DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE AUSLANDER-BRIDGER TRANSPOSE

by
Gülizar GÜNAY

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İZMİR

ON THE AUSLANDER-BRIDGER TRANSPOSE

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We have read the thesis entitled "**ON THE AUSLANDER-BRIDGER TRANSPOSE**" completed by **GÜLİZAR GÜNAY** under supervision of **ASSOC. PROF. DR. ENGİN MERMUT** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

....

Assoc. Prof. Dr. Engin MERMUT

Supervisor

....

Prof. Dr. Selcuk Demir

Jury Member

....

Assoc. Prof. Dr. Engin BÜYÜKAŞIK

Jury Member

Prof. Dr. Ayşe OKUR
Director
Graduate School of Natural and Applied Sciences

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Gülizar GÜNAY

ON THE AUSLANDER-BRIDGER TRANSPOSE

ABSTRACT

Let R be a ring with unity. A module M is said to be stable if it has no nonzero projective direct summand. It is well-known that over a semiperfect ring R, taking the Auslander-Bridger transpose establishes a one to one correspondence between the isomorphism classes of finitely presented stable right R-modules and the isomorphism classes of finitely presented stable left R-modules. Over a general ring R, Facchini and Girardi extend this one to one correspondence by taking an Auslander-Bridger transpose between two isomorphism classes of some modules which they call Auslander-Bridger right R-modules and Auslander-Bridger left R-modules. In this thesis, we give some ring classes over which every finitely presented module can be decomposed into a direct sum of a projective submodule and a stable submodule. If the ring R is semilocal has finite hollow dimension as a right R-module or left semihereditary, then every finitely presented right R-module has such a decomposition. We give some ring examples over which there exists a finitely generated module or a finitely presented module where such a decomposition fails. Another problem that we deal with in this thesis is the Auslander-Bridger transpose of finitely presented simple modules. It is shown by Türkoğlu in his master thesis that if R is a commutative ring where every maximal ideal is finitely generated and projective, then an Auslander-Bridger transpose of every simple R-module is isomorphic to itself, and so simple. A related result is given by Haack. We give the details of the proof of Haack's result: for an Artinian ring R, the Auslander-Bridger transpose of every nonprojective simple right R-module is simple if and only if R is a serial ring. This also gives us some relations between the proper classes of short exact sequences of R-modules that are projectively generated or flatly generated by simple modules.

Keywords: Auslander-Bridger transpose, stable module, decompositions into projective and stable submodules, finitely presented module, simple module, Artinian serial ring, hereditary ring, semihereditary ring, semiartinian V-ring.

AUSLANDER-BRIDGER TRANSPOZU ÜZERİNE

ÖZ

R birimli bir halka olsun. R üstünde bir M modülünün 0'dan farklı projektif bileşeni yoksa bu modüle kararlı modül deriz. Yarı-mükemmel halkalar üzerinde, Auslander-Bridger transpozu alma işleminin sağ kararlı modül sınıfları ile sol kararlı modül sınıfları arasında birebir eşleme kurduğu bilinmektedir. Facchini ve Girardi, genel R halkaları üstünde Auslander-Bridger modüllerine Auslander-Bridger transpoz alma işlemi uygulayarak, sağ ve sol Auslander-Bridger R-modül sınıfları arasında birebir eşleme kurmuşlardır. Bu tezde, sonlu sunulmuş modülleri projektif ve kararlı alt modül olarak ayrıştırabildiğimiz bazı halka sınıflarını ve sonlu üretilen ya da sonlu sunulmuş modüllerin bu ayrışımı sağlamadığı bazı halka örnekleri verdik. Sağ R modül olarak sonlu oyuk boyutu olan yarı-yerel R halkası üzerinde ya da sol yarıkalıtlı R halkası üzerinde, her sonlu sunulmuş modül böyle bir ayrışıma sahiptir. Bu tez de ilgilendiğimiz diğer bir problemde sonlu sunulmuş basir modüllerin Auslander-Bridger transpozudur. Her maksimal idealin sonlu üretilmiş ve projektif olduğu değişmeli halkalarda, basit modüllerin bir Auslander-Bridger transpozunun da kendisine izomorf ve bu yüzden basit olduğu Türkoğlu'nun yüksek lisans tezinde gösterilmiştir. Bununla ilgili bir sonuç Haack tarafından verilmiştir. Biz ek olarak Haack'ın makalesinde geçen bir çift taraflı Artin R halkası üzerinde, projektif olmayan basit sağ R-modüllerin Auslander-Bridger transpozunun basit sol R-modül olması ancak ve ancak R halkasının serisel olması ile mümkündür sonucunun detaylı ispatını verdik. Bu da bize basit modüller tarafından projektif olarak veya düz olarak üretilen modüllerin kısa tam dizilerinin öz sınıfları arasında bazı ilişkiler verir.

Anahtar kelimeler: Auslander-Bridger transpoz, kararlı modül, projektif ve kararlı altmodüllere ayrışma, sonlu sunulmuş modül, basit modül, Artinian serisel halka, kalıtsal halka, yarı-kalıtsal halka, yarı-Artin V-halkası.

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CHAPTER ONE

INTRODUCTION

In this introductory chapter, we shall explain the motivation for the problems that we deal in this thesis and what we have done. Throughout this thesis, R denotes an arbitrary ring with unity and an R-module or module means a unital *right* R-module unless otherwise stated. The Jacobson radical of R is denoted by $\mathrm{J} = \mathrm{J}(R)$.

A concept that we shall frequently use in this thesis is the notion of stable R-modules. An R-module M is said to be a *stable R-module* if M has no nonzero projective direct summand. This is not a quite standard terminology, we follow this terminology as used in the article by Martsinkovsky & Zangurashvili (2015). See Chapter 2 for any undefined term and for the summary of some concepts and results with related references for module and ring theory.

1.1 Motivation

In this thesis, we study about two problems related with the Auslander-Bridger transpose of finitely presented R-modules. The first one is about the decomposition of a finitely presented R-module into a direct sum of a projective submodule and a stable submodule. The second one is the Auslander-Bridger transpose of finitely presented simple modules.

The decomposition into projective and stable submodules is related with the stable category of finitely presented R-modules where two modules M and N are considered to be isomorphic in this category if $M \oplus P \cong N \oplus Q$ for some projective R-modules P and Q. We have two main kind of problems related with decompositions into a direct sum of a projective submodule and a stable submodule:

A. For which kind of rings R, and for which class of R-modules, does every R-module M in that class has a decomposition $M = P \oplus N$, where P is a projective R-module and N is a stable R-module? Find examples of such rings. Can we characterize such rings?

B. What properties should a ring R have if every R-module M in a given class of R-modules does *not* have a decomposition $M = P \oplus N$, where P is projective and N is stable? Can we characterize such rings?

More precisely, in the discussions with Noyan Er and my advisor, we started to ask the following questions and obtained some answers and examples for some of these questions:

1. For which rings R, does every *finitely presented* R-module M has a decomposition $M = P \oplus N$, where P is a projective submodule and N is a stable submodule? It is easily seen that if we have some finiteness assumption on the ring R (like being Noetherian or Artinian, or having a finite uniform dimension or finite hollow dimension), then such a decomposition surely exists; see Sections 3.2 and 3.3. One hard, maybe unreasonable question, is to characterize all such rings. The following questions seem more reasonable at first.

2. Give examples of rings R, where there exists a *cyclic* (or *finitely generated*) R-module M that does not have a decomposition $M = P \oplus N$, where P is a projective submodule and N is a stable submodule. See Section 3.4 for some examples that we have given.

3. Give examples of rings R, where there exists a *finitely presented* R-module M or a *cyclically presented* R-module M (that is, $M \cong R/aR$ for some $a \in R$) that does not have a decomposition $M = P \oplus N$, where P is a projective submodule and N is a stable submodule. Noyan Er gives such an example of a *cyclically presented* R-module, see Example 3.4.23.

4. Characterize the rings R, where *every* R-module M has a decomposition $M = P \oplus N$, where P is a projective submodule and N is a stable submodule. The motivaton is that the dual statement characterizes right Noetherian rings (see Section 3.1): A ring R is right Noetherian if and only if every right R-module M has a decomposition $M = E \oplus N$, where E is injective and N has no nonzero injective submodules.

My advisor has discussed about finitely presented modules over Leavitt path algebras with Kulumani Rangaswamy, Murat Özaydın and Pere Ara in the workshop CIMPA (2015). Pere Ara has referred in his article Ara & Brustenga (2007) to a result in the book Lück (2002) for some class of rings which are semihereditary rings. He suggested that the decomposition of a finitely presented module into a direct sum of a projective submodule and a stable submodule should hold over semihereditary rings. The results given in Lück (2002, Theorem 6.7) motivated the proof for semihereditary rings given in Section 3.3. In our proof, we shall use the Auslander-Bridger transpose of finitely presented modules; their properties that we shall need are given in the next section. We showed that the decomposition holds for finitely presented right modules over left semihereditary rings (semihereditary condition for the ring is on the opposite side of the module). On the positive side, the result also holds over semilocal rings; see Corollary 3.2.3.

An effective tool in the representation theory of Artinian rings and Artin algebras is the transpose of Auslander and Bridger; see Auslander & Bridger (1969a, Section 4) and Auslander et al. (1997). The best setting for this is over semiperfect rings. See Anderson & Fuller (1992, Section 32) for the results given in Section 1.3. The duality given by the Auslander-Bridger tranpose is best formulated using the stable categories of finitely presented left and right R-modules. See the next section.

Facchini & Girardi (2012) introduced the concept of Auslander-Bridger left and right R-modules over any ring R so that similar properties of the finitely presented modules over semiperfect rings hold for the Auslander-Bridger modules. See Section 1.4. When R is a semiperfect ring, it is well-known that every finitely presented R-module can be decomposed into a direct sum of a projective submodule and a stable submodule uniquely up to isomorphism; see Theorem 1.3.5. This property holds for the Auslander-Bridger modules over any ring; see Section 1.4. Thus a natural question for us was to have some examples where this decomposition fails for some finitely presented modules and some rings where this decomposition always holds for all finitely presented modules. In the discussions with Noyan Er and my advisor, a counter example where such a decomposition fails for a finitely generated module was given by Noyan Er. He also gave some examples where the decomposition fails for some finitely generated modules (but not for finitely presented ones) over semiartinian V-rings that are not semisimple which are a class of von Neumann regular rings (see Baccella (1995) for semiartinian V-rings). If R is a right hereditary ring which is not right perfect or left coherent, then we have shown that the decomposition fails for the torsionless module $M = \prod_{i \in I} R$, where I is an infinite index set, which is not finitely generated. If R is a right semihereditary ring that is not right Baer then we have shown that the decomposition fails for some finitely generated torsionless module. See Section 3.4 for these examples.

In Türkoğlu (2013), it was shown that if R is a commutative ring where every maximal ideal is finitely generated and projective, then an Auslander-Bridger transpose of every simple R-module is isomorphic to itself, and so simple. A result along these lines is the characterization given by Haack (1984, Theorem 1) for Artinian serial rings: For an artinian ring R, we have that R is a serial ring if and only if the Auslander-Bridger tranpose of every nonprojective simple left (respectively right) R-module is a simple right (respectively left) R-module. We have given a detailed proof of this in Section 4.3 after summarizing the properties of the serial rings that we use in Sections 4.1 and 4.2. Using this we describe in Section 4.4 some relations among the proper classes of short exact sequences of modules that are projectively or injectively or flatly generated by simple left or right modules as has been dealt in the master thesis Türkoğlu (2013). A natural question to ask is over which rings R, $\mathrm{Tr}(M)$ is projectively equivalent to a simple left (right) R-module for every nonprojective finitely presented simple right (left) R-module M.

1.2 Auslander-Bridger Transpose

Let M be a finitely presented right R-module defined by m generators and s relations. Then M_R has a free presentation:

$$R_R^s \xrightarrow{\lambda_A} R_R^m \longrightarrow M_R \longrightarrow 0.$$

The elements of R_R^s and R_R^m can be viewed as column vectors, that is, as $s\times 1$ and $m\times 1$ matrices respectively, so that the morphism $\lambda_A : R_R^s \to R_R^m$ is left multiplication by an $m\times s$ matrix A. Then we can apply the functor $(-)^* = \mathrm{Hom}_R(-,R) : \mathrm{Mod}\text{-}R \to R\text{-}\mathrm{Mod}$ to this free presentation . Thus we obtain an exact sequence of left R-modules:

$$0 \longrightarrow M^* = \mathrm{Hom}_R(M_R, R_R) \longrightarrow {}_RR^m \xrightarrow{\rho_A} {}_RR^s.$$

The elements of the left R-modules ${}_RR^m \cong \mathrm{Hom}_R(R_R^m, R_R)$ and ${}_RR^s \cong \mathrm{Hom}_R(R_R^s, R_R)$ are considered as row vectors, that is, as $1\times m$ and $1\times s$ matrices respectively, and the morphism $\rho_A : {}_R R^m \to {}_R R^s$ is right multiplication by the same $m\times s$ matrix A. We will call the cokernel of ρ_A, which is a finitely presented left R-module with a presentation with s generators and m relations, *an Auslander-Bridger transpose* of M_R. Of course, this depends on the chosen presentation of M.

The name transpose comes when we consider the case where R is a commutative ring. In this case, there is no need to distinguish R^m and R^s as a right or left R-module, and we consider its elements as column vectors in any case, as in linear algebra. In this case, take a finitely presented R-module M, defined by m generators and s relations, with a free presentation:

$$R^s \xrightarrow{\lambda_A} R^m \longrightarrow M \longrightarrow 0.$$

The morphism $\lambda_A : R^s \to R^m$ is given by left multiplication with an $m\times s$ matrix A. Apply the functor $(-)^* = \mathrm{Hom}_R(-,R) : \mathrm{Mod}\text{-}R \to R\text{-}\mathrm{Mod}$ to this presentation:

$$0 \longrightarrow M^* = \mathrm{Hom}_R(M,R) \longrightarrow R^m \xrightarrow{\rho_A} R^s.$$

Then this time the morphism $\rho_A : R^m \to R^s$ is given by left multiplication with the $s\times m$ matrix A^T which is the transpose of the $m\times s$ matrix A.

More generally we consider projective presentations of a finitely presented R-module M, that is, we take the first two terms of a projective resolution of M. Let M be a *finitely presented* right R-module. Take a *projective presentation* of it, that is, take an exact sequence

$$\gamma : \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

where P_0 and P_1 are *finitely generated projective* modules. Apply the functor $(-)^* = \mathrm{Hom}_R(-,R) : \mathrm{Mod}\text{-}R \to R\text{-}\mathrm{Mod}$ to this presentation:

$$0 \longrightarrow \mathrm{Hom}_R(M,R) \xrightarrow{g^*} \mathrm{Hom}_R(P_0,R) \xrightarrow{f^*} \mathrm{Hom}_R(P_1,R)$$

where $\mathrm{Hom}_R(P_0,R) = P_0^*$ and $\mathrm{Hom}_R(P_1,R) = P_1^*$. Fill the right side of this sequence of *left* R-modules by the module $\mathrm{Tr}_\gamma(M) = \mathrm{Coker}(f^*) = P_1^* / \mathrm{Im}(f^*)$ to obtain the exact sequence

$$\gamma^* : \qquad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}_\gamma(M) \longrightarrow 0, \tag{1.1}$$

where σ is the canonical epimorphism. Since the modules P_0^* and P_1^* are finitely generated projective left R-modules, the exact sequence (1.1) is a projective presentation for the finitely presented *left* R-module $\mathrm{Tr}_\gamma(M)$ which is called the *Auslander-Bridger transpose* of the finitely presented right R-module M with respect to the presentation γ. If there is another projective presentation δ for the finitely presented right R-module M, then $\mathrm{Tr}_\gamma(M)$ and $\mathrm{Tr}_\delta(M)$ are projectively equivalent, that is,

$$\mathrm{Tr}_\gamma(M) \oplus P \cong \mathrm{Tr}_\delta(M) \oplus Q$$

for some projective modules P and Q. So an Auslander-Bridger transpose of the finitely presented R-module M is unique up to projective equivalence. See Auslander & Bridger (1969b) or Maşek (2000, §1) or Türkoğlu (2013, Theorem 4.2.4) for a proof of this and for some other properties that we shall use. We shall just write $\mathrm{Tr}(M)$ for an Auslander-Bridger transpose of the finitely presented R-module M keeping in mind that this is unique up to projective equivalence. Moreover $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong M$. If we drop the subscript for the dependent presentations γ^* and γ in $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M))$, then we can only say that $\mathrm{Tr}(\mathrm{Tr}(M))$ is projectively equivalent to M. Note that

$\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) = \mathrm{Coker}(f^{**})$ is defined by the exact sequence:

$$\gamma^{**}: \qquad P_1^{**} \xrightarrow{f^{**}} P_0^{**} \xrightarrow{\sigma'} \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \longrightarrow 0,$$

where σ' is the canonical epimorphism. On the other hand, applying the functor $(-)^*$ to the exact sequence (1.1), we obtain the following exact sequence:

$$0 \longrightarrow (\mathrm{Tr}_\gamma(M))^* \xrightarrow{\sigma^*} P_1^{**} \xrightarrow{f^{**}} P_0^{**}.$$

Since we have natural isomorphisms $P \cong P^{**}$ for every finitely generated projective R-module P, we obtain $(\mathrm{Tr}_\gamma(M))^* \cong \mathrm{Im}(\sigma^*) = \mathrm{Ker}(f^{**}) \cong \mathrm{Ker}(f)$. This proves:

Proposition 1.2.1. *(Angeleri Hügel & Bazzoni, 2010, Lemma 6.1-(2)) For a finitely presented R-module M,* $\mathrm{pd}(M) \leq 1$ *if and only if there exists a presentation*

$$\gamma: \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

of M such that (f is monic and) $(\mathrm{Tr}_\gamma(M))^* = 0$.

See Türkoğlu (2013, Theorem 4.2.6 and Theorem 4.2.7) for detailed proofs of the following two theorems.

Theorem 1.2.2. *(Sklyarenko, 1978, Proposition 5.1, Remarks 5.1 and 5.2) Let M be a* finitely presented *R-module and let γ be a presentation of M:*

$$\gamma: \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

(i) For every R-module N, there is a monomorphism $\mathrm{Ext}_R^1(M,N) \to N \otimes_R \mathrm{Tr}_\gamma(M)$ *and an epimorphism* $\mathrm{Hom}_R(\mathrm{Tr}_\gamma(M),N) \to \mathrm{Tor}_1^R(M,N)$. *Both are natural in N.*

(ii) If $\mathrm{pd}(M) \leq 1$, *then the map* $f: P_1 \to P_0$ *in the above presentation γ can be taken to be a monomorphism and in this case the monomorphism and epimorphism in the previous part become isomorphisms. Moreover by taking $N = R$, we obtain*

$$\mathrm{Tr}_\gamma(M) \cong \mathrm{Ext}_R^1(M,R) \qquad \textit{and} \qquad (\mathrm{Tr}_\gamma(M))^* = \mathrm{Hom}_R(\mathrm{Tr}_\gamma(M),R) = 0$$

for the presentation γ of M where the map $f : P_1 \to P_0$ is a monomorphism.

(iii) If $\mathrm{pd}(M) \leq 1$, *then* $\mathrm{Tr}(M)$ *is projectively equivalent to* $\mathrm{Ext}^1_R(M,R)$.

(iv) If $M^* = \mathrm{Hom}_R(M,R) = 0$, *then in the presentation*

$$\gamma^* : \qquad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}_\gamma(M) \longrightarrow 0,$$

we necessarily have that $f^* : P_0^* \to P_1^*$ *is a monomorphism and so* $\mathrm{pd}(\mathrm{Tr}_\gamma(M)) \leq 1$ *which implies*

$$M \cong \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong \mathrm{Ext}^1_R(\mathrm{Tr}_\gamma(M), R).$$

(v) If M is not *projective, then* $\mathrm{Tr}_\gamma(M) \neq 0$.

