1-\nu)(1-\nu^2)}P_5+\frac{12-7\nu}{15(1-\nu)(1-\nu^2)}P_3+\frac{2n^2(1-n^2)}{3(1-\nu^2)}W^{(2)}$$
$$+\frac{1}{1+\nu}\Omega_0^2W^{(2)}+\frac{n(1+n^2)}{1+\nu}\Omega_0^2V^{(1)}-\frac{n}{1+\nu}\Omega_2^2V^{(0)}$$
$$+\frac{n(10-11\nu)-n^3(2-\nu)-6n^5(1-\nu)}{6(1-\nu^2)}\Omega_0^2V^{(0)}. \tag{4.134}$$

Comparison of (4.133) and (4.134), we arrive at

$$
\begin{aligned}
\frac{1+n^2}{1+\nu}\Omega_2^2 V^{(0)} = &-\frac{1}{1+\nu}\left(\frac{\partial^2 V^{(2)}}{\partial \xi^2} - n\frac{\partial U^{(2)}}{\partial \xi}\right) + \frac{(1-n^2)(4n^2-3+\nu)}{3(1-\nu^2)}\frac{\partial^2 V^{(0)}}{\partial \xi^2} \\
&- \frac{2}{3(1-\nu^2)}Q_6 - \frac{2}{1-\nu^2}Q_4 - \frac{4}{3(1-\nu^2)}Q_2 - \frac{2}{3(1-\nu^2)}P_8 \\
&- \frac{48-43\nu}{15(1-\nu)(1-\nu^2)}P_6 - \frac{68-58\nu}{15(1-\nu)(1-\nu^2)}P_4 - \frac{6-5\nu}{3(1-\nu)(1-\nu^2)}P_2 \\
&- \frac{2n(1-n^2)^2}{3(1-\nu^2)}W^{(2)} + \frac{2n^2(1-n^2)^2}{15(1-\nu^2)}V^{(0)} + \frac{n}{1+\nu}\Omega_0^2 W^{(2)} \\
&- \frac{1}{1+\nu}\Omega_0^2 V^{(2)} - \frac{2+n^2-n^4}{1+\nu}\Omega_0^2 V^{(1)} \\
&- \frac{24(1-\nu)+6n^6(1-\nu)-n^2(8-7\nu)-n^4(10-11\nu)}{6(1-\nu^2)}\Omega_0^2 V^{(0)}.
\end{aligned}
\tag{4.135}
$$

Finally, with taking into account the relations (4.63), (4.96) and (4.98) into latter equation, we arrive at

$$
\begin{aligned}
\frac{n^2(1+n^2)}{1+\nu}\Omega_2^2 V^{(0)} = &\, 2\frac{\partial^4 V^{(0)}}{\partial \xi^4} - \frac{2n^2(1-n^2)^2(2n^2+1-2\nu)}{3(1-\nu^2)(1+n^2)}\frac{\partial^2 V^{(0)}}{\partial \xi^2} \\
-&\frac{2n^4(1-n^2)^2\left(17(1-\nu)-n^2(9+\nu)+11n^4(1-\nu)+n^6(17-7\nu)\right)}{45(1-\nu)(1-\nu^2)(1+n^2)^2}V^{(0)}.
\end{aligned}
\tag{4.136}
$$

The sum of equation (4.81) and (4.136), and taking into account (4.30) and (4.43), we arrive at

$$
\begin{aligned}
&\frac{2\eta^2(1+\nu)}{n^2(1+n^2)}\frac{\partial^4 W^{(0)}}{\partial \xi^4} - \frac{2\eta^2(1-n^2)^2(2n^2+1-2\nu)}{3(1-\nu)(1+n^2)^2}\frac{\partial^2 W^{(0)}}{\partial \xi^2} + \frac{2n^2(1-n^2)^2}{3(1-\nu)(1+n^2)} \\
&\quad\times\left(1-\eta^2\frac{17(1-\nu)-n^2(9+\nu)+11n^4(1-\nu)+n^6(17-7\nu)}{15(1-\nu)(1+n^2)^2}\right)W^{(0)} \\
&\quad -\Omega^2 W^{(0)} = 0.
\end{aligned}
\tag{4.137}
$$

4.7. Discussion

Introducing $W = RW^{(0)}$ and setting $\omega^2 W = -\partial^2 W/\partial t^2$, and keeping terms of $O(\eta^2)$ the last equation in the earlier section can be written in the original variables as

$$
\gamma R^2 \frac{\partial^4 W}{\partial \alpha_1^4} - 4\delta\frac{h^2}{R^2}\frac{\partial^2 W}{\partial \alpha_1^2} + \frac{1-\nu}{2}\frac{h^2}{R^4}\left(\Omega_0^2 + \frac{h^2}{R^2}\Omega_2^2\right)W + \frac{\rho(1-\nu^2)}{E}\frac{\partial^2 W}{\partial t^2} = 0.
\tag{4.138}
$$

where

$$
\gamma = \frac{1-\nu^2}{n^2(1+n^2)}, \qquad \delta = \frac{(1-n^2)^2(2n^2+1-2\nu)}{12(1+n^2)^2},
\tag{4.139}
$$

with

$$\Omega_2^2 = -\frac{2n^2(1-n^2)^2\left(17(1-\nu) - n^2(9+\nu) + 11n^4(1-\nu) + n^6(17-7\nu)\right)}{45(1-\nu)^2(1+n^2)^3}, \tag{4.140}$$

and Ω_0^2 is given by (4.82). The graph of the coefficients given by (4.82), (4.139) and (4.140) versus Poisson's ratio ν are displayed in figures for $n = 2, 3, 4$.

The solutions of equation (4.138) can be substituted into the formulae in the previous two sections in order to derive the expressions for stress and strain components. As an example, we present the asymptotic estimations for the quantities e and s, see (4.13), at the mid-surface $\zeta = 0$. We stress once more that these quantities are key ones for a cylindrical shell. Starting from formulae (4.98) and (4.105), along with other intermediate relations, we readily deduce, at leading order,

$$e = \frac{\nu R^2}{n^2}\frac{\partial^2 W}{\partial \alpha_1^2} - \frac{(1-n^2)(2+\nu-n^2(2-5\nu))}{6(1-\nu)(1+n^2)}\frac{h^2}{R^2}W \tag{4.141}$$

and

$$s = \frac{2(1+\nu)}{n}\frac{R^2}{h}\left(\frac{R^2}{n^2}\frac{\partial^3 W}{\partial \alpha_1^3} + \frac{(1-n^2)(1-3n^2)}{6(1-\nu)(1+n^2)}\frac{h^2}{R^2}\frac{\partial W}{\partial \alpha_1}\right). \tag{4.142}$$

We also present the expressions in terms of W for the tangential stress resultants, see (2.72) defined as

$$T_i = \int_{-h}^{h}\left(1 + \frac{\alpha_3}{R_j}\right)s_{ii}\, d\alpha_3, \qquad S_{ij} = \int_{-h}^{h}\left(1 + \frac{\alpha_3}{R_j}\right)s_{ij}\, d\alpha_3, \tag{4.143}$$

where $i \neq j = 1, 2$. Let us set, according to (4.29)

$$s_{kl} = \frac{Eh}{R}\left(s_{kl}^{(0)} + \frac{h}{R}s_{kl}^{(1)} + \cdots\right), \quad k, l = 1, 2. \tag{4.144}$$

First, insert the latter into the first formula in (4.143) taking into consideration (4.67), (4.68) and (4.99), (4.100) together with (4.43) and (4.44), we obtain at leading order

$$T_1 = -\frac{2Eh}{R}\left(\frac{R^2}{n^2}\frac{\partial^2 W}{\partial \alpha_1^2} - \frac{2\nu n^2(1-n^2)}{3(1-\nu^2)(1+n^2)}\frac{h^2}{R^2}W\right) \tag{4.145}$$

and

$$T_2 = \frac{4Eh^3}{3(1-\nu^2)R^3}\frac{n^2(1-n^2)}{1+n^2}W. \tag{4.146}$$

Similarly, substituting equations (4.75) and (4.106) in the second equation of (4.143), we arrive at the following leading order expressions

$$S_{12} = \frac{2Eh}{3(1-\nu^2)}\left(\frac{3(1-\nu^2)R^2}{n^3}\frac{\partial^3 W}{\partial \alpha_1^3} + \frac{(1-n^2)(\nu - n^2(2+\nu))}{n(1+n^2)}\frac{h^2}{R^2}\frac{\partial W}{\partial \alpha_1}\right) \tag{4.147}$$

and

$$S_{21} = \frac{2Eh}{3(1-\nu^2)}\left(\frac{3(1-\nu^2)R^2}{n^3}\frac{\partial^3 W}{\partial \alpha_1^3} + \frac{(1-n^2)(1 - n^2(1+2\nu))}{n(1+n^2)}\frac{h^2}{R^2}\frac{\partial W}{\partial \alpha_1}\right). \tag{4.148}$$

It is important to compare the formulae above in this section with the prediction of the 2D Sanders-Koiter theory in [44]. First of all, we note that all coefficients in the equations of motion (4.138) apart from the correction to the cut-off frequency Ω_2^2, which is denoted in the paper [44] by Ω_1^2, see formula (3.6) there, are identical to those predicted by Sanders-Koiter theory. At the same time, the value Ω_2^2 in (4.140) is the same as in the plane problem studied in previous chapter. For the quantities e and s, we start from the refined formulae (5.1) and (5.2) derived in [44] for mid-surface displacements in terms of the notation of the current chapter they can be rewritten as

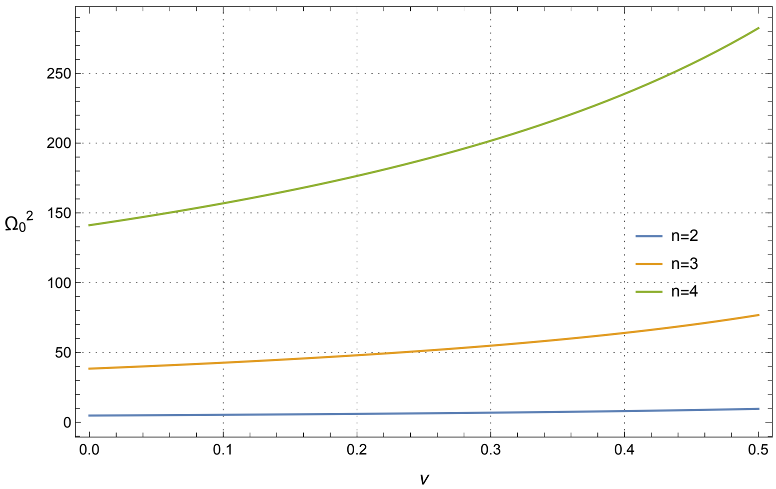

Figure 4.3. *The dimensionless frequency* Ω_0^2 *given by* (4.82) *versus the Poisson's ratio* ν *for* $n = 2, 3, 4$.

