

Market Clearing Models in European Day Ahead Electricity Markets

by

Nermin Elif Kurt Şahin

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Market Clearing Models in European Day Ahead Electricity Markets

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Graduate School of Sciences and Engineering

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Nermin Elif Kurt Şahin

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Committee Members:

Prof. Emre Alper Yıldırım

Prof. Lerzan Örmeci

Prof. Metin Türkay

Prof. Alp Atakan

Prof. İlker Birbil

Asst. Prof. Esra Koca

Date: _____



ABSTRACT

Electricity is a commodity that must be constantly kept in balance. Electricity markets have therefore evolved accordingly. The day-ahead electricity market (DAM) is a spot market where the electricity is traded one day before the actual delivery date. Each day, there is a two sided closed auction in a DAM and the corresponding market clearing problem needs to be solved in a limited amount of time due to the strict regulations.

In this thesis, we study the European DAM clearing problem. We consider hourly, block, linked block, and flexible orders in our model. We first start with a stylized market clearing problem, which is a total surplus maximization problem with hourly orders that can be represented by piecewise linear functions. The total surplus is the sum of the producer and consumer surpluses. In the stylized problem, consumer and producer surpluses are determined by using the demand and the supply curves of the hourly orders. We propose an optimization model that instead employs total curves represented by the difference between the demand and supply curves. We show that the set of optimal solutions of the proposed model is precisely the same as that of the stylized problem.

We next incorporate block, linked block and flexible orders and present an optimization model, denoted by (MC), based on the total curves. The model (MC) is a mixed integer quadratic programming problem. By fixing the binary variables of the model (MC) to their optimal values, we investigate the optimality conditions of the resulting problem. We identify two sets of optimality conditions. We show that these two sets of conditions may not simultaneously be satisfied. By imposing each of these two sets of conditions in the model (MC), we obtain two different variants,

one of which is already used in the European power exchanges and the other one is a new contribution to the literature.

For the two problem variants, we propose several solution methodologies. Our computational experiments illustrate that the proposed solution methods are capable of computing optimal solutions on real and synthetic data sets in a reasonably small amount of time in most of the cases.



ÖZETÇE

Elektirik arzı ve talebinin sürekli dengede olması gereken bir metadır. Elektrik marketleri bu özelliğe göre şekillenmiştir. Gün öncesi piyasası (GÖP), elektriğin tedarikinden bir gün önce ticaretinin yapıldığı bir piyasadır. Hergün GÖP’te elektrik alıcı ve satıcılarının katılabildiği kapalı ihale gerçekleşir. Bu ihalenin kazanan belirleme probleminin (piyasa takas problemi) sonucu kısa sürede belirlenmelidir. Bu tezde, içinde saatlik, blok, bağlı blok ve esnek tekliflerin olduğu Avrupa GÖP’lerinin piyasa takas problemlerini sunacağız. Öncelikle sadece saatlik tekliflerin kullanıldığı toplam fazla maksimizasyonu yapan basitleştirilmiş piyasa takas modelini inceleyeceğiz. Toplam fazla, üretici ve tüketici fazlasının toplamına eşittir. Basitleştirilmiş piyasa takas probleminde üretici ve tüketici fazlasını hesaplamak için saatlik tekliflerin arz ve talep eğrileri kullanılmaktadır. Saatlik tekliflerin arz ve talep eğrilerinin ayrı ayrı kullanıldığı bu model ile, talep eğrisinin arz eğrisinden farkı olan toplam eğrinin kullanıldığı modelin aynı optimal sonucu verdiğini göstereceğiz. Toplam eğrinin kullanıldığı bu yeni modele, blok, bağlı blok ve esnek teklifleri ekleyerek bir ikinci dereceden bir karma tam sayılı programlama modeli olan model (MC)’yi oluşturacağız. Bu modelin ikili değişkenlerinin optimal değerlerini sabitleyerek elde ettiğimiz modelin optimallik koşullarını incelediğimizde, ortaya iki farklı kural kümesi çıkacaktır. Bu kuralların hepsini model (MC)’ye eklendiğimizde, model (MC)’nin her zaman olurlu bir çözüm bulamayacağından, model (MC)’ye bu kümeleri ayrı ayrı ekleyip iki farklı model elde edeceğiz. Bu modellerden biri, Avrupa marketlerinde kullanılırken, diğeri bizim önerdiğimiz yeni bir model olacak. Bu modelleri hızlı bir şekilde çözen farklı çözüm yöntemleri sunacağız. Yöntemlerimizin performanslarını gerçek ve sentetik veriler ile göstereceğiz.

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TABLE OF CONTENTS

List of Figures	xiii
List of Tables	xv
Nomenclature	1
Chapter 1: Introduction	3
1.0.1 General Market Structure	4
1.0.2 Electricity Market in the US	6
1.0.3 Electricity Market in Europe	7
Chapter 2: Economic Background	9
2.1 Market Formation	9
2.2 Auction Mechanism	12
2.3 Non-convex Markets and Combinatorial Auctions	13
2.4 European Day-Ahead Electricity Markets and Their Auction Mechanism	14
2.4.1 Locational Pricing in Europe	14
2.4.2 Order Types in European DAM Auctions	16
2.5 Evaluation of the Orders	22
Chapter 3: Literature Review	28
3.1 Combinatorial Auctions in Electricity Markets	28
3.1.1 Pricing Models in Non-Convex Electricity Markets	29
3.2 Solution Methodologies for Electricity Market Models	32

3.2.1	Unit Commitment Problems	32
3.2.2	Exchange Models	33
3.3	Our Contributions	37
Chapter 4: Stylized Supply Demand Balance Problem with Hourly Orders		39
4.1	Mathematical Model for Supply Demand Balance Problem with Hourly Orders (SDB-H)	40
4.1.1	Hourly Orders	40
4.1.2	Notation to Represent Hourly Orders	47
4.1.3	Mathematical Model, (SDB-H)	52
4.1.3.1	Objective Function	53
4.1.4	Mathematical Properties of the model (SDB-H)	57
4.2	Equivalent Formulation of Supply Demand Balance Problem With Hourly Orders by Using Aggregated Total Curves	65
4.2.1	Model (MC-H)	68
4.3	Summary	70
Chapter 5: Supply Demand Balance Problem		73
5.1	Mathematical Model for Supply Demand Balance Problem (MC) . . .	73
5.1.1	Block and Flexible Orders	74
5.1.2	Mathematical Model	78
5.1.3	Objective Function	80
5.1.3.1	Consumer and Producer Surplus Arising from Hourly Orders	80
5.1.3.2	Consumer and Producer Surplus Arising from Block and Flexible Orders	81
5.1.3.3	The Total Surplus	82

5.2	Dorn's Dual for a Convex Quadratic Program	85
5.3	Quadratic Programming Relaxation of the Model (MC)	87
5.3.1	Properties of an Optimal Solution of the model (QPR-MC) . .	89
5.3.1.1	New Contributions About the Aggregated Total Curve Segments	89
5.3.1.2	New Results About Linked Block Orders	93
5.3.1.3	Our Contributions About Flexible Orders	97
5.4	Restricted Supply Demand Balance Problem	100
5.5	Summary	106
Chapter 6:	Variants of Supply Demand Balance Problem	110
6.1	Supply Demand Balance Problem Variants	111
6.1.1	Supply Demand Balance Problem Without PRBs	111
6.1.1.1	Mathematical Model of the Supply Demand Balance Problem Without PRB with a Quadratic Constraint	113
6.1.1.2	Mathematical Model of the Supply Demand Balance Problem Without PRB with Linear Constraints . . .	116
6.1.1.3	Economic Interpretation	130
6.1.2	Supply Demand Balance Problem Without PABs	132
6.1.2.1	Mathematical Model of the Supply Demand Balance Problem Without PAB with a Quadratic Constraint	134
6.1.2.2	Mathematical Model of the Supply Demand Balance Problem Without PAB with Linear Constraints . . .	136
6.2	Summary	147
Chapter 7:	Solution Methodologies	149
7.1	Adding Valid Inequalities To The Model	150
7.2	Approximations of the Objective Function	154

7.2.1	Piecewise Linear Approximations	155
7.2.2	Approximations	156
7.3	Decomposition Method	164
7.3.1	Decomposition Procedure to Solve the Model (MC-WPRB) . .	165
7.3.1.1	Sub-Problem	165
7.3.1.2	Cutting the current node solution	167
7.3.2	Decomposition Procedure to Solve the Model (MC-WPAB) . .	172
7.3.2.1	Sub-problem	172
7.3.2.2	Cutting the Current Node Solution	174
7.3.3	Heuristic Cuts	177
7.3.4	Cutting Alternative Solutions For Flexible Orders	180
7.4	Summary	180
Chapter 8:	Numerical Experiments	181
8.1	Data	181
8.1.1	Order Types in Turkish DAM	181
8.1.2	Evaluation of the Orders in Turkish DAM	183
8.1.3	Descriptive Statistics	183
8.2	Test Environment and Experiment Set-up	184
8.2.1	Test Environment	184
8.2.2	Solver Parameters	185
8.2.3	Numerical Problems	187
8.2.4	Tests and Performance Measures	189
8.3	Results	189
8.3.1	Performance of the Models	190
8.3.2	Performance of the Approximate Models	192
8.3.3	Performance of Adding Warm Start	197

8.3.4	Performance of the Compact Procedure	201
8.3.4.1	When the Time Limit is 25 seconds	201
8.3.4.2	When the Time Limit is 550 seconds	204
8.3.5	Performance of the Decomposition Procedure	207
8.4	Summary	208
Chapter 9:	Scenario Analysis	210
9.1	Parameters	211
9.2	Data	212
9.3	Tests and Performance Measures	214
9.4	Results	215
9.4.1	Impact of the Number of Periods	215
9.4.2	Impact of the Number of Block Orders	218
9.4.3	Impact of the Number of Flexible Orders	220
9.4.4	Impact of the Number of Linked Orders	222
9.5	Summary	223
Chapter 10:	Discussion and Conclusion	225
10.1	Managerial and Practical Insights	226
10.2	Future Directions	227
Appendix A:	Changed constraints in the models (MC-WPRB) and (MC-WPAB) when SDB is used as a base model	229
Appendix B:	Supplementary Results	232

LIST OF FIGURES

1.1	Electricity Generation, Transmission and Distribution (source: www.instituteforenergyresearch.org)	3
1.2	Unbundling process of Turkish electricity system	4
2.1	Supply and demand curves	10
2.2	Electricity Markets Using Price Coupling Regions (source: N-Side) . .	15
2.3	Step functions of the hourly orders in Table 2.1	17
2.4	Piecewise linear functions of the hourly orders in Table 2.2	18
2.5	Aggregated supply and demand curves for a period to show the impact of accepting/rejecting a sales block order	24
2.6	Supply and demand curves for a period to show the impact of accept- ing/rejecting a purchase block order	25
4.1	Total curves of the hourly orders 1, 2, and 3 from Table 4.1	41
4.2	Supply and demand curves of the hourly orders 1, 2 and 3	43
4.3	Aggregated supply, demand and total curves in period 1, by considering the example hourly orders 1, 2, and 3	46
4.4	Breakpoints of the aggregated supply and demand curves after adding the artificial breakpoints for curtailment	49
4.5	Segments of the aggregated supply and demand curves after adding the artificial breakpoints for curtailment	50
4.6	Producer and consumer surplus of a supply and demand segment k in period t	54

4.7	Consumer and producer surplus arising from aggregated hourly curves	56
4.8	Paired segments	59
4.9	Aggregated total curve obtained by two different approaches	66
4.10	Notation used for the aggregated total curve in Figure 4.9 (b)	67
5.1	Possible Linked Block Order Combinations	76
5.2	Producer surplus arising from a sales block order	82
7.1	Stepwise approximations of the original piecewise function c	155
7.2	Flows of the Algorithms 3-5	162
8.1	Cumulative percentage of solution times of optimal instances for the models (SDB-WPRB), (MC-WPRB), (MC-WPRB-Q), (SDB-WPAB), (MC-WPAB), and (MC-WPAB-Q)	191
8.2	Comparison of the optimal solution of the models (UB-P-WPRB) and (LB-P-WPRB)	194
8.3	Comparison of the optimal solutions of the models (UB-P-WPAB) and (LB-P-WPAB)	195
8.4	Cumulative Percentage Of Solution Times	196
B.1	Relative gap values for 10 randomly selected instances	233
B.2	Relative gap values for 10 randomly selected instances	233

LIST OF TABLES

2.1	An hourly order example	17
2.2	Another hourly order example	18
2.3	An hourly sales order to show linear cost structure with a piece-wise linear marginal cost curve	19
2.4	A sales block order with a constant quantity	20
2.5	Order types in European DAMs	22
4.1	Example hourly orders 1, 2, and 3 for the first period	41
4.2	Hourly orders 1, 2, and 3 where purchase and sales quantities are sep- arated	42
4.3	Aggregated quantities for the first period	46
4.4	Breakpoints of the aggregated curves for the first period	48
4.5	Breakpoints of the aggregated curves for the first period after adding the artificial breakpoints to handle curtailment	49
4.6	Size of the models we proposed in this chapter	71
4.7	Sets, indices, parameters and variables used in this chapter	72
5.1	Parent orders and their children for families 1, 2 and 3 given in Figure 5.1	76
5.2	Set of subfamilies originated from each block order in families 1, 2 and 3 given in Figure 5.1	76
5.3	Descendants and ancestors of each block order in families 1, 2 and 3 given in Figure 5.1	77

5.4	Sets, indices, parameters and variables used in this chapter	107
6.1	Sets, indices, parameters and variables used in this chapter	148
7.1	The models (P), (LB-P), and (UB-P)	157
8.1	Average number of orders per day	183
8.2	Average number price levels per day after aggregation	184
8.3	Solver tolerance values	185
8.4	Performance of the models we proposed for the problem variants . . .	191
8.5	Performance of the approximation procedures	193
8.6	Performance of adding warm starts	198
8.7	Comparison of cold started MC-WPRB and warm stated MC-WPRB	199
8.8	Comparison of cold started (MC-WPAB) and warm stated (MC-WPAB)	200
8.9	Performance of the procedure when we allocate 25 seconds to the heuristics	202
8.10	Comparison of cold started models and procedure in Algorithm 5 when we allocate 25 seconds to the heuristics	203
8.11	Quality of the initial solutions	204
8.12	Performance of the procedure when we allocate 550 seconds for the heuristics	205
8.13	Comparison of cold started models and procedure in Algorithm 5 when we allocate 550 seconds for the heuristics	206
8.14	Change in the optimality percentages with the relative gap values . .	206
8.15	Performance of decomposition procedures with exact cuts	207
8.16	Performance of decomposition procedures with heuristic cuts	208
9.1	Impact of the number of periods: Results when the model (MC-WPRB) is solved via Algorithm 5	216

9.2	Impact of the number of periods: Results when the model (MC-WPAB) is solved via Algorithm 5	216
9.3	Impact of the number of periods: Paired student t-test results when the models are solved with the exact and heuristic algorithms	217
9.4	Impact of the number of block orders: Results when the model (MC-WPRB) is solved via Algorithm 5	218
9.5	Impact of the number of block orders: Results when the model (MC-WPAB) is solved via Algorithm 5	219
9.6	Impact of the number of block orders: Paired student t-test results when the models are solved with the exact and heuristic algorithms .	220
9.7	Impact of the Number of Flexible Orders: Results when the model (MC-WPRB) is solved via Algorithm 5	221
9.8	Impact of the Number of Flexible Orders: Results when the model (MC-WPAB) is solved via Algorithm 5	221
9.9	Impact of the Number of Linked Block Orders: Results when the model (MC-WPRB) is solved via Algorithm 5	222
9.10	Impact of the Number of Linked Block Orders: Results when the model (MC-WPAB) is solved via Algorithm 5	223
B.1	Size of the models when the real data is used	232

NOMENCLATURE

AE	Approximation Error
AMCP	Average Market Clearing Price
ATC	Available Transmission Capacities
ATM	At-the-money
DAM	Day-ahead Market
EMRA	Energy Market Regulatory Authority
EUPHEMIA	Pan-European Hybrid Electricity Market Integration Algorithm
EÜAŞ	Turkish Electricity Generation Corporation
EXIST	Energy Exchange Istanbul
ISO	Independent System Operator
ITM	In-the-money
LCL	Lower Confidence Limit
MC	Market Clearing Model
MCP	Market Clearing Price
MILP	Mixed Integer Linear Programming
MIP	Mixed Integer Programming
MIQCQP	Mixed Integer Quadratically Constrained Quadratic Programming
MIQP	Mixed Integer Quadratic Programming
MWh	Megawatt Hour
NN	Average number of nodes discovered

NS	The instances that no feasible solution is found within time limit
OTC	Over the Counter
OTM	Out-of-the-money
OS	The instances that an optimal solution is found
PAB	Paradoxically Accepted Bid
PCR	Price Coupling Region
PJM	Pennsylvania, New Jersey, Maryland
PM	Performance Metric
PRB	Paradoxically Rejected Bid
PX	Power Exchange
RG	Average relative gap for TLS
QP	Quadratic Programming
SDB	Supply Demand Balance Model
SO	System Operator
TEAŞ	Turkish Electricity Generation and Transmission Corporation
TEDAS	Turkish Electricity Distribution Corporation
TEK	Turkish Electricity Corporation
TEİAŞ	Turkish Electricity Transmission Corporation
TETAŞ	Electricity Trading and Contracting Corporation
TL	Turkish Lira
TLS	The instances that at least one feasible solution is found but we could not prove its optimality due to time limit
TSO	Transmission System Operators
TTL	Average time for the instances which are solved to optimality
UC	Unit Commitment
UCL	Upper Confidence Limit

Chapter 1

INTRODUCTION

Electricity is an indispensable commodity in our lives. It is generated in power plants, which may be far from the consumers, converted into high voltage and transmitted through a grid to a distribution substations. Then, it is re-converted to low voltage to be consumed. In case of disruption in electricity generation, our hands are tied, because today's electricity storage technologies are not economically viable due to high cost and technical requirements. Although we can store electrical energy in other energy forms (e.g. potential energy, thermal energy or chemical energy), they are neither efficient nor adequate. Therefore, electricity is generated and consumed instantly. In addition, the voltage and frequency of the electricity must be kept between certain limits during its journey to our homes. Otherwise, the grid may collapse which results in long-term power outages.



Figure 1.1: Electricity Generation, Transmission and Distribution (source: www.instituteforenergyresearch.org)

In early stages of electricity power industry, generation, transmission and distribution are controlled by a central authority, which is usually the government. After the industry reaches maturity, the operations are unbundled and firms are privatized.

Turkish electricity market has gone through such an unbundling and a detailed Turkish electricity market liberalization history is given in Figure 1.2.

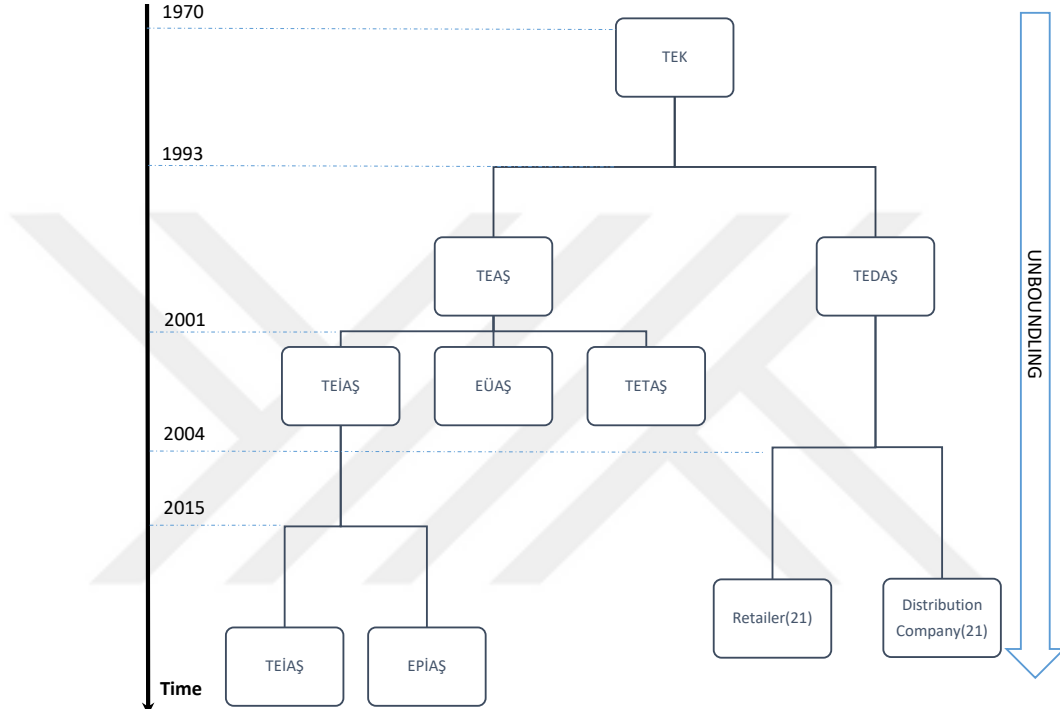


Figure 1.2: Unbundling process of Turkish electricity system

As we see in the example of Turkey, production and consumption are controlled by separate entities after vertical unbundling. Therefore, electricity becomes a commodity that can be bought and sold between these entities, which leads to a competitive electricity market. Before explaining electricity markets in detail, we will briefly discuss general market structure.

1.0.1 General Market Structure

Market is a place where a commodity is purchased and sold. Delivery time of the commodity may be different than the actual financial transaction. Markets are divided into future markets and spot markets in terms of delivery time of the commodity. In

spot markets, the commodity is delivered immediately, whereas in future exchange markets, future contracts are traded and the commodity is delivered at a future predetermined time. In both types of markets, prices are determined according to the current market price. In future markets, the price of the contracted commodity is derived from the movement of the underlying asset. Hence, these types of contracts are also called derivatives.

Spot markets are also called physical markets, and they are examined under two main categories: (1) exchange and (2) over-the-counter (OTC). Exchange is a market where financial instruments and commodities are traded. There are certain regulations or rules in exchange markets. On the other hand, an OTC trade is based on contracts between two parties and no specific rules are defined for this market. Electricity markets have also the same market categories.

Electricity has a distinctive balance characteristic due to the instant generation and consumption requirement. System safety and balance are very crucial in electricity systems and markets have evolved accordingly. Electricity is traded in three different spot markets to preserve system safety. The first one is *day-ahead market* (DAM), where the electricity is traded a day before the actual transmission. Therefore, the approximate consumption and production is known one day before the actual delivery. The second market is *intraday* where market participants join this market to cover unpredictable electricity needs during the day. In addition, the *balance market* aims to keep the physical system in balance in time of delivery. There are also ancillary services, where active and reactive power are controlled to keep frequency and voltage in the desired interval.

We can divide the spot market designs into two in terms of their planning levels: (1) pool market and (2) exchange market. A pool electricity market considers unit based requirements in the trade. On the other hand, in exchange electricity markets there is a high level of planning where scheduling of the portfolios of each participant is the participants' duty. A pool model is adopted in most of the US electricity

markets, whereas the European markets use the exchange market model. We will cover the general US electricity market structure next and discuss the European market afterwards.

1.0.2 Electricity Market in the US

In the US Electricity market, The Federal Energy Regulatory Commission (or the Commission) has a surveillance role in market design decisions and issues orders. According to Orders 888 and 889, Independent System Operators (ISOs) are defined as organizations which have authority for coordinating, controlling and monitoring the operation of an electrical power system. In the US, there are ISO and non-ISO markets. In non-ISO markets, electricity is traded by bilateral contracts, where generation and transmission are operated by vertically integrated companies. In ISO markets, power exchanges (PXs) and ISOs are working either in a centralized or a decentralized manner. PXs are responsible for organizing forward auctions, whereas ISOs run DAM and real-time electricity markets, and ensure the reliability of the system by short-term ancillary services (voltage support, operating reserves, and automatic generation control), and longer term capacity products. Pennsylvania-New Jersey-Maryland (PJM) interconnection system, New York and New England use a centralized unit commitment auction market. On the other hand, California has a decentralized system, i.e., there is no central planner for scheduling of production units in the system. Hence, scheduling coordinators of the generation companies self-commit their units.

In this type of market, electricity trading should be accomplished by the pool, where the power pool is defined by Sivanagaraju (2009) as interconnected utilities funded upon agreements to exchange power through tie lines, which connect different regions. In most of the cases, only selling electricity is allowed in the market to satisfy the fixed estimated demand. Due to the nature of the pool markets, the cost of electricity generators, such as start-up costs, shut down costs, ramp rates, etc.,

are considered by the central planner. In pool markets, finding generation quantities becomes a *unit-commitment* (UC) problem.

The UC problem determines the schedule of generation units within a power system over multi-hour to multi-day frame. Generally, UC problems minimize operating costs subject to operating constraints (i.e. ramp-rate limits, minimum up or down times, etc.) and system constraints (reserve and energy requirements and transmission constraints). The resulting problem is a large scale nonlinear mixed integer program. The UC models can be used by individual firms, centralized auctioneers, decentralized aggregators of generation schedules, and transmission system operators. As a centralized auctioneer PJM and New England solve unit commitment model in their DAMs to satisfy system reliability.

1.0.3 Electricity Market in Europe

Most of the European electricity markets have an exchange market structure. There are transmission system operators (TSO) in Europe which are responsible for balancing electricity in real time and grid safety. They determine the network boundary conditions to guarantee system feasibility and reliability and act as the system operator. One of the boundary conditions is the available transmission capacities (ATCs) of the transmission lines between regions. ATCs are delivered to independent PXs (i.e. market operators) who conduct spot market auctions subject to these capacities. The spot market auctions provide the electricity companies the total amount of electricity the company must dispatch to its power plants or provide to its consumers. Nord Pool Spot (Norway, Sweden, Finland, Estonia, Latvia, Lithuania, and Denmark), EPEX Spot (Germany, France, the United Kingdom, the Netherlands, Belgium, Austria, Switzerland and Luxembourg), GME (Italy), OMIE (Spain and Portugal), and EXIST (Turkey) are some PXs operated in continental Europe.

In this study, we will focus on the market clearing problem of European DAMs. Hence, we will cover DAM structure of European PXs in a detailed way in the next

chapter. We also provide the background information about economic interpretation of the markets. The rest of this study is organized as follows: In Chapter 3, a comprehensive literature review about pool and exchange market models is presented. We present the stylized supply demand balance problem in exchange type DAMs in Chapter 4, supply demand problem in Chapter 5 and its variants in Chapter 6. We present the solution methodologies for the proposed problems in Chapter 7. We conduct some numerical experiments in Chapter 8 with real data. We present our scenario analysis in Chapter 9. Finally, we conclude the thesis in Chapter 10.

Chapter 2

ECONOMIC BACKGROUND

In this chapter, we will give some background information about market design and auction mechanisms. Then, we will focus on European DAM and its auction mechanisms.

2.1 Market Formation

According to Gans et al. (2011), a market consists of a product/service and buyers and sellers of this product/service. Market players trade a commodity for a particular unit price. The price of the commodity is affected by the amount of the traded product.

Each market player may have a different level of willingness to pay or sell for that good. Jules Dupuit discovered that total quantity purchased increases when price decreases (Houghton, 1958). The relation between the unit price and supply/demand quantity is represented by supply and demand curves. Dupuit's law of demand determines downward sloping demand curves with respect to price. Moreover, marginal cost of production increases when quantity increases; therefore, supply curves are upward sloping. Figure 2.1 presents an example of supply and demand curves. It is conventional in economics to depict the price on y -axis and the quantity on x -axis although quantity is dependent on price.

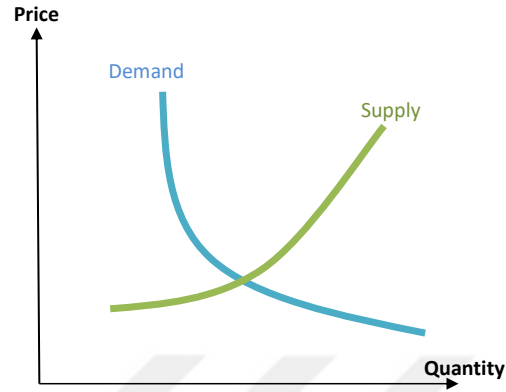


Figure 2.1: Supply and demand curves

Convexity is a basic assumption in microeconomic analysis, and the budget sets and preference curves are assumed to be convex. For a market, convexity ensures that market equilibrium occurs, which is defined as follows:

Definition 1. *Given price p^* and announced actions Γ_e^* for market player e , a market is in equilibrium if*

- *There is no other action than Γ_e^* that gives better profit or benefit for e given price p^* .*
- *Market clears (supply must be equal to the demand).*

Equilibrium is a desired property in market designs so that players have an incentive to play in the game. The participants may not want to trade in a market where the equilibrium is not satisfied, which may result in the crash of the market in the long term.

In a competitive market, the price of the commodity settles at a level where supply and demand are in balance. This price is called the *market clearing price* (MCP). MCP is also called *equilibrium price*, because the market clearing price satisfies the conditions stated in Definition 1.

Another desirable property of a competitive market satisfying convexity assumptions is *efficiency*. The first welfare theorem states that the market equilibrium leads Pareto efficient outcome which is defined as follows:

Definition 2. *A market outcome is Pareto efficient if it is impossible to make one player better off without having to make other player worse off.*

Market efficiency is important because we determine the future price of the goods by looking at the MCPs. Therefore, MCPs show true signals for investment decisions in efficient markets.

We measure efficiency in terms of total surplus. Total surplus is defined as the sum of the consumer surplus and the producer surplus (Mas-Colell et al., 1995). The difference between MCP and willingness to pay/sell is consumer's/producer's surplus (benefit) from one unit of good. Total surplus is the total net benefit so that it measures size of the "pie". The market is inefficient if we can increase the pie. The market is efficient if we cannot increase the size of the pie further. Therefore, we aim to maximize total surplus to obtain a competitive market.

Marshall et al. (1961) define consumer surplus as the area between MCP and the demand curve, and producer surplus as the area between the supply curve and the clearing point. If the demand is assumed to be price inelastic in a market, then there is no need to include the consumer surplus into surplus maximization problem because the corresponding value is fixed. Instead, we face with only the producer surplus, which corresponds to the total profit for all producers under the assumption that supply curve represents the **true** marginal costs.

Efficiency occurs when the marginal cost equals marginal benefit in the market. Otherwise, there is dead weight loss (loss of surplus) in the market. Due to surplus loss, we leave money on the table in an inefficient market.

There are two common pricing rules in markets: *uniform pricing* and *discriminatory pricing*.

Definition 3. *Uniform pricing is a pricing rule in a market where each market player trades the same commodity with the same price. If the price of the good changes for different players it is called discriminatory pricing.*

For instance, the price of a theater ticket may change when the buyer is a student or an adult, which is an example of price discrimination. The goods or services provided by the government have uniform prices in most of the cases. To illustrate, government owned bread seller in Turkey which is called “halk ekmek” sells bread for the same price all over the country to any consumer.

2.2 Auction Mechanism

Auction is a market mechanism where an auctioneer takes the bids from the participants and announces the winners of the auction in terms of some pre-defined rules.

The most common auction mechanism is *open ascending price auction*, where the bidders submit their bids sequentially. Each bid must have a higher price than the previous one. The winner of the auction is the player who offers the highest bid price. There are other pricing mechanisms used in the auctions, such as *pay-as-bid-auction*, or discriminatory pricing auction where each participant has different prices for the product. *Walrasian auction* is another example where auctioneer takes bids and offers from both buyers and sellers and then announces a uniform price where supply and demand are in balance.

There is a general convention that a *bid price* represents the participant’s willingness to buy and an *offer price* shows the minimum price that a seller wants to receive. Although there are some studies which use “bid” and “offer” to indicate the direction of the orders, we do not consider such a difference in this study.

2.3 Non-convex Markets and Combinatorial Auctions

A market is called non-convex if the convexity assumptions are violated. In such a market, equilibrium may be inefficient or supply demand balance may not be satisfied.

Combinatorial auction is a type of auction where discrete items, which do not have continuous quantity, are offered. Combinatorial auctions give rise to a non-convex market structure. Hence, we cannot guarantee an auction design which enjoys all the desirable properties we observe in a convex market. However, according to Pekeč and Rothkopf (2003) the following properties are desired in a combinatorial auction:

- *Allocative efficiency* is achieved when the total value of the bidding item to the auction winners is maximized. Pareto efficiency described in Definition 2 can also be used as allocative efficiency in an auction.
- *Revenue maximization (or cost minimization)* is the main goal of the auctions where one-sided auction (only buyers or sellers submits bids) is used. Total surplus can be used in two-sided auctions as an objective function.
- *Fairness* is another property that an auction designer should consider. Equal treatment of the participants irrespective of their market sizes impacts willingness to participate to the auction.
- *Low transaction costs* reflects the cost for participating the auction should be low. When the participation costs are high, participation rate will be low and it impacts fairness.
- Delay in concluding the auction may lead to opportunity cost for the participants, therefore, *auction speed* is important.
- *Failure freeness* is a property that shows that auction must be run in even

“extreme conditions”. We can also interpret this property as winner of the auction must be announced in any conditions.

Winner determination problem is a process of finding the optimal allocations of the participants. Combinatorial auctions have emerged to satisfy the need for all-or-nothing bids, where partial allocation of the bid is not allowed. This creates complexity in winner determination problem.

In the next section, we cover European day-ahead electricity market, which is a real life example of a non-convex market where combinatorial auction is used.

2.4 European Day-Ahead Electricity Markets and Their Auction Mechanism

In this section, we cover the basic structure of European day-ahead electricity markets (DAM). DAM is a type of electricity market in which electricity is traded one day before its actual delivery. Therefore, electricity balance is achieved in one day before.

On each day, there is a two-sided auction where both suppliers and retailers can place orders for the next day. Order submission is the only way to participate in the DAM. Participating in the DAM is not obligatory. For instance, the participant can always trade electricity through bilateral contracts. After gathering the orders from participants, an optimization tool evaluates the orders and finds the unit price of the electricity for each hour of the upcoming day. This price is called the *market clearing price* (MCP), which is the base price for the stock market.

2.4.1 Locational Pricing in Europe

Transmission lines have a finite capacity. Therefore, the cheapest electricity may not be transmitted to a region where MCPs are higher. *Locational pricing* is the differentiation of the clearing prices of the electricity in different areas because of transmission limits.

European Union (EU) has a goal of having a single market in all over EU. Electricity, which is also a commodity, is also affected by this goal. Countries in EU started to integrate their electricity markets to increase the total liquidity and grid safety. Integration first started in DAMs. Before 2010, European Market Coupling Company (EMCC) collected all offers from the PXs and received ATC information from TSO's and computed power flows between adjacent markets and prices. This is called the *volume coupling problem*. After net import-export was determined by EMCC, the PXs run their own algorithm to determine the MCP in their region. This created a bi-level solving procedure for the coupling problem. In 2010, with the initiative of seven PXs in Europe, Power Coupling of Regions (PCR) were formed in order to solve the coupling problem by a common algorithm in a single level. Today, the coupling problem of multi regional coupling (MRC) members, which is more comprehensive than PCR, is solved by a common algorithm called *EUPHEMIA* (EU Pan-European Hybrid Electricity Market Integration Algorithm). *EUPHEMIA* solves a mixed integer quadratic programming model which maximizes total surplus subject to the constraints of each region. After *EUPHEMIA* solves the problem, the resulting flows and price signals are submitted back to the PXs and TSOs so that the necessity for resolving the problem in each PX vanishes.

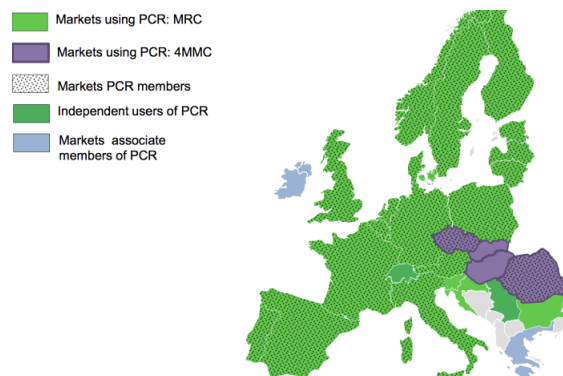


Figure 2.2: Electricity Markets Using Price Coupling Regions (source: N-Side)

2.4.2 Order Types in European DAM Auctions

The order types in European DAM Auctions share some common properties. Each order consists of a price and a corresponding quantity information. The price of the order reflects the unit price of the electricity a participant wants to purchase or sell. An order can be either in sales (supply) or purchase (demand) direction. Quantities may have positive or negative values, where a negative quantity represents sell and a positive one represents buy direction. The unit of quantity is “*MWh*”.

A day is divided into 24 hour periods, given by 00:00 - 01:00, ..., 23:00-00:00. On the days where there is a transition between day-light saving time, the number of period would be 23 or 25.

A supply order with unit price p and quantity q in period t indicates that the participant is willing to sell q units of electricity to the system if the MCP is greater than equal to p in the corresponding period. Similarly, if it is a purchase order, then the participant is willing to buy q units of electricity from the system if the MCP is less than or equal to p .

There are seven types of orders in European DAM auctions. We next review each type in detail.

- **Hourly Order (Single Order):** This type of an order can be placed in a single period. For each period, participants may order different price levels for different quantities in both sales and purchase directions. Each breakpoint consists of a price level and corresponding quantity. In each order, price levels must be in ascending order, whereas corresponding quantities are given in descending order.

Each breakpoint is linearly interpolated to find the corresponding quantities. Then the resulting curves are either step functions or piece-wise linear functions. Step functions arise if there exist two consecutive same price values with different quantities or consecutive quantities with different prices. For a piece-wise linear

function, the submitted prices must be strictly increasing where corresponding quantities are non-increasing. Hourly orders which result in step functions are used in EPEX Spot, whereas Nord Pool Spot and EXIST use piece-wise linear functions. An example is presented in Table 2.1. We also present the resulting step functions after the linear interpolation in Figure 2.3.

Table 2.1: An hourly order example

Price (TL/MWh)/Period	0	130	130	250	250	2000
00:00-01:00	-1000	-1000	-1500	-1500	-1600	-1600
01:00-02:00	200	200	150	150	50	50

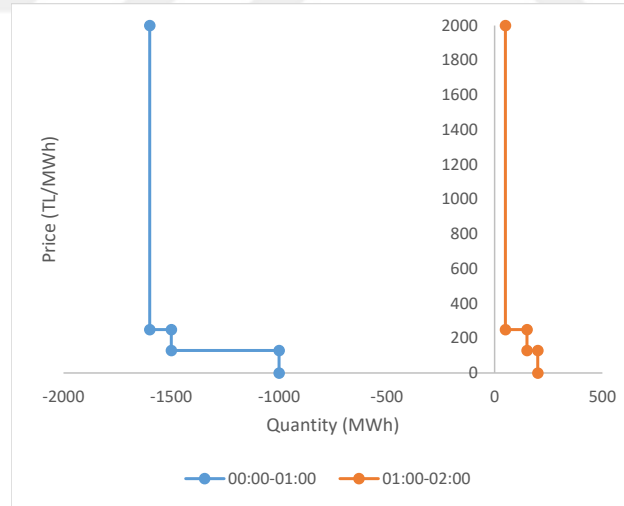


Figure 2.3: Step functions of the hourly orders in Table 2.1

Table 2.2 presents an hourly order with 5 price levels in sales direction for the first period, and 3 levels in sales and 2 levels in purchase directions for the second period. For example, a participant with the given order is willing to sell 150

MWh if the unit price is at least 250 TL/MWh in the first period and is willing to buy 15 MWh if the price is at most 130 TL/MWh in the second period. Figure 2.4 presents the resulting piece-wise linear functions of the hourly order given in Table 2.2 for each period.

Table 2.2: Another hourly order example

Price (TL/MWh)/Period	0	130	170	250	2000
00:00-01:00	0	-100	-120	-150	-160
01:00-02:00	20	15	-10	-16	-30

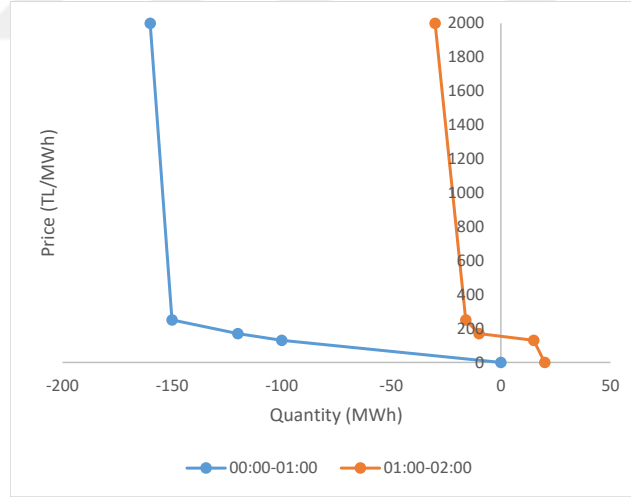


Figure 2.4: Piecewise linear functions of the hourly orders in Table 2.2

We now explain how linear interpolation is employed in piece-wise linear functions. Suppose that the price-quantity pair for a participant be (p_1, q_1) and the other one be (p_2, q_2) . For a price p , where $p_1 \leq p \leq p_2$, the participant is willing

to sell or purchase q units of electricity, where $q = q_2 + \frac{q_2 - q_1}{p_2 - p_1}(p - p_2)$, with $q_2 \leq q_1$ in piece-wise linear hourly orders.

Each single order must include minimum and maximum price levels. The minimum price limit is negative in Europe, so that some producers may be willing to pay to the system to continue producing electricity. Moreover, a maximum price limit is defined by value of loss load (voll). The price limits may change among different countries.

Since the prices reflect the marginal cost or benefit, step-wise hourly orders are used to represent linear costs, whereas piece-wise hourly orders shows quadratic cost structure. The participants with linear cost/benefit structure, may offer piece-wise linear curves which are almost step functions in a country that uses piece-wise linear curves only. Table 2.3 presents this type of orders. A fixed quantity is quoted in an interval of prices, e.g. 1000 MWh offered in the range of prices [130, 169.99] and 1500 MWh in [170, 1999.99].

Table 2.3: An hourly sales order to show linear cost structure with a piece-wise linear marginal cost curve

Price (TL/MWh)	0	129.99	130	169.99	170	1999.99	2000
Quantity(Lot)	0	0	-1000	-1000	-1500	-1500	-1600

Definition 4. *A price independent order has a fixed quantity and the participant is willing to sell/purchase the electricity no matter what the announced price is.*

Some of the participants may want to sell the electricity for any price. Hence, they offer price independent orders, where the same non-zero quantity is offered for minimum and maximum price levels. This type of orders cause non-zero quantities in the price limits.

- **Block Order:** Thermal power plants should continue production for a certain time period to gain positive profit since start-up costs of such plants are high. A block order meets this need by its all-or-nothing property.

A block order is valid in a certain number of consecutive periods. If accepted, its quantity must be allocated in every period in its interval limit, or the order is entirely rejected. There is a single price for the order which shows the volume weighted average price that the participant offers over the time window of the order. The volume weighted average price is the ratio of the total cost/benefit of the order over the total quantity offered in the time window.

A block order owner may submit a single quantity for the period during which the order is active. This shows that the participant wants to buy/sell that quantity in each period. On the other hand, this property may result in inefficiency because of ramping constraints of power plants. Some power plants such as thermal power plants cannot generate electricity with a full capacity in an hour if the plant was in off state before. Hence, in a *profile block order*, where the bidder can submit different quantities per period to reflect its ramp-up and ramp-down constraints.

Table 2.4: A sales block order with a constant quantity

Period	Price	Quantity
18:00-23:00	100	-2000

In Table 2.4, a supplier offers 2000 MWh per period at the price of 100 TL/MWh during 18:00-23:00.

There are some power plants where the second turbine of the plant cannot run unless the first one is operational for a time. To capture this need, there is a

special type of block order which is called the *linked block order*. A block order can be linked to another block order, which affects the acceptance condition of the block orders.

Another block order type is *curtailable block order*, where the block owner provides minimum acceptance ratio (MAR) for the block order. Then, the order is either rejected, or the accepted portion of the order is between MAR and 1. MAR is the same in all periods during which the block order is valid. Curtailable block orders may have link relations and their quantities may be different in its interval.

- **Flexible Order:** This order type is similar to block orders. They are comprised of a price, a quantity, number of consecutive periods (duration), and an interval limit where the order must be accepted. The main difference is that the size of time interval of the flexible order is strictly larger than its duration. Thus, they do not contain any information on which hour it must be accepted. The period where the flexible order is accepted is determined by the auctioneer.
- **Exclusive Group:** Sometimes a bidder may want to order several block orders and may request that only one of them is activated. This need is satisfied by exclusive groups.
- **Prezzo Unico Nazionale (PUN) Order:** These are hourly purchase orders which are matched at the same PUN price although the orders are offered for different price regions.
- **Minimum Income Condition (MIC):** The marginal cost/benefit can be reflected in aforementioned order types. On the other hand, power plants have fixed costs, which should be covered. Fixed cost and variable costs are reflected in MIC orders in such a way that the bidder gets at least its submitted minimum income when the order is accepted.

We summarize which PX uses which order type in Table 2.5.

Table 2.5: Order types in European DAMs

	EPEX	Nord Pool	EXIST	GME	OMIE
Hourly	Step-wise	Piece-wise	Piece-wise	Step-wise	—
Block	✓	✓	✓	-	-
Profile Block	✓	✓	-	-	-
Linked Block	✓	✓	✓	-	-
MAR	-	✓	-	-	-
Exclusive	✓	✓	-	-	-
Flexible	-	✓	✓	-	-
PUN	-	-	-	✓	-
MIC	-	-	-	-	✓

In this study, we focus on piece-wise hourly, block and flexible orders as in EXIST. There may be more than one piece-wise hourly order in a period. In next section, we discuss these orders and explain a method to aggregate these orders.

In the next section, we discuss the evaluation of the bids and the resulting market clearing problem.

2.5 Evaluation of the Orders

At the end of the market clearing process in DAMs, each participant must know how much electricity they produce or sell in each period, which is determined by their bids. A participant's order is called *matched (accepted)*, if he produces/consumes electricity due to that order after market clearing. The amount of electricity the participant sells/buys is called *matching quantity* and the unit price for this matching is called *matching price*.

Since European DAM is an exchange market, they adopt some terms used in stock exchanges.

Definition 5. *Moneyness is the relative position of the current price of an asset with respect to the strike price of an option. Moneyness is classified as: in-the-money, out-of-the-money and at-the-money.*

In power markets, order matchings depend on participants' incentives, which are defined as the benefit gained from the outcome. A participant has an incentive if the price of his sales order is less than the MCP or his purchase order has a price which is larger than the MCP. The orders whose incentive constraints are satisfied are called *in-the-money* (ITM) and the orders whose incentive conditions are not satisfied are called *out-of-the-money* (OTM). If the price of the order is the same as the MCP, then the order is *at-the-money* (ATM).

In order to determine allocations of each order type, the market clearing problem needs to be solved within 10 minutes (Euphemia, 2016). The objective is to maximize the total surplus while ensuring that supply and demand must be equal. Solving the winner determination problem in a short time is important as we discussed in the auction speed. Moreover, finding the optimal solution of the problem is also important to have efficient outcomes. Due to the combinatorial structure of the market clearing problem, it is not always easy to find an optimal solution in a short time. This challenge is one of the main motivations of our study.

Another interesting point in the DAM is that there may be some ITM orders which are rejected. However, the participant would have an incentive for her order to be accepted because he/she would have gained a positive profit if her order were accepted. Moreover, it is also possible to have an accepted order which is OTM. The reason of this phenomenon is the non-convex nature of the market.

Definition 6. *The orders which are rejected although they are ITM are called paradoxically rejected bids (PRBs). Similarly, the OTM orders which are accepted are called paradoxically accepted bids (PABs).*

We will illustrate the PAB by an example. Suppose that the aggregated demand curve and the aggregated supply curve are the same for all periods as presented in Figure 2.5 (a) for one period. Also suppose that there is a sales block order with quantity of 50 *MWh* with a price of 170 TL, which means the block order owner is willing to sell the specified quantity if and only if the average unit price of the electricity for the periods where the block is active is higher than 170 TL.

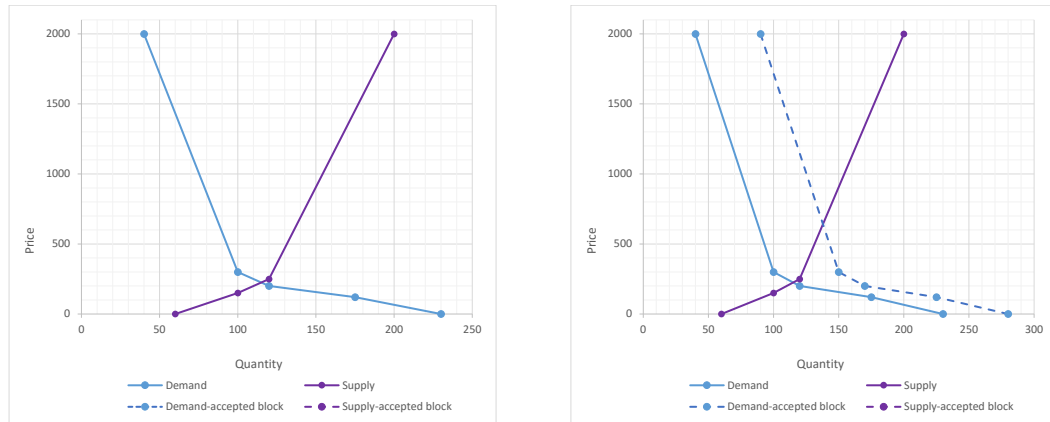


Figure 2.5: Aggregated supply and demand curves for a period to show the impact of accepting/rejecting a sales block order

If we reject the block order, the MCP for each period is equal to 225 TL, which is the point where supply and demand curves intersect in Figure 2.5 (a). In this case, the MCP is higher than the price of the block order. Hence, the block order is rejected although it is ITM and it becomes a PRB. On the other hand, if we accept the block order then the supply curve is shifted by 50 *MWh* to the right as depicted in Figure 2.5 (b) because we introduce extra sales quantity to the system. Then, MCP is 154,03 TL, which is less than the price of the block orders. In this case, the block order is accepted although it is OTM and an additional payment has to be made to the block

order owner to compensate this loss since otherwise he would make a loss and prefer not to sell his matching quantity.

Let us consider another example in which aggregated demand and supply curves are as in previous example. Instead of sales block order, suppose that there is a purchase block order with a quantity of 50 MWh with a price of 250 TL. In other words the block order owner is willing to buy the specified quantity if and only if the average unit price of the electricity for the periods where the order is active is lower than 250 TL. If we reject the purchase block order, the MCP for each period becomes 225 TL, which is lower than the price of the block order. Hence, the block order is rejected although it is ITM and we end up with a PRB. On the other hand, if we accept the bid, the MCP is the demand curve is shifted by 50 MWh to the right as depicted in Figure 2.6 (b) because we introduce extra purchase quantity to the system. Then, MCP is 642 TL, which is higher than the price of the block order. In this case, the purchase block order is accepted although it is OTM and an additional payment has to be made to the block order owner.



(a) Hourly orders when we reject the purchase block bid (b) Hourly orders after accepting the purchase block bid

Figure 2.6: Supply and demand curves for a period to show the impact of accepting/rejecting a purchase block order

As illustrated by the above example, it may not be possible to have a solution with no PABs or PRBs. There may be some additional payment to compensate for the paradoxical bids to support the bidders.