Note that the isomorphisms above containing Ext *or* Tor *are abelian group isomorphisms but because of the naturality in (i), when the ring R is a commutative ring, then all these isomorphisms become R-module isomorphisms. Even when R is not commutative,* $\mathrm{Ext}^1_R(-,R)$ *has a right or left R-module structure using the bimodule structure ${}_RR_R$ and the isomorphisms containing those are left or right R-module isomorphisms.*

Theorem 1.2.3. *(Maşek, 2000, Proposition 5) Let M be a* finitely presented *(right) R-module. Let $\sigma_M : M \to M^{**}$ be the natural map into the double dual. Let* $K_M =$ $\mathrm{Ker}(\sigma_M)$ *and* $C_M = \mathrm{Coker}(\sigma_M)$. *Then we have natural isomorphisms*

$$K_M \cong \mathrm{Ext}^1_R(\mathrm{Tr}(M), R) \qquad \textit{and} \qquad C_M \cong \mathrm{Ext}^2_R(\mathrm{Tr}(M), R).$$

Note that the right sides do not depend on which presentation of M is used to obtain $\mathrm{Tr}(M)$. *That is, we have the following exact sequence:*

$$0 \longrightarrow \mathrm{Ext}^1_R(\mathrm{Tr}(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow \mathrm{Ext}^2_R(\mathrm{Tr}(M), R) \longrightarrow 0,$$

Note that the Ext groups here are (right) R-modules and the isomorphisms are R-module isomorphisms.

Auslander-Bridger transpose is not unique, to solve this problem the stable category is used. For any two (right) R-modules M and N, let $\mathfrak{P}_{\mathfrak{R}}(M,N)$ be the set of module homomorphisms $f : M \to N$ that factor through a projective module, that is $f = g \circ h$ for some homomorphisms $g : P \to N$ and $h : M \to P$, where P is a projective module. Then $\mathfrak{P}_{\mathfrak{R}}$ is an ideal of Mod-R and the *stable category* is the factor category $\underline{\text{Mod}}$-R = (Mod-R)/$\mathfrak{P}_{\mathfrak{R}}$. If mod-$R$ is the full subcategory of Mod-R that consists of all finitely presented right R-modules, then $\underline{\text{mod}}$-R = (mod-R)/$\mathfrak{P}_{\mathfrak{R}}$. Two R-modules M and N are said to be stably isomorphic if they are isomorphic objects in the stable category $\underline{\text{Mod}}$-R. It is known that two R-modules M and N are stably isomorphic if and only if they are projectively equivalent, that is, $M \oplus P \cong N \oplus Q$ for some projective R-modules P and Q. Since Tr(M) is unique up to projective equivalence, the duality given by the Auslander-Bridger transpose is best formulated using the stable categories of finitely presented left and right R-modules. Similarly, the stable category R-$\underline{\text{mod}}$ is defined. See Auslander et al. (1997, Section IV.1) for the stable categories.

Theorem 1.2.4. *(Facchini & Girardi, 2012, Theorem 6.2) For any ring R, the Auslander-Bridger transpose is a duality* Tr : $\underline{\text{mod}}$-$R \to R$-$\underline{\text{mod}}$ *of the stable category* $\underline{\text{mod}}$-R *of finitely presented right R-modules into the stable category R-*$\underline{\text{mod}}$ *of finitely presented left R-modules.*

1.3 Auslander-Bridger Transpose over Semiperfect Rings

For semiperfect rings, see Section 2.2. Let R be a semiperfect ring. Then every finitely generated R-module has a projective cover. Let M be a finitely presented module. Then it has a projective cover $p : P \to M$. Now take a projective cover of Ker(p), say $g : Q \to$ Ker(p). Let $q : Q \to P$ be this map g with range enlarged to P. Then we say that the presentation

$$\gamma : Q \xrightarrow{q} P \xrightarrow{p} M \longrightarrow 0$$

where Q and P are finitely generated projective modules is a *minimal presentation* of the finitely presented R-module M. Over a semiperfect ring R, every finitely generated projective R-module is isomorphic to a direct sum of indecomposable projective modules. If δ is another minimal presentation of M, then this presentation γ is unique up to isomorphism by the below Lemma 1.3.2. Therefore over a semiperfect ring R, a finitely presented R-module M has a unique minimal projective presentation up to isomorphism of complexes of modules. In this section for the following theorems, when taking the Auslander-Bridger transpose of an R-module M, we take a minimal projective presentation of M and $\mathrm{Tr}(M)$ denotes the Auslander-Bridger transpose of M with respect to this minimal presentation of M (Note that the precise definition of a minimal projective presentation of a minimal projective presentation is as follows:

Definition 1.3.1. An exact sequence of R-modules

$$Q \xrightarrow{q} P \xrightarrow{p} M \longrightarrow 0$$

is said to be a *minimal projective presentation* of a finitely presented R-module M if Q and P are finitely generated projective modules, $\mathrm{Ker}(q) \ll Q$ and $\mathrm{Im}(q) \ll P$.

Lemma 1.3.2. *(Anderson & Fuller, 1992, Lemma32.11) If M and N have minimal projective presentations*

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0 \quad \textit{and} \quad Q_1 \xrightarrow{g} Q_0 \longrightarrow N \longrightarrow 0,$$

then $M \cong N$ if and only if there are isomorphisms ϕ_1 and ϕ_0 making the diagram

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "\phi_1"] & P_0 \arrow[d, "\phi_0"] \\ Q_1 \arrow[r, "g"] & Q_0 \end{tikzcd}$$

commutative.

Theorem 1.3.3. *(Anderson & Fuller, 1992, Theorem 32.13) Let R be a semiperfect ring. If M is a right (left)* stable *R-module with a* minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0,$$

then the exact sequence

$$P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}(M) \longrightarrow 0$$

is a minimal projective presentation *of the left (right) R-module* $\mathrm{Tr}(M) = \mathrm{Coker}(f^*) = P_1^*/\mathrm{Im}(f^*)$. *Moreover,* $\mathrm{Tr}(M)$ *is a* stable *R-module.*

Over a semiperfect ring R, applying the Auslander-Bridger transpose gives a one to one correspondence between the finitely presented stable right R-modules and the finitely presented stable left R-modules. For a basic set of primitive idempotents $e_1,..,e_n$ of a semiperfect ring R, $e_iR \mapsto Re_i \cong (e_iR)^*$ is a one to one correspondence between the isomorphism types of indecomposable projective right and left R-modules.

Corollary 1.3.4. *(Anderson & Fuller, 1992, Corollary 32.14) If R is a semiperfect ring, then the Auslander-Bridger transpose mappings* Tr *(with respect to minimal presentations) between the classes of the finitely presented stable right and left R-modules, satisfy the following for all finitely presented stable right and left R-modules M and N:*

(i) $\mathrm{Tr}(0) = 0$,

(ii) $\mathrm{Tr}(M) \cong \mathrm{Tr}(N)$ *if and only if* $M \cong N$,

(iii) $\mathrm{Tr}(M \oplus N) \cong \mathrm{Tr}(M) \oplus \mathrm{Tr}(N)$,

(iv) $\mathrm{Tr}(\mathrm{Tr}(M)) \cong M$.

Over a semiperfect ring R, every finitely generated R-module has a decomposition into a direct sum of a projective submodule and a stable submodule which are unique up to isomorphism by the following theorem:

Theorem 1.3.5. *(Warfield (1975, Theorem 1.4) or Facchini (1998, Theorem3.15)) Let M be a finitely generated R-module over a semiperfect ring R. Then $M = P \oplus N$, where P is a projective submodule and N is a stable submodule of M. This decomposition is unique up to isomorphism, in the sense that if $M = P' \oplus N'$ is another decomposition with P' a projective submodule and N' a stable submodule, then $P \cong P'$ and $N \cong N'$.*

Proof. We shall give the details of the proof given by Warfield (1975, Theorem 1.4). Let M be a finitely generated R-module over a semiperfect ring R and let J be the Jacobson radical of R.

Existence: We shall prove by induction on the composition length $\mathrm{cl}(M/M\mathrm{J})$ of $(M/M\mathrm{J})$. If $\mathrm{cl}(M/M\mathrm{J}) = 0$, then $M/M\mathrm{J} = 0$, that is $M = M\mathrm{J}$. Since M is a finitely generated R-module, $M = 0$ by Nakayama's Lemma and we are done. Now, suppose that $\mathrm{cl}(M/M\mathrm{J}) > 0$ and the induction hypothesis is true for all modules N such that $\mathrm{cl}(N/N\mathrm{J}) < \mathrm{cl}(M/M\mathrm{J})$. If M is stable, we are done. Assume that, M has a nonzero projective direct summand, say $M = P \oplus N$ where P is a nonzero projective submodule and N is a submodule of M. Since R is a semiperfect ring, every finitely generated projective module is a direct sum of hollow projective modules (see Proposition 2.2.11) each of which always has a local endomorphism ring by Proposition 1.4.1. Say $P = \bigoplus_{i=1}^{n} P_i$ for some $n \in \mathbb{Z}^+$ and submodules $P_1, P_2, ..., P_n$ where each P_i has a local endomorphism ring for $i = 1, 2, ..., n$ and $P_i \cong e_i R$ for some nonzero idempotent $e_i \in R$. Then $\mathrm{cl}(P/P\mathrm{J}) > 0$ because if $\mathrm{cl}(P/P\mathrm{J})$ were 0, then we would have $P = P\mathrm{J}$ which would imply that $P = 0$ would hold by Nakayama's Lemma and this contradict $P \neq 0$. Since

$$M/M\mathrm{J} \cong (P/P\mathrm{J}) \oplus (N/N\mathrm{J}), \quad \text{and} \quad \mathrm{cl}(P/P\mathrm{J}) > 0,$$

we have $\mathrm{cl}(N/N\mathrm{J}) < \mathrm{cl}(M/M\mathrm{J})$. By induction hypothesis, there exists a decomposition $N = P' \oplus N'$ for some submodules P' and N' of N where P' is projective and N' is stable. Therefore,

$$M = P \oplus N = P \oplus (P' \oplus N') = (P \oplus P') \oplus N'$$

is a decomposition of M where $P \oplus P'$ is projective and N' is stable.

Uniqueness: Suppose that $M = P \oplus N = P' \oplus N'$ for some submodules P, P', N, N' of M such that P, P' are projective and N, N' are stable. The finitely generated projective R-modules P and P' are finite direct sums of finitely generated hollow R-modules by Proposition 2.2.11. The endomorphism ring of a hollow projective R-module is local by Proposition 1.4.1. These imply that every finitely generated projective R-module has the exchange property by Warfield (1969b, Proposition 1). Applying the exchange property for P, there are submodules $P'_0 \leq P'$ and $N'_0 \leq N'$ such that $M = P \oplus (P'_0 \oplus N'_0)$. Since

$$M = P \oplus (P'_0 \oplus N'_0) = P \oplus N,$$

we obtain that $N \cong P'_0 \oplus N'_0$. But N has no nonzero projective direct summand and so $P'_0 = 0$ which gives $N \cong N'_0$ and $M = P \oplus N'_0$. We have $N'_0 \leq N' \leq P \oplus N'_0$. Then

$$N' = N' \cap (P + N'_0) = N' \cap P + N'_0.$$

So we can decompose $N' = (N' \cap P) \oplus N'_0$ and obtain

$$M = P' \oplus (N' \cap P) \oplus N'_0 = P \oplus N'_0.$$

Thus $P \cong P' \oplus (N' \cap P)$ since both sides are isomorphic to M/N'_0. Direct summand of a projective R-module is a projective R-module and so $N' \cap P$ is projective. But it is also a direct summand of N' and N' is stable. Hence $N' \cap P = 0$. Therefore $N' = N'_0$. We obtain that $N \cong N'$ and $P \cong P'$. $\square$

Proposition 1.3.6. *(Anderson & Fuller, 1992, Proposition 27.10) For a semiperfect ring R, there exists a complete set $e_1, ..., e_n$ of pairwise orthogonal primitive idempotents, so $R = e_1R + ... + e_nR$ and $e_iR/e_i\mathrm{J}$ is simple and each e_iRe_i is a local ring.*

Now, let $e_1, ..., e_n$ and $f_1, ..., f_m$ be primitive idempotents in a semiperfect ring R and suppose that M has a minimal projective presentation:

$$Q = \bigoplus_{j=1}^{m} f_jR \xrightarrow{\psi} P = \bigoplus_{i=1}^{n} e_iR \longrightarrow M \longrightarrow 0.$$

The elements of $Q = \bigoplus_{j=1}^{m} f_jR$ and $P = \bigoplus_{i=1}^{n} e_iR$ can be viewed as column vectors, that is, as $m \times 1$ and $n \times 1$ matrices respectively, so that the morphism $\psi : Q \to P$ is left multiplication by an $n \times m$ matrix $A = (a_{ij})_{i,j=1}^{n,m}$ such that $a_{ij} \in e_iRf_j$: For all $r_1, ..., r_m \in R$, we have

$$\begin{bmatrix} f_1r_1 \\ . \\ . \\ . \\ f_mr_m \end{bmatrix} \xmapsto{\psi} A \begin{bmatrix} f_1r_1 \\ . \\ . \\ . \\ f_mr_m \end{bmatrix} = \begin{bmatrix} e_1s_1 \\ . \\ . \\ . \\ e_ns_n \end{bmatrix}$$

where $\sum_{j=1}^{m} a_{ij} f_j r_j = s_i \in R$ for $i = 1,..,n$. Then we can apply to this minimal projective presentation of M, the functor $(-)^* = \mathrm{Hom}_R(-,R) : \mathrm{Mod}\text{-}R \to R\text{-}\mathrm{Mod}$. Thus we obtain an exact sequence of left R-modules

$$0 \longrightarrow M^* \longrightarrow P^* \cong \bigoplus_{i=1}^{n} Re_k \xrightarrow{\psi^*} Q^* \cong \bigoplus_{j=1}^{m} Rf_j \longrightarrow \mathrm{Tr}(M) \longrightarrow 0.$$

This time the elements of $P^* \cong \bigoplus_{i=1}^{n} Re_k$ and $Q^* \cong \bigoplus_{j=1}^{m} Rf_j$ can be viewed as row vectors, that is, as $1 \times n$ and $1 \times m$ matrices respectively, so that the morphism $\psi^* : P^* \to Q^*$ is right multiplication by the same $n \times m$ matrix A. That is,

$$\begin{bmatrix} s_1e_1 & . & . & . & s_ne_n \end{bmatrix} \overset{\psi^*}{\longmapsto} \begin{bmatrix} s_1e_1 & . & . & . & s_ne_n \end{bmatrix} A = \begin{bmatrix} r_1f_1 & . & . & . & r_mf_m \end{bmatrix},$$

where $r_j = \sum_{i=1}^{n} s_i e_i a_{ij}$ for $j = 1,..,m$ and $\mathrm{Tr}(M)$ is the cokernel of ψ^*.

1.4 Auslander-Bridger Modules

We saw in Section 1.3 that over a semiperfect ring R, taking the Auslander-Bridger transpose establishes a one to one correspondence between the isomorphism classes of finitely presented stable right R-modules and the isomorphism classes of finitely presented stable left R-modules. Over a general ring R, Facchini & Girardi (2012) extend this one to one correspondence between the two isomorphism classes of modules which they call Auslander-Bridger right and left R-modules by taking an Auslander-Bridger transpose with respect to a minimal presentation. To define the Auslander-Bridger modules, firstly note the following for projective modules:

Lemma 1.4.1. *(Facchini & Girardi, 2012, Lemma 2.1) The following conditions are equivalent for a projective (right) module P_R over an arbitrary ring R:*

(i) The module P_R is hollow, that is, it is nonzero and the sum of any two proper submodules of P_R is a proper submodule of P_R,

(ii) The endomorphism ring $\mathrm{End}_R(P_R)$ of P_R is a local ring,

(iii) The module P_R is the projective cover of a simple module,

(iv) There exist an idempotent $e \in R$ with $P_R \cong eR$ and eRe is a local ring,

(v) The module P_R is a local module,

(vi) The module P_R is finitely generated, nonzero, and all its proper submodules are small.

Moreover, if these equivalent conditions hold, then $P^* = \mathrm{Hom}_R(P_R, R)$ is a hollow projective left R-module.

Proposition 1.4.2. *(Facchini & Girardi, 2012, Proposition 2.4) The following conditions are equivalent for a projective right module $P_R \neq 0$ over an arbitrary ring R:*

(i) The module P_R is a direct sum of finitely many hollow submodules,

(ii) The module P_R is finitely generated, $P_R/P_R\,\mathrm{J}(R)$ is semisimple, and every quotient of P_R has a projective cover,

(iii) P_R is the projective cover of a finitely generated semisimple module T_R, and every direct summand of T_R has a projective cover,

(iv) The endomorphism ring $\mathrm{End}_R(P_R)$ of P_R is a semiperfect ring.

Let $\mathbf{P}_R$ be the class of projective (right) R-modules satisfying the equivalent properties of the above proposition:

$\mathbf{P}_R = \{P \mid P$ is a projective right R-module whose endomorphism ring is semiperfect$\}$.

Denote $\mathbf{P}_R$ shortly by $\mathbf{P}$. An R-module M is said to be $\mathbf{P}$-*finitely generated* if there exists a surjective morphism $P_0 \to M$ with $P_0 \in \mathbf{P}$, and we say that M is $\mathbf{P}$-*finitely presented* if there exists an exact sequence

$$P_1 \xrightarrow{q} P_0 \xrightarrow{p} M \longrightarrow 0$$

with both P_1 and P_0 in $\mathbf{P}$. Every $\mathbf{P}$-finitely presented R-module M has a *minimal projective presentation* as above with both P_1 and $P_0 \in \mathbf{P}$ by Facchini & Girardi (2012, Proposition 3.3). It is shown in Facchini & Girardi (2012, Proposition 2.4) that for a ring R, every finitely presented R-module is $\mathbf{P}$-finitely presented if and only if R is a semiperfect ring. Facchini and Girardi extend the properties satisfied by all finitely presented modules over a semiperfect ring to some of the finitely presented modules which they call Auslander-Bridger modules over an arbitrary ring:

Definition 1.4.3. An R-module M is said to be an *Auslander-Bridger module* if it is $\mathbf{P}$-finitely presented and stable.

Facchini & Girardi (2012) show that every $\mathbf{P}$-finitely generated module can be decomposed into a direct sum of a projective submodule and a stable submodule uniquely up to isomorphism, and obtain some results like the one over semiperfect rings:

Proposition 1.4.4. *(Facchini & Girardi, 2012, Proposition 3.5) Let X be a $\mathbf{P}$-finitely generated R-module. Then X decomposes as $X = P \oplus N$, where $P \in \mathbf{P}_R$ and N is a stable module. Moreover, if $X = P' \oplus N'$ is another decomposition of X with $P' \in \mathbf{P}_R$ and N' stable, then $P \cong P'$ and $N \cong N'$.*

Proposition 1.4.5. *(Facchini & Girardi, 2012, Proposition 3.7) Let M and N be $\mathbf{P}$-finitely presented modules. Then M and N are stably isomorphic if and only if $M \oplus P_0 \cong N \oplus P_1$ for suitable $P_1, P_0 \in \mathbf{P}_R$.*

Corollary 1.4.6. *(Facchini & Girardi, 2012, Corollary 3.8)*

(i) Every $\mathbf{P}$-finitely presented module is the direct sum of an Auslander-Bridger module and a module in $\mathbf{P}_R$.

(ii) Every $\mathbf{P}$-finitely presented module is stably isomorphic to an Auslander-Bridger module.

(iii) Two Auslander-Bridger modules M and N are stably isomorphic if and only if they are isomorphic.

1.5 Main Results of This Thesis

The main results obtained in this thesis are the following:

(i) If R is a *left* semihereditary ring and M_R is a finitely presented *right* R-module then M_R has a decomposition $M = P \oplus N$ for some projective submodule P and a stable submodule N of M. See Example3.3.5.