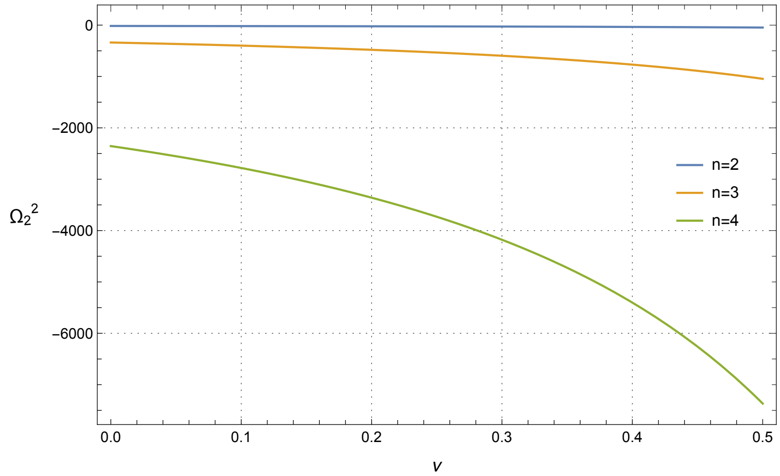

Figure 4.4. *The dimensionless frequency* Ω_2^2 *given by* (4.140) *versus the Poisson's ratio* ν *for* $n = 2, 3, 4$.

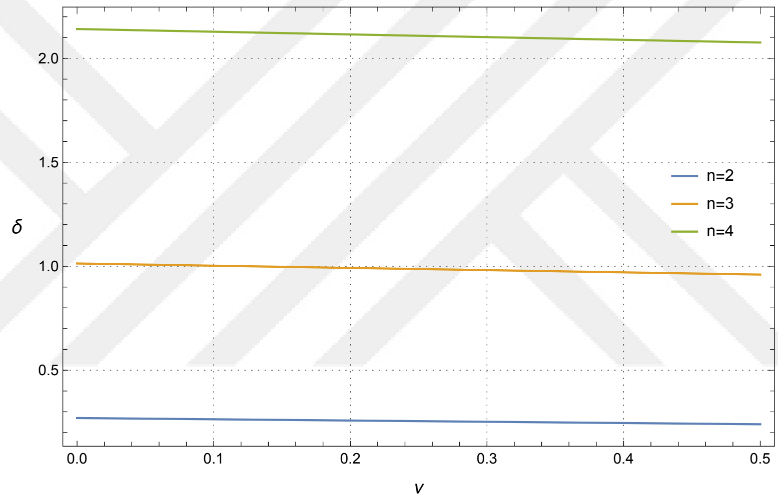

Figure 4.5. *The coefficient* δ *given by* (4.139) *versus the Poisson's ratio* ν *for* $n = 2, 3, 4$.

$$u_2 + \frac{1}{n} u_3 = \frac{\nu R^2}{n^3} \frac{\partial^2 u_3}{\partial \alpha_1^2} + \frac{2n(1-n^2)}{3(1+n^2)} \frac{h^2}{R^2} u_3, \tag{4.149}$$

$$u_1 + \frac{R}{n^2} \frac{\partial u_3}{\partial \alpha_1} = \frac{R}{(1-\nu)n^2} \left(\frac{(\nu(1+\nu)-2) R^2}{n^2} \frac{\partial^3 u_3}{\partial \alpha_1^3} - \frac{(1-n^2)(1+\nu-n^2(5+\nu))}{3(1+n^2)} \frac{h^2}{R^2} \frac{\partial u_3}{\partial \alpha_1} \right). \tag{4.150}$$

These relations are similar but not identical to the formulae (4.98) and (4.105). Inserting (4.149) and (4.150) into (4.13), we have

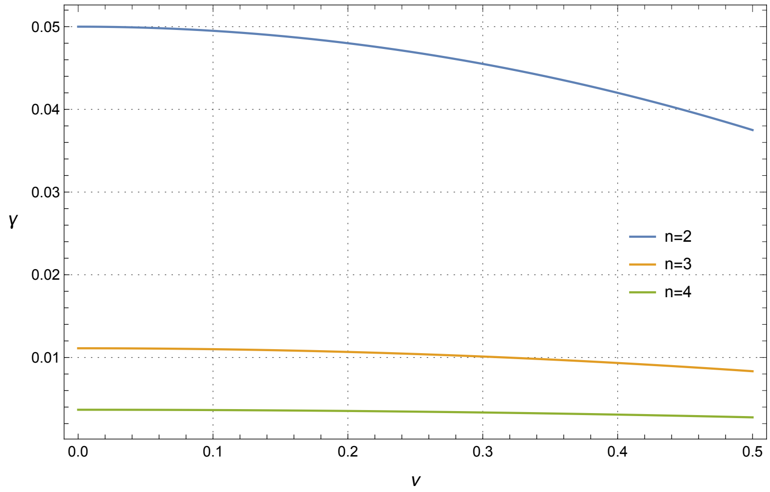

Figure 4.6. *The coefficient γ given by* (4.139) *versus the Poisson's ratio ν for $n = 2, 3, 4$.*

$$e_M = \frac{\nu R^2}{n^2}\frac{\partial^2 W}{\partial \alpha_1^2} + \frac{2n^2(1-n^2)}{3(1+n^2)}\frac{h^2}{R^2}W, \tag{4.151}$$

$$s_M = \frac{2(1+\nu)}{n}\frac{R^2}{h}\left(\frac{R^2}{n^2}\frac{\partial^3 W}{\partial \alpha_1^3} + \frac{(1-n^2)(1-3n^2)}{6(1-\nu)(1+n^2)}\frac{h^2}{R^2}\frac{\partial W}{\partial \alpha_1}\right). \tag{4.152}$$

Thus, the coefficients in formula (4.152) coincide with that given by (4.142) and (4.151) only for the highest order derivatives of (4.141) and the last term in the right hand side of this formula is identical to that in [44]. Now, substitute (4.149), (4.150) into the traditional expressions for tangential stress couples, e.g. see [24, 42], given by

$$T_1 = \frac{2Eh}{1-\nu^2}\left(\frac{\partial u_1}{\partial \alpha_1} + \frac{\nu e}{R}\right), \tag{4.153}$$

$$T_2 = \frac{2Eh}{1-\nu^2}\left(\frac{e}{R} + \nu\frac{\partial u_1}{\partial \alpha_1}\right) \tag{4.154}$$

and

$$S_{12} = \frac{Eh^2}{(1+\nu)R^2}\left[s + \frac{2}{3}h\left(\frac{\partial u_2}{\partial \alpha_1} + n\frac{\partial u_3}{\partial \alpha_1}\right)\right], \tag{4.155}$$

$$S_{21} = \frac{Eh^2}{(1+\nu)R^2}s. \tag{4.156}$$

As a result, the formula for T_i are identical to (4.145) and (4.146) while for S_{ij} these coincide with (4.147), (4.148) only with respect to the coefficients at the highest order derivatives taking the form

$$S_{12} = \frac{2Eh}{n}\left(\frac{R^2}{n^2}\frac{\partial^3 W}{\partial \alpha_1^3} - \frac{(1-n^2)(1-3\nu+n^2(5+\nu))}{6(1-\nu^2)(1+n^2)}\frac{h^2}{R^2}\frac{\partial W}{\partial \alpha_1}\right), \tag{4.157}$$

$$S_{21} = \frac{2Eh}{n}\left(\frac{R^2}{n^2}\frac{\partial^3 W}{\partial \alpha_1^3} + \frac{(1-n^2)(1-3n^2)}{6(1-\nu)(1+n^2)}\frac{h^2}{R^2}\frac{\partial W}{\partial \alpha_1}\right). \tag{4.158}$$

It is worth noting that the deviation between two above mentioned sets of formulae occurs despite the same expression for the tangential shear s given by (4.142). This happens due to the presence of the ζ^2 term in (4.106), see also (4.143) and (4.144). Thus, the coincidence of the tangential shear deformations along the mid-surface does not guarantee asymptotic correct expressions for the tangential shear stress resultants calculated from the classical 2D shell theory.

5. ASYMPTOTIC DERIVATION OF 2D DYNAMIC EQUATIONS OF MOTION FOR INHOMOGENEOUS ELASTIC SHELL

In this chapter, the low-frequency vibration of a thin-walled functionally graded cylinder, that is characterized by a continuous through thickness variation of material properties through-thickness, is considered within the plane strain framework. The procedure for calculating a 1D approximate equation of motion from dynamic equations at the mid-surface is similar to that described in previous chapters, but the mechanical parameters vary across the thickness. The effect of the transverse inhomogenety manifests in an explicit manner through the constant coefficients in the aforementioned equation. A simple explicit formula for natural frequencies emerging from the derived equation of motion is validated by comparison with previous numerical predictions. Also useful expansions for the variation of all stress components across the thickness are obtained.