Definition 7. *The financial compensation paid to the participants for giving them some incentives to accept the resulting allocation is called uplift, or side payment.*

Although most of the electricity market models use MCP as a pricing strategy, some of the markets combine MCP and pay-as-bid analogy together. For example, the electricity market in New York announces MCP and can accept some offers which are financially compensated. This financial compensation is called uplift and such a system is referred to as a nonlinear price regime because some of the participants get paid according to the MCP while some other players may receive a different price than MCP as a result of the total surplus maximizing solution. Uplift may be necessary in order to achieve Walrasian (competitive) equilibrium, where each participant is happy. Although this payment generates a socially maximized welfare, it also has some drawbacks. First of all, participants do not bid their true costs so that market can be manipulated if it is not strictly regulated. Uplift payments change the cost structure of the participants. Hence, the bids of the participants may no longer depend on the marginal cost/benefit, which results in false signals to market. The other disadvantage of having an uplift payment is collecting this additional money. On the other hand, EU adopts a linear (or uniform) price scheme that proposes a sub-optimal value, but all participants are paid according to one price at the same location and time.

In the ideal case, there is no PAB and PRB in an optimal result of the clearing problem. However, if we do not allow having PABs and PRBs, the problem may become infeasible due to the existence of the block orders as illustrated by the previous examples. We will also focus on this problem, and propose a market design which does not have any PRB in the subsequent chapters.

In this chapter, we give some background information about the economic inter-

pretation of the markets, especially the non-convex markets, and define some terms used in this study. In next section, we will focus on the literature review and discuss our contributions to the literature.



Chapter 3

LITERATURE REVIEW

In this section, we will review studies on economic basis of markets and define several basic concepts that will be used in our analysis of electricity markets. Later on, we will present studies conducted on exchange and pool electricity markets.

The electricity market literature is divided into two: (1) The electricity markets on the rule-follower perspective, i.e., plant scheduling, strategic decision making of the plant owner, price prediction, etc., (2) the electricity market problem from a rule maker point. Market design options and their economical investigations, nodal-zonal pricing mechanisms and their determination, financial risks of the markets are several research topics that address to find solutions for the rule maker's problems. We focus on combinatorial electricity market auctions in this study and present a comprehensive literature review about market clearing mechanisms in day-ahead electricity markets. First, we review the combinatorial auctions. Then, we focus on market clearing mechanisms for day-ahead electricity markets. Finally, we focus on the solution methodologies and state our contributions.

3.1 Combinatorial Auctions in Electricity Markets

Both pool and exchange markets depend on an auction mechanism. There are two different pricing systems in auctions: market clearing price and pay-as-bid price. In pay-as-bid system, each bidder is settled with the price she has offered. On the other hand, in MCP schemes, participants take the same clearing price. According to María (2010), in the auctions where MCP is used as the pricing scheme, the price is used as a right signal about dispatch and investment recovery as well as generation adequacy.

On the other hand, in a pay-as-bid system, the settled price cannot be a reference price for the future market as in MCP. Actually, the pay-as-bid system does not have any advantage according to the comparison in María (2010). This system is used in Panama and Mexico (Maurer and Barroso, 2011).

In the literature, several ways of determining MCP have been investigated. In the next section, we cover the pricing models in electricity markets.

3.1.1 Pricing Models in Non-Convex Electricity Markets

In the electricity markets, the participants have non-convex cost structures due to start-up and minimum run time requirements as in thermal plants as discussed in Arroyo and Conejo (2000). Therefore, orders with all-or-nothing constraints are used in the electricity markets which create non-convexities in the market design. However, there may be cases where linear equilibrium prices do not exist in case of non-convexities, which may effect the competitive market structure (Motto and Galiana, 2002; O'Neill et al., 2005; Hobbs et al., 2006). Different pricing mechanisms are proposed to deal with this problem.

Hogan and Ring (2003) present an uplift minimizing auction problem to find the optimal prices. They formulate three models which form a competitive partial equilibrium. The first one is the market operator's profit maximization problem which contains uplifts. The remaining models formulate the problems faced by each generator unit and load. The authors indicate that uplifts can be negative or positive; however, negative uplifts -which means participants should pay money- decrease the generator profit to zero. Instead, they should be positive, therefore the authors suggest to formulate a minimum uplift equilibrium model where the uplift is always greater than or equal to zero to capture the incentive effects in a better way.

O'Neill et al. (2005) present a pricing model which depends on solving the welfare maximization problem with non-convexities where the dual of the clearing constraint yields the efficient prices. The work starts with analyzing the example studied in

Scarf (1994). Then, they propose a method which assumes a production plan is known beforehand and formulates an MILP problem and an LP problem. The discrete decision variables of the MILP problems are fixed to their optimal values and it results in the LP problem. By using this idea, they treat binary variables as “additional commodities”. Then, they take the dual of the LP model and find the optimal prices for all commodities. The shadow prices which correspond to binary decision variables are the uplifts. These optimal prices represent the contracts of each participant which forms Walrasian decentralized price directed equilibrium. The pricing method proposed in O’Neill et al. (2005) is called *IP pricing*. We will use a variant of IP pricing in this study.

Gribik et al. (2007) propose convex hull pricing model, which minimizes the total uplift corresponding to the solution of the dual of the non-convex unit commitment model. They show that the duality gap of non-convex unit commitment model equals the total uplifts. Observing zero duality gap in a case means that there is no uplift at the optimal solution of that problem. When the duality gap is zero, both the financial balance and Walrasian equilibrium are satisfied. The results obtained in Gribik et al. (2007) show that the optimal prices in Hogan and Ring (2003) are the Lagrangian multipliers of the problem defined in Gribik et al. (2007).

Oneill et al. (2007) propose a multi-part discriminatory pricing mechanism and a settlement procedure for European Exchanges where block orders and limit orders are considered. They propose a simple surplus maximization problem where supply and demand are in balance, with all-or-nothing property of the block orders and stepwise hourly order functions. They first aim to find a linear pricing result for the problem. Since no such price may exist in general, they propose a model which minimizes the total discriminatory payment. They also suggest an alternative pro-rata procedure by paying a uplift for the orders which are accepted although they have negative surplus and paying additional payment for the orders which are rejected although they have positive surplus.

Bjørndal and Jörnsten (2008) use the reformulation proposed in O'Neill et al. (2005) as a Benders sub-problem. In their formulation, the dual variables of equality constraints are the coefficients in the Benders cut. Then, they validate their results by using the data provided in Scarf (1994).

Van Vyve et al. (2011) compares pool and exchange models by considering truth-telling, missing money problem, Walrashian equilibrium, and fairness. They argue that pool models have truth-telling and missing money problems due to additional financial compensations. They state that European market model lacks Walrashian equilibrium and fairness. Then, they propose a market design for European markets which is mixture of exchange and pool models. In the first step, they solve a surplus maximization problem with basic conditions such as supply demand balance, all-or-nothing constraints, and ramping of the lines. Then, by fixing the quantities coming from the first problem, they compute a price that supports financial balance and surplus incentives by minimizing uplift in the second stage.

Classical pool models maximize total surplus by considering supply demand balance of the electricity, constraints on transmission lines, and constraints of the plants. Ruiz et al. (2012) propose a model that minimizes the difference between the total surplus and the objective function of the dual of the pool model while ensuring primal and dual constraints. They argue that the proposed model does not cause uplifts since dual variables are explicitly taken into account. However, they do not discuss whether there exists a solution for the resulting model.

Kuang et al. (2017) extend the IP formulation proposed by O'Neill et al. (2005) by considering quadratic commodity costs of ramping constraints of renewable energy sources. Madani et al. (2018) reformulate IP pricing, convex hull pricing and European market model when start up costs, minimum power output and ramping constraints are considered. They compare the results of the three models.

3.2 Solution Methodologies for Electricity Market Models

The proposed solution methodologies for the market models are problem specific. There are two main problem types in the day-ahead electricity markets: Unit commitment models and exchange models. We review the solution methodologies for each problem type.

3.2.1 Unit Commitment Problems

The UC problem has been studied over the decades. Since the UC problem is a large-scale problem with nonlinear constraints, heuristic approaches are widely used in the literature due to the smaller computational effort and their simplicity. Genetic algorithm (Arroyo and Conejo, 2002; Kumar and Mohan, 2010), simulated annealing (Simopoulos et al., 2006), evolutionary algorithms (Georgopoulou and Giannakoglou, 2010; Chung et al., 2011) and particle swarm optimization (Jeong et al., 2010; Kuo et al., 2011) are commonly used heuristic methods to solve the UC problem.

Mathematical programming techniques are also used to solve the UC problem. For instance, dynamic programming (DP) approach is employed in several studies (Rong et al., 2009; Singhal and Sharma, 2011). However, the curse of the dimensionality is main challenge in DP. Due to the improvement in MIP solvers, MILP formulations of the UC problem have recently been proposed (Ostrowski et al., 2012; Morales-España et al., 2013, 2014; Zheng et al., 2016; Yang et al., 2017). Although the MILP solvers can solve these problems to optimality, it is still a large scale problem that needs special attention. Lagrangian relaxation, Benders decomposition, second order cone programming, outer approximation methods are also used as solution approaches for the UC problem. We refer the reader to detailed survey papers (Padhy, 2004; Tahanan et al., 2015) for further information about the solution approaches for the UC problem.

3.2.2 Exchange Models

Exchange models rely on a two sided auction system. Participants provide continuous hourly order curves as well as orders that may lead to non-convexities. After taking the orders, and solving a combinatorial market clearing problem, the market operator announces a clearing price. The main difference between the US and European models is that orders used in the US contain technical information about the generation units. Instead, the European model adopts a portfolio-based bidding, so that after MCP and the corresponding quantities for all members are announced, each member decides which generation units to use. Moreover, the European market adopts strict linear pricing, which means there are no uplifts as in the US case. Non-convexities exist due to the all-or-nothing nature of block orders, which reflects start-up costs and a minimum run level of a generator. These non-convexities cause paradox as we discussed in the previous chapter. The European model prefers to reject this kind of orders. In the US model the bidders are financially compensated to achieve Walrasian equilibrium. Therefore, we can conclude that the European system is not based on competitive equilibrium.

Meeus et al. (2009a) model the Dutch DAM where the objective is to maximize total trade gain. They work with real Dutch data and solve 200 different days by using CPLEX. They focus on the statistical part of their results and observe the following: (1) There is no correlation between the volume of the order and PRBs. (2) There is no correlation between the number of block orders and the calculation time. (3) The number of PRBs increases with the number of block orders

Zak et al. (2012) present a stylized model for the European DAM. Their modeling approach is based on bilevel programming where the upper level is an MIP model and the lower level is a parametric dual linear program when the binary variables of the upper level models are fixed. They form a relation between the condition of accepting block orders and the decision variables of the lower level model. Finally, they can convert this bilevel program into a single-level MIP. A toy example is created and

solved in less than 16 seconds.

Martin et al. (2014) refer to PRBs as missing not-realized profit because the blocks are in-the-money and can be accepted. They propose a model which solves a multi-regional European market clearing problem. The change between consecutive hours in a cross border transmission line, which is called the ramp rate, has an upper bound and the prices in each region are given. The objective of this model is to maximize the social welfare. The authors define a mixed integer quadratic programming (MIQP) problem, and then present a relaxed version of the problem. This model relaxes the incentive constraints so that a participant may incur a loss, and contains only ramp rates and supply demand balance constraint whereas the values of binary decision variables are given exogenously. Then, the authors prove that the optimal dual variable of the balance constraint gives the clearing price for the relaxed problem. They propose a bid-cut heuristic and a branch-and-cut decomposition method to solve the problem. The heuristic terminates in 1 minute whereas the branch-and-cut decomposition terminates in 9 seconds when it starts with the output of the heuristic.

Each PRB gives rise to an opportunity cost, which is given by the volume of the PRB times the difference between the MCP and the price of the PRB. Madani and Van Vyve (2014) propose an approach in which the total opportunity cost flourish from PRBs is minimized. They show that the total surplus maximization may not necessarily minimize the opportunity cost on a toy example. They use stepwise hourly order curves and propose an MIP model by only considering the balance constraints while maximizing the total surplus. They split block variables into two sets to obtain an LP, where the first set represents accepted blocks, and the other one represents rejected blocks. They show that the value corresponding to the the dual variable of accepted blocks constraint is an upper bound on the loss of accepting that order whereas the value of the dual variable associated with the rejected blocks constraint is an upper bound on the opportunity cost. When the problem is converted into an opportunity cost minimization problem, it is numerically observed that the trade-off

between welfare maximization and opportunity cost minimization is very small.

Madani and Van Vyve (2015) propose two models for European power exchanges. The first one uses stepwise hourly order curves and block orders, leading to an MILP model. The second one uses piece-wise linear hourly orders and block orders yielding an MIQP problem. They develop a Benders-like decomposition for single-region one-period linear and quadratic problems. The cuts are stronger than the cuts defined in Martin et al. (2014). Furthermore, Madani and Van Vyve (2018) apply the modeling and solution techniques proposed in Madani and Van Vyve (2015) to a new order type called the MP (minimum profit/maximum payment) order by which a participant may bid its fixed cost and variable cost together. They are similar to MIC orders since a participant signals minimum profit/maximum payment required for the execution of the order. MP orders differ from MIC orders in terms of their contribution to the objective function.

Derinkuyu (2015) formulates the problem as a MIP problem. Besides all types of orders in used in the Turkish system, he also introduces flexible orders which are valid not only in supply but also in demand direction. Special-ordered set of type-2 (SOS2) variables are used to define hourly orders in the model. The objective function is the sum of the hourly MCPs of a given day. In order to decrease the number of decision variables used in the model, all hourly orders are aggregated into a single order for each hour. Then, he uses a variable elimination method to decrease the problem size and the solution time. In this method, lower and upper bounds of MCPs are determined for each hour. The lower bound is found by accepting all sales block orders while rejecting all purchase block orders and removing the block orders which do not satisfy the average price criterion iteratively. The reverse of this method gives an upper bound. After variable elimination, he suggests a heuristic method to find an initial solution, which depends on iteratively accepting block and flexible orders. Then, he proposes a very large neighborhood search, which starts with a feasible solution and solves the reduced problem iteratively until no further improvement is

observed. In each iteration MCP ranges are reduced by a perturbation parameter, and the variable elimination algorithm is applied. According to the computational results on real-life data, all of the cases are solved in one hour and most of them are terminated in less than one minute.

Yörükoğlu (2015) extends and modifies the Central Western Europe (CWE) market coupling algorithm defined by Djabali et al. (2011) to adopt the problem into the Turkish DAM. First, he adds linked blocks and flexible orders to solve a single-zone problem. Moreover, the quadratic mixed integer problem suggested in the study maximizes welfare, and there is a linear approximation of the objective function of the problem. The proposed model includes constraints for accepting and rejecting a block order. The author develops an iterative bi-criteria algorithm to deal with PRB. He iteratively changes the maximum number of PRBs used and obtains an efficient frontier between the number of PRBs and social welfare. The model is tested on a synthetic data set. The numerical results illustrate that the linear approximated problem deviates from the optimal solution by around 10% and it takes longer to solve the approximated LP problem than the original one.

Chatzigiannis et al. (2016) formulate the market clearing problem of block, linked block, flexible, exclusive group orders and MIC orders with network constraints such as PTDF and ramping. They propose a heuristic solution to solve the proposed model.

Ceyhan et al. (2017) empirically compare three different market models for European electricity markets in terms of some performance measures. All of the models maximize the total surplus using piece-wise hourly orders and block orders. One of their models forces block orders with positive surplus to be accepted, whereas another model forces the block orders with negative surplus to be rejected.

Kurt et al. (2018) apply an adaptive tabu search algorithm for Turkish electricity market with price regions. They use a synthetic dataset to show the performance of their algorithm and compare their results with the heuristic decomposition proposed

in Martin et al. (2014). According to the results, their algorithm finds better results than the heuristic decomposition when the ATC limits are low between the price regions.

3.3 Our Contributions

In this study we focus on exchange models with piece-wise hourly, block, linked block and flexible orders. We first examine the properties of a stylized supply demand balance model where there are only supply and demand hourly order curves. We show that instead of using supply and demand curves separately, we can use their summations and the resulting model still generate the same optimal solution with the stylized model with supply and demand curves. The new market clearing model has less number of decision variables than the stylized supply demand balance model.

Secondly, we investigate the properties of an optimal solution of the supply demand balance problem of the power exchange, which is concerned with surplus maximization with supply demand balance constraint, all-or-nothing conditions of block and flexible orders, and link condition of block orders. To the best of our knowledge, this is the first study that investigates the optimality conditions of piece-wise linear hourly, linked block and flexible orders.

As discussed in Van Vyve et al. (2011), Walrasian equilibrium fails in the market clearing problem in European power exchanges. Moreover, the resulting MCPs of the market clearing problem in the Turkish DAM also suffer from equilibrium. In this study, we propose a new IP pricing mechanism with additional constraints that supports Walrasian equilibrium when there exist piece-wise hourly, block, linked block and flexible orders with additional uplift.

We model both the European exchange model of NordPool Spot, where there exist piece-wise hourly, block, linked block and flexible orders, and our proposed market clearing problem. Since we use piece-wise linear hourly orders, the surplus function, which is the objective function, has quadratic terms. Therefore, decreasing the num-

ber quadratic terms improves the solution times. Hence, we propose a new market clearing model for the supply demand balance problem with a fewer number of decision variables and we provide a new interpretation of the surplus maximization problem. Since the supply demand balance problem is used with additional constraints in both the European exchange model of Nord Pool Spot and our proposed clearing model, we use the market clearing model in these formulations. We derive new valid strong cuts for the resulting problems.

Solving the market clearing problems to the optimality is important to find the reference prices. We propose solution methodologies for both our market clearing model which does not yield any PRBs and the market clearing model without PABs used in European power exchanges. We conduct computational experiments to assess the performance of our models using the data procured from the Turkish DAM operator.

Chapter 4

STYLIZED SUPPLY DEMAND BALANCE PROBLEM WITH HOURLY ORDERS

Electricity is generated when it is consumed because storing electricity to balance the entire grid is not cost effective (Divya and Østergaard, 2009). The aim of the day-ahead electricity markets is to balance supply and demand one day before the actual delivery. Therefore, the generic form of the market clearing problem is a supply demand balance problem, where only the fundamental features of the orders and periodic balance of supply and demand quantities are considered.

In this chapter, we start with the simplest market design where there are only hourly orders in an attempt to investigate the resulting mathematical model efficiently. Model (SDB-H), or Supply Demand Balance with Hourly orders, is the well-known formulation used in the literature where only hourly orders are considered (Martin et al., 2014; Madani and Van Vyve, 2015). Since the hourly orders can be partially accepted, the model (SDB-H) is a quadratic programming (QP) problem due to the quadratic objective function and linear constraints. By investigating the optimality conditions of the model (SDB-H), first we show the basic market equilibrium conditions for the hourly orders. This enables us to propose a new equivalent model with the same optimal solution with the model (SDB-H), which we denote by the model (E-SDB-H). The size of the model (E-SDB-H) is about the half of the size of the model (SDB-H). To represent the model (E-SDB-H) in a simpler and a compact form, we present a new model and refer to the model as the market clearing model with hourly orders, (MC-H). The size of the model (MC-H) is less than or equal to the size of the model (E-SDB-H). Hence, we will use the model (MC-H) as a base

model for the rest of this thesis.

4.1 Mathematical Model for Supply Demand Balance Problem with Hourly Orders (SDB-H)

In this section, we first explain the hourly orders and introduce the notation to represent these orders. Afterwards, we present the Supply Demand Balance Problem with Hourly orders and present a well known model, referred to as the model (SDB-H), which is also covered in Martin et al. (2014) and Madani and Van Vyve (2015). We establish new properties of the optimal solution of the model (SDB-H) in Theorems 1 and 2. We use these theorems to derive a new equivalent model (E-SDB-H), whose problem size is almost half of the problem size of the model (SDB-H).

4.1.1 Hourly Orders

Hourly orders can be offered in a single period. For each period, participants may bid different price levels for different quantities in both sales and purchase directions. These quantities can be partially accepted, where only some portion of the offered quantities is matched in the auction.

In each period, each participant may offer at most one hourly order. Table 4.1 illustrates the hourly orders of three participants for the first period, where the sales volumes are represented by negative values and purchase volumes are represented by positive values. There are four price levels for hourly order 1 and two price levels for hourly orders 2 and 3.

Table 4.1: Example hourly orders 1, 2, and 3 for the first period

(a) Hourly order 1					(b) Hourly order 2		
Price (TL/MWh)	0	300	700	2000	Price (TL/MWh)	0	2000
Quantity	100	10	-10	-200	Quantity	150	150

(c) Hourly order 3		
Price (TL/MWh)	0	2000
Quantity	-125	-125

Although the participants only submit a price level and the corresponding quantity in their hourly orders, they also agree to buy or sell the quantities between these price levels. Therefore, we form piecewise functions by employing a linear interpolation. We call these functions as *total quantity curves*, or briefly *total curves*, because they map the prices to both purchase and sales quantities. The total curve of hourly order h in period t is a function of price p , denoted by $TQ_{ht}(p)$. We plot the total curves of hourly orders 1, 2, and 3 in period 1 in Figure 4.1.

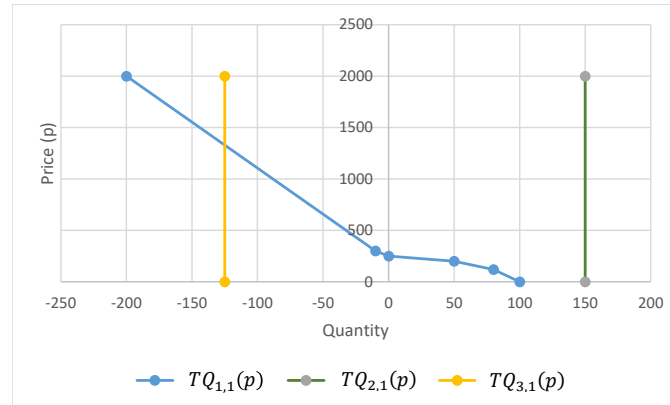


Figure 4.1: Total curves of the hourly orders 1, 2, and 3 from Table 4.1

We first separate the sales and purchase quantities of these hourly orders in Table 4.2 and we use positive quantities to represent them. The quantity that separates

the sales and purchase volumes is zero. If this quantity does not exist in the hourly order with both purchase and sales quantities, the corresponding price is found by interpolation. Zero quantity that is offered for a price level means there is no power produced or consumed for that price. Since hourly order 1 contains both sales and purchase quantities, we find the price that separates purchase and sales volume, which is 500 TL as seen on Figure 4.1. We add this price-quantity pair in both purchase and sales directions. Up to 500 TL, a participant with hourly order 1 bids purchase quantities and above 500 TL the participant offers sales quantities. There is no sales quantity for the hourly order 2 and hourly order 3 does not contain any purchase quantity.

Table 4.2: Hourly orders 1, 2, and 3 where purchase and sales quantities are separated

(a) Hourly order 1						(b) Hourly order 2		
Price (TL/MWh)	0	300	500	700	2000	Price (TL/MWh)	0	2000
Purchase Quantity	100	10	0	0	0	Purchase Quantity	150	150
Sales Quantity	0	0	0	10	200	Sales Quantity	0	0

(c) Hourly order 3		
Price (TL/MWh)	0	2000
Purchase Quantity	0	0
Sales Quantity	125	125

Finding the price level that corresponds to zero quantity for an hourly order is important to separate purchase and sales quantities properly. To illustrate, the participant with hourly order 1 would pay 300 TL to purchase 10 MWh and pay 700 TL for 0 MWh. If we separated the purchase and sales quantities without adding 500 TL - zero quantity pair, then the participant would have willingness to purchase 5 MWh for 500 TL, which is wrong. Therefore, when we separate the purchase and sales quantities of an order, it is important to add the price level that corresponds to zero quantity if the participant offers both purchase and sales quantities in his

hourly order. Since we calculate the price level that corresponds to zero quantity by interpolation, the price may be represented by a high precision decimals. This may cause some numerical problems in real life examples, because the computers work with a finite precision.

Hourly orders consist of supply and demand curves, which represent marginal cost and marginal benefit, respectively. Supply curves are the piecewise functions where sales quantities in hourly orders are used and denoted by $S_{ht}(p)$. Therefore, the function $S_{ht}(p)$ represents the sales quantity at price p offered with hourly order h in period t . Demand curves consist only of purchase quantities and denoted by $D_{ht}(p)$ which defines the purchase quantity at price p offered with hourly order h in period t . Since, supply and demand curves are obtained by decomposing $TQ_{ht}(p)$, there is a relation between demand, supply and total curves of each hourly order function h in period t given by $TQ_{ht}(p) = D_{ht}(p) - S_{ht}(p)$. We plot supply and demand curves of hourly orders 1, 2, and 3 in Figure 4.2.

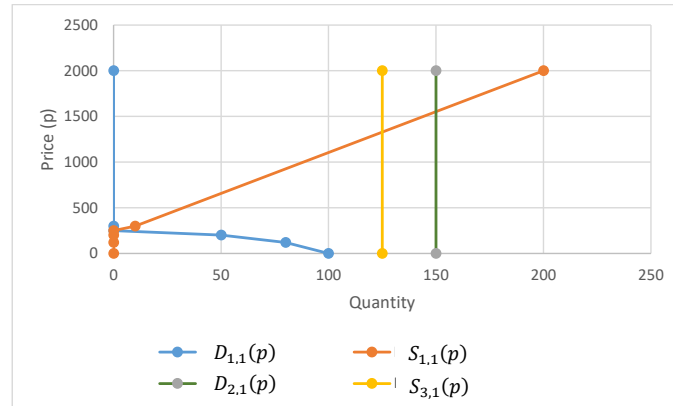


Figure 4.2: Supply and demand curves of the hourly orders 1, 2 and 3

Participants may have similar marginal costs and benefits. Therefore, they may offer the same price levels. To eliminate the repeated price levels, we use an *aggregation* procedure. Since we use supply and demand curves in our study, we will explain

the aggregation procedure for supply and demand curves in a detailed way. As a result of this procedure, we obtain an aggregated supply and an aggregated demand curve for each period. Although the aggregated supply and demand curves are used in Nord Pool Spot¹, there is no detailed explanation about how they apply the aggregation. Therefore, it is our contribution to the literature to propose an aggregation procedure to obtain the aggregated supply and demand curves.

Aggregation is the procedure consisting of summing the piecewise linear functions of each hourly order in a period. These functions are either supply curve, $S_{ht}(p)$, or demand curve, $D_{ht}(p)$, of the hourly order h in period t . Let H represent the set of hourly orders. Then, the aggregated demand curve in period t is $D_t(p) = \sum_{h \in H} D_{ht}(p)$ and the aggregated supply curve in period t is $S_t(p) = \sum_{h \in H} S_{ht}(p)$. This procedure decreases the problem size when the hourly orders contain the same price levels. In practice, each hourly order must contain the minimum and maximum price levels. Therefore, we expect the aggregation procedure decrease the problem size.

Price levels and the corresponding quantities form the breakpoints of functions $D_{ht}(p)$ and $S_{ht}(p)$. Since it is sufficient to represent a piecewise linear function with its breakpoints, we find the total quantities at each price level by Algorithm 1. As a result we obtain the aggregated supply and demand curves.

This procedure ensures that the aggregated supply and demand curves have the same price levels. Although it may increase the problem size, it will help us to represent the curves in a simple way.

When we apply Algorithm 1 for the data in Table 4.1, the unique price levels for these hourly orders are $PL_t = \{0, 300, 700, 2000\}$. Since the participant with hourly order 1 offers both purchase and sales quantities, we add 500 TL, which is the price that the participant offers zero quantity, to the set PL_t . The ordered PL_t becomes $\{0, 300, 500, 700, 2000\}$. For 0 TL, the participant with hourly order 1 offers 100 MWh purchase quantity and 0 MWh sales quantity, the participant with

¹<https://www.nordpoolgroup.com/trading/Day-ahead-trading/Price-calculation/>

Algorithm 1 Aggregation of hourly orders for a period

Initialization: $S_t = \emptyset$, $D_t = \emptyset$, $PL_t = \emptyset$ Find the unique price levels of the hourly orders for that period add them to PL_t .**for** each hourly order h in period t **do** **if** h contains both positive and negative quantities and does not have price level corresponding to zero quantity **then** Find the price level p that corresponds to zero quantity for h by using interpolation. Add p to PL_t if PL_t does not contain p . **end if****end for**

Sort price levels from smallest to largest.

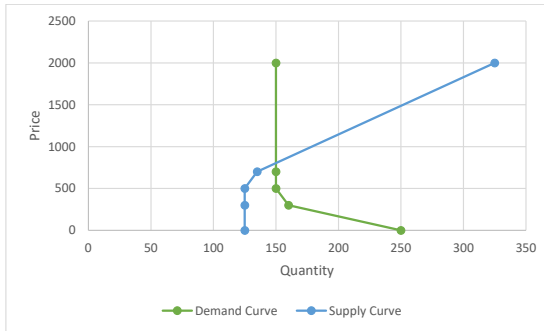
for each price level p in PL_t **do** Total sales quantity corresponding price p is $Q^S \leftarrow 0$ Total purchase quantity corresponding price p is $Q^D \leftarrow 0$ **for** each hourly order h **do** Calculate $q = TQ_{h,t}(p)$ **if** $q \leq 0$ **then** $Q^S \leftarrow Q^S - q$ **end if** **if** $q \geq 0$ **then** $Q^D \leftarrow Q^D + q$ **end if** **end for** Add (p, Q^S) to S_t and add (p, Q^D) to D_t **end for***Output:* S_t and D_t .

hourly order 2 offers 150 MWh purchase quantity and 0 MWh sales quantity and the participant with hourly order 3 offers 0 MWh purchase quantity and 125 MWh sales quantity. Then, aggregated purchase quantity at 0 TL is 250 MWh ($100 + 150 + 0$) and aggregated sales quantity is 125 MWh ($0 + 0 + 125$). We continue this procedure for price levels 300, 500, 700, and 2000. Table 4.3 shows aggregated purchase, sales and total quantities given by subtracting the aggregated sales quantity from the aggregated purchase quantity.

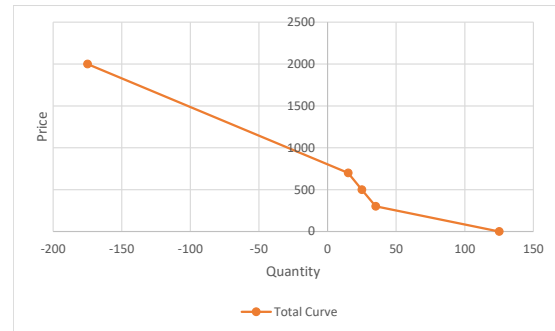
Table 4.3: Aggregated quantities for the first period

Price (TL)	0	300	500	700	2000
Aggregated Purchase Quantity	250	160	150	150	150
Aggregated Sales Quantity	125	125	125	135	325
Aggregated Total Quantity	125	35	25	15	-175

By drawing lines between the consecutive breakpoints of the aggregated purchase and sales quantities, we obtain an aggregated demand curve and an aggregated supply curve for a period.



(a) Supply and demand curves



(b) Total curve

Figure 4.3: Aggregated supply, demand and total curves in period 1, by considering the example hourly orders 1, 2, and 3

Figure 4.3 shows (a) the aggregated supply and demand curves and (b) the aggregated total curve obtained from Table 4.3. There are five breakpoints in these curves.

4.1.2 Notation to Represent Hourly Orders

Aggregated supply and demand curves have all the necessary information. Hence, instead of using each hourly order separately, we use these aggregated curves in our study. We denote by T the set that contains all possible time periods for t . The aggregated supply and demand curves have the same number of breakpoints in a period t , which is denoted by K_t for all $t \in T$, by construction. Price levels are represented by $P_{k,t}^D$ and $P_{k,t}^S$ in the demand and supply curves, respectively and the corresponding aggregated purchase and sales quantities are denoted by $Q_{k,t}^D$ and $Q_{k,t}^S$, respectively where $k \in \{1, 2, \dots, K_t\}$. Let $(P_{k,t}^D, Q_{k,t}^D)$, and $(P_{k,t}^S, Q_{k,t}^S)$ be the breakpoints for demand and supply curves, respectively.

We start enumerating the breakpoints of the demand curve from the highest price: The first breakpoint of the demand curve has the highest price level, i.e., $P_{1,t}^D$ is the highest price in period t and $P_{K_t,t}^D$ is the lowest price in period t , for all $t \in T$. However, the breakpoints of the supply curve start from the lowest price level, i.e. $P_{1,t}^S$ is the lowest price in period t and $P_{K_t,t}^S$ is the highest price in period t , for all $t \in T$.

In our example in Table 4.3, there are 5 breakpoints in period 1. Therefore, $K_1 = 5$. In Table 4.4, we present the breakpoints of the aggregated supply and demand curves with the price and quantity values, i.e., $(P_{k,t}^D, Q_{k,t}^D)$ and $(P_{k,t}^S, Q_{k,t}^S)$ for the demand and supply curves, respectively.

Table 4.4: Breakpoints of the aggregated curves for the first period

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$(P_{k,1}^D, Q_{k,1}^D)$	(2000,150)	(700,150)	(500,150)	(300,160)	(0,250)
$(P_{k,1}^S, Q_{k,1}^S)$	(0,125)	(300, 125)	(500, 125)	(700,135)	(2000,325)

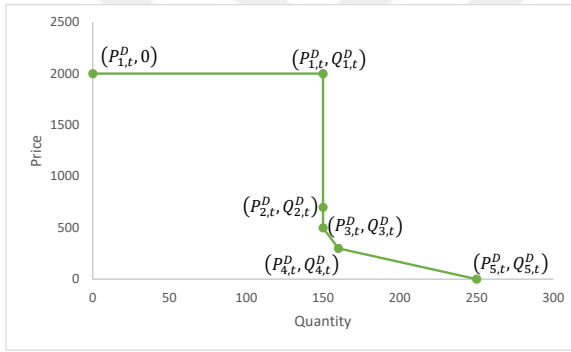
Due to price independent orders (see Definition 4 in Chapter 2) in the supply direction, there may be some non-zero quantity in the aggregated supply curve at minimum price or some non-zero quantity in the aggregated demand curve at the maximum price due to price independent orders in the demand direction. These quantities are curtailable, i.e., the participants are willing to purchase or sell a portion of the quantities they offer at the maximum or the minimum price, respectively. For the example given in Table 4.4, the purchase quantity 150 MWh can be curtailed when the price is 2000 TL and the sales quantity 125 MWh is the curtailable quantity when the price is 0 TL.

We add an artificial breakpoint with zero quantity-minimum price, $P_{1,t}^S$, to the supply curve and zero quantity-maximum price, $P_{1,t}^D$, to the demand curve as in Yörükoğlu (2015) to curtail hourly orders in imbalance. Then, the artificial breakpoint that will be added to the demand curve is $(P_{1,t}^D, 0)$ whereas the artificial breakpoint that will be added to the supply curve is $(P_{1,t}^S, 0)$ for all $t \in T$.

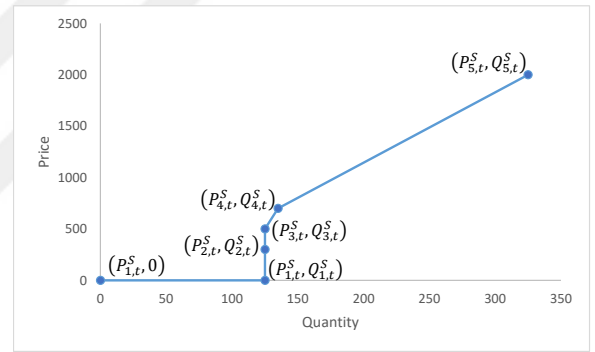
After adding the artificial breakpoints, there are $K_t + 1$ breakpoints in the demand or supply curves. The breakpoints of the aggregated supply and demand curves after adding the artificial breakpoints are presented in Table 4.5 for our sample aggregated curves in Table 4.3. Moreover, we present these breakpoints on the aggregated demand and supply curves as shown in Figure 4.4.

Table 4.5: Breakpoints of the aggregated curves for the first period after adding the artificial breakpoints to handle curtailment

	<i>Artificial</i>	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$(P_{k,1}^D, Q_{k,1}^D)$	(2000,0)	(2000,150)	(700,150)	(500,150)	(300,160)	(0,250)
$(P_{k,1}^S, Q_{k,1}^S)$	(0,0)	(0,125)	(300, 125)	(500, 125)	(700,135)	(2000,325)



(a) Demand



(b) Supply

Figure 4.4: Breakpoints of the aggregated supply and demand curves after adding the artificial breakpoints for curtailment

Each piece formed by two consecutive breakpoints k and $k + 1$ in the piecewise linear supply or demand curves in period t is defined by the *segment* formed by breakpoint k , or concisely *segment k* . There are K_t segments in each of the supply and demand curves in period t after adding the artificial breakpoints. The quantity difference between any two consecutive breakpoints are allowed to be accepted partially. We represent these quantity differences by $\Delta Q_{k,t}^D$ and $\Delta Q_{k,t}^S$ for demand and supply curves, respectively. We define $\Delta Q_{k,t}^D = Q_{(k+1),t}^D - Q_{k,t}^D$ and $\Delta Q_{k,t}^S = Q_{(k+1),t}^S - Q_{k,t}^S$ such that $\Delta Q_{k,t}^D \geq 0$ and $\Delta Q_{k,t}^S \geq 0$ for all $k \in \{1, \dots, K_t - 1\}$. To represent these indices in a simple way, we define $I_t = \{1, \dots, K_t - 1\}$. We also represent the price difference of

two consecutive breakpoints in the demand and supply direction by $\Delta P_{k,t}^D$ and $\Delta P_{k,t}^S$, respectively. To have non-negative price differences we define: $\Delta P_{k,t}^D = P_{k,t}^D - P_{(k+1),t}^D$ and $\Delta P_{k,t}^S = P_{(k+1),t}^S - P_{k,t}^S$ for all $k \in I_t$, $t \in T$.

We call the special segment that is formed after adding the artificial breakpoint $(P_{1,t}^D, 0)$ *curtailable demand* segment. Similarly, we call the special segment between the breakpoints $(P_{1,t}^S, 0)$ and $(P_{1,t}^S, Q_{1,t}^S)$ as *curtailable supply* segment. We present the segments for our sample aggregated curves in Figure 4.5.

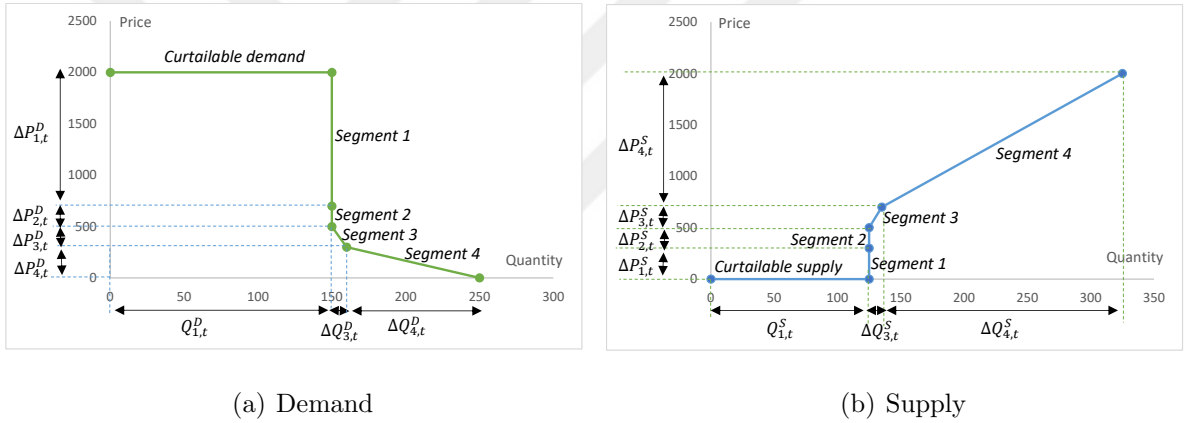


Figure 4.5: Segments of the aggregated supply and demand curves after adding the artificial breakpoints for curtailment

Let $\delta_{k,t}$ and $\sigma_{k,t}$ be the accepted fraction of a segment k of demand and supply curves in period t , respectively, where $k \in I_t$, $t \in T$. We can accept ($\delta_{k,t} = 1$ or $\sigma_{k,t} = 1$), partially accept ($0 < \delta_{k,t} < 1$, or $0 < \sigma_{k,t} < 1$), or reject ($\delta_{k,t} = 0$ or $\sigma_{k,t} = 0$) a segment. Hence $0 \leq \delta_{k,t} \leq 1$ and $0 \leq \sigma_{k,t} \leq 1$ for all $k \in I_t$. Similarly, δ_t^c and σ_t^c represent the accepted fraction of curtailable demand and supply segments, where $0 \leq \delta_t^c \leq 1$ and $0 \leq \sigma_t^c \leq 1$ for all $t \in T$.

This problem has a set of continuous variables. Let δ_t be a row vector which contains values of δ_t^c and $\delta_{k,t}$ in period t , i.e, $\delta_t = [\delta_t^c \ \delta_{1,t} \dots \delta_{K_t,t}]$. We denote by vector δ which contains δ_t for all $t \in T$ such that $\delta = [\delta_1 \ \delta_2 \dots \delta_{|T|}]^T$ where $[\cdot]^T$ defines the

transpose operator. Moreover, let σ_t be a row vector which contains the values of σ_t^c and $\sigma_{k,t}$ in period t , i.e., $\sigma_t = [\sigma_t^c \ \sigma_{1,t} \dots \sigma_{K_t,t}]$ for all $t \in T$ and $\sigma = [\sigma_1 \ \sigma_2 \dots \sigma_{|T|}]^T$. We present the decision variables in vector format, i.e., δ, σ .

Before introducing the model, we summarize the sets, indices and decision variables related to aggregated supply and demand curves as follows:

Sets Related with aggregated supply and demand curves

t, T	Time period index and set of time periods.
K_t	The number of breakpoints in the aggregated supply or demand curves in period t , $t \in T$ before adding the artificial breakpoints.
I_t	Set of segments that are formed between the two consecutive breakpoints $k + 1$ and k , i.e., $I_t = \{1, \dots, K_t - 1\}$.

Parameters Related with aggregated supply and demand curves

$P_{k,t}^D, Q_{k,t}^D$	Price and purchase quantity of breakpoint k of the demand curve in period t , where $k \in \{1, 2, \dots, K_t\}$, $t \in T$.
$P_{k,t}^S, Q_{k,t}^S$	Price and sales quantity of breakpoint k of the supply curve in period t , where $k \in \{1, 2, \dots, K_t\}$, $t \in T$.
$\Delta P_{k,t}^D, \Delta Q_{k,t}^D$	The price and quantity difference of segment k in demand curve in period t for all $k \in I_t$, $t \in T$ where $\Delta Q_{k,t}^D = Q_{(k+1),t}^D - Q_{k,t}^D \geq 0$, and $\Delta P_{k,t}^D = P_{k,t}^D - P_{(k+1),t}^D \geq 0$.
$\Delta P_{k,t}^S, \Delta Q_{k,t}^S$	The price and quantity difference of segment k in supply curve in period t for all $k \in I_t$, $t \in T$ where $\Delta Q_{k,t}^S = Q_{(k+1),t}^S - Q_{k,t}^S \geq 0$, and $\Delta P_{k,t}^S = P_{(k+1),t}^S - P_{k,t}^S \geq 0$.
$P_{1,t}^D, P_{1,t}^S$	Maximum and minimum price allowed in period t , where $P_{1,t}^D = P_{K_t,t}^S$ and $P_{K_t,t}^D = P_{1,t}^S$.
$Q_{1,t}^D, Q_{1,t}^S$	Curtailable quantity in the maximum and minimum price levels in period t .

Decision Variables	Related with aggregated supply and demand curves
$\delta_{k,t}$	Accepted fraction of segment k of the demand curve in period t , where $k \in I_t, t \in T$.
$\sigma_{k,t}$	Accepted fraction of segment k of the supply curve in period t , where $k \in I_t, t \in T$.
δ_t^c, σ_t^c	Accepted fraction of the curtailable demand and supply segments in period t , where $k \in I_t, t \in T$.
(δ, σ)	The solution vector that contains all decision variables $\delta_{k,t}, \sigma_{k,t}, \delta_t^c$, and σ_t^c for all $k \in I_t, t \in T$.

4.1.3 Mathematical Model, (SDB-H)

In the literature, most of the studies start with supply demand balance problem, which maximizes the total surplus while considering supply demand balance and fundamental features of the orders (Martin et al., 2014; Madani and Van Vyve, 2015). To efficiently investigate the problem, we focus on the stylized supply demand balance problem, where we only consider the aggregated supply and demand curves. The objective of the problem is to maximize the total surplus, which is denoted by the function $g(\delta, \sigma)$. We present the regular mathematical model which we refer to as the Supply Demand Balance model with only Hourly orders (SDB-H) below.

(SDB-H)

$$\max g(\delta, \sigma)$$

s.t.

$$\sum_{k \in I_t} (\Delta Q_{k,t}^D \delta_{k,t} - \Delta Q_{k,t}^S \sigma_{k,t}) + Q_{1,t}^D \delta_t^c - Q_{1,t}^S \sigma_t^c = 0, \quad \forall t \in T, \quad (4.1)$$

$$0 \leq \delta_{k,t} \leq 1, \quad \forall k \in I_t, t \in T, \quad (4.2)$$

$$0 \leq \sigma_{k,t} \leq 1, \quad \forall k \in I_t, t \in T, \quad (4.3)$$

$$0 \leq \delta_t^c \leq 1, \quad \forall t \in T, \quad (4.4)$$

$$0 \leq \sigma_t^c \leq 1, \quad \forall t \in T, \quad (4.5)$$

where g is defined as in (4.9). We will cover the objective function in the next section in a detailed way. The total demand equals the total supply for all periods by (4.1). The second and the third inequalities set upper and lower limits on the accepted fraction of the demand and supply segment quantities, respectively. Moreover, the last two inequalities ensure the limits on the accepted fraction of the curtailable demand and supply segments. The following table shows the number of constraints and nonnegative continuous variables of the model (SDB-H).

Size of the the model (SDB-H)	
Number of constraints	$ T + \sum_{t \in T} 2K_t.$
Number of nonnegative continuous variables	$\sum_{t \in T} 2K_t$

Now we derive the mathematical formulation for the objective function $g(\delta, \sigma)$.

4.1.3.1 Objective Function

Total surplus is used as an objective function in almost all electricity market problems including DAMs due to the advantages described in María (2010). Total surplus is the sum of producer and consumer surplus (see Mas-Colell et al. (1995)).

The *producer surplus* is the difference between the total revenue earned by a producer in return for producing a particular amount of electricity and the total cost incurred by generating this amount of electricity. Similarly, the *consumer surplus* is the difference of total willingness-to-pay of a consumer to consume a particular amount of electricity and the total cost of purchasing this amount of electricity. Since it is assumed that the bidders submit their marginal cost or benefits through their orders in DAM auction (Van Vyve et al., 2011), we can calculate the total surplus if we know the matching price and quantity of each order type.

Hourly orders contain segments. If we know the surplus arising from each segment, we can find the surplus arising from the hourly orders by summing over the segments. Therefore, we first define the surplus arising from a demand and a supply segment.

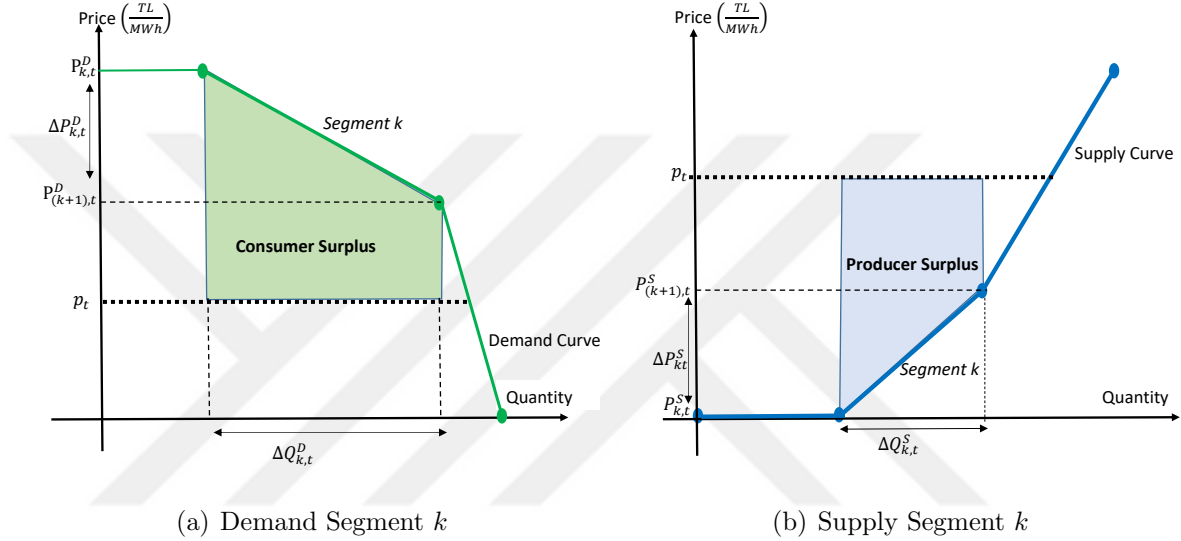


Figure 4.6: Producer and consumer surplus of a supply and demand segment k in period t

Hourly orders that are valid in period t are matched at a price denoted by p_t . In Figure 4.6 we show producer and consumer surplus arising from a demand segment and a supply segment as an example.