(ii) If R is a right semiartinian right V-ring which is not a semisimple ring, then there exists a cyclic right R-module M that cannot be decomposed into a direct sum of a projective module and a stable module, see Theorem 3.4.5 (Noyan Er). Such a module M can not be finitely presented module because right semiartinian right V-rings are von Neumann regular, and over a von Neumann regular ring, every finitely presented module is projective.

(iii) If R is a right hereditary ring that is not right perfect or left coherent, for some infinite index set I the right R-module $M = \prod_{i \in I} R_i$ cannot be decomposed into a direct sum of a projective module and a stable module. See Theorem 3.4.13. In this result, we use the fact, over a right hereditary ring R, a module M_R is stable if and only if $M^* = 0$.

(iv) If R is a right semihereditary ring that is not right Baer, then there exists a finitely generated torsionless R-module M that cannot be decomposed into a direct sum of a projective module and a stable module. See Proposition 3.4.21. In this example, we use the fact that over a right semihereditary ring, a finitely generated module M_R is stable if and only if $M^* = 0$.

(v) There exists a commutative ring R, over which there exists a finitely presented R-module actually a cyclically presented R-module M that cannot be decomposed into a direct sum of a projective module and a stable module. See Example 3.4.23 (Noyan Er).

CHAPTER TWO
PRELIMINARIES

In this chapter, we will summarize the main results that we use from module and ring theory, and homological algebra. In the second section, we will give some properties of local, semilocal and semiperfect rings. In the sections four and five, we will give some properties of uniform dimension and hollow dimension of R-modules. In the third section, we will give the Krull-Remak-Schmidt theorems.

2.1 Preliminaries, Terminology and Notation

Throughout this thesis, by a ring we mean an associative ring with unity; R will denote such a general ring, unless otherwise stated. So, if nothing is said about R in the statement of a theorem, proposition, etc., then that means R is just an arbitrary ring. We consider unital right R-modules; an R-module will mean a right R-module always. We denote the abelian group by $\mathcal{A}b$. Mod-R denotes the category of all right R-modules. R-Mod denotes the category of all left R-modules. cl(M) denotes the composition length of an R-module M. For an R-module M, Rad(M) denotes the radical of M, that is, the intersection of all maximal submodules of M. For the ring R, J(R) denotes the Jacobson radical and U(R) denotes the group of units in R. Endomorphism ring of the R-module M is denoted by $\mathrm{End}_R(M)$ and the injective hull of the R-module M is denoted by E(M). All definitions not given here can be found in Anderson & Fuller (1992), Lam (1999) and Kasch (1982). We do not give the details of definitions of every term in modules, rings and homological algebra. Essentially, we accept fundamentals of module theory, categories, the Hom functors and tensor product, projective modules, injective modules, the functor $\mathrm{Ext}^1_R : R\text{-}\mathrm{Mod}\times R\text{-}\mathrm{Mod} \to \mathcal{A}b$, projective dimension of a module are known. For more details in homological algebra see the books Alizade & Pancar (1999) and Rotman (1979). For modules and rings, see the books Anderson & Fuller (1992), Lam (1999, 2001), Facchini (1998) and Kasch (1982). We will explain most of the terms and summarize the necessary concepts in the next sections.

2.2 Local, Semilocal and Semiperfect Rings

Definition 2.2.1. An R-module M is said to be *indecomposable* if it is nonzero and cannot be written as a direct sum of two non-zero submodules of M.

Definition 2.2.2. Let R be a ring. An element $e \in R$ is said to be an *idempotent* if $e^2 = e$. Two idempotents e and f of R are said to be *orthogonal* if $fe = ef = 0$. An idempotent $e \in R$ is said to be *primitive* if e has no decomposition $e = e_1 + e_2$ as a sum of nonzero orthogonal idempotents e_1, e_2 in R.

For any ring R, $\mathrm{U}(R)$ denotes the group of units of R.

Definition 2.2.3. (Lam, 2001, Section 20)A ring R is said to be a *local* ring if any of the following equivalent conditions hold:

(i) R has a unique maximal right ideal;

(ii) R has a unique maximal left ideal;

(iii) $R/\mathrm{J}(R)$ is a division ring;

(iv) $R \setminus \mathrm{U}(R)$ is an ideal of R;

(v) $R \setminus \mathrm{U}(R)$ is a group under addition.

If R is a local ring, then R has no nontrivial idempotents different from 0 and 1.

Definition 2.2.4. A finitely generated R-module M is said to be *local module* if M contains a largest proper submodule. The largest proper submodule of M coincides with $\mathrm{Rad}(M)$ and with the set of nongenerators of M. For a local module M, $M = mR$ for every $m \in M \setminus \mathrm{Rad}(M)$, in particular, M is cyclic.

Definition 2.2.5. A ring R is said to be *semilocal* if $R/\mathrm{J}(R)$ is a right Artinian ring, or equivalently, if $R/\mathrm{J}(R)$ is a semisimple ring.

Lemma 2.2.6. *(Facchini, 1998, Lemma 1.14) Let R be a semilocal ring and let M be a finitely generated right R-module. Then:*

(i) $M/M\mathrm{J}$ *is a semisimple R-module of finite length;*

(ii) *M is local if and only if* $M/M\mathrm{J}$ *is simple;*

(iii) *M is a direct sum of indecomposable modules.*

Definition 2.2.7. A ring R is said to be *semiperfect* if R is semilocal and idempotents of $R/\mathrm{J}(R)$ can be lifted to R (that is, for every idempotent b in $R/\mathrm{J}(R)$, there exists an idempotent e in R such that $b = e + \mathrm{J}(R)$).

Definition 2.2.8. A subset A of a ring R is said to be *right (left) T-nilpotent*, if for any sequence $(a_i)_{i=1}^{\infty} \in A$, there exists an integer $n \geq 1$ such that $a_1 a_2 \cdots a_n = 0$ (resp. $a_n a_{n-1} \cdots a_1 = 0$).

Definition 2.2.9. A ring R is said to be *right (left) perfect* if $R/\mathrm{J}(R)$ is semisimple and $\mathrm{J}(R)$ is right (left) T-nilpotent.

Proposition 2.2.10. *(Facchini, 1998, Proposition 3.14) Let M be a right R-module and let* $E = \mathrm{End}_R(M_R)$ *be its endomorphism ring. The following conditions are equivalent:*

(i) M_R *is a finite direct sum of R-modules with local endomorphism rings;*

(ii) E_E *is a finite direct sum of E-modules with local endomorphism rings.*

(iii) *E is semiperfect.*

Proposition 2.2.11. *(Lam, 2001, Corollary 24.14)*

(i) *If R is a semiperfect ring, then any finitely generated projective module* P_R *is isomorphic to a finite direct sum* $\bigoplus_{i \in I} e_i R$*, where* $\{e_i\}$ $i \in I$ *are orthogonal local idempotents and I is a finite indexing set.*

(ii) *If R is right perfect ring, then any projective module* P_R *is isomorphic to a direct sum* $\bigoplus_{i \in I} e_i R$*, where* $\{e_i\}$ $i \in I$ *are orthogonal local idempotents for some finite index set I.*

Proposition 2.2.12. *(Anderson & Fuller, 1992, Theorem 27.6) The following are equivalent for a ring R:*

(i) *R is semiperfect;*

(ii) $R = e_1R \oplus ... \oplus e_nR$ *for (pairwise orthogonal) idempotents* $e_1, \cdots, e_n \in R$ *such that every R-module* e_iR *is local;*

(iii) $R = Re_1 \oplus ... \oplus Re_n$, *for some idempotents* $e_1, \cdots, e_n \in R$ *such that every R-module* Re_i *is local;*

(iv) *R contains a complete system of orthogonal idempotents* $e_1, ..., e_n$ *such that any ring* $e_iRe_i \cong \mathrm{End}_R(e_iR) \cong \mathrm{End}_R(Re_i)$ *is local;*

(v) *Every finitely generated right (left) R-module over R has a projective cover.*

A decomposition $R = e_1R \oplus ... \oplus e_nR$ for a complete set of orthogonal primitive idempotents $e_1, ..., e_n$ of a semiperfect ring R is unique in view of the Krull-Remak-Schmidt Theorem given in Section 2.3, but more can be said:

Theorem 2.2.13. *(Anderson & Fuller, 1992, Theorem7.9) Let R be a ring whose identity can be written as a sum*

$$1 = e_1 + ... + e_n$$

of pairwise orthogonal primitive idempotents $e_1, \cdots, e_n \in R$. *Let* $f_1, f_2..., f_m$ *be the block idempotents (central primitive idempotents) of R determined by* $E = \{e_1, .., e_n\}$. *Then* $f_1, .., f_m$ *are pairwise orthogonal central idempotents, with*

$$1 = f_1 + ... + f_m.$$

Moreover, each block f_iRf_i $(i = 1, ..m)$ *is an indecomposable ring, and*

$$R = (f_1Rf_1) + (f_2Rf_2) + ... + (f_mRf_m)$$

is a (necessarily unique) decomposition of R into indecomposable rings.

An idempotent e of a semiperfect ring R is said to be a *basic idempotent* of R if $e = e_1 + ... + e_m$ of primitive idempotents of R. A set $\{e_1, \cdots, e_m\}$ of idempotents of semiperfect ring R is said to be *basic* if it is pairwise orthogonal and $e_1R,\cdots e_mR$ is a complete irredundant set of representatives of the primitive right R-modules. Since a basic set of primitive idempotents are pairwise orthogonal, every basic set sums to a basic idempotent. If $e = e_1 + ... e_m$ and $f_1 + ... + f_m$ are basic idempotents for R, then $eR = e_1R + ... + e_nR \cong f_1R + ... + f_mR = fR$ so as rings $eRe \cong \mathrm{End}_R(eR_R) \cong \mathrm{End}_R(fR_R) \cong fRf$. A ring S is a *basic ring* for R if $S \cong eRe$ for some basic idempotent $e \in R$. Thus for each semiperfect ring R a basic ring exists and uniquely defined within isomorphism.

Theorem 2.2.14. *(Anderson & Fuller, 1992, Theorem 27.16) Every semiperfect ring R has a block decomposition. If $e_1, ...e_m$ is basic set of idempotents for R, then two primitive idempotents e and f of R belong to the same block if and only if there exist idempotents $e_{i_1}, ..., e_{i_t}$ in that basic set such that $eR \cong e_{i_k}R$, $fR \cong e_{i_k}R$ and the members of each consecutive pair $e_{i_j}R$, $e_{i_{j+1}}R$ for $j = 1, ..., k-1$ have a common composition factor.*

Proposition 2.2.15. *(Anderson & Fuller, 1992, Corollary 27.18) Let R be a semiperfect with* $\mathrm{J} = \mathrm{J}(R)$ *and* $\bigcap_{n=1}^{\infty} \mathrm{J}^n = 0$ *(e.g. Let R be a right Artinian). Then R is indecomposable if and only if* R/J^2 *is indecomposable.*

2.3 Krull-Remak-Schmidt Theorems

We shall give below some of the results that we use in this thesis related with Krull-Remak-Schmidt Theorems. If A, B, C are submodules of a module M and $C \le A$, then the modular law says that $A \cap (B + C) = (A \cap B) + C$.

Lemma 2.3.1. *(Facchini, 1998, Lemma2.1) If $A \le B \le A \oplus C$ for some submodules A, B, C of a module, then $B = A \oplus D$, where $D = B \cap C$.*

Definition 2.3.2. Given a cardinal number $\mathbf{N}$, an R-module M is said to have the $\mathbf{N}$-*exchange property* if for any R-module G and any two (internal) direct sum decompositions

$$G = M' \oplus N = \bigoplus_{i \in I} A_i, \quad \text{where} \quad M' \cong M \quad \text{and} \quad |I| \le \mathbf{N},$$

there are R-submodules B_i of A_i for all $i \in I$ such that

$$G = M' \oplus \left(\bigoplus_{i \in I} B_i \right).$$

In this case, an application of previous lemma to the submodules

$$B_i \leq A_i \leq B_i \oplus \left(M' \oplus \left(\bigoplus_{j \neq i} B_j \right) \right)$$

gives $A_i = B_i \oplus D_i$, where $D_i = A_i \cap \left(M' \oplus \left(\bigoplus_{j \neq i} B_j \right) \right)$. Hence the submodules B_i of A_i are direct summands of A_i.

Proposition 2.3.3. *(Warfield, 1969b, Proposition 1) An indecomposable module has the exchange property if and only if its endomorphism ring is local.*

Lemma 2.3.4. *(Kasch, 1982, Lemma 7.3.2) Let M be an R-module and $\{M_i \mid i \in I\}$ be a collection of submodules of M. Suppose*

$$M = \bigoplus_{i \in I} M_i, \quad \textit{where } \operatorname{End}_R(M_i) \textit{ is local for all } i \in I,$$

and

$$\sigma, \tau \in \operatorname{End}_R(M) \textit{ with } 1_M = \sigma + \tau.$$

Then to every $j \in I$ there exists a $U_j \subseteq M$ and an isomorphism $\varphi_j : M_j \to U_j$ which is induced by σ or τ (i.e $\varphi_j(x) = \sigma(x)$ for all $x \in M_j$ or $\varphi_j(x) = \tau(x)$ for all $x \in M_j$), so that we have

$$M = U_j \oplus \left(\bigoplus_{i \in I, i \neq j} M_i \right).$$

Lemma 2.3.5. *(Kasch, 1982, Lemma 7.3.3) Assumptions as in Lemma 2.3.4. Let further $E = \{i_1, i_2, ..., i_t\} \subseteq I$. Then there are $C_{i_j} \subseteq M$, $j = 1, ..., t$, and isomorphisms*

$$\gamma_{i_j} : M_{i_j} \to C_{i_j},$$

which are induced either by σ or τ, so that we have:

$$M = C_{i_1} \oplus C_{i_2} \oplus \ldots \oplus C_{i_t} \oplus \left(\bigoplus_{i \in I, i \notin E} M_i \right).$$

Lemma 2.3.6. *(Kasch, 1982, Lemma 7.3.4) Let M be an R-module and $\{M_i \mid i \in I\}$ be a collection of submodules of M. Suppose*

$$M = \bigoplus_{i \in I} M_i, \qquad \textit{where } \operatorname{End}_R(M_i) \textit{ is local for all } i \in I,$$

and $M = A \oplus B$ for some submodules A, B of M where A is nonzero and indecomposable. Let $\pi' : M \to A$ the corresponding projection. Then a $k \in I$ exists so that π' induces an isomorphism of M_k onto A and $M = M_k \oplus B$ holds.

Theorem 2.3.7. *(Kasch, 1982, Theorem 7.3.1) Let M be an R-module and $\{M_i \mid i \in I\}$ be a collection of submodules of M. Suppose*

$$M_R = \bigoplus_{i \in I} M_i, \qquad \textit{where } \operatorname{End}_R(M_i) \textit{ is local for all } i \in I$$

and $M_R = \bigoplus_{j \in J} N_j$ where N_j is an indecomposable submodule of M and nonzero for all $j \in J$. Then a bijection $\beta : I \to J$ exists with $M_i \cong N_{\beta(i)}$ for all $i \in I$.

Corollary 2.3.8. *(Kasch, 1982, Corollary 7.3.5) Let an R-module $M = \bigoplus_{i \in I} M_i$ where* $\operatorname{End}_R(M_i)$ *is local for all $i \in I$. Suppose $N = \bigoplus_{j \in J} N_j$ where N_j is indecomposable and nonzero for all $j \in J$ and $M \cong N$. Then a bijection $\beta : I \to J$ exists with $M_i \cong N_{\beta(i)}$ for all $i \in I$.*

Proposition 2.3.9. *(Lam, 2001, Proposition 19.20) Let R be any ring, and M be a right R-module whose submodules satisfy either the ascending chain condition or the descending chain condition. Then M can be decomposed into a finite direct sum of indecomposable submodules.*

Theorem 2.3.10. *(Kasch, 1982, Theorem 7.2.8) If $Q_R \neq 0$ is a nonzero indecomposable injective module, then* $\operatorname{End}_R(Q)$ *is local.*

Definition 2.3.11. A ring E is said to have *left stable range 1* if whenever $Ea+Eb=E$ where $a,b \in E$, there exists $e \in E$ such that $a+eb \in \mathrm{U}(E)$ (the group of units of E is denoted by $\mathrm{U}(E)$). See Lam (2001, Section:20); semilocal rings have stable range 1.

Theorem 2.3.12. *(Lam, 2001, Theorem 20.11) Let R be a ring, and A,B,C be right R-modules. Suppose $E=\mathrm{End}_R(A_R)$ has left stable range 1 (e.g. E is semilocal). Then $A \oplus B \cong A \oplus C$ as R-modules implies that $B \cong C$.*

Theorem 2.3.13. *(Lam, 2001, Theorem 20.13) Let R be a ring which has stable range 1 (e.g. R is a semilocal ring). Let A,B,C be right R-modules such that A_R is finitely generated and projective. Then $A \oplus B \cong A \oplus C$ implies that $B \cong C$.*

2.4 Uniform Dimension

Definition 2.4.1. A submodule K of M is said to be *essential* in M, if for every submodule $L \leq M$, $L \cap K = 0$ implies $L=0$. We denote it by $K \leq_e M$.

Definition 2.4.2. A module M is said to be *uniform module* if intersection of any two nonzero submodules is nonzero, that is, every nonzero submodule of M is an *essential* submodule.

Definition 2.4.3. An R-module M_R has *uniform dimension n*, where $n>0$, if there is an essential submodule $N \leq_e M$ such that N is the direct sum of n nonzero uniform submodules (if $M=0$, then $n=0$). If, on the other hand, no such integer n exists, we write $\mathrm{u.dim}(M)=\infty$.

Proposition 2.4.4. *(Lam, 1999, Proposition 6.3) Suppose $\mathrm{u.dim}(M)=n<\infty$. Then, any finite direct sum of nonzero submodules $N=N_1 \oplus N_2 \oplus N_3 \oplus \ldots N_k$ of M has $k \leq n$ summands.*

Proposition 2.4.5. *(Lam, 1999, Proposition 6.10)*

(i) $\mathrm{u.dim}\left(\bigoplus_{i=1}^{k} M_i\right)=\sum_{i=1}^{k}\mathrm{u.dim}(M_i)$ *for modules $M_1, M_2, \ldots, M_k$, where $k \in \mathbb{Z}^+$.*

(ii) Suppose $N \leq M$. Then $\mathrm{u.dim}(N) \leq \mathrm{u.dim}(M)$, *with equality when $N \leq_e M$. If N is not essential in M, then* $\mathrm{u.dim}(N) < \mathrm{u.dim}(M)$, *unless* $\mathrm{u.dim}(N) = \mathrm{u.dim}(M) = \infty$.

Proposition 2.4.6. *(Lam, 1999, Proposition 6.4)* $\mathrm{u.dim}(M) = \infty$ *if and only if M contains an infinite direct sum of nonzero submodules.*

Corollary 2.4.7. *(Lam, 1999, Corollary 6.7) Suppose a module M has (finite) composition length n. Then* $\mathrm{u.dim}(M) \leq n$ *if and only if M is semisimple.*

2.5 Hollow Dimension

Definition 2.5.1. A submodule K of M is said to be *small* or *superfluous* in M if for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

Definition 2.5.2. An R-module M is called *hollow* (or *couniform*) if $M \neq 0$ and every proper submodule N of M is small in M. We shall use the hollow terminology in this thesis.

Proposition 2.5.3. *(Wisbauer, 1991, 41.4) Hollow modules are indecomposable modules and every factor module of a hollow module is hollow.*

Definition 2.5.4. Let M be a right R-module. A finite set $\{N_i \mid i \in I\}$ of proper submodules of M is said to be *coindependent* if $N_i + \left(\bigcap_{j \neq i} N_j\right) = M$ for every $i \in I$, or, equivalently, if the canonical injective mapping

$$M / \bigcap_{i \in I} N_i \to \bigoplus_{i \in I} M/N_i$$

is bijective.