5.1. Statement of the Problem

In this section, we consider the problem discussed in chapter (3) for an FGM thin-walled cylinder. Let us consider a thin elastic cylinder made of FGM with having thickness of $2h$ for which the geometrical parameter $\eta = h/R$ is small compared to unity, where R is the radius of curvature of the mid-surface of the shell. The mechanical parameters are considered as a function of thickness variable α_3 $(-h \le \alpha_3 \le h)$, i.e., $E = E(\alpha_3)$, $\nu = \nu(\alpha_3)$, and $\rho = \rho(\alpha_3)$, where E, ν and ρ denotes Young's modulus, Poisson's ratio and density, respectively.
We define the curvilinear coordinates α_2 and α_3 as

$$\alpha_2 = R\theta, \qquad \alpha_3 = h\zeta, \tag{5.1}$$

where $\theta(0 \le \theta < 2\pi)$ is the angular coordinate and $\zeta(-1 \le \zeta \le 1)$ is the transverse coordinate, while the coordinate α_1 along the longitudinal axis of the cylinder is not shown in figure (5.1).

The 3D equations of motion can be reduced to 2D plane strain equations of motion as, see section (2.4),

$$\frac{R}{R+\alpha_3}\frac{\partial\sigma_{22}}{\partial\alpha_2} + \frac{\partial\sigma_{32}}{\partial\alpha_3} + \frac{2}{R+\alpha_3}\sigma_{32} - \rho\frac{\partial^2 v_2}{\partial t^2} = 0, \tag{5.2}$$

$$\frac{R}{R+\alpha_3}\frac{\partial\sigma_{32}}{\partial\alpha_2} + \frac{\partial\sigma_{33}}{\partial\alpha_3} - \frac{1}{R+\alpha_3}\sigma_{22} + \frac{1}{R+\alpha_3}\sigma_{33} - \rho\frac{\partial^2 v_3}{\partial t^2} = 0, \tag{5.3}$$

with

$$\sigma_{22} = \frac{E}{1-\nu^2}\left(\frac{R}{R+\alpha_3}\frac{\partial v_2}{\partial \alpha_2} + \frac{1}{R+\alpha_3}v_3\right) + \frac{\nu}{1-\nu}\sigma_{33}, \tag{5.4}$$

$$E\frac{\partial v_3}{\partial \alpha_3} = (1-\nu^2)\sigma_{33} - \nu(1+\nu)\sigma_{22}, \tag{5.5}$$

$$\sigma_{32} = \frac{E}{2(1+\nu)}\left(\frac{R}{R+\alpha_3}\frac{\partial v_3}{\partial \alpha_2} + \frac{\partial v_2}{\partial \alpha_3} - \frac{1}{R+\alpha_3}v_2\right), \tag{5.6}$$

where σ_{ij} are the stress components, v_i are displacements with $i, j = 2, 3$, and t denotes the time variable.

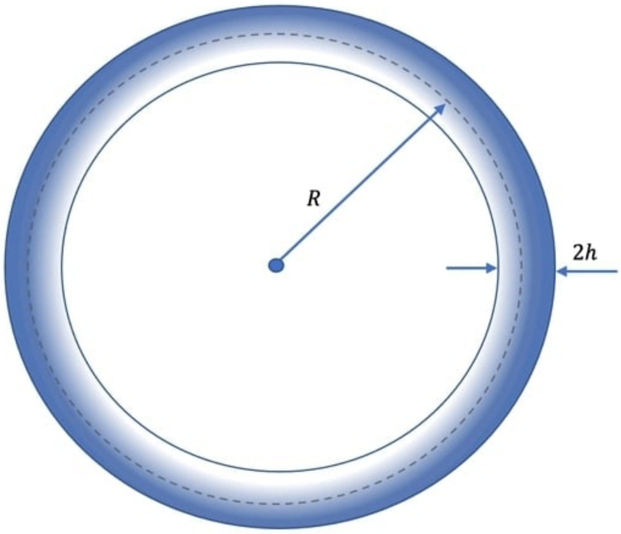

Figure 5.1. *A thin-Walled FGM cylinder.*

The above equations are same as we derived for the plane strain problem for a homogeneous thin-walled elastic cylinder, see chapter (3).
For purposes of simplicity, we consider traction-free boundary conditions along the faces $\alpha_3 = \pm h$, that is

$$\sigma_{3i} = 0. \tag{5.7}$$

The main focus of this chapter is to reduce the formulated $2D$ problem to a $1D$ one over the low-frequency range defined through the dimensionless time variable τ as

$$\tau = \frac{\eta c_{20}}{R}t, \qquad c_{20} = \sqrt{\frac{E_0}{2(1+\nu_0)\rho_0}}, \tag{5.8}$$

where $E_0 = E(0)$, $\nu_0 = \nu(0)$ and $\rho_0 = \rho(0)$.

Let us write equations (5.2) – (5.6) in the nondimensional coordinates

$$\frac{1}{1+\eta\zeta}\frac{\partial\sigma_{22}}{\partial\theta} + \frac{1}{\eta}\frac{\partial\sigma_{32}}{\partial\zeta} + \frac{2}{1+\eta\zeta}\sigma_{32} - \rho\frac{\eta^2 c_{20}^2}{R}\frac{\partial^2 v_2}{\partial\tau^2} = 0, \tag{5.9}$$

$$\frac{1}{1+\eta\zeta}\frac{\partial\sigma_{32}}{\partial\theta} + \frac{1}{\eta}\frac{\partial\sigma_{33}}{\partial\zeta} - \frac{1}{1+\eta\zeta}\sigma_{22} + \frac{1}{1+\eta\zeta}\sigma_{33} - \rho\frac{\eta^2 c_{20}^2}{R}\frac{\partial^2 v_2}{\partial\tau^2} = 0, \tag{5.10}$$

$$\sigma_{22} = \frac{E}{1-\nu^2}\frac{1}{R}\left(\frac{1}{1+\eta\zeta}\frac{\partial v_2}{\partial\theta} + \frac{1}{1+\eta\zeta}v_3\right) + \frac{\nu}{1-\nu}\sigma_{33}, \tag{5.11}$$

$$\frac{E}{R\eta}\frac{\partial v_3}{\partial\zeta} = (1-\nu^2)\sigma_{33} - \nu(1+\nu)\sigma_{22}, \tag{5.12}$$

$$\sigma_{32} = \frac{E}{2(1+\nu)}\frac{1}{R}\left(\frac{1}{1+\eta\zeta}\frac{\partial v_3}{\partial\theta} + \frac{1}{\eta}\frac{\partial v_2}{\partial\zeta} - \frac{1}{1+\eta\zeta}v_2\right), \tag{5.13}$$

with the boundary conditions at $\zeta = \pm 1$

$$\sigma_{3i} = 0, \quad i = 2, 3. \tag{5.14}$$

5.2. Asymptotic Scaling

Let us introduce dimensionless stresses and displacements by

$$\sigma_{11} = E_0\eta\sigma_{11}^*, \qquad \sigma_{22} = E_0\eta\sigma_{22}^*,$$
$$\sigma_{33} = E_0\eta^2\sigma_{33}^*, \qquad \sigma_{32} = E_0\eta^2\sigma_{32}^*, \tag{5.15}$$

and

$$v_2 = R v_2^*, \qquad v_3 = R v_3^*, \tag{5.16}$$

with all starred quantities to be of order unity. Above scaling is motivated by that for a homogeneous cylinder in section (3.2) of chapter (3).

Now, insert the scaling given by formulae (5.15) and (5.16) into the equations of motion (5.9)–(5.13), yields

$$\frac{1}{1+\eta\zeta}\frac{\partial\sigma_{22}^*}{\partial\theta}+\frac{\partial\sigma_{32}^*}{\partial\zeta}+\frac{2\eta}{1+\eta\zeta}\sigma_{32}^*-\rho_*\frac{\eta}{2(1+\nu_0)}\frac{\partial^2 v_2^*}{\partial\tau^2}=0, \tag{5.17}$$

$$\frac{\eta}{1+\eta\zeta}\frac{\partial\sigma_{32}^*}{\partial\theta}+\frac{\partial\sigma_{33}^*}{\partial\zeta}-\frac{1}{1+\eta\zeta}\sigma_{22}^*+\frac{\eta}{1+\eta\zeta}\sigma_{33}^*-\rho_*\frac{\eta}{2(1+\nu_0)}\frac{\partial^2 v_3^*}{\partial\tau^2}=0, \tag{5.18}$$

$$\eta\sigma_{22}^*=\frac{E_*}{1-\nu^2}\left(\frac{1}{1+\eta\zeta}\frac{\partial v_2^*}{\partial\theta}+\frac{1}{1+\eta\zeta}v_3^*\right)+\frac{\nu}{1-\nu}\eta^2\sigma_{33}^*, \tag{5.19}$$

$$E_*\frac{\partial v_3^*}{\partial\zeta}=(1-\nu^2)\eta^3\sigma_{33}^*-\nu(1+\nu)\eta^2\sigma_{22}^*, \tag{5.20}$$

$$\eta^3\sigma_{32}^*=\frac{E_*}{2(1+\nu)}\left(\frac{\eta}{1+\eta\zeta}\frac{\partial v_3^*}{\partial\theta}+\frac{\partial v_2^*}{\partial\zeta}-\frac{\eta}{1+\eta\zeta}v_2^*\right), \tag{5.21}$$

where

$$E_*(\zeta)=\frac{E(\zeta)}{E_0},\quad \rho_*(\zeta)=\frac{\rho(\zeta)}{\rho_0}. \tag{5.22}$$

The boundary conditions become

$$\sigma_{3i}^*=0,\quad \zeta=\pm 1. \tag{5.23}$$

In addition, we suppose nearly inextensible condition along the circumferential direction, that is,

$$\frac{\partial v_2^*}{\partial\theta}+v_3^*=\eta e_*. \tag{5.24}$$

It arises from mid-surface inextensibility in the semi membrane shell theory. Such asymptotic behaviour is also observed within the framework of plane strain problem for a homogeneous cylinder, see chapter (3). Taking into account the last formula, equation (5.19) may be rewritten as

$$\sigma_{22}^*=\frac{E_*}{1-\nu^2}\frac{e_*}{1+\eta\zeta}+\frac{\nu}{1-\nu}\eta\sigma_{33}^*. \tag{5.25}$$