Now we derive the expression for the consumer surplus arising from a demand segment k . If a demand segment k is not accepted, i.e., $\delta_{k,t} = 0$, then the surplus arising from a segment is zero. If a demand segment k is fully accepted, i.e., $\delta_{k,t} = 1$, then the surplus of the segment is $(P_{k,t}^D - p_t)\Delta Q_{k,t}^D - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D}{2}$ as shown in Figure 4.6 (a). If a demand segment is partially accepted, i.e., $0 < \delta_{k,t} = \alpha < 1$ then the matching quantity arising from the segment is $\Delta Q_{k,t}^D \alpha$. Moreover, the surplus arising from the triangle area is $\frac{\Delta Q_{k,t}^D \alpha \times \Delta P_{k,t}^D \alpha}{2}$. Then, the producer surplus of a partially accepted segment k is, $(P_{k,t}^D - p_t)\Delta Q_{k,t}^D \alpha - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \alpha^2}{2}$. We present the consumer

surplus arising from a demand segment k below:

$$(P_{k,t}^D - p_t)\Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2}, \quad \forall k \in I_t, t \in T. \quad (4.6)$$

We can represent the producer surplus arising from a supply segment k similarly: The surplus arising from a supply segment is zero if the segment is not accepted. If a supply segment k is fully accepted, i.e., $\sigma_{k,t} = 1$, then the surplus of the segment is $(p_t - P_{k,t}^S)\Delta Q_{k,t}^S - \frac{\Delta P_{k,t}^S \Delta Q_{k,t}^S}{2}$ as shown in Figure 4.6 (b). If a supply segment is partially accepted, i.e., $\sigma_{k,t} = \alpha$, then the matching quantity arising from the segment is $\Delta Q_{k,t}^S \alpha$. Moreover, the surplus arising from the triangle area becomes $\Delta Q_{k,t}^S \alpha \times \Delta P_{k,t}^S$. Then, the producer surplus of an partially accepted segment k is, $(p_t - P_{k,t}^S)\Delta Q_{k,t}^S \alpha - \frac{\Delta P_{k,t}^S \Delta Q_{k,t}^S \alpha^2}{2}$. We present the producer surplus arising from a supply segment k below:

$$(p_t - P_{k,t}^S)\Delta Q_{k,t}^S \sigma_{k,t} - \frac{\Delta P_{k,t}^S \Delta Q_{k,t}^S \sigma_{k,t}^2}{2}, \quad \forall k \in I_t, t \in T. \quad (4.7)$$

The total surplus arising from aggregated hourly orders is the sum of the consumer and the producer surpluses arising from aggregated hourly orders for all periods, which is given by:

$$\begin{aligned} g(\delta, \sigma) = & \sum_{t \in T} (P_{1,k}^D - p_t) Q_{1,t}^D \delta_t^c + \sum_{t \in T} \sum_{k \in I_t} \left((P_{k,t}^D - p_t) \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) \\ & + \sum_{t \in T} (p_t - P_{1,k}^S) Q_{1,t}^S \sigma_t^c + \sum_{t \in T} \sum_{k \in I_t} \left((p_t - P_{k,t}^S) \Delta Q_{k,t}^S \sigma_{k,t} - \frac{\Delta P_{k,t}^S \Delta Q_{k,t}^S \sigma_{k,t}^2}{2} \right) \end{aligned} \quad (4.8)$$

We illustrate the producer and consumer surplus for a period in Figure 4.7

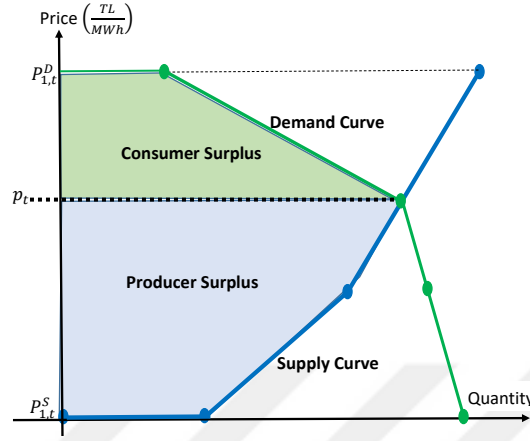


Figure 4.7: Consumer and producer surplus arising from aggregated hourly curves

The objective function has quadratic terms with $p_t \sigma_{k,t}$, $p_t \delta_{k,t}$, $p_t \sigma_t^c$, $p_t \delta_t^c$, $\sigma_{k,t}^2$, and $\delta_{k,t}^2$. On the other hand, there is a commonly used simplification of the objective function, as in Madani and Van Vyve (2015), where terms $p_t \sigma_{k,t}$, $p_t \delta_{k,t}$, $p_t \sigma_t^c$, and $p_t \delta_t^c$ are eliminated as follows:

We write the total surplus and we combine the terms with p_t , i.e.,

$$\begin{aligned}
 g(\delta, \sigma) &= \sum_{t \in T} (P_{1,t}^D - p_t) Q_{1,t}^D \delta_t^c + \sum_{t \in T} \sum_{k \in I_t} \left((p_t - P_{k,t}^S) \Delta Q_{k,t}^S \sigma_{k,t} - \frac{\Delta P_{k,t}^S \Delta Q_{k,t}^S \sigma_{k,t}^2}{2} \right) \\
 &\quad + \sum_{t \in T} (p_t - P_{1,t}^S) Q_{1,t}^S \sigma_t^c + \sum_{t \in T} \sum_{k \in I_t} \left((P_{k,t}^D - p_t) \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) \\
 g(\delta, \sigma) &= \sum_{t \in T} \sum_{k \in I_t} \left(P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} - P_{k,t}^S \Delta Q_{k,t}^S \sigma_{k,t} - \frac{\Delta P_{k,t}^S \Delta Q_{k,t}^S \sigma_{k,t}^2}{2} \right) \\
 &\quad + \sum_{t \in T} (P_{1,t}^D Q_{1,t}^D \delta_t^c - P_{1,t}^S Q_{1,t}^S \sigma_t^c) \\
 &\quad - \sum_{t \in T} p_t \left(\sum_{k \in I_t} (\Delta Q_{k,t}^D \delta_{k,t} - \Delta Q_{k,t}^S \sigma_{k,t}) + Q_{1,t}^D \delta_t^c - Q_{1,t}^S \sigma_t^c \right),
 \end{aligned}$$

where the last term is zero due to (4.1). Then, total surplus is a quadratic function,

where the quadratic term comes from interpolation, i.e. $\sigma_{k,t}^2$ and $\delta_{k,t}^2$ and is given by;

$$g(\delta, \sigma) = \sum_{t \in T} \sum_{k \in I_t} \left(\Delta Q_{k,t}^D P_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} - \Delta Q_{k,t}^S P_{k,t}^S \sigma_{k,t} - \frac{\Delta P_{k,t}^S \Delta Q_{k,t}^S \sigma_{k,t}^2}{2} \right) + \sum_{t \in T} (P_{1,t}^D Q_{1,t}^D \delta_t^c - P_{1,t}^S Q_{1,t}^S \sigma_t^c), \quad (4.9)$$

which is the well-known concave surplus function used in the literature. Since the aggregated curves are represented by piecewise linear functions, g is a quadratic function. It follows that the model (SDB-H) is a QP problem.

4.1.4 Mathematical Properties of the model (SDB-H)

In this section, we first present a condition in Lemma 1 that an optimal solution of the model (SDB-H) must satisfy, which is proved by Madani and Van Vyve (2015). We use this lemma and establish a new property of the optimal solution of the model (SDB-H) in Theorem 1. By using this property, we present a simplification of the objective function and the supply demand balance constraints in Theorem 2. Then, we show that the simplified objective function and the supply demand balance constraints depend only on the aggregated total curves and construct a new model, (E-SDB-H), by using the aggregated total curves.

Mathematical properties of an optimal solution of the model (SDB-H) can be found by the primal-dual relations. In the literature, Madani and Van Vyve (2015) derived the Dorn's dual of this problem when there are only block and hourly orders and showed the following result about hourly orders.

Lemma 1. *Let the shadow prices of the constraints (4.1) be p_t for all $t \in T$. In any optimal solution of the model (SDB-H), the optimal values of primal variables $\delta_{k,t}^*$ and $\sigma_{k,t}^*$ for all $k \in I_t$, $t \in T$, where $\Delta Q_{k,t}^D \neq 0$, $\Delta Q_{k,t}^S \neq 0$, and the optimal values of dual variables p_t^* satisfy the following relations:*

$$\begin{aligned}
\delta_{k,t}^* = 1 & \Leftrightarrow p_t^* \leq P_{(k+1),t}^D, & k \in I_t, t \in T. \\
\delta_{k,t}^* = 0 & \Leftrightarrow P_{k,t}^D \leq p_t^*, & k \in I_t, t \in T. \\
0 < \delta_{k,t}^* = \alpha < 1 & \Leftrightarrow p_t^* = \alpha P_{(k+1),t}^D + (1 - \alpha) P_{k,t}^D, & k \in I_t, t \in T. \\
\sigma_{k,t}^* = 1 & \Leftrightarrow P_{(k+1),t}^S \leq p_t^*, & k \in I_t, t \in T. \\
\sigma_{k,t}^* = 0 & \Leftrightarrow p_t^* \leq P_{k,t}^S, & k \in I_t, t \in T. \\
0 < \sigma_{k,t}^* = \alpha < 1 & \Leftrightarrow p_t^* = \alpha P_{(k+1),t}^S + (1 - \alpha) P_{k,t}^S, & k \in I_t, t \in T.
\end{aligned}$$

The detailed proof of this lemma can be found in Madani and Van Vyve (2015). We only re-arrange and simplify the results in Lemma 1. Note that, if we did not aggregate the hourly orders, there would be a relation between each hourly order segment and p_t .

Lemma 1 creates a relation between the optimal values of the solution (δ, σ) and the dual price p_t . We will show that we can establish a relation between δ and σ when the supply and demand curves have the same price levels as in the construction of the aggregated curves. For this purpose, we define the so called *paired* segments which is depicted in Figure 4.8

Definition 8. *Segment k of the demand curve and segment $\hat{k}$ of the supply curve are called paired if they have the same price levels, i.e.,*

$$\begin{aligned}
P_{k,t}^D &= P_{(\hat{k}+1),t}^S, & \hat{k} &= 1 + K_t - k \text{ for all } k \in I_t, t \in T, \\
P_{(k+1),t}^D &= P_{\hat{k},t}^S, & \hat{k} &= 1 + K_t - k \text{ for all } k \in I_t, t \in T.
\end{aligned} \tag{4.10}$$

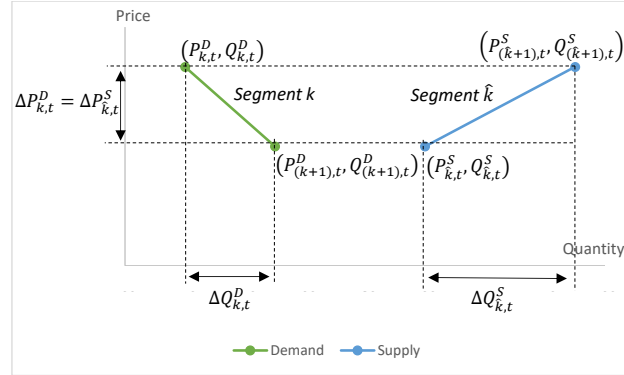


Figure 4.8: Paired segments

Since our aggregation procedure ensures that aggregated supply and demand curves have the same price levels, each segment k of the demand curve is paired with the segment $\hat{k}$ of the supply curve. Then, an optimal solution of the model (SDB-H) satisfies the relation stated in Theorem 1.

Theorem 1. *Let demand segment k and supply segment $\hat{k}$ be paired for all $k \in I_t$, $t \in T$, where $\hat{k} = 1 + K_t - k$. Assume that the feasible region is nonempty. Then, there exists an optimal solution of the model (SDB-H) which satisfies $\delta_{k,t}^* + \sigma_{\hat{k},t}^* = 1$ for all $k \in I_t, t \in T$.*

Proof. To prove this theorem, we use the relation between p_t^* , $\sigma_{k,t}^*$ and $\delta_{k,t}^*$ given by Lemma 1 in any optimal solution for the model (SDB-H).

• **Case 1:** $\delta_{k,t}^* = 1$

$\delta_{k,t}^* = 1$, if and only if $p_t^* \leq P_{(k+1),t}^D$. Since k and $\hat{k}$ are paired, $P_{(k+1),t}^D = P_{\hat{k},t}^S$. Then,

$$\begin{aligned} \delta_{k,t}^* = 1 &\Leftrightarrow p_t^* \leq P_{(k+1),t}^D \\ &\Leftrightarrow p_t^* \leq P_{\hat{k},t}^S \\ &\Leftrightarrow \sigma_{\hat{k},t}^* = 0. \end{aligned}$$

- **Case 2:** $\delta_{k,t}^* = 0$

Since k and $\hat{k}$ are paired, $P_{k,t}^D = P_{(\hat{k}+1),t}^S$. Then,

$$\begin{aligned}\delta_{k,t}^* = 0 &\Leftrightarrow P_{k,t}^D \leq p_t^* \\ &\Leftrightarrow P_{(\hat{k}+1),t}^S \leq p_t^* \\ &\Leftrightarrow \sigma_{\hat{k},t}^* = 1.\end{aligned}$$

- **Case 3:** $0 < \delta_{k,t}^* < 1$

Since k and $\hat{k}$ are paired, $P_{(k+1),t}^D = P_{\hat{k},t}^S$ and $P_{k,t}^D = P_{(\hat{k}+1),t}^S$. Then,

$$\begin{aligned}0 < \delta_{k,t}^* < 1 &\Leftrightarrow p_t^* = \delta_{k,t}^* P_{(k+1),t}^D + (1 - \delta_{k,t}^*) P_{k,t}^D \\ &\Leftrightarrow p_t^* = \delta_{k,t}^* P_{\hat{k},t}^S + (1 - \delta_{k,t}^*) P_{(\hat{k}+1),t}^S \\ &\Leftrightarrow \sigma_{\hat{k},t}^* = 1 - \delta_{k,t}^*\end{aligned}$$

Then, $\delta_{k,t}^* + \sigma_{\hat{k},t}^* = 1$ for all $k \in I_t, t \in T$.

Note that these results hold for an optimal solution of the model (SDB-H) when $\Delta Q_{k,t}^S \neq 0$ and $\Delta Q_{k,t}^D \neq 0$ due to the assumptions in Lemma 1. For a segment l whose quantity difference is zero, i.e., $\Delta Q_{l,t}^D = 0$, $\delta_{l,t}$ can take any value between 0 and 1 because it no longer has an impact on the problem (in both the objective function and the balance constraint $\Delta Q_{l,t}^D \delta_{l,t}$ term exists, which is canceled out in case of $\Delta Q_{l,t}^D = 0$). Therefore, $\delta_{l,t} = 1 - \sigma_{(K_t+1-l),t}$ is still valid for a segment l where $\Delta Q_{l,t}^D = 0$. A similar argument holds if $\Delta Q_{l,t}^S = 0$. $\square$

By Theorem 1, we can substitute $\sigma_{\hat{k},t} = 1 - \delta_{k,t}, \forall \hat{k} \in I_t$ into the objective function and the first constraint to obtain a simpler formulation. Substitution of $\sigma_{\hat{k},t} = 1 - \delta_{k,t}$ in $g(\delta, \sigma)$ is as follows:

$$\begin{aligned}g(\delta, \sigma) &= \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) + P_{1,t}^D Q_{1,t}^D \delta_t^c \right. \\ &\quad \left. - \sum_{\hat{k} \in I_t} \left(P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S \sigma_{\hat{k},t} + \frac{\Delta P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S \sigma_{\hat{k},t}^2}{2} \right) - P_{1,t}^S Q_{1,t}^S \sigma_t^c \right)\end{aligned}$$

$$\begin{aligned}
g(\delta, \sigma) &= \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) + P_{1,t}^D Q_{1,t}^D \delta_t^c \right. \\
&\quad \left. - \sum_{k=K_t+1-\hat{k}, \hat{k} \in I_t} \left(P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S (1 - \delta_{k,t}) + \frac{\Delta P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S (1 - \delta_{k,t})^2}{2} \right) - P_{1,t}^S Q_{1,t}^S \sigma_t^c \right) \\
&= \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) + P_{1,t}^D Q_{1,t}^D \delta_t^c \right. \\
&\quad \left. - \sum_{k=K_t+1-\hat{k}, \hat{k} \in I_t} \left(P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S (1 - \delta_{k,t}) + \frac{\Delta P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S (1 - 2\delta_{k,t} + \delta_{k,t}^2)}{2} \right) \right. \\
&\quad \left. - P_{1,t}^S Q_{1,t}^S \sigma_t^c \right) \\
&= \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) + P_{1,t}^D Q_{1,t}^D \delta_t^c \right. \\
&\quad \left. + \sum_{k=K_t+1-\hat{k}, \hat{k} \in I_t} \left((P_{\hat{k},t}^S + \Delta P_{\hat{k},t}^S) \Delta Q_{\hat{k},t}^S \delta_{k,t} - \frac{\Delta P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S \delta_{k,t}^2}{2} \right) - P_{1,t}^S Q_{1,t}^S \sigma_t^c \right. \\
&\quad \left. - \sum_{i \in I_t} \left(P_{i,t}^S \Delta Q_{i,t}^S + \frac{\Delta P_{i,t}^S \Delta Q_{i,t}^S}{2} \right) \right) \\
&= \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) + P_{1,t}^D Q_{1,t}^D \delta_t^c \right. \\
&\quad \left. + \sum_{k=K_t+1-\hat{k}, \hat{k} \in I_t} \left(P_{(\hat{k}+1),t}^S \Delta Q_{\hat{k},t}^S \delta_{k,t} - \frac{\Delta P_{\hat{k},t}^S \Delta Q_{\hat{k},t}^S \delta_{k,t}^2}{2} \right) - P_{1,t}^S Q_{1,t}^S \sigma_t^c \right. \\
&\quad \left. - \sum_{i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \right) \\
&= \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{k,t}^D \delta_{k,t}^2}{2} \right) + P_{1,t}^D Q_{1,t}^D \delta_t^c \right. \\
&\quad \left. + \sum_{k=K_t+1-\hat{k}, \hat{k} \in I_t} \left(P_{k,t}^D \Delta Q_{\hat{k},t}^S \delta_{k,t} - \frac{\Delta P_{k,t}^D \Delta Q_{\hat{k},t}^S \delta_{k,t}^2}{2} \right) - P_{1,t}^S Q_{1,t}^S \sigma_t^c \right. \\
&\quad \left. - \sum_{i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \right)
\end{aligned}$$

$$\begin{aligned}
g(\delta, \sigma) = & \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D (\Delta Q_{k,t}^D + \Delta Q_{k,t}^S) \delta_{k,t} - \frac{\Delta P_{k,t}^D (\Delta Q_{k,t}^D + \Delta Q_{k,t}^S) \delta_{k,t}^2}{2} \right) + P_{1t}^D Q_{1,t}^D \delta_t^c \right. \\
& \left. - P_{1,t}^S Q_{1,t}^S \sigma_t^c - \sum_{i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \right)
\end{aligned} \tag{4.11}$$

The following steps are followed in the derivation of (4.11): First, we substitute $\sigma_{\hat{k},t} = 1 - \delta_{k,t}$, where k and $\hat{k}$ are paired, i.e., $k = K_t + 1 - \hat{k}$. We use distributive property of multiplication and re-arrange terms until the fourth equation. We use $\Delta P_{\hat{k},t}^S = P_{(\hat{k}+1),t}^D - P_{\hat{k},t}^S$ and obtain the fifth equation. In the sixth equation, we use $P_{(\hat{k}+1),t}^S = P_{k,t}^D$, $\Delta P_{\hat{k},t}^S = \Delta P_{k,t}^S$ and re-arrange the terms in the seventh equation. Substitution of $\sigma_{k,t} = 1 - \delta_{\hat{k},t}$ in the supply demand balance constraint is as follows:

$$\begin{aligned}
\sum_{k \in I_t} (\Delta Q_{k,t}^D \delta_{k,t} - \Delta Q_{k,t}^S \sigma_{k,t}) + Q_{1,t}^D \delta_t^c - Q_{1,t}^S \sigma_t^c &= 0 \quad \forall t \in T. \\
\sum_{k \in I_t} \Delta Q_{k,t}^D \delta_{k,t} - \sum_{k=K_t+1-\hat{k}, \hat{k} \in I_t} \Delta Q_{\hat{k},t}^S (1 - \delta_{k,t}) + Q_{1,t}^D \delta_t^c - Q_{1,t}^S \sigma_t^c &= 0 \quad \forall t \in T. \\
\sum_{k \in I_t, \hat{k}=K_t+1-k} (\Delta Q_{k,t}^D + \Delta Q_{\hat{k},t}^S) \delta_{k,t} + Q_{1,t}^D \delta_t^c - Q_{1,t}^S \sigma_t^c &= \sum_{i \in I_t} \Delta Q_{i,t}^S \quad \forall t \in T.
\end{aligned}$$

We present the resulting simplifications in Theorem 2.

Theorem 2. When segment k and segment $\hat{k}$ are paired as presented in (4.10), the objective function can be written as

$$\begin{aligned}
g(\delta, \sigma) = & \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D (\Delta Q_{k,t}^D + \Delta Q_{k,t}^S) \delta_{k,t} - \frac{\Delta P_{k,t}^D (\Delta Q_{k,t}^D + \Delta Q_{k,t}^S) \delta_{k,t}^2}{2} \right) + P_{1t}^D Q_{1,t}^D \delta_t^c \right. \\
& \left. - P_{1,t}^S Q_{1,t}^S \sigma_t^c - \sum_{i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \right)
\end{aligned} \tag{4.12}$$

and the supply demand balance constraint can be written as:

$$\sum_{k \in I_t, \hat{k}=K_t+1-k} (\Delta Q_{k,t}^D + \Delta Q_{\hat{k},t}^S) \delta_{k,t} + Q_{1,t}^D \delta_t^c - Q_{1,t}^S \sigma_t^c = \sum_{i \in I_t} \Delta Q_{i,t}^S \quad \forall t \in T. \tag{4.13}$$

We next investigate the implication of the relations in Theorem 2. In (4.12) and (4.13), there are terms that contain $(\Delta Q_{k,t}^D + \Delta Q_{\hat{k},t}^S)$. We know that $\Delta Q_{k,t}^D = Q_{(k+1),t}^D - Q_{k,t}^D$ and $\Delta Q_{\hat{k},t}^S = Q_{(\hat{k}+1),t}^S - Q_{\hat{k},t}^S$, by definition. Summing these terms yields $\Delta Q_{k,t}^D + \Delta Q_{\hat{k},t}^S = (Q_{(k+1),t}^D - Q_{\hat{k},t}^S) - (Q_{k,t}^D - Q_{(\hat{k}+1),t}^S)$, where;

$$\begin{aligned} Q_{(k+1),t}^D &= D_t(P_{(k+1),t}^D), \\ Q_{\hat{k},t}^S &= S_t(P_{\hat{k},t}^S), \\ Q_{k,t}^D &= D_t(P_{k,t}^D), \\ Q_{(\hat{k}+1),t}^S &= S_t(P_{(\hat{k}+1),t}^S). \end{aligned}$$

Since k and $\hat{k}$ are paired, $P_{(k+1),t}^D = P_{\hat{k},t}^S = M$ and $P_{k,t}^D = P_{(\hat{k}+1),t}^S = N$, where M and N are scalar values for a given k and $\hat{k}$. Then,

$$\begin{aligned} (Q_{(k+1),t}^D - Q_{\hat{k},t}^S) &= D_t(M) - S_t(M) \\ &= TQ_t(M), \end{aligned}$$

is the total quantity offered at price $P_{(k+1),t}^D = P_{\hat{k},t}^S = M$. Similarly,

$$\begin{aligned} (Q_{k,t}^D - Q_{(\hat{k}+1),t}^S) &= D_t(N) - S_t(N) \\ &= TQ_t(N), \end{aligned}$$

is the total quantity offered at price $P_{k,t}^D = P_{(\hat{k}+1),t}^S = N$. Then, we can state the following corollary.

Corollary 1. *The term $\Delta Q_{k,t}^D + \Delta Q_{\hat{k},t}^S$ in (4.12) and (4.13) is the quantity difference on the aggregated total curve between the two consecutive breakpoints with prices $P_{(k+1),t}^D = P_{\hat{k},t}^S$ and $P_{k,t}^D = P_{(\hat{k}+1),t}^S$, i.e., $\Delta Q_{k,t}^D + \Delta Q_{\hat{k},t}^S = TQ_t(P_{(k+1),t}^D) - TQ_t(P_{k,t}^D)$.*

Proof. The result follows from the above derivation. □

We add the following remark about the objective function as follows:

Remark 1. The term $\sum_{i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2}$ denotes the area under the aggregated supply curve and it has a fixed value. Therefore, omitting this term does not affect the optimization problem.

By using Corollary 1 and Remark 1, we next present an equivalent model of (SDB-H), which we call (E-SDB-H). To derive the model (E-SDB-H), we first replace the term $\Delta Q_{k,t}^D + \Delta Q_{k,t}^S$ with $TQ_t(P_{(k+1),t}^D) - TQ_t(P_{k,t}^D)$ by Corollary 1, and omit the term $\sum_{i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2}$ by Remark 1.

(E-SDB-H)

$$\max \sum_{t \in T} \left(\sum_{k \in I_t} \left(P_{k,t}^D \left(TQ_t(P_{(k+1),t}^D) - TQ_t(P_{k,t}^D) \right) \delta_{k,t} - \frac{\Delta P_{k,t}^D \left(TQ_t(P_{(k+1),t}^D) - TQ_t(P_{k,t}^D) \right) \delta_{k,t}^2}{2} \right) + P_{1t}^D Q_{1,t}^D \delta_t^c - P_{1,t}^S Q_{1,t}^S \sigma_t^c \right)$$

s.t.

$$\sum_{k \in I_t} \left(TQ_t(P_{(k+1),t}^D) - TQ_t(P_{k,t}^D) \right) \delta_{k,t} + Q_{1,t}^D \delta_t^c - Q_{1,t}^S \sigma_t^c = \sum_{i \in I_t} \Delta Q_{i,t}^S \quad \forall t \in T.$$

$$0 \leq \delta_{k,t} \leq 1, \quad \forall k \in I_t, t \in T,$$

$$0 \leq \delta_t^c \leq 1, \quad \forall t \in T,$$

$$0 \leq \sigma_t^c \leq 1, \quad \forall t \in T.$$

The first two terms in the objective function correspond to the area under the $TQ_t(p)$ and the last two terms together give the area under the curtailable demand and supply curves.

Size of the the model (E-SDB-H)

Number of constraints	$ T + \sum_{t \in T} (K_t + 1).$
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Number of nonnegative continuous variables	$\sum_{t \in T} (K_t + 1)$
--	----------------------------

The number of constraints in the model (SDB-H) is $|T| + \sum_{t \in T} 2K_t$ and the number of nonnegative continuous variable is $\sum_{t \in T} 2K_t$. We remark that the size of the model (E-SDB-H) is almost half of the size of the model (SDB-H). Therefore, it may be computationally more appealing to use the model (E-SDB-H) instead of the model (SDB-H).

In this section, we present the stylized supply demand balance problem with only hourly orders. In this problem, we use aggregated demand and supply curves to find the consumer and producer surplus, which is the objective function of the model (SDB-H). Then, we show that instead of using aggregated demand and supply curves, we can use the total curves and present an equivalent model for the model (SDB-H). Since the model (E-SDB-H) uses the aggregated total curves, we will focus on the aggregated total curves in the next section and present a compact formulation for the model (E-SDB-H).

4.2 Equivalent Formulation of Supply Demand Balance Problem With Hourly Orders by Using Aggregated Total Curves

In this section, we provide a model which is similar to the model (E-SDB-H) by using aggregated total curves. Note that, the aggregated total curve in the model (E-SDB-H) is obtained by subtracting the aggregated supply curve from the aggregated demand curve in period t , i.e. $TQ_t(p) = D_t(p) - S_t(p)$. On the other hand, we can obtain the aggregated total curve by summing the total curve of each hourly order h , i.e.,

$$\begin{aligned} TQ_t(p) &= D_t(p) - S_t(p) \\ &= \sum_{h \in H} (D_{ht}(p) - S_{ht}(p)) \\ &= \sum_{h \in H} TQ_{ht}(p) \end{aligned}$$

The advantage of creating the total curve by summing the total curve of each

hourly order is that we do not need to add the additional breakpoints that separate the purchase and sales curves of the hourly orders. In our example, this breakpoint is 500 TL-0 MWh for the hourly order 1. Then, the aggregated total curve has five breakpoints if we derive TQ_t as $D_t - S_t$ and it has four breakpoints if we use $TQ_t(p) = \sum_{h \in H} TQ_{h,t}(p)$ shown in Figure 4.9.

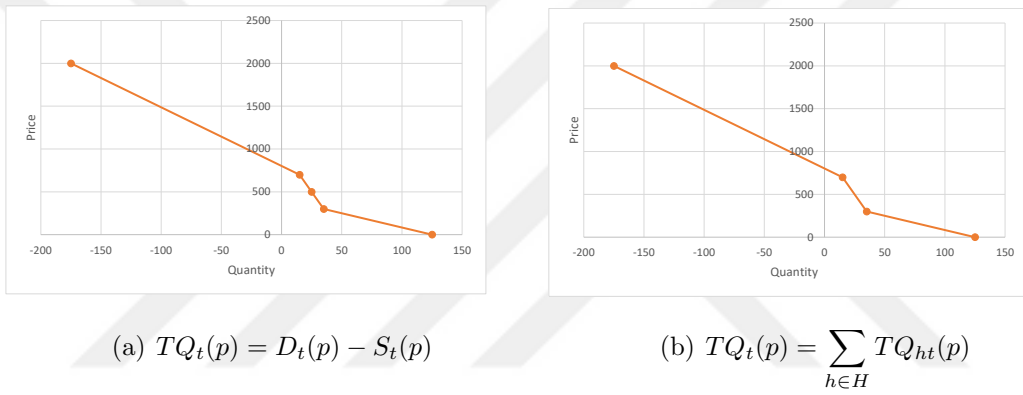


Figure 4.9: Aggregated total curve obtained by two different approaches

Assume that there are N_t breakpoints in the aggregated total curve when we derive TQ_t from TQ_{ht} . Since we do not add the additional breakpoints which separate the purchase and sales quantities of the hourly orders, we have $N_t \leq K_t$ for all $t \in T$.

We use the aggregated total curve obtained by summing the total curve of each hourly order. Total curves are monotonically decreasing piecewise linear functions. We denote by $P_{j,t}$ and $Q_{j,t}$ the price and quantity of the breakpoint j in the aggregated total curve in period t for all $j \in \{1, \dots, N_t\}$, respectively.

The piece between the breakpoint j and $j + 1$ is called segment j . We start enumerating total curve segments from the highest price. There are $N_t - 1$ segments in period t for each $t \in T$. Then, $\{1, \dots, N_t - 1\}$ is the set of indices for the segments of the aggregated total curve in period t . Then, we obtain $P_{j,t} \geq P_{(j+1),t}$ and $Q_{j,t} \leq Q_{(j+1),t}$ for all $j \in \{1, \dots, N_t - 1\}$.

Let J_t be the set of segments in aggregated total curve for all $t \in T$. The price difference of segment j at time t is represented by $\Delta P_{j,t}$, where $\Delta P_{j,t} = P_{j,t} - P_{(j+1),t} \geq 0$, for all $j \in J_t, t \in T$. Similarly, $\Delta Q_{j,t}$ is the quantity difference of segment j in period t , i.e., $\Delta Q_{j,t} = Q_{(j+1),t} - Q_{j,t} \geq 0$, where $j \in J_t, t \in T$.

To obtain a compact notation, we also add the curtailable supply and demand segments to the index set. We create segment 0 by adding a breakpoint $(P_{0,t}, Q_{0,t})$ where $P_{0,t} = P_{1,t}$ and $Q_{0,t} = Q_{1,t} - Q_{1,t}^D$ so that $\Delta Q_{0,t} = Q_{1,t}^D$ and $\Delta P_{0,t} = 0$. Similarly, we add an artificial breakpoint $(P_{N_t,t}, Q_{N_t,t} - Q_{1,t}^S)$ to represent the curtailable supply segment such that $\Delta Q_{N_t,t} = -Q_{1,t}^S$ and $P_{N_t,t} = P_{(N_t-1),t} = P_{1,t}^S$. Then, we define $J_t = \{0, 1, \dots, N_t\}$. Figure 4.10 shows a sample aggregated total curve.

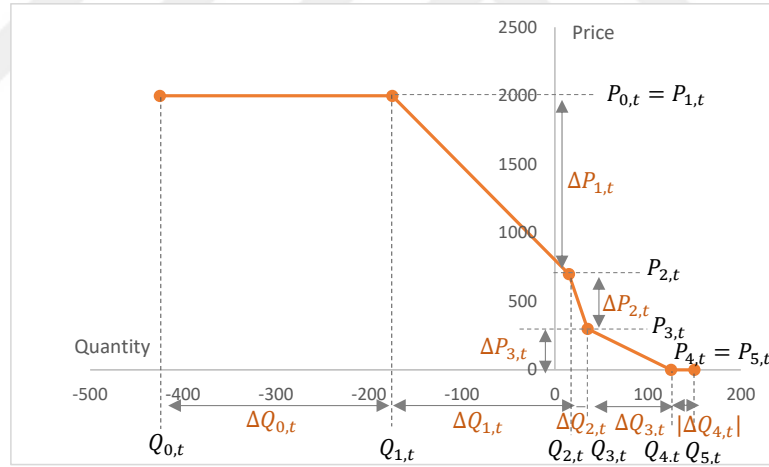


Figure 4.10: Notation used for the aggregated total curve in Figure 4.9 (b)

We denote by $x_{j,t}$ the accepted fraction of the segment j for all $j \in J_t, t \in T$. We use the row vector $\mathbf{x}_t$ which contains values of $x_{j,t}$, i.e., $\mathbf{x}_t = [x_{0,t} \ x_{1,t} \dots x_{N_t,t}]$ for all $t \in T$. Let $\mathbf{x}$ be a vector of $\mathbf{x}_t$ such that $\mathbf{x} = [\mathbf{x}_1 \ \mathbf{x}_2 \dots \mathbf{x}_T]^T$. We use $\mathbf{x}$ to present the decision variables in vector format.

4.2.1 Model (MC-H)

We are going to present a model, denoted by (MC-H), which maximizes the area under the aggregated total curve and curtailable demand and supply segments while ensuring that the accepted total quantity from the hourly orders is equal to zero. Therefore, the model (MC-H) is equivalent to the model (E-SDB-H). The differences between the models (E-SDB-H) and (MC-H) are as follows: (1) We construct the model (MC-H) using the aggregated total curve obtained by summing $TQ_{h,t}$ functions. However, the model (E-SDB-H) is constructed by the aggregated total curve obtained by subtracting the aggregated demand and supply curves. Therefore, the size of the model (MC-H) is at most as the size of the model (E-SDB-H). (2) The model (MC-H) employs a simpler notation than the model (E-SDB-H).

(MC-H)

$$\max c(x) = \sum_{t \in T} \sum_{j \in J_t} \left(P_{j,t} \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right)$$

s.t.

$$\sum_{j \in J_t} \Delta Q_{j,t} x_{j,t} = Q_t, \quad \forall t \in T, \quad (4.14)$$

$$0 \leq x_{j,t} \leq 1, \quad \forall j \in J_t, t \in T, \quad (4.15)$$

where c is the sum of the area under the aggregated total curves and the area under the demand curtailable segments excluding the area under the supply curtailable segments, and $Q_t = \sum_{i \in I_t} \Delta Q_{i,t}^S$ is the difference between the supply quantity offered at the maximum price and the supply quantity offered at the minimum price. The remaining constraints specify the ranges of decision variables x .

In the cases where there are sufficient supply and demand quantities in the system, we do not observe a curtailment. If there is no curtailment, we have $x_{0,t} = x_{N_t,t} = 1$,

in which case (4.14) can be written as:

$$Q_{1,t} + \sum_{j=1}^{N_t-1} \Delta Q_{j,t} x_{j,t} = 0 \quad (4.16)$$

which states that the total quantity offered to the system by the hourly orders must be zero.

We summarize the model size of the model (MC-H) below.

	Size of the model (MC-H)
Number of constraints	$ T + \sum_{t \in T} (N_t + 1).$
Number of nonnegative continuous variables	$\sum_{t \in T} (N_t + 1)$

The objective function $c(\mathbf{x})$ no longer corresponds to the total surplus arising from hourly orders as we derived c by removing the fixed term from the surplus. Since we will use the notation we used for the total curve in the next chapter, we need to express total surplus function by using the notation we defined for aggregated total curves. In Lemma 2, we state the total surplus arising from hourly orders.

Lemma 2. *The total surplus arising from the hourly orders is $g(\mathbf{x})$ and can be written as;*

$$\begin{aligned} g(\mathbf{x}) &= c(\mathbf{x}) - \sum_{t \in T, i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \\ &= \sum_{t \in T, j \in J_t} \left((P_{j,t} - p_t) \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) \end{aligned} \quad (4.17)$$

$$+ \sum_{t \in T, i \in I_t} \left(p_t - \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \right) \Delta Q_{i,t}^S, \quad (4.18)$$

where p_t is the matching price of the hourly orders.

Proof. We add $p_t \left(\sum_{j \in J_t} \Delta Q_{j,t} - Q_t \right) = 0$ to the equality and re-arrange the terms.

$$\begin{aligned}
g(\mathbf{x}) &= c(\mathbf{x}) - \sum_{t \in T, i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \\
&= \sum_{t \in T, j \in J_t} \left(P_{j,t} \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) - \sum_{t \in T, i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \\
&= \sum_{t \in T, j \in J_t} \left(P_{j,t} \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) - \sum_{t \in T, i \in I_t} \frac{(P_{i,t}^S + P_{(i+1),t}^S) \Delta Q_{i,t}^S}{2} \\
&\quad - \sum_{t \in T} p_t \left(\sum_{j \in J_t} \Delta Q_{j,t} x_{j,t} - Q_t \right) \\
&= \sum_{t \in T, j \in J_t} \left((P_{j,t} - p_t) \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) \\
&\quad + \sum_{t \in T, i \in I_t} \left(p_t - \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \right) \Delta Q_{i,t}^S
\end{aligned}$$

□

We will use the model (MC-H) as our base model in this thesis because: (1) Its size is less than or equal to the size of the model (E-SDB-H) since $N_t \leq K_t$ for all $t \in T$ (2) If there exist breakpoints that are added to the hourly orders to separate the purchase and demand quantities, the model (E-SDB-H) may contain the price values that have high precision values. High precision values may lead to numerical problems when we try to solve the model as we discussed before.

4.3 Summary

In this chapter, we first introduce the hourly orders and show how we obtain the aggregated supply and demand curves from these orders. Afterwards, we present a special case of the well-known the supply demand balance problem with only hourly orders, whose optimization model is referred to as the model (SDB-H). We show that instead of using supply and demand curves we can use the total curves to reformulate

the model (SDB-H). We propose an equivalent model to the model (SDB-H), which we denote by (E-SDB-H), whose problem size is almost half of that of the model (SDB-H). The model (E-SDB-H) employs the aggregated total curves. We propose a simpler and more compact version of the model (E-SDB-H), and denote this model by (MC-H). We summarize the problem size for the models (SDB-H), (E-SDB-H), and (MC-H) in the following table, where $N_t \leq K_t$

Table 4.6: Size of the models we proposed in this chapter

	(SDB-H)	(E-SDB-H)	(MC-H)
Number of constraints	$ T + \sum_{t \in T} 2K_t$	$ T + \sum_{t \in T} (K_t + 1)$	$ T + \sum_{t \in T} (N_t + 1)$
Number of nonnegative continuous variables	$\sum_{t \in T} 2K_t$	$\sum_{t \in T} (K_t + 1)$	$\sum_{t \in T} (N_t + 1)$

Problem size reduction may lead to improved solution times. In this chapter three different problem size reduction techniques are implemented: (1) Aggregation of supply and demand curves, initially proposed by Derinkuyu (2015) for total curves, (2) Eliminating the supply segments by using the relations given by Theorem 1, (3) Using the total curve by summing the total curves of each hourly order. To the best of our knowledge, the second and the third techniques have not presented in the literature before.

Our other contributions to the literature are as follows: (1) We propose an equivalent model to the model (SDB-H) whose size is less than (SDB-H). (2) In order to show the relation between the models (MC-H) and (SDB-H), we state two new results in Theorems 1 and 2.

In the next chapter, we introduce the block and flexible orders to our problem. We summarize the symbols we define in this section in Table 4.7 because for reference in the subsequent chapters.

Table 4.7: Sets, indices, parameters and variables used in this chapter

Sets and Indices	
t, T	Time period index and set of time periods.
j, J_t	Segment index and set of segments for total curve in period t , $\forall t \in T$, where $N_t + 1 = J_t $.
$j = 0$	The index that represents the curtailable demand segment.
$j = N_t$	The index that represents the curtailable supply segment.
Parameters	
$P_{j,t}, Q_{j,t}$	The price and quantity of the breakpoint j in period t , where $j \in \{1, \dots, N_t\}$, $t \in T$.
$\Delta P_{j,t}$	Price difference of segment j at time t , i.e., $\Delta P_{j,t} = P_{j,t} - P_{(j+1),t}$, where $j \in J_t, t \in T$.
$\Delta Q_{j,t}$	Quantity difference of segment j at time t , i.e., $\Delta Q_{j,t} = Q_{(j+1),t} - Q_{j,t}$, where $j \in J_t, t \in T$.
Q_t	The difference of the maximum and minimum quantity offered by the supply curve in period t for all $t \in T$.
$P_{1,t}$	The maximum price in period t and $P_{1,t} = P_{0,t}$, for all $t \in T$.
$P_{N_t,t}$	The minimum price in period t and $P_{N_t,t} = P_{(N_t+1),t}$, for all $t \in T$.
Variables	
$x_{j,t}$	Accepted fraction of segment j in period t , where $j \in J_t, t \in T$.

Chapter 5

SUPPLY DEMAND BALANCE PROBLEM

In the previous chapter, we start with the supply demand balance problem which has only hourly orders and propose an equivalent version of the model. In this chapter, we extend the supply demand balance problem to introduce block and flexible orders.

We start with explaining the supply demand balance problem and provide the corresponding model (MC) which is a mixed integer quadratic programming (MIQP) problem. As we discussed in Chapter 2, convexity of the feasible region of the model (MC) ensures the existence of the market equilibrium. Therefore, we analyze the problem structure by investigating primal-dual results arising from quadratic programming (QP) relaxation of the model (MC), where convexity holds. We show the equilibrium conditions by using QP relaxation of the model (MC). However, the model (MC) has a non-convex feasible region due to block and flexible orders. Therefore, we also investigate the restricted supply demand balance problem where the integer variables of the model (MC) are fixed to some values. We show how the equilibrium conditions change when there are integer variables in the problem.

We contribute to the literature by filling the gaps in the results about aggregated total curves, linked block and flexible orders in the primal-dual analysis.

5.1 Mathematical Model for Supply Demand Balance Problem (MC)

In this section, we present the supply demand balance problem, which is the generic form of the market clearing problem with hourly, block and flexible orders. We have just explained the hourly orders in Chapter 4. In this section, we first describe the new order types, i.e., block and flexible orders. Then, we provide the mathematical model

for supply demand balance problem, which we call Market Clearing model (MC). Afterwards, we explain the objective function of the model (MC) which corresponds to total surplus.

We contribute to the literature by proposing a supply demand balance problem with total aggregate curves. We show in Lemma 3 that the total surplus function can be simplified.

5.1.1 Block and Flexible Orders

Some power plants with high start-up or shut-down costs prefer to produce the electricity in consecutive periods. Block orders are used to satisfy this need in DAM auction. They are valid in specific periods. Flexible orders, on the other hand, are flexible in terms of their execution times. They are preferred by hydro power plants whose production times are flexible.

Block and flexible orders consist of a price, a quantity and a time window, which are respectively represented by $P_b, Q_{b,t}, T_b$ for a block order b and P_f, Q_f, T_f for a flexible order f . Sets of sales and purchase block orders are represented by B^S, B^D , respectively where the set of all block orders is $B = B^S \cup B^D$. Similarly, F^S, F^D and F respectively denote the sets of sales, purchase and all flexible orders. Block and flexible orders are valid for a prespecified number of consecutive periods (duration). However, the time window for a flexible orders must be higher than its duration. The time window that a block order is valid is denoted by T_b , where $|T_b| = N_b$ and $Q_{b,t} = 0$ for all $t \in T \setminus T_b$. Similarly, the time window that a flexible order can be accepted is T_f . There is no limitation on time windows for the flexible orders in European power exchanges, in this study we assume that $T_f = T$ for all $f \in F$ and the duration of the order is one for the sake of simplicity. Our proposed models and results can be easily extended for a general T_f and higher duration.

The block and flexible orders cannot be partially accepted. Hence we use binary variables y_b to indicate that block order b is accepted or not for all $b \in B$ and $y_{f,t}$ to

show that the flexible order f is accepted in period t , for all $t \in T_f$, $f \in F$.

A block order may be linked to another block order, which ensures that acceptance of a block order depends on the acceptance of the other. This type of block orders are called *linked* block orders. There is a common parent-child analogy used to describe linked block orders. A *child* block order is a linked block order which cannot be accepted unless its *parent* block order is accepted. For instance, there are some power plants where the second turbine of the plant cannot run unless the first one is operational for a period of a time. Linked block orders are used to represent this need in DAM auction. In both European and Turkish DAMs, a block order may have at most one parent block order. On the other hand, a parent block order may have multiple children. A *family* is a combination of linked block orders, and it forms a connected graph. We define sets to show relations in the connected graph. A *subfamily* of a block order b , denoted by ϕ_b , contains the block order and a subtree of the connected graph originated from the block order. Set of all subfamilies of block order b is denoted by Φ_b . The *descendants* of a block order b consist of all block orders which must be rejected if b is rejected and are denoted by ξ_b . Similarly, the set of *ancestors* of b is the set of all block orders which must be accepted if b is accepted. Moreover, Θ_b shows the immediate child/children of the block order b . If b has a parent but not a child order, then $\Theta_b = \emptyset$. The *first generation* orders are the ones without any ancestors but with children. For the remaining orders in a family, an i^{th} generation order must have $(i - 1)^{th}$ generation parent order.

Figure 5.1 shows three possible family combinations. Tables 5.1- 5.3 clarify these relations. According to Table 5.1, $\Theta_{b_1} = \{b_2\}$ and $\Theta_{b_2} = \{b_3\}$ for family 1, $\Theta_{b_1} = \{b_2, b_4\}$ and $\Theta_{b_2} = \{b_3\}$ for family 2 and $\Theta_{b_1} = \{b_2\}$ and $\Theta_{b_2} = \{b_3, b_4\}$ for family 3. Set of subfamilies are presented in Table 5.2 and descendants and ancestors of each order can be defined as in Table 5.3.

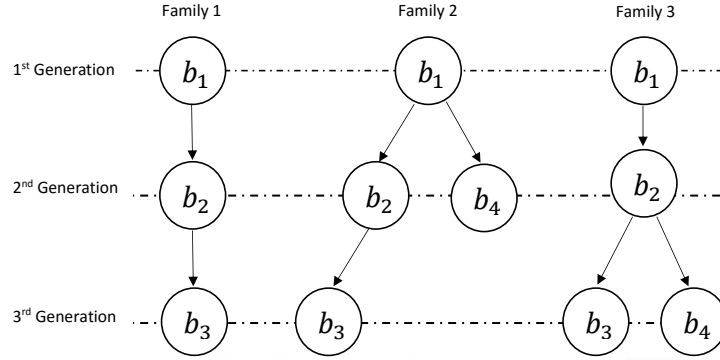


Figure 5.1: Possible Linked Block Order Combinations

Table 5.1: Parent orders and their children for families 1, 2 and 3 given in Figure 5.1

Family 1		Family 2		Family 3	
parent order	child orders	parent order	child orders	parent order	child orders
b_1	b_2	b_1	b_2, b_4	b_1	b_2
b_2	b_3	b_2	b_3	b_2	b_3, b_4

Table 5.2: Set of subfamilies originated from each block order in families 1, 2 and 3 given in Figure 5.1

	Family 1	Family 2	Family 3
originated order	set of subfamily	set of subfamily	set of subfamily
b_1	$\Phi_{b_1} = \{\{b_1\}, \{b_1, b_2\}, \{b_1, b_2, b_3\}\}$	$\Phi_{b_1} = \{\{b_1\}, \{b_1, b_2\}, \{b_1, b_2, b_3\}, \{b_1, b_4\}, \{b_1, b_2, b_3, b_4\}\}$	$\Phi_{b_1} = \{\{b_1\}, \{b_1, b_2\}, \{b_1, b_2, b_3\}, \{b_1, b_2, b_4\}\}$
b_2	$\Phi_{b_2} = \{\{b_2\}, \{b_2, b_3\}\}$	$\Phi_{b_2} = \{\{b_2\}, \{b_2, b_3\}\}$	$\Phi_{b_2} = \{\{b_2\}, \{b_2, b_3\}, \{b_2, b_4\}, \{b_2, b_3, b_4\}\}$
b_3	$\Phi_{b_3} = \{\{b_3\}\}$	$\Phi_{b_3} = \{\{b_3\}\}$	$\Phi_{b_3} = \{\{b_3\}\}$
b_4	-	$\Phi_{b_4} = \{\{b_4\}\}$	-

Table 5.3: Descendants and ancestors of each block order in families 1, 2 and 3 given in Figure 5.1

	Family 1		Family 2		Family 3	
block order	descendants	ancestors	descendants	ancestors	descendants	ancestors
b_1	b_2, b_3	-	b_2, b_3, b_4	-	b_2, b_3, b_4	-
b_2	b_3	b_1	b_3	b_1	b_3, b_4	b_1
b_3	-	b_1, b_2	-	b_1, b_2	-	b_1, b_2
b_4	-	-	-	b_1	-	b_1, b_2

We summarize the sets, indices, parameters and decision variables related to block and flexible orders as follows:

Sets and Indices Related with Block and Flexible orders

b, B	Block order index and set of block orders.
B^S, B^D	Sets of sales and purchase block orders, i.e., $B = B^S \cup B^D$.
Θ_b	Set of child orders of block order b , where $b \in B$.
ϕ_b	Set of block orders in subfamily of a block order b , where $b \in B$.
Φ_b	Set of subfamily sets of block order b , $b \in B$.
ξ_b	Set of orders that shows descendants of b , $b \in B$.
B^c, B^m	Set of child block orders and parent block orders.
f, F	Flexible order index and set of flexible orders.
F^S, F^D	Set of sales and purchase flexible orders, i.e., $F = F^S \cup F^D$.
t, T_b, T_f	Time period index and set of time periods that block order b is valid and flexible order f can be accepted where $T_f = T$, respectively.

Parameters	Related with Block and Flexible orders
$P_b, Q_{b,t}$	Price and quantity of block order b in period t , where $b \in B$, $t \in T$. $Q_{b,t} < 0$ for $b \in B^S$ and $Q_{b,t} > 0$ for $b \in B^D$, $t \in T_b$.
P_f, Q_f	Price and quantity of flexible order f , where $f \in F$. $Q_f < 0$ for $f \in F^S$ and $Q_f > 0$ for $f \in F^D$.
N_b	Number of periods for which block order b is valid, i.e. $ T_b $ for all $b \in B$.

Decision Variables	Related with Block and Flexible orders
y_b	$\begin{cases} 1, & \text{if block order } b \text{ is accepted} \\ 0, & \text{otherwise} \end{cases}$ for all $b \in B$.
$y_{f,t}$	$\begin{cases} 1, & \text{if flexible order } f \text{ is accepted in period } t \\ 0, & \text{otherwise} \end{cases}$ for all $f \in F$, $t \in T$.

In our problem we have a set of binary decision variables. Let $\mathbf{y}$ be a vector which contains the values of y_b and $y_{f,t}$. We define $\mathbf{y} = [\mathbf{y}^B \mathbf{y}^F]^T$ where $\mathbf{y}^B = [y_{b_1} \dots y_{b_{|B|}}]$ and $\mathbf{y}^F = [\mathbf{y}_{f_1} \dots \mathbf{y}_{f_{|F|}}]$, $\mathbf{y}_f = [y_{f,t_1} \dots y_{f,t_{|T|}}]$ for all $f \in F$. For the rest of the work, we present the decision variables in vector format, i.e., $\mathbf{x}$ and $\mathbf{y}$, which is the compact form of the solution of the problem.

5.1.2 Mathematical Model

Supply demand balance problem maximizes the total surplus while considering supply demand balance and the fundamental features of the orders. The basic properties of the offers are about the limits on the accepted fraction of hourly order segments, all-or-nothing nature of the block and flexible orders, link conditions for block orders

and the validity of flexible orders in a single period.