Theorem 2.5.5. *(Facchini, 1998, Theorem 2.40) The following conditions are equivalent for a right R-module M:*

(i) There do not exist an infinite coindependent set of proper submodules of M.

(ii) *There exists a finite coindependent set $\{N_1, N_2, N_3, ..., N_n\}$ of n elements of proper submodules of M with M/N_i hollow for all i and $N_1 \cap N_2 \cap N_3 ... \cap N_n$ small in M.*

(iii) *For a non-negative integer m, the cardinality of every coindependent set of proper submodules of M is $\leq m$.*

(iv) *If $N_0 \supseteq N_1 \supseteq N_2 \supseteq ...$ is a descending chain of submodules of M, then there exists $i \geq 0$ such that N_i/N_j is small in M/N_j for every $j \geq i$.*

Moreover, if these equivalent conditions hold and $\{N_1, N_2, ..., N_n\}$ is a finite coindependent set of proper submodules of M with M/N_i hollow for all i and $N_1 \cap N_2 \cap ... \cap N_n$ small in M, then every other coindependent set of proper submodules of M has cardinality $\leq n$.

Remark 2.5.6. An R-module M has finite hollow dimension n if and only if there exists a finite coindependent set $\{N_1, N_2, ..., N_n\}$ of proper submodules of M with M/N_i hollow for all i and $N_1 \cap N_2 \cap ... \cap N_n$ small in M. An R-module M has *infinite hollow dimension* if there exists an infinite coindependent set of proper submodules of M.

Proposition 2.5.7. *(Facchini, 1998, Proposition 2.42) Let M be an R-module.*

(i) $\mathrm{h.dim}(M) = 0$ *if and only if* $M = 0$.

(ii) $\mathrm{h.dim}(M) = 1$ *if and only if M is hollow.*

(iii) *If $N \leq M$ and M has finite hollow dimension, then M/N has finite hollow dimension and* $\mathrm{h.dim}(M/N) \leq \mathrm{h.dim}(M)$.

(iv) *If M has finite hollow dimension and $N \leq M$, then* $\mathrm{h.dim}(M/N) = \mathrm{h.dim}(M)$ *if and only if N is small in M.*

(v) *If M and M' are modules of finite hollow dimension, then $M \oplus M'$ is a module of finite hollow dimension and* $\mathrm{h.dim}(M \oplus M') = \mathrm{h.dim}(M) + \mathrm{h.dim}(M')$.

Semilocal rings are exactly the rings with finite hollow dimension as a right or left module:

Proposition 2.5.8. *(Facchini, 1998, Proposition 2.43) The following conditions are equivalent for a ring R:*

(i) The ring R is semilocal;

(ii) The right R-module R_R has finite hollow dimension;

(iii) The left R-module ${}_RR$ has finite hollow dimension.

Moreover, if these equivalent conditions hold,

$$\mathrm{h.dim}(R_R) = \mathrm{h.dim}({}_RR) = \dim(R/\mathrm{J}(R))$$

CHAPTER THREE
DECOMPOSITIONS INTO PROJECTIVE AND STABLE SUBMODULES

In Sections 3.2 and 3.3, we will give some classes of rings R, where every finitely presented or finitely generated R-module M has a decomposition $M = P \oplus N$ for some submodules P and N, where P is a projective R-module and N is a stable R-module. The motivation of this question comes from characterization of Noetherian rings, that is, a ring R is right Noetherian if and only if every right R-module M has a decomposition $M = E \oplus N$, where E is injective and N has no nonzero injective submodules. In the first section, for completeness, we will give the proof of this characterization of right Noetherian rings. In Section 3.4, we will give some ring examples over which we cannot decompose some modules into a direct sum of a projective submodule and a stable submodule.

3.1 Decompositions into Injective and Reduced Submodules

In abelian groups, we have that every abelian group G can be decomposed as $G = D \oplus C$ where D is the unique maximal divisible subgroup of G and C is a reduced group, that is, C has no nonzero divisible subgroup. Following this terminology, let us call an R-module M *reduced* if M has no nonzero injective submodule. The above property for abelian groups also hold for modules over a right Noetherian ring in the sense given in Theorem 3.1.2. Firstly note the following well-known characterization of the right Noetherian rings:

Theorem 3.1.1. *(Kasch, 1982, Theorem 6.5.1) The following conditions are equivalent for a ring R,*

(i) *R_R is Noetherian,*

(ii) *Every direct sum of injective (right) R-modules is injective,*

(iii) *Every countable direct sum of injective hulls of simple (right) R-modules is injective.*

We shall give the detailed proof of the following theorem: the (*i*) $\Rightarrow$ (*ii*) part is well-known; for completeness, we also give the proof of (*ii*) $\Rightarrow$ (*i*) part.

Theorem 3.1.2. *For a ring R, the following are equivalent:*

(i) R is a right Noetherian ring.

(ii) For every right R-module M, there is a decomposition $M = E \oplus N$ where E is an injective right R-module and N is a reduced, that is, N has no nonzero injective right R submodule.

Proof. (*i*) $\Rightarrow$ (*ii*) : Let

$$\Gamma = \left\{ \Lambda \subseteq \{\text{all injective submodules of M}\} : \sum_{U \in \Lambda} U = \bigoplus_{U \in \Lambda} U \right\}.$$

Then Γ is ordered by inclusion and $\Gamma \neq \emptyset$ because $\{0\} \in \Gamma$. For any H totally ordered subset H of Γ, $\Omega = \bigcup\limits_{\Lambda \in H} \Lambda$ is an upper bound of H in Γ. Then by Zorn's Lemma, there is a maximal subset $\Lambda_0 \subseteq \Gamma$ such that $\sum\limits_{U \in \Lambda_0} U = \bigoplus\limits_{U \in \Lambda_0} U$. Since R is right Noetherian, every direct sum of injective right R-modules is injective by Theorem 3.1.3, and so $E = \bigoplus\limits_{U \in \Lambda_0} U$ is an injective R-module. Since E is an injective R-module, E is a direct summand of M, say $M = E \oplus N$ for some submodule N of M. It remains to show that N is reduced. If $0 \neq E' \leq N$, where E' is an injective R-module, then $E \oplus E'$ is also injective and this contradicts with the maximality of Λ_0. Hence, N has no nonzero injective submodule, that is, N is reduced.

(*ii*) $\Rightarrow$ (*i*) : Assume that for every right R-module M, we have a decomposition $M = E \oplus N$ where E is an injective R-module and N has no nonzero injective submodules. To show R is a right Noetherian ring, it suffices to prove that, every countable direct sum of injective hulls of simple (right) R-modules is injective (see Theorem 3.1.3). Let $\{S_i\}_{i=1}^{\infty}$ be a sequence of simple R-modules. Clearly $S_i \leq \mathrm{Soc}(\mathrm{E}(S_i))$ since S_i is a simple submodule of $\mathrm{E}(S_i)$. Since we also have $\mathrm{Soc}(\mathrm{E}(S_i)) = \bigcap\limits_{H \leq_e \mathrm{E}(S_i)} H$ is the intersection of all essential submodules of $\mathrm{E}(S_i)$ and since $S_i \leq \mathrm{E}(S_i)$, we also have $\mathrm{Soc}(\mathrm{E}(S_i)) \leq S_i$. So we obtain that $\mathrm{Soc}(\mathrm{E}(S_i)) = S_i$. Now, suppose that, $\mathrm{E}(S_i) = X \oplus Y$ for some submodules X and Y of $\mathrm{E}(S_i)$. Since $\mathrm{E}(S_i)$ is injective, its direct summand X and Y are injective R-modules. Taking the socle of both sides, we obtain

$$S_i = \mathrm{Soc}(\mathrm{E}(S_i)) = \mathrm{Soc}(X) \oplus \mathrm{Soc}(Y).$$

Since S_i is a simple R-module, $\mathrm{Soc}(X) = S_i$ or $\mathrm{Soc}(Y) = S_i$. If $\mathrm{Soc}(Y) = S_i$, then $S_i \leq Y \leq \mathrm{E}(S_i)$ and Y is injective imply that $Y = \mathrm{E}(S_i)$. Similarly, if $\mathrm{Soc}(X) = S_i$, then $X = \mathrm{E}(S_i)$. Thus $\mathrm{E}(S_i)$ is indecomposable. $\mathrm{E}(S_i)$ is an injective and indecomposable R-module implies that $\mathrm{End}_R(\mathrm{E}(S_i))$ is local by Theorem 2.3.10. Suppose that $M = \bigoplus_{i\in 1}^{\infty} E_i$ is an internal direct sum, where $E_i = \mathrm{E}(S_i)$ with S_i a simple submodule of M for every $i \in \mathbb{Z}^+$. By hypothesis $M = E \oplus N$ for some injective submodule E and reduced submodule N of M. Let σ, $\tau \in \mathrm{End}(M)$ be the projection maps onto E and N:

$$\begin{aligned} \sigma : M = E \oplus N &\rightarrow M, \\ m = e + n &\mapsto e \quad \text{for every} \quad e \in E,\ n \in N. \end{aligned}$$

has $\mathrm{Ker}(\sigma) = N$ and $\mathrm{Im}(\sigma) = E$, and

$$\begin{aligned} \tau : M = E \oplus N &\rightarrow M, \\ m = e + n &\mapsto n \quad \text{for every} \quad e \in E,\ n \in N. \end{aligned}$$

has $\mathrm{Ker}(\tau) = E$ and $\mathrm{Im}(\tau) = N$. Clearly we have $1_M = \sigma + \tau$. Let $t \in \mathbb{Z}^+$ and let $E' = \{i_1, i_2, \cdots, i_t\} \subseteq \mathbb{Z}^+$. From Lemma 2.3.5, there are inclusions $C_{i_j} \rightarrow M$ and isomorphisms

$$\gamma_{i_j} : E_{i_j} \rightarrow C_{i_j} \quad \text{for} \quad j = 1, 2, 3, \cdots, t,$$

which are induced either σ or τ, that is, for each $j \in \{1, 2, \cdots, t\}$, either $\gamma_{i_j}(x) = \sigma(x)$ for all $x \in E_{i_j}$ or $\gamma_{i_j}(x) = \tau(x)$ for all $x \in E_{i_j}$ and $M = C_{i_1} \oplus C_{i_2} \oplus C_{i_3} \oplus \cdots \oplus C_{i_t} \oplus \left(\bigoplus_{i \in \mathbb{Z}^+ \setminus E} E_i \right)$. If $\gamma_{i_j}(x) = \tau(x)$ for all $x \in E_{i_j}$, then $E_{i_j} \cong \gamma_{i_j}(E_{i_j}) = \tau(E_{i_j}) \subseteq \mathrm{Im}(\tau) = N$ would hold which contradicts with N having no nonzero injective submodule. So we must have $\gamma_{i_j}(x) = \sigma(x)$ for all $x \in E_{i_j}$ and $j = 1, \cdots, t$. Thus $\sigma(E_{i_j}) = \gamma_{i_j}(E_{i_j}) = \mathrm{Im}(\gamma_{i_j}) = C_{i_j}$ for all $j = 1, \cdots, t$. Now, we assert that

$$\sigma : M = \bigoplus_{i=1}^{\infty} E_i \rightarrow M \text{ is one to one.}$$

Suppose that, for $t \in \mathbb{Z}^+$ and $i_1, i_2, \cdots, i_t \in \mathbb{Z}^+$, $\sigma\left(\sum_{j=1}^{t} x_{i_j}\right) = 0$ where $x_{i_j} \in E_{i_j}$ for $j = 1, \cdots, t$. Then we have bt the above notation in the argument for $E = \{i_1, \cdots, i_t\}$ that

$$0 = \sigma\left(\sum_{j=1}^{t} x_{i_j}\right) = \sum_{j=1}^{t} \sigma(x_{i_j}) = \sum_{j=1}^{t} \gamma_{i_j}(x_{i_j}) \in \bigoplus_{j=1}^{t} C_{i_j}$$

since $\gamma_{i_j}(x_{i_j}) \in C_{i_j}$, for each $j = 1, \cdots, t$, and so $\gamma_{i_j}(x_{i_j}) = 0$ for $j = 1, 2, \cdots, t$. Since $\gamma'_{i_j}s$ are isomorphisms, $x_{i_j} = 0$ for all $j = 1, \cdots, t$. Then σ is one to one. So $N = \ker(\sigma) = 0$ and $M = E \oplus N = E$ is injective. Thus every countable direct sum of injective hull of simple modules is injective. So R is a right Noetherian ring by Theorem 3.1.3. □

Theorem 3.1.3. *(Kasch, 1982, Theorem 6.5.1) The following conditions are equivalent for a ring R,*

(i) *R_R is Noetherian,*

(ii) *Every direct sum of injective right R-modules is injective,*

(iii) *Every countable direct sum of injective hulls of simple right R-modules is injective.*

3.2 Decompositions into Projective and Stable Submodules

Theorem 3.2.1. *If an R-module M cannot be decomposed as $M = P \oplus N$ where P is a projective submodule and N is a stable submodule, then there exists a sequence $(P_k)_{k=1}^{\infty}$ of nonzero proper projective submodules of M and a sequence $(N_k)_{k=1}^{\infty}$ of nonzero proper submodules of M such that for every $k \in \mathbb{Z}^+$,*

$$M = N_k \oplus P_k \oplus P_{k-1} \oplus \cdots \oplus P_1 \quad \textit{with} \quad N_k = N_{k+1} \oplus P_{k+1},$$

and so $\mathrm{u.dim}(M) = \infty = \mathrm{h.dim}(M)$.

Proof. If M were a projective R-module or a stable R-module, then it would have a decomposition of the required form trivially. So M must be an R-module which is not projective and it can be decomposed as $M = P \oplus N$ for some submodules P and N where P is a nonzero projective R-module. Then N is not projective since otherwise

$M = P \oplus N$ would be a projective R-module. In particular, $N \neq 0$. If N were stable, then $M = P \oplus N$ would be a decomposition into a direct sum of a projective submodule and a stable submodule. So N is not a stable R-module. Thus N is neither projective nor stable. Now argue as for M. The R-module N should have a nonzero projective direct summand, that is, $N = P_1 \oplus N_1$ for some nonzero projective submodule P_1 and a submodule N_1 which is not projective and not stable. Continuing in this way by induction, we obtain a sequence $(P_k)_{k=1}^{\infty}$ of nonzero proper projective submodules of M and a sequence $(N_k)_{k=1}^{\infty}$ of nonzero proper submodules of M such that

$$M = N_k \oplus P_k \oplus P_{k-1} \oplus \cdots \oplus P_1 \quad \text{with} \quad N_k = N_{k+1} \oplus P_{k+1} \quad \text{for all} \quad k \in \mathbb{Z}^+.$$

Since $P_i \neq 0$ for all $i \in I$, $\operatorname{u.dim}(P_i) \geq 1$ and so for every $n \in \mathbb{Z}^+$, $\operatorname{u.dim}(M) = \operatorname{u.dim}(N_n \oplus P_n \oplus P_{n-1} \oplus \cdots \oplus P_1) \geq n$. Therefore $\operatorname{u.dim}(M) = \infty$. Similarly $\operatorname{h.dim}(M) = \infty$ should hold. □

Corollary 3.2.2. *If* $\operatorname{u.dim}(M) < \infty$ *or* $\operatorname{h.dim}(M) < \infty$ *then M can be decomposed as $M = P \oplus N$ for submodules P and N of M where P is projective and N is stable.*

If R is a semilocal ring, then R_R and ${}_RR$ have finite a hollow dimension by Proposition 2.5.8.

Corollary 3.2.3. *If R is a semilocal ring, then a finitely generated right R-module M has finite hollow dimension and so it can be decomposed as $M = P \oplus N$ for submodules P and N of M, where P is projective and N is stable.*

Corollary 3.2.4. *If M_R is either a Noetherian or an Artinian module, then* $\operatorname{u.dim}(M) < \infty$*, and so M can be decomposed as $M = P \oplus N$ for some submodules P and N of M, where P is projective and N is stable.*

Corollary 3.2.5. *(i) If R is a right Noetherian ring and M is a finitely generated right R-module, then M is a Noetherian right R-module and so M can be decomposed as $M = P \oplus N$ for some submodules P and N of M, where P is projective and N is stable.*

(ii) If R is a right Artinian ring and M is a finitely generated right R-module, then M is an Artinian right R-module and so M can be decomposed as $M = P \oplus N$ for some submodules P and N of M, where P is projective and N is stable.

3.3 Decompositions over Semihereditary Rings

Observe that if P is a finitely generated non-zero projective R-module, then $P^* \neq 0$, because otherwise $P \cong P^{**} = 0^* = 0$ would hold. This gives us the following proposition:

Proposition 3.3.1. *If $M^* = 0$ for a finitely generated right R-module M, then M is stable, that is, it has* no *nonzero projective direct summand.*

Proof. If $M = P \oplus N$ for submodules P and N of M where P is a projective module, then $0 = M^* \cong P^* \oplus N^*$ gives $P^* = 0$ which implies $P = 0$ by the above observation. □

Note that Theorem 1.2.2-(ii) gives:

Proposition 3.3.2. *If M is a* finitely presented *(right) R-module with* $\mathrm{pd}(M) \leq 1$*, then* $\mathrm{Ext}^1_R(M,R)$ *is a finitely presented* left *R-module and it is stable since its dual is zero:* $(\mathrm{Ext}^1_R(M,R))^* = 0$.

For a finitely generated nonzero projective R-module P, we have $\mathrm{Tr}(P)$ is projectively equivalent to the zero module and so it is projective; in this case $\mathrm{pd}(\mathrm{Tr}(P)) = 0 \leq 1$ but $P^* \neq 0$. In Angeleri Hügel & Bazzoni (2010, Lemma 6.1-(1)), it is written that for a finitely presented R-module U, $\mathrm{pd}(\mathrm{Tr}(U)) \leq 1$ if and only if $U^* = 0$. For the "only if" part, we need to assume that U has no non-zero projective direct summand:

Theorem 3.3.3. *Let M be a finitely presented R-module.*

(i) If $M^* = 0$, then $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$.

(ii) If $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$*, then we have:*

(a) M^* is a projective and finitely generated left *R-module.*

(b) *M^{**} is a projective and finitely generated (right) R-module.*

(c) *The (right) R-module* $\mathrm{Ext}^1_R(\mathrm{Tr}(M), R)$ *is a stable R-module, that is, it has no nonzero projective direct summand, since its dual is zero:* $(\mathrm{Ext}^1_R(\mathrm{Tr}(M), R))^* = 0$.

(d) $M \cong M^{**} \oplus \mathrm{Ext}^1_R(\mathrm{Tr}(M), R)$ *gives a decomposition of the (right) R-module M into a direct sum of a projective R-module and a stable R-module.*

(iii) If $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$ and *M has* no *non-zero projective direct summand, then* $M^* = 0$.

Proof. (i) is just Theorem 1.2.2-(4). See the proof of Angeleri Hügel & Bazzoni (2010, Lemma 6.1-(1)). Suppose $\mathrm{Tr}(M)$ has projective dimension at most 1. Let γ be a presentation of M:

$$\gamma : \qquad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0.$$

Then we have the following exact sequence

$$0 \longrightarrow M^* \xrightarrow{g^*} P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}_\gamma(M) \longrightarrow 0$$

and so $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$ implies that M^* is projective. Indeed $M^* \cong \mathrm{Im}(g^*) = \mathrm{Ker}(f^*)$ is a direct summand of P_0^* (since $\mathrm{Im}(f^*)$ is projective as $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$). Since P_0^* is finitely generated, so it is direct summand. By Theorem 1.2.3, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_R(\mathrm{Tr}(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow \mathrm{Ext}^2_R(\mathrm{Tr}(M), R) \longrightarrow 0,$$

The last term $\mathrm{Ext}^2_R(\mathrm{Tr}(M), R) = 0$ since $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$. Since M^* is finitely generated and projective, M^{**} is also projective and so the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_R(\mathrm{Tr}(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow 0,$$

splits which gives $M \cong M^{**} \oplus \mathrm{Ext}^1_R(\mathrm{Tr}(M), R)$. Let $N = \mathrm{Tr}(M)$. By the left version of Proposition 3.3.2 for the finitely presented left R-module N that satisfies $\mathrm{pd}(N) \leq 1$, we have that $\mathrm{Ext}^1_R(N,R)$ is a finitely presented (right) R-module and it is stable since its dual is zero. For (iii), if we further assume that M has no nonzero projective direct summand, the projective direct summand $M^{**} = 0$ must hold. For the projective left R-module M^* this gives $M^* \cong M^{***} = 0^* = 0$, that is, $M^* = 0$. □

Definition 3.3.4. A ring R is said to be *right semihereditary (resp. left)* if every finitely generated right ideal (resp. left ideal) of R is projective as a right R-module (resp. left R-module).