In order to derive 1D robust approximation of the original 2D problem, we expand all the starred quantities in asymptotic series as

$$f_i^*=f_i^{(0)}+\eta f_i^{(1)}+\eta^2 f_i^{(2)}+\cdots. \tag{5.26}$$

5.3. The Leading-Order Approximation

Let us substitute equation (5.26) into equations (5.17) – (5.21), and (5.24) with discarding $O(\eta)$ terms, yields

$$\frac{\partial \sigma_{22}^{(0)}}{\partial \theta} + \frac{\partial \sigma_{32}^{(0)}}{\partial \zeta} = 0, \tag{5.27}$$

$$\frac{\partial \sigma_{33}^{(0)}}{\partial \zeta} - \sigma_{22}^{(0)} = 0, \tag{5.28}$$

$$\sigma_{22}^{(0)} = \frac{E_*}{1-\nu^2} e^{(0)}, \tag{5.29}$$

$$\frac{\partial v_3^{(0)}}{\partial \zeta} = \frac{\partial v_2^{(0)}}{\partial \zeta} = 0 \tag{5.30}$$

and

$$\frac{\partial v_2^{(0)}}{\partial \theta} + v_3^{(0)} = 0, \tag{5.31}$$

with the boundary conditions at $\zeta = \pm 1$

$$\sigma_{3i}^{(0)} = 0, \qquad i = 2, 3. \tag{5.32}$$

Integrating equations (5.30) with respect to the thickness variable ζ, it is straightforward that

$$v_2^{(0)} = V_2^{(0)}(\theta, \tau), \qquad v_3^{(0)} = V_3^{(0)}(\theta, \tau), \tag{5.33}$$

where the functions $V_2^{(0)}$ and $V_3^{(0)}$ are related by equation (5.31) as

$$V_3^{(0)} = -\frac{\partial V_2^{(0)}}{\partial \theta}. \tag{5.34}$$

As might be expected, the last formula corresponds to circumferential inextensibility of the midplane in the semi-membrane theory for thin elastic shells.

Next, integrating equation (5.28) with respect to ζ and employing the boundary condition (5.32) on the upper boundary we obtain

$$\frac{\partial \sigma_{33}^{(0)}}{\partial \zeta} = \sigma_{22}^{(0)},$$

$$\sigma_{33}^{(0)} = \int_0^{\zeta} \sigma_{22}^{(0)}(s)\, ds + s_{33}^{(0)},$$

$$\sigma_{33}^{(0)} = -\int_{\zeta}^{1} \sigma_{22}^{(0)}(s)\, ds. \tag{5.35}$$

Thus, as it follows from the boundary condition (5.32) at $\zeta = -1$, we have

$$\int_{-1}^{1} \sigma_{22}^{(0)}(s)\, ds = 0. \tag{5.36}$$

Solving equation (5.27) for $\sigma_{32}^{(0)}$, we get below relations

$$\frac{\partial \sigma_{32}^{(0)}}{\partial \zeta} = -\frac{\partial \sigma_{22}^{(0)}}{\partial \theta},$$

$$\sigma_{32}^{(0)} = -\int_0^{\zeta} \sigma_{22}^{(0)}\, ds + s_{32}^{(0)}.$$

Applying the boundary condition (5.32) on the upper and lower surfaces $\zeta = \pm 1$ in the latter equation, we obtain, respectively,

$$\sigma_{32}^{(0)} = \int_{\zeta}^{1} \frac{\partial \sigma_{22}^{(0)}(s)}{\partial \theta}\, ds \tag{5.37}$$

and

$$\int_{-1}^{1} \frac{\partial \sigma_{22}^{(0)}(s)}{\partial \theta}\, ds = 0. \tag{5.38}$$

The relations above do not form a close set of equations and also do not incorporate inertial terms, this motivates to proceed to the first order approximation.

5.4. First-Order Approximation

At first order, let us substitute equation (5.26) into equations (5.17) – (5.21), and (5.24), with Keeping $O(\eta)$ terms, yields

$$\frac{\partial \sigma_{22}^{(1)}}{\partial \theta} - \zeta \frac{\partial \sigma_{22}^{(0)}}{\partial \theta} + \frac{\partial \sigma_{32}^{(1)}}{\partial \zeta} + 2\sigma_{32}^{(0)} - \frac{\rho_*}{2(1+\nu_0)} \frac{\partial^2 v_2^{(0)}}{\partial \tau^2} = 0, \tag{5.39}$$

$$\frac{\partial \sigma_{32}^{(0)}}{\partial \theta} + \frac{\partial \sigma_{33}^{(1)}}{\partial \zeta} - \sigma_{22}^{(1)} + \zeta \sigma_{22}^{(0)} + \sigma_{33}^{(0)} - \frac{\rho_*}{2(1+\nu_0)} \frac{\partial^2 v_3^{(0)}}{\partial \tau^2} = 0, \tag{5.40}$$

$$\sigma_{22}^{(1)} = \frac{E_*}{1-\nu^2}\left(e^{(1)} - \zeta e^{(0)}\right) + \frac{\nu}{1-\nu}\sigma_{33}^{(0)}, \tag{5.41}$$

$$\frac{\partial v_3^{(1)}}{\partial \zeta} = 0, \tag{5.42}$$

$$\frac{\partial v_3^{(0)}}{\partial \theta} + \frac{\partial v_2^{(1)}}{\partial \zeta} - v_2^{(0)} = 0 \tag{5.43}$$

and

$$e^{(0)} = \frac{\partial v_2^{(1)}}{\partial \theta} + v_3^{(1)}, \tag{5.44}$$

with the boundary conditions at $\zeta = \pm 1$

$$\sigma_{3i}^{(1)} = 0, \qquad i = 2, 3. \tag{5.45}$$

On integrating (5.42) and (5.43) along the thickness and using the functions (5.33), we obtain

$$v_3^{(1)} = V_3^{(1)}(\theta, \tau), \tag{5.46}$$

$$v_2^{(1)} = \zeta P_0 + V_2^{(1)}(\theta, \tau), \tag{5.47}$$

where, here and henceforth, operator notation (3.55) is valid.

On employing equations (5.46) and (5.47) in the inextensible condition (5.44) we obtain, at first order,

$$e^{(0)} = \zeta P_1 + \frac{\partial V_2^{(1)}}{\partial \theta} + V_3^{(1)}. \tag{5.48}$$

It is worth nothing that the latter feature is usually outside of the limits of the traditional shell theory.

Making use of the last equation in (5.29), we have

$$\sigma_{22}^{(0)} = \frac{E_*}{1-\nu^2}\left(\zeta P_1 + \frac{\partial V_2^{(1)}}{\partial\theta} + V_3^{(1)}\right). \tag{5.49}$$

Now, integrating this equation from -1 to 1 and on employing (5.36), we get

$$\frac{\partial V_2^{(1)}}{\partial\theta} + V_3^{(1)} = -\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}P_1, \tag{5.50}$$

from the last two equations, we conclude

$$\sigma_{22}^{(0)} = -\frac{1}{1-\nu^2}\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}E_*(\zeta) - \zeta E_*(\zeta)\right)P_1, \tag{5.51}$$

where we introduce the notation

$$\mathcal{E}_i^* = \int_{-1}^{1} \zeta^i \frac{E_*(\zeta)}{1-\nu^2(\zeta)}\, d\zeta, \quad i = 0, 1, 2, \cdots. \tag{5.52}$$

Comparing equation (5.29) with (5.51), it is easy to see that (5.48) may conveniently be rewritten as

$$e^{(0)} = -\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*} - \zeta\right)P_1. \tag{5.53}$$

We may now obtain the first order displacement component $V_3^{(1)}$ using (5.46), (5.47) and (5.53) in the inextensible condition (5.24) to give

$$V_3^{(1)} + \frac{\partial V_2^{(1)}}{\partial\theta} = -\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}P_1. \tag{5.54}$$

It is now a straightforward matter to obtain leading order stress components $\sigma_{32}^{(0)}$ and $\sigma_{33}^{(0)}$, inserting equation (5.51) in equations (5.37) and (5.35), respectively, we get

$$\sigma_{32}^{(0)} = -\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{\zeta}^{1} \frac{E_*(s)}{1-\nu^2(s)}\,ds - \int_{\zeta}^{1} s\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_2, \tag{5.55}$$

$$\sigma_{33}^{(0)} = \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{\zeta}^{1} \frac{E_*(s)}{1-\nu^2(s)}\,ds - \int_{\zeta}^{1} s\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_1. \tag{5.56}$$

Integrating (5.40) along the thickness and using (5.51), (5.55) and (5.56), gives

$$\begin{aligned}
\sigma_{33}^{(1)} &= \int_{0}^{\zeta} \sigma_{22}^{(1)}\,ds + \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{0}^{\zeta}\int_{s}^{1} \frac{E_*(z)}{1-\nu^2(z)}\,dzds - \int_{0}^{\zeta}\int_{s}^{1} \frac{zE_*(z)}{1-\nu^2(z)}\,dzds\right)P_3 \\
&- \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{0}^{\zeta}\int_{s}^{1} \frac{E_*(z)}{1-\nu^2(z)}\,dzds - \int_{0}^{\zeta}\int_{s}^{1} \frac{zE_*(z)}{1-\nu^2(z)}\,dzds\right)P_1 \\
&+ \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{0}^{\zeta} \frac{sE_*(s)}{1-\nu^2(s)}\,ds - \int_{0}^{\zeta} \frac{s^2E_*(s)}{1-\nu^2(s)}\,ds\right)P_1 + \frac{1}{2(1+\nu_0)}\int_{0}^{\zeta} \rho_*(s)\,ds\frac{\partial^2 V_3^{(0)}}{\partial\tau^2} + s_{33}^{(1)},
\end{aligned} \tag{5.57}$$

on employing the boundary condition at $\zeta = 1$ we get the constant of integration, and reducing the double integral to single one, we get

$$\begin{aligned}
\sigma_{33}^{(1)} &= -\int_{\zeta}^{1} \sigma_{22}^{(1)}\,ds - \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{\zeta}^{1} (s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds - \int_{\zeta}^{1} s(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_3 \\
&+ \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{\zeta}^{1} (s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds - \int_{\zeta}^{1} s(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_1 \\
&- \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{\zeta}^{1} s\frac{E_*(s)}{1-\nu^2(s)}\,ds - \int_{\zeta}^{1} s^2\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_1 - \frac{1}{2(1+\nu_0)}\frac{\partial^2 V_3^{(0)}}{\partial\tau^2}\int_{\zeta}^{1} \rho_*(s)\,ds.
\end{aligned} \tag{5.58}$$