We will use the aggregated total curve instead of aggregated supply and demand curves, because it provides a simple representation without changing the optimal solution as we discussed in the previous chapter. Block and flexible orders are considered as we discussed before. Then, the supply demand balance problem can be modeled as follows:

(MC)

$$\max g(\mathbf{x}, \mathbf{y})$$

s.t.

$$\sum_{j \in J_t} x_{j,t} \Delta Q_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} = Q_t, \quad \forall t \in T, \quad (5.1)$$

$$y_b \leq y_{b_m}, \quad \forall b \in \Theta_{b_m}, b_m \in B^m, \quad (5.2)$$

$$\sum_{t \in T} y_{f,t} \leq 1, \quad \forall f \in F, \quad (5.3)$$

$$x_{j,t} \leq 1, \quad \forall i \in J_t, t \in T, \quad (5.4)$$

$$x \geq 0, \quad y \in [0, 1]^n, \quad (5.5)$$

where n is the dimension of $\mathbf{y}$ and is given by $n = |B| + |T| \times |F|$ and the total surplus function g is defined in Lemma 3. We will cover the objective function in the next section in a detailed way. The total demand equals the total supply for all periods by (5.1). The second inequality ensures that a parent block order is accepted if its child block order is accepted. The third and the fourth inequalities set upper limits on the periods that a flexible order can be accepted and the accepted fraction of hourly order segments, respectively. The following table shows the number of constraints, nonnegative and binary variables of the model (MC).

	Size of the (MC)
Number of constraints	$ T + B^c + F + \sum_{t \in T} (N_t + 1).$
Number of nonnegative continuous variables	$\sum_{t \in T} (N_t + 1)$
Number of binary variables	$ B + T \times F .$

Now we derive the mathematical formulation for the objective function $g(\mathbf{x}, \mathbf{y})$.

5.1.3 Objective Function

Each order type we introduce in the previous chapter has its own surplus formulation. Hence, we first define producer and consumer surplus arising from hourly, block and flexible orders and then present the overall objective function, g . Then, we show that we can simplify g as in Lemma 3.

5.1.3.1 Consumer and Producer Surplus Arising from Hourly Orders

In the previous chapter, we presented how consumer and producer surplus arising from the hourly orders are calculated. Then, we showed that we can use the aggregated total curve instead of aggregated demand and supply curves. In Lemma 2, we showed that the total surplus arising from the hourly orders can be written by using aggregated total curve as follows:

$$\sum_{t \in T} \left(\sum_{j \in J_t} (P_{j,t} - p_t) \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} + \sum_{i \in I_t} (p_t - \frac{P_{i,t}^S + P_{(i+1),t}^S}{2}) \Delta Q_{i,t}^S \right),$$

where the first and the second terms are the surplus arising from the total curve segment, and the last term is the difference between surplus of accepted supply segments loss arising from the rejected supply segments.

5.1.3.2 Consumer and Producer Surplus Arising from Block and Flexible Orders

The surplus arising from an accepted block or flexible order per unit time is the area between the matching price and P_b (P_f). The order gains zero surplus if it is rejected. For a market design which uses uniform pricing (see Definition 3 in Chapter 2), (MC)P of the periods where a block order is active is the matching price of the block order. For a market design where the discriminatory pricing with an uplift payment is applied, matching price of the order is determined in such a way that the payment is sufficient enough not to incur a loss. However, in this design the uplift is collected from all participants and it causes a surplus reduction.

We are going to explain discriminatory pricing with an uplift payment with an example. Suppose we have an accepted sales block order as we show in Figure 5.6. The matching price of the block order is the maximum of its price and the average (MC)P for the periods during which the block order is valid, i.e., $MP = \max \left\{ P_b, \frac{\sum_{t \in T_b} p_t}{N_b} \right\}$. Therefore, a participant with a sales block order cannot gain any surplus if his matching price is P_b . On the other hand, to satisfy the financial balance the uplift payment which equals to $\sum_{t \in T_b} (p_t - MP)Q_{b,t}$ must be collected. This amount vanishes when $MP = p_t$ for all $t \in T_b$ but has a non-negative value when $MP = P_b$. Then, the surplus of the block minus the uplift payment arising from that block is given by $\sum_{t \in T_b} ((P_b - MP)Q_{b,t} - (p_t - MP)Q_{b,t})$, which is nothing but $\sum_{t \in T_b} (P_b - p_t)Q_{b,t}$. It follows that, the total surplus arising from block order and uplift is the same as the surplus arising from a block order in a market where the uniform pricing is applied (see Definition 3 in Chapter 2).

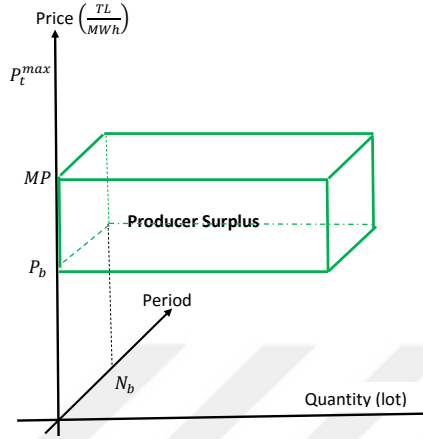


Figure 5.2: Producer surplus arising from a sales block order

We can follow the same logic and show that the surplus of the block orders minus the uplift payment (if exists) can be represented as follows :

$$\sum_{b \in B, t \in T_b} Q_{b,t} (P_b - p_t) y_b, \quad (5.6)$$

when the uplift is enough to cover the loss of the block orders to the break even point.

Flexible orders are special cases of block orders that the execution time of the order is not specified by the bidder. Hence, if it is rejected or accepted with an uplift payment, the surplus of the order is zero. For the rest of the cases it is $(P_f - MP)Q_f$. We can follow the same arguments as in block orders and the surplus arising from flexible orders minus the uplift payment (zero if uniform pricing is applied) can be represented as follows:

$$\sum_{t \in T} \sum_{f \in F} (P_f - p_t) Q_f y_{f,t}, \quad (5.7)$$

5.1.3.3 The Total Surplus

The total surplus is the sum of surpluses arising from aggregated hourly orders as in (4.8) and surpluses arising from block and flexible orders as defined in (5.6) and (5.7).

Then the total surplus is as follows:

$$\begin{aligned}
 g(\mathbf{x}, \mathbf{y}) = & \sum_{t \in T} \left(\sum_{j \in J_t} (P_{j,t} - p_t) \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} + \sum_{i \in I_t} (p_t - \frac{P_{i,t}^S + P_{(i+1),t}^S}{2}) \Delta Q_{i,t}^S \right) \\
 & + \sum_{b \in B, t \in T} (P_b - p_t) Q_{b,t} y_b \sum_{f \in F, t \in T} (P_f - p_t) Q_f y_{f,t}. \tag{5.8}
 \end{aligned}$$

The total surplus function g has quadratic parts due to the terms $x_{j,t}^2$, $y_b p_t$, and $y_{f,t} p_t$. On the other hand, we can simplify the total function by eliminating some of the quadratic terms as in Lemma 3 and propose an equivalent function c which omits the fixed terms in function g .

Lemma 3. *Total surplus is a quadratic function, where the quadratic term comes from interpolation, i.e. $x_{j,t}^2$ and can be written as;*

$$\begin{aligned}
 g(\mathbf{x}, \mathbf{y}) = & \sum_{t \in T} \left(\sum_{i \in J_t} \left(\Delta Q_{j,t} P_{j,t} x_{j,t} - \frac{x_{j,t}^2 \Delta P_{j,t} \Delta Q_{j,t}}{2} \right) - \sum_{i \in I_t} \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \Delta Q_{i,t}^S \right. \\
 & \left. + \sum_{b \in B} P_b N_b Q_{b,t} y_b + \sum_{f \in F} P_f Q_f y_{f,t} \right). \tag{5.9}
 \end{aligned}$$

Moreover, instead of using the total surplus function g , we can use the following function c , which omits fixed term representing the area under the aggregated supply curve.

$$c(\mathbf{x}, \mathbf{y}) = \sum_{t \in T} \left(\sum_{i \in J_t} \left(\Delta Q_{j,t} P_{j,t} x_{j,t} - \frac{x_{j,t}^2 \Delta P_{j,t} \Delta Q_{j,t}}{2} \right) + \sum_{b \in B} P_b N_b Q_{b,t} y_b + \sum_{f \in F} P_f Q_f y_{f,t} \right). \tag{5.10}$$

Proof. To derive the resulting function g , we first re-organize the terms. Then, we use the relation $Q_t = \sum_{i \in I_t} \Delta Q_{i,t}^S$ and the term with p_t because the terms in the parenthesis is zero by supply demand balance equality.

$$\begin{aligned}
g(\mathbf{x}, \mathbf{y}) &= \sum_{t \in T} \left(\sum_{j \in J_t} \left((P_{j,t} - p_t) \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) \right. \\
&\quad \left. + \sum_{i \in I_t} \left(p_t - \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \right) \Delta Q_{i,t}^S \right) \\
&\quad + \sum_{b \in B, t \in T} (P_b - p_t) Q_{b,t} y_b + \sum_{f \in F, t \in T} (P_f - p_t) Q_f y_{f,t} \\
&= \sum_{t \in T} \left(\sum_{j \in J_t} \left(P_{j,t} \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) + \sum_{i \in I_t} \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \Delta Q_{i,t}^S \right) \\
&\quad + \sum_{b \in B, t \in T} P_b Q_{b,t} y_b + \sum_{f \in F, t \in T} P_f Q_f y_{f,t} \\
&\quad - \sum_{t \in T} p_t \left(\sum_{j \in J_t} \Delta Q_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} - \sum_{i \in I_t} \Delta Q_{i,t}^S \right) \\
&= \sum_{t \in T} \left(\sum_{j \in J_t} \left(P_{j,t} \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) + \sum_{i \in I_t} \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \Delta Q_{i,t}^S \right) \\
&\quad + \sum_{b \in B, t \in T} P_b Q_{b,t} y_b + \sum_{f \in F, t \in T} P_f Q_f y_{f,t} \\
&\quad - \sum_{t \in T} p_t \left(\sum_{j \in J_t} \Delta Q_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} - Q_t \right) \\
&= \sum_{t \in T} \left(\sum_{j \in J_t} \left(P_{j,t} \Delta Q_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \right) + \sum_{i \in I_t} \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \Delta Q_{i,t}^S \right. \\
&\quad \left. + \sum_{b \in B} P_b Q_{b,t} y_b + \sum_{f \in F} P_f Q_f y_{f,t} \right)
\end{aligned}$$

□

The third term, $\sum_{i \in I_t, t \in T} \frac{P_{i,t}^S + P_{(i+1),t}^S}{2} \Delta Q_{i,t}^S$, in (5.9) does not depend on any decision variables. Hence, we will use the constant β to represent this term so that $g(\mathbf{x}, \mathbf{y}) = c(\mathbf{x}, \mathbf{y}) + \beta$. We will omit the term β and use $c(\mathbf{x}, \mathbf{y})$ as a objective function in the following sections.

Decision variables have finite upper and lower bounds and the constraints are linear. Hence, the constraint set is closed and bounded. However, the feasible region

is not a convex set due to integer decision variables. Rather than investigating the model (MC) directly, we instead focus on two different problems originating from (MC), each of which has a convex constraint set in the subsequent sections. We investigate the optimality conditions of the resulting problems by using Dorn's duality (Dorn, 1960). Hence, we first focus on the Dorn's dual and then propose two different problems originating from the model (MC).

5.2 Dorn's Dual for a Convex Quadratic Program

We present the Dorn's dual and basic facts resulting from the Dorn's dual by using compact vector notation in Lemma 4. The proof of the lemma can be found in Dorn (1960); Madani and Van Vyve (2015).

Lemma 4. *The following primal convex quadratic program has the Dorn's dual*

Primal

$$\max \frac{1}{2} \mathbf{x}^T Q \mathbf{x} + e_1^T \mathbf{x} + e_2^T \mathbf{y}$$

s.t.

$$A_1 \mathbf{x} + A_2 \mathbf{y} \leq b$$

$$\mathbf{x} \geq 0$$

$$\mathbf{y} \geq 0$$

when Q is a negative semidefinite matrix. If $(\mathbf{x}^*, \mathbf{y}^*)$ and $(\mathbf{s}^*, \mathbf{u}^*)$ are primal and dual feasible solutions, then;

Dorn's Dual

$$\min b^T \mathbf{s} - \frac{1}{2} \mathbf{u}^T Q \mathbf{u}$$

s.t.

$$A_1^T \mathbf{s} - Q \mathbf{u} \geq e_1$$

$$A_2^T \mathbf{s} \geq e_2$$

$$\mathbf{s} \geq 0,$$

I $(\mathbf{x}^*, \mathbf{y}^*)$ and $(\mathbf{s}^*, \mathbf{u}^*)$ are optimal and there exist a solution such that $\mathbf{u}^* = \mathbf{x}^*$ holds.

II The objective functions of primal and dual problems are equal, i.e., $\frac{1}{2} \mathbf{x}^{*T} Q \mathbf{x}^* + e_1^T \mathbf{x}^* + e_2^T \mathbf{y}^* = b^T \mathbf{s}^* - \frac{1}{2} \mathbf{x}^{*T} Q \mathbf{x}^*$

III Complementarity constraints hold, i.e.,

$$\mathbf{s}^{*T}(A_1\mathbf{x} + A_2\mathbf{y} - b) = 0$$

$$\mathbf{x}^{*T}(A_1^T\mathbf{s} - Q\mathbf{u} - e_1^T) = 0$$

$$\mathbf{y}^{*T}(A_2^T\mathbf{s} - e_2^T) = 0$$

Dorn's dual is valid for only quadratic programming problems and presents practical results about primal and dual problems. On the other hand Lagrangian duality is also applicable for other types of problems. Geoffrion (1971) shows how we can obtain the Dorn's dual by using Lagrangian duality.

According to Lemma 4, if Q is negative semi-definite for a convex QP problem, then we can find the Dorn's dual and necessary conditions for a solution in QP problem to be optimal. Since we focus on a QP problem, Q is the Hessian matrix. The following lemma states that the objective function c is concave in $(\mathbf{x}, \mathbf{y})$.

Lemma 5. *The objective function $g(\mathbf{x}, \mathbf{y}) = c(\mathbf{x}, \mathbf{y}) + \beta$ defined in Lemma 3 has a negative semi-definite Hessian matrix, therefore, it is concave in $(\mathbf{x}, \mathbf{y})$.*

Proof. By Lemma 3, the only quadratic term in the objective function arises from the terms $x_{j,t}^2$, which are the only non-zero elements in the Hessian matrix of the objective function. The partial derivatives of $c(\mathbf{x}, \mathbf{y})$ with respect to $x_{j,t}$ are:

$$\frac{\partial^2 c}{\partial x_{j,t}^2} = -\Delta P_{j,t} \Delta Q_{j,t}, \quad \forall i \in J_t, t \in T, \quad (5.11)$$

where $-\Delta P_{j,t} \Delta Q_{j,t} \leq 0$. The Hessian of c is a diagonal matrix, whose main diagonal entries are non-positive. Therefore, it is negative semidefinite. Thus, the objective function is concave. $\square$

By Lemmas 4 and 5, we have shown that we can use Dorn's dual for a convex QP problem whose objective is c . However, the model (MC) is not a convex QP problem due to the integrality constraints. In the next section, we present QP relaxation of (MC), which is a convex QP problem.

5.3 Quadratic Programming Relaxation of the Model (MC)

In economic theory, linearity and convexity assumptions are widely used to construct useful models (O'Neill et al., 2005). For linear and convex models, the equilibrium prices are the shadow prices of market clearing constraint.

In this section, we relax the all-or-nothing constraints on block and flexible orders in the model (MC), i.e., a block or a flexible order may be partially accepted. The resulting problem is equivalent to the quadratic programming relaxation of the model (MC). Hence, we investigate the economical insights from its optimal solutions by using duality. We first formulate the model (QPR-MC).

$$\max \sum_{t \in T} \sum_{j \in J_t} \left(\Delta Q_{j,t} P_{j,t} x_{j,t} - \frac{\Delta P_{j,t} \Delta Q_{j,t}}{2} x_{j,t}^2 \right) + \sum_{b \in B, t \in T_b} P_b Q_{b,t} y_b + \sum_{f \in F, t \in T} P_f Q_f y_{f,t}$$

$$[p_t] \quad \sum_{j \in J_t} \Delta Q_{j,t} x_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} = Q_t, \quad \forall t \in T, \quad (5.12)$$

$$[s_{j,t}] \quad x_{j,t} \leq 1, \quad \forall j \in J_t, t \in T, \quad (5.13)$$

$$[s_b] \quad y_b \leq 1, \quad \forall b \in B, b \notin B^c, \quad (5.14)$$

$$[s_b] \quad y_b \leq y_{b_m}, \quad \forall b \in \Theta_{b_m}, b_m \in B^m, \quad (5.15)$$

$$[s_{f,t}] \quad y_{f,t} \leq 1, \quad \forall f \in F, t \in T, \quad (5.16)$$

$$[s_f] \quad \sum_{t \in T} y_{f,t} \leq 1, \quad \forall f \in F, \quad (5.17)$$

$$x, y \geq 0. \quad (5.18)$$

This is a QP problem whose objective function is concave and all decision variables are continuous. Let the shadow prices of the constraints be defined as in brackets. There is also a dual variable $u_{j,t}$ for each quadratic term, however, as it is shown in Lemma 4, $u^* = x^*$ in the optimal solution. We replace, $u^* = x^*$ and write the equivalent Dorn's dual as:

$$\begin{aligned} \min \quad & \sum_{j \in J_t, t \in T} s_{j,t} + \sum_{b \in B, b \notin B^c} s_b + \sum_{f \in F} s_f + \sum_{f \in F, t \in T} s_{f,t} \\ & - \sum_{t \in T} Q_t p_t + \sum_{t \in T, j \in J_t} \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \end{aligned}$$

$$[x_{j,t}] \quad s_{j,t} \geq \Delta Q_{j,t}(P_{j,t} - p_t) - \Delta Q_{j,t} \Delta P_{j,t} x_{j,t}, \quad \forall i \in J_t, t \in T, \quad (5.19)$$

$$[y_b] \quad s_b \geq \sum_{t \in T_b} Q_{b,t}(P_b - p_t), \quad \forall b \in B, b \notin B^m, \quad (5.20)$$

$$[y_b] \quad s_b - \sum_{b_i \in \Theta_b} s_{b_i} \geq \sum_{t \in T_b} Q_{b,t}(P_b - p_t), \quad \forall b \in B^m, \quad (5.21)$$

$$[y_{f,t}] \quad s_{f,t} + s_f \geq Q_f(P_f - p_t), \quad \forall f \in F, t \in T, \quad (5.22)$$

$$s_b \geq 0, \quad \forall b \in B,$$

$$s_{f,t} \geq 0 \quad \forall t \in T, f \in F. \quad (5.23)$$

Complementary slackness conditions are also satisfied as follows:

$$p_t \left(\sum_{j \in J_t} x_{j,t} \Delta Q_{j,t} + \sum_{b \in B} y_b Q_{b,t} + \sum_{f \in F} y_{f,t} Q_f - Q_t \right) = 0, \quad \forall t \in T, \quad (5.24)$$

$$s_{j,t}(1 - x_{j,t}) = 0, \quad \forall i \in J_t, t \in T, \quad (5.25)$$

$$s_b(1 - y_b) = 0, \quad \forall b \in B, b \notin B^c, \quad (5.26)$$

$$s_b(y_b - y_{b_m}) = 0, \quad \forall b \in B^c, b \in \Theta_{b_m}, \quad (5.27)$$

$$s_{f,t}(1 - y_{f,t}) = 0, \quad \forall f \in F, t \in T, \quad (5.28)$$

$$s_f(1 - \sum_{t \in T} y_{f,t}) = 0, \quad \forall f \in F, \quad (5.29)$$

$$x_{j,t}(s_{j,t} - \Delta Q_{j,t}(P_{j,t} - p_t) + \Delta Q_{j,t} \Delta P_{j,t} x_{j,t}) = 0, \quad \forall i \in J_t, t \in T, \quad (5.30)$$

$$y_b \left(s_b - \sum_{t \in T_b} Q_{b,t}(P_b - p_t) \right) = 0, \quad \forall b \in B, b \notin B^m, \quad (5.31)$$

$$y_b \left(s_b - \sum_{b_i \in \Theta_b} s_{b_i} - \sum_{t \in T_b} Q_{b,t}(P_b - p_t) \right) = 0, \quad \forall b \in B^m, \quad (5.32)$$

$$y_{f,t}(s_{f,t} + s_f - Q_f(P_f - p_t)) = 0, \quad \forall f \in F, t \in T. \quad (5.33)$$

In the next section, we are going to use the primal feasibility, dual feasibility and complementary slackness conditions in this section to establish several properties of any optimal solution of the model (QPR-MC).

5.3.1 Properties of an Optimal Solution of the model (QPR-MC)

The model (QPR-MC) is a concave maximization problem with a convex feasible region. Therefore, KKT conditions provide sufficient conditions for an optimal solution of the model (QPR-MC). In this section, we investigate optimality conditions of the model (QPR-MC) for each order type. We start with the optimality conditions of aggregated total curve segments. We re-state the results about block orders which are already shown in the literature (see Madani and Van Vyve (2014)). We derive the results about linked block orders and flexible orders. We explain the economic interpretation of the results. To the best of our knowledge, this is the first study where the optimality conditions of the piece-wise hourly order segments, linked and flexible orders are investigated.

5.3.1.1 New Contributions About the Aggregated Total Curve Segments

We introduce the optimality conditions of aggregated total curve segments. To the best of our knowledge, this is the first study that proposes to use the total curves in the supply demand balance problems. Theorem 3 and Lemma 6 are our contributions to the literature.

Theorem 3 relates the optimal values of the dual variable p_t with the optimal values of the x variables.

Theorem 3. *For any optimal solution (x^*, y^*) of the model (MC), the following conditions are satisfied for all $j \in J_t$, $t \in T$ where $\Delta Q_{j,t} \neq 0$:*

$$x_{0,t}^* = 1 \Rightarrow p_t^* \leq P_{0,t}, \quad t \in T,$$

$$p_t^* < P_{0,t} \Rightarrow x_{0,t}^* = 1, \quad t \in T,$$

$$x_{0,t}^* < 1 \Rightarrow P_{0,t} \leq p_t^*, \quad t \in T,$$

$$\begin{aligned}
x_{j,t}^* &= 1 \Leftrightarrow p_t^* \leq P_{(j+1),t}, \quad j \in \{1, \dots, N_t - 1\}, t \in T, \\
x_{j,t}^* &= 0 \Leftrightarrow P_{j,t} \leq p_t^*, \quad j \in \{1, \dots, N_t - 1\}, t \in T, \\
0 < x_{j,t}^* &= \alpha < 1 \Leftrightarrow p_t^* = \alpha P_{(j+1),t} + (1 - \alpha)P_{j,t}, \quad j \in \{1, \dots, N_t - 1\}, t \in T,
\end{aligned}$$

$$x_{N_t,t}^* = 1 \Rightarrow P_{N_t,t} \leq p_t^*, \quad t \in T,$$

$$P_{N_t,t} < p_t^* \Rightarrow x_{N_t,t}^* = 1, \quad t \in T,$$

$$x_{N_t,t}^* < 1 \Rightarrow p_t^* \leq P_{N_t,t}, \quad t \in T,$$

where $x_{j,t}^*$ is the accepted fraction of segment j , $j = 0$ is the index for the curtailable demand segment, $j = N_t$ is the index for curtailable supply segment, and p_t^* is the optimal dual price in period t .

Proof. We will use primal feasibility, dual feasibility and complementary slackness constraints in (5.12)-(5.33).

Assume that $\Delta Q_{j,t} \neq 0, \forall j \in J_t, t \in T$.

1. From complementary slackness condition (5.25), $s_{j,t} = 0$ for all j and t such that $x_{j,t}^* < 1$ where $j \in J_t, t \in T$.

- When $x_{j,t}^* = 0$, dual feasibility (5.19) states that $0 \geq \Delta Q_{j,t}(P_{j,t} - p_t^*)$ for $j \in J_t$. For $j \in \{0, 1, \dots, N_t - 1\}$ $\Delta Q_{j,t}$ is non-negative and $\Delta Q_{N_t,t}$ is non-positive. Therefore, $P_{j,t} \leq p_t^*$ for all $j \in \{0, 1, \dots, N_t - 1\}$ and $p_t^* \leq P_{N_t,t}$.
- If $0 < x_{j,t}^* = \alpha < 1$, complementary slackness condition (5.30) shows that $s_{j,t} = \Delta Q_{j,t}(P_{j,t} - p_t^*) - \Delta Q_{j,t}\Delta P_{j,t}\alpha$ for $j \in J_t$. For $j = 0$ and $j = N_t$, $\Delta P_{j,t} = 0$. Therefore, $p_t^* = P_{j,t}$ when $j = 0$ or $j = N_t$. For $j \in \{1, \dots, N_t - 1\}$, $p_t^* = P_{j,t} - \alpha\Delta P_{j,t} = (1 - \alpha)P_{j,t} + \alpha P_{(j+1),t}$.

2. If $x_{j,t}^* = 1$, $s_{j,t} = \Delta Q_{j,t}(P_{j,t} - p_t^* - \Delta P_{j,t}) = \Delta Q_{j,t}(P_{(j+1),t} - p_t^*)$ for all $j \in J_t, t \in T$ by (5.30). For all $j \in \{0, 1, \dots, N_t - 1\}$; $\Delta Q_{j,t} \geq 0$ and $P_{(j+1),t} - p_t^* \geq 0$ by (5.19). Moreover, $P_{(N_t+1),t} - p_t^* \leq 0$ by (5.19) since $\Delta Q_{j,t} \leq 0$. Furthermore, $P_{N_t,t} = P_{(N_t+1),t}$ and $P_{0,t} = P_{1,t}$ result in $P_{N_t} \leq p_t^*$ and $p_t^* \leq P_0$.

So far, we show that:

$$\begin{aligned}
 x_{0,t}^* = 1 &\Rightarrow p_t^* \leq P_{0,t} \quad t \in T. \\
 x_{0,t}^* < 1 &\Rightarrow P_{0,t} \leq p_t^*, \quad t \in T, \\
 x_{j,t}^* = 1 &\Rightarrow p_t^* \leq P_{(j+1),t}, \quad j \in \{1, \dots, N_t - 1\}, t \in T, \\
 x_{j,t}^* = 0 &\Rightarrow P_{j,t} \leq p_t^*, \quad j \in \{1, \dots, N_t - 1\}, t \in T, \\
 0 < x_{j,t}^* = \alpha < 1 &\Rightarrow p_t^* = \alpha P_{(j+1),t} + (1 - \alpha) P_{j,t}, \quad j \in \{1, \dots, N_t - 1\}, t \in T, \\
 x_{N_t,t}^* = 1 &\Rightarrow P_{N_t,t} \leq p_t^*, \quad t \in T, \\
 x_{N_t,t}^* < 1 &\Rightarrow p_t^* \leq P_{N_t,t}, \quad t \in T,
 \end{aligned} \tag{5.34}$$

- Let $p_t^* \leq P_{(j+1),t}$ for $j \in \{1, \dots, N_t - 1\}$, $t \in T$. Assume that $x_{j,t}^* \neq 1$ for the same segment j . Then from (5.34), $P_{j,t} \leq p_t^*$ when $x_{j,t}^* = 0$ and $p_t^* = P_{(j+1),t}x_{j,t}^* + P_{j,t}(1 - x_{j,t}^*) > P_{(j+1),t}$ when $x_{j,t}^* \in (0, 1)$. This is a contradiction because $p_t^* \leq P_{(j+1),t} < P_{j,t}$.
- Let $P_{j,t} \leq p_t^*$ for $j \in \{1, \dots, N_t - 1\}$, $t \in T$. Assume that $x_{j,t}^* \neq 0$ for the same segment j . Then from (5.34), $p_t^* \leq P_{j,t}$ when $x_{j,t}^* = 1$ and $p_t^* = P_{(j+1),t}x_{j,t}^* + P_{j,t}(1 - x_{j,t}^*) < P_{j,t}$ when $x_{j,t}^* \in (0, 1)$. This is a contradiction because $P_{(j+1),t} < P_{j,t} \leq p_t^*$.
- Let $P_{(j+1),t}\alpha + P_{j,t}(1 - \alpha) = p_t^*$ for $j \in \{1, \dots, N_t - 1\}$, $t \in T$ and $\alpha \in (0, 1)$. Assume that $x_{j,t}^* = 1$ for the same segment j and from (5.34), $p_t^* \leq P_{(j+1),t}$, which is a contradiction because $P_{(j+1),t} < p_t^* < P_{j,t}$. Then $x_{j,t}^* \neq 1$. Assume that $x_{j,t}^* = 0$ for the same segment j and from (5.34), $P_{j,t} \leq p_t^*$, which is a contradiction because $P_{(j+1),t} < p_t^* < P_{j,t}$. Then, $x_{j,t}^* \neq 0$.
- Let $P_{N_t,t} \leq p_t^*$ for $t \in T$. Assume that $x_{N_t,t}^* \neq 1$. Then from (5.34), $p_t^* \leq P_{N_t,t}$, which is a contradiction.
- Let $p_t^* < P_{N_t}$ for $t \in T$. Assume that $x_{N_t,t}^* = 1$. Then from (5.34), $P_{N_t,t} \leq p_t^*$, which is also a contradiction.

- Let $p_t^* = P_{N_t,t}$ for $t \in T$. Then from (5.34), $x_{N_t,t}^* \leq 1$.
- Let $P_{0,t} < p_t^*$ for $t \in T$. Assume that $x_{0,t}^* = 1$. Then from (5.34), $p_t^* \leq P_{0,t}$, which is a contradiction.
- Let $p_t^* < P_{0,t}$ for $t \in T$. Assume that $x_{0,t}^* < 1$. Then from (5.34), $P_{0,t} \leq p_t^*$, which is also a contradiction.
- Let $p_t^* = P_{0,t}$ for $t \in T$. Then from (5.34), $x_{0,t}^* \leq 1$.

□

In an optimal solution of the model (QPR-MC), we fully accept ITM segments, i.e., $s_{j,t}^* > 0$, reject the segments which are OTM, i.e., $s_{j,t}^* < 0$ and partially reject the ATM, i.e., $s_{j,t}^* = 0$ for all $j \in 1, \dots, N_t - 1$. Moreover, if the curtailable demand or supply segment is ATM, then the corresponding variable x may take any value.

We will define a special segment next. We will use this definition to relate components of x to each other.

Definition 9. *A partially accepted segment is called a decisive segment.*

There cannot be more than one decisive segment in one period. If segment $j + 1$ is accepted in period t , then segment j must also be accepted at that period. Similarly, if segment j in period t is rejected, then segment $j + 1$ in period t must also be rejected. We show the mathematical relations that the variables x must satisfy in any optimal solution of the model (MC) in Lemma 6 and then present the corresponding result afterwards.

Lemma 6. *In any optimal solution (x^*, y^*) of the model (MC), we have $x_{j,t}(1 - x_{(j-1),t}) = 0$ for all $j \in \{1, \dots, N_t - 1\}$ and $(1 - x_{N_t,t})(1 - x_{(N_t-1),t}) = 0, t \in T$.*

Proof. Assume that $x_{j,t} = 1$ and $x_{(j-1),t} \neq 1$ for all $j \in \{1, \dots, N_t - 1\}$ and $t \in T$. By Theorem 3, $x_{(j-1),t} \neq 1$ implies $P_{(j-1),t} \leq p_t$, and, $x_{j,t} = 1$ implies $p_t \leq P_{(j+1),t}$ for

$j \in \{1, \dots, N_t - 1\}$, $t \in T$. This results in $P_{(j-1),t} \leq p_t \leq P_{(j+1),t}$, but $P_{(j+1),t} < P_{j,t} < P_{(j-1),t}$ by definition, which is a contradiction. Thus, if $x_{j,t} = 1$ then $x_{(j-1),t} = 1$ for all $j \in J_t \setminus \{n_t^D + n_t^S + 3\}$ and $t \in T$.

Assume that $x_{(j-1),t} = 0$ and $x_{j,t} \neq 0$ for all $j \in \{1, \dots, N_t - 1\}$ and $t \in T$. By Theorem 3; $x_{j,t} \neq 0$ implies $p_t \leq P_{(j+1),t} < P_{j,t}$ and $x_{(j-1),t} = 0$ implies $P_{(j-1),t} \leq p_t$ for $j \in \{1, \dots, N_t - 1\}$, $t \in T$. This results in $P_{(j-1),t} \leq p_t \leq P_{(j+1),t}$, which is a contradiction. Thus, if $x_{(j-1),t} < 1$ then $x_{j,t} = 0$ for $j \in \{1, \dots, N_t - 1\}$, $t \in T$.

By Theorem 3, $x_{N_t,t} < 1$ implies $p_t \leq P_{N_t,t}$. Then, $x_{(N_t-1),t}$ must be 1 to satisfy $p_t \leq P_{(N_t-1),t}$. $\square$

Lemma 6 establishes a result about consecutive segments. By direct consequence of Lemma 6, we have the following result.

Theorem 4. *Decisive segments form a threshold that $x_{j,t}$ should be equal to one until the decisive segment, and zero afterwards in the model (MC) in any period t .*

5.3.1.2 New Results About Linked Block Orders

In this part, we will present some results about block orders. Madani and Van Vyve (2015) show these results for a block order b which is neither a parent nor a child block order, i.e., $b \in B, b \notin B^c, b \notin B^m$ as in Lemma 9. We extend these results for the linked block orders in Lemmas 7 and 8. To the best of our knowledge, there are no complete results about linked block orders in the literature.

At most three generations are allowed for a link family in the current European market rules^{1,2}. We show our results for a family with three generations, but the conditions can be extended for a block family with more than three generations.

We start with stating the moneyiness (see Definition 5 in Chapter 2) conditions for block orders.

¹<https://www.nordpoolgroup.com/TAS/Day-ahead-trading/Order-types/Block-bid/>

²<https://www.epias.com.tr/en/day-ahead-market/matching>

Definition 10. A block order b for a price p_t is called ITM if $\sum_{t \in T_b} Q_{b,t}(P_b - p_t) > 0$. Likewise, it is OTM if $\sum_{t \in T_b} Q_{b,t}(P_b - p_t) < 0$. When $\sum_{t \in T_b} Q_{b,t}(P_b - p_t) = 0$, b is called ATM.

We use Definition 10 in Lemmas 7 and 8 and interpret the optimal solution of the model (QPR-MC) economically.

The results about block orders reveal that the descendants of a linked block order have an impact on the acceptance of the order. We will show that ITM block orders may be rejected due to their parents and OTM block orders may be accepted due to their children in Lemmas 7 and 8.

Lemma 7. In any optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ of the model (QPR-MC), let a block order be ITM, i.e., $\sum_{t \in T_b} Q_{b,t}(P_b - p_t^*) > 0$.

- (a) if the block order b is not a child block order, then the block order b is accepted, i.e., $y_b^* = 1$ for $b \in B, b \notin B^c$;
- (b) if the block order b is a child order, then b has the same decision with its parent, i.e., $y_b^* = y_{b_m}^*$ for $b \in B^c$.

Proof. (a) The dual feasibility conditions (5.20) and (5.21) state that $s_b^* > 0$ when

$\sum_{t \in T_b} Q_{b,t}(P_b - p_t^*) > 0$, since $\sum_{b_i \in \Theta_b} s_{b_i}^* \geq 0$. Then, complementary slackness condition (5.26) indicates that $y_b^* = 1$ for all $b \in B, b \notin B^c$.

- (b) When $\sum_{t \in T_b} Q_{b,t}(P_b - p_t^*) > 0$, then the complementary slackness condition (5.27) indicates that $y_b^* = y_{b_m}^*$, where $b \in \Theta_{b_m}$.

□

By Lemma 7, a child block order may be rejected although it is ITM since its parent block order is rejected. Furthermore, this lemma also reveals that if each

order in a family is ITM, then all of the members in the family are accepted in any optimal solution of the model (QPR-MC).

In the following lemma, we show the results if a block order is OTM.

Lemma 8. Let $\sum_{t \in T_b} Q_{b,t}(P_b - p_t^*)$ be denoted by $\bar{s}_b^*$. In any optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ of the model (QPR-MC), if a block order b is OTM, i.e., $\bar{s}_b^* < 0$, then

(a) If b is a first generation order:

- (i) $y_b^* = 1$, when $\bar{s}_b^* + \sum_{b_i \in \Theta_b} \max\{0, \bar{s}_{b_i}^* + \sum_{b_j \in \Theta_{b_i}} \max\{0, \bar{s}_{b_j}^*\}\} > 0$ if $b \in B^m$ and $b \notin B^c$,
- (ii) $y_b^* = 0$, when $\bar{s}_b^* + \sum_{b_i \in \Theta_b} \max\left\{0, \bar{s}_{b_i}^* + \sum_{b_j \in \Theta_{b_i}} \max\{0, \bar{s}_{b_j}^*\}\right\} < 0$ if $b \in B^m$ and $b \notin B^c$.

(b) If b is not a first generation order:

- (i) $y_b^* = y_{b_m}^*$, when $\bar{s}_b^* + \sum_{b_i \in \Theta_b} \max\{0, \bar{s}_{b_i}^*\} > 0$, if $b \in B^c$ and $b \in B^m$, $b \in \Theta_{b_m}$,
- (ii) $y_b^* = 0$, when $\bar{s}_b^* + \sum_{b_i \in \Theta_b} \max\{0, \bar{s}_{b_i}^*\} < 0$ if $b \in B \setminus \{B^m \setminus B^c\}$,

Proof. For a third generation or a regular block order, (b)(ii) states that $\bar{s}_b^* < 0$ since $\theta = \emptyset$. Then, $y_b^* = 0$ by (5.31).

For a second generation order b with child block order b_i ; $s_{b_i}^* \geq \max\{0, \bar{s}_{b_i}^*\}$ since $s_{b_i}^*$ is a non-negative variable and $s_{b_i}^* \geq \bar{s}_{b_i}^*$. Then, (5.21) becomes $s_b^* \geq \bar{s}_b^* + \sum_{b_i \in \Theta_b} \max\{0, \bar{s}_{b_i}^*\}$. If this sum is positive (i), $s_b^* > 0$ and (5.27) indicates that $y_b^* = y_{b_m}^*$. If the sum is negative as in (ii), then (5.32) forces $y_b^* = 0$ and (ii) holds.

We can continue this argument for a first generation order b and show the results in (a). □

Lemma 7 states that the last generation order follows the same decision with its parent order, if it is ITM. We have also shown in Lemma 8 that if the third generation order is OTM then the order is rejected. There is no direct result about what happens when the order is at-the-money, where the bidder is indifferent between actions.

The following lemma is a special case of Lemma 7 (a) and Lemma 8 (b). These conditions are also covered in Madani and Van Vyve (2015).

Lemma 9. *In any optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ of the model (QPR-MC), for all $b \in B, b \notin B^c, b \notin B^m$;*

$$(a) \text{ If } \sum_{t \in T_b} Q_{b,t}(P_b - p_t^*) > 0, \text{ then } y_b^* = 1 .$$

$$(b) \text{ If } y_b^* \neq 0, \text{ then } \sum_{t \in T_b} Q_{b,t}(P_b - p_t^*) \geq 0.$$

In other words by Lemma 9: (a) If a block order without a link relation is ITM then it is (fully) accepted in the optimal solution (If a block order b is not (fully) accepted then it is not ITM, it could be ATM if it is partially accepted, or OTM if it is rejected) and (b) A block order which is not the parent of another block order is rejected if it is OTM.

We demonstrate a toy example to clarify the rules in Lemma 8. We use family 3 we presented in Figure 5.1 as our example. Suppose that $\bar{s}_b^*$ is -1, -1, 5, and -20 for b_1, b_2, b_3 , and b_4 , respectively. Then, Lemma 8 (a-i) states $y_{b_1} = 1$ since $\bar{s}_{b_1} + \max\{0, \bar{s}_{b_2} + \max\{0, \bar{s}_{b_3}\} + \max\{0, \bar{s}_{b_4}\}\} = -1 + \max\{0, -1 + \max\{0, -1\} + \max\{0, 5\}\} = -1 + 4 = 3 > 0$. Similarly, $y_{b_1} = y_{b_2} = y_{b_4} = 1$ since $\bar{s}_{b_2} + \max\{0, \bar{s}_{b_3}\} + \max\{0, \bar{s}_{b_4}\} = -1 + 5 = 4 > 0$ by Lemma 8 (b-i). Finally, $y_{b_4} = 0$ by Lemma 8 (b-ii).

The loss of the first generation block order is compensated by its descendants in Lemmas 8. Properties in 8(a-ii) and (b-ii) are the rules used by European power exchanges, which are interpreted as “the child can save the parent with its surplus but not the opposite” as stated in Euphemia (2016).

Suppose there exists another allocation than the conditions in Lemma 7 and 8 that results in a better profit. Then, there must be at least one block order which is ITM but it is rejected. This happens when the parent order is rejected. However, we know that a parent order is rejected when its children could not save the parent by Lemma 8. Hence, the total profit obtained from the parent is negative. Then, there does not exist another profit maximizing allocation than the conditions in Lemmas 7 and 8.

5.3.1.3 Our Contributions About Flexible Orders

In this part, we show the optimality conditions for the remaining order type. To the best of our knowledge, this is the first study that derives the optimality conditions of flexible orders in the model (QPR-MC). We again start with the moneyiness (see Definition 5 in Chapter 2) of the flexible orders.

Definition 11. A flexible order is OTM in period t if $(P_f - p_t)Q_f < 0$, and ATM if $(P_f - p_t)Q_f = 0$. Similarly, it is ITM if $(P_f - p_t)Q_f > 0$ holds.

We provide the economic interpretations of the optimal values of flexible orders in the model (QPR-MC). By Lemma 10, a flexible order f is rejected in $\bar{t}$ if it is OTM in $\bar{t}$.

Lemma 10. In any optimal solution $(\mathbf{x}, \mathbf{y})$ of the model (QPR-MC), if a flexible order f is not rejected in $\bar{t}$, i.e., $y_{f\bar{t}} \neq 0$ then it is not OTM in $\bar{t}$, i.e., $Q_f(P_f - p_{\bar{t}}) \geq 0$.

Proof. We know that $y_{f\bar{t}} \neq 0$ and by (5.33), $s_{f\bar{t}} + s_f = Q_f(P_f - p_{\bar{t}})$, which is non-negative by (5.23). Hence, f is not OTM in $\bar{t}$. $\square$

The following lemma reveals the conditions about an optimal solution when the flexible order is not fully accepted over the periods, i.e., $\sum_{t \in T} y_{f,t} \neq 1$.

Lemma 11. *In any optimal solution $(\mathbf{x}, \mathbf{y})$ of the model (QPR-MC), if $\sum_{t \in T} y_{f,t} \neq 1$ for a flexible order f , then $Q_f(P_f - p_t) \leq 0$ for all $t \in T$. In other words, if a flexible order is not fully accepted in any of the periods, it is not ITM in all periods.*

Proof. We know that $\sum_{t \in T} y_{f,t} \neq 1$, i.e., $y_{f,t} \neq 1, \forall t \in T$. By (5.28), we obtain $s_{f,t} = 0$ for all $t \in T$, and (5.29) implies $s_f = 0$. By (5.22),

$$0 = s_{f,t} + s_f \geq Q_f(P_f - p_t), \forall t \in T.$$

□

By Lemma 11, a flexible order f is not accepted over at least one period, if it is ITM in at least one of the periods. This lemma does not provide any condition about the period that f is fully accepted, but we know that $\sum_{t \in T} y_{f,t} = 1$ when it is ITM for some periods. The following lemma gives a characterization of periods during which f must not be rejected. However, it still does not provide a condition by which f is fully accepted in a period.

Lemma 12. *Let T_i be the set of periods where flexible order f is ITM, i.e., $Q_f(P_f - p_t) > 0$ for all $t \in T_i$ where $T_i \subset T$. In any optimal solution $(\mathbf{x}, \mathbf{y})$ of the model (QPR-MC), f is rejected in $T \setminus T_i$ (i.e., $y_{f,t} = 0, t \in T \setminus T_i$) and not rejected in at least one period in T_i (i.e., $\sum_{t \in T_i} y_{f,t} > 0$).*

Proof. If f is ITM in T_i , then it is not ITM in $T \setminus T_i$, i.e., $Q_f(P_f - p_t) > 0, \forall t \in T_i$ and $Q_f(P_f - p_t) \leq 0, \forall t \in T \setminus T_i$. By (5.22), $s_{f,t} + s_f > 0$, for all $t \in T_i$. Then, either $s_{f,t} = 0$ for all $t \in T_i$ or $s_{f\bar{t}} > 0$ for a particular $\bar{t} \in T_i$.

1. If $s_{f,t} = 0$ for all $t \in T_i$, then $s_f > 0$ to satisfy $s_{f,t} + s_f > 0$, for all $t \in T_i$. Assume that $0 < y_{f\bar{t}} \leq 1$ and $\bar{t} \in T \setminus T_i$. Then, $s_{f\bar{t}} + s_f = Q_f(P_f - p_{\bar{t}})$ by Eq. (5.33). Since $s_f > 0$, then $Q_f(P_f - p_{\bar{t}}) > 0$ which happens only when $\bar{t} \in T_i$, which is a contradiction. Then, $y_{f,t} = 0$ for all $t \in T \setminus T_i$ and $y_{f,t} \neq 0$ for at least one $t \in T_i$ because $\sum_{t \in T} y_{f,t} = 1$.

2. If $s_{f\bar{t}} > 0$ for a particular $\bar{t} \in T_i$, then $y_{f\bar{t}} = 1$ and $y_{f,t} = 0$ for all $t \in T \setminus \{\bar{t}\}$

□

The following lemma shows that if a flexible order is accepted then it is accepted in a period where it brings the maximum gain.

Lemma 13. *In any optimal solution $(\mathbf{x}, \mathbf{y})$ of the model (QPR-MC), if a flexible sales (purchase) order f is accepted in a period $\bar{t}$ so that $y_{f\bar{t}} = 1$, then $\bar{t}$ has the highest (lowest) (MC)P.*

Proof. For a problem where there are m periods, (5.22) yields:

$$\begin{aligned} s_{ft_1} + s_f &\geq Q_f(P_f - p_{t_1}) \\ s_{ft_2} + s_f &\geq Q_f(P_f - p_{t_2}) \\ &\dots \\ s_{ft_m} + s_f &\geq Q_f(P_f - p_{t_m}) \end{aligned} \tag{5.35}$$

Let $\bar{t}$ be the period where f is accepted, i.e., $y_{f\bar{t}} = 1$ and $y_{f,t} = 0 \forall t \in T, t \neq \bar{t}$. Then, $s_{f,t} = 0$ for $t \in T \setminus \{\bar{t}\}$ by (5.28). The inequalities in (5.35) can be simplified as $s_f \geq \max_{t \in T \setminus \{\bar{t}\}} Q_f(P_f - p_t)$. Then $Q_f(P_f - p_{\bar{t}}) \geq \max_{t \in T \setminus \{\bar{t}\}} Q_f(P_f - p_t)$ because $s_{f\bar{t}} + s_f = Q_f(P_f - p_{\bar{t}})$ by (5.33) and $s_{f,t} \geq 0$ for all $t \in T$. Therefore, f is fully accepted in a period where it has the highest surplus, which is a period with the highest and the lowest (MC)P for sales and purchase order f , respectively. □

Now, we explain the meaning of the variables s in the dual problem.

Lemma 14. $\sum_{t \in T} s_{f,t} + s_f$ is the total surplus of flexible order f .

Proof. • Assume that f is fully accepted in $\bar{t}$, i.e., $y_{f\bar{t}} = 1$. By (5.33), $s_{f\bar{t}} + s_f = Q_f(P_f - p_{\bar{t}})$, which defines the surplus of f . It is clear that $s_{f,t} = 0$ for $t \in T \setminus \{\bar{t}\}$ by (5.28).

- Assume that f is partially accepted in T_i . Then, by (5.28), $s_{f,t} = 0 \forall t \in T_i$. By (5.33), $s_{f,t} + s_f = s_f = Q_f(P_f - p_t)$, $\forall t \in T_i$ which defines the surplus of f .
- Assume that f is rejected, i.e., $y_{f,t} = 0$, $\forall t \in T$ and $\sum_{t \in T} y_{f,t} = 0$. The surplus of b must be zero. By (5.28) and (5.29), $s_f = 0$ and $s_{f,t} = 0$.

Hence, $\sum_{t \in T} s_{f,t} + s_f$ is the total surplus of f . $\square$

Note that, the proof of Lemma 14 also shows that when f is partially accepted, p_t values are the same in at least two periods.

In this section, we show the relation between the optimal solution $(\mathbf{x}, \mathbf{y})$ and dual variable p_t . We know that the constraint given by (5.1) clears the market for all $t \in T$. The shadow price p_t of each balance constraint reflects the clearing price. We discussed that the optimality conditions of the model (QPR-MC) provide profit maximizing allocation for order owners. Hence, the market equilibrium (see Definition 1) is achieved in the model (QPR-MC) under the assumption that the bidders submit their marginal costs and benefits. Another desirable property of the model (QPR-MC) is that there is no PRB or PAB (see Definition 6). However, integrality conditions are essential in electricity markets. Therefore, the model (QPR-MC) is not a realistic model but it provides an economic interpretation of the model without any integrality constraints. This will help us to define variants of supply demand balance problem in the next chapter.

5.4 Restricted Supply Demand Balance Problem

In the previous section, we showed a model where Walrasian equilibrium is achieved and market clearing prices are the shadow prices of supply-demand balance constraints. However, non-convexities are inevitable in day-ahead markets due to start-up and ramping constraints of some power plants as we discussed in Chapter 2. There are plenty of ways to deal with these non-convexities in the literature (See O'Neill et al. (2005), Bjørndal and Jörnsten (2008), Van Vyve et al. (2011), Ruiz et al. (2012)).

An approach is fixing binary variables to certain values and obtaining a convex feasible region. O'Neill et al. (2005) and Bjørndal and Jörnsten (2008) fix the binary variables to their optimal values and show that the shadow prices for the clearing constraint is the market clearing price. Hogan and Ring (2003) refer this approach as “IP pricing”. These studies focus on unit commitment models. Zak et al. (2012) and Madani and Van Vyve (2015) extend these results for European exchange models.

In this section, we introduce the restricted supply demand balance problem, where we fix binary variables to certain values and investigate its properties.

First, we fix variables y_b to 1 or 0 by preserving link feasibility and flexible order acceptance constraints. Let B_1 be the set of block orders which are forced to 1 and B_0 be the set of block orders which are fixed to 0. Likewise, F_1 is the set of flexible orders which are accepted, and F_0 is the set of flexible orders which are rejected. Finally, we let T_{1f} be the set of periods in which flexible order f is accepted and T_{0f} be the set of periods in which flexible order f is rejected. The corresponding formulation is provided below.

(R-MC)

$$\max c(\mathbf{x}, \mathbf{y})$$

$$s.t.$$

$$[v_{b_1}] \quad 1 \leq y_{b_1}, \quad \forall b_1 \in B_1, \quad (5.36)$$

$$[v_{b_0}] \quad y_{b_0} \leq 0, \quad \forall b_0 \in B_0, \quad (5.37)$$

$$[v_{f_1, t_1}] \quad 1 \leq y_{f_1, t_1}, \quad \forall t_1 \in T_{1f_1}, f_1 \in F_1, \quad (5.38)$$

$$[v_{f, t_0}] \quad y_{f, t_0} \leq 0, \quad \forall t_0 \in T_{0f}, f \in F, \quad (5.39)$$

$$(\mathbf{x}, \mathbf{y}) \in C_1 \quad (5.40)$$

where C_1 is the feasible region of the model (QPR-MC) defined by (5.12)-(5.18) and the function c is as defined in Lemma 3. The model (R-MC) is a QP problem, whose objective function is concave and all decision variables are continuous. If the shadow prices of the constraints are as in the brackets, then the equivalent Dorn's

dual problem can be defined as follows.