If R is a left semihereditary ring, then every finitely presented left R-module has projective dimension ≤ 1 and so for every finitely presented (right) R-module M, the finitely presented left R-module $\mathrm{Tr}(M)$ satisfies $\mathrm{pd}(\mathrm{Tr}(M)) \leq 1$ which gives us the following corollary:

Corollary 3.3.5. *If R is a* left semihereditary *ring and M is a* finitely presented (right) *R-module, then $M = P \oplus N$ for some* projective *submodule P of M and* stable *submodule N of M.*

3.4 Examples where the Decomposition Fails

Definition 3.4.1. An R-module M is said to be *semiartinian* if its all nonzero quotients have a simple submodule.

Definition 3.4.2. A ring R is said to be a *right (resp. left) semiartinian ring* if every nonzero right (resp. left) R-module has a simple submodule. A ring R is said to be a *semiartinian ring* if it is right and left semiartinian.

Proposition 3.4.3. *(Stenstrom, 2012, Proposition 2.5) The following properties of a ring R are equivalent:*

(i) R is right semiartinian,

(ii) Every R-module is semiartinian,

(iii) Every nonzero R-module has nonzero socle,

(iv) Every R-module is an essential extension of its socle.

Definition 3.4.4. A ring R is called a right *V-ring* if every simple right R-module is injective.

The following example where the decomposition into a direct sum of a projective and a stable module fails for a cyclic module was given by Noyan Er.

Theorem 3.4.5. *(Noyan Er) If R is a right semiartinian right V-ring that is not a semisimple ring, then there is a cyclic right R-module M that cannot be decomposed as $M = P \oplus N$ such that P is projective and N is stable.*

Proof. Firstly; since R is a right semiartinian ring, $\operatorname{Soc}(R_R) \neq 0$ and $\operatorname{Soc}(R_R) \leq_e R$. Note that $\operatorname{Soc}(R_R)$ cannot be finitely generated. Because if $\operatorname{Soc}(R_R)$ were finitely generated, then it would be a direct sum of finitely many simple right R-modules which are injective as R is right V-ring and so $\operatorname{Soc}(R_R)$ would be an injective R-module since a finite direct sum of injective R-modules is injective. The injective submodule $\operatorname{Soc}(R_R)$ of R_R would then be a direct summand of R_R, that is, $R_R = \operatorname{Soc}(R_R) \oplus U$ for some submodule U of R_R. Then

$$\operatorname{Soc}(R_R) = \operatorname{Soc}(\operatorname{Soc}(R_R)) \oplus \operatorname{Soc}(U) = \operatorname{Soc}(R_R) \oplus \operatorname{Soc}(U),$$

which implies $\operatorname{Soc}(U) = 0$. Hence $U \cong R/\operatorname{Soc}(R_R)$ would have no nonzero simple R-module. Since R is right semiartinian, $U = 0$ would hold. Therefore $R = \operatorname{Soc}(R_R)$ would hold, that is, R would be a semisimple ring which contradicts our hypothesis. Since R is not a semisimple ring, $R/\operatorname{Soc}(R_R) \neq 0$. Since R is a right semiartinian ring, the right R-module $R/\operatorname{Soc}(R_R)$ must have a simple submodule $C/\operatorname{Soc}(R_R)$ where $\operatorname{Soc}(R_R) \subseteq C \subseteq R$. Let $c \in C \setminus \operatorname{Soc}(R_R)$. Then

$$C/\operatorname{Soc}(R_R) = (c + \operatorname{Soc}(R_R))R = (cR + \operatorname{Soc}(R_R))/\operatorname{Soc}(R_R)).$$

Let $D = cR$. We have that D is not semisimple since otherwise $D = cR \subseteq \operatorname{Soc}(R_R)$ would hold which contradicts $c \notin \operatorname{Soc}(R_R)$. So, $\operatorname{Soc}(D) \neq D$ and $\operatorname{Soc}(D) \neq 0$ cannot be finitely generated because then it would be injective (as R is a right V-ring) and so it would be a direct summand of D which contradicts $\mathrm{Soc}(D) \leq_e D$ (since R is right semiartinian).

Now consider $D/\mathrm{Soc}(D)$;

$$D/\mathrm{Soc}(D) = cR/\mathrm{Soc}(cR) = cR/cR \cap \mathrm{Soc}(R_R) \cong (cR + \mathrm{Soc}(R_R))/\mathrm{Soc}(R_R) = C/\mathrm{Soc}(R_R)$$

is a simple R-module. So $\mathrm{Soc}(D)$ is a maximal submodule of D. Since $\mathrm{Soc}(D)$ is not finitely generated, consider a decomposition of it of the form $\mathrm{Soc}(D) = A \oplus B$ where both A and B are semisimple submodules of D that are not finitely generated. Let $M = D/A$. Our claim is that M has no decomposition into a direct sum of a projective submodule and a stable submodule. Suppose for the contrary that $M = D/A = (P/A) \oplus (N/A)$ for some submodules P, N of D such that $A \subseteq P$, $N \subseteq D = cR \subseteq R$, where P/A is projective and N/A has no nonzero projective direct summand.

Firstly we must have $N/A \neq 0$ because otherwise $M = D/A = P/A$ would be projective, and then A would be a direct summand of $P = D = cR$, which must be finitely generated (being a direct summand of the finitely generated module $D = cR$) but A is not finitely generated. Our claim is that $P \not\supseteq \mathrm{Soc}(D)$. Suppose for the contrary that $P \supseteq \mathrm{Soc}(D)$. Since $\mathrm{Soc}(D) \subseteq P \subseteq D$ and $\mathrm{Soc}(D)$ is a maximal submodule of D, we must have either $P = \mathrm{Soc}(D)$ or $P = D$. As seen above $P \neq D$ (since $N/A \neq 0$). So we must then have $P = \mathrm{Soc}(D) = A \oplus B$. In this sum $B \cong P/A$ is a cyclic R-module since it is a quotient of the cyclic R-module $D/A = cR/A$ (because P/A is a direct summand of D/A). But by our choice, B is not finitely generated. This contradiction shows that $P \not\supseteq \mathrm{Soc}(D) = A \oplus B$. We already have $A \subseteq P$. Since $A \oplus B \not\subseteq P$, there exists a simple submodule $S \subseteq B$ such that $S \not\subseteq P$. So $S \cap P = 0$. We then have $S \cong (S \oplus A)/A \subseteq \mathrm{Soc}(M) = \mathrm{Soc}(P/A) \oplus \mathrm{Soc}(N/A)$. Since $((S \oplus A)/A) \cap \mathrm{Soc}(P/A) = 0$, the submodule $((S \oplus A)/A) \cap \mathrm{Soc}(P/A)$ must be a direct summand of $\mathrm{Soc}(M) = \mathrm{Soc}(P/A) \oplus \mathrm{Soc}(N/A)$. So $\mathrm{Soc}(M) = \mathrm{Soc}(P/A) \oplus ((S \oplus A)/A) \oplus (U/A)$ for some submodule U/A of $\mathrm{Soc}(M)$ where $A \subseteq U \subseteq D$. Then $(S \oplus A)/A$ is isomorphic to a direct summand of $\mathrm{Soc}(M)/\mathrm{Soc}(P/A) \cong \mathrm{Soc}(N/A)$. Therefore N/A should have a simple submodule $T/A \cong (S \oplus A)/A \cong S$. But S is a simple R-module in $B \subseteq \mathrm{Soc}(D) \subseteq D = cR \subseteq R$, that is, S is a simple R-module in R_R. Since R is a right V-ring, S is injective and so S is a direct summand of R_R. So S is also projective. Thus N/A has an injective and projective simple submodule S which is a direct summand of N/A since S is injective. This then contradicts with the assumption that N/A has no nonzero projective direct summand. This contradiction shows that the cyclic R-module $M = D/A = cR/A$ does not have a decomposition into a direct sum of a projective submodule and a stable submodule. □

Right semiartinian V-rings form a special class of von Neumann regular rings, see Baccella (1995). The following example was first presented by Faith (1982) as an example of a right semiartinian right V-ring that is not a semisimple ring.

Example 3.4.6. (Goodearl, 1991, Example 5.15) Let D be a field and V_D be an infinite dimensional vector space over D. Set $T = \mathrm{End}_D(V_D)$ and $K = \mathrm{Soc}(T)$. Let R be the subring of T generated by K and the scalar transformations $1_V d$ for all $d \in D$. Then R is von Neumann regular, and right and left semiartinian since $\mathrm{Soc}(R) = K$ and $R/K = D$. It is also a right V-ring. But, it is not semisimple due to $\mathrm{Soc}(R_R) \neq R$.

Over a right semiartinian right V-ring R that is not a semisimple ring, we cannot find a finitely presented R-module M that does not have a decomposition into a direct sum of a projective submodule and a stable submodule because such rings are von Neumann regular and every finitely presented module over a von Neumann regular ring is projective as is noted below:

Proposition 3.4.7. *By Goodearl (1991, Theorem 1.11) If R is a von Neumann regular ring and M is a finitely presented R-module, then M is projective.*

Proof. Let M be a finitely presented R-module over a von Neumann regular ring R. Then $M \cong R^n/K$ for some finitely generated submodule K of R^n. Since R^n is a finitely generated projective R-module, by Goodearl (1991, Theorem 1.11), all finitely generated submodules of R^n are direct summands of R^n. So K is a direct summand of R^n. Say $R^n = K \oplus U$ for some submodule U of R^n. Then $M \cong R^n/K \cong U$ is also projective as U is a direct summand of the projective module R^n. □

Definition 3.4.8. An R-module M is said to be *torsionless* if M can be embedded as an R-submodule into a direct product $\prod_{i \in I} R_R$ for some index set I.

By Kasch (1982, Remark 4.65), an R-module M is torsionless if and only if for every $m \neq 0$ in M, there exist a homomorphism $f \in M^* = \mathrm{Hom}_R(M, R)$ such that $f(m) \neq 0$. Thus, an R-module M is torsionless if and only if the natural map

$$\begin{array}{rcl} \iota : M & \to & M^{**} \text{ defined for all } m \in M \text{ by} \\ m & \mapsto & \iota(m): \; M^* \to R, \quad \iota(m)(f) = f(m) \text{ for all } \quad f: M \to R \text{ in } M^* \end{array}$$

is injective. For properties of torsionless modules, see Kasch (1982, Section 4). Remember that for a ring R, a direct sum of injective right R-module is injective if and only if R is a right Noetherian ring by Theorem 3.1.2. The dual problem was solved by Chase (1960):

Theorem 3.4.9. *(Chase, 1960, Theorem 3.3) For any ring R, the following statements are equivalent:*

(i) The direct product of any family of projective right R-modules is projective.

(ii) The direct product of any family of copies of R is projective as a right R-module.

(iii) R is right perfect and left coherent.

Observe that any submodule of a free right R-module is torsionless. Any projective module P_R is a direct summand of a free R-module and so P_R is torsionless. Conversely, assume that all torsionless right R-modules are projective. For all left modules ${}_RN$, N^* is a torsionless right R-module by Kasch (1982, Remarks 4.65(f)). In particular, let ${}_RN = \bigoplus_{i \in I} R$ where I is an infinite index set. Then

$$({}_RN)^* = \left(\bigoplus_{i \in I} ({}_RR) \right)^* = \prod_{i \in I} ({}_RR)^* = \prod_{i \in I} R_R$$

is torsionless and so by hypothesis it is projective. Then R is a right perfect and a left coherent ring by Theorem 3.4.9. Hence, if R is not *right perfect or left coherent*, then there exist a torsionless (right) R-module M_R that is not projective. As seen above, such an M_R can be taken to be $\prod_{i \in I} R_R$ for some infinite index set I.

Definition 3.4.10. A ring R is said to be a *right hereditary (resp. left hereditary)* if every right ideal (resp.left ideal) of R is projective as a right R-module (resp. left R-module).

Proposition 3.4.11. *(Martsinkovsky & Zangurashvili, 2015, Lemma 2.6)*

(i) If R is a right hereditary ring, then for an R-module M_R, $M^* = 0$ if and only if M is a stable R-module.

(ii) If R is a right semihereditary ring, then for a finitely generated R-module M, $M^* = 0$ if and only if M is a stable R-module.

Proof. (i) Suppose that $M^* \neq 0$. Then there exists $0 \neq f \in \mathrm{Hom}_R(M,R)$. Take the onto map $f' : M \to \mathrm{Im}(f)$ defined by $f'(x) = f(x)$ for all $x \in M$. $\mathrm{Im}(f) \neq 0$, since $f \neq 0$. Since R is right hereditary, the submodule $\mathrm{Im}(f')$ of R_R is projective. So the epimorphism f' splits. Then M has a nonzero projective direct summand isomorphic to $\mathrm{Im}(f)$. Therefore M is not stable. Conversely suppose that M is not stable. Let $0 \neq P$ be a nonzero projective direct summand of M, say $M = P \oplus N$ for some submodule N of M. Then $M^* = P^* \oplus N^* \neq 0$ because for a nonzero projective R-module P, $P^* \neq 0$ by below Proposition 3.4.12.

(ii) This is proved similarly as in part (i). Just note that if M_R is a finitely generated module, then for any $f \in M^*$. $\mathrm{Im}(f)$ is a finitely generated submodule of R_R. For every nonzero projective R-module P, $P^* \neq 0$ follows from the following: □

Proposition 3.4.12. *(Osborne, 2000, Proposition 4.16) Suppose $P \in M_R$. The following are equivalent:*

(i) P is projective,

(ii) If P is generated by $\{s_i \mid i \in I\}$, then there exists $\phi_i \in P^*$, $i \in I$ such that for all $x \in P$, $\{i \in I : \phi_i(x) \neq 0\}$ is finite, and $x = \sum \phi_i(x)s_i$,

(iii) There exists a generating set $\{s_i \mid i \in I\}$ of P for which there exist $\phi \in P^*$, for every $i \in I$ such that for all $x \in P$, $\{i \in I : \phi_i(x) \neq 0\}$ is finite, and $x = \sum_{i \in I} \phi_i(x)s_i$.

Theorem 3.4.13. *If R is a right hereditary ring that is not right perfect or left coherent, then there is a module M_R such that M does not have a decomposition $M = P \oplus N$ for some submodules P and N, where P is projective and N is stable.*

Proof. By the argument before Definition 3.4.10, there exists a torsionless (right) R-module M_R that is not projective. Suppose for the contrary that M has a decomposition $M = P \oplus N$ for some submodules P and N, where P is projective and N is stable. Since M is not projective, $N \neq 0$. Since M is torsionless, its nonzero submodule N is also torsionless. Hence $N^* \neq 0$. But then by Proposition 3.4.11, N is not a stable module. This contradiction ends the proof. □

Example 3.4.14. The ring $\mathbb{Z}$ is a hereditary Noetherian commutative domain which is not a perfect ring. See Lam (2001, Theorem 23.24). The $\mathbb{Z}$-module

$$M = \left(\bigoplus_{i=1}^{\infty} \mathbb{Z}\right)^* = \prod_{i=1}^{\infty} \mathbb{Z}^* = \prod_{i=1}^{\infty} \mathbb{Z}$$

is torsionless but not projective as is well-known in abelian groups (see Lam (1999, Example 2.8)). We don't have a decomposition $M = P \oplus N$ such that P is a projective $\mathbb{Z}$-module and N is a stable $\mathbb{Z}$-module by the proof of Theorem 3.4.13.

Proposition 3.4.15. *(Small, 1967, Corollary 2) If R is a right perfect ring in which principal right ideals are projective, then R is a semiprimary ring and principal left ideals of R are projective. In addition, a right hereditary right perfect ring is also left hereditary.*

By this proposition, if R is not a left hereditary ring, then R is not right perfect or R is not right hereditary. Hence a right hereditary ring that is not left hereditary, is not right perfect. So they give examples of rings where we need in Theorem 3.4.13 (a right hereditary ring which is not right perfect or left coherent). The below example is an example of a right hereditary ring that is not left hereditary.

Example 3.4.16. (Lam, 1999, Small's Example 2.33). The triangular ring

$$R = \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$$

is right hereditary but not left hereditary. By the above arguments for some infinite index set I, $M = \prod_{i \in I}^{\infty} R_R$ cannot be decomposed as $M = P \oplus N$ for some projective R-module P and stable R-module N.

Note that in the above examples the modules for which the decomposition into projective and stable submodules fails is not finitely generated. Now we shall construct another finitely generated example for which the decomposition fails.

Definition 3.4.17. A ring R is said to be *right Baer* if every right annihilator of every subset of R is of the form eR for some idempotent e in R. A ring R is said to be a *left Baer* if every left annihilator of every subset of R is of the form Re for some idempotent e in R.

Definition 3.4.18. A ring R is said to be *right Rickart* if the right annihilator of every element of R is of the form eR for some idempotent e in R. A ring R is said to be *left Rickart* if the left annihilator of every element of R is of the form Re for some idempotent e in R.

Clearly, a right (left) Baer ring is always a right (left) Rickart ring. See Lam (1999, Section 7D).

Proposition 3.4.19. *(Lam, 1999, Proposition 7.46) A ring R is right Baer if and only if it is left Baer.*

Proposition 3.4.20. *(Lam, 1999, Proposition 7.48) A ring R is right Rickart if and only if every principal right ideal in R is projective.*

Thus, right semihereditary rings are right Rickart.

Proposition 3.4.21. *If R is a right semihereditary ring that is not a right Baer ring, then there is a finitely generated torsionless R-module M which cannot be decomposed as $M = P \oplus N$ for some submodules P and N of M such that P is projective and N is stable.*

Proof. Let R be a right semihereditary ring. Then it is a right Rickart ring. Assume that R is not a left Rickart ring. Then R is not a left Baer ring. So R is not a right Baer ring by Proposition 3.4.19. Since R is not right Baer, there is a right annihilator of a subset S of R which is not a direct summand of R. That is, the right annihilator

$$I = \mathrm{r.ann}_R(S) = \{r \in R \mid Sr = 0\}$$

of S is not a direct summand of R. Say $S = \{a_\lambda \mid \lambda \in \Lambda\}$ and let $a = (a_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} R_R$. The module $\prod_{\lambda\in\Lambda} R_R = \left(\bigoplus_{\lambda\in\Lambda} R_R\right)^*$ is a torsionless R-module and so its cyclic submodule aR is also torsionless. The cyclic module $aR \cong R/\mathrm{ann}_R(a) = R/I$ and $aR \cong R/I$ is not projective since I is not direct summand of R. Now aR is a torsionless, finitely generated but not projective right R-module. This module aR cannot be decomposed as $aR = P \oplus N$ such that P is a projective R-module and N is a stable R-module by the same argument as in the proof of Theorem 3.4.13. □

As shown in the below example, there are rings which are right semihereditary but not right Baer as needed in the above proposition.

Example 3.4.22. Let S be a von Neumann regular ring with an ideal I such that as a submodule of S_S, I is not direct summand of S_S. Let $R = S/I$. Then R is also a von Neumann regular ring. As a right S-module, R is not projective since I is not a direct summand. Viewing R as an $(R\text{-}S)$-bimodule, we can form the triangular ring $T = \begin{bmatrix} R & 0 \\ R & S \end{bmatrix}$. Then T is right semihereditary but not left semihereditary. So T is right Rickart. Indeed in the proof T being not left semihereditary, it has been shown that there exists a principal left ideal that is not left projective. Thus it was shown that R is not left Rickart. So T is a right semihereditary ring which is not left Rickart. Then T is not left Baer and thus also T is not right Baer by Proposition 3.4.19. See Lam (1999, Chase's Example 2.34).