Similarly, applying the boundary condition at $\zeta = -1$, we obtain

$$\int_{-1}^{1}\sigma_{22}^{(1)}\,ds=-\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{-1}^{1}(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds-\int_{-1}^{1}s(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_3$$
$$+\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{-1}^{1}(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds-\int_{-1}^{1}s(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_1$$
$$-\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{-1}^{1}s\frac{E_*(s)}{1-\nu^2(s)}\,ds-\int_{-1}^{1}s^2\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_1-\frac{1}{2(1+\nu_0)}\frac{\partial^2V_3^{(0)}}{\partial\tau^2}\int_{-1}^{1}\rho_*(s)\,ds. \qquad (5.59)$$

Simplifying the last equation and employing the notation (5.52), we arrive at

$$\int_{-1}^{1}\sigma_{22}^{(1)}\,ds=\left(\mathcal{E}_2^*-\frac{\mathcal{E}_1^{*2}}{\mathcal{E}_0^*}\right)P_3-\frac{1}{2(1+\nu_0)}\frac{\partial^2V_3^{(0)}}{\partial\tau^2}\int_{-1}^{1}\rho_*(s)\,ds, \qquad (5.60)$$

defining the resultant for the circumferential normal stress, which is zero at leading order according to (5.36).

Integrating (5.39) along the thickness and using (5.51), (5.55), gives

$$\sigma_{32}^{(1)}=-\int_{0}^{\zeta}\frac{\partial\sigma_{22}^{(1)}}{\partial\theta}\,ds-\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{0}^{\zeta}\frac{sE_*(s)}{1-\nu^2(s)}\,ds-\int_{0}^{\zeta}\frac{s^2E_*(s)}{1-\nu^2(s)}\,ds\right)P_2$$
$$+2\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{0}^{\zeta}\int_{s}^{1}\frac{E_*(z)}{1-\nu^2(z)}\,dzds-\int_{0}^{\zeta}\int_{s}^{1}\frac{zE_*(z)}{1-\nu^2(z)}\,dzds\right)P_2+\frac{1}{2(1+\nu_0)}\frac{\partial^2V_2^{(0)}}{\partial\tau^2}\times$$
$$\times\int_{0}^{\zeta}\rho_*(s)\,ds+s_{32}^{(1)}, \qquad (5.61)$$

and on employing the boundary condition at $\zeta=1$ we get the constant of integration, and reducing double integral to single one, we get

$$\sigma_{32}^{(1)}=\int_{\zeta}^{1}\frac{\partial\sigma_{22}^{(1)}}{\partial\theta}\,ds+\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{\zeta}^{1}s\frac{E_*(s)}{1-\nu^2(s)}\,ds-\int_{\zeta}^{1}s^2\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_2$$
$$-2\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{\zeta}^{1}(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds-\int_{\zeta}^{1}s(s-\zeta)\frac{E_*(s)}{1-\nu^2(s)}\,ds\right)P_2$$
$$-\frac{1}{2(1+\nu_0)}\frac{\partial^2V_2^{(0)}}{\partial\tau^2}\int_{\zeta}^{1}\rho_*(s)\,ds. \qquad (5.62)$$

Here, applying the boundary condition at $\zeta = -1$, we obtain

$$\int_{-1}^{1} \frac{\partial \sigma_{22}^{(1)}}{\partial \theta}\, ds = -\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*} \int_{-1}^{1} s \frac{E_*(s)}{1-\nu^2(s)}\, ds - \int_{-1}^{1} s^2 \frac{E_*(s)}{1-\nu^2(s)}\, ds \right) P_2$$
$$+ 2\left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*} \int_{-1}^{1} (s-\zeta) \frac{E_*(s)}{1-\nu^2(s)}\, ds - \int_{-1}^{1} s(s-\zeta) \frac{E_*(s)}{1-\nu^2(s)}\, ds \right) P_2$$
$$+ \frac{1}{2(1+\nu_0)} \frac{\partial^2 V_2^{(0)}}{\partial \tau^2} \int_{-1}^{1} \rho_*(s)\, ds. \tag{5.63}$$

Simplifying the last equation and employing the notation (5.52), we arrive at

$$\int_{-1}^{1} \frac{\partial \sigma_{22}^{(1)}}{\partial \theta}\, ds = -\left(\mathcal{E}_2^* - \frac{\mathcal{E}_1^{*2}}{\mathcal{E}_0^*} \right) P_2 + \frac{1}{2(1+\nu_0)} \frac{\partial^2 V_3^{(0)}}{\partial \tau^2} \int_{-1}^{1} \rho_*(s)\, ds. \tag{5.64}$$

Now, comparing equations (5.60) and (5.64) , we conclude that

$$\frac{\mathcal{R}_*}{2(1+\nu_0)} \left(\frac{\partial^2 V_2^{(0)}}{\partial \tau^2} - \frac{\partial^4 V_2^{(0)}}{\partial \theta^2 \partial \tau^2} \right) = -\left(\frac{\mathcal{E}_1^{*2}}{\mathcal{E}_0^*} - \mathcal{E}_2^* \right) (P_4 + P_2)\,,$$

using the adapted notation (3.55), we arrive at

$$\frac{\mathcal{E}_1^{*2} - \mathcal{E}_0^* \mathcal{E}_2^*}{\mathcal{R}_* \mathcal{E}_0^*} \left(\frac{\partial^6 V_2^{(0)}}{\partial \theta^6} + 2\frac{\partial^4 V_2^{(0)}}{\partial \theta^4} + \frac{\partial^2 V_2^{(0)}}{\partial \theta^2} \right) + \frac{1}{2(1+\nu_0)} \left(\frac{\partial^2 V_2^{(0)}}{\partial \tau^2} - \frac{\partial^4 V_2^{(0)}}{\partial \theta^2 \partial \tau^2} \right) = 0, \tag{5.65}$$

where

$$\mathcal{R}_* = \int_{-1}^{1} \rho_*(\zeta)\, d\zeta. \tag{5.66}$$

This 1D equation governs low-frequency plane strain motion of a thin-walled FGM cylinder.

5.5. Discussion

Writing equation (5.65) in terms of the original variables given by (5.1), (5.8), we arrive at

$$\rho_0 \frac{\mathcal{E}_1^2 - \mathcal{E}_0 \mathcal{E}_2}{\mathcal{R} \mathcal{E}_0 E_0} \left(R^4 \frac{\partial^6 v_2}{\partial \alpha_2^6} + 2R^2 \frac{\partial^4 v_2}{\partial \alpha_2^4} + \frac{\partial^2 v_2}{\partial \alpha_2^2} \right) + \frac{1}{2(1+\nu_0)} \left(\frac{R^2}{h^2} \frac{2(1+\nu_0)\rho_0}{E_0} \frac{\partial^2 v_2}{\partial t^2} - \right.$$
$$\left. - \frac{R^4}{h^2} \frac{2(1+\nu_0)\rho_0}{E_0} \frac{\partial^4 v_2}{\partial \alpha_2^2 \partial t^2} \right) = 0,$$

the convenient form of latter is

$$\frac{h^2}{R^2}\frac{\mathcal{E}_1^2-\mathcal{E}_0\mathcal{E}_2}{\mathcal{R}\mathcal{E}_0}\left(1+2R^2\frac{\partial^2}{\partial\alpha_2^2}+R^4\frac{\partial^4}{\partial\alpha_2^4}\right)\frac{\partial^2 v_2}{\partial\alpha_2^2}+\left(1-R^2\frac{\partial^2 v_2}{\partial\alpha_2^2}\right)\frac{\partial^2 v_2}{\partial t^2}=0, \tag{5.67}$$

where $v_2 = RV_2^{(0)}$, $\mathcal{R} = \rho_0\mathcal{R}_*$ and $\mathcal{E}_n = E_0\mathcal{E}_n^*$, $n = 1, 2, \ldots$.

For a traveling wave $v_2 = \exp[i(n\alpha_2/R - \omega t)]$, where n is wavenumber and ω is angular frequency, we have from equation (5.67) for natural frequencies,

$$\Omega^2 = 2(1+\nu_0)\frac{\mathcal{E}_0^*\mathcal{E}_2^* - \mathcal{E}_1^{*2}}{\mathcal{R}_*\mathcal{E}_0^*}\frac{n^2(1-n^2)^2}{1+n^2}, \tag{5.68}$$

where $\Omega = \eta^{-1}\omega R/c_{20}$ with $\eta = \frac{h}{R} \ll 1$ and c_{20} defined in (5.8).