(DR-MC)

$$\begin{aligned} \min \quad & \sum_{j \in J_t, t \in T} s_{j,t} + \sum_{b \in B, b \notin B^c} s_b + \sum_{f \in F} s_f + \sum_{f \in F, t \in T} s_{f,t} - \sum_{b \in B_1} v_b - \sum_{f \in F_1, t \in T_{1f_1}} v_{f,t} \\ & - \sum_{t \in T} Q_t p_t + \sum_{t \in T, j \in J_t} \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \end{aligned}$$

$$[x_{j,t}] \quad s_{j,t} \geq \Delta Q_{j,t}(P_{j,t} - p_t) - \Delta Q_{j,t} \Delta P_{j,t} x_{j,t}, \quad \forall i \in J_t, t \in T, \quad (5.41)$$

$$[y_{b_1}] \quad s_{b_1} - \sum_{b_i \in \Theta_{b_1}} s_{b_i} - v_{b_1} \geq \sum_{t \in T} Q_{b_1,t}(P_{b_1} - p_t), \quad \forall b_1 \in B_1, \quad (5.42)$$

$$[y_{b_0}] \quad s_{b_0} - \sum_{b_i \in \Theta_{b_0}} s_{b_i} + v_{b_0} \geq \sum_{t \in T} Q_{b_0,t}(P_{b_0} - p_t), \quad \forall b_0 \in B_0, \quad (5.43)$$

$$[y_{f_1, t_1}] \quad s_{f_1, t_1} + s_{f_1} - v_{f_1, t_1} \geq Q_{f_1}(P_{f_1} - p_{t_1}), \quad \forall f_1 \in F_1, t_1 \in T_{1f_1}, \quad (5.44)$$

$$[y_{f, t_0}] \quad s_{f, t_0} + s_f + v_{f, t_0} \geq Q_f(P_f - p_{t_0}), \quad \forall f \in F, t_0 \in T_{0f}, \quad (5.45)$$

$$s_b \geq 0, v_b \geq 0 \quad \forall b \in B, \quad (5.46)$$

$$s_{f,t} \geq 0, s_f \geq 0, v_{f,t} \geq 0, \quad \forall f \in F, t \in T. \quad (5.46)$$

Complementary slackness conditions must hold:

$$p_t \left(\sum_{j \in J_t} x_{j,t} \Delta Q_{j,t} + \sum_{b \in B} y_b Q_{b,t} + \sum_{f \in F} y_{f,t} Q_f - Q_t \right) = 0, \quad \forall t \in T, \quad (5.47)$$

$$s_{j,t}(1 - x_{j,t}) = 0, \quad \forall j \in J_t, t \in T, \quad (5.48)$$

$$s_b(1 - y_b) = 0, \quad \forall b \in B \setminus B^c, \quad (5.49)$$

$$s_b(y_b - y_{b_m}) = 0, \quad \forall b \in \Theta_{b_m}, b_m \in B^m, \quad (5.50)$$

$$s_{f,t}(1 - y_{f,t}) = 0, \quad \forall f \in F, t \in T, \quad (5.51)$$

$$s_f \left(1 - \sum_{t \in T} y_{f,t} \right) = 0, \quad \forall f \in F, \quad (5.52)$$

$$v_{b_1}(1 - y_{b_1}) = 0, \quad \forall b_1 \in B_1, \quad (5.53)$$

$$v_{b_0}y_{b_0} = 0, \quad \forall b_0 \in B_0, \quad (5.54)$$

$$v_{f_1,t_1}(1 - y_{f_1,t_1}) = 0, \quad \forall f_1 \in F_1, t_1 \in T_{1f_1}, \quad (5.55)$$

$$v_{f,t_0}y_{f,t_0} = 0, \quad \forall f \in F, t_0 \in T_{0f_1}, \quad (5.56)$$

$$x_{j,t}(s_{j,t} - \Delta Q_{j,t}(P_{j,t} - p_t) + \Delta Q_{j,t}\Delta P_{j,t}x_{j,t}) = 0, \quad \forall j \in J, t \in T, \quad (5.57)$$

$$y_{b_1} \left(s_{b_1} - v_{b_1} - \sum_{b_i \in \Theta_{b_1}} s_{b_i} - \sum_{t \in T} Q_{b_1,t}(P_{b_1} - p_t) \right) = 0, \quad \forall b_1 \in B_1, \quad (5.58)$$

$$y_{b_0} \left(s_{b_0} + v_{b_0} - \sum_{b_i \in \Theta_{b_0}} s_{b_i} - \sum_{t \in T} Q_{b_0,t}(P_{b_0} - p_t) \right) = 0, \quad \forall b_0 \in B_0, \quad (5.59)$$

$$y_{f_1,t_1}(s_{f_1,t_1} + s_{f_1} - v_{f_1,t_1} - Q_{f_1}(P_{f_1} - p_{t_1})) = 0, \quad \forall f_1 \in F, t_1 \in T_{1f_1}, \quad (5.60)$$

$$y_{f,t_0}(s_{f,t_0} + s_f + v_{f,t_0} - Q_f(P_f - p_{t_0})) = 0, \quad \forall f \in F, t_0 \in T_{0f_1}. \quad (5.61)$$

It can be easily proven that Theorem 3 and Lemma 6 are still valid for the model (R-MC). In the next theorem, we show that there exists an optimal solution for the restricted problem where the optimal values for s_b and v_b variables are complementary for all $b \in B$. Similarly, the product of the optimal values for $s_{f,t}$ and $v_{f,t}$ is zero.

Theorem 5. *For a feasible restricted problem defined by B_0 , B_1 , F_0 , F_1 , T_{1f} and T_{0f} for all $f \in F$, there exists a dual optimal solution which satisfies the following relations:*

$$s_b = \begin{cases} 0, & \text{if } b \in B_0, \\ \max \left\{ 0, \sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} \right\}, & \text{if } b \in B_1, \end{cases} \quad (5.62)$$

$$s_f = \begin{cases} 0, & \text{if } f \in F_0, \\ \max \left\{ 0, \max_{t \in T_{0f}} \{(P_f - p_t)Q_f\} \right\}, & \text{if } f \in F_1, \end{cases} \quad (5.63)$$

$$s_{f,t} = \begin{cases} 0, & \text{if } t \in TF_{0f}, \\ \max \{0, (P_f - p_t)Q_f - s_f\}, & \text{if } t \in TF_{1f}, \end{cases}, \forall f \in F. \quad (5.64)$$

$$v_b = \begin{cases} \max \left\{ 0, \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} \right\}, & \text{if } b \in B_0, \\ \max \left\{ 0, - \left(\sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} \right) \right\}, & \text{if } b \in B_1, \end{cases} \quad (5.65)$$

$$v_{f,t} = \begin{cases} 0, & \text{if } f \in F_1, t \in TF_{0f} \\ \max \{0, s_f - (P_f - p_t) Q_f\}, & \text{if } f \in F_1, t \in TF_{1f}, \\ \max \{0, (P_f - p_t) Q_f\}, & \text{if } f \in F_0, t \in T, \end{cases} \quad (5.66)$$

Proof. Taking the maximum of the terms and zero ensures that variables s and v are non-negative. Now, we build the solution given in the theorem.

- Consider $b \in B_0$. Then, $y_b = 0$. Hence, (5.49) implies that $s_b = 0$ for $b \notin B^c$. For a child order b whose parent is b_m , $y_{b_m} = 1$ and $y_b = 0$, yielding $s_b = 0$ by (5.50). When $y_{b_m} = y_b = 0$, s_b can take any non-negative value, therefore, $s_b = 0$ is valid. This justifies (5.62) for $b \in B_0$.
- We have shown that $s_b = 0$ is a valid optimal solution for $b \in B_0$. If $\sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}$ is positive, v_b must be at least $\sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}$ to satisfy (5.43). Otherwise, v_b can take any non-negative value. Hence, (5.65) holds for $b \in B_0$.
- Consider $b \in B_1$. Then, $y_b = 1$. Therefore, (5.58) forces $s_b - v_b$ to $\sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}$. If $\sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} < 0$, then v_b must be positive. There are infinitely many solutions that satisfy this condition and $s_b = 0$ and $v_b = - \left(\sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} \right)$ is one of them. If $\sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} > 0$, then $v_b = 0$ and $s_b = \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}$ yield a valid solution. These justify (5.62) and (5.65) for $b \in B_1$.

- Consider $f \in F_0$, i.e. $y_{f,t} = 0$ for all $t \in T$, (5.52) forces s_f to zero. So, (5.63) holds for $f \in F_0$.
- Consider $f \in F_1$, i.e. $\sum_{t \in T} y_{f,t} = 1$. Then, $y_{f,t} = 1$ for $t \in TF_{1f}$ and $y_{f,t} = 0$ for all $t \in TF_{0f}$. Hence, $s_{f,t} = 0$ for all $t \in TF_{0f}$ by (5.51). So, (5.64) holds for $t \in TF_{0f}$.
- Consider, $f \in F_1$ and $t \in TF_{1f}$. Then, $y_{f,t} = 1$ and $s_f + s_{f,t} - v_{f,t} = (P_f - p_t)Q_f$ by (5.60). Assume that $(P_f - p_t)Q_f - s_f > 0$. This yields that $v_{f,t} = (P_f - p_t)Q_f - s_f$. So, (5.65) holds for $f \in F_1$, $t \in TF_{1f}$. Now consider, $f \in F_0$. It follows that $s_f = 0$, $s_{f,t} = 0$ and $v_{f,t} = \max\{0, (P_f - p_t)Q_f\}$ by (5.45). Hence, (5.65) holds for $f \in F_0$. Furthermore, $v_{f,t} = 0$ for $t \in TF_{0f}$ is satisfied. So, (5.65) holds for $f \in F_1$ and $t \in TF_{0f}$.
- We have shown that $s_{f,t} = 0$, $v_{f,t} = 0$ for all $t \in TF_{1f}$ in an optimal solution when $f \in F_1$. The equality $s_f = \max_{t \in T_{0f}} \{(P_f - p_t)Q_f\}$ satisfies (5.45) for all $t \in TF_{0f}$ for all $f \in F_1$. This justifies (5.63) for $f \in F_1$.

□

There may be alternative optimal solutions with noncomplementary s and v values. However, we can always modify them so that the complementarity holds in the optimal solution.

By Theorem 5, dual variable s is related with the surplus of an order. First, variable s_b corresponds to the surplus of the block orders. The surplus of a parent order is affected by the surplus arising from its children. If an order is rejected, its surplus is zero, as expected. For an accepted flexible order f , s_f shows the maximum non-negative opportunity cost that f can gain in a period other than the one where it is accepted. Similarly, $s_{f,t}$ is the additional surplus that order has gained when it is accepted in t .

The dual variables v represent the loss of an accepted order or opportunity cost of a rejected order. Madani and Van Vyve (2014) also state that v_b is a bound on the loss or opportunity cost of a block order. We extend this relation for linked block and flexible orders.

When v is positive, the corresponding order is either PAB or PRB. Therefore, there may be PRB or PAB in an optimal solution of the model (R-MC). In addition, strong duality holds by Lemma 4.

Each problem in an incumbent node of the branch-and-bound tree for the model (MC) can be modeled by (R-MC). Note that we can show the equivalence by fixing B_0/B_1 (F_0/F_1) to the set of rejected/accepted block (flexible) orders in the incumbent solution. T_{1f} is given by the set consisting of the accepted period of an existing flexible f in the incumbent solution and T_{0f} shows the rest of the periods for f . Therefore, the results for any optimal solution of the model (R-MC) are also valid for any optimal solution of the model (MC). Any optimal solution of the model (MC) may contain PRBs and PABs, which distorts Walrasian equilibrium. Thus, energy exchanges in Europe embrace additional rules to overcome this problem. We will cover this topic in Chapter 6.

5.5 Summary

In this chapter, we define a supply demand balance problem and propose an MIQP model for this problem, which we denoted by (MC). Since there are integer variables in the model (MC), the feasible region is not convex. We present two models with a convex feasible. We use QP relaxation and fixing binary variables to some binary values to convert the non-convex feasible region of the model (MC) into a convex set. We investigate the properties of an optimal solution of the resulting problems by using duality.

We show that there may be PAB and PRB in an optimal solution of the model (MC), where Walrasian equilibrium fails. Power exchanges adopt some additional

rules arising from properties of QP relaxation of the model (MC) to reach a competitive market environment. In the next chapter, present the problem variants of supply demand balance problem. We summarize the symbols we defined in this section in Table 5.4 for future reference.

Table 5.4: Sets, indices, parameters and variables used in this chapter

Sets and Indices	
b, B	Block order index and set of block orders.
B^S, B^D	Sets of sales and purchase block orders, i.e., $B = B^S \cup B^D$.
Θ_b	Set of child orders of block order b , where $b \in B$.
ϕ_b	Set of block orders in subfamily of a block order b , where $b \in B$.
Φ_b	Set of subfamily sets of block order b , $b \in B$.
ξ_b	Set of orders that shows descendants of b , $b \in B$.
B^c, B^m	Set of child block orders and parent block orders.
B_0, B_1	Set of block orders which are fixed to reject and accept, respectively.
f, F	Flexible order index and set of flexible orders.
F^S, F^D	Set of sales and purchase flexible orders, i.e., $F = F^S \cup F^D$.
F_0, F_1	Set of flexible orders which are fixed to reject and accept, respectively.
t, T_b	Time period index and set of time periods that block order b is valid.
TF_{0f}, TF_{1f}	Set of periods that flexible order f is fixed to reject and accept, respectively.

Parameters

$P_b, Q_{b,t}$	Price and quantity of block order b , where $b \in B$. $Q_{b,t} < 0$ for $b \in B^S$ and $Q_{b,t} > 0$ for $b \in B^D$.
P_f, Q_f	Price and quantity of flexible order f , where $f \in F$. $Q_f < 0$ for $f \in F^S$ and $Q_f > 0$ for $f \in F^D$.
N_b	Number of periods for which block order b is valid, i.e. $N_b = T_b $ for all $b \in B$.

Decision Variables

$y_b =$	$\begin{cases} 1, & \text{if block order } b \text{ is accepted} \\ 0, & \text{otherwise} \end{cases}$ for all $b \in B$.
$y_{f,t}$	$\begin{cases} 1, & \text{if flexible order } f \text{ is accepted in period } t \\ 0, & \text{otherwise} \end{cases}$ for all $f \in F, t \in T$.
(x, y)	Vector of decision variables, where the dimension of y is n
p_t	Market clearing price for period $t, t \in T$. (Dual variable of supply demand balance constraint)
$s_{j,t}$	Dual variable of bounding constraint of variable $x_{j,t}, j \in J_t, t \in T$.
s_b	Surplus of block order $b, b \in B$. (Dual variable of bounding constraint of variable y_b).
$s_{f,t}$	Additional surplus coming from period t if it is accepted in period t , for all $f \in F, t \in T$ (Dual variable of bounding constraint of variable f in period t).

- s_f Opportunity cost of flexible order f coming from the all periods except for the one where it is accepted, for all $f \in F$ (Dual variable of the constraint showing the flexible order starting period limit).
- v_b Opportunity cost/loss due to acceptance/rejection of block order b , $b \in B$ (Dual variable of fixing constraint of variable y_b).
- $v_{f,t}$ Opportunity cost of a rejected flexible order f or loss of an accepted flexible order if f is accepted in period for all t , $f \in F$, $t \in T$ (Dual variable of fixing constraint of variable $y_{f,t}$).

Chapter 6

VARIANTS OF SUPPLY DEMAND BALANCE PROBLEM

The supply demand balance problem is a kind of combinatorial auction mechanism. An optimal solution of this problem may contain both PABs and PRBs, which are undesired outcomes. It is important to prevent these solutions if it is possible. We can restrict the problem to have at most one undesired outcome. As we discussed in Chapter 2, if the market design avoids both PABs and PRBs, the resulting problem may be infeasible, which leads us to two different variants of the supply demand balance problem.

In this chapter, we focus on problem variants of the supply demand balance problem. We first propose a new variant of the supply demand balance problem and present a corresponding optimization formulation. We name this problem as the *supply demand balance problem without PRBs*. It is a new combinatorial auction mechanism which is in Walrasian equilibrium. This mechanism differs from Turkish DAM and the related literature in terms of matching and uplift (side payment) rules.

We also provide another optimization model for a different variant of the supply demand balance problem, which is currently used in the European power exchanges such as Nord Pool Spot. We refer to this problem as the *supply demand balance problem without PABs*, where the equilibrium fails. Although Martin et al. (2014) consider block, linked block and flexible orders as we do, their formulation does not reflect the same rules used in the Nord Pool Spot in terms of matching conditions of linked block orders.

We use the model (MC) as a base model for our problem variants, which is based

on aggregated total curves. For each of the problem variants, we propose two different models. The first one includes a quadratic constraint, whereas the second one contains only linear constraints. We extend the formulation used in Madani and Van Vyve (2015) to introduce both linked block and flexible orders to model the supply demand balance problem without PAB.

6.1 Supply Demand Balance Problem Variants

Proposing pricing mechanisms for non-convex DAMs where strong duality fails is a challenging task and extensive research has been conducted on this topic. The supply demand balance problem we discussed in the previous chapter is not a sufficient market clearing mechanism because an optimal solution may contain both PRBs and PABs. Unfortunately, there is no market design known so far which eliminates both PABs and PRBs while satisfying the feasibility under any parameter set. Hence, there are two possible variants arising from this market model, where the first one eliminates PRBs and the other one avoids PABs.

The market design that avoids PABs is being used in European power exchanges. However, to the best of our knowledge, there is no market design currently used which eliminates PRBs while satisfying the Walrasian equilibrium. Therefore, in this section we first propose a market design which eliminates PRBs and show that this market design satisfies the Walrasian equilibrium. We refer to this market design as the *supply demand balance problem without PRBs*. Moreover, we also present the market design used in European power exchanges that avoids PABs, which we refer to as *the supply demand balance problem without PABs*.

6.1.1 Supply Demand Balance Problem Without PRBs

In this section, we introduce the supply demand balance problem without PRBs. Then, we present two different formulations to model the problem: the first one contains a quadratic constraint whereas the other one does not.

To obtain a market design that does not have PRBs, the dual problem arising from the QP relaxation of the model (MC) is used. The properties obtained from the primal-dual relation are incorporated to the supply demand balance problem.

The supply demand balance problem without PRBs is a surplus maximization problem. In addition to the constraints defining the supply demand balance problem, a feasible solution of the supply demand balance problem without PRBs must satisfy the following conditions.

1. A segment is fully accepted if it is ITM, rejected if it is OTM, and partially accepted if it is ATM (Theorem 3).
2. A block order which is not a child of another order is accepted if it is ITM (Lemma 7(a)).
3. A child block order must follow the same decision with its parent if the child order is ITM (Lemma 7(b)) or the children of the child block order can save that child with their positive surplus (Lemma 8(b)-i).
4. A first generation block order is accepted if its descendants save the first generation order with their positive surplus (Lemma 8(a)-i).
5. A flexible order is accepted, if the order is ITM in at least one of the periods during which it is valid (Lemma 11).

Each of the above conditions is actually a property that any optimal solution of the model (QPR-MC) must satisfy. In the previous chapter, we established these conditions in the lemmas that are shown in the brackets. However, when integrality conditions are imposed on the block and flexible orders, the conditions defined in 2-5 may not be satisfied as we discussed in Section 5.4. Thus, we add these conditions to the supply demand balance problem to ensure that each feasible solution of the problem does not yield a PRB.

In this model formulation, there may be accepted orders which are OTM. Hence, we propose compensating the loss of accepted block and flexible orders with an uplift in our market design. We define uplift for each order as defined in Theorem 5. We propose to collect this uplift from the market participants as a pro-rata¹ payment according to their earnings in this auction.

In next section, we formulate the problem proposed in this section.

6.1.1.1 Mathematical Model of the Supply Demand Balance Problem Without PRB with a Quadratic Constraint

The supply demand balance problem without PRBs stems from the supply demand balance problem. We showed that we can solve this problem by employing the model (MC). Therefore, we add the constraints corresponding to the conditions 1-5 to the model (MC). In order to convert these conditions into the inequalities, we focus on the problem where these conditions are derived. This problem is restricted supply demand balance problem we modeled in Chapter 5 as the model (R-MC).

The model (R-MC) relies on the assumption that we know the accepted and rejected block and flexible orders a priori. The model (DR-MC) yields the optimal price and the opportunity cost of rejected orders and loss of accepted orders. We add all of these dual constraints to the model (MC). However, we do not want any block or flexible order which is rejected with a positive opportunity cost in the variant of the problem without PRB. Hence, we force the decision variable corresponding to the opportunity cost to zero when the order is rejected. Recall that the strong duality holds in the restricted problem. We link the primal variable with the dual variable of the model (R-MC) by adding the strong duality condition of the model (R-MC) to the model (MC). This is a similar formulation to the one proposed by Madani and Van Vyve (2015). In contrast to our model, Madani and Van Vyve (2015) use another constraint set and do not use the model (MC) as a base model.

¹Latin, proportion

We next present the our model (MC-WPRB-Q). In this model the variables v_b and $v_{f,t}$ represent the financial compensation needed for all $b \in B$, and $f \in F$, $t \in T$.

(MC-WPRB-Q)

$$\max c(x, y)$$

s.t.

$$\sum_{j \in J_t} \Delta Q_{j,t} x_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} = Q_t, \quad \forall t \in T, \quad (6.1)$$

$$y_b \leq y_{b_m}, \quad \forall b \in B^c, b \in \Theta_{b_m}, \quad (6.2)$$

$$\sum_{t \in T} y_{f,t} \leq 1, \quad \forall f \in F, \quad (6.3)$$

$$x_{j,t} \leq 1, \quad \forall j \in J_t, t \in T, \quad (6.4)$$

$$s_{j,t} \geq \Delta Q_{j,t}(P_{j,t} - p_t) - \Delta Q_{j,t} \Delta P_{j,t} x_{j,t}, \quad \forall j \in J_t, t \in T, \quad (6.5)$$

$$s_b - v_b \geq \sum_{t \in T_b} Q_{b,t}(P_b - p_t), \quad \forall b \in B \setminus B^m, \quad (6.6)$$

$$s_b - \sum_{b_i \in \Theta_b} s_{b_i} - v_b \geq \sum_{t \in T_b} Q_{b,t}(P_b - p_t), \quad \forall b \in B^m, \quad (6.7)$$

$$s_{f,t} + s_f - v_{f,t} \geq Q_f(P_f - p_t), \quad \forall f \in F, t \in T, \quad (6.8)$$

$$\begin{aligned} c(x, y) \geq & \sum_{j \in J_t, t \in T} s_{j,t} + \sum_{b \in B, b \notin B^c} s_b + \sum_{f \in F} s_f \\ & + \sum_{f \in F, t \in T} s_{f,t} - \sum_{b \in B} v_b - \sum_{f \in F, t \in T} v_{f,t} \\ & - \sum_{t \in T} p_t Q_t + \sum_{t \in T, j \in J_t} \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \end{aligned} \quad (6.9)$$

$$M_b y_b \geq v_b, \quad \forall b \in B \quad (6.10)$$

$$M_{f,t} y_{f,t} \geq v_{f,t}, \quad \forall f \in F, t \in T \quad (6.11)$$

$$s_{j,t} \geq 0, x_{j,t} \geq 0, \quad \forall t \in T, j \in J_t, \quad (6.12)$$

$$y_b \in \{0, 1\}, s_b \geq 0, v_b \geq 0, \quad \forall b \in B, \quad (6.13)$$

$$y_{f,t} \in \{0, 1\}, s_f \geq 0, s_{f,t} \geq 0, v_{f,t} \geq 0, \quad \forall f \in F, t \in T. \quad (6.14)$$

In the model (MC-WPRB-Q), (6.1) represents the supply demand balance in each period and (6.2) ensures the link order relations. Flexible orders are accepted in at most one period via (6.3). Moreover, (6.4) ensures that segments can be partially accepted. These constraints are also valid for the model (MC). The constraints (6.5)-(6.8) are the dual feasibility constraints of the model (R-MC) whose dual variables are x , y_b , $y_{f,t}$, respectively. The strong duality condition of the fixed problem is represented by (6.9). Since we do not allow any opportunity cost arising from the rejected orders, (6.10)-(6.11) force v to zero if the order is rejected. Finally, (6.12)-(6.14) are the non-negativity constraints. The big- M values are as follows:

$$M_b = \sum_{t \in T_b} Q_{b,t}(P_b - P_{N_t,t}), \quad \forall b \in B^D, \quad (6.15)$$

$$M_b = \sum_{t \in T_b} Q_{b,t}(P_b - P_{0,t}), \quad \forall b \in B^S, \quad (6.16)$$

$$M_{f,t} = Q_f(P_f - P_{N_t,t}), \quad \forall f \in F^D, t \in T, \quad (6.17)$$

$$M_{f,t} = Q_f(P_f - P_{0,t}), \quad \forall f \in F^S, t \in T. \quad (6.18)$$

We can interpret this formulation as follows: (6.1)-(6.4) find a feasible y combination and corresponding surplus maximizing x values. Then, (6.5) - (6.8) find the corresponding surplus maximizing price values with the help of (6.9). Finally, we check whether this price yields an opportunity cost for a rejected order by (6.10) and (6.11). The following table presents the size of the model (MC-WPRB-Q).

	Size of the model (MC-WPRB-Q)
Number of linear constraints	$ T + B^c + F + \sum_{t \in T} 2(N_t + 1) + 2 B + 2 T \times F .$
Number of quadratic constraints	1
Number of nonnegative continuous variables	$2 \sum_{t \in T} (N_t + 1 + F) + F + 2 B $
Number of binary variables	$ B + T \times F .$

The formulation is a mixed integer quadratically-constrained quadratic programming problem (MIQCQP). There is a quadratic constraint arising from the strong duality condition. In the quadratically-constrained quadratic programming (QCQP) relaxation of the problem, the quadratic constraint can be converted into a second order cone constraint. Commercial solvers (e.g. IBM Ilog CPLEX) can solve this problem. As we will show in Chapter 8, the performance of this model is not satisfactory. Hence, in the next section we present an equivalent formulation for the model (MC-WPRB-Q) with only linear constraints.

6.1.1.2 Mathematical Model of the Supply Demand Balance Problem Without PRB with Linear Constraints

In this section, we propose an equivalent formulation for the supply demand balance problem without PRBs using an MIQP model. Afterwards, we investigate some properties of an optimal solution of the resulting model.

Instead of using the strong duality condition in the constraints as we did in the model (MC-WPRB-Q), we can directly model the conditions 1-5 with the help of auxiliary integer variables and big-M type constraints. This is similar to the approach used in Meeus et al. (2009a), Ceyhan et al. (2017) and Derinkuyu et al. (2019). For this aim we define new binary variables and constraints.

We formulate the conditions stated in Theorem 3 which proposes a relation between $x_{j,t}$ and p_t for the model (MC). For this purpose, we need to first formulate the conditions stated in Lemma 6, which ensure that there cannot be more than one decisive segment in each period. Let $w_{j,t}$ be a binary variable for $j \in \{0, \dots, N_t - 1\}$ and $t \in T$ that ensures the segment j in period t is fully accepted. We define a new set W_t , where $W_t = \{0, \dots, N_t - 1\}$ for all $t \in T$.

$$w_{0,t} \leq x_{0,t} \leq 1, \quad \forall t \in T \quad (6.19)$$

$$w_{j,t} \leq x_{j,t} \leq w_{(j-1),t}, \quad \forall j \in \{1, \dots, N_t - 1\}, t \in T \quad (6.20)$$

$$1 - w_{(N_t-1),t} \leq x_{N_t,t} \leq 1, \quad \forall t \in T \quad (6.21)$$

Note that, when the total demand offered at the minimum price is not adequate to balance supply quantity at the same price level (such that *demand deficiency* occurs in the system) in period t , the price must be at the minimum limit and we have $x_{N_t,t} < 1$, and $x_{j,t} = 1$ for all $j \in W_t$ in that period. For the rest of the cases we have $x_{N_t,t} = 1$. Therefore, the forcing variable $w_{(N_t-1),t}$ and $x_{N_t,t}$ cannot be zero at the same time as we show in (6.21). Moreover, when the total supply quantity offered at the maximum price is not adequate to balance demand quantity at the same price level (such that *supply deficiency* occurs in the system) in period t , the price must be at the maximum limit and we have $x_{0,t} < 1$, and $x_{j,t} = 0$ for all $j \in \{1, \dots, N_t - 1\}$ for all $t \in T$. Hence, the constraint defined in (6.19) is satisfied. In the presence of (6.19)-(6.21), the following equalities express the relations between variables p_t and $x_{j,t}$ stated in Theorem 3.

$$\begin{aligned} p_t = P_{0,t}(1 - x_{0,t}) + \sum_{j \in \{1, \dots, N_t\}} P_{j,t}(x_{(j-1),t} - x_{j,t}) \\ + P_{N_t,t}x_{N_t,t}, \quad \forall t \in T, \end{aligned} \quad (6.22)$$

whenever $N_t \geq 2$, otherwise this constraint forces p_t to its upper limit that may not be true. This constraint is valid when there is at least one segment in the aggregated total curve in period t , otherwise we omit the constraint.

By Lemma 9, for a block order b such that $b \notin B^c$ and $b \notin B^m$, b is fully accepted if it is ITM. We formulate this condition by the following constraint:

$$\sum_{t \in T} (P_b - p_t) Q_{b,t} \leq y_b M_b, \quad \forall b \in B, b \notin B^c, b \notin B^m \quad (6.23)$$

where the values of M_b are given by (6.15) and (6.16). Note that, these big-M values are smaller than the ones defined in Ceyhan et al. (2017). In (6.23), when $\sum_{t \in T} (P_b - p_t) Q_{b,t}$ is positive (b is ITM) then y_b must be 1. If it is negative (b is OTM), y_b may be 0 or 1. Hence, (6.23) may result in a PAB in an optimal solution.

We remark that the conditions in Lemma 7, 8(a)-i, (b)-i are equivalent to the following condition: a block order having at least one subfamily which is ITM must be accepted or follow the same decision of its parent, if it exists. We use big-M approach and convert this implication into the following constraints:

$$\begin{aligned} (a) \quad & \sum_{t \in T_{b_i}, b_i \in \phi_b} (P_{b_i} - p_t) Q_{b_i,t} \leq y_b \sum_{b_i \in \phi_b} M_{b_i}, \quad \forall \phi_b \in \Phi_b, b \notin B^c, b \in B^m \setminus B^c, \\ (b) \quad & \sum_{t \in T_{b_i}, b_i \in \phi_b} (P_{b_i} - p_t) Q_{b_i,t} \leq (1 + y_b - y_{b_m}) \sum_{b_i \in \phi_b} M_{b_i}, \quad \forall \phi_b \in \Phi_b, b \in \Theta_{b_m}, b_m \in B^m, \end{aligned} \quad (6.24)$$

where M_{b_i} is defined in (6.15) or (6.16) depending on the direction of the block order b . If y_{b_m} is 1, then y_b is accepted if any subfamily of b is ITM. Otherwise, this constraint is redundant.

The following constraint formulates the conditions stated in Lemma 11.

$$(P_f - p_t) Q_f \leq \sum_{\bar{t} \in T} y_{f\bar{t}} M_{f,t}, \quad \forall f \in F, t \in T, \quad (6.25)$$

where $M_{f,t}$ are defined in (6.17) and (6.18) for all $f \in F$.

We next present the alternative model, denoted by (MC-WPRB):

(MC-WPRB)

$$\max c(x, y)$$

s.t.

$$\sum_{j \in J} \Delta Q_{j,t} x_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} = Q_t, \quad \forall t \in T, \quad (6.26)$$

$$y_b \leq y_{b_m}, \quad \forall b \in B^c, b \in \Theta_{b_m}, \quad (6.27)$$

$$\sum_{t \in T} y_{f,t} \leq 1, \quad \forall f \in F, \quad (6.28)$$

$$w_{0,t} \leq x_{0,t} \leq 1, \quad \forall t \in T, \quad (6.29)$$

$$w_{j,t} \leq x_{j,t} \leq w_{(j-1),t}, \quad \forall j \in \{1, \dots, N_t - 1\}, t \in T, \quad (6.30)$$

$$1 - w_{(N_t-1),t} \leq x_{N_t,t} \leq 1, \quad \forall t \in T, \quad (6.31)$$

$$p_t = P_{0,t} (1 - x_{0,t}) + P_{N_t,t} x_{N_t,t} + \sum_{j \in \{1, \dots, N_t\}} P_{j,t} (x_{(j-1),t} - x_{j,t}), \quad \forall t \in T \text{ if } N_t \geq 2, \quad (6.32)$$

$$\sum_{t \in T} (P_b - p_t) Q_{b,t} \leq y_b M_b, \quad \forall b \in B \setminus \{B^c \cup B^m\}, \quad (6.33)$$

$$\sum_{t \in T, b_i \in \phi_b} (P_{b_i} - p_t) Q_{b_i,t} \leq y_b \sum_{b_i \in \phi_b} M_{b_i}, \quad \forall \phi_b \in \Phi_b, b \in B^m \setminus B^c, \quad (6.34)$$

$$\sum_{b_i \in \phi_b, t \in T_{b_i}} (P_{b_i} - p_t) Q_{b_i,t} \leq (y_b - y_{b_m}) \sum_{b_i \in \phi_b} M_{b_i} + \sum_{b_i \in \phi_b} M_{b_i}, \quad \forall \phi_b \in \Phi_b, b \in \Theta_{b_m}, b_m \in B^m, \quad (6.35)$$

$$(P_f - p_t) Q_f \leq \sum_{t \in T} y_{f,t} M_{f,t}, \quad \forall f \in F, t \in T, \quad (6.36)$$

$$x_{j,t} \geq 0, \quad \forall j \in J, t \in T, \quad (6.37)$$

$$y_b, y_{f,t} \in \{0, 1\}, \quad \forall b \in B, f \in F, t \in T, \quad (6.38)$$

$$w_{j,t} \in \{0, 1\}, \quad \forall j \in W_t, t \in T. \quad (6.39)$$

Recall that (6.26)-(6.28) are the primal feasibility constraints of the model (MC).

(6.29)-(6.31) ensure that the aggregated total curve segments are accepted in a specific order: The segments until the decisive segment must be fully accepted, and the segments after the decisive segment must be fully rejected in a period. This is a property of the model (MC) as defined in Lemma 6. (6.32) defines (MC)P in each period. (6.33) forces block orders without a link relation to be accepted when they are ITM (Lemma 9(a)). (6.34) forces the first generation block orders to be accepted when at least one of their subfamilies is ITM and (6.35) forces child block orders to receive the same decision of its parent order when there exists at least one subfamily of the child that is ITM. (6.34) and (6.35) reflect the property of any optimal solution of the model (QPR-MC) as stated in Lemma 7, 8 (a)-i, (b)-i. (6.36) ensures that ITM flexible orders to be accepted as in Lemma 11 and (6.37)-(6.39) define the ranges of the variables. The size of the model is summarized below:

Size of the model (MC-WPRB)

Number of linear constraints	$ T + B^c + F + \sum_{t \in T} \left(2(N_t + 1) + \mathbb{1}_{N_t \geq 2} \right)$ $+ B \setminus \{B^c \cup B^m\} + \sum_{b \in B^c \cup B^m} \Phi_b $ $+ T \times F $
Number of nonnegative continuous variables	$\sum_{t \in T} (N_t + 2)$
Number of binary variables	$ B + T \times F + \sum_{t \in T} N_t.$

The model (MC-WPRB-Q) has more continuous variables than the model (MC-WPRB). On the other hand, it has less number of integer variables than the model (MC-WPRB). The number of linear constraints in the model (MC-WPRB) depends on the set of subfamily sets of block orders. Hence, it is possible that the number of linear constraints in the model (MC-WPRB) is higher than the ones in the model (MC-WPRB-Q). However, the model (MC-WPRB-Q) has a quadratic constraint, which may affect the performance of the model (MC-WPRB-Q) in a bad way. Note that, if

we did not aggregate the hourly orders, there would be a relation between each hourly order segment and p_t . It would increase the problem size and we would formulate the relation between segments and p_t as in Meeus et al. (2009b).

Properties of the model (MC-WPRB)

In this section, we define strong valid inequalities for the model (MC-WPRB). Before presenting these theorems, we introduce a set definition, which will be used in these theorems.

Let k be a block or flexible order with price P_k which is valid in the period set T_k . For all $t \in T_k$ define the set of segments $J_{k,t}$ for a sales order k as

$$J_{k,t} = \{j \in J_t | P_{(j+1),t} \leq P_k < P_{j,t}\}, \quad k \in B^S \cup F^S \quad (6.40)$$

and for a purchase order k ,

$$J_{k,t} = \{j \in J_t | P_{(j+1),t} < P_k \leq P_{j,t}\}, \quad k \in B^D \cup F^D. \quad (6.41)$$

Recall that there is a strict order in price levels for the aggregated total curve segments, i.e., $P_{(j+1),t} < P_{j,t}$. Moreover, $|J_{k,t}| = 0$ when $P_k = P_{N_t,t}$ for a purchase order k and $P_k = P_{0,t}$ for a sales order k . Hence, the cardinality of $J_{k,t}$ is at most one for all $t \in T_k$, $k \in B$ or $k \in F$.

Theorems 6 and 7 creates a relation between variables y and w for demand and sales orders, respectively. They basically force y_b to 1 when it is ITM in each period for all $b \in B \setminus B^c$, $y_b = y_b^m$ if it is ITM in each period for all $b \in B^c$, and $y_{f,t} = 1$ if f is ITM in at least one $t \in T$.

Theorem 6. *Let k be a block or flexible purchase order with price P_k which is valid in the period set T_k where $|T_k| = N_k$. Then, the following inequality is a strong valid*

inequality for the model (MC-WPRB).

$$\sum_{t \in T_b, j \in J_{b,t}} w_{j,t} \leq (N_b - 1) + y_b, \quad \forall b \in B^D \setminus B^c, \quad (6.42)$$

$$\sum_{t \in T_b, j \in J_{b,t}} w_{j,t} \leq N_b - y_{b_m} + y_b, \quad \forall b \in \Theta_{b_m}, b_m \in B^m \text{ and } b \in B^D, \quad (6.43)$$

$$w_{j,t} \leq \sum_{t \in T_f} y_{f,t}, \quad \forall j \in J_{f,t}, t \in T_f, f \in F^D. \quad (6.44)$$

Proof. We divide the proof of the theorem into two parts: In the first part we will show that (6.42)-(6.44) are feasible for the model (MC-WPRB). Next, we show that these inequalities are strong.

PART 1: Validity

Recall that the cardinality of set $J_{k,t}$ is at most one for all $t \in T_k$, $k \in B \cup F$. Since $w_{j,t}$ is a binary variable, (6.42)-(6.44) are redundant if y_b or $y_{f,t}$ takes its upper bound, i.e. $y_b = 1$, $y_b = y_{b_m}$, or $y_{f,t} = 1$ for all $t \in T$, $b \in B^D$, $f \in F^D$. Moreover, these constraints are redundant when $P_k = P_{N_t,t}$ for an order $k \in B^D \cup F^D$ in at least one $t \in T_k$. On the other hand, if $\sum_{t \in T_b, j \in J_{b,t}} w_{j,t} = N_b$, then (6.42) forces y_b to be equal to one for $b \in B^D \setminus B^c$. Similarly, if $\sum_{t \in T_b, j \in J_{b,t}} w_{j,t} = N_b$, then $y_b = y_{b_m}$ for $b \in \Theta_{b_m}$ and $b_m \in B^m$ by (6.43). Finally, if $w_{j,t} = 1$ then (6.44) forces $\sum_{t \in T_f} y_{f,t}$ to be equal to one $\forall j \in J_{f,t}, t \in T_f, f \in F^D$.

Now we make the following assumption:

Assumption A: $w_{j,t} = 1 \implies (P_k - p_t)Q_k > 0$ for $j \in J_{k,t}, t \in T_k$ and $k \in B^D \cup F^D$.

Next, we show that $w_{j,t} = 1$ implies $(P_k - p_t)Q_k > 0$ for $j \in J_{k,t}, t \in T_k$ and $k \in B^D \cup F^D$.

$$w_{j,t} = 1 \implies x_{j,t} = 1, \quad \forall j \in J_{k,t} \text{ and } t \in T_k, \text{ for } k \in B^D \cup F^D,$$

$$\implies p_t \leq P_{(j+1),t}, \quad \forall j \in J_{b,t} \text{ and } t \in T_b, \text{ for } b \in B^D \setminus \{B^c \cup B^m\}, \text{ by Theorem 3}$$

By definition of $J_{k,t}$, $P_{(j+1),t} < P_k$ for all $j \in J_{k,t}, t \in T_k, k \in B^D \cup F^D$. Thus, $p_t < P_k$.

Let Q_k be the quantity of order k . We know that $Q_k > 0$ for all $k \in B^D \cup F^D$. Then,

$\sum_{t \in T_k} (P_k - p_t) Q_k > 0$ for all $B^D \cup F^D$. Hence Assumption A holds. We will use this assumption below.

- $\sum_{t \in T_b, j \in J_{b,t}} w_{j,t} = N_b$ implies $w_{j,t} = 1$ for all $j \in J_{b,t}, t \in T_b$. By Assumption A, $\sum_{t \in T_b} (P_b - p_t) Q_{b,t} > 0$ for $b \in B^D \setminus B^c$, which forces y_b to 1 by (6.33) and (6.34) in the model (MC-WPRB). Furthermore, by Assumption A, $y_b = y_{b_m}$ by (6.35) for $b \in B^c$ in the model (MC-WPRB).
- By Assumption A, $w_{j,t} = 1$ implies $(P_f - p_t) Q_f > 0$ for $j \in J_{f,t}, t \in T_f, F \in F^D$. This yields $\sum_{t \in T_f} y_{f,t} = 1$ by (6.36) in the model (MC-WPRB).

Thus, (6.42)-(6.44) are also valid for the model (MC-WPRB).

PART 2: Strength

Now we show that there exists a feasible solution of the QP relaxation of the model (MC-WPRB) that violates (6.42)-(6.44). For this aim construct a sample instance, which we will use in the next proofs.

Example Set 1: Suppose that there are 5 block orders, i.e., $B = \{b_1, b_2, \dots, b_5\}$, which are valid in T with the same quantity Q and the same price P . b_2 is the parent of b_3 and b_4 is parent of b_5 , i.e., $\Theta_{b_2} = \{b_3\}$ and $\Theta_{b_4} = \{b_5\}$. Moreover, there is a flexible order f_1 with the same price P and the same quantity Q . Furthermore, the quantity differences in each segment are $\Delta Q_{l,t} = |Q|$ for all $l \in J_t$ and $t \in T$. Since the price of the orders are the same and they have the same quantity, we have $J_{b_1,t} = J_{b_2,t} = \dots = J_{b_5,t} = J_{f_1,t} = J_{k,t}$.

We assume that all orders in B and f_1 are purchase orders. Moreover, we assume that P satisfies the following condition: $P < (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t} < P_{(j-1),t}$ for all $j \in J_{k,t}, k \in B \cup \{f_1\}, t \in T$ where ϵ satisfies $0 < 1 - \frac{1}{N} < \epsilon < 1$. Furthermore, $Q_t = Q(j - 1 + 2\epsilon)$ where $j \in J_{k,t}, k \in B \cup f, t \in T$. Moreover, we assume that $P \neq P_{N_t,t}$, therefore, the inequalities (6.42)-(6.44) are not redundant.

We now construct a solution (x, y) as follows: Let $x_{it} = w_{it} = 1$, $x_{j,t} = x_{(j-1),t} = w_{j,t} = w_{(j-1),t} = \epsilon$ and $x_{k,t} = w_{k,t} = 0$ for all $i, j, k \in J_t$, $t \in T$ such that $i < j - 1 < j < k$ and $k \neq N_t$ where $j \in J_{k,t}$ for all $k \in B \cup \{f\}$. Furthermore, let $x_{N_t,t} = w_{N_t,t} = 1$ for all $t \in T$ and $y_b = 0$ for all $b \in B \setminus \{b_4\}$ where $y_{b_4} = 1$ and $y_{f,t} = 0$ for all $t \in T$.

The inequalities stated in Theorem 6 are violated by the solution (x, y) , since:

- $N\epsilon = \sum_{t \in T_{b_1}, j \in J_{b_1,t}} w_{j,t} \leq N - 1$ by (6.42), but $1 - \frac{1}{N} < \epsilon$,
- $N\epsilon = \sum_{t \in T_{b_2}, j \in J_{b_2,t}} w_{j,t} \leq N - 1$ by (6.42), but $1 - \frac{1}{N} < \epsilon$,
- $N\epsilon = \sum_{t \in T_{b_5}, j \in J_{b_5,t}} w_{j,t} \leq N - 1$ by (6.43), but $1 - \frac{1}{N} < \epsilon$,
- $\epsilon = w_{j,t} \leq 0$ by (6.44), but $0 < \epsilon$,

by definition. However, (x, y) is a feasible solution of the QP relaxation of the model (MC-WPRB) as we will show next.

The supply-demand balance is one of the basic constraints in the model (MC-WPRB), which is given by (6.26). Note that (x, y) satisfies this constraint as we show below:

$$\begin{aligned}
 Q_t &= \sum_{l \in J_t} x_{l,t} \Delta Q_{l,t} + \sum_{b \in B} y_b Q_{b,t} + \sum_{f \in F} y_{f,t} Q_f, \\
 &= \sum_{l \in J_t} x_{l,t} Q + 0 \times Q + 1 \times Q, \quad \forall t \in T \\
 &= Q \sum_{l \in J_t} x_{l,t} + Q, \quad \forall t \in T \\
 &= Q \sum_{i < (j-1), i \in J_t} x_{it} + Q x_{(j-1),t} + Q x_{j,t} + Q \sum_{k > j, k \in J_t, k \neq N_t} x_{k,t} - Q x_{N_t,t} + Q, \quad \forall t \in T \\
 &= Q \sum_{i < (j-1), i \in J_t} 1 + \epsilon Q + \epsilon Q + Q \sum_{k > i, k \in J_t} 0, \quad \forall t \in T \\
 &= Q(j - 2 + 1 + 2\epsilon) = Q_t, \quad \forall t \in T
 \end{aligned}$$

Note that (x, y) satisfies (6.27) and (6.28). For the x and w values defined above, (6.29)-(6.31) are also satisfied. Moreover, (6.32) can be written as follows:

$$\begin{aligned}
 p_t = & P_{0,t}(1 - x_{0,t}) + (x_{0,t} - x_{1,t})P_{1,t} \\
 & + (x_{1,t} - x_{2,t})P_{2,t} \\
 & + (x_{2,t} - x_{3,t})P_{3,t}^L + \dots \\
 & + (x_{(j-2),t} - x_{(j-1),t})P_{(j-1),t} + (x_{(j-1),t} - x_{j,t})P_{j,t} \\
 & + (x_{j,t} - x_{(j+1),t})P_{(j+1),t} + (x_{(j+1),t} - x_{(j+2),t})P_{(j+2),t} + \dots \\
 & + (x_{(N_t-1),t} - x_{N_t,t})P_{N_t,t} + x_{N_t,t}P_{N_t,t},
 \end{aligned}$$

where $j \in J_{k,t} \subset J_t$, $k \in B \cup \{f_1\}$ for all $t \in T$.

$$\begin{aligned}
 p_t = & \cancel{(1-1)P_{0,t}} + \cancel{(1-1)P_{1,t}} + \cancel{(1-1)P_{2,t}} \\
 & + \cancel{(1-1)P_{3,t}} + \dots + (1 - \epsilon)P_{(j-1),t} \\
 & + \cancel{(\epsilon - \epsilon)P_{j,t}} + (\epsilon - 0)P_{(j+1),t} + \cancel{(0 - 0)P_{(j+1),t}} + \dots \\
 & + \cancel{(0 - 1)P_{N_t,t}} + P_{N_t,t} \\
 = & (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t}, \quad j \in J_{k,t} \subset J_t, k \in B \cup \{f_1\}, t \in T.
 \end{aligned}$$

By the last part of the proof, since b is OTM for each $b \in B \setminus \{b_5\}$, $y_b = 0$ is a feasible solution when $p_t = (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t}$ where $j \in J_{b,t}$. We know that $P < (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t} = p_t$ by definition. When we re-arrange the terms, we obtain $P - p_t < 0$ for all $t \in T$ which is also equivalent to $\sum_{t \in T} (P - p_t)Q < 0$ since $Q > 0$. Therefore, (6.33)-(6.35) are satisfied. We can follow a similar argument to show that (6.36) is satisfied.

Therefore, all constraints are satisfied and the solution (x, y) defined above is feasible for the QP relaxation of the model (MC-WPRB). However, (6.42)-(6.44) are violated, which are valid for the model (MC-WPRB). Hence, (6.42)-(6.44) are strong valid inequalities. $\square$

Theorem 6 shows strong valid cuts for the model (MC-WPRB) considering purchase orders by exploiting a relation between the variables $w_{j,t}$ and y_b . The following

theorem also relates the variables $w_{j,t}$ and y_b where $y_b \in B^S \cup F^S$.

Theorem 7. *Let k be a block or flexible sales order with price P_k which is valid in the period set T_k where $|T_k| = N_k$. Then, each of the following inequalities is a strong valid inequality for the model (MC-WPRB).*

$$1 - y_b \leq \sum_{t \in T_b, j \in J_{b,t}} w_{(j-1),t}, \quad \forall b \in B^S \setminus B^c. \quad (6.45)$$

$$y_{b_m} - y_b \leq \sum_{t \in T_b, j \in J_{b,t}} w_{(j-1),t}, \quad \forall b \in \Theta_{b_m}, b_m \in B^m \cap B^S, \quad (6.46)$$

$$1 - \sum_{t \in T_f} y_{f,t} \leq w_{(j-1),t}, \quad \forall j \in J_{f,t}, t \in T_f, f \in F^S. \quad (6.47)$$

Proof. We divide the proof into two parts. In the first part we show the validity of the (6.45)-(6.47) and in the second part we show that these inequalities are strong.

PART 1: Validity

We start the proof by showing that any feasible solution of the model (MC-WPRB) satisfies (6.45)-(6.47). Initially, $w_{(j-1),t}$ is well-defined because $w_{j,t}$ is defined for all j in $J_{k,t}$. When y_b or $y_{f,t}$ is at its upper bound, these inequalities are redundant. Moreover, these constraints are redundant if P_k equals $P_{0,t}$ in at least one $t \in T_k$ for $k \in B^S \cup F^S$. On the other hand, if the right hand side of the inequalities are zero, then variables y are forced to their upper bound values.

If $w_{(j-1),t} = 0$ implies $P_k < p_t$ or equivalently $(P_k - p_t)Q_k > 0$ for $k \in B^S \cup F^S$, then (6.33)-(6.36) in the model (MC-WPRB) force the corresponding variables y to be equal to their upper bound values (i.e. 1 or y_{b_m} depending on whether there is a link relation or not). Then, (6.45)-(6.47) are satisfied. In the following lines, we use direct proof and show that $w_{(j-1),t} = 0$ implies that $P_k < p_t$ for $k \in B^S \cup F^S$ and $j \in J_{k,t}$.

$$w_{(j-1),t} = 0 \Rightarrow x_{j,t} = 0, \quad \forall j \in J_{k,t} \text{ and } t \in T_k, \text{ for } b \in B^S \cup F^S$$

$$\text{since } w_{it} \leq x_{it} \leq w_{i-1,t} \text{ for } i \in J_t \supset J_{b,t},$$

$$\Rightarrow P_{j,t} \leq p_t, \quad \text{if } j \in J_{k,t} \text{ and } t \in T_k, \text{ by Theorem 3}$$

By definition, $P_{(j+1),t} \leq P_k < P_{j,t}$ for all $j \in J_{k,t}$, $t \in T_k$, $k \in B^S \cup F^S$. Hence, $P_k < p_t$ for all $t \in T_k$, $b \in B^S \cup F^S$.