We have seen examples of finitely generated modules that has no decomposition into a direct sum of a projective submodule and a stable submodule. In the following last example, we will see a finitely presented module over a commutative ring that has no decomposition into a direct sum of a projective module and a stable module. This important example has been given by Noyan Er.

Example 3.4.23. (*Noyan Er*) Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$. Consider the commutative ring

$$R = \prod_{i=1}^{\infty} \mathbb{Z}_2, \quad \text{and its ideal} \quad D = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2 \subseteq R.$$

Let T be a maximal ideal of R such that $D \subseteq T \subseteq R$. Then $S = R/T$ is a simple R-bimodule. Then D annihilates S, that is, $SD = DS = 0$. Let A be the following matrix ring:

$$A = \begin{bmatrix} R & & S \\ & \backslash & \\ 0 & & R \end{bmatrix} = \left\{ \begin{bmatrix} r & s \\ 0 & r \end{bmatrix} : r \in R, s \in S \right\}.$$

Observe that A is a commutative ring. Then we shall show below that the Jacobson radical of the commutative ring A is $\mathrm{Rad}(A_A) = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$, and the module $M = A/\mathrm{Rad}(A_A)$ is a cyclically presented right A-module that cannot be decomposed into a direct sum of a projective submodule and a stable submodule.

A matrix $\begin{bmatrix} r & s \\ 0 & r \end{bmatrix}$ in the ring A is given by the pair (r, s) where $r \in R$ and $s \in S$. So indeed we have the ring $R \times S$ as a set, where addition is componentwise but multiplication is given by $(r, s)(r', s') = (rr', rs' + sr')$ for all $(r, s), (r', s') \in R \times S$ since

$$\begin{bmatrix} r & s \\ 0 & r \end{bmatrix} \begin{bmatrix} r' & s' \\ 0 & r' \end{bmatrix} = \begin{bmatrix} rr' & rs' + sr' \\ 0 & rr' \end{bmatrix}.$$

It is actually known, see Huckaba (1988), that the maximal ideals of this ring $R \times S$ are of the form $B \times S$ where B is a maximal ideal of R. We show this below directly. For any ideal I of the ring $A = \begin{bmatrix} R & & S \\ & \backslash & \\ 0 & & R \end{bmatrix}$, it is easily verified that

$$B = \left\{ b \in R : \begin{bmatrix} b & m \\ 0 & b \end{bmatrix} \in I \text{ for some } m \in S \right\}$$

is an ideal of R and

$$X = \left\{ x \in S : \begin{bmatrix} r & x \\ 0 & r \end{bmatrix} \in I \text{ for some } r \in R \right\}$$

is a submodule of S. But S is a simple R-module hence $X = 0$ or $X = S$. Since

$$\begin{bmatrix} 0 & bs \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b & m \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} \text{ for all } b \in B \text{ and } s \in S,$$

$BS \subseteq X$ and $\begin{bmatrix} 0 & BS \\ 0 & 0 \end{bmatrix} \subseteq I$. We have two cases $B \nsubseteq T$ or $B \subseteq T$.

Suppose $B \nsubseteq T$. Then $0 \neq BS \leqslant S$ and S is a simple module implies that $BS = S$ and so $\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \subseteq I$. Thus for each $b \in B$, since $\begin{bmatrix} b & m \\ 0 & b \end{bmatrix} \in I$ for some $m \in S$, and $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in I$ as $\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \subseteq I$, we obtain

$$\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} b & m \\ 0 & b \end{bmatrix} - \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in I.$$

So $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \subseteq I$ also. We have

$$\begin{bmatrix} B & & S \\ & \diagdown & \\ 0 & & B \end{bmatrix} = \begin{bmatrix} B & & 0 \\ & \diagdown & \\ 0 & & B \end{bmatrix} + \begin{bmatrix} 0 & & S \\ & \diagdown & \\ 0 & & 0 \end{bmatrix} \subseteq I \subseteq \begin{bmatrix} B & & S \\ & \diagdown & \\ 0 & & B \end{bmatrix}$$

where the last inclusion follows by the definition of B and S. So $I = \begin{bmatrix} B & & S \\ & \diagdown & \\ 0 & & B \end{bmatrix}$. Note that, in this case if $B = R$, then $I = A$. So if I is a proper ideal of A, then B is also a proper ideal of R.

Now suppose $B \subseteq T$. Then $I \subseteq \begin{bmatrix} T & & S \\ & \setminus & \\ 0 & & T \end{bmatrix}$.

So in any case, if I is a proper ideal of A, then I is contained in an ideal of the form $\begin{bmatrix} B & & S \\ & \setminus & \\ 0 & & B \end{bmatrix}$ where B is a proper ideal of R.

Conversely, if B is a proper ideal of R, then $\begin{bmatrix} B & & S \\ & \setminus & \\ 0 & & B \end{bmatrix}$ is a proper ideal of A. From these arguments, we obtain that each maximal ideal of A is of the form $\begin{bmatrix} B & & S \\ & \setminus & \\ 0 & & B \end{bmatrix}$ for some maximal ideal B of R. Then the Jacobson radical of the ring A is

$$\mathrm{Rad}(A_A) = \bigcap_{B \subseteq_{max} R} \begin{bmatrix} B & & S \\ & \setminus & \\ 0 & & B \end{bmatrix} = \begin{bmatrix} \mathrm{Rad}(R_R) & & S \\ & \setminus & \\ 0 & & \mathrm{Rad}(R_R) \end{bmatrix} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$

This holds because the commutative ring $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$ has $\mathrm{Rad}(R_R) = 0$ since for every $i \in \mathbb{Z}^+$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times Z_2 \times 0 \times \mathbb{Z}_2 \times \cdots$ is a maximal ideal of R where 0 is in the i-th coordinate and all other coordinates are $\mathbb{Z}_2$. Thus $\mathrm{Rad}(A_A) = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$. Let $M_A = A/\mathrm{Rad}(A_A)$. We claim that the right A-module M cannot be decomposed as a direct sum of a projective submodule and a stable submodule. Since A_A is a finitely generated A-module, $\mathrm{Rad}(A_A)$ is small in A and so it cannot be a direct summand of A. Hence $M = A/\mathrm{Rad}(A_A)$ is not a projective A-module. Suppose that $M = P \oplus N$ for some submodules P and N of M_A such that P is projective and N is stable. N cannot be projective (or 0) since M is not projective. So $N \neq 0$. Say $N = Y/\mathrm{Rad}(A_A)$ where $\mathrm{Rad}(A_A) \subsetneq Y \subseteq A$. Then Y has an element $y = \begin{bmatrix} a & s \\ 0 & a \end{bmatrix}$ that is not in $\mathrm{Rad}(A_A)$, where

$s \in S$ and $0 \neq a \in R$. Since $0 \neq a = (a_i)_{i=1}^{\infty} \in R = \prod_{i=1}^{\infty} \mathbb{Z}_2$, there exists $n \in \mathbb{Z}^+$ such that $a_n = \overline{1}$. Let $z = (\overline{0}, \cdots, \overline{0}, \overline{1}, \overline{0}, \cdots) \in R$ be the sequence whose $n'th$ coordinate is $\overline{1}$ and all other coordinates $\overline{0}$. Then

$$y\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} a & s \\ 0 & s \end{bmatrix}\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} az & sz \\ 0 & az \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

since $az = z$ and $sz = 0$ as $Sz \in SD = 0$.

$$L = \begin{bmatrix} 0\times\cdots\times 0\times\mathbb{Z}_2\times 0\cdots & 0 \\ & \backslash \\ 0 & 0\times\cdots\times 0\times\mathbb{Z}_2\times 0\cdots \end{bmatrix} \leq Y$$

where in $0\times\cdots\times 0\times\mathbb{Z}_2\times 0\times\cdots$, the n-th coordinate is $\mathbb{Z}_2$ and all other coordinates are 0. L is an ideal of A since $DS = SD = 0$. Furthermore, $L \oplus C = A_A$ for

$$C = \begin{bmatrix} \mathbb{Z}_2\times\cdots\times\mathbb{Z}_2\times 0\times\mathbb{Z}_2\times\cdots & S \\ & \backslash \\ 0 & \mathbb{Z}_2\times\cdots\times\mathbb{Z}_2\times 0\times\mathbb{Z}_2\times\cdots \end{bmatrix}$$

where in $\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2\times 0\times\mathbb{Z}_2\times\cdots$, the $n'th$ coordinate is 0 and all other coordinates are $\mathbb{Z}_2$. Thus L_A is projective since L_A is a direct summand of the right A-module A_A. Since $L \subseteq Y \subseteq A_A$, L is also a direct summand of Y. Since $\mathbb{Z}_2\times..\times\mathbb{Z}_2\times 0\times\mathbb{Z}_2\times\cdots$, is a maximal ideal of $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$, C is a maximal ideal of A_A. Observe that $\mathrm{Rad}(C_A) = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \mathrm{Rad}(A_A)$. Then from the decomposition $L \oplus C = A$, we obtain that,

$$(L \oplus \mathrm{Rad}(A_A))/\mathrm{Rad}(A_A)) \oplus (C/\mathrm{Rad}(A_A)) = A/\mathrm{Rad}(A_A)$$

and by intersecting with $N = Y/\mathrm{Rad}(A_A)$ (since $L \oplus \mathrm{Rad}(A_A) \subseteq Y$), we obtain by modular law that

$$[(L \oplus \mathrm{Rad}(A_A))/\mathrm{Rad}(A_A)] \oplus [(C/\mathrm{Rad}(A_A)) \cap (Y/\mathrm{Rad}(A_A))] = Y/\mathrm{Rad}(A_A) = N.$$

Thus $0 \neq L_A \cong (L \oplus \mathrm{Rad}(A_A)) / \mathrm{Rad}(A_A)$ is a direct summand of N and L_A is a projective A-module. This contradicts with N being a stable A-module. Therefore M has no decomposition into a direct sum of a projective submodule and a stable submodule.

CHAPTER FOUR
AUSLANDER-BRIDGER TRANSPOSE OF SIMPLE MODULES

In this chapter, we will give the detailed proof of Haack (1984, Theorem 1): for an Artinian ring R, we have that R is a serial ring if and only if the Auslander-Bridger tranpose of every nonprojective simple (right) R-module is a simple left R-module. The properties of the serial rings that we use are summarized in the first two sections using Anderson & Fuller (1992, section 32). In the last section, using Haack's result, for Artinian serial rings we give some relations between the proper classes of short exact sequences of R-modules that are projectively generated or flatly generated by simple R-modules.

4.1 Serial Rings

Definition 4.1.1. An R-module M is said to be *uniserial* if for any submodules A and B of M we have either $A \subseteq B$ or $B \subseteq A$, that is, an R-module M is uniserial if and only if its submodules are linearly ordered under set inclusion.

Definition 4.1.2. R_R is right uniserial module (resp. ${}_RR$ is left uniserial module),if every right submodules (resp. left submodules) A and B of R either $A \subseteq B$ or $B \subseteq A$ hold. A ring R is said to be *uniserial* if R_R and ${}_RR$ are uniserial R-modules.

Definition 4.1.3. An R-module M is said to be *serial* if it is a direct sum of uniserial R-modules. A ring R is said to be *right serial (resp. left serial)* if it is a serial module as a right R-module (resp. left R-module) over itself. A ring R is said to be *serial* if it is left and right serial.

Every serial ring R can be represented in the form

$$R = e_1R \oplus \cdots \oplus e_nR \;\; (R = Re_1 \oplus \cdots \oplus Re_n)$$

where every R-module e_iR (Re_i) is uniserial. Since e_i is not in $\mathrm{Rad}(e_iR)$, every R-module e_iR is local. Therefore every serial ring is semiperfect by Proposition 2.2.12. In particular, every uniserial ring is local, and a serial ring is uniserial if and only if it is local.

Definition 4.1.4. A ring R for which $\mathrm{Rad}(R)$ is nilpotent and $\overline{R} = R/\mathrm{J}(R)$ is a semisimple ring is said to be a *semiprimary* ring.

Definition 4.1.5. For each nonzero R-module M over a semiprimary ring R, there is a smallest positive integer ℓ such that $\mathrm{J}^{\ell} M = 0$. This number ℓ is said to be the *Loewy length* of M and we write $\mathrm{L}(M) = \ell$. If M=0, then we take $\mathrm{L}(M) = 0$. The *upper Loewy series*, or *radical series*, for M is

$$M > M\mathrm{J} > \cdots > M\mathrm{J}^{\ell} = 0.$$

The *lower Loewy series or socle series*, for M is

$$0 < \mathrm{Soc}(M) < \cdots < \mathrm{Soc}^{\ell}(M) = M.$$

Let $\mathbf{r}_M(\mathrm{J}^{\ell})$ denote the maximal semisimple submodule of M. $\mathbf{r}_M(\mathrm{J}^k)/\mathbf{r}_M\,\mathrm{J}^{k-1}$ is the k-th lower factor of M. Each of these factors are semisimple and nonzero unless $M = 0$. Then $\mathbf{r}_M(\mathrm{J}^k)/\mathbf{r}_M(\mathrm{J}^{k-1}) = \mathrm{Soc}(M/\mathbf{r}_M(\mathrm{J}^{k-1}))$ for $k = 1, \cdots n$. So $\mathrm{Soc}^k(M) = \mathbf{r}_M(\mathrm{J}^k)$.
Theorem 4.1.6. *(Lam, 2001, Theorem 4.15) (Hopkins-Levitzki Theorem) Let R be a semiprimary ring. Then for any R-module M, the following statements are equivalent:*

(i) M is Noetherian.

(ii) M is Artinian.

(iii) M has a composition series.

A ring R is right Artinian if and only if it is a right Noetherian and semiprimary; any finitely generated right module over a right Artinian ring has a composition series.

If R is a right Artinian serial ring, then R is a left Artinian ring by Puninski (2012, Corollary 1.31). Moreover a right serial ring (resp. left serial ring) R with nilpotent Jacobson radical M is right Artinian (resp. left Artinian). So for a serial ring R; R is right Artinian implies it is left Artinian, and R is left Artinian implies R is right Artinian. Thus for a serial ring R, R is right or left Artinian implies that R is an Artinian serial ring.

Lemma 4.1.7. *(Anderson & Fuller, 1992, Lemma 32.1) The following statements about a module $M_R \neq 0$ over a semiprimary ring R are equivalent:*

(i) *M is uniserial;*

(ii) *M has a unique composition series;*

(iii) *The upper Loewy series*

$$M > M\mathrm{J} > .. > M\mathrm{J}^{\ell} = 0$$

is a composition series for M;

(iv) *The lower Loewy series*

$$0 < \mathrm{Soc}(M) < \mathrm{Soc}^2 M < \cdots < \mathrm{Soc}^{\ell}(M) = M$$

is a composition series for M.

Theorem 4.1.8. *(Anderson & Fuller, 1992, Theorem 32.2) The following statements about a right Artinian ring R with $\mathrm{J} = \mathrm{J}(R)$ are equivalent:*

(i) *R is a serial ring;*

(ii) *R/J^2 is a serial ring.*

If R is an Artinian ring with $\mathrm{J} = \mathrm{J}(R)$, then by the above theorem R is a serial ring if and only if R/J^2 is a serial ring (see Thrall (1948, Lemma 3) where the older terminology generalized uniserial ring is used instead of Artinian serial ring).

Theorem 4.1.9. *(Anderson & Fuller, 1992, Theorem 32.3) If R is a right Artinian ring, then the following are equivalent:*

(i) *R is a serial ring;*

(ii) *Every right R-module is a direct sum of uniserial R-modules;*

(iii) *Every finitely generated indecomposable right R-module is uniserial;*

(iv) *The projective cover and the injective envelope of every simple right R-module are uniserial.*

4.2 Kupisch Series

Let R be a serial ring with a basic set of primitive idempotents $e_1, \cdots, e_n$ and $\mathrm{J} = \mathrm{J}(R)$. For each $i = 1, \cdots, n$, set $S_i = e_iR/e_i\mathrm{J}$ and $T_i = Re_i/\mathrm{J}e_i$. Thus $S_1, \cdots, S_n$ and $T_1, \cdots, T_n$ are complete irredundant sets of simple right and left R-modules, respectively. The (right) *quiver* of R is the directed graph $\mathrm{Q}(R)$ with vertex set $\{e_1, , e_n\}$ and with an arrow $e_i \to e_j$ if and only if $e_i\mathrm{J}e_j \nsubseteq \mathrm{J}^2$. Equivalently,

$$\begin{aligned} e_i \to e_j \in \mathrm{Q}(R) &\Leftrightarrow e_i\mathrm{J}/e_i\mathrm{J}^2 \cong T_j \\ &\Leftrightarrow \mathrm{J}e_j/\mathrm{J}^2e_j \cong S_i \\ &\Leftrightarrow e_jR \to e_i\mathrm{J} \to 0 \ \ is\ \ a\ \ projective\ \ cover \\ &\Leftrightarrow Re_i \to \mathrm{J}e_j \to 0 \ \ is\ \ a\ \ projective\ \ cover \end{aligned}$$

By Proposition 2.2.15, R is indecomposable if and only if $\mathrm{Q}(R)$ is connected as an undirected graph. Now since R is serial, for all $i = 1, .., n$, the modules $e_i\mathrm{J}$ and $\mathrm{J}e_i$ have unique projective covers of the form e_jR and Re_j, so in $\mathrm{Q}(R)$ in and out of any valence of any vertex is at most 1, that is, neither

$$e_i \to e_j \leftarrow e_k \quad \text{nor} \quad e_i \leftarrow e_j \to e_k$$

can occur in $\mathrm{Q}(R)$. Thus if R is also indecomposable, so that $\mathrm{Q}(R)$ is either a single directed path of length n or a single cycle of length n. That is,

Theorem 4.2.1. *(Anderson & Fuller, 1992, Theorem 32.4) If R is an indecomposable serial ring with* $\mathrm{J} = \mathrm{J}(R)$*, then a basic set of idempotents can be numbered so that the* *quiver* Q(R) *of R is either*

$$e_1 \to e_2 \to \cdots \to e_n \quad \textit{or} \quad \begin{array}{ccccc} & & e_1 & & \\ & \nearrow & & \searrow & \\ e_n & & & & e_2 \\ \vdots & & & & \downarrow \\ & & & & e_3 \\ & \ddots & & \swarrow & \\ & & \cdots & & \end{array}$$

so that there are projective covers

$$e_{i+1}R \to e_i \mathrm{J} \to 0 \ \ (i = 2,..,n) \quad \textit{and} \quad e_1 R \to e_n \mathrm{J} \quad \textit{if} \quad e_n \mathrm{J} \neq 0.$$

The list of indecomposable projective R-modules $e_1R, \cdots, e_nR$ of Theorem 4.2.1 is said to be the *Kupisch series* for R. It is unique, except for cyclic permutation of $\{1,..,n\}$ when $e_n \mathrm{J} \neq 0$. Also, if $\mathrm{cl}(e_iR) = c_i$, then the numbers $c_1, \cdots, c_n$ of composition lengths of the indecomposable projective R-modules is called an *admissible sequence* for R.

Lemma 4.2.2. *(Anderson & Fuller, 1992, Lemma 32.5) Let R be an indecomposable serial ring with Kupisch series $e_1R, \cdots, e_nR$ and* $\mathrm{J} = \mathrm{J}(R)$. *Then there exist*

$$a_{i-1} \in e_i \mathrm{J} e_{i-1} \ \textit{for} \ i = 2, \cdots, n \ \textit{and} \ a_n \in e_1 \mathrm{J} e_n$$

such that, for any $k > 0$ and any $i \in \{1, \cdots, n\}$,

$$e_i \mathrm{J}^k = a_i a_{[i+1]} \cdots a_{[i+k-1]} R \ \textit{and} \ \mathrm{J}^k e_i = R a_{[i-k]} \cdots a_{[i-2]} a_{[i-1]}$$

where $[m]$ denotes the least positive residue of m modulo n for $m \in \mathbb{Z}$.