It might easily be verified that $\mathcal{E}_2^*\mathcal{E}_0^* - \mathcal{E}_1^{*2} \geq 0$. Because using the notation (5.52) and the Holder's inequality, we can prove the above inequality

$$\begin{aligned}
|\mathcal{E}_1^*| &= \left|\int_{-1}^{1}\frac{\zeta E_*(\zeta)}{1-\nu^2(\zeta)}\,d\zeta\right| \\
&\leq \int_{-1}^{1}|\zeta|\left|\frac{E_*(\zeta)}{1-\nu^2(\zeta)}\right|d\zeta = \int_{-1}^{1}|\zeta|\sqrt{\frac{E_*(\zeta)}{1-\nu^2(\zeta)}}\sqrt{\frac{E_*(\zeta)}{1-\nu^2(\zeta)}}\,d\zeta \\
&\leq \left(\int_{-1}^{1}\frac{E_*(\zeta)}{1-\nu^2(\zeta)}\,d\zeta\right)^{1/2}\left(\int_{-1}^{1}\frac{\zeta^2 E_*(\zeta)}{1-\nu^2(\zeta)}\,d\zeta\right)^{1/2} \\
&= \mathcal{E}_0^{*1/2}\cdot\mathcal{E}_2^{*1/2}
\end{aligned}$$

hence

$$\mathcal{E}_1^{*2} \leq \mathcal{E}_0^*\cdot\mathcal{E}_2^*.$$

It is also clear from equation (5.68) that the effect of inhomogeneity, due to the presence of the integral quantities $\mathcal{E}_n$ and $\mathcal{R}$, does not depend on the mode number n. For a homogeneous cylinder, when $\mathcal{E}_0 = 2E_0/(1-\nu^2)$, $\mathcal{E}_1 = 0$, $\mathcal{E}_2 = 2E_0/3(1-\nu^2)$, $\mathcal{R} = 2\rho_0$, the last formula agrees with previous results, e.g., see chapter (3) and [44]. For $n \sim 1$, (5.68) gives generally $\Omega \sim 1$. However, this formula is also valid at $n \ll \eta^{-1/2}$ when $\Omega \ll \eta^{-1}$.

It is worth mentioning that the leading order stress and displacement components can be expressed through the solution of equation (5.67). In particular, we obtain from equation (5.34)

$$\frac{\partial v_2}{\partial\alpha_2}+\frac{v_3}{R}=0, \tag{5.69}$$

where $v_3 = RV_3^{(0)}$. Also, the stress $\sigma_{22}^{(0)}$ in equation (5.51) can be written in original variables as

$$\frac{1}{E_0\eta}\sigma_{22} = -\frac{1}{(1-\nu^2)E_0}\left(\frac{\partial v_2}{\partial \alpha_2} - R\frac{\partial^2 v_3}{\partial \alpha_2^2}\right)\left(\frac{\mathcal{E}_1}{\mathcal{E}_0}E(\alpha_3) - \frac{\alpha_3}{h}E(\alpha_3)\right),$$

simplifying the latter, we obtain

$$\sigma_{22} = \frac{h}{R}\frac{E}{(1-\nu^2)}\left(\frac{\alpha_3}{h} - \frac{\mathcal{E}_1}{\mathcal{E}_0}\right)\left(\frac{\partial v_2}{\partial \alpha_2} - R\frac{\partial^2 v_3}{\partial \alpha_2^2}\right). \tag{5.70}$$

In similar manner, the stresses $\sigma_{32}^{(0)}$, $\sigma_{33}^{(0)}$ in (5.55) and (5.56) can be written in original variables as

$$\frac{1}{E_0\eta^2}\sigma_{3k} = (-1)^{k+1}\left(\frac{1}{E_0}\frac{\mathcal{E}_1}{\mathcal{E}_0}\int_{\alpha_3/h}^{1}\frac{E(s)}{1-\nu^2(s)}\,ds - \frac{1}{E_0}\int_{\alpha_3/h}^{1}\frac{sE(s)}{1-\nu^2(s)}\,ds\right)\times$$
$$\times\frac{1}{R}\left(\frac{\partial^{4-k}v_2}{\frac{1}{R^{4-k}}\partial\alpha_2^{4-k}} - \frac{\partial^{5-k}v_3}{\frac{1}{R^{5-k}}\partial\alpha_2^{5-k}}\right)$$

Simplifying the latter we get

$$\sigma_{3k} = (-1)^{k+1}\frac{h^2}{R^{k-1}}\left(\frac{\mathcal{E}_1}{\mathcal{E}_0}\int_{\alpha_3/h}^{1}\frac{E(s)}{1-\nu^2(s)}\,ds - \int_{\alpha_3/h}^{1}\frac{sE(s)}{1-\nu^2(s)}\,ds\right)\left(\frac{\partial^{4-k}v_2}{\partial\alpha_2^{4-k}} - R\frac{\partial^{5-k}v_3}{\partial\alpha_2^{5-k}}\right). \tag{5.71}$$

here $k = 2, 3$.

At the same time, the first order corrections to stresses and displacements do not follow from the formulae in the previous sections necessitating analysis at higher orders. For example, we may only calculate the resultant of the normal circumferential stress starting from formula (5.60) but not its variation across the thickness.

In contrast to a homogeneous cylinder, the leading order stresses do not demonstrate a simple polynomial variation in thickness. Consequently, they cannot be restored from the associated stress resultants and stress couples as in the classical shell theory, e.g., see [27]. In spite of this observation, we present below the one dimensional equations of motion expressed in terms of the stress resultants and stress couples following a long term tradition.

5.6. Stress Resultants and Stress Couples

Let us write the basic formulae in the section (5.3) and (5.4) as:

$$\frac{\partial T_2}{\partial \alpha_2} + \frac{N_2}{R} - h \int_{-1}^{1} \rho(\zeta)\, d\zeta \frac{\partial^2 v_2}{\partial t^2} = 0, \tag{5.72}$$

$$\frac{\partial G_2}{\partial \alpha_2} - N_2 = 0, \tag{5.73}$$

$$\frac{\partial N_2}{\partial \alpha_2} - \frac{T_2}{R} - h \int_{-1}^{1} \rho(\zeta)\, d\zeta \frac{\partial^2 v_3}{\partial t^2} = 0, \tag{5.74}$$

with

$$G_2 = h^3 \frac{\mathcal{E}_1^2 - \mathcal{E}_0 \mathcal{E}_2}{\mathcal{E}_0} \frac{\partial}{\partial \alpha_2} \left(\frac{\partial v_3}{\partial \alpha_2} - \frac{v_2}{R} \right), \tag{5.75}$$

the above equations are written in terms of the mid-surface displacements v_2 and v_3 and longitudinal stress resultant T_2, shear force resultant N_2 and stress couple G_2. Above mentioned resultants and couple are derived from (2.72) – (2.74) as

$$T_2 = \int_{-h}^{h} \sigma_{22}\, d\alpha_3, \qquad G_2 = \int_{-h}^{h} \sigma_{22} \alpha_3\, d\alpha_3, \qquad N_2 = \int_{-h}^{h} \sigma_{32}\, d\alpha_3. \tag{5.76}$$

Now we derive the above equations one by one with the help of the previous sections.

Derivation of (5.72)

Integrating the equation (5.39) with respect to ζ from -1 to 1, we have

$$\frac{1}{\eta} \int_{-1}^{1} \frac{\partial}{\partial \theta} \left(\sigma_{22}^{(0)} + \eta \sigma_{22}^{(1)} \right) d\zeta + \int_{-1}^{1} \sigma_{32}^{(0)} d\zeta - \frac{1}{2(1+\nu_0)} \frac{\partial^2 v_2^{(0)}}{\partial \tau^2} \int_{-1}^{1} \rho_*(\zeta)\, d\zeta = 0, \tag{5.77}$$

rewriting the last equation back in form of original variables (i.e., (5.1) and (5.2)), one obtain

$$\frac{R}{h} \frac{1}{E_0 \eta^2} \int_{-h}^{h} \frac{\partial \sigma_{22}}{\partial \alpha_2}\, d\alpha_3 + \frac{1}{h} \frac{1}{E_0 \eta^2} \int_{-h}^{h} \sigma_{32} d\alpha_3 - \frac{R}{E_0 \eta^2} \int_{-1}^{1} \rho(\zeta)\, d\zeta \frac{\partial^2 v_2}{\partial t^2} = 0, \tag{5.78}$$

On considering (5.76), the latter can be written as hence

$$\frac{\partial T_2}{\partial \alpha_2} + \frac{N_2}{R} - h\int_{-1}^{1} \rho(\zeta)\, d\zeta \frac{\partial^2 v_2}{\partial t^2} = 0. \tag{5.79}$$

Derivation of (5.73)

Multiply equation (5.27) by ζ and integrate with respect to ζ form -1 to 1, we have

$$\frac{\partial}{\partial \theta}\int_{-1}^{1} \zeta \sigma_{22}^{(0)}\, d\zeta - \int_{-1}^{1} \sigma_{32}^{(0)}\, d\zeta = 0,$$

returning back to the original variables, we get

$$\frac{R}{h^2}\frac{1}{E_0\eta}\frac{\partial}{\partial \alpha_2}\int_{-h}^{h} \sigma_{22}\alpha_3\, d\alpha_3 - \frac{1}{h}\frac{1}{E_0\eta^2}\int_{-h}^{h} \sigma_{32}\, d\alpha_3 = 0,$$

hence

$$\frac{\partial G_2}{\partial \alpha_2} - N_2 = 0. \tag{5.80}$$

Derivation of (5.74)