PART 2: Strength

Now we show that there exists a feasible solution for the QP relaxation of the model (MC-WPRB) that violates the constraints (6.45)-(6.47).

Suppose that we use the same sample instance and the same solution as in the proof of Theorem 6. We assume that all orders in B and f are sales orders, i.e., $Q < 0$. Moreover, we assume that P satisfies the following condition: $P > (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t}$ for all $j \in J_{k,t}$ and $t \in T_k$ where ϵ also satisfies $0 < \epsilon < \frac{1}{N} < 1$. Since $\epsilon \neq 0$ and $P_{(j-1),t} > P_{(j-1),t}^L = P_{j,t} > P_{(j+1),t}$ by definition, $P_{(j+1),t} < (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t} < P_{(j-1),t}$ holds. Thus, $(1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t} < P < P_{j,t} < P_{(j-1),t}^H$ is satisfied for $j \in J_{k,t}$, $k \in B \cup \{f_1\}$. Furthermore, $Q_t = Q(j - 2 + 2\epsilon)$.

The inequalities given in (6.45)-(6.47) are violated by the solution (x, y) , since:

- $1 \leq N\epsilon = \sum_{t \in T_{b_1}, j \in J_{b_1,t}} w_{(j-1),t}$, by (6.45), but $\epsilon < \frac{1}{N}$,
- $1 \leq N\epsilon = \sum_{t \in T_{b_2}, j \in J_{b_2,t}} w_{(j-1),t}$, by (6.45), but $\epsilon < \frac{1}{N}$,
- $1 \leq N\epsilon = \sum_{t \in T_{b_5}, j \in J_{b_5,t}} w_{(j-1),t}$, by (6.46), but $\epsilon < \frac{1}{N}$,
- $1 \leq \epsilon = w_{(j-1),t}$, by (6.47), but $\epsilon < 1$,

by definition. However, (x, y) is a feasible solution for the QP relaxation of the model (MC-WPRB), which we show next.

We can follow the same steps as in the proof of Theorem 6 and show that $Q_t = Q(j - n_t^D - n_t^S - 5 + 2\epsilon)$. Since the solution (x, y) is the same as the solution in the proof of Theorem 6 and we showed that (x, y) satisfies (6.27)-(6.32), we do not repeat the same steps in here. Instead, we show that $y_b = 0$ is a feasible solution when $p_t = (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t}$ where $j \in J_{b,t}$ for $b \in B \setminus \{b_5\}$. We know that

$P > (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t} = p_t$ by definition. When we re-arrange the terms, we obtain $P - p_t > 0$ for all $t \in T$ which is also equivalent to $\sum_{t \in T} (P - p_t)Q < 0$ since $Q < 0$. Therefore, (6.33)-(6.35) are satisfied. In a similar fashion, after following the same steps we can show that (6.36) is satisfied when $\sum_{t \in T} y_{f_1,t} = 0$.

It follows that all constraints are satisfied and the solution (x, y) is feasible for the QP relaxation of the model (MC-WPRB) but violates (6.45)-(6.47). Thus, the inequalities defined in Theorem 7 are strong valid inequalities. $\square$

Remark 2. Theorems 6 and 7 are also valid for a block order which has a different quantity for each period during which it is active.

Remark 3. For a case where a block order b must be accepted if it is ATM as in the Turkish DAM, Theorems 6 and 7 are still valid for $b \in B^D \setminus \{B^c \cup B^m\}$, $J_{b,t} = \{j \in J_t | P_{(j+1),t} \leq P_b < P_{j,t} \text{ and } 0 \text{ when } P_{0,t} = P_b\}$ and for $b \in B^S \setminus \{B^c \cup B^m\}$ $J_{b,t} = \{j \in J_t | P_{(j+1),t} < P_b \leq P_{j,t} \text{ and } N_t \text{ when } P_{N_t,t} = P_b\}$.

The following theorem a sales and a purchase block order to each other.

Theorem 8. Suppose that there are two block orders b_1 and b_2 where $b_1 \in B^D \setminus B^c$ and $b_2 \in B^S \setminus B^c$. If $P_{b_1}N_{b_1} - P_{b_2}N_{b_2} + \sum_{t \in T_{b_2} \setminus T_{b_1}} P_{N_t,t} - \sum_{t \in T_{b_1} \setminus T_{b_2}} P_{0,t} > 0$, then

$$y_{b_1} + y_{b_2} \geq 1, \quad (6.48)$$

is a strong valid inequality.

Proof. **PART 1: Validity**

The block orders b_1 and b_2 must satisfy the following constraints in a feasible solution of the model (MC-WPRB):

$$\begin{aligned} \sum_{t \in T_{b_1}} (P_{b_1} - p_t)Q_{b_1} &\leq M_{b_1}y_{b_1} \\ \sum_{t \in T_{b_2}} (P_{b_2} - p_t)Q_{b_2} &\leq M_{b_2}y_{b_2} \end{aligned}$$

These inequalities are also equivalent to

$$\begin{aligned}\sum_{t \in T_{b_1}} (P_{b_1} - p_t) &\leq \frac{M_{b_1}}{Q_{b_1}} y_{b_1} \\ \sum_{t \in T_{b_2}} (p_t - P_{b_2}) &\leq \frac{M_{b_2}}{Q_{b_2}} y_{b_2},\end{aligned}$$

since $Q_{b_1} > 0 > Q_{b_2}$. If we sum these inequalities we obtain

$$\begin{aligned}\sum_{t \in T_{b_1}} (P_{b_1} - p_t) + \sum_{t \in T_{b_1}} (P_{b_1} - p_t) &\leq \frac{M_{b_1}}{Q_{b_1}} y_{b_1} + \frac{M_{b_1}}{Q_{b_1}} y_{b_2}, \\ P_{b_1} N_{b_1} - P_{b_2} N_{b_2} + \sum_{t \in T_{b_2} \setminus T_{b_1}} p_t - \sum_{t \in T_{b_1} \setminus T_{b_2}} p_t &\leq \frac{M_{b_1}}{Q_{b_1}} y_{b_1} + \frac{M_{b_1}}{Q_{b_1}} y_{b_2}\end{aligned}$$

If the left hand side of the above inequality is positive, then at least one of the variables y must be positive. This is satisfied if the minimum possible value of the left hand side is positive, i.e., if $P_{b_1} N_{b_1} - P_{b_2} N_{b_2} + \sum_{t \in T_{b_2} \setminus T_{b_1}} P_{N_t, t} - \sum_{t \in T_{b_1} \setminus T_{b_2}} P_{0, t} > 0$.

PART 2: Strength Assume that $T_{b_1} = T_{b_2} = T$ and $P_{b_1} > p_t > P_{b_2}$ for all $t \in T$ and $Q_{b_1} = 1$, $Q_{b_2} = -1$. Furthermore, let $P_{b_1} = (1 - \beta)P_{b_2} + \frac{\beta P_{0, t}}{|T|}$ and $P_{b_2} = (1 - \alpha)P_{b_1} + \frac{\alpha P_{N_t, t}}{|T|}$. Also, we assume that supply balance is satisfied. Let $y_{b_1} = \alpha$ and $y_{b_2} = \beta$, where $\alpha \in [0, 1]$, $\beta \in [0, 1]$, and $\alpha + \beta < 1$, therefore, (6.48) is violated. However, this solution is feasible for the model (MC-WPRB) because;

$$\begin{aligned}\sum_{t \in T_{b_1}} (P_{b_1} - p_t) &\leq M_{b_1} y_{b_1} = (P_{b_1} - P_{N_t, t}) \alpha, \\ \sum_{t \in T_{b_2}} -(P_{b_2} - p_t) &\leq -M_{b_2} y_{b_2} = (P_{b_1} - P_{0, t}) \beta,\end{aligned}$$

equivalent to $(1 - \beta)P_{b_2} + \frac{\beta P_{0, t}}{|T|} = P_{b_1} > p_t > P_{b_2} = (1 - \alpha)P_{b_1} + \frac{\alpha P_{N_t, t}}{|T|}$. $\square$

Complexity of finding b_1 and b_2 block order pairs given in Theorem 8 is the at most $O(N^2)$.

We can use the strong valid inequalities to reduce the feasible region of the problem solved in each node of the branch and bound tree.

6.1.1.3 Economic Interpretation

In this section, we define the desired properties of the market designs and investigate whether an optimal solution of our proposed mechanism supports these properties or not.

We assume that the price of the orders represents the marginal cost or benefit of the bidders as discussed in Van Vyve et al. (2011). A bidder is assumed indifferent between two actions when the profit earned from these two actions is the same. Moreover, profit coming from the rejection of an order is zero, so that there is no start-up cost for an order.

We determine the uplifts of the block orders from the complementary slackness constraints. Hence, the profit earned from an accepted block order with an uplift is zero. A bidder is indifferent between the rejection of his block order and acceptance with an uplift.

Equilibrium which is defined as in Definition 1, is a desirable property in a market design to prevent the collapse. In the next theorem, we establish that our proposed auction mechanism is in equilibrium.

Theorem 9. *A market design defined by the supply demand balance problem without PRBs is in equilibrium.*

Proof. Let (x^*, y^*) be an optimal solution of our proposed mechanism. The second condition of the equilibrium is always satisfied by (x^*, y^*) , because it is the first constraint of the model (MC-PRB). To investigate the first condition, we focus on the profit maximization for block, linked block and flexible order given p_t^* for all $t \in T$.

- Profit maximization for a block order b where $b \in B \setminus B^c \cup B^m$ is defined as follows:

Suppose that $\bar{s}_b = \sum_{t \in T_b} (P_b - p_t^*) Q_b$ for a given p_t^* and action space of a participant which offers block order b is $\Gamma_b = \{A, R, AU\}$. Action A represents the order acceptance, R stands for rejection and AU shows that the order is accepted

with an uplift. Then the following problem shows the optimal action for the block order b given price p_t^* .

$$\max\left\{\underbrace{0}_{R,AU}, \underbrace{\bar{s}_b}_A\right\}.$$

If $\bar{s}_b > 0$ then the optimal action for the participant is A , otherwise the participant is indifferent among R , AU , A . There is no action for the participant that results in a better profit than the one corresponding to y_b^* for $b \in B \setminus B^c \cup B^m$. To illustrate, if $\bar{s}_b > 0$, the block order b is ITM. Therefore, $y_b^* = 1$ is the only possible solution for the model (MC-PRB), which implies that the order is accepted. For the other cases, the model (MC-PRB) does not force y_b^* to a particular value, which is compatible with the participant's optimal action.

- Profit maximization for a linked block order b where $b \in B^c \cup B^m$ can be defined by a dynamic programming problem, where k represents the generation of a child order whose parent order is b and $\Pi_{k,b}(\gamma_{b_m})$ represents the profit function for the k^{th} generation block order b when the action for its parent is γ_{b_m} . Since the first generation block orders are not the children of any other block order, the profit function for the first generation block order is defined as $\Pi_{1,b}$.

$$\begin{aligned} \Pi_{1,b} &= \max\left\{\underbrace{0}_{R,AU}, \underbrace{\bar{s}_b + \sum_{b_i \in \Theta_b} \Pi_{2,b_i}(\gamma_b)}_A\right\}, & b \in B^m \setminus B^c \\ \Pi_{k,b}(\gamma_{b_m}) &= \max\left\{0, \left(\bar{s}_b + \sum_{b_i \in \Theta_b} \Pi_{k+1,b_i}(\gamma_b)\right) \mathbb{1}_{\gamma_{b_m} \neq R}\right\}, & k \geq 2, b \in B^c. \end{aligned}$$

Assume that the value of y_b^* for a linked block order does not correspond to a profit maximizing strategy. Then, the optimal action for b must be A , which means the order is ITM. However, if b is ITM, then $y_b^* = 1$ for a feasible solution, which implies that the order is accepted.

- Profit maximization for a flexible order f , where $f \in F$ is defined as follows:
Let $\bar{s}^{max} = \max_{t \in T} (P_f - p_t^*) Q_f$ for a given p_t^* . Then the optimal action for the flexible order is reject or accept in a period by paying uplift when $\bar{s}^{max} \leq 0$. When $\bar{s}^{max} > 0$, then the profit maximizing action is to accept the flexible order in a period since the auctioneer offers an uplift to take maximum profit. Profit maximizing actions are compatible with an optimal solution of the model (MC-WPRB).

□

We discussed the properties that an auction design should satisfy in Chapter 2 as efficient, revenue adequate, fairness, and failure freeness. It is a desirable property for an equilibrium to be *efficient*. In our design, the equilibrium is usually efficient. The only exception is the case in which the allocations are on the price limits. We provide the acceptance guarantee for the orders if they are ITM. Hence, we provide equal conditions for the participants which can offer a low quantity (e.g., a small company) or a high quantity (e.g., a large company). Hence, it is a *fair* design. The uplift is collected from the participants, therefore, the design is *revenue adequate*.

If there are no hourly orders in an auction, supply demand balance may not be satisfied by block or flexible orders. Thus, there may not be a feasible solution in our system. However, if there exists sufficient curtailable quantity (See Chapter 4), i.e., if $Q_{N,t} > \sum_{b \in B^D} Q_{b,t}$ and $Q_{0,t} > - \sum_{b \in B^S} Q_{b,t}$ for all $t \in T$, then there always exists a feasible price at the price limits because we allow rejecting ATM block or flexible orders. Hence, we can guarantee *failure freeness* in some special cases.

6.1.2 Supply Demand Balance Problem Without PABs

The market clearing mechanism used in European PXs including EPEX Spot (France, Germany, Austria, Belgium, The Netherlands, Switzerland and the UK) and Nord Pool Spot (Norway, Sweden, Finland, Denmark, Latvia, Estonia, Lithuania) results

in ITM rejected orders and leads to linear pricing for each participant. Flexible orders are allowed in Nord Pool Spot instead of exclusive groups, which are used in EPEX Spot (see Chapter 2). It is the only difference between DAMs operated by these PXs.

In this section, we name the clearing problem of DAM in Nord Pool Spot as the *supply demand balance problem without PAB*. We develop an optimization formulation for this problem. We first define the matching rules used in the Nord Pool Spot to eliminate PABs. Matching rules for hourly order curves are the same as those for the problem without PRBs, but the rules for block, linked block and flexible orders are slightly different.

1. A block order which is not a parent of another order (but may be a third generation child order) is rejected if it is OTM (Lemma 9(b)).
2. An OTM parent order is rejected if the sum of positive surplus coming from its descendants cannot save the parent (Lemma 8(a)-ii,(b)-ii).
3. A flexible order is rejected in a period if the order is OTM in that period (Lemma 10).

As we discussed before, the lemmas shown in the brackets defining the matching rules are the properties that any optimal solution of the model (QPR-MC) must satisfy.

The supply demand balance problem without PABs are not in equilibrium, because there exists PRB orders whose owners prefer that these orders to be accepted. The system is revenue adequate. The problem may yield rejection of the deep ITM orders, which means that the price of the supply/purchase order is significantly below/above the (MC)P. In most of the cases, *deep in-the-money* orders are the ones with small quantity (N-Side, 2016). Thus, the supply demand balance problem without PRBs treats the small companies, which cannot offer high quantities, unfairly. To hinder this problem, the Euphemia algorithm does a post-processing where the deep ITM

orders are accepted according to a “delta price rule” (Euphemia, 2016). There is no publicly available data reveals that the exact cut-off value to define deep ITM orders, however, there are some days where the price difference between the rejected order and the (MC)P may be as large as 100€(PCR, 2017). In our study, we do not consider the acceptance of deep ITM orders, because it may render the problem infeasible. In the next section, we focus on the mathematical model of the problem.

6.1.2.1 Mathematical Model of the Supply Demand Balance Problem Without PAB with a Quadratic Constraint

We model the problem by formulating the optimality conditions as in Madani and Van Vyve (2015). The authors propose a model for the supply demand balance problem without PAB for European PXs. They use hourly and block orders in their configuration with matching rule 1. The proposed model contains primal constraints as in the model (MC) defined in the previous chapter, dual feasibility constraints of the model (QPR-MC) with small modifications to eliminate PABs and the strong duality constraint to link primal variables to dual variables. We extend this formulation by adding linked block and flexible orders as well as using a different supply demand balance model. We use the model (MC) along with the dual feasibility constraints of its QP relaxation instead of using the model (MC) because in the previous chapter we show that these two models have the same set of optimal solutions. Then, the resulting model for formulating the problem of Nord Pool Spot is as follows:

(MC-WPAB-Q)

$$\max c(x, y)$$

s.t.

$$\sum_{j \in J} \Delta Q_{j,t} x_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} = Q_t, \quad \forall t \in T, \quad (6.49)$$

$$y_b \leq y_{b_m} \quad \forall b \in B^c, b \in \Theta_{b_m}, \quad (6.50)$$

$$\sum_{t \in T} y_{f,t} \leq 1, \quad \forall f \in F, \quad (6.51)$$

$$x_{j,t} \leq 1, \quad \forall j \in J_t, t \in T, \quad (6.52)$$

$$s_{j,t} \geq \Delta Q_{j,t}(P_{j,t} - p_t) - \Delta Q_{j,t} \Delta P_{j,t} x_{j,t}, \quad \forall j \in J_t, t \in T, \quad (6.53)$$

$$s_b \geq \sum_{t \in T_b} Q_{b,t}(P_b - p_t) - M_b(1 - y_b), \quad \forall b \in B \setminus B^m, \quad (6.54)$$

$$s_b - \sum_{b_i \in \Theta_b} s_{b_i} \geq \sum_{t \in T_b} Q_{b,t}(P_b - p_t) - M_b(1 - y_b), \quad \forall b \in B^m, \quad (6.55)$$

$$s_{f,t} + s_f \geq Q_f(P_f - p_t) - M_{f,t}(1 - y_{f,t}), \quad \forall f \in F, t \in T, \quad (6.56)$$

$$\begin{aligned} c(x, y) \geq & \sum_{j \in J_t, t \in T} s_{j,t} + \sum_{b \in B, b \notin B^c} s_b + \sum_{f \in F} s_f + \sum_{f \in F, t \in T} s_{f,t} \\ & - \sum_{t \in T} p_t Q_t + \sum_{t \in T, j \in J_t} \frac{\Delta P_{j,t} \Delta Q_{j,t} x_{j,t}^2}{2} \end{aligned} \quad (6.57)$$

$$s_{j,t} \geq 0, x_{j,t} \geq 0 \quad \forall t \in T, j \in J_t, \quad (6.58)$$

$$s_b \geq 0, y_b \in \{0, 1\} \quad \forall b \in B, \quad (6.59)$$

$$s_f \geq 0, s_{f,t} \geq 0, y_{f,t} \in \{0, 1\} \quad \forall f \in F, t \in T. \quad (6.60)$$

The objective function c is as defined in Lemma 3. The constraints defined by (6.49)-(6.53) are the same as the constraints (6.1)-(6.5) in the model (MC-WPRB-Q). (6.53)-(6.56) are the feasibility constraints of the dual problem of the QP relaxation of the model (MC) whose corresponding dual variables are x , y_b for $b \in B \setminus B^m$, y_b for $b \in B^m$ and $y_{f,t}$, respectively. The dual feasibility constraints are modified to allow rejection of some ITM block or flexible orders by employing the big-M values defined in (6.15)-(6.18). (6.57) links the primal and dual variables by forcing the strong duality condition of the QP relaxation of the mode (MC). Finally, (6.58)-(6.60) correspond to the ranges of the decision variables. We present the size of the model in the following table.

	Size of the the model (MC-WPAB-Q)
Number of constraints	$ T + B^c + F + \sum_{t \in T} 2(N_t + 1) + B $ $+ T \times F .$
Number of quadratic constraints	1
Number of nonnegative continuous variables	$\sum_{t \in T} (2(N_t + 1) + F) + F + B $
Number of binary variables	$ B + T \times F .$

The formulation is a MIQCQP problem. This formulation also does not exhibit a satisfactory performance in practice as we will show in Chapter 8. In the next section we provide a MIQP equivalent of the model (MC-WPAB-Q), which yields a more competitive alternative in practice.

6.1.2.2 Mathematical Model of the Supply Demand Balance Problem Without PAB with Linear Constraints

In this section, we propose an equivalent formulation for the supply demand balance problem without PABs. Later, we investigate some properties of an optimal solution of the resulting model.

In the model (MC-WPAB-Q) we use the strong duality condition of the model (R-MC) to cover complementary slackness conditions of the same model. Instead, we directly formulate the complementary slackness conditions to formulate the supply demand balance problem without PABs, thereby avoiding the quadratic constraint as in the model (MC-WPAB-Q). However, we need to define additional auxiliary binary variables and additional constraints. Next, we discuss these constraints and variables.

Constraints (6.19)-(6.21) are still valid in our new formulation to represent the conditions in Theorem 3. By Lemma 9(b) and Lemma 8 (a)-ii, (b)-ii, a block order b

is rejected if it is OTM. We formulate this condition by the following constraint:

$$\underline{M}_b(1 - y_b) \leq \sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}, \quad \forall b \in B, \quad (6.61)$$

where s_{b_i} denotes the non-negative surplus contribution of the child order b_i to its parent b and the big-M values given by:

$$\begin{aligned} \underline{M}_b &= \sum_{t \in T_b} (P_b - P_{0,t})Q_{b,t}, \text{ if } b \in B^D, \\ \underline{M}_b &= \sum_{t \in T_b} (P_b - P_{N_t,t})Q_{b,t}, \text{ if } b \in B^S. \end{aligned} \quad (6.62)$$

If $\sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}$ is negative then y_b must be 0 by (6.61). If it is positive, y_b may be 0 or 1. Hence, (6.61) may allow PRB in an optimal solution.

If a block order does not have any child, i.e., $\Theta_b = \emptyset$, the variables s vanish in (6.61). For the rest of the cases, we need to force the variable s_b to its optimal value in the model (MC), which is defined in Theorem 5. Note that the relation is nonlinear, i.e., $s_b = y_b \left(\sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} \right)$ for $s_b \geq 0$. We can linearize s_b for all $b \in B^c$ as follows.

$$s_b \leq \overline{M}_b y_b, \quad \forall b \in B^c, \quad (6.63)$$

$$s_b \leq \sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} + \overline{M}_b(1 - y_b), \quad \forall b \in B^c, \quad (6.64)$$

$$s_b \geq \sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} - \overline{M}_b(1 - y_b), \quad \forall b \in B^c, \quad (6.65)$$

$$s_b \geq 0, \quad \forall b \in B^c, \quad (6.66)$$

where $\overline{M}_b = \sum_{b_i \in \xi_b} M_{b_i} + M_b$. If $y_b = 0$, (6.63) and (6.66) force s_b to zero. Similarly, if b is accepted, i.e., if $y_b = 1$, s_b is forced to $\sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}$ by (6.64) and (6.65). The inequality (6.66) ensures that the total surplus of a block order is non negative if it is accepted. Hence, we can remove the constraints defined in (6.61) for $b \in B^c$ because (6.63)-(6.66) ensure the solution does not have PAB or sets.

Flexible orders are rejected in a period, if they are OTM in the period by Lemma 10. The following constraint ensures this condition.

$$\underline{M}_{f,t}(1 - y_{f,t}) \leq (P_f - p_t)Q_f, \quad \forall t \in T, f \in F, \quad (6.67)$$

where $\underline{M}_{f,t} = (P_f - P_{N_t,t})Q_f$ for $f \in F^S$ and $\underline{M}_{f,t} = (P_f - P_{0,t})Q_f$ for $f \in F^D$.

We present the clearing model we formulate for the supply demand balance problem without PAB below:

(MC-WPAB)

$$\max c(x, y)$$

s.t.

$$\sum_{j \in J} \Delta Q_{j,t} x_{j,t} + \sum_{b \in B} Q_{b,t} y_b + \sum_{f \in F} Q_f y_{f,t} = Q_t, \quad \forall t \in T, \quad (6.68)$$

$$y_b \leq y_{b_m}, \quad \forall b \in B^c, b \in \Theta_{b_m}, \quad (6.69)$$

$$\sum_{t \in T} y_{f,t} \leq 1, \quad \forall f \in F, \quad (6.70)$$

$$w_{0,t} \leq x_{0,t} \leq 1, \quad \forall t \in T, \quad (6.71)$$

$$w_{j,t} \leq x_{j,t} \leq w_{(j-1),t}, \quad \forall j \in \{1, \dots, N_t - 1\}, t \in T, \quad (6.72)$$

$$1 - w_{(N_t-1),t} \leq x_{N_t,t} \leq 1, \quad \forall t \in T, \quad (6.73)$$

$$p_t = P_{0,t}(1 - x_{0,t}) + P_{N_t,t}x_{N_t,t} + \sum_{j \in \{1, \dots, N_t\}} P_{j,t}(x_{(j-1),t} - x_{j,t}) \quad \forall t \in T \text{ if } N_t \geq 2, \quad (6.74)$$

$$(1 - y_b)\underline{M}_b \leq \sum_{t \in T_b} (P_b - p_t)Q_{b,t} + \sum_{b \in \Theta_b} s_{b_i}, \quad \forall b \in B \setminus B^c, \quad (6.75)$$

$$s_b \leq \overline{M}_b y_b, \quad \forall b \in B^c, \quad (6.76)$$

$$s_b \leq \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} + \bar{M}_b (1 - y_b), \quad \forall b \in B^c, \quad (6.77)$$

$$s_b \geq \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} - \bar{M}_b (1 - y_b), \quad \forall b \in B^c, \quad (6.78)$$

$$\underline{M}_{f,t} (1 - y_{f,t}) \leq (P_f - p_t) Q_f, \quad \forall f \in F, t \in T, \quad (6.79)$$

$$x_{j,t} \geq 0, \quad \forall j \in J_t, t \in T, \quad (6.80)$$

$$y_b, y_{f,t} \in \{0, 1\} \quad \forall b \in B, f \in F, t \in T, \quad (6.81)$$

$$w_{j,t} \in \{0, 1\}, \quad \forall j \in W_t, t \in T, \quad (6.82)$$

$$s_b \geq 0, \quad \forall b \in B^c. \quad (6.83)$$

The constraints (6.68)-(6.74) are the same as in the model (MC-WPRB), which are discussed in the previous section. (6.75) forces block orders to be rejected when they are OTM, which is a property of any optimal solution of the model (QPR-MC) (Lemma 9 (b) and Lemma 8 (a)-ii,(b)-ii). (6.76)-(6.78) ensure that there is no accepted subfamily which is OTM. (6.79) ensures that OTM flexible orders are rejected. The size of the model is summarized below.

Size of the the model (MC-WPAB)

Number of constraints	$ T + B^c + F + \sum_{t \in T} 2(N_t + 1) + T \mathbb{1}_{N_t \geq 2} + B + 2 B^c + T \times F .$
Number of nonnegative continuous variables	$\sum_{t \in T} (N_t + 2) + B^c $
Number of binary variables	$\sum_{t \in T} N_t + T \times F + B .$

The model (MC-WPAB-Q) has more continuous variables than the model (MC-WPAB). On the other hand, it has less number of integer variables than the model (MC-WPAB). The number of linear constraints in the model (MC-WPAB) is not less than the ones in the model (MC-WPAB-Q). However, (MC-WPAB-Q) has a quadratic

constraint and it may deteriorate the performance of the model (MC-WPAB-Q).

Properties of the model (MC-WPAB)

In this section, we define strong valid inequalities for the model (MC-WPAB). Before presenting these valid inequalities, we introduce a set definition, which is used in these inequalities.

Let k be a block or flexible order with price P_k which is valid in the period set T_k . We define the set of segments $J_{k,t}$ for all $t \in T_k$. For a sales order k :

$$J_{k,t} = \{j \in J_t | P_{(j+1),t} < P_k \leq P_{j,t}\}, \quad k \in B^S \cup F^S \quad (6.84)$$

and for a purchase order k ,

$$J_{k,t} = \{j \in J_t | P_{(j+1),t} \leq P_k < P_{j,t}\}, \quad k \in B^D \cup F^D. \quad (6.85)$$

Set $J_{k,t}$ which is defined in (6.84) and valid for sales orders, is the same as the set defined in (6.41) for purchase orders. Similarly, the set defined in (6.85) is the same as the one in (6.40), but the former is valid for purchase orders and the latter is defined for sales orders. Recall that, there is a strict order in price levels for aggregated total curve segments, i.e., $P_{(j+1),t} < P_{j,t}$. Moreover, $|J_{k,t}| = 0$ when $P_k = P_{N_t,t}$ for a sales order k and $P_k = P_{0,t}$ for a purchase order k . Hence, the cardinality of set $J_{k,t}$ is at most one for all $t \in T_k$, $k \in B$ or $k \in F$.

In Theorems 10 and 11, we relate variables y and w such that $y_b = 0$ if b and all its descendants are OTM in all periods and $y_{f,t} = 0$ if it is OTM in period t .

Theorem 10. *Let k be a purchase block or a flexible order with price P_k which is valid in period set T_k and $J_{k,t}$ is defined as in (6.85). The following are valid inequalities*

for the model (MC-WPAB).

$$y_b \leq \sum_{t \in T_b, j \in J_{b,t}} w_{(j-1),t}, \quad \forall b \in B^D \setminus B^m. \quad (6.86)$$

$$y_b \leq \sum_{t \in T_{b_i}, j \in J_{b_i,t}, b_i \in \xi_b \cup \{b\}} w_{(j-1),t}, \quad \forall b \in B^m \text{ and } b \in B^D. \quad (6.87)$$

$$y_{f,t} \leq w_{(j-1),t}, \quad \forall j \in J_{f,t}, \quad f \in F^D, \quad t \in T_f. \quad (6.88)$$

Proof. We again cover the proof in two parts: in the first one we show that (6.86)-(6.88) are valid for the model (MC-WPAB), and in the second part we show that these inequalities are strong.

PART 1: Validity

Since w is a binary variable, (6.86)-(6.88) are redundant if the left hand side of the inequalities are zero. On the other hand, if $w_{(j-1),t} = 0$ for all $j \in J_{b,t}$, $t \in T_b$, then (6.86) forces y_b to zero. Similarly, if $w_{(j-1),t} = 0$ for all $j \in J_{b_i,t}$, $t \in T_{b_i}$ for all $b_i \in \Theta_b \cup \{b\}$ then (6.87) forces y_b to zero. Finally, if $w_{(j-1),t} = 0$ for a $j \in J_{f,t}$, $t \in T_f$, then (6.88) forces $y_{f,t}$ to zero.

If $w_{(j-1),t} = 0$ implies $P_k - p_t < 0$ for $k \in B^D \cup F^D$ then (6.75)-(6.79) in the model (MC-WPAB) force the corresponding variable y to 0. Then, (6.86)-(6.88) are satisfied. Next, we show that $w_{(j-1),t} = 0$ implies $P_k - p_t < 0$ for $k \in B^D \cup F^D$ for $j \in J_{k,t}$.

$$w_{(j-1),t} = 0 \Rightarrow x_{j,t} = 0, \text{ for all } j \in J_t \text{ when } k \in B^D \cup F^D,$$

$$\text{since } w_{k,t} \leq x_{k,t} \leq w_{(k-1)t} \quad \forall k \in J_t \supset J_{k,t} \text{ by (6.71)-(6.73).}$$

$$\Rightarrow P_{j,t} \leq p_t, \quad \text{if } j \in J_{b,t} \text{ and } t \in T_b, \text{ by Theorem 3}$$

By definition, $P_{(j+1),t} \leq P_k < P_{j,t}$ for all $j \in J_{k,t}$, $t \in T_k$, $k \in B^D \cup F^D$. Hence, $P_k < p_t$, $\forall k \in B^D \cup F^D$. Hence, $P_k - p_t < 0$, $\forall B^D \cup F^D$.

PART 2: Strength

Now we show that there may be a feasible solution for the QP relaxation of the model (MC-WPRB) that violates (6.86)-(6.88).

Suppose that we use the same instance example set 1 in the proof of Theorem 6. We assume that all orders in B and f_1 are purchase orders such that $Q > 0$. We assume that $(1-\epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t} < P$ for all $j \in J_{k,t}$ and $t \in T$ where ϵ also satisfies the following condition $0 < \epsilon < \frac{1}{2N} < \frac{1}{N} < 1$. Moreover, $Q_t = Q(j + 2 + \frac{1}{N} + 2\epsilon)$ for all $t \in T$.

Let us construct a solution (x, y) for the QP relaxation of the model (MC-WPAB) model as follows:

Example Solution 1: Let $x_{j,t} = x_{(j-1),t} = w_{j,t} = w_{(j-1),t} = \epsilon$, $x_{it} = w_{it} = 1$, $x_{k,t} = w_{k,t} = 0$ for all $i, j, k \in J_t$, $t \in T$ such that $i < j - 1 < j < k < N_t$ and $j \in J_{k,t}$. We have $x_{N_t,t} = w_{N_t,t} = 1$ for all $t \in T$ and $y_b = 1$ for all $b \in B \setminus \{b_5\}$, $y_{b_5} = 0$ and $y_{f_1,t} = \frac{1}{N}$ for all $t \in T$.

The inequalities given in (6.86)-(6.88) are violated by the solution (x, y) , since:

- $N\epsilon = \sum_{t \in T, j \in J_{b_3,t}} w_{(j-1),t} = \sum_{t \in T, j \in J_{b_1,t}} w_{(j-1),t} < 1 = y_{b_3} = y_{b_1}$ (Violates (6.86)),
- $2N\epsilon = \sum_{b_i \in \{b_2, b_3\}, j \in J_{b_i,t}, t \in T} w_{(j-1),t} < 1 = y_{b_2}$ (Violates (6.87)),
- $2N\epsilon = \sum_{b_i \in \{b_4, b_5\}, j \in J_{b_i,t}, t \in T} w_{(j-1),t} < 1 = y_{b_4}$ (Violates (6.87)),
- $\epsilon = w_{(j-1),t} < \frac{1}{N} = y_{f_1,t}, \forall j \in J_{f_1,t}, t \in T$ (Violates (6.88)),

when $0 < \epsilon < \frac{1}{2N} < \frac{1}{N} < 1$. However, (x, y) is a feasible solution for the QP relaxation of the model (MC-WPAB) model, which we show next.

Note that (x, y) satisfies the supply-demand balance constraint (6.68) as follows :

$$\begin{aligned}
Q_t &= \sum_{l \in J_t} x_{l,t} \Delta Q_{l,t} + \sum_{b \in B} y_b Q_{b,t} + \sum_{f \in F} y_{f,t} Q_f, \\
&= \sum_{l \in J_t} x_{l,t} Q + 4 \times Q + \frac{1}{N} Q, \\
&= Q \sum_{i < (j-1), i \in J_t} x_{it} + Q x_{(j-1),t} + Q x_{j,t} + Q \sum_{N_t > k > j, k \in J_t} x_{k,t} - Q + 4Q + \frac{1}{N} Q, \quad \forall t \in T. \\
&= Q \sum_{i < (j-1), i \in J_t} 1 + \epsilon Q + \epsilon Q + Q \sum_{N_t > k > i, k \in J_t} 0 + 3Q + \frac{1}{N} Q, \quad \forall t \in T. \\
&= Q(j - 2 + 1 + 3 + \frac{1}{N} + 2\epsilon) = Q_t, \quad \forall t \in T
\end{aligned}$$

Recall that the constructed solution satisfies (6.69) and (6.70). Furthermore we have shown that (x, w) satisfies (6.71)-(6.74) in the proof of Theorem 6. We know that $P > p_t = (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t}$ by definition for all $j \in J_{k,t}$, $t \in T$. After we re-arrange the terms, we obtain $(P - p_t)Q > 0$ for all $t \in T$ since $Q > 0$. Then, $\sum_{t \in T} (P - p_t)Q > 0$, which implies that b_1, b_2, b_3 and b_4 are ITM. Hence, (6.75)-(6.78) are satisfied. Finally, f_1 is ITM in all periods because $(P - p_t)Q > 0$ for all $t \in T$ so that (6.79) is satisfied.

Therefore, all constraints are satisfied and the solution (x, y) defined above is feasible for the model (MC-WPAB) model. On the other hand, (6.86)-(6.88) are violated. Hence, these inequalities are valid and strong. $\square$

Theorem 11. *Let k be a block or a flexible sales order with price P_k which is valid in period set T_k , where $|T_{k,t}| = N_k$. We define the set of segments $J_{k,t}$ for all $t \in T_k$ as in (6.84). The following inequalities are valid and strong for the model (MC-WPAB).*

$$\sum_{t \in T_b, j \in J_{b,t}} w_{j,t} \leq N_b - y_b, \quad \forall b \in B^S \setminus B^m. \quad (6.89)$$

$$\sum_{t \in T_{b_i}, j \in J_{b_i,t}, b_i \in \xi_b \cup \{b\}} w_{j,t} \leq \sum_{b_i \in \xi_b \cup \{b\}} N_{b_i} - y_b, \quad \forall b \in B^m \text{ and } b \in B^S. \quad (6.90)$$

$$w_{j,t} \leq 1 - y_{f,t}, \quad \forall j \in J_{f,t}, \quad f \in F^S, \quad t \in T_f. \quad (6.91)$$

Proof. We also divide the proof into two parts consisting of the validity and strength of the inequalities.

PART 1: Validity

Since w is a binary variable and $|J_{k,t}|$ is at most 1 for all $t \in T_k$, (6.89)-(6.91) are redundant if y_b or $y_{f,t}$ are 0. On the other hand, if $w_{j,t} = 1$ for all $j \in J_{b,t}$, $t \in T_b$, then (6.89) forces y_b to zero. Similarly, if $w_{j,t} = 1$ for all $j \in J_{b_i,t}$, $t \in T_{b_i}$, $b_i \in \xi_b \cup \{b\}$, then (6.90) forces y_b to zero. Finally, if $w_{j,t} = 1$ for a $j \in J_{f,t}$, $t \in T_f$, then (6.91) forces $y_{f,t}$ to zero.

If $w_{j,t} = 1$ implies that $(P_k - p_t)Q_k < 0$ for $j \in J_{k,t}$, $t \in T_k$ and $k \in B^S \cup F^S$, then each order which is subject to the inequalities in this theorem is OTM. Thus, (6.75)-(6.79) in the model (MC-WPAB) force the corresponding y variables to zero and (6.89)-(6.91) are also satisfied. In the following lines, we use direct proof and show that $w_{j,t} = 1$ implies $(P_k - p_t)Q_k < 0$ for $j \in J_{k,t}$, $t \in T_k$ and $k \in B^S \cup F^S$ in the model (MC-WPAB).

$$w_{j,t} = 1 \Rightarrow x_{j,t} = 1, \text{ for all } j \in J_t \text{ when } k \in B^S \cup F^S,$$

$$\text{since } w_{k,t} \leq x_{k,t} \leq w_{(k-1),t} \quad \forall k \in J_t \supset J_{k,t}$$

$$\text{by (6.71)-(6.73).}$$

$$\Rightarrow p_t \leq P_{(j+1),t}, \quad \text{if } j \in J_{b,t} \text{ and } t \in T_b, \text{ by Theorem 3.}$$

By definition, $P_{(j+1),t} < P_k \leq P_{j,t}$ for all $j \in J_{k,t}$, $t \in T_k$, $k \in B^S \cup F^S$. Hence, $P_k > p_t$, $\forall k \in B^S \cup F^S$. Let Q_k be the quantity of order k , where $k \in B^S \cup F^S$. We know that $Q_k < 0$ for all $k \in B^S \cup F^S$. Then, $(P_k - p_t)Q_k < 0$, $\forall B^S \cup F^S$. Hence our assumption holds.

PART 2: Strength

Now we can construct an instance such that there exists a feasible solution for the QP relaxation of the model (MC-WPAB) that is infeasible for (6.89)-(6.91).

Suppose that we use the same parameter set we defined in the example set 1 in the proof of Theorem 6 and the same solution as the example solution 1 in Theorem 10.

We assume that all block orders in B and f_1 are sales orders. Moreover, we assume that $P < (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t}$ for all $j \in J_{k,t}$ and $t \in T$, $k \in B \cup \{f_1\}$ where ϵ also satisfies the condition $1 - \frac{1}{N} < 1 - \frac{1}{2N} < \epsilon < 1$. Moreover, $Q_t = Q(j - 6 + \frac{1}{N} + 2\epsilon)$ for all $t \in T$, $j \in J_{k,t}$.

If we substitute the values of w and y into the (6.89)-(6.91) we can show that the resulting inequalities contradict $1 - \frac{1}{N} < 1 - \frac{1}{2N} < \epsilon < 1$. Therefore, the inequalities defined in (6.89)-(6.91) are violated by the solution (x, y) . However, (x, y) is a feasible solution for the QP relaxation of the model (MC-WPAB).

The solution (x, y) satisfies the supply-demand balance constraint (6.68) as follows:

$$\begin{aligned}
 Q_t &= \sum_{l \in J_t} x_{l,t} \Delta Q_{l,t} + \sum_{b \in B} y_b Q_{b,t} + \sum_{f \in F} y_{f,t} Q_f, \\
 &= \sum_{l \in J_t} x_{l,t} Q - 4 \times Q - \frac{1}{N} Q, \quad \forall t \in T. \\
 &= Q \sum_{i < (j-1), i \in J_t} x_{it} + Q x_{(j-1),t} + Q x_{j,t} + Q \sum_{N_t > k > j, k \in J_t} x_{k,t} - Q - 4Q - \frac{1}{N} Q, \quad \forall t \in T. \\
 &= Q \sum_{i < (j-1), i \in J_t} 1 + \epsilon Q + \epsilon Q + Q \sum_{k > i, k \in J_t} 0 - 5Q - \frac{1}{N} Q, \quad \forall t \in T. \\
 &= Q(j - 2 + 1 - 5 - \frac{1}{N} + 2\epsilon) = Q_t, \quad \forall t \in T
 \end{aligned}$$

We showed that (6.68) is satisfied by the solution (x, y) and in the proof of Theorem 10 we show (6.69) and (6.70) are also satisfied. Since the values of x and w values are the same with the ones defined in the proof of Theorem 6, (6.71)-(6.74) are satisfied as we illustrated before. Finally, we know that $P < p_t = (1 - \epsilon)P_{(j-1),t} + \epsilon P_{(j+1),t}$ by definition for all $j \in J_{k,t}$, $t \in T$. After we re-arrange the terms, we obtain $-(P - p_t)Q > 0$ for all $t \in T$ since $Q > 0$. Then, (6.75)-(6.79) are satisfied.

Therefore, (x, y) is a feasible solution of the QP relation of the model (MC-WPAB) but it violates (6.89)-(6.91). Thus, the inequalities are valid and strong. $\square$

Remark 4. In Appendix A, we show the equivalence of the conditions in Theorem 6-11 if supply and demand curves are separately used as in Chapter 5.

Theorem 12. Suppose that there are two block orders b_1 and b_2 where $b_1 \in B^D \setminus B^c$ and $b_2 \in B^S \setminus B^c$. If $P_{b_1}N_{b_1} - P_{b_2}N_{b_2} + \sum_{t \in T_{b_2} \setminus T_{b_1}} P_{0,t} - \sum_{t \in T_{b_1} \setminus T_{b_2}} P_{N_t,t} > 0$, then

$$y_{b_1} + y_{b_2} \leq 1 \quad (6.92)$$

is a valid inequality for the model (MC-WPAB).

Proof. PART 1: Validity The block orders b_1 and b_2 must satisfy the following constraints in a feasible solution of the model (MC-WPAB):

$$M_{b_1}(y_{b_1} - 1) \leq \sum_{t \in T_{b_1}} (P_{b_1} - p_t)Q_{b_1}$$

$$M_{b_2}(y_{b_2} - 1) \leq \sum_{t \in T_{b_2}} (P_{b_2} - p_t)Q_{b_2},$$

where $Q_{b_1} > 0 > Q_{b_2}$. When we divide these inequalities respectively by Q_{b_1} and Q_{b_2} and sum the resulting inequalities we obtain

$$\begin{aligned} \frac{M_{b_1}}{Q_{b_1}}(y_{b_1} - 1) + \frac{M_{b_2}}{Q_{b_2}}(y_{b_2} - 1) &\leq \sum_{t \in T_{b_1}} (P_{b_1} - p_t) - \sum_{t \in T_{b_2}} (P_{b_2} - p_t), \\ \frac{M_{b_1}}{Q_{b_1}}(y_{b_1} - 1) + \frac{M_{b_2}}{Q_{b_2}}(y_{b_2} - 1) &\leq P_{b_1}N_{b_1} - P_{b_2}N_{b_2} + \sum_{t \in T_{b_2} \setminus T_{b_1}} p_t - \sum_{t \in T_{b_1} \setminus T_{b_2}} p_t \end{aligned}$$

If the right hand side of the above inequality is negative, then at least one of the y variables must be zero. This is satisfied if the maximum possible value of the right hand side is negative, i.e., $P_{b_1}N_{b_1} - P_{b_2}N_{b_2} + \sum_{t \in T_{b_2} \setminus T_{b_1}} P_{0,t} - \sum_{t \in T_{b_1} \setminus T_{b_2}} P_{N_t,t} < 0$.

PART 2: Strength

Assume that $T_{b_1} = T_{b_2} = T$ and $P_{b_2} > p_t > P_{b_1}$ for all $t \in T$ and $Q_{b_1} = 1$, $Q_{b_2} = -1$. Furthermore, $P_{b_1} = (1 - \beta)P_{b_2} + \frac{\beta P_{0,t}}{|T|}$ and $P_{b_2} = (1 - \alpha)P_{b_1} + \frac{\alpha P_{N_t,t}}{|T|}$. Also we assume that supply balance is satisfied. Let $y_{b_1} = \alpha$ and $y_{b_2} = \beta$, where $\alpha \in [0, 1]$, $\beta \in [0, 1]$ and $\alpha + \beta > 1$, therefore, (6.92) is violated. However, this solution is feasible

for the model (MC-WPRB) because;

$$\begin{aligned} \sum_{t \in T_{b_1}} (P_{0,t} - P_{b_1})(\alpha - 1) &= M_{b_1}(y_{b_1} - 1) \leq \sum_{t \in T_{b_1}} (P_{b_1} - p_t)1, \\ - \sum_{t \in T_{b_1}} (P_{N_t,t})(\beta - 1) &= -M_{b_2}(y_{b_2} - 1) \leq \sum_{t \in T_{b_2}} -(P_{b_2} - p_t), \end{aligned}$$

equivalent to $\alpha P_{b_1} + \frac{(1-\alpha)P_{0,t}}{|T|} = P_{b_2} > p_t > P_{b_1} = \beta P_{b_1} + \frac{(1-\beta)P_{N_t,t}}{|T|}$. $\square$

We will investigate the impact of these cuts on the model in Chapter 8.

6.2 Summary

The supply demand balance problem may result in PABs or PRBs. In this chapter, we propose a new variant of the supply demand balance problem which does not contain PRBs in its feasible region. We show that the variant we propose is at Walrasian equilibrium. We also state the supply demand balance problem without PABs which is used in Nord Pool Spot.

We formulate two different variants of the market clearing problem as MIQCQP, which contains only one quadratic constraint. We extend the formulation used in Madani and Van Vyve (2015) for different order types and market rule settings. To the best of our knowledge there is no model in the literature which covers block, linked block and flexible orders while supporting all the properties coming from the duality for problems without PRB. We also formulate the variants with linear constraints and present strong valid cuts for these models. In the next chapter, we will propose exact solution methodologies to solve the optimization models presented in this chapter.

We summarize the new symbols we define in this chapter in Table 6.1.

Table 6.1: Sets, indices, parameters and variables used in this chapter

Sets and Indices

W_t	Set of index values, where $W_t = \{0, \dots, N_t - 1\}$
$J_{k,t}$	Set of segments that $J_{k,t} = \{j \in J_t P_{(j+1),t} < P_k \leq P_{j,t}\}$ for the model (MC-WPRB) and $k \in B^D \cup F^D$ or for the model (MC-WPAB) and $k \in B^S \cup F^S$. Similarly, $J_{k,t} = \{j \in J_t P_{(j+1),t} \leq P_k < P_{j,t}\}$ for the model (MC-WPRB) and $k \in B^S \cup F^S$ or for the model (MC-WPAB) and $k \in B^D \cup F^D$.

Variables

$w_{j,t}$	$\begin{cases} 1, & \text{if segment } j \text{ is fully accepted,} \\ 0, & \text{other wise,} \end{cases}$
	for all $j \in W_t, t \in T$.

Chapter 7

SOLUTION METHODOLOGIES

The electricity market of interest in this thesis is a daily based market and its results affect the other mechanisms such as intra-day electricity market, futures contracts and day ahead grid security analysis. The earliest mechanism that requires the matching results of the DAM is the day ahead grid security analysis, which is performed by the system operators (SOs). In day-ahead security analysis, SOs take the matching quantities resulting from DAM and bilateral agreements to plan the electricity that could flow safely in the next day in the grid. Therefore, the process requires an input solution from the models proposed in the Chapter 6.

However, integer variables, quadratic objective function, quadratic constraints and big-M constraints give rise to a complicated optimization model. The integer variables are needed in the model to represent the block and flexible orders. Hence, we cannot avoid the integer variables. We can eliminate the quadratic constraints by adding additional constraints and auxiliary binary variables to the models as we did in the previous chapter. However, this method increases the problem size and requires additional constraints and variables when the flow constraints are added to the problem as in the European DAMs.

In this chapter, we propose several approaches to overcome problems arising from auxiliary binary variables, big-M type constraints and quadratic objective functions. We first identify a relation that is satisfied by any optimal solution of the model (MC). We approximate the objective function with linear functions so that we convert the MIQP models into MILPs. We present conditions under which the proposed method yields an optimal solution of the problems proposed in Chapter 6. We also decompose

the models (MC-WPRB) and (MC-WPAB) into two and solve the resulting problems with Bender's decomposition, which eliminates the big-M constraints and auxiliary integer variables of the models (MC-WPRB) and (MC-WPAB).

Our contributions are as follows: (1) We identify a new cut which is valid for any optimal solution of (MC). (2) We propose a new approach for solving the models (MC-WPRB) and (MC-WPAB) proposed in Chapter 6 by approximating the quadratic objective function by a piecewise linear function. (3) We propose a new decomposition procedure whose cuts are stronger than those proposed in Madani and Van Vyve (2015)

7.1 Adding Valid Inequalities To The Model

Adding valid inequalities to a model may improve the solution time. In the previous chapter, we show some strong valid cuts for the models (MC-WRPB) and (MC-WPAB). In this section, we show some properties of the optimal solution of the model (MC). By using these inequalities we derive a cut in Lemma 18, which provides a necessary condition for the optimality in the model (MC). Since, it is an optimality condition for the model (MC), it is also valid for any problem variants. To the best of our knowledge, this is the first such inequality.

We now focus on how the objective function changes as one moves from one feasible solution to another one in the model (MC). Recall that $TQ_t(p)$ is the aggregated total quantity function in period t . Since supply demand balance must hold, $TQ_t(p_t) = -\sum_{b \in B} y_b Q_{b,t} - \sum_{b \in F} \bar{y}_{f,t} Q_f - Q_t$, where p_t is the clearing price for period t . We know that, TQ_t is a non-increasing function of p , which yields the following lemma.