For a detailed information about quivers of rings, see Hazewinkel et al. (2006, Chapters 11-13).

4.3 Auslander-Bridger Transpose of Simple Modules

Theorem 4.3.1. *(Haack, 1984, Theorem 1) Let R be an Artinian ring. The Auslander-Bridger transpose* $\mathrm{Tr}(S)$ *(taken with respect to a minimal projective presentation of S) is simple for every nonprojective simple (right) R-module S if and only if R is a serial ring.*

Proof. Suppose that $\mathrm{Tr}(S)$ is simple for every nonprojective simple (right) R-module S. Since R is Artinian, it is also semiperfect. So there exist a complete orthogonal set of nonzero primitive idempotents $\{e_1, \cdots, e_n\}$ for the ring R. Then $e_1 + \mathrm{J}^2, \cdots, e_n + \mathrm{J}^2$ is a complete orthogonal set of nonzero primitive idempotents of the factor ring R/J^2, because e_i's are not in J^2. For $e_i \in J^2$, we have $\mathrm{J}^2 \subseteq \mathrm{J}$ is nilpotent because R is Artinian. Then e_i is both nilpotent and idempotent implies it is zero. So we can write

$$R/\mathrm{J}^2 = \bigoplus_{i=1}^{n} (e_i/\mathrm{J}^2)(R + \mathrm{J}^2) = \bigoplus_{i=1}^{n} (R/\mathrm{J}^2)(e_i + \mathrm{J}^2).$$

Since R is Artinian, to prove that R is a serial ring, we need to show R/J^2 is a serial ring by Theorem 4.1.8. Now, to show that R/J^2 is a serial ring, for each primitive idempotent $e = e_i$ for $i \in \{1, \cdots, n\}$, we must prove that both $(R/\mathrm{J}^2)(e + \mathrm{J}^2)$ and $(e + \mathrm{J}^2)(R/\mathrm{J}^2)$ are uniserial left and right R/J^2-modules respectively. It suffices to show that they must have unique composition series by Lemma 4.1.7.

$$(R/\mathrm{J}^2)(e + \mathrm{J}^2) = (Re + \mathrm{J}^2)/\mathrm{J}^2 \cong Re/(Re \cap \mathrm{J}^2) = Re/\mathrm{J}^2 e.$$

We have $\mathrm{J}^2 e \subseteq Re \cap \mathrm{J}^2$ since $\mathrm{J}^2 e \subseteq Re$ and $\mathrm{J}^2 e \subseteq \mathrm{J}$ as J and J^2 are two sided ideals. For the other side take $x \in Re \cap \mathrm{J}^2$. Then $x = re \in \mathrm{J}^2$ and $xe = re^2 = re = x \in \mathrm{J}^2 e$. Thus we obtain that $Re \cap \mathrm{J}^2 = \mathrm{J}^2 e$. Similarly, $(e + \mathrm{J}^2)(R/\mathrm{J}^2) = eR/e\mathrm{J}^2$. Next, we must prove that both $Re/\mathrm{J}^2 e$ and $eR/e\mathrm{J}^2$ have unique composition series.

For the first part, we claim that $M = Re/\mathrm{J}^2 e$ has a unique composition series. By Lemma 4.1.7, it suffices to prove that

$$M > (\mathrm{J}/\mathrm{J}^2)M = \mathrm{J}e/\mathrm{J}^2 e > (\mathrm{J}/\mathrm{J}^2)^2 M = 0$$

is a composition series if $\mathrm{J}e/\mathrm{J}^2e \neq 0$ since $(\mathrm{J}/\mathrm{J}^2)^2M = 0$ already. That is we must prove that

$$Re/\mathrm{J}^2e > \mathrm{J}e/\mathrm{J}^2e > \mathrm{J}^e/\mathrm{J}^2e = 0$$

is a composition series if $\mathrm{J}e/\mathrm{J}^2e \neq 0$. Since the ring R is semiperfect, $S = Re/\mathrm{J}e$ is a simple R-module and

$$Re \xrightarrow{\varphi} Re/\mathrm{J}e = S \longrightarrow 0$$

is a projective cover, where φ is the canonical epimorphism. If $\mathrm{J}e/\mathrm{J}^2e = 0$, then $Re/\mathrm{J}e > \mathrm{J}e/\mathrm{J}^2e = 0$ is a composition series since $(Re/\mathrm{J}^2e)/(\mathrm{J}e/\mathrm{J}^2e) = Re/\mathrm{J}e$ is simple. Assume that $\mathrm{J}e/\mathrm{J}^2e \neq 0$. We have then;

$$Re/\mathrm{J}^2e > \mathrm{J}e/\mathrm{J}^2e > 0$$

and $(Re/\mathrm{J}^2e)/(\mathrm{J}e/\mathrm{J}^2e) \cong Re/\mathrm{J}e$ is simple. So we need to show that $\mathrm{J}e/\mathrm{J}^2e$ is simple. Consider the projective cover of $\mathrm{J}e$ (since R is Artinian, it is semiprimary, so R is perfect. Then every R-module has a projective cover, and so does $\mathrm{J}e$). If R is Artinian, then it is Noetherian by Theorem 4.1.6. Thus J is finitely generated. Hence $\mathrm{J}e$ is a finitely generated R-module. Then it has a finitely generated projective cover which is a direct sum of principal indecomposable projective R-modules e_iR's by Proposition 2.2.12. Since R is perfect, every projective R-module can be written as a direct sum $\bigoplus_{i\in I}(Rf_i)$ where I is finite index set and $f_i \in \{e_1, \cdots, e_n\}$ for every $i \in I$ by Corollary 2.2.11. Say

$$\bigoplus_{j=1}^{m} Rf_j \xrightarrow{\psi} \mathrm{J}e \longrightarrow 0$$

is a projective cover of $\mathrm{J}e$ where $f_j \in \{e_1, \cdots, e_n\}$ for every $j \in \{1, \cdots, m\}$. Hence we have a minimal presentation of $S = Re/\mathrm{J}e$:

$$\bigoplus_{j=1}^{m} Rf_j \xrightarrow{\psi} Re \longrightarrow S = Re/\mathrm{J}e \longrightarrow 0$$

where $\operatorname{Im}(\psi) = \mathrm{J}e \ll Re$. Here ψ can be represented by an $m \times 1$ matrix $A = \begin{bmatrix} a_{11} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix}$ where $a_{j1} \in f_jRe$ and so $a_{j1} = f_j a_{j1} e$ for $j = 1, \cdots, m.$. Hence for every $r_1, \cdots, r_m \in R$,

$$\begin{bmatrix} r_1 f_1 & . & . & . & r_m f_m \end{bmatrix} \overset{\psi}{\longmapsto} \begin{bmatrix} r_1 f_1 & . & . & . & r_m f_m \end{bmatrix} A;$$

and

$$\begin{bmatrix} r_1 f_1 & . & . & . & r_m f_m \end{bmatrix} \begin{bmatrix} a_{11} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix} = \sum_{j=1}^{m} r_j f_j a_{j1} = \sum_{j=1}^{m} r_j f_j a_{j1} e.$$

Since $\operatorname{Im}(\psi) = \mathrm{J}e$, we have that for each $j \in \{1, \cdots, m\}$,

$$\psi(f_j) = f_j f_j a_{j1} e = f_j a_{j1} e = a_{j1}$$

and so, $a_{j1} = \psi(f_j) \in \operatorname{Im}(\psi) = \mathrm{J}e$ implies that $a_{j1} \in \mathrm{J}e \subseteq \mathrm{J}$ for every $j = 1, \cdots, n$ since J is a two sided ideal of R. That is, the entries of the $m \times 1$ matrix A are in J. Applying the contravariant functor $(-)^* = \operatorname{Hom}_R(-, R)$, we obtain

$$(eR)^* \cong Re \overset{\psi^*}{\to} \left(\bigoplus_{j=1}^{m} Rf_j \right)^* \cong \bigoplus_{j=1}^{m} f_j R \to \operatorname{Tr}(S) \to 0$$

and $\operatorname{Tr}(S) = (\bigoplus_{j=1}^{m} f_j R)/\operatorname{Im}(\psi^*)$. Here, ψ^* is again represented by the same matrix A whose entries are in J, but this time it is given by left multiplication with A: for every

$r \in R$,

$$\begin{bmatrix} er \end{bmatrix} \overset{\psi^*}{\mapsto} A \begin{bmatrix} er \end{bmatrix} = \begin{bmatrix} a_{11}e_r \\ \cdot \\ \cdot \\ \cdot \\ a_{m1}e_r \end{bmatrix}.$$

Since J is a two sided ideal, all entries $a_{j1}er \in \mathrm{J}$, for $j = 1, .., m$, because $a_{j1} \in \mathrm{J}$. Thus $\mathrm{Im}(\psi^*) \subseteq \bigoplus_{j=1}^{m} a_{j1}eR = \bigoplus_{j=1}^{m} f_j a_{j1}eeR = \bigoplus_{j=1}^{m} f_j f_j a_{j1}eR = \bigoplus_{j=1}^{m} f_j a_{j1}R \subseteq \bigoplus_{j=1}^{m} f_j \mathrm{J}$ since $a_j \in \mathrm{J}$ for each $j \in \{1, \cdots, m\}$. Therefore $\mathrm{Im}(\psi^*) \subseteq \bigoplus_{j=1}^{m} f_j \mathrm{J}$. So $\mathrm{Tr}(S) = \left(\bigoplus_{j=1}^{m} f_j R\right) / \mathrm{Im}(\psi^*)$ and $\mathrm{Im}(\psi^*) \subseteq \bigoplus_{j=1}^{m} f_j \mathrm{J}$. Then

$$1 = \mathrm{cl}(\mathrm{Tr}(S)) \geq \mathrm{cl}\left(\bigoplus_{j=1}^{m} f_j R / \bigoplus_{j=1}^{m} f_j \mathrm{J}\right) = m > 0$$

since $\left(\bigoplus_{j=1}^{m} f_j R / \bigoplus_{j=1}^{m} f_j \mathrm{J}\right) \cong \bigoplus_{j=1}^{m} (f_j R / f_j \mathrm{J})$ is a semisimple R-module and $\mathrm{Tr}(S)$ is a simple module by hypothesis. Thus we obtain that $m = \mathrm{cl}(\mathrm{Tr}(S)) = 1$. Hence the sequence

$$Rf_1 \xrightarrow{\psi} Re \longrightarrow S = Re/\mathrm{J}e \longrightarrow 0$$

is exact and $\mathrm{Ker}(\psi) \ll Rf_1$. So $\mathrm{Ker}(\psi) \subseteq \mathrm{Rad}(Rf_1) = \mathrm{J}f_1$. We have $R\psi(f_1) = \psi(Rf_1) = \mathrm{J}e$, and

$$\psi(\mathrm{J}f_1) = \mathrm{J}\psi(f_1) = \mathrm{J}R\psi(f_1) = \mathrm{J}\mathrm{J}e = \mathrm{J}^2 e.$$

So

$$\mathrm{J}e/\mathrm{J}^2 e = \psi(Rf_1)/\psi(\mathrm{J}f_1) \cong (Rf_1/\mathrm{Ker}(\psi))/(\mathrm{J}f_1/\mathrm{Ker}(\psi)) \cong Rf_1/\mathrm{J}f_1$$

is simple. Thus $\mathrm{J}e/\mathrm{J}^2 e \cong Rf_1/\mathrm{J}f_1$ is simple. This ends the proof of the claim that

$$Re/\mathrm{J}e > \mathrm{J}e/\mathrm{J}^2 e > \mathrm{J}^2 e/\mathrm{J}^2 e = 0$$

is a composition series if $\mathrm{J}e/\mathrm{J}^2 e \neq 0$.

Now, let's prove that

$$eR/e\,\mathrm{J}^2 > e\,\mathrm{J}\,/e\,\mathrm{J}^2 > e\,\mathrm{J}^2\,/e\;J^2 = 0$$

is a composition series if $e\,\mathrm{J}\,/e\,\mathrm{J}^2 \neq 0$. If $e\,\mathrm{J}\,/e\,\mathrm{J}^2$ is zero, then $(eR/e\,\mathrm{J}^2)/(e\,\mathrm{J}\,/e\,\mathrm{J}^2) \cong eR/e\,\mathrm{J}$ is simple, and we are done. Hence, suppose that $e\,\mathrm{J}\,/e\,\mathrm{J}^2 \neq 0$. To prove that $eR/e\,\mathrm{J}^2 > e\,\mathrm{J}\,/e\,\mathrm{J}^2 > e\,\mathrm{J}^2\,/e\;J^2 = 0$ is a composition series, we need to show that $e\,\mathrm{J}\,/e\,\mathrm{J}^2$ is simple, because $(eR/e\,\mathrm{J}^2)/(e\,\mathrm{J}\,/e\,\mathrm{J}^2) \cong eR/e\,\mathrm{J}$ is already simple. The module $e\,\mathrm{J}\,/e\,\mathrm{J}^2$ is annihilated by J. So $e\,\mathrm{J}\,/e\,\mathrm{J}^2$ is also an $R/\,\mathrm{J}$-module. Since R is semiperfect, $R/\,\mathrm{J}$ is semisimple. Thus $e\,\mathrm{J}\,/e\,\mathrm{J}^2$ is a semisimple $R/\,\mathrm{J}$-module. So the nonzero semisimple $R/\,\mathrm{J}$-module $e\,\mathrm{J}\,/e\,\mathrm{J}^2$ has a simple direct summand that is isomorphic to $fR/f\,\mathrm{J}$ for some primitive idempotent $f \in \{e_1, \cdots, e_n\}$. Then we have a nonzero homomorphism

$$\varphi : fR \to e\,\mathrm{J}\,/e\,\mathrm{J}^2 \;\text{ with }\; \mathrm{Ker}(\varphi) = f\,\mathrm{J}.$$

Since $\varphi \neq 0$, we must have $\varphi(f) \neq 0$. Then $e\,\mathrm{J}\,f \nsubseteq \mathrm{J}^2$. Because if $e\,\mathrm{J}\,f \subseteq \mathrm{J}^2$, then $e\,\mathrm{J}\,f \subseteq e\,\mathrm{J}^2$ and $0 \neq \varphi(f) = \varphi(f^2) = \varphi(f)f \in (e\,\mathrm{J}\,f + e\,\mathrm{J}^2)/e\,\mathrm{J}^2 = 0$ since $\varphi(f) \in e\,\mathrm{J}\,/e\,\mathrm{J}^2$. This is a contradiction. Then there exists $a \in \mathrm{J}$ such that eaf is not in J^2 and so $eaf \notin \mathrm{J}^2 f \subseteq \mathrm{J}$. Consider the left R-module homomorphism defined for all $t = r \in Re$, $r \in R$, by

$$\begin{array}{rccl} \gamma: & Re & \to & \mathrm{J}\,f/\,\mathrm{J}^2 f \\ & t & \mapsto & taf + \mathrm{J}^2 f \\ & re & \mapsto & reaf + \mathrm{J}^2 f \end{array}$$

Since $\gamma(e) = eaf + \mathrm{J}^2 f \neq 0$ (as $eaf \notin \mathrm{J}^2 f$), γ is a nonzero homomorphism. Since $eaf \in \mathrm{J}\,f$, we have $\gamma(\mathrm{J}\,e) = \mathrm{J}\,\gamma(e) = 0$ because $\gamma(e) \in \mathrm{J}\,f/\,\mathrm{J}^2 f$. Then $\mathrm{J}\,e \subseteq \mathrm{Ker}(\gamma)$. So the map defined for all $t + \mathrm{J}\,e \in Re/\,\mathrm{J}\,e$, $t \in Re$, by

$$\begin{array}{rcl} \widetilde{\gamma} : Re/\,\mathrm{J}\,e & \to & \mathrm{J}\,f/\,\mathrm{J}^2 f \\ t + \mathrm{J}\,e & \mapsto & \gamma(t) \end{array}$$

is well-defined, and $\widetilde{\gamma} \neq 0$ since $\widetilde{\gamma}(e + \mathrm{J}\,e) = \gamma(e) \neq 0$. Since $Re/\,\mathrm{J}\,e$ is a simple R-module and $\widetilde{\gamma}$ is a non-zero homomorphism, we obtain $\mathrm{Ker}(\widetilde{\gamma}) = 0$. Thus $\widetilde{\gamma}$ is one to one, and so $\mathrm{J}f/\mathrm{J}^2 f$ has a simple submodule isomorphic to $Re/\mathrm{J}e$. We have already showed that $\mathrm{J}f/\mathrm{J}^2 f$ is a simple left R-module. Thus we obtain that $\mathrm{J}f/\mathrm{J}^2 f \cong Re/\mathrm{J}e$. Then $\widetilde{\gamma}$ is an isomorphism and so γ is onto. For $x = eaf$, $exf = x$ and $\gamma(re) = rx + \mathrm{J}^2 f$ for all $r \in R$. Since γ is onto, $\gamma(Re) = \mathrm{J}f/\mathrm{J}^2 f$. Thus $Rx + \mathrm{J}^2 f = \mathrm{J}f$. Also $\mathrm{J}(\mathrm{J}f) \ll \mathrm{J}f$ by Anderson & Fuller (1992, Lemma 28.3) (because R is an Artinian ring, and so a perfect ring). Then $Rx = \mathrm{J}f$ and $x = exf$ implies that $Rexf = \mathrm{J}f$. Now let $T = Rf/\mathrm{J}f$. We have

$$Re \to Rf \to T = Rf/\mathrm{J}f \to 0$$

and

$$\begin{aligned} \psi : Re &\to Rf, \\ [y] &\mapsto [y]A = [yexf] \text{ for all } y \in Re. \end{aligned}$$

ψ is only given by the 1×1 matrix $A = [exf]$. So we have for $\mathrm{Tr}(T)$:

$$fR \xrightarrow{\psi^*} eR \longrightarrow \mathrm{Tr}(T) \longrightarrow 0, \text{ where } \psi^* \text{ is given by:}$$

$$\begin{aligned} \psi^* : fR &\to eR, \\ [y] &\mapsto [exf][y] \quad \text{for all} \quad y \in fR. \end{aligned}$$

Then $\mathrm{Im}(\psi^*) = exfR$ since $f^2 = f$. Thus $\mathrm{Tr}(T) \cong eR/exfR$. By our hypothesis $\mathrm{Tr}(T)$ is simple, so $eR/exfR$ is simple. If $a \in \mathrm{J}$ and $x = eaf$, then $x \in \mathrm{J}$, too. Then $exfR \subseteq e\mathrm{J} \subseteq eR$ and $eR/e\mathrm{J}$ is known to be a simple R-module. We have also showed that $eR/exfR$ is simple. These imply that, $exfR = e\mathrm{J}$. Hence it remains to obtain that $e\mathrm{J}/e\mathrm{J}^2 \cong fR/f\mathrm{J}$ is simple. By similar arguments as above define $x = eaf$ and $exf = x$, and

$$\begin{aligned} \theta \quad : fR &\to e\mathrm{J}/e\mathrm{J}^2, \\ y &\mapsto exfy + e\mathrm{J}^2 \text{ for all } y \in eR, \\ fr &\mapsto exfr + e\mathrm{J}^2 \text{ for all } r \in R \end{aligned}$$

We have $\theta \neq 0$ since $\theta(f) = exf + e\mathrm{J}^2 \neq 0$, because $exf = x = eaf$ is not in J^2, so it is not in $e\mathrm{J}^2 \subseteq \mathrm{J}$. And $\theta(f\mathrm{J}) = \theta(f)\mathrm{J} = 0$ since $\theta(f) \in e\mathrm{J}/e\mathrm{J}^2$. So $f\mathrm{J} \subseteq \mathrm{Ker}(\theta)$. Thus we have a homomorphism

$$\begin{aligned}
\widetilde{\theta}: fR/f\mathrm{J} &\rightarrow e\mathrm{J}/e\mathrm{J}^2, \\
y + f\mathrm{J} &\mapsto \theta(y) = exfy + e\mathrm{J}^2 \quad \text{for all} \quad y \in fR, \\
fr + f\mathrm{J} &\rightarrow exfr + e\mathrm{J}^2 \quad \text{for all} \quad r \in R.
\end{aligned}$$

which is onto since $e\mathrm{J} = exfR$. An onto nonzero homomorphism from a simple module must be also one to one. Then $\widetilde{\theta}$ is an isomorphism, that is,

$$fR/f\mathrm{J} \cong e\mathrm{J}/e\mathrm{J}^2,$$

and so $e\mathrm{J}/e\mathrm{J}^2$ is simple R-module.