Integrating the equation (5.40) with respect to ζ from -1 to 1, we get

$$\int_{-1}^{1} \frac{\partial \sigma_{32}^{(0)}}{\partial \theta}\, d\zeta + \int_{-1}^{1} \frac{\partial \sigma_{33}^{(1)}}{\partial \zeta}\, d\zeta - \frac{1}{\eta}\int_{-1}^{1} \left(\sigma_{22}^{(0)} + \eta\sigma_{22}^{(1)}\right) d\zeta + \int_{-1}^{1} \zeta\sigma_{22}^{(0)}\, d\zeta + \int_{-1}^{1} \sigma_{33}^{(0)}\, d\zeta$$
$$- \frac{1}{2(1+\nu_0)}\int_{-1}^{1} \rho_*(\zeta)\, d\zeta \frac{\partial^2 v_3^{(0)}}{\partial \tau^2} = 0, \tag{5.81}$$

simplifying this relation, we get

$$\frac{\partial}{\partial \theta}\int_{-1}^{1} \sigma_{32}^{(0)}\, d\zeta - \int_{-1}^{1} \sigma_{22}^{(1)}\, d\zeta - \frac{1}{2(1+\nu_0)}\int_{-1}^{1} \rho_*(\zeta)\, d\zeta \frac{\partial^2 v_3^{(0)}}{\partial \tau^2} = 0, \tag{5.82}$$

rewriting the last equation back in form of original variables, we get

$$\frac{R}{h}\frac{1}{E_0\eta^2}\frac{\partial}{\partial \alpha_2}\int_{-h}^{h} \sigma_{32}\, d\alpha_3 - \frac{1}{h}\frac{1}{E_0\eta^2}\int_{-h}^{h} \sigma_{22}\, d\alpha_3 - \frac{R}{E_0\eta^2}\int_{-1}^{1} \rho(\zeta)\, d\zeta \frac{\partial^2 v_3}{\partial t^2} = 0, \tag{5.83}$$

hence

$$\frac{\partial N_2}{\partial \alpha_2} - \frac{T_2}{R} - h\int_{-1}^{1} \rho(\zeta)\, d\zeta \frac{\partial^2 v_3}{\partial t^2} = 0. \tag{5.84}$$

Derivation of (5.75)

Multiply equation (5.28) by ζ and integrate with respect to ζ from -1 to 1, we have

$$\int_{-1}^{1} \sigma_{33}^{(0)}\, d\zeta + \int_{-1}^{1} \zeta\sigma_{22}^{(0)}\, d\zeta = 0, \tag{5.85}$$

for evaluation of the first integral of this equation, we use (5.56), that is

$$\int_{-1}^{1} \sigma_{33}^{(0)}\, d\zeta = \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{-1}^{1}\int_{s}^{1} \frac{E_*(z)}{1-\nu^2(z)}\, dzds - \int_{-1}^{1}\int_{s}^{1} \frac{zE_*(z)}{1-\nu^2(z)}\, dzds\right) P_1. \tag{5.86}$$

By change of a variable, we reduce the double integral to a single one as

$$\int_{-1}^{1} \sigma_{33}^{(0)}\, d\zeta = \left(\frac{\mathcal{E}_1^*}{\mathcal{E}_0^*}\int_{-1}^{1} (1+z)\frac{E_*(z)}{1-\nu^2(z)}\, dz - \int_{-1}^{1} z(1+z)\frac{E_*(z)}{1-\nu^2(z)}\, dz\right) P_1. \tag{5.87}$$

$\Rightarrow$

$$\int_{-1}^{1} \sigma_{33}^{(0)}\, d\zeta = \left(\frac{\mathcal{E}_1^{*2}}{\mathcal{E}_0^*} - \mathcal{E}_2^*\right) P_1, \tag{5.88}$$

by using this equation in (5.85), we get

$$\left(\frac{\mathcal{E}_1^{*2}}{\mathcal{E}_0^*} - \mathcal{E}_2^*\right) P_1 + \int_{-1}^{1} \zeta\sigma_{22}^{(0)}\, d\zeta = 0. \tag{5.89}$$

Next, returning back to the original coordinates, we have

$$\left(\frac{\partial v_2}{\partial \alpha_2} - R\frac{\partial^2 v_3}{\partial \alpha_2^2}\right)\frac{1}{E_0}\left(\frac{\mathcal{E}_1^2}{\mathcal{E}_0} - \mathcal{E}_2\right) + \frac{1}{h^2 E_0 \eta}\int_{-h}^{h} \alpha_3\sigma_{22}\, d\zeta = 0, \tag{5.90}$$

G_2 can be written as

$$G_2 = h^3 \frac{\mathcal{E}_1^2 - \mathcal{E}_0 \mathcal{E}_2}{\mathcal{E}_0} \frac{\partial}{\partial \alpha_2} \left(\frac{\partial v_3}{\partial \alpha_2} - \frac{v_2}{R} \right). \tag{5.91}$$

In conclusion, remark that for a homogeneous cylinder the formulae in this section coincide with those within the traditional semi-membrane shell theory adopted for plane strain [14].

5.7. Numerical Examples

First, we present two examples related to two special types of Young's modulus variable. For the sake of simplicity, density and Poisson's ratio are assumed to be uniform across the thickness. We restrict ourselves to the squared natural frequencies of the FGM cylinder δ normalized by Ω_H^2 denoting those of a homogeneous cylinder of the same size characterized by the material parameters E_0, ρ_0 and ν_0, see [14, 44]. In this case,

$$\delta = \frac{\Omega^2}{\Omega_H^2} = 3(1 - \nu_0^2) \frac{\mathcal{E}_0^* \mathcal{E}_2^* - \mathcal{E}_1^{*2}}{\mathcal{R}_* \mathcal{E}_0^*}, \tag{5.92}$$

where

$$\Omega_H^2 = \frac{2}{3(1 - \nu_0)} \frac{n^2(1 - n^2)^2}{1 + n^2}. \tag{5.93}$$

First, consider the Young's modulus given by

$$E(\alpha_3) = E_0(1 + \alpha \sin(\beta \alpha_3 / h)), \quad |\alpha| < 1 \tag{5.94}$$

Inserting (5.94) into (5.92) and taking into account (5.52) and (5.66),

$$\delta = 1 - \frac{3\alpha^2(\beta \cos\beta - \sin\beta)^2}{\beta^4}, \tag{5.95}$$

Figure (5.2) displays the normalized squared frequencies δ against α for several values of β. The variation of δ against β is presented in figure (5.3).

It can easily be verified that δ given by (5.92) does not depend on the Poisson's ratio and density, see (5.52) and (5.66). The derivation from the $\delta = 1$ corresponding to a homogeneous cylinder, illustrates the effect of FGM. Figures (5.3) and (5.4) show that the natural frequencies may both decrease or stay the same within the considered parameter range.

Formula (5.92) is also valid for layered cylinders with piecewise uniform Young's modulus. As an example, consider a three -layered cylinder with the outer layers having the same material properties and thicknesses. In this case

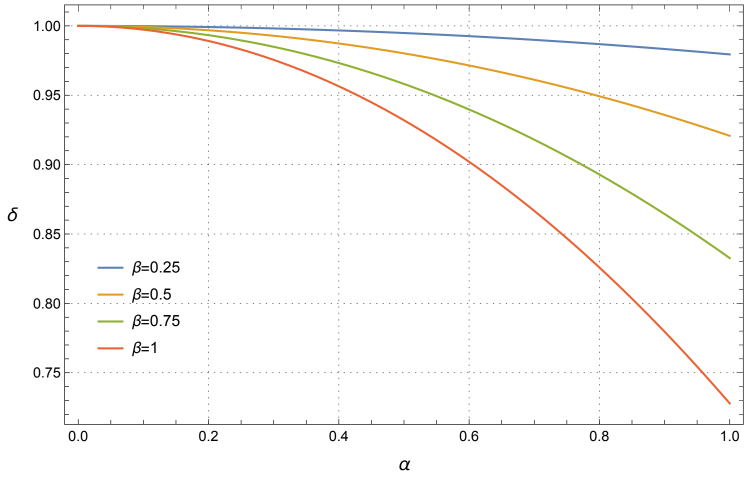

Figure 5.2. *The ratio δ against α for several values of β.*

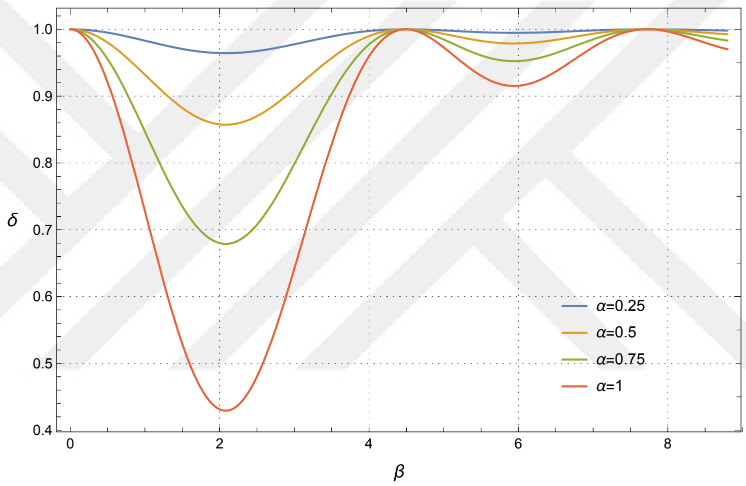

Figure 5.3. *The ratio δ against β for several values of α.*

$$E(\alpha_3) = \begin{cases} E_0, & -h_1 \le \alpha_3 \le h_1 \\ E_1, & -h \le \alpha_3 \le -h_1, \; h_1 \le \alpha_3 \le h \end{cases} \tag{5.96}$$

where E_0 and E_1 are the Young's modulus of the inner and outer layers, respectively, and h_1 is the half thickness of the inner layer.

once again, inserting (5.96) into (5.92), we obtain

$$\delta = h_*^3 + E_*(1 - h_*^3), \tag{5.97}$$

where $h_* = h_1/h$ and $E_* = E_1/E_0$. The numerical results are given in figures (5.4) and (5.5) adapting the similar layout for illustrating two parametric formulae as in previous two figures above with the exception of an expected increase in natural frequencies for a stiffer cylinder displayed in figure (5.5).