Lemma 15. *Suppose there are two solutions denoted by $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ and $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ where their dual prices are $\bar{p}_t$ and $\underline{p}_t$, respectively. Then, the following relation always holds:*

$$(\bar{p}_t - \underline{p}_t)q_t \leq 0, \quad \forall t \in T \quad (7.1)$$

where $q_t = TQ_t(\bar{p}_t) - TQ_t(\underline{p}_t)$.

The following lemma identifies some cases with PRB or PAB.

Lemma 16. *Consider two feasible solutions for the model (MC) denoted by $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ and $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, where a block/flexible order b is accepted in $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ and b is rejected in $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, the remaining block and flexible order decisions are the same for solution $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ and solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.*

- If block/flexible order b is ITM in $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ then it is PRB in $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$.
- If block/flexible order b is OTM in $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ then it is PAB in $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

Proof. We prove the assertion for a block order b . We can use similar arguments for a flexible order f . Suppose $\underline{p}_t$ is the MCP for $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ and $\bar{p}_t$ is the MCP for $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ for all $t \in T$. By Lemma 15, $-(\bar{p}_t - \underline{p}_t)Q_{b,t} \leq 0$ for all $t \in T_b$. By adding and subtracting the term $P_b Q_{b,t}$ we obtain $(P_b - \underline{p}_t)Q_{b,t} \geq (P_b - \bar{p}_t)Q_{b,t}$ for all $t \in T_b$. By summing these terms for all $t \in T_b$, we obtain, $\sum_{t \in T_b} (P_b - \underline{p}_t)Q_{b,t} \geq \sum_{t \in T_b} (P_b - \bar{p}_t)Q_{b,t}$.

- If b is ITM in $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, then $\sum_{t \in T_b} (P_b - \underline{p}_t)Q_{b,t} \geq \sum_{t \in T_b} (P_b - \bar{p}_t)Q_{b,t} > 0$. Then, the order is also ITM in $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$. Since the order b is rejected in solution $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, it is a PRB.
- If b is OTM in $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, then $0 > \sum_{t \in T_b} (P_b - \underline{p}_t)Q_{b,t} \geq \sum_{t \in T_b} (P_b - \bar{p}_t)Q_{b,t}$. Then, the order is also OTM in $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Hence, the accepted order b causes a PAB.

□

In the following lemma, we compare the objective functions of two solutions.

Lemma 17. *Consider two feasible solutions for the model (MC) as $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ and $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, where $\underline{p}_t$ denotes MCP for $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ and $\bar{p}_t$ denotes MCP for $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ for all $t \in T$. If $c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \geq c(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, then the following inequality holds.*

$$0 \geq \sum_{b \in B, t \in T_b} (P_b - \underline{p}_t)Q_{b,t}(\underline{y}_b - \bar{y}_b) + \sum_{f \in F, t \in T_f} (P_f - \underline{p}_t)Q_{f,t}(\underline{y}_{f,t} - \bar{y}_{f,t}) \quad (7.2)$$

Proof. Define $q_t = TQ_t(\bar{p}_t) - TQ_t(\underline{p}_t)$.

$$\begin{aligned}
c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) &= \sum_{t \in T, j \in J_t} P_{jt} \Delta Q_{jt} \bar{x}_{jt} - \frac{\Delta P_{jt} \Delta Q_{jt} \bar{x}_{jt}^2}{2} + \sum_{b \in B, t \in T_b} P_b Q_{b,t} \bar{y}_b + \sum_{f \in F, t \in T_f} P_f Q_f \bar{y}_{f,t} \\
&\geq c(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \\
&= \sum_{t \in T, j \in J_t} P_{jt} \Delta Q_{jt} \underline{x}_{jt} - \frac{\Delta P_{jt} \Delta Q_{jt} \underline{x}_{jt}^2}{2} + \sum_{b \in B, t \in T_b} P_b Q_{b,t} \underline{y}_b + \sum_{f \in F, t \in T_f} P_f Q_f \underline{y}_{f,t} \\
&\geq \sum_{t \in T, j \in J_t} P_{jt} \Delta Q_{jt} \bar{x}_{jt} - \frac{\Delta P_{jt} \Delta Q_{jt} \bar{x}_{jt}^2}{2} + \sum_{b \in B, t \in T_b} P_b Q_{b,t} \underline{y}_b + \sum_{f \in F, t \in T_f} P_f Q_f \underline{y}_{f,t} \\
&\quad - \sum_{t \in T} \underline{p}_t q_t,
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{b \in B, t \in T_b} P_b Q_{b,t} \bar{y}_b + \sum_{f \in F, t \in T_f} P_f Q_f \bar{y}_{f,t} &\geq \sum_{b \in B, t \in T_b} P_b Q_{b,t} \underline{y}_b + \sum_{f \in F, t \in T_f} P_f Q_f \underline{y}_{f,t} - \sum_{t \in T} \underline{p}_t q_t. \\
0 &\geq \sum_{b \in B, t \in T_b} P_b Q_{b,t} (\underline{y}_b - \bar{y}_b) \\
&\quad + \sum_{f \in F, t \in T_f} P_f Q_f (\underline{y}_{f,t} - \bar{y}_{f,t}) - \sum_{t \in T} \underline{p}_t q_t. \tag{7.3}
\end{aligned}$$

From supply demand balance equality; $TQ_t(\underline{p}_t) + \sum_{b \in B} Q_{b,t} \underline{y}_b + \sum_{f \in F} Q_f \underline{y}_{f,t} = 0$ and

$TQ_t(\bar{p}_t) + \sum_{b \in B} Q_{b,t} \bar{y}_b + \sum_{f \in F} Q_f \bar{y}_{f,t} = 0$. Hence, q_t is given by;

$$\begin{aligned}
q_t &= \sum_{b \in B} Q_{b,t} \underline{y}_b + \sum_{f \in F} Q_f \underline{y}_{f,t} - \sum_{b \in B} Q_{b,t} \bar{y}_b - \sum_{f \in F} Q_f \bar{y}_{f,t} \\
&= \sum_{b \in B} Q_{b,t} (\underline{y}_b - \bar{y}_b) + \sum_{f \in F} Q_f (\underline{y}_{f,t} - \bar{y}_{f,t}).
\end{aligned}$$

Substituting q_t into (7.3) yields the following inequality.

$$\begin{aligned}
0 &\geq \sum_{b \in B, t \in T_b} P_b Q_{b,t}(\underline{y}_b - \bar{y}_b) + \sum_{f \in F, t \in T_f} P_f Q_f(\underline{y}_{f,t} - \bar{y}_{f,t}) \\
&\quad - \sum_{t \in T} \underline{p}_t \left(\sum_{b \in B} Q_{b,t}(\underline{y}_b - \bar{y}_b) + \sum_{f \in F} Q_f(\underline{y}_{f,t} - \bar{y}_{f,t}) \right) \\
&= \sum_{b \in B, t \in T_b} (P_b - \underline{p}_t) Q_{b,t}(\underline{y}_b - \bar{y}_b) + \sum_{f \in F, t \in T_f} (P_f - \underline{p}_t) Q_f(\underline{y}_{f,t} - \bar{y}_{f,t})
\end{aligned}$$

□

Lemma 17 presents a useful result that can be used for comparing any two feasible solutions of the model (MC). The following lemma provides a necessary condition for optimality in the model (MC). It is a result from Lemma 17.

Lemma 18. *Consider a feasible solution for the model (MC) denoted by $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, where $\underline{p}_t$ is MCP of this solution. The following holds for any optimal solution $(\mathbf{x}, \mathbf{y})$ of the model (MC)*

$$\begin{aligned}
&\sum_{b \in B, t \in T_b, \underline{y}_b = 0} (P_b - \underline{p}_t) Q_{b,t} \underline{y}_b + \sum_{f \in F, t \in T_f, \underline{y}_{f,t} = 0} (P_f - \underline{p}_t) Q_f \underline{y}_{f,t} \\
&\geq \sum_{b \in B, t \in T_b, \underline{y}_b = 1} (P_b - \underline{p}_t) Q_{b,t} (1 - \underline{y}_b) \\
&\quad + \sum_{f \in F, t \in T_f, \underline{y}_{f,t} = 1} (P_f - \underline{p}_t) Q_f (1 - \underline{y}_{f,t}) \tag{7.4}
\end{aligned}$$

We remark that the condition given in Lemma 18 is satisfied by the solution $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, which may be optimal. Hence, this inequality does not cut any optimal solution but may cut some feasible solutions which cannot be optimal.

If there exists a better solution than $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ then this solution must satisfy the

following inequality:

$$\begin{aligned}
& \sum_{b \in B, \underline{y}_b = 0, \sum_{t \in T_b} (P_b - \underline{p}_t) Q_{b,t} > 0} y_b + \sum_{f \in F, t \in T_f, \underline{y}_{f,t} = 0, (P_f - \underline{p}_t) Q_{f,t} > 0} y_{f,t} \\
& + \sum_{b \in B, \underline{y}_b = 1, \sum_{t \in T_b} (P_b - \underline{p}_t) Q_{b,t} < 0} (1 - y_b) \\
& + \sum_{f \in F, t \in T_f, \underline{y}_{f,t} = 1, (P_f - \underline{p}_t) Q_{f,t} < 0} (1 - y_{f,t}) \geq 1 \quad (7.5)
\end{aligned}$$

where $\underline{p}_t$ is the market clearing price of the solution $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$. This inequality is a direct result of the condition given in Lemma 18. It implies that if there exists a better solution than the current one, one of the rejected ITM block orders or one of the rejected ITM flexible order must be accepted or one of the accepted OTM block orders or one of the accepted OTM flexible order must be rejected.

This inequality can be used in a local search method to create a neighborhood around solution $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$. Actually, there are some studies which use a similar logic (Kurt et al., 2018; Derinkuyu et al., 2019). Our result explains why these methods give a good performance.

In this study, we test the performance of adding the inequality in Lemma 18 to the model (MC) (variants).

7.2 Approximations of the Objective Function

The objective function used in our formulations has quadratic terms due to piecewise linear curves. However, there are power exchanges in Europe (e.g. EPEX, OMIE, GME, etc.) which use stepwise curves, whose market clearing models result in a linear objective function. Models with linear objective functions are usually easier to solve than the models which have quadratic objectives.

In this section, we propose two different piecewise linear approximations of the objective function $c(\cdot)$. By using these functions, we present three models (two of them arising from the approximations, the other one is a model with the original objective

function c). We find an upper bound on the approximation error arising from the linearizations. We also present a condition obtained by the optimal solutions of the linearizations, that is the sufficient condition for the original problem to be optimal.

Our contributions to the literature are as follows: (1) Although there is a work which uses a linear function to find a feasible solution for the market clearing problem with quadratic objective function, there is no information about how far we can improve the objective function when we use the solution of the linearizations. This is the first study which evaluates the quality of the solutions of the linearizations. (2) By using the approximation error, we propose a sufficient condition for the original problem to be optimal. (3) We propose a new algorithm to illustrate a way of solving the original problem by using approximations.

7.2.1 Piecewise Linear Approximations

In this section we provide two linear approximations of the objective function c as in Figure 7.1.

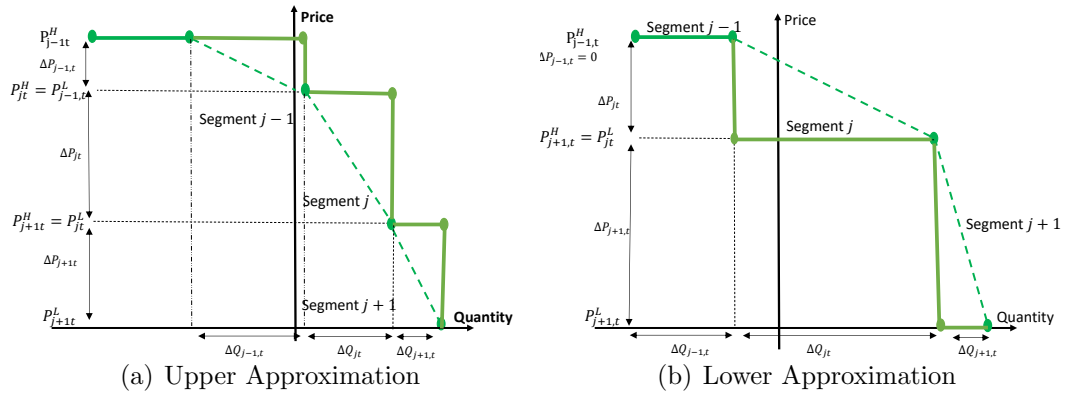


Figure 7.1: Stepwise approximations of the original piecewise function c

The approximate functions in Figure 7.1 are formulated as follows:

$$\begin{aligned}\bar{c}(x, y) &= \sum_{j \in J_t, t \in T} P_{jt} \Delta Q_{jt} x_{j,t} + \sum_{b \in B, t \in T} P_b Q_{b,t} y_b + \sum_{f \in F, t \in T} P_f Q_{f,t} y_{f,t} \\ &= c(x, y) + f(x)\end{aligned}\quad (7.6)$$

$$\begin{aligned}\underline{c}(x, y) &= \sum_{j \in J_t, t \in T} P_{jt}^L \Delta Q_{jt} x_{j,t} + \sum_{b \in B, t \in T} P_b Q_{b,t} y_b + \sum_{f \in F, t \in T} P_f Q_{f,t} y_{f,t} \\ &= c(x, y) - h(x) + f(x)\end{aligned}\quad (7.7)$$

,

where $h(x)$ and $f(x)$ are:

$$f(x) = \sum_{t \in T} \sum_{j \in J_t} \frac{\Delta P_{jt} \Delta Q_{jt} x_{jt}^2}{2} \quad (7.8)$$

$$h(x) = \sum_{t \in T} \sum_{j \in J_t} \Delta P_{jt} \Delta Q_{jt} x_{jt} \quad (7.9)$$

It is straightforward to verify that $c(x, y) \leq \bar{c}(x, y)$, $\forall (x, y) \in C$ by (7.6) because $0 \leq f(x)$. This shows that $\bar{c}(x, y)$ is an overestimating function of $c(x, y)$ on C , i.e., it always exceeds $c(x, y)$ on C . Since $0 \leq f(x) \leq h(x)$, $\underline{c}(x, y)$ is an underestimating function of $c(x, y)$ on C , i.e., it never exceeds $c(x, y)$ on C . Moreover, both $\underline{c}(x, y)$ and $\bar{c}(x, y)$ are linear functions.

We can summarize the relations between the objective functions via:

$$\underline{c}(x, y) \leq c(x, y) \leq \bar{c}(x, y), \quad \forall (x, y) \in C.$$

We will use these linear functions to form approximations in the next section.

7.2.2 Approximations

In this section we define the approximate problems and investigate their relations.

We first construct three different models as (P), (LB-P), and (UB-P) as in Table 7.1. These models have the same feasible region but different objective functions as c , $\underline{c}$, and $\bar{c}$. Then, (UB-P) gives an upper bound on (P) whereas (LB-P) provides lower bound.

Table 7.1: The models (P), (LB-P), and (UB-P)

(P)	$(LB-P)$	$(UB-P)$
$\max c(x, y)$	$\max \underline{c}(x, y)$	$\max \bar{c}(x, y)$
<i>s.t.</i>	<i>s.t.</i>	<i>s.t.</i>
$(x, y) \in C$	$(x, y) \in C$	$(x, y) \in C$

We next derive error bounds on the deviations of the optimal values of (LB-P) and (UB-P) from the optimal value of (P). First, we propose a lemma that shows how an optimal solution of a problem changes if we know the optimal solutions of its subproblems.

Lemma 19. *If $(\bar{x}, \bar{y})$ is an optimal solution for the approximate problems $\max a(x, y)$ subject to $(x, y) \in C$ and $\max b(x, y)$ subject to $(x, y) \in C$, then it is also the best feasible solution for $\max a(x, y) + b(x, y)$ subject to $(x, y) \in C$.*

Proof. Assume that $(\mathbf{x}^*, \mathbf{y}^*)$ is the best feasible solution for $\max a(x, y) + b(x, y)$ subject to $(x, y) \in C$ such that that $(\bar{x}, \bar{y}) \neq (\mathbf{x}^*, \mathbf{y}^*)$ and $a(\bar{x}, \bar{y}) + b(\bar{x}, \bar{y}) < a(\mathbf{x}^*, \mathbf{y}^*) + b(\mathbf{x}^*, \mathbf{y}^*)$. Since $(\bar{x}, \bar{y})$ is the best feasible solution for the approximate problems, we can write,

$$a(\mathbf{x}^*, \mathbf{y}^*) \leq a(\bar{x}, \bar{y})$$

$$b(\mathbf{x}^*, \mathbf{y}^*) \leq b(\bar{x}, \bar{y})$$

After summing the inequalities we obtain that $a(\mathbf{x}^*, \mathbf{y}^*) + b(\mathbf{x}^*, \mathbf{y}^*) \leq a(\bar{x}, \bar{y}) + b(\bar{x}, \bar{y})$, which is a contradiction. $\square$

We set $a(x, y) = \bar{c}$ and $b(x, y) = \underline{c}^Q(x, y)$, where $\underline{c}^Q(x, y) = c(x, y) - f(x)$. We define a model called (QLB-P) which maximizes a quadratic function $\underline{c}^Q(x, y) = c(x, y) - f(x)$ subject to $(x, y) \in C$. We will use this model as an intermediate step

for analyzing the quality of the optimal solutions of (LB-P) and (UB-P). Corollary 2 reveals a relation among the optimal solutions of (QLB-P), (UB-P) and (P).

Corollary 2. *If $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is an optimal solution for (QLB-P) and (UB-P), then it is also the best feasible solution for the (P).*

Corollary 2 states that an optimal solution for (P) is available if (UB-P) and (QLB-P) have the same optimal solution. However, in order to check this condition we need to solve (QLB-P) whose objective function has quadratic terms. Instead, we aim to solve “easier” problems as (LB-P) and (UB-P). In the following corollary, we present a set of models whose optimal solutions may be known by solving only (LB-P) and (UB-P).

Lemma 20. *If $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is the best solution for (LB-P) and (UB-P), then it is also the best feasible solution for the $\max c(x, y) + f(x) - \frac{h(x)}{2^n}$, subject to $(x, y) \in C$ for any choice of $n \in \mathbb{Z}^+$.*

Proof. We can follow the argument in Lemma 19 and show that if (LB-P) and (UB-P) have the same optimal solution, then it is also an optimal solution for $\max c(x, y) + f(x) - h(x)/2$ s.t. $(x, y) \in C$. When we continue this argument, we can generate different models which has the same optimal solution with (LB-P) and (UB-P) such that $\max c(x, y) + f(x) - \frac{h(x)}{2^n}$, s.t. $(x, y) \in C$ for any choice of $n \in \mathbb{Z}^+$ $\square$

The optimal solutions of (LB-P) and (UB-P) yield a pair of feasible solutions of (P). We next focus on the quality of these feasible solutions. We can measure the deviations of the optimal values of (LB-P) and (UB-P) from the optimal value of (P), which is defined as approximation error (AE). The following two lemmas give upper bound on AE when (UB-P) and (LB-P) have different and the same optimal solutions.

Lemma 21. *Let $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ be an optimal solution for (LB-P) and $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ be an optimal*

solution for (UB-P) such that they are not the same. Then, AE can be at most,

$$f(\bar{x}) = \sum_{j \in J_t, t \in T} \frac{\bar{x}_{jt}^2 \Delta P_{jt} \Delta Q_{jt}}{2}.$$

Proof. Assume that $(\mathbf{x}^*, \mathbf{y}^*)$ is an optimal solution for (P). We define approximation error of $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ as $c(\mathbf{x}^*, \mathbf{y}^*) - c(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = AE_{(\underline{\mathbf{x}}, \underline{\mathbf{y}})}$. From the optimality conditions we can write $\underline{c}(\mathbf{x}^*, \mathbf{y}^*) \leq \underline{c}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ and $\bar{c}(\mathbf{x}^*, \mathbf{y}^*) \leq \bar{c}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Or by using (7.6) and (7.7):

$$\begin{aligned} c(\mathbf{x}^*, \mathbf{y}^*) + f(x^*) - h(x^*) &\leq c(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + f(\underline{x}) - h(\underline{x}) \\ c(\mathbf{x}^*, \mathbf{y}^*) + f(x^*) &\leq c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) + f(\bar{x}) \end{aligned}$$

Therefore we obtain:

$$\begin{aligned} AE_{(\underline{\mathbf{x}}, \underline{\mathbf{y}})} &= c(\mathbf{x}^*, \mathbf{y}^*) - c(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \leq h(x^*) - f(x^*) - (h(\underline{x}) - f(\underline{x})) \\ AE_{(\bar{\mathbf{x}}, \bar{\mathbf{y}})} &= c(\mathbf{x}^*, \mathbf{y}^*) - c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \leq f(\bar{x}) - f(x^*) \end{aligned} \tag{7.10}$$

Since $0 \leq f(x)$, $AE_{(\bar{\mathbf{x}}, \bar{\mathbf{y}})} \leq f(\bar{x})$. □

One observation coming from the proof of Lemma 21 is that, the upper bound problem limits the AE. Hence, the closer overestimating function we have, the lower AE we get. The following lemma shows the best AE we can obtain when (UB-P) and (LB-P) have the same best solutions.

Lemma 22. *Let $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ be an optimal solution for both (UB-P) and (LB-P). Then, AE can be at most $f(\bar{x}) - h(\bar{x})/2^n$ where n is the smallest integer that $f(\bar{x}) - h(\bar{x})/2^n > 0$.*

Proof. Lemma 20 states that, when (UB-P) and (LB-P) have the same optimal solution, then $\max c(x, y) + f(x) - h(x)/2^n$ subject to $x \in C$ has the same optimal solution. If n is the smallest integer number that $f(\bar{x}) - \frac{h(\bar{x})}{2^n} > 0$ then $c(x, y) + \frac{h(\bar{x})}{2^n}$ is an upper approximating function of c . By following the same arguments in Lemma 21 we can show that $f(\bar{x}) - \frac{h(\bar{x})}{2^n}$ is the AE for the current solution. □

We mark each solution in relative gap as optimal. Therefore, if $AE_{(x, y)}$ is sufficiently low, (x, y) is optimal for (P) as we stated in the following theorem.

Theorem 13. *Let $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ be an optimal solution for both (UB-P) and (LB-P). Then, it is also optimal solution for (P) if $\frac{AE}{c(\bar{\mathbf{x}}, \bar{\mathbf{y}})}$ is less than the relative gap, where AE is calculated as in Lemma 22.*

Up to this point, we have presented several relations among (P), (LB-P) (UB-P). Since we show our results on a generic feasible region C , we can use these results for any model whose objective function is c .

We focus on solving the models (MC-WPAB) and (MC-WPRB) such that (P) corresponds to either the model (MC-WPAB) or the model (MC-WPRB). We propose an algorithm to find at least one good feasible solution for these models within limited time in Algorithm 2.

In Algorithm 2, we basically solve (UB-P) and (LB-P) simultaneously in two different machines. If AE calculated by Lemma 22 is less than the absolute gap then we say the solution is “Optimal”, otherwise we return the solution with the best objective value with status “Unknown”. The solution status “Unknown” is a feasible solution to the model (P). However, we do not know whether this solution is optimal for the model (P) or not. Hence, we will add this solution as a warm start to the model (P) if the optimality status is “Unknown”.

Algorithm 2 does not use the machines efficiently because after we set the optimality status, one machine becomes idle and the other one is allocated to solve (P) with warm start. In order to fully utilize the machines, we will present another set of algorithms 3-5 that also aim to solve the model (P) to optimality in a short time.

Figure 7.2 shows the how we use Algorithms 3-5. Algorithm 5 basically starts and ends the overall procedure. By applying Algorithm 4, in one machine we solve (UB-P-X) and (P) and in another machine we solve (LB-P-X) and (P). X is either WPRB or WPAB which represents the feasible region of the model (MC-WPRB) or the model (MC-WPAB), respectively. The solutions of the models (P), (LB-P-X) and (UB-P-X) are converted into an “output” which is described in Algorithm 3. Then, we send the outputs of the models to a listener which follows Algorithm 5 that shows

Algorithm 2 Approximate Problem Algorithms

Initialization: Allocate the same number of threads for each problem

Best solution of (LB-P) we found so far = $(\underline{x}, \underline{y}) \leftarrow \emptyset$

Best solution of (UB-P) we found so far = $(\overline{x}, \overline{y}) \leftarrow \emptyset$

Best solution of (P) we found so far = $(\mathbf{x}^*, \mathbf{y}^*) \leftarrow \emptyset$

Optimality Status $\in \{Optimal, Unknown\}$

Solve (LB-P) and (UB-P) at the same time.

$AE = f(\overline{x})$

if (LB-P) and (UB-P) are solved to optimality **then**

if $(\underline{x}, \underline{y}) = (\overline{x}, \overline{y})$ **then**

 Find the set of n such that $AE = f(\overline{x}) - h(\overline{x})/2^n > 0$ and pick the one with the smallest AE.

end if

end if

if $c(\overline{x}, \overline{y}) \leq c(\underline{x}, \underline{y})$ **then**

$(\mathbf{x}^*, \mathbf{y}^*) = (\underline{x}, \underline{y})$

else

$(\mathbf{x}^*, \mathbf{y}^*) = (\overline{x}, \overline{y})$

end if

if $\frac{AE}{c((\mathbf{x}^*, \mathbf{y}^*))} < \text{Relative Gap}$ **then**

 Optimality Status $\leftarrow$ Optimal

else

 Optimality Status $\leftarrow$ Unknown

end if

Output: $(\mathbf{x}^*, \mathbf{y}^*)$, optimality status and AE.

under which conditions we announce which output.

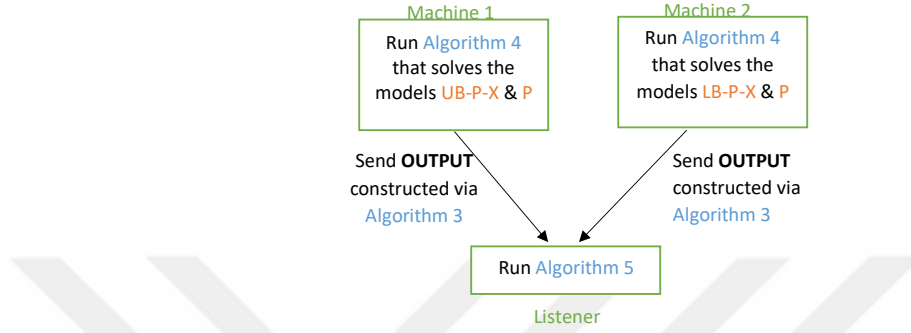


Figure 7.2: Flows of the Algorithms 3-5

We first describe the details of the output preparation in Algorithm 3. Output can be generated by machine 1 or 2. It consists of the current solution of the model and the status of the solution. The solution may arise from the models UB-P-X, LB-P-X or P, and status of the solution can be VALID-UB-P, VALID-LB-P or VALID. These statuses will be used in Algorithm 5.

Algorithm 3 Output Preparation

OUTPUT consists of the current solution from the solver and a status.

if The solver solves UB-P- and does not hit the time limit **then**

Set Status as VALID-UB-P.

else if The solver solves LB-P- and does not hit the time limit **then**

Set Status as VALID-LB-P.

else if The solver solves P **then**

Set Status as VALID.

end if

Algorithm 4 explains the steps that must be followed in each machine. After the first output is generated, the solver is allocated to solve the model P, which is either

the model (MC-WPRB) or the model (MC-WPAB). Therefore, the machines are fully utilized. Since the branch-and-bound tree is preserved, the final output of Algorithm 4 has always greater than or equal objective than the output generated in the middle of the algorithm (STEP 2).

Algorithm 4 Sub Tasks

Input: Model M, time limit for heuristic, time limit for overall.

STEP 1: Set the time limit for the solver to the given heuristic time limit. Solve M.

STEP 2: Run Algorithm 3 and obtain OUTPUT.

STEP 3: Send OUTPUT.

STEP 4: Change the objective to function c .

Set solver time limit to the remaining time limit from the overall time.

Resolve the problem.

STEP 5: Run Algorithm 3 and obtain OUTPUT.

STEP 6: Send OUTPUT.

STEP 7: STOP.

Algorithm 5 shows a procedure to solve model (P). It starts Algorithm 4 in two different machines simultaneously. It waits for the outputs from Algorithm 4 and announce the solution.

The procedure described in Algorithm 5 uses the machines more efficiently than the procedure described in Algorithm 2. Both procedures uses two machines so there is no difference between these procedures in terms of machine cost.

In this section, we propose two linear approximations of $c(\cdot)$, which are less complicated than $c(\cdot)$ and they are easier to optimize over C . We formulate three different models (P), (LB-P), and (UB-P) which have the same feasible region (i.e., $(x, y) \in C$) but different objective functions given by c , an underestimation of c , and an overestimation of c , respectively. Then, we show that by solving only the models (LB-P),

Algorithm 5 Procedure to solve the model (P)

Input: Time limit for heuristic, time limit for overall, Model Name.

Start Algorithm 4 at the same time in two different machines with model LB-P-X and UB-P-X in each. Set the heuristic and overall time limit as given.

WAIT for an OUTPUT.

if The OUTPUT status is **Valid then**

 Announce the solution inside the OUTPUT.

STOP.

else if The OUTPUT status is **Valid-LB-P** and an OUTPUT with status **Valid-UB-P** has been received **then**

if Conditions in Theorem 13 are satisfied **then**

 Announce the solution inside the OUTPUT.

STOP.

else

 Return **WAIT** state.

end if

end if

and (UB-P), we can find a lower bound on (P) and an upper bound on the gap (approximation error). Although Yörükoğlu (2015) uses a linear objective function $(c(x, y) + f(x) - h(x))/2$ to solve his proposed model, he did not use two different objective functions to take advantage of their optimal solution as we did in this section. Also, he does not show a theoretical result about the approximation error of his proposed model.

7.3 Decomposition Method

Decomposition methods are widely used in the literature to solve large scale problems, where the number of decision variables or constraints are large. The main motivation

in decomposition methods is to divide a large problem into smaller parts so that easier problems can be obtained or the relaxation of the original problem may be improved.

In decomposition methods the problem is divided into two: the master problem and the sub-problem. The sub-problems provide information to the master problem so that the sequence of optimal solutions converges to a good feasible solution for the original problem. If the decomposition method is an exact solution method, then the sequence converges to an optimal solution for the original problem.

Our problem is suitable for decomposition methods, because the models proposed in Chapter 6 depend on the model (MC). In this section, we will decompose the models (MC-WPRB) and (MC-WPAB) into two, such that the model (MC) is the master problem and the remaining constraints construct the sub-problem.

7.3.1 Decomposition Procedure to Solve the Model (MC-WPRB)

We solve the model (MC) as a master model in the decomposition procedure. If there is an integer feasible solution in the branch and bound (BB) tree of the model (MC), we solve a sub-problem which basically checks whether the node solution is feasible for the model (MC-WPRB) or not. According to the results of the sub-problem, we fathom the node or continue with branching. Next, we describe each step in a detailed way.

7.3.1.1 Sub-Problem

Constructing a sub-problem which provides information about our current integer solution is our primal goal. The models (MC-WRPB-Q) and (MC-WPRB) formulate the same problem. Hence, we can decompose both of the models. If we decompose the model (MC-WPRB-Q), the sub-problem has a constraint with quadratic terms. A similar approach is applied in Madani and Van Vyve (2015). However, such a sub-problem is not beneficial in terms of its solution times. Hence, we decompose the model (MC-WPRB) such that the constraints defined by (6.29)-(6.39) form the

sub-problem.

Let the solution $(\mathbf{x}^*, \mathbf{y}^*)$ be an integer feasible solution for the model (MC) at node n . If the sub-problem (Sub-WPRB) is feasible, the solution $(\mathbf{x}^*, \mathbf{y}^*)$ is feasible for the model (MC-WPRB).

(Sub-WPRB)

$$p_t = P_{0,t} (1 - x_{0,t}^*) + P_{N_t,t} x_{N_t,t}^* + \sum_{j \in \{1, \dots, N_t\}} P_{j,t} (x_{(j-1),t}^* - x_{j,t}^*) \quad \forall t \in T \text{ if } N_t \geq 2, \quad (7.11)$$

$$\sum_{t \in T} (P_b - p_t) Q_{b,t} \leq y_b^* M_b, \quad \forall b \in B \setminus \{B^c \cup B^m\}, \quad (7.12)$$

$$\sum_{t \in T, b_i \in \phi_b} (P_{b_i} - p_t) Q_{b_i,t} \leq y_b^* \sum_{b_i \in \phi_b} M_{b_i}, \quad \forall \phi_b \in \Phi_b, b \in B^m \setminus B^c, \quad (7.13)$$

$$\begin{aligned} \sum_{b_i \in \phi_b, t \in T_{b_i}} (P_{b_i} - p_t) Q_{b_i,t} &\leq (y_b^* - y_{b_m}^*) \sum_{b_i \in \phi_b} M_{b_i} \\ &+ \sum_{b_i \in \phi_b} M_{b_i} \quad \forall \phi_b \in \Phi_b, b \in \Theta_{b_m}, b_m \in B^m, \end{aligned} \quad (7.14)$$

$$(P_f - p_t) Q_f \leq \sum_{\bar{t} \in T} y_{f\bar{t}}^* M_{f,t}, \quad \forall f \in F, t \in T, \quad (7.15)$$

$$p_t \in [P_{0,t}, P_{N_t,t}], \quad \forall t \in T. \quad (7.16)$$

We do not include the constraints defined by (6.29)-(6.31) because the solution $(\mathbf{x}^*, \mathbf{y}^*)$ already satisfies these constraints by Lemma 6. The only unknown in (7.11)-(7.16) is p_t . We can find the value of p_t via (7.11). Then, it is easy to check whether the constraints defined by (7.12)-(7.16) are satisfied. We can simplify these constraints by defining some sets, which we present in the following lemma.

Lemma 23. *An integer feasible solution $(\mathbf{x}^*, \mathbf{y}^*)$ at node n of the model (MC) is*

infeasible for the model (MC-WPRB), if at least one of the following sets is nonempty.

$$\begin{aligned}
 B_{prb} &= \left\{ b \in B_0^n \setminus B^c : \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} > 0 \right\} \\
 C_{prb} &= \left\{ b \in B_0^n, b \in \Theta_{b_m}, b_m \in B_1^n : \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} \right. \\
 &\quad \left. + \sum_{b_i \in \Theta_b} \max \left\{ 0, \sum_{t \in T_{b_i}} (P_{b_i} - p_t^*) Q_{b_i,t} \right\} > 0 \right\} \\
 F_{prb} &= \left\{ f \in F_0^n : \max_{t \in T_f} (P_f - p_t^*) Q_f > 0, \right\},
 \end{aligned}$$

where $B_0^n = \{b \in B : y_b^* = 0\}$, $B_1^n = \{b \in B : y_b^* = 1\}$, $F_0^n = \{f \in F : \sum_{t \in T_f} y_{f,t}^* = 0\}$, and p_t^* is defined by (7.11).

These sets contain the block, child block and flexible orders that are PRB. Hence, if these sets are empty, then it means there is no PRB order in the current solution. In the following section, we focus on how we cut a current node solution.

7.3.1.2 Cutting the current node solution

We need to cut the current integer node solution if the condition on Lemma 23 is satisfied. The straightforward principle is to propose a “no good cut”. It is an inequality that cuts off only the current integer feasible point among all possible combinations because the integer solution is not feasible for the model (MC-WPRB). This method is proposed by Martin et al. (2014). Let $(\mathbf{x}^*, \mathbf{y}^*)$ denote the integer feasible solution at node n for the model (MC). Then, the following inequality cuts the current node solution.

$$\sum_{b \in B: y_b^* = 0} y_b + \sum_{b \in B: y_b^* = 1} (1 - y_b) + \sum_{f \in F, t \in T: y_{f,t}^* = 0} y_{f,t} + \sum_{f \in F, t \in T: y_{f,t}^* = 1} (1 - y_{f,t}) \geq 1. \quad (7.17)$$

It is straightforward to verify that y^* violates (7.17) and it only cuts y^* . We know that, y^* is not feasible for the model (MC-WPRB), hence (7.17) is a globally valid

cut. Since it only cuts one integer point from the feasible region, we expect that this cut will not result in a fast solution method.

We know that a feasible solution must satisfy $\sum_{t \in T_f} y_{f,t} \leq 1$. We change the third term in (7.17) and obtain (7.18). It is stronger than the cut in (7.17) and still cuts off the current solution.

$$\sum_{b \in B: y_b^* = 0} y_b + \sum_{b \in B: y_b^* = 1} (1 - y_b) + \sum_{f \in F, t \in T: \sum_{t \in T} y_{f,t}^* = 0} y_{f,t} + \sum_{f \in F, t \in T: y_{f,t}^* = 1} (1 - y_{f,t}) \geq 1. \quad (7.18)$$

We know that rejecting an ITM block order results in an infeasible solution by Lemma 16. Hence, we use this argument and we propose two new cuts in the following lemma. To obtain these cuts, we basically divide (7.18) into two: In the cut defined by (7.19), we do not include the ITM sales block and flexible orders which are ITM. Because, if an ITM sales order is rejected, it becomes PRB. In the cut defined by (7.20), we can follow the same statements for purchase orders.

Lemma 24. *Let $(\mathbf{x}^*, \mathbf{y}^*)$ denote an integer feasible solution at node n^* for the model (MC) where $\mathbf{y}^*$ is infeasible for the model (MC-WPRB). Both of the following inequalities cut off the current solution.*

$$\begin{aligned} & \sum_{b \in B: y_b^* = 0} y_b + \sum_{b \in B^S: y_b^* = 1, \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} < 0} (1 - y_b) + \sum_{b \in B^D: y_b^* = 1} (1 - y_b) \\ & + \sum_{f \in F, t \in T: y_{f,t}^* = 0} y_{f,t} + \sum_{f \in F^S, t \in T: y_{f,t}^* = 1, \max_{t \in T_f} (P_f - p_t^*) Q_f < 0} (1 - y_{f,t}) \\ & + \sum_{f \in F^D, t \in T: y_{f,t}^* = 1} (1 - y_{f,t}) \geq 1. \end{aligned} \quad (7.19)$$

$$\begin{aligned} & \sum_{b \in B: y_b^* = 0} y_b + \sum_{b \in B^D: y_b^* = 1, \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} < 0} (1 - y_b) + \sum_{b \in B^S: y_b^* = 1} (1 - y_b) \\ & + \sum_{f \in F, t \in T: y_{f,t}^* = 0} y_{f,t} + \sum_{f \in F^D, t \in T: y_{f,t}^* = 1, \max_{t \in T_f} (P_f - p_t^*) Q_f < 0} (1 - y_{f,t}) \\ & + \sum_{f \in F^S, t \in T: y_{f,t}^* = 1} (1 - y_{f,t}) \geq 1. \end{aligned} \quad (7.20)$$

Proof. Let p_t^* be the market clearing price that solution $(\mathbf{x}^*, \mathbf{y}^*)$ yields. Assume that a feasible solution (x, y) of the model (MC-WPRB) violates (7.19). Then, the following inequality must hold:

$$\begin{aligned}
& \sum_{b \in B: y_b^* = 0} y_b + \sum_{b \in B^S: y_b^* = 1, \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} < 0} (1 - y_b) + \sum_{b \in B^D: y_b^* = 1} (1 - y_b) \\
& + \sum_{f \in F, t \in T: y_{f,t}^* = 0} y_{f,t} + \sum_{f \in F^S, t \in T: y_{f,t}^* = 1, \max_{t \in T_f} (P_f - p_t^*) Q_f < 0} (1 - y_{f,t}) \\
& + \sum_{f \in F^D, t \in T: y_{f,t}^* = 1} (1 - y_{f,t}) = 0. \tag{7.21}
\end{aligned}$$

Since (x, y) is feasible it must satisfy (7.18). Then, the following inequality must hold:

$$\begin{aligned}
& \sum_{b \in B^S: y_b^* = 1, \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} > 0} (1 - y_b) \\
& + \sum_{f \in F^S, t \in T: y_{f,t}^* = 1, \max_{t \in T_f} (P_f - p_t^*) Q_f > 0} (1 - y_{f,t}) \geq 1. \tag{7.22}
\end{aligned}$$

To satisfy this inequality at least one of the following must hold:

- A block order b is rejected, i.e. $y_b = 0$ for a b satisfying $\sum_{t \in T_b} (P_b - p_t) Q_{b,t} > 0$. We know from Lemma 16 that if we reject an ITM block order, the order would be still ITM. Rejecting an ITM block order which is not a child results in an infeasible solution. Hence $y_b = 0$ is not feasible.
- Similarly, rejecting an ITM child order is not allowed if its parent b_m is accepted, i.e. $y_{b_m} = 1$. Suppose, b_m is OTM, then $y_{b_m} = 1$ from (7.21). Suppose, b_m is ITM, then b_m must follow its parent block order. If its parent order is ITM then the parent must be accepted and it yields $y_{b_m} = 1$. Otherwise, it satisfies (7.21) and $y_{b_m} = 1$.
- A flexible order f is rejected, i.e., $y_{f,t_1} = 0$ where f satisfying $\max_{t \in T_f} (P_f - p_t^*) Q_f > 0$ and $y_{f,t_1}^* = 1$. From (7.21), $y_{f,t} = 0$ for $t \in T$ and $t \neq t_1$. We know

from Lemma 16 that if we reject an ITM flexible order, the order would be still ITM. Therefore, this solution is not feasible.

A solution $(\mathbf{x}, \mathbf{y})$ that satisfies (7.22) is infeasible, which is a contradiction. We can follow the same arguments to show that a feasible solution for the model (MC-WPRB) must satisfy (7.20). $\square$

The inequalities given in Lemma 24 are stronger than (7.17) and valid for the model (MC-WPRB). The similar inequalities will be constructed for the model (MC-WPAB) in (7.34) and (7.35).

We propose a cut in the following theorem which is valid in the subtree of node n .

Theorem 14. *Suppose $(\mathbf{x}^*, \mathbf{y}^*)$ is an integer feasible solution for the model (MC) at node n that is infeasible for the model (MC-WPRB). The following inequality is valid in the subtree.*

$$\sum_{b \in B_0^n} y_b + \sum_{(f,t) \in TF_0^n} y_{f,t} \geq 1, \quad (7.23)$$

where $B_0^n = \{b \in B : y_b^* = 0\}$ and $TF_0^n = \{(f, t) : y_{f,t}^* = 0, f \in F, t \in T\}$.

Proof. It is straightforward that the inequality defined in (7.23) cuts $(\mathbf{x}^*, \mathbf{y}^*)$. Next, we show that it does not cut off any the optimal solution of the model (MC-WPRB).

Let the solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ be optimal for the model (MC-WPRB) and it is the integer feasible solution arising from node $\bar{n}$. Assume that, $\bar{n}$ is a leaf node originating from n , therefore, $c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \leq c(\mathbf{x}^*, \mathbf{y}^*)$. Then, by using Lemma 17, we can write the following inequality.

$$0 \geq \sum_{b \in B, t \in T_b} (P_b - \bar{p}_t) Q_{bt} (\bar{y}_b - y_b^*) + \sum_{f \in F, t \in T_f} (P_f - \bar{p}_t) Q_{ft} (\bar{y}_{f,t} - y_{f,t}^*) \quad (7.24)$$

where $\bar{p}_t$ is the market clearing price for $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

Assume that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ violates (7.23). Then, it follows that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies:

$$\sum_{b \in B_0^n} \bar{y}_b + \sum_{(f,t) \in TF_0^n} \bar{y}_{f,t} = 0.$$

Then, $\bar{y}_b = 0$ for all $b \in B_0^n$ and $\bar{y}_{f,t} = 0$ for all $(f, t) \in TF_0^n$. Therefore, $B_0^n \subseteq B_0^{\bar{n}}$ and $TF_0^n \subseteq TF_0^{\bar{n}}$. Then, (7.24) can be written as:

$$\sum_{b \in B_1^n \cap B_0^{\bar{n}}, t \in T_b} (P_b - \bar{p}_t) Q_{bt} + \sum_{(f,t) \in TF_1^n \cap TF_0^{\bar{n}}} (P_f - \bar{p}_t) Q_f \geq 0,$$

where $B_1^n = \{b \in B : y_b^* = 1\}$ and $TF_1^n = \{(f, t) : y_{f,t}^* = 1, t \in T, f \in F\}$. Then, at least one of the following inequalities is satisfied:

$$\begin{aligned} \sum_{t \in T_b} (P_b - \bar{p}_t) Q_{bt} &\geq 0 \quad b \in B_1^n \cap B_0^{\bar{n}} \\ (P_f - \bar{p}_t) Q_f &\geq 0, \quad (f, t) \in TF_1^n \cap TF_0^{\bar{n}} \end{aligned}$$

Then, it means at least one order is rejected although it is ITM in solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Hence, the solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is infeasible for the model (MC-WPRB), which is a contradiction. $\square$

The cut defined in Theorem 14 is a modified version of the cut defined in Madani and Van Vyve (2015) for the supply demand balance problems without PRB. This cut is only valid in the subtree. We define the following cut which is a complementary of the cut defined in Theorem 14 and it is globally valid.

Theorem 15. *Suppose an integer feasible solution $(\mathbf{x}^*, \mathbf{y}^*)$ for the model (MC) found at n that is infeasible for the model (MC-WPRB). The following is a valid inequality for the model (MC-WPRB) and cuts off the current node solution.*

$$\begin{aligned} \sum_{b \in B_1^n, \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} < 0} (1 - y_b) \\ + \sum_{(f,t) \in TF_1^n, (P_f - p_t^*) Q_f < 0} (1 - y_{f,t}) + \sum_{b \in B_0^n} y_b + \sum_{(f,t) \in TF_0^n} y_{f,t} \geq 1. \end{aligned} \quad (7.25)$$

Proof. An optimal solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ for the model (MC-WPRB) can be found: (1) At a leaf node of n , where $c(\mathbf{x}^*, \mathbf{y}^*) > c(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, (2) In another branch of BB tree than node n so that $c(\mathbf{x}^*, \mathbf{y}^*) < c(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, and (3) In another branch than n , where

$c(\mathbf{x}^*, \mathbf{y}^*) \geq c(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. If the optimal solution of the model (MC-WPRB) is located on (1), then it is sufficient to add the cut on Theorem 14 which is stronger than (7.25). If $c(\mathbf{x}^*, \mathbf{y}^*) < c(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, then the results on (7.5) is stronger than (7.25). Finally, if $c(\mathbf{x}^*, \mathbf{y}^*) \geq c(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ holds, then from Lemma 17, (7.25) must hold. $\square$

In this section, we present some inequalities that cut the incumbent node solution if it is infeasible for the model (MC-WPRB). We first present (7.17), which is proposed by Martin et al. (2014). Then, we propose a cut which is stronger than (7.17) in (7.18). These inequalities only cut off one feasible solution and valid for both models (MC-WPRB) and (MC-WPAB). We propose an stronger cut than (7.18) in Lemma 24, which is only valid for the model (MC-WPRB). Then, we modify the cut proposed in Madani and Van Vyve (2015) in terms of order types and problem type in Theorem 14. It is a local cut and we also present another cut in Theorem 15 which is the similar to the one in Theorem 14 but globally valid. Since the supply demand balance problem without (PRBs) is our proposed market design, these cuts are adapted to our problem from literature. We are going to use these cuts as a benchmark. In next section, we present the decomposition procedure to solve the model (MC-WPAB).

7.3.2 Decomposition Procedure to Solve the Model (MC-WPAB)

Decomposition procedure to solve the model (MC-WPAB) is similar to the one described in the previous section. We also solve (MC) and if there is an integer feasible solution $(\mathbf{x}, \mathbf{y})$ at node n of BB tree, we solve a sub-problem. If this sub-problem yields that $(\mathbf{x}, \mathbf{y})$ is an infeasible solution for the model (MC-WPAB), then we add an inequality to the problem which cuts off the current node solution. First, we cover the sub-problem.

7.3.2.1 Sub-problem

We decompose the model (MC-WPAB) such a way that the constraints defined by (6.71)-(6.83) form the sub-problem.

Let the solution $(\mathbf{x}^*, \mathbf{y}^*)$ be the integer feasible solution for the model (MC) at node n . If the model (Sub-WPAB) is feasible then $(\mathbf{x}^*, \mathbf{y}^*)$ is feasible for the model (MC-WPAB), which is defined as follows:

(Sub-WPAB)

$$p_t = P_{0,t} (1 - x_{0,t}^*) + P_{N_t,t} x_{N_t,t}^* + \sum_{j \in \{1, \dots, N_t\}} P_{j,t} (x_{(j-1),t}^* - x_{j,t}^*) \quad \forall t \in T \text{ if } N_t \geq 2, \quad (7.26)$$

$$(y_b^* - 1) \underline{M}_b \leq \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}, \quad \forall b \in B \setminus B^c, \quad (7.27)$$

$$s_b \leq \overline{M}_b y_b^*, \quad \forall b \in B^c, \quad (7.28)$$

$$s_b \leq \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} + \underline{M}_b (1 - y_b^*), \quad \forall b \in B^c, \quad (7.29)$$

$$s_b \geq \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i} - M_b (1 - y_b^*), \quad \forall b \in B^c, \quad (7.30)$$

$$\underline{M}_{f,t} (y_{f,t}^* - 1) \leq (P_f - p_t) Q_f, \quad \forall f \in F, t \in T, \quad (7.31)$$

$$p_t \in [P_{N_t,t}, P_{0,t}], \quad \forall t \in T, \quad (7.32)$$

$$s_b \geq 0, \quad \forall b \in B^c. \quad (7.33)$$

We omit (6.71)-(6.73) in the model (Sub-WPAB) because the solution $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies these conditions by Lemma 6. We can find the value of p_t via (7.26). Also, we know that $s_b = 0$ if $y_b^* = 0$ and $s_b = \sum_{t \in T_b} (P_b - p_t) Q_{b,t} + \sum_{b_i \in \Theta_b} s_{b_i}$. Then, solving (7.27)-(7.33) is not complicated. We can simplify these constraints by defining some sets, which we present in the following lemma.

Lemma 25. *An integer feasible solution $(\mathbf{x}^*, \mathbf{y}^*)$ at node n of the model (MC) is infeasible for the model (MC-WPAB), if at least one of the following sets is nonempty.*

$$\begin{aligned}
B_{pab} &= \left\{ b \in B_1^n \setminus B^m : \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} < 0 \right\} \\
M_{pab} &= \left\{ b \in B_1^n, b \in \Theta_{b_m}, b_m \in B^m : \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} \right. \\
&\quad \left. + \sum_{b_i \in \Theta_b} \max \left\{ 0, \sum_{t \in T_{b_i}} (P_{b_i} - p_t^*) Q_{b_i,t} \right\} < 0 \right\} \\
TF_{pab} &= \left\{ (f, t) \in TF_1^n : (P_f - p_t^*) Q_f < 0 \right\},
\end{aligned}$$

where $B_1^n = \{b \in B : y_b^* = 0\}$, $TF_1^n = \{(f, t) : y_{f,t}^* = 1, f \in F, t \in T\}$ and p_t^* equals to (7.26).

These sets contain the block, child block and flexible orders that are PAB. Hence, if these sets are empty, then it means there is no PAB order in the current solution. In the following section, we focus on cutting the current node solution.