For the converse part we use Proposition 4.3.2. Since R is an Artinian serial ring, it is also a semiperfect ring. Then R has a block decomposition by Theorem 2.2.14. So R has a decomposition $R = e_1Re_1 \oplus e_2Re_2 \oplus \cdots \oplus e_nRe_n$ for some indecomposable Artinian serial rings e_iRe_i, where $\{e_1, e_2, \cdots, e_n\}$ are pairwise orthogonal central idempotents. If $f_j \in R$ for $j = 1, \cdots, n$ is a primitive idempotent, then a primitive idempotent $f_j \in R$ exactly one $f_je_i \neq 0$. So each primitive idempotent belongs to one block. Let $S = f_iR/f_i\mathrm{J}$ is a nonprojective simple module for an idempotent $f_i \in R$. Then this idempotent belongs to one block. Then by Proposition 4.3.2, $\mathrm{Tr}(f_jR/f_j\mathrm{J}) \cong (Rf_{j+1}/\mathrm{J}f_{j+1})$ is simple. □

Proposition 4.3.2. *(Haack, 1984, Proposition 2) Let R be an indecomposable serial ring with Kupisch series $e_1R, \cdots, e_nR$. Then*

$$\mathrm{Tr}(e_{i-1}R/e_{i-1}\mathrm{J}) \cong Re_i/\mathrm{J}e_i \ \ (i = 2, \cdots, n),$$

and either e_nR is simple or $\mathrm{Tr}(e_nR/e_n\mathrm{J}) \cong Re_1/\mathrm{J}e_1$.

Proof. Let $i \in \{2, \cdots, n\}$ Then e_iR is a projective cover of $e_{i-1}\mathrm{J}$ by Theorem 4.2.1 So there is a minimal projective presentation for the simple R-module $S_{i-1} = e_{i-1}R/e_{i-1}\mathrm{J}$

such that

$$e_iR \xrightarrow{\psi} e_{i-1}R \longrightarrow e_{i-1}R/e_{i-1}\,\mathrm{J} \longrightarrow 0$$

$\mathrm{Im}(\psi) = e_{i-1}\,\mathrm{J} \ll e_{i-1}R$ and ψ is given by the 1×1 matrix $A = [a]$ where $a = e_{i-1}ae_i$ for some $a \in R$:

$$\begin{aligned} \psi \quad : \quad e_iR &\rightarrow e_{i-1}R, \\ [y] &\mapsto A[y] = [e_{i-1}ae_iy] \text{ for all } y \in e_iR. \end{aligned}$$

Then $e_{i-1}\,\mathrm{J} = \mathrm{Im}(\psi) = e_{i-1}ae_ie_iR = aR$ and this implies $a \in aR = e_{i-1}\,\mathrm{J} \subseteq \mathrm{J}$ since J is a two sided ideal. Thus $\mathrm{Tr}(S_{i-1}) = \mathrm{Coker}(\psi^*)$ is given by the exact sequence

$$(e_{i-1}R)^* \cong Re_{i-1} \xrightarrow{\psi^*} Re_i \longrightarrow \mathrm{Tr}(S_{i-1}) \longrightarrow 0$$

where

$$\psi^* : [y] \rightarrow [y]A = [ye_{i-1}ae_i] \text{ for all } y \in Re_{i-1}, \text{ and}$$

$\mathrm{Im}(\psi^*) = Re_{i-1}ae_i = Ra$. Now, we claim that $\mathrm{Im}(\psi^*) = Ra = \mathrm{J}\,e_i$. By Lemma 4.2.2, for $k = 1$, there exists $a = a_{i-1} \in e_{i-1}\,\mathrm{J}\,e_i$ such that $\mathrm{J}\,e_i = Ra_{i-1} = Ra$. Therefore $\mathrm{Tr}(S_{i-1}) = Re_i/\mathrm{Im}(psi^*) = Re_i/Ra = Re_i/\,\mathrm{J}\,e_i$. In conclusion, we obtain that $\mathrm{Tr}(e_{i-1}R/e_{i-1}\,\mathrm{J}) \cong Re_i/\,\mathrm{J}\,e_i$. □

4.4 Proper Classes Generated by Simple Modules

A short exact sequence $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of R-modules is said to be a *neat-exact sequence* if for every simple R-module S, the sequence

$$\mathrm{Hom}_R(S,\mathbb{E}) : 0 \longrightarrow \mathrm{Hom}_R(S,A) \xrightarrow{f^*} \mathrm{Hom}_R(S,B) \xrightarrow{g^*} \mathrm{Hom}_R(S,C) \longrightarrow 0$$

is exact. Observe that $\mathbb{E}$ is a neat-exact sequence if and only if $\mathrm{Im}(f)$ is a neat submodule of B. A submodule A of an R-module B is said to be *neat* in B if for every simple R-module S, the sequence $\mathrm{Hom}_R(S,B) \rightarrow \mathrm{Hom}_R(S,B/A) \rightarrow 0$ obtained by applying the functor $\mathrm{Hom}_R(S,-)$ to the canonical epimorphism $B \rightarrow B/A$ is exact. We denote the proper class of neat-exact sequences of R-modules by $\mathcal{N}eat_R$. (See

Türkoğlu (2013, Section 1,3.3) for this proper class $\mathcal{N}eat_R$ of short exact sequences of R-modules that is projectively generated by all simple R-modules.

An ideal P in R is said to be a *left primitive ideal (resp. right primitive ideal)* if P is the annihilator of a simple left R-module (resp. right R-module). If S is a simple R-module and $P = \mathrm{ann}(S)$, then we can write $S = R/M$ for some maximal right ideal M of R. If R is a commutative ring, then necessarily $M = P$. Denote by $\mathcal{P}$ the collection all left primitive ideals of the ring R. A short exact sequence $\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is said to be ${}_R\mathcal{P}$-pure-exact if for every left primitive ideal P of R, the sequence

$$(R/P) \otimes_R \mathbb{E} : 0 \longrightarrow A \otimes_R (R/P) \longrightarrow B \otimes_R (R/P) \longrightarrow C \otimes_R (R/P) \longrightarrow 0$$

is exact, that is, $\mathbb{E} \in \tau^{-1}(\{R/P \mid P \in \mathcal{P}\}) =_R \mathcal{P}\text{-}\mathcal{P}ure$ (see Türkoğlu (2013, Section 3.3)). In general, over a commutative ring, Fuchs (2012) showed that the proper classes $\mathcal{N}eat$ and $\mathcal{P}\text{-}\mathcal{P}ure$ are not equal. Fuchs proved that for a commutative domain R, $\mathcal{N}eat$ and $\mathcal{P}\text{-}\mathcal{P}ure$ coincide if and only if all the maximal ideals of the commutative domain R are (finitely generated) projective modules. From that motivation we wish to extend this result to a class of commutative rings. We prove that if R is a commutative Artinian serial ring, then $\mathcal{N}eat = \mathcal{P}\text{-}\mathcal{P}ure$. The known main result we shall use is that a proper class projectively generated by a set of finitely presented R-modules is flatly generated by the Auslander-Bridger transpose of these finitely presented R-modules (see Sklyarenko (1978, Section 8)). For some results that we use and a detailed information about the proper classes $\mathcal{N}eat$ and $\mathcal{P}\text{-}\mathcal{P}ure$, see the master thesis Türkoğlu (2013, Chapters 3-4).

Theorem 4.4.1. *(Sklyarenko, 1978, Corollary 5.1) Let M be a finitely presented right R-module and E a short exact sequence of R-modules. Then the sequence $M \otimes_R E$ is exact if and only if* $\mathrm{Hom}_R(\mathrm{Tr}(M), E)$ *is exact.*

This gives us that a proper class that is projectively generated by a set of finitely presented R-modules is flatly generated by the Auslander-Bridger transpose of these finitely presented R-modules.

Theorem 4.4.2. *(Sklyarenko, 1978, Theorem 8.3) Let $\mathcal{M}$ be a set of finitely presented right R-modules. Let* $\mathrm{Tr}(\mathcal{M}) = \{\mathrm{Tr}(M) \mid M \in \mathcal{M}\}$. *We may assume that* $\mathrm{Tr}(\mathrm{Tr}(\mathcal{M})) = \mathcal{M}$. *Then we have*

$$\tau^{-1}(\mathcal{M}) = \pi^{-1}(\mathrm{Tr}(\mathcal{M})) \quad \textit{with} \quad \pi^{-1}(\mathcal{M}) = \tau^{-1}(\mathrm{Tr}(\mathcal{M}))$$

The following proper classes are projectively or flatly generated by simple (right or left) modules.

(i) $\mathcal{N}eat_R = \pi^{-1}(\{\text{all simple } R\text{-modules}\})$.

(ii) $\mathcal{N}eat_R^{\pi} = \pi^{-1}(\{R/P \mid P$ is maximal right ideal of $R\})$ is the proper class projectively generated by all simple right R-modules.

(iii) $\mathcal{N}eat_R^{\tau} = \tau^{-1}(\{R/P \mid P$ is maximal left ideal of $R\})$ is the proper class flatly generated by all simple left R-modules.

(iv) $\mathcal{P}\text{-}\mathcal{P}ure = \tau^{-1}(\{R/P \mid P \in \mathcal{P}\})$ where $\mathcal{P}$ is the collection of all left primitive ideals of R.

For a commutative ring R, primitive ideals are just maximal ideals and so $\mathcal{N}eat^{\tau} = \mathcal{P}\text{-}\mathcal{P}ure$. Türkoğlu (2013) extends the results in Fuchs (2012) for the Auslander-Bridger transpose of finitely presented simple modules of projective dimension ≤ 1 over a commutative domain to commutative rings:

Theorem 4.4.3. *(Türkoğlu, 2013, 4.4.4) Let R be a commutative ring and P be a* finitely generated *maximal ideal of R that is projective. Take the following presentation of the simple R-module $S = R/P$ (where f is the inclusion monomorphism and g is the natural epimorphism):*

$$\gamma : \qquad P \xrightarrow{f} R \xrightarrow{g} S \longrightarrow 0$$

1. If S is projective, then $S^* \neq 0$ *and* $\mathrm{Tr}_{\gamma}(S) = 0$.

2. *If S is* not *projective, then $S^* = 0$ and* $\mathrm{Tr}_\gamma(S) \cong \mathrm{Ext}^1_R(S,R) \cong S$.

3. *S is projective if and only if $S^* \neq 0$.*

For the proof of the theorem see Türkoğlu (2013, Theorem 4.4.4). Using Theorem 4.4.3, they obtain the following sufficient condition for $\mathcal{N}eat = \mathcal{P}\text{-}\mathcal{P}ure$ over commutative rings as in Fuchs' characterization of N-domains:

Theorem 4.4.4. *(Türkoğlu, 2013, Theorem 4.5.1) If R is a* commutative *ring such that every maximal ideal of R are finitely generated and projective, then $\mathcal{N}eat = \mathcal{P}\text{-}\mathcal{P}ure$.*

Proof. Since the maximal ideals of the commutative ring R are finitely generated and projective, the simple R-modules are finitely presented. So using Theorem 4.4.2, we obtain:

$$\begin{aligned}
\mathcal{P}\text{-}\mathcal{P}ure &= \tau^{-1}(\{\text{all simple } R\text{-modules}\}) \\
&= \tau^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R\}) \\
&= \tau^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R \text{ and } R/P \text{ is not projective}\}) \\
&= \pi^{-1}(\{\mathrm{Tr}(R/P) \mid P \text{ is a maximal ideal of } R \text{ and } R/P \text{ is not projective}\}) \\
&= \pi^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R \text{ and } R/P \text{ is not projective}\}) \\
&= \pi^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R\}) \\
&= \mathcal{N}eat
\end{aligned}$$

□

Theorem 4.4.5. *If R is an Artinian serial ring, then*

$$\pi^{-1}(\{\textit{all simple left R-modules}\}) = \tau^{-1}(\{\textit{all simple (right) R-modules}\})$$

Proof. Let $0_R S$ be a simple nonprojective left R-module. Then the Auslander-Bridger transpose $\mathrm{Tr}({}_R S)$ (with respect to a minimal presentation of S)is nonprojective. Because if $\mathrm{Tr}({}_R S)$ were projective, then $S = 0$ would hold since $S \cong \mathrm{Tr}(\mathrm{Tr}({}_R S)) \cong \mathrm{Ext}^1_R(\mathrm{Tr}({}_R S), R)$ by Theorem 1.2.2 and $\mathrm{Ext}^1_R(\mathrm{Tr}({}_R S), R) = 0$ for a projective module

$\mathrm{Tr}({}_RS)$. From Theorem 4.3.1, $\mathrm{Tr}({}_RS) = U_R$ for some nonzero nonprojective simple right R-module U_R. Now, let $A = \{S_i \mid i \in I\}$ be the set of representatives of all simple left R-modules up to isomorphism that are not projective. Then $B = \{\mathrm{Tr}({}_RS_i) \mid i \in I\}$ is a set of representatives of all simple right R-modules up to isomorphism that are not projective. Note that, since R is Artinian, these simple modules are also finitely presented. Then

$$\tau^{-1}(A) = \pi^{-1}(\mathrm{Tr}(A)) \quad \text{and} \quad \pi^{-1}(A) = \tau^{-1}(\mathrm{Tr}(A))$$

by Theorem 4.4.2. □

In the projectively or flatly generated classes, there is no need to put the projectives in the generating class. So if R is Artinian serial ring, then purity and neatness coincide for proper classes generated by (nonprojective) simple modules.

Corollary 4.4.6. *If R is a commutative Artin serial ring, then $\mathcal{N}eat = \mathcal{P}\text{-}\mathcal{P}ure$.*

Proof.

$$\begin{aligned}
\mathcal{P}\text{-}\mathcal{P}ure &= \tau^{-1}(\{S_i \mid i \in I\}) \\
&= \pi^{-1}(\{\mathrm{Tr}(S_i) \mid S_i \text{ is not projective}\}) \\
&= \pi^{-1}(\{\mathrm{Tr}(S_i) \mid S_i \ \ i \in I\}) \\
&= \mathcal{N}eat
\end{aligned}$$

□

Corollary 4.4.7. *If R is a commutative Artinian valuation ring, then $\mathcal{N}eat = \mathcal{P}\text{-}\mathcal{P}ure$.*

Proof. Since commutative valuation rings are uniserial, the result follows from the same arguments. □

The converse of the theorem 4.4.4 is not true;

Example 4.4.8. Consider the Artinian uniserial ring $R = \mathbb{Z}/p^n\mathbb{Z}$ for $n > 2$. Then it has one maximal ideal $p\mathbb{Z}/p^n\mathbb{Z}$ which is a principal ideal. Every maximal ideal is a principal ideal implies that $\mathcal{N}eat_R = \mathcal{P}\text{-}\mathcal{P}ure$ by (Warfield, 1969a, Proposition 2). On $\mathbb{Z}/p^n\mathbb{Z}$, $\mathcal{N}eat_R = \mathcal{P}\text{-}\mathcal{P}ure$ since it has only one maximal ideal, that is, $p\mathbb{Z}/p^n\mathbb{Z}$. But $p\mathbb{Z}/p^n\mathbb{Z}$ is not projective. Because

$$\gamma : \mathbb{Z}/p^n\mathbb{Z} \to p\mathbb{Z}/p^n\mathbb{Z} \text{ defined as } k + p^n\mathbb{Z} \mapsto pk + p^n\mathbb{Z}.$$

If $p\mathbb{Z}/p^n\mathbb{Z}$ were projective, then γ would split and so $p\mathbb{Z}/p^n\mathbb{Z}$ would be isomorphic to a direct summand of $\mathbb{Z}/p^n\mathbb{Z}$. Since $\mathbb{Z}/p^n\mathbb{Z}$ is uniserial, this is a contradiction.

CHAPTER FIVE
CONCLUSION

We asked for which kind of rings R, and for which class of R-modules, does every R-module M in that class has a decomposition $M = P \oplus N$, where P is projective and N is stable? We have seen that if the ring R is semiperfect or left semihereditary or right Artinian or right Noetherian ring, we have this decomposition for all finitely presented R modules. Also all finitely presented modules over the rings which have a finite hollow dimension or a finite uniform dimension satisfy this decomposition into a projective submodule and a stable submodule. We have also given examples of rings and finitely generated or finitely presented modules or cyclically presented modules over that ring for which the decomposition fails. For the main results of this thesis, see Section 1.5. So the questions given in the motivation (Section 1.1) have not been fully answered and there seems to be more work left for them with more refined questions such as the following:

(i) Characterize the rings R, where *every* R-module M has a decomposition $M = P \oplus N$, where P is a projective submodule and N is a stable submodule of M.

(ii) Over which rings R *every cyclically presented module* has a decomposition $M = P \oplus N$, where P is a projective submodule and N is a stable submodule of M?

(iii) Give example of rings R, where *every finitely presented* R-module M has a quotient K such that K does not have a decomposition $K = P \oplus N$, where P is a projective submodule and N is a stable submodule of K.

(iv) Characterize the rings R, where *every finitely presented* R-module M has a decomposition $M = P \oplus N$, where P is a projective module and N is a stable module?

We also have the general problem left related with the Auslander-Bridger transpose of finitely presented simple R-modules: over which rings R, the Auslander-Bridger transpose of every nonprojective finitely presented simple right R-module is projectively equivalent to a simple left R-module?

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APPENDICES

NOTATIONS

R	an associative ring with unity
M_R	a right R-module
$_RM$	a left R-module
$\mathbb{Z}$	the ring of integers
$\mathbb{R}$	the field of real numbers
$\leq$	submodule
$\ll$	small (or superfluous) submodule
$\leq_e$	essential submodule
$\cong$	isomorphic
$M \times N$	the direct product of the R-modules M and N
$M \oplus N$	the direct sum of the R-modules M and N
$M \approx N$	the R-module M is projectively equivalent to the R-module N
$\mathrm{ann}(M)$	the annihilator of the R-module M
$\mathrm{pd}(M)$	the projective dimension of the R-module M
$\mathrm{Ker}(f)$	the kernel of the map f
$\mathrm{Im}(f)$	the image of the map f
$\mathrm{Soc}(M)$	the socle of the R-module M
$\mathrm{Soc}(R_R)$	the socle of the ring R consider as a *right* R-module
$\mathrm{Soc}(_RR)$	the socle of the ring R consider as a *left* R-module
$\mathrm{cl}(M)$	the composition length of the R-module M
$\mathrm{u.dim}(M)$	the uniform dimension of the R-module M
$\mathrm{h.dim}(M)$	the hollow dimension of the R-module M
$\mathrm{End}_R(M)$	the endomorphism ring of the R-module M and $(f \cdot g)(x) = f(g(x))$
$\mathrm{E}(M)$	the injective hull of the R-module M
$\mathrm{Hom}_R(M,N)$	all R-module homomorphisms from the R-module M to the R-module N
$\mathrm{Mod}\text{-}R$	the category of all right R-modules

R-Mod	the category of all left R-modules
$\underline{\mathrm{Mod}}$-R	the stable category of all right R-modules
R-$\underline{\mathrm{Mod}}$	the stable category of all left R-modules
$\underline{\mathrm{mod}}$-R	the stable category of all finitely presented right R-modules
R-$\underline{\mathrm{mod}}$	the stable category of all finitely presented left R-modules

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