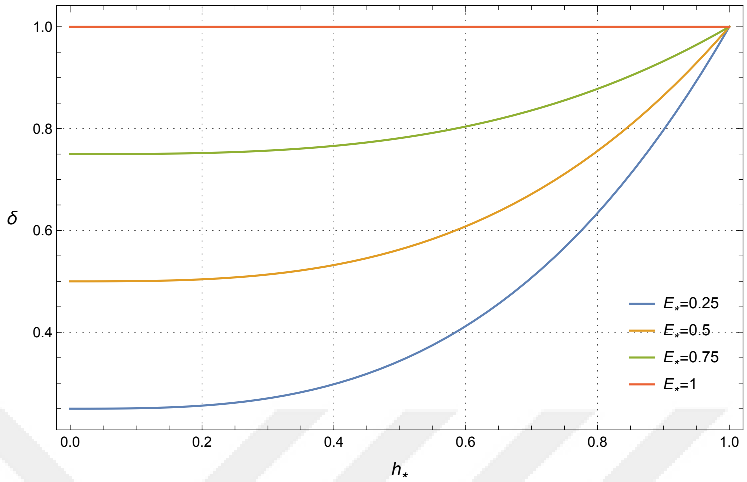

Figure 5.4. *The nondimensional frequency* (5.92) *against* h_* *for several values of* E_*

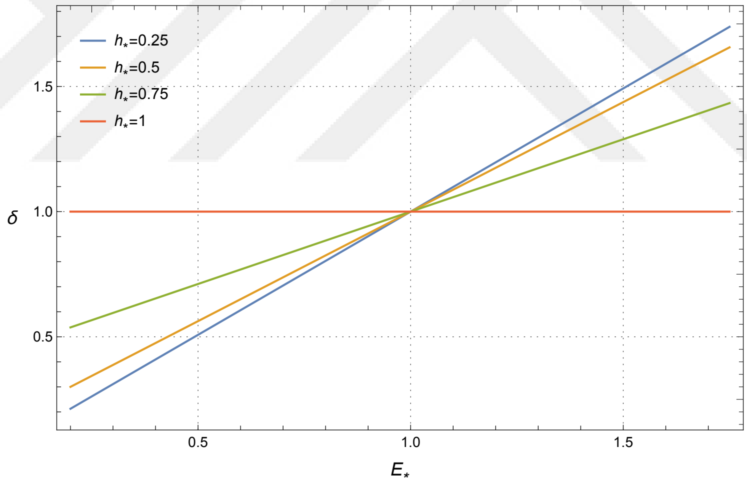

Figure 5.5. *The nondimensional frequency* (5.92) *against* E_* *for several values of* h_*.

The proposed asymptotic formula is also tested against the results in [54] obtained using the full Kirchhoff-Love theory adopted for an FGM cylindrical shell. The FGM is assumed to be composed of two constituent materials with Young's modulus, Poisson's ratio and mass density expressed, respectively, as

$$E = (E_{out} - E_{in})\left(\frac{\zeta + 1}{2}\right)^p + E_{in}, \tag{5.98}$$

$$\nu = (\nu_{out} - \nu_{in})\left(\frac{\zeta + 1}{2}\right)^p + \nu_{in} \tag{5.99}$$

and

$$\rho = (\rho_{out} - \rho_{in}) \left(\frac{\zeta + 1}{2} \right)^p + \rho_{in}, \tag{5.100}$$

where p is a non-negative real number denoting the volume fraction exponent, see [54], and also [36]; $-1 \leq \zeta \leq 1$ as above. The values of the natural frequencies $\gamma = \Omega c_{20}/(2\pi R)(Hz)$ versus the circumferential wavenumber n are presented in Table 1. The data in the second and fourth columns in the table correspond to formula (31) in [54] using the constants $C_{ij}, i, j = 1, 2$ and 6 defined in the Appendix of the mentioned reference at $m = 0$ associated with plane strain. The results in the third and fifth columns in the table are calculated from formula (5.68) evaluating numerically integrals (5.52) and (5.66) with $\mathcal{R} = \rho_0 \mathcal{R}_*$ and $\mathcal{E}_n = E_0 \mathcal{E}_n^*$, $n = 1, 2, \ldots$ and $c_{20} = \sqrt{E_0/(2(1 + \nu_0)\rho_0)}$. The material properties of the cylinder vary from Nickel at the inner surface $\zeta = -1$ with $E_{in} = 223.95 \times 10^9 N/m^2$, $\nu_{in} = 0.31$ and $\rho_{in} = 8900 kg/m^3$ to stainless steel at the outer surface $\zeta = 1$ with $E_{out} = 201.04 \times 10^9 N/m^2$, $\nu_{out} = 0.3262$ and $\rho_{out} = 8166 kg/m^3$. The half-thickness is $h = 0.001m$ and the radius is $R = 1m$. Excellent agreement with the results for the Kirchhoff-Love theory suggests that the latter may be substituted by the derived shortened asymptotic formulation oriented to the low-frequency range.

Table 5.1. *Variation of fundamental natural frequency $\gamma(Hz)$ versus the wavenumber n for $h/R = 0.002$ with $m = 0$*

n	**P = 1**		**P = 15**	
	Ref. [54]	**Current**	**Ref. [54]**	**Current**
2	1.29727	1.29724	1.29534	1.29533
3	3.66925	3.66915	3.66378	3.66375
4	7.03548	7.03527	7.02499	7.02492
5	11.3779	11.3775	11.3609	11.3608
6	16.6911	16.6906	16.6662	16.6661
7	22.9732	22.9725	22.9389	22.9387
8	30.2233	30.2223	30.1782	30.1779
9	38.4408	38.4397	38.3835	38.3831
10	47.6258	47.6243	47.5547	47.5542

6. CONCLUSSIONS

In any asymptotic analysis, the problem is normalized, i.e., defined in terms of dimensionless variables whose typical scale is of the order of one, and the relative magnitude of different physical effects is measured by dimensionless parameters and non-dimensional groups [4].

In this thesis, first the related problem is normalized, then the 1D equation of motion is derived from reduced 2D equations of motion starting from 3D equations for a transverse cross-section (annulus) of both homogeneous and FGM thin elastic circular cylindrical shells within a plane strain framework. Mechanical parameters are assumed to be constant for homogeneous problems and vary along a thickness coordinate in FGM. Also, the consistent 1D equation of motion is derived from full 3D equations of motion for a homogeneous, thin, elastic cylindrical shell. These 1D equations of motion govern the vibration of a cylindrical shell. In order to obtain the expected results, we assumed the deformations to be inextensible, which means that any line drawn on the mid-surface remains unchanged during vibrations of minor amplitude. Stretching of the shell is ignored, while bending is accepted. The inextensible conditions are written in the form of formulae in each chapter. Let's wrap up each chapter separately so that we have a better understanding of it.

In Chapter 3, low-frequency vibrations of a thin elastic annulus are considered. The reduced 2D equations of motion are subjected to asymptotic treatment for leading and next-order (second-order) approximations. All the dimensionless stresses and displacements are expanded into an asymptotic series expressed in terms of relative thickness. The relative thickness of the annulus is assumed to be small. The leading-order low-frequency approximations correspond to the so-called semi-membrane theory of shells. The derived next-order low-frequency approximation to the original 2D equations has never been deducted from full equations in linear elasticity. We dealt with the low-frequency vibrations of an elastic annulus with traction-free inner and outer contours, and the obtained results may also be generalized to the loaded inner and outer contours of the annulus.

As a result of the asymptotic treatment of plane strain equations, a 1D refined low-frequency model for a thin annulus with frequencies of the order of relative thickness is obtained. These frequencies are also the cutoffs for a thin cylindrical shell. The investigated problem may be treated as a plane strain problem for a cylindrical shell. The presented analysis is a significant preliminary step for developing 2D asymptotic equations for a cylindrical shell, starting from 3D elasticity. Neither leading-order nor next-order natural frequencies are identical to the formula expressed in [44]. Finally, the asymptotic technique adapted in the study is not restricted to a circular cylinder; it can be applied to non-circular cylindrical shells.

In Chapter 4, a novel special scaling is introduced for the first time, in order to derive an asymptotically consistent 1D equation for a cylindrical shell from a 3D framework. Comparisons with previous results obtained using the popular 2D Sanders-Koiter shell theory [44] is presented, where in [44] the equations of motion are derived using a straightforward asymptotic procedure in the framework of the Sanders-Koiter version of 2D shell theory. Asymptotic corrections are deduced for the fourth-order equation of low-frequency motion and some other relations. Most of the coefficients in the established low-frequency equation are identical to their counterparts in [44]. The essential correction to the leading order estimation for the cut-off frequency, which coincides with that obtained in [9, 14] from the related plane strain problem, is the exception. At the same time, the leading-order expressions for the mid-surface circumferential extension and the tangential shear stress resultants are different from those in [44], whereas the tangential normal stress resultants and mid-plane shear deformation are the same. The established asymptotic formulation is also valid over a broader parameter range.

In Chapter 5, low-frequency vibration of a thin-walled functionally graded cylinder is considered within the plane strain framework. The procedure of attaining a new asymptotically consistent approximate equation of motion on the mid-surface from dynamic equations subject to asymptotic analysis over a cylinder cross-section is very similar to that of homogenization techniques generalized to thin structures. The derived 1D equation governs the low-frequency motion of a thin-walled FGM cylinder under plane strain conditions. The governing equations for stress resultants and stress couples are also presented.
The squared natural frequencies of the FGM cylinder, which are generated from the 1D equation and govern low-frequency plane strain motion in a thin-walled FGM cylinder, are normalized by a natural frequency, those of a homogeneous cylinder. This normalized equation does not depend on the Poisson's ratio and density. This means that these parameters are uniform across the thickness. The predictions by the useful asymptotic formula for natural frequencies derived from 1D refined equations are in excellent agreement with the calculations according to the setup developed in [36, 54] on the basis of Kirchhoff-Love theory. The elegant formula for natural frequencies obtained is tested by comparison with numerical data in [54]. The results obtained here have the potential to be implemented for the interpretation of numerical and experimental data. The obtained results may be extended to the derivation of higher-order low-frequency approximations for a 2D plane strain problem as well as to a full 3D problem for an FGM cylinder.

The proposed asymptotic formulation may be generalized to an FGM cylindrical shell within a 3D framework, and the established approach is also valid over a broader parameter range. Also, it exhibits that this asymptotic treatment is not limited to cylindrical shells only, which means that it may be applied to non-cylindrical shells.

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