7.3.2.2 Cutting the Current Node Solution

Decomposition is used to solve supply demand balance problem without PRB in the literature where SDB is solved as a master problem. There are two different “infeasibility” cuts proposed by Martin et al. (2014) and Madani and Van Vyve (2015). We update these cuts for our model in this section.

We present the cut proposed by Martin et al. (2014) in the previous section, that is valid for the both supply demand balance problem variants. We first extend this cut in the following lemma.

Lemma 26. *Let $(\mathbf{x}^*, \mathbf{y}^*)$ denote an integer feasible solution at node n^* , which is infeasible for the model (MC-WPAB). The following inequalities cut off the current*

solution.

$$\begin{aligned}
& \sum_{b \in B: y_b^* = 1} (1 - y_b) + \sum_{b \in B^S: y_b^* = 0, \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} > 0} y_b + \sum_{b \in B^D: y_b^* = 0} y_b \\
& + \sum_{f \in F^S, t \in T: y_{f,t}^* = 0, (P_f - p_t^*) Q_{b,t} > 0} y_{f,t} + \sum_{f \in F^D, t \in T: y_{f,t}^* = 0} y_{f,t} \\
& + \sum_{f \in F, t \in T: y_{f,t}^* = 1} (1 - y_{f,t}) \geq 1. \tag{7.34}
\end{aligned}$$

$$\begin{aligned}
& \sum_{b \in B: y_b^* = 1} (1 - y_b) + \sum_{b \in B^D: y_b^* = 0, \sum_{t \in T_b} (P_b - p_t^*) Q_{b,t} > 0} y_b + \sum_{b \in B^S: y_b^* = 0} y_b \\
& + \sum_{f \in F^D, t \in T: y_{f,t}^* = 0, (P_f - p_t^*) Q_{b,t} > 0} y_{f,t} + \sum_{f \in F^S, t \in T: y_{f,t}^* = 0} y_{f,t} \\
& + \sum_{f \in F, t \in T: y_{f,t}^* = 1} (1 - y_{f,t}) \geq 1. \tag{7.35}
\end{aligned}$$

The proof is similar to the proof of Lemma 24. Hence, we omit the proof.

The cut defined in Madani and Van Vyve (2015) only considers block orders, which we extend in Lemma 27 to cover linked block and flexible orders.

Lemma 27. Suppose $(\mathbf{x}^*, \mathbf{y}^*)$ denote an integer feasible solution for the model (MC) at node n that is infeasible for the model (MC-WPAB). Then, the following is valid in the subtree.

$$\sum_{b \in B_1^n} (1 - y_b) + \sum_{(f,t) \in TF_1^n} (1 - y_{f,t}) \geq 1. \tag{7.36}$$

Proof. The inequality (7.36) cuts the current integer feasible solution at node n . We follow the similar steps in the proof of Lemma 14.

Let the solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ be optimal for the model (MC-WPAB) and it is the integer feasible solution arising from node $\bar{n}$. Assume that, $\bar{n}$ is a leaf node originating from n , therefore, $c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \leq c(\mathbf{x}^*, \mathbf{y}^*)$. From the condition on Lemma 17, since $c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \leq$

$c(\mathbf{x}^*, \mathbf{y}^*)$.

$$0 \geq \sum_{b \in B, t \in T_b} (P_b - \bar{p}_t) Q_{bt} (\bar{y}_b - y_b^*) + \sum_{f \in F, t \in T_f} (P_f - \bar{p}_t) Q_f (\bar{y}_{f,t} - y_{f,t}^*) \quad (7.37)$$

where $\bar{p}_t$ is the market clearing price for $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

Assume that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ violates (7.36). Then, it follows that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies:

$$\sum_{b \in B_1^n} (1 - \bar{y}_b) + \sum_{(f,t) \in TF_1^n} (1 - \bar{y}_{f,t}) = 0.$$

Then, $\bar{y}_b = 1$ for all $b \in B_1^n$ and $\bar{y}_{f,t} = 1$ for all $(f, t) \in TF_1^n$. Therefore, $B_1^n \subseteq B_1^{\bar{n}}$ and $TF_1^n \subseteq TF_1^{\bar{n}}$. Then, (7.37) can be written as:

$$0 \geq \sum_{b \in B_0^n \cap B_1^{\bar{n}}, t \in T_b} (P_b - \bar{p}_t) Q_{bt} + \sum_{(f,t) \in TF_0^n \cap TF_1^{\bar{n}}} (P_f - \bar{p}_t) Q_f,$$

where $B_0^n = \{b \in B : y_b^* = 0\}$, and $TF_0^n = \{(f, t) : y_{f,t}^* = 0, t \in T, f \in F\}$. Then, at least one of the following inequalities is satisfied:

$$\begin{aligned} 0 &\geq \sum_{t \in T_b} (P_b - \bar{p}_t) Q_{bt} \quad b \in B_0^n \cap B_1^{\bar{n}}, \\ 0 &\geq (P_f - \bar{p}_t) Q_f, \quad (f, t) \in TF_0^n \cap TF_1^{\bar{n}}. \end{aligned}$$

Then, it means at least one order is accepted although it is OTM in solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Hence, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is infeasible for the model (MC-WPAB), which is a contradiction. $\square$

This cut is valid for sub-branches of the node n . It changes at least one of the block or flexible order decisions which are 1 for the integer solution at the node n .

In this section, we present some inequalities that cut the incumbent node solution if it is infeasible for the model (MC-WPAB). We first propose a cut in Lemma 26, which is only valid for the model (MC-WPAB) and stronger than the cut proposed by Martin et al. (2014). This lemma is equivalent to Lemma 24 used to solve (MC-WPRB). Secondly, we modify the cut proposed in Madani and Van Vyve (2015) in terms of order types in Theorem 27. These inequalities do not cut off any optimal solution. We will use these inequalities as a benchmark. In next section, we present

heuristic cuts which may cut off the optimal solution of the model we want to solve. We will use these cuts as an alternative solution methodology to the ones we present in Section 7.1 and 7.2.

7.3.3 Heuristic Cuts

In this section we focus on constructing a heuristic cut which may cut the optimal solution of the model we want to solve. The purpose of proposing this cut is to find a solution for the problem in a short time. We will use these cuts as an alternative solution methodology to the ones we present in Section 7.1 and 7.2.

Heuristic cut should not yield an infeasible solution for the models. Hence, we need to cut the infeasible solutions. First, we identify the set of variables which creates infeasibility at node n^* . Then, we propose an inequality for each order whose decision variable value creates infeasibility in Lemmas 28 and 29.

If a solution $(\mathbf{x}^*, \mathbf{y}^*)$ at node n^* with market clearing price p_t^* in period t is infeasible for the model (MC-WPRB) then at least one of the sets $B_{prb}^{n^*}, C_{prb}^{n^*}, F_{prb}^{n^*}$ must be nonempty. Lemma 28 presents a heuristic cut for the model (MC-WPRB).

Lemma 28. *Suppose $(\mathbf{x}^*, \mathbf{y}^*)$ is an integer feasible solution for the model (MC) at node n that is infeasible for the model (MC-WPRB). The following cuts the current node solution.*

- For all $b_i \in B_{prb}^n$ and $b_i \in C_{prb}^n$.

– If $Q_{b_i,t} < 0$

$$\begin{aligned}
 & \sum_{b \in B_{pab}^n \cup M_{pab}^n, 0 > -Q_{b,t} \geq Q_{b_i,t}} (1 - y_b) + \sum_{(f,t) \in TF_{pab}^n, 0 > -Q_f \geq Q_{b_i,t}} (1 - y_{f,t}) \\
 & + \sum_{b \in B_0^n, 0 > Q_{b,t} \geq Q_{b_i,t}} y_b + \sum_{(f,t) \in TF_0^n, 0 > Q_f \geq Q_{b_i,t}} y_{f,t} \geq 1.
 \end{aligned} \tag{7.38}$$

– If $Q_{b_i,t} > 0$

$$\begin{aligned} & \sum_{b \in B_{pab}^n \cup M_{pab}^n, 0 > Q_{b,t} \geq -Q_{b_i,t}} (1 - y_b) + \sum_{(f,t) \in TF_{pab}^n, 0 > Q_f \geq -Q_{b_i,t}} (1 - y_{f,t}) \\ & + \sum_{b \in B_0^n, 0 > -Q_{b,t} \geq -Q_{b_i,t}} y_b + \sum_{(f,t) \in TF_0^n, 0 > -Q_{b,t} \geq -Q_{b_i,t}} y_{f,t} \geq 1. \end{aligned} \quad (7.39)$$

• For all $f_i \in F_{prb}^n$

– If $Q_{f_i} < 0$

$$\begin{aligned} & \sum_{b \in B_{pab}^n \cup M_{pab}^n, 0 > -Q_{b,t} \geq Q_{f_i}} (1 - y_b) + \sum_{(f,t) \in TF_{pab}^n, 0 > -Q_f \geq Q_{f_i}} (1 - y_{f,t}) \\ & + \sum_{b \in B_0^n, 0 > Q_{b,t} \geq Q_{f_i}} y_b + \sum_{(f,t) \in TF_0^n, 0 > Q_f \geq Q_{f_i}} y_{f,t} \geq 1. \end{aligned} \quad (7.40)$$

– If $Q_{f_i} > 0$

$$\begin{aligned} & \sum_{b \in B_{pab}^n \cup M_{pab}^n, 0 > Q_{b,t} \geq -Q_{f_i}} (1 - y_b) + \sum_{(f,t) \in TF_{pab}^n, 0 > Q_f \geq -Q_{f_i}} (1 - y_{f,t}) \\ & + \sum_{b \in B_0^n, 0 > -Q_{b,t} \geq -Q_{f_i}} y_b + \sum_{f,t \in TF_0^n, 0 > -Q_{b,t} \geq -Q_{f_i}} y_{f,t} \geq 1. \end{aligned} \quad (7.41)$$

If a solution $(\mathbf{x}^*, \mathbf{y}^*)$ at node n^* with market clearing price p_t^* on period t is infeasible for the model (MC-WPAB) then at least one of sets $B_{pab}, M_{pab}, F_{pab}$ must be nonempty. Lemma 29 presents a heuristic cut for the model (MC-WPAB).

Lemma 29. Suppose $(\mathbf{x}^*, \mathbf{y}^*)$ denotes an integer feasible solution for the model (MC) at node n that is infeasible for the model (MC-WPAB). The following cuts off the current node solution.

• For all $b_i \in B_{pab}^{n^*}$ and $b \in M_{pab}^{n^*}$

– If $Q_{b_i,t} < 0$

$$\begin{aligned} & \sum_{b \in B_1^n, 0 > Q_{b,t} \geq Q_{b_i,t}} (1 - y_b) + \sum_{(f,t) \in TF_1^n, 0 > Q_f \geq Q_{b_i,t}} (1 - y_{f,t}) \\ & + \sum_{b \in B_{prb}^n \cup C_{prb}^n, 0 > -Q_{b,t} \geq Q_{b_i,t}} y_b + \sum_{f \in F_{prb}^n, y_{f,t}^* = 0, 0 > -Q_f \geq Q_{b_i,t}} y_{f,t} \geq 1. \end{aligned} \quad (7.42)$$

– If $Q_{b_i,t} > 0$

$$\begin{aligned} & \sum_{b \in B_1^n, 0 > -Q_{b,t} \geq -Q_{b_i,t}} (1 - y_b) + \sum_{(f,t) \in TF_1^n, 0 > -Q_f \geq -Q_{b_i,t}} (1 - y_{f,t}) \\ & + \sum_{b \in B_{prb}^n \cup C_{prb}^n, 0 > Q_{b,t} \geq -Q_{b_i,t}} y_b + \sum_{f \in F_{prb}^n, y_{f,t}^* = 0, 0 > Q_{b,t} \geq -Q_{b_i,t}} y_{f,t} \geq 1. \end{aligned} \quad (7.43)$$

• For all $(f_i, t) \in TF_{pab}^n$

– If $Q_{f_i} < 0$

$$\begin{aligned} & \sum_{b \in B_1^n, 0 > Q_{b,t} \geq Q_{f_i}} (1 - y_b) + \sum_{(f,t) \in TF_1^n, 0 > Q_f \geq Q_{f_i}} (1 - y_{f,t}) \\ & + \sum_{b \in B_{prb}^n \cup C_{prb}^n, 0 > -Q_{b,t} \geq Q_{f_i}} y_b + \sum_{f \in F_{prb}^n, y_{f,t}^* = 0, 0 > -Q_f \geq Q_{f_i}} y_{f,t} \geq 1. \end{aligned} \quad (7.44)$$

– If $Q_{f_i} > 0$

$$\begin{aligned} & \sum_{b \in B_1^n, 0 > -Q_{b,t} \geq -Q_{f_i}} (1 - y_b) + \sum_{(f,t) \in TF_1^n, 0 > -Q_f \geq -Q_{f_i}} (1 - y_{f,t}) \\ & + \sum_{b \in B_{prb}^n \cup C_{prb}^n, 0 > Q_{b,t} \geq -Q_{f_i}} y_b + \sum_{f \in F_{prb}^n, y_{f,t}^* = 0, 0 > -Q_{b,t} \geq -Q_{f_i}} y_{f,t} \geq 1. \end{aligned} \quad (7.45)$$

The inequalities in Lemmas 28 and 29 are globally valid and may cut the optimal solution of the models we want to solve. Hence, we call them heuristic cuts. We will use these cuts as an alternative solution methodology. In the following section, we propose a valid inequality to cut alternative solutions.

7.3.4 Cutting Alternative Solutions For Flexible Orders

Each flexible order is represented by 24 binary variables since $|T_f| = 24$ for $f \in F$ in Turkish DAM. Thus, they increase the size of the problem. Suppose that there are two flexible orders f_1 and f_2 with the same quantity Q and the same price P . Let $y_{f_1, t_1}^* = 1$ and $y_{f_2, t_2}^* = 1$ at node n^* where $t_1 \neq t_2$. Then, there exist another solution in branch and bound tree which gives the same objective value with the objective value on n^* where $y_{f_1, t_2} = 1$ and $y_{f_2, t_1} = 1$. This is an alternative solution and we need to cut such cases to decrease the solution time. We will add the following cuts to decrease number of nodes discovered at node n^* .

$$y_{f_1, t_2} + y_{f_2, t_1} \leq 1 \quad (7.46)$$

We will use this inequality in each node which results in an integer feasible solution for the model (MC).

7.4 Summary

In this chapter, we propose three different solution methodologies. First one is a cut which is valid for any optimal solution for the model (MC). The second method is based on approximating the objective function, which does not always provide an optimal solution for the model (MC-WPRB) or the model (MC-WPAB). The last one is a solution methodology which is based on decomposition. In next chapter, we will conduct numerical experiments to show the performance of the solution methodologies we proposed.

Chapter 8

NUMERICAL EXPERIMENTS

In this chapter, we are going to test our models and solution methodologies with real data submitted to the Turkish DAM. First, we introduce the data and give some descriptive statistics. Then, we explain the test environment we use. Afterwards, we provide the experiment set-up and give the corresponding results.

8.1 Data

We use a real data set of the Turkish DAM that covers 365 days, i.e., starting from 1 March 2017 to 28 February 2018. Before providing the descriptive statistics about the data set, we first explain the order types in the Turkish DAM and some rules that an order must follow to enter the system.

8.1.1 Order Types in Turkish DAM

According to Energy Market Regulatory Authority (2016a), there are three types of orders in Turkish DAM: Hourly (single), block and flexible. Each order consists of a price and a corresponding quantity information. Quantities have positive or negative values, where negative quantity represents sales and the positive one represents purchase orders. The unit of quantity is “lot” and 1 lot equals 0.1 MWh. The minimum price in Turkish system is 0 and the maximum price is 2000 TL. The maximum order quantity is 100,000 lots. A day is divided into 24 hour periods, given by 00:00 - 01:00, ..., 23:00-00:00. Except flexible orders, all orders must contain period information.

An hourly order is given for each period. For each time slot, Turkish DAM participants can offer at most 32 levels in both sales and purchase directions. Each level

consists of a price and a corresponding quantity. In each order, price levels must be in strictly increasing order, whereas corresponding quantities are given in descending order. Hence, the resulting curves are piece-wise linear functions. Each hourly order must include minimum and maximum price levels.

Some participants may have marginal costs/revenues which can be demonstrated by stepwise functions. In that instance, the participants may offer an hourly order curve with small absolute slopes. In Table 2.3 in Chapter 2, an illustrative example is presented, in which there exist some segments with almost zero slopes, i.e., $2 \times 10^{-3} = \frac{170-169.99}{1500-1000}$ or $10^{-4} = \frac{2000-1999.99}{1600-1500}$.

The aggregated supply demand curves are given in the web-site of EXIST¹. We can observe segments with almost zero slopes in these curves too. Moreover, there exist some segments whose quantity differences are too high, i.e, 18,000 MWh, and some of those whose quantity difference is too low, i.e, 1 MWh. Hence we may encounter numerical difficulties while solving the corresponding models².

Block orders are valid for a minimum of 3 and a maximum of 24 hours in a day. A participant may offer at most 50 block orders in a day. At most 6 block orders can link to each other with at most three generations. There must be at most 3 orders in each generation. Under these rules, there exist at most 30 paths per link family, which is the size of $|\Phi_b|$.

Only sales flexible orders are allowed in the Turkish DAM. Their interval limit is 24 hours with a validity duration of one period. Each participant can submit at most 10 flexible orders. The maximum quantity that can be offered by a block or flexible order is 600 MWh or 6000 lot. Currently, there are no profile orders in the Turkish DAM.

¹<https://seffaflik.epias.com.tr/transparency/piyasalar/gop/arz-talep.xhtml>

²https://www.ibm.com/support/knowledgecenter/SSSA5P_12.6.1/ilog.odms.cplex.help/CPLEX/UsrMan/topics/cont_optim/simplex/20_num_difficulty.html

8.1.2 Evaluation of the Orders in Turkish DAM

The matching rules in Turkish DAM are defined in Energy Market Regulatory Authority (2016b). The difference between the rules in Turkish DAM and supply demand balance problem without PRBs is stated as follows:

- A block order must be accepted if its price is the same as MCP.
- There is no forcing constraint on child block orders.
- There is financial compensation for a parent block order even its children can save the parent with their surplus values.
- There are no uplift payments for the accepted sales flexible orders if they are not accepted in the highest MCP.

Since the Turkish DAM has different rules than our proposed market design and supply demand balance problem without PABs, we will consider the impact of the bidding strategies on the complexity of the problem.

8.1.3 Descriptive Statistics

In this section we present the descriptive statistics of the dataset. Table 8.1 shows the average number of hourly, block and flexible orders per day where 17% of the block orders have a link relation.

Table 8.1: Average number of orders per day

Hourly orders	Block orders	Flexible orders
15,797	148	4

After applying the aggregation procedure as described in Chapter 4, we obtain a demand, a supply, and a total curve for each period. Table 8.2 presents the average number of price levels in supply and demand curves together and total curve per day.

Table 8.2: Average number price levels per day after aggregation

Supply + Demand Curve	Total Curve
13,039	9,353

According to Table 8.2, we can reduce the problem size by 28% if we use total quantity curve instead of supply and demand curves.

8.2 Test Environment and Experiment Set-up

In this section, we provide the information about our test environment and give experiment set-up details such as solver parameters, the tests we have done and the performance measures we are going to use in our experiments.

8.2.1 Test Environment

We run our experiments on a computer provided by EXIST to ensure the data safety. Computational experiments are coded in Java. We use IBM ILOG CPLEX 12.8. solver with ILOG Concert Technology Interface on a computer running Windows 10 Pro (64 bits) with Intel Core i7-4790 CPU @ 3.60 GHz and 32 GB RAM.

In some test instances, we need to solve two different models at the same time. Running parallel models in CPLEX on the same machine has some disadvantages: (1) We need to divide the total number of the cores by the number of models we want to solve to evenly distribute the jobs among threads. This decreases the number of threads per model, which may result in longer solution times. (2) Running parallel CPLEX objects causes longer solution times to satisfy thread safety. Hence, it is

better to run two different models in two different virtual machines on the same platform. The cost of this implementation is to have a CPLEX license for 16-core machines instead of 8 and additional RAM. The test environment provided by EXIST to us does not satisfy this property. So we could not test our models in a distributed environment. Instead, we sequentially run the models and assume that they are solved in parallel on different machines.

8.2.2 Solver Parameters

There are parameter values that can be set in IBM ILOG CPLEX. In this section we present the parameter values we change in the solver.

Both the Turkish DAM and European DAMs must find a solution within 10 minutes. Hence, the time limit for the solver is set to 10 minutes (600 seconds) unless stated otherwise. Moreover, we set numerical emphasis to true because the data is numerically sensitive as we stated before. Numerically sensitive data may cause numerical problems while solving the problem. We also use the primal simplex algorithm for the root node relaxation problem except for the model with the quadratic constraint. Otherwise, the problem may be deemed to be infeasible by the solver or the basis matrix may be deemed to be singular due to numerical problems. The other solver parameters are shown in Table 8.3.

Table 8.3: Solver tolerance values

Relative MIP Gap	Simplex Feasibility	Integrality	Simplex Optimality
1e-7	1e-8	1e-8	1e-8

Relative gap is the ratio of the difference between the best bound on the problem and the best objective value to the best objective value. The relative MIP gap is a tolerance value such that an integer feasible solution becomes an optimal solution if its relative gap is less than or equal to the tolerance value. We assume that the

market can tolerate a surplus loss of at most 100 TL and we pick the relative gap accordingly (The objective function is around 10^9 . Setting relative MIP gap to 10^{-7} gives a difference of at most 999 between the objective values, which results in at most 99.9 TL loss because the unit of the objective is TL/MWh*lot).

A real number is deemed to be an integer even if it varies from the closest integer with a tolerance value, which is called the integrality tolerance. We use a tight integrality tolerance because Madani and Van Vyve (2015) report issues due to choosing loose integrality tolerance values. There is a relation between the integrality tolerance value and the relative MIP gap. Assume that we set the integrality tolerance to 10^{ϵ_i} . The maximum $P_b Q_b N_b$ value that a block order can get is $2.88 \cdot 10^8$ ($2000 \times 6000 \times 24$). The maximum error we can make per order is $2.88 \times 10^8 \times 10^{\epsilon_i}$. We have an average of 150 block orders per day. Hence, the maximum error arising from block orders is $2.88 \times 150 \times 10^8 \times 10^{\epsilon_i}$, which is around $10^{11+\epsilon_i}$. We want this error to be less than the absolute gap tolerance, which we set to be 999 or roughly 10^3 . Thus, $\epsilon_i \leq -8$. We can neglect the error coming from the flexible orders because $P_f Q_f$ value that a flexible order can get is at most $1.2 \cdot 10^7$ (2000×6000). The maximum error for flexible orders is 4.8×10^7 . The error term we calculated for block orders also covers the amount coming from the flexible orders.

Simplex feasibility tolerance reflects the tolerance from the bound of a basic variable. Setting this value too small may cause infeasibility even if the problem is not infeasible. Setting this value larger than the integrality tolerance may result in a feasible solution that is not integer feasible. This may lead to a sub-optimal solution or to additional post processing. Hence, we set this value to the same value as the integrality tolerance.

The simplex optimality tolerance has an impact of the tolerance on the reduced costs that should enter the basis. The reduced cost of a variable equals the slack variable of the constraint in the dual problem for that variable. Hence, the simplex optimality has an impact on whether strong duality holds or not. The simplex opti-

optimality tolerance must be less than or equal to the relative MIP gap. As a remark, a too low simplex optimality tolerance may lead to infeasible solutions due to numerical problems.

8.2.3 Numerical Problems

The data is numerically sensitive as we discussed earlier. We can also observe this feature by studying the MIP Kappa statistics of our models. Due to numerical difficulties, CPLEX may erroneously report infeasibility. Hence, we aim to take some precautions against this problem.

1. We set numerical emphasis parameter to true. This forces CPLEX solver for paying more attention to the numerical precision. Using this setting is not sufficient to eliminate all numerical problems. Moreover, it may increase the solution time.
2. We use the “Primal Simplex” algorithm in root node relaxation. Otherwise, CPLEX may deem the problem to be infeasible in root node and may not start the branching.
3. We divide the forcing constraints, which eliminate PAB or PRB depending on which model we use, by the quantity of the order. We will explain why this creates a problem and why dividing every term by the same quantity eliminates the infeasibilities.

Suppose that $y_b = 1$ for a linked block order b in the model (MC-WPAB). Then, we expect that s_b equals $\sum_{t \in T_b} (P_b - p_t)Q_b + \sum_{b_i \in \Theta_b} s_{b_i}$ by (6.63) -(6.66). However, in practice, the machines do not store $y_b = 1$ exactly, instead it may be approximately equal to one. We use 10^{-8} as the integrality tolerance. So, CPLEX assumes $y_b = 1 + \epsilon \times 10^{-8}$ as 1 where $-0.\bar{9} \leq \epsilon \leq 0.\bar{9}$. The following inequality is a direct result of (6.64) and (6.65).

$$-M_b(1 - y_b) \leq \underline{M}_b(1 - y_b)$$

When we insert the big-M values defined in (6.15), (6.16), and (6.62) we obtain the following relation.

$$-\sum_{t \in T_b} (P_b - P_t^{max}) Q_b (1 - y_b) \leq -\sum_{t \in T_b} (P_b - P_t^{min}) Q_b (1 - y_b), b \in B^S,$$

which is also equivalent to

$$\sum_{t \in T_b} P_t^{max} Q_b (1 - y_b) \leq \sum_{t \in T_b} P_t^{min} Q_b (1 - y_b), b \in B^S.$$

For $y_b = 1 + \epsilon 10^{-8}$, the above inequality may take $24 \times 2000 \times (-6000) \times (-\epsilon) \times 10^{-8} \leq 0$, which is $2.88 \times \epsilon \leq 0$. This inequality may be violated when the feasibility tolerance is 10^{-8} and $\epsilon > 0$. We can follow the same argument for a purchase block order.

After dividing the inequalities by $Q_b(Q_f)$, we obtain $-4.8 \times \epsilon \times 10^{-4} \geq 0$ for a sales order. This inequality is feasible when the feasibility tolerance is set to 10^{-3} or higher. However, setting the feasibility tolerance to 10^{-3} causes problems which we discuss in the next item. We observe numerical problems if we do not divide the forcing constraints by the quantity of orders, but we do not observe numerical issues if we divide the forcing constraints by the quantity of the order.

4. There is another option for setting a higher value for the feasibility tolerance. However, it results in imbalance, which we explain in the following example.

Suppose we set the feasibility tolerance to 10^{-3} . Then there is no difference between $x_{jt} = 0.0001$ and $x_{jt} = 0.0009$. From the supply-demand curve in the web-site of EXIST, there exist segments whose quantity difference is around 60,000 lot. Although supply-demand balance constraint is satisfied, there may

be 6 to 54 lots difference per segment. This yields a supply-demand imbalance. Hence, we set the feasibility tolerance to 10^{-8}

8.2.4 Tests and Performance Measures

We solve each mathematical model we proposed in Chapter 6 and implement each solution methodology we proposed in Chapter 7. We compare the efficiency of a model or a solution methodology in terms of performance metrics.

Performance measures we use in this study are: (1) percentage of instances that an optimal solution is found (OS), (2) percentage of the instances that no feasible solution is found within the time limit (NS), (3) percentage of the instances that a feasible solution is found but we could not prove its optimality due to time limit (TLS), (4) Average time for the instances which are solved to optimality (TTL), (5) average relative gap for the instances where a feasible solution is found but its optimality could not be proven within the given time limit (RG).

EXIST announces MCPs on every day. Hence, the percentage of the instances where there is no solution is zero for EXIST. Moreover, their optimization engine can find the optimal solution in 96% of the days³. Hence, we aim to obtain better results in terms of these metrics.

We use paired t-test to show that a model or a solution methodology is better than the other one in terms of elapsed time and objective function if there is no obvious improvement.

8.3 Results

By using the real data set, we run the models we proposed in Chapter 6 we present the performance of the models in Section 8.3.1. Afterwards, we apply the solution methods we present in Chapter 7.

³EPIAS 2016 Yili Elektrik Piyasasi Ozet Bilgiler Raporu

8.3.1 Performance of the Models

In this section, we present the performance of the models for the problem variants. The models proposed for supply demand balance problem without PRBs are (SDB-WPRB), (MC-WPRB), and (MC-WPRB-Q). The models we presented to formulate supply demand balance problem without PABs are (SDB-WPAB), (MC-WPAB), and (MC-WPAB-Q). We are going to show the performance of these models on the Turkish dataset.

Table 8.4 presents the performances of the models we proposed for the problem variants. We know that the model (MC) has fewer constraints and decision variables than SDB. Therefore, it is an expected result that the models which use (MC) as a base model, such that (MC-WPRB) and (MC-WPAB), perform better than the models (SDB-WPRB) and (SDB-WPAB) which use (SDB) as a base model. Although the models (MC-WPRB-Q) and (MC-WPAB)-Q use (MC) as a base model, their performance is worse than (SDB-WPRB) and (SDB-WPAB), respectively, because of the quadratic constraints in these models. As a result, (MC-WPRB) and (MC-WPAB) are the most promising models for solving supply demand balance problem without PRBs and PABs, respectively. Hence, we use model (MC) as a base model and focus on the models (MC-WPRB) and (MC-WPAB) in our experiments.

We also present the cumulative percentage of solution times of the optimal instances for the instances in Figure 8.1. For the models (MC-WPRB) and (MC-WPAB) around 60% of the instances are solved within 100 seconds. On the other hand, within the same amount of time, we can only find the optimal solutions for 30% of the instances for the model (SDB-WPRB) and 25% for the model (SDB-WPAB).

Table 8.4: Performance of the models we proposed for the problem variants

	OS	TLS	NS	TTL	RG
	(%)	(%)	(%)	(secs.)	(%)
(SDB-WPRB)	49.3	26.6	24.1	135	0.610
(MC-WPRB)	82.2	17.3	0.5	63	0.358
(MC-WPRB-Q)	17.0	45.8	37.2	80	0.111
(SDB-WPAB)	60	24.1	15.9	121	0.255
(MC-WPAB)	80.5	18.6	0.8	64	0.132
(MC-WPAB-Q)	1.9	30.7	67.4	221	1.768

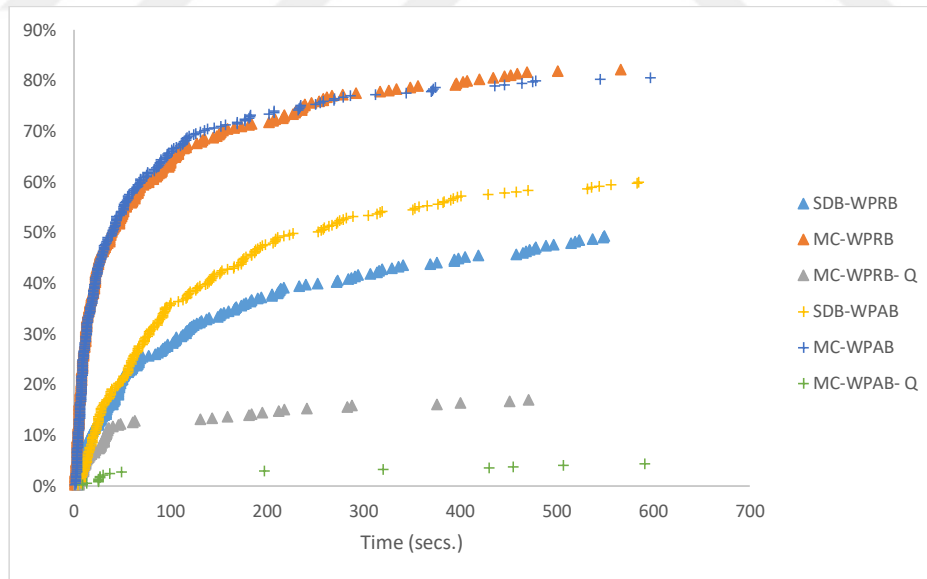


Figure 8.1: Cumulative percentage of solution times of optimal instances for the models (SDB-WPRB), (MC-WPRB), (MC-WPRB-Q), (SDB-WPAB), (MC-WPAB), and (MC-WPAB-Q)

For both problem variants, the best formulation is the model (MC-WPRB) because it has the lowest TLS and NS and highest OS. There are instances where there is no solution, which is an undesired thing in a DAM. Therefore, using only the solver as a solution method is not adequate. We next focus on finding at least one feasible solution for the problem variants.

8.3.2 Performance of the Approximate Models

In this section, our aim is to find a good feasible solution to our problem variants in 10 minutes. Therefore, we run the approximate models whose feasible solutions are also feasible for the models (MC-WPRB) or (MC-WPAB) depending on which problem variant we want to solve. On the other hand, we do not guarantee the optimality of the best solutions of the approximate models for the problem variants.

We use LB-P and UB-P for both problem variants, which we discussed in Chapter 7. The objective function defined by (7.6) is used in UB-P and (7.7) is used in LB-P. We call the models we are going to solve depending on whether we use the constraints of the models (MC-WPRB) or (MC-WPAB).

We run the models (UB-P-WPRB), (LB-P-WPRB), (UB-P-WPAB) and (LB-P-WPAB), where the objective function of the models (UB-P-WPRB) and (UB-P-WPAB) is the upper approximation function of c , and the models (LB-P-WPRB) and (LB-P-WPAB) have the lower approximation function of c as an objective. Moreover, the constraint sets of the models (MC-WPRB), (UB-P-WPRB) and (LB-P-WPRB) are the same. Finally, the constraint sets of the models (MC-WPAB), (UB-P-WPAB) and (LB-P-WPAB) are the same.

Table 8.5 presents the performance of the models (UB-P-WPRB), (LB-P-WPRB), (UB-P-WPAB) and (LB-P-WPAB). There is not any instance with no solution. We solve 99.2% of the instances to optimality in 19 seconds on average by the models (UB-P-WPRB) and (LB-P-WPRB). The model (UB-P-WPAB) solved 99.7% of the instances to optimality in 18 seconds on average and we solve all instances to opti-

mality in 19 seconds with the model (LB-P-WPAB). These results show us that, if step-wise curves were used in hourly orders, the problem would become much simpler. As a remark, the relative gap values do not reflect the relative distance between the optimal solution of the approximate problems and the optimal solution of the models (MC-WPRB) or (MC-WPAB). They are just relative gaps of the heuristic models.

Table 8.5: Performance of the approximation procedures

	OS	TLS	TTL	RG
	(%)	(%)	(secs.)	(%)
(UB-P-WPRB)	99.2	0.8	19	0.002
(LB-P-WPRB)	99.2	0.8	19	0.002
(UB-P-WPAB)	99.7	0.3	18	0.042
(LB-P-WPAB)	100	-	19	-

If we simultaneously solve the models (UB-P-WPRB) and (LB-P-WPRB) in different machines with 8 cores, we can find an optimal solution of the model (MC-WPRB) in 110 instances by Algorithm 2. Figure 8.2 presents the percentage of instances where the models (UB-P-WPRB) and (LB-P-WPRB) have the same optimal solution but; approximation error (AE) is too large so we cannot find the optimal solution for the model (MC-WPRB) (**Case 1**), the models (UB-P-WPRB) and LB-P-WRB have different optimal solutions (**Case 2**), the models (UB-P-WPRB) and (LB-P-WPRB) result in an optimal solution for the model (MC-WPRB) and we have found that solution by solving the model (MC-WPRB) in Table 8.4 (**Case 3**), the models (UB-P-WPRB) and (LB-P-WPRB) result in an optimal solution for the model (MC-WPRB) and we could not find the optimal solution by solving the model (MC-WPRB) in Table 8.4 due to the time limit (**Case 4**). For cases 1 and 2, we cannot find the optimal solution of (MC-WPRB) by using Algorithm 2. However, for cases

3 and 4 we can find the optimal solution for the model (MC-WPRB) by solving the models (LB-P-WPRB) and (UB-P-WPAB). According to Figure 8.2, in 8% of the instances we find the optimal solutions of the model (MC-WPRB) by using Algorithm 2, which we could not find by solving the model (MC-WPRB). Therefore, we can find the optimal solution for the model (MC-WPRB) in 90.2% ($82.2 + 8$) of the instances.

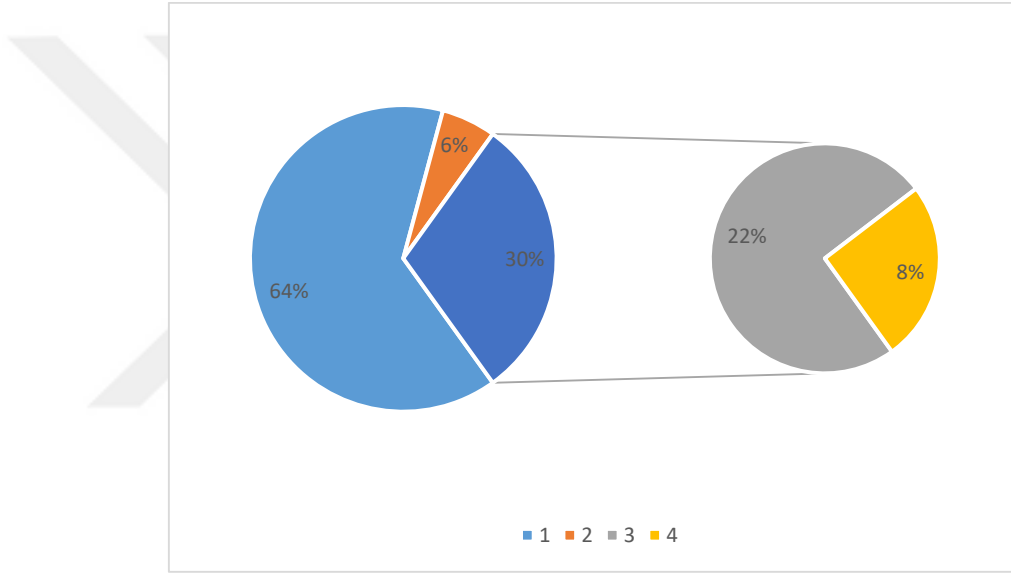


Figure 8.2: Comparison of the optimal solution of the models (UB-P-WPRB) and (LB-P-WPRB)

The similar results are also valid for the model (MC-WPAB). We can prove the optimality of solutions coming from the models (UB-P-WPAB) and (LB-P-WPAB) for the model (MC-WPAB) in 111 instances via Algorithm 2. Figure 8.3 shows the comparison of the optimal solutions of the models (UB-P-WPAB) and (LB-P-WPAB). The meaning of the colors are similar with the ones in Figure 8.2. According to Figure 8.3, in 63% of the instances the models (UB-P-WPAB) and (LB-P-WPAB) have the same optimal solution but AE is not low enough so that we cannot ensure it is the optimal solution for the model (MC-WPAB) by Algorithm 2. The models (UB-P-

WPAB) and (LB-P-WPAB) have different optimal solutions in 7% of the instances. Moreover, in 30% of the instances we can find the optimal solution for the model (MC-WPAB) by Algorithm 2. Finally, although we could not find the optimal solution of the model (MC-WPAB) by directly solving it, but we can find the optimal solution of the same problem in 11% of the instances by only solving the models (UB-P-WPAB) and (LB-P-WPAB). Hence, we know the optimal solution for the model (MC-WPAB) in 91.5% ($80.5 + 11$) of the instances.

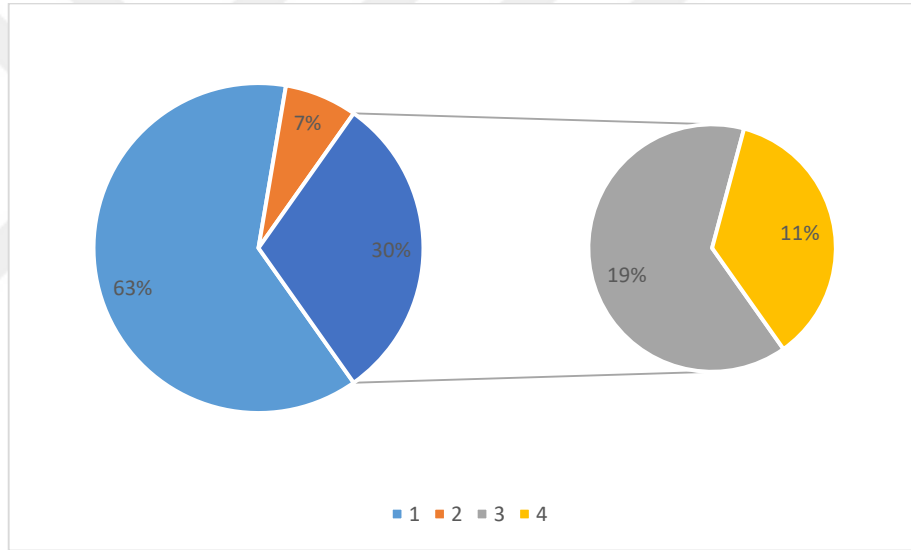


Figure 8.3: Comparison of the optimal solutions of the models (UB-P-WPAB) and (LB-P-WPAB)

In most of the instances, we cannot say that the results of the approximate models are optimal for the models (MC-WPRB) and (MC-WPAB). We still need to continue searching for the optimality of the main problems after solving the approximate models. Hence, we need to focus on the solution times of the approximate models to check whether the remaining time is sufficient for investigating the optimal solution for the problem variants.

We can find the optimal solutions of the approximate models in a short time. Figure 8.4 shows the cumulative frequency plot of the solution times for solving the models (UB-P-WPRB), (LB-P-WPRB), (UB-P-WPAB) and (UB-P-WPAB).

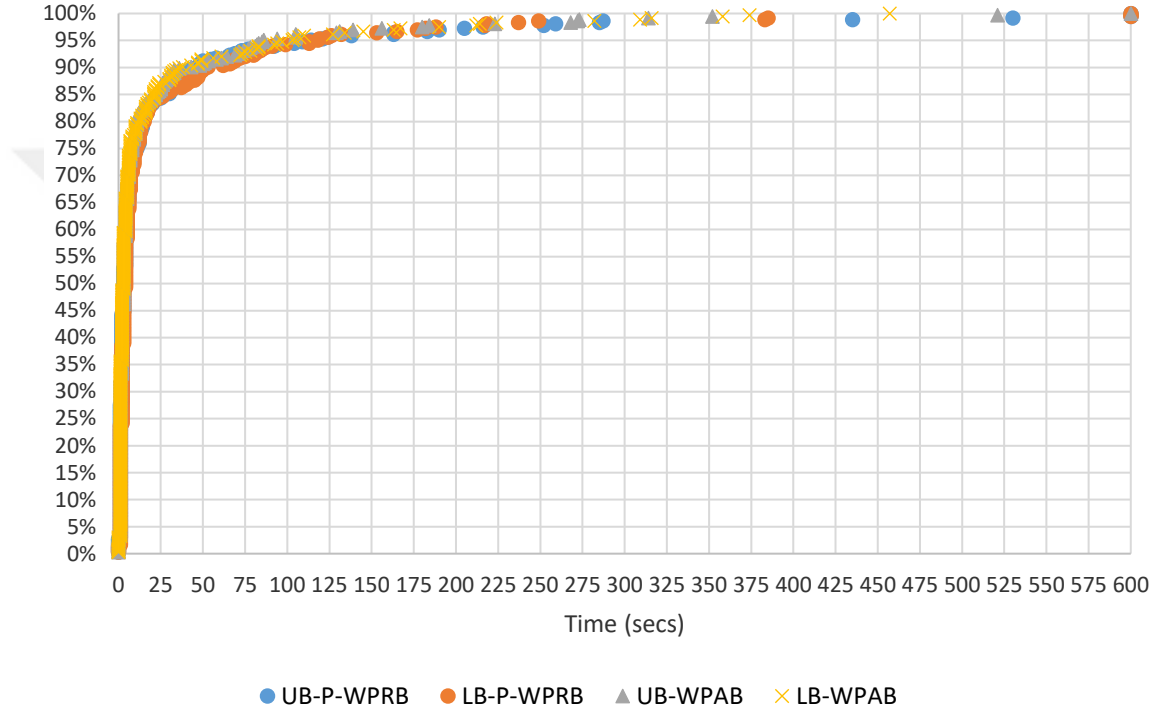


Figure 8.4: Cumulative Percentage Of Solution Times

According to Figure 8.4, we can find the optimal solution of the approximate models in less than 25 seconds in %85 of the instances. However, there are some instances where the solution times of the approximate models could be reached within 10 minutes. We know that the optimal solutions of these problems are also feasible for the models (MC-WPRB) or (MC-WPAB). It motivates us using these models to obtain an initial solution for the models (MC-WPAB) and (MC-WPRB), which we cover next.

8.3.3 Performance of Adding Warm Start

In this section, we will present the performance of adding warm starts to the models (MC-WPRB) and (MC-WPAB). First, we present the methodology for obtaining a warm start solution as fast as we can. Second, we present the performance of adding warm starts.

In order to allocate more time to solve the problem variants, we want to spend as short time as possible to find an initial solution. We are going to solve the approximate models (UB-P-WPRB), (LB-P-WPRB) and (UB-P-WPAB), (LB-P-WPAB) to find an initial solution for the models (MC-WPRB) and (MC-WPAB), respectively.

According to Figure 8.4, the optimal solution for the approximate problems can be found in 25 seconds for 85% of the instances. Therefore, we solve the approximate models and set the time limit to 25 seconds. If there is no feasible solution for a instance, we give another 25 seconds until there is a feasible solution. Then, we provide the solution as a warm start to the models (MC-WPRB) or (MC-WPAB) depending on which problem variant we want to solve. We give the remaining time to the solver to solve the models (MC-WPRB)/(MC-WPAB). In only 1 instance, we could not find a feasible solution for the model (LB-P-WPAB) in 25 seconds, instead we find a feasible solution between 25 and 50 seconds.

Table 8.6 shows the performance of the warm start procedure. UB+MC-WPRB represents the configuration where we solve the model (MC-WPRB) by warm-starting the solution arising from the model (UB-P-WPRB). Similarly, LB+MC-WPRB shows the configuration where the initial solution for the model (MC-WPRB) is arising from the model (LB-P-WPRB). The remaining configurations are UB+MC-WPAB and LB+MC-WPAB where UB and LB indicate whether we solve the models (UB-P-WPAB) or (LB-P-WPAB) to provide a warm start for the model (MC-WPAB).

According to Table 8.6, the performance of the models (MC-WPRB) and (MC-WPAB) improves over against solving the models without a warm start. First of all, there exists at least one solution for all models which is not the case in Table

8.4. Adding a warm start to the model (MC-WPRB) improves OS from 82.5% to 88.5% when the warm start is obtained via solving the model (LB-P-WPRB), which is a 6.3% improvement. Moreover, the average solution times to the instances solved to optimality decreases from 63 seconds to 53 seconds. RG reduces from 0.358 to 0.036. Similar results are valid for the model (MC-WPAB). OS increases from 80.5% to 85.2% by adding a warm start obtain from the model (LB-P-WPRB) to the model (MC-WPAB). Furthermore, average time for solving the optimal instances decreases from 64 seconds to 46 seconds. Finally, the average relative gap value decreases from 0.132 to 0.020.

Table 8.6: Performance of adding warm starts

	OS	TLS	TTL	RG
	(%)	(%)	(secs.)	(%)
UB + MC-WPRB	86.0	14.0	44	0.006
LB + MC-WPRB	88.5	11.5	53	0.036
UB + MC-WPAB	85.2	14.8	46	0.020
LB + MC-WPAB	84.9	15.1	45	0.006

In Table 8.7, we compare the model (MC-WPRB) with warm started (MC-WPRB). There are 297 instances that both cold started model (MC-WPRB) and UB+MC-WPRB find an optimal solution within the time limit. For those instances, UB+MC-WPRB solves significantly faster than the cold stated (MC-WPRB) on the average where 95% confidence interval is given by 35 ± 9.4 seconds. In 46 instances both the model (MC-WPRB) with cold start and UB+MC-WPRB hit the time limit with at least one feasible solution. For those instances, the average relative gap of the model (MC-WPRB) is significantly more than the average relative gap of UB+MC-WPRB, where 95% confidence interval is given by $0.043\% \pm 0.03\%$. In 17 instances, the model

(MC-WPRB) with cold start hits the time limit with a solution but UB+MC-WPRB finds an optimal solution. Moreover, there are three instances that (MC-WPRB) finds an optimal solution but UB+MC-WPRB hits the time limit. Similarly, there are 298 instances that both cold started (MC-WPRB) and LB+MC-WPRB find an optimal solution within the time limit. For those instances, LB+MC-WPRB is significantly faster than the model (MC-WPRB) on the average, where 95% confidence interval is given by 38 ± 9.5 seconds. In 38 instances both the model (MC-WPRB) with cold start and LB+MC-WPRB hit the time limit with at least one feasible solution. For those instances, the average relative gap value of the model (MC-WPRB) has significantly more than the average relative gap of LB+MC-WPRB, where 95% confidence interval is given by $0.035\% \pm 0.029\%$. In 25 instances, the model (MC-WPRB) is TLS, but LB+MC-WPRB is OS. Moreover, there are two instances that (MC-WPRB) is OS but UB+MC-WPRB is TLS. Also, there are two instances that the model (MC-WPRB) with cold start could not find a solution within the time limit but UB+MC-WPRB and LB+MC-WPRB find them as TLS.

Table 8.7: Comparison of cold started MC-WPRB and warm stated MC-WPRB

		UB+MC-WPRB		LB+MC-WPRB	
		OS	TLS	OS	TLS
(MC-WPRB)	OS	297	3	298	2
	TLS	17	46	25	38
	NS	-	2	-	2

In Table 8.8 we make a similar comparison as we did before. There are 292 instances that both the model (MC-WPAB) with cold start and UB+MC-WPAB find an optimal solution within the time limit. Let the difference between the average solution time between the model (MC-WPAB) and UB+MC-WPAB be denoted by

Δ_{MU} . For the instances where UB+MC-WPAB and the model (MC-WPAB) are OS, the 95% confidence interval on Δ_{MU} is given by (25 ± 7) seconds, showing that the UB+MC-WPAB solves significantly faster than the model (MC-WPAB). Similarly, there are 291 instances that both the model (MC-WPAB) with cold start and LB+MC-WPAB find an optimal solution within the time limit. For those instances, LB+MC-WPAB is significantly faster than the model (MC-WPAB) where the 95% confidence interval is given by (25 ± 7) seconds. In 50 instances both the model (MC-WPAB) with cold start, UB+MC-WPAB and LB+MC-WPAB hit the time limit with at least one feasible solution. For those instances, the average relative gap of the model (MC-WPAB) is significantly larger than the average relative gap of UB+MC-WPAB, where 95% confidence interval is $0.009 \% \pm 0.008\%$. Moreover, for the 50 instances that are TLS for both (MC-WPAB), the average relative gap of the model (MC-WPAB) is significantly larger than the average relative gap of LB+MC-WPAB, where 95% confidence interval is $0.014 \% \pm 0.011\%$. As a result adding warm start decreases average solution times and average relative gaps.

Table 8.8: Comparison of cold started (MC-WPAB) and warm stated (MC-WPAB)

		UB+MC-WPAB		LB+MC-WPAB	
		OS	TLS	OS	TLS
(MC-WPAB)	OS	292	2	291	3
	TLS	18	50	18	50
	NS	1	2	1	2

Although the warm starts are feasible, CPLEX solver may consider the initial solution as infeasible for ill-conditioned models. There is not such an instance when we solve the model (MC). On the other hand, if we use the model (SDB-WPAB), there are only 4 instances that the model gets the warm start and in 361 instances

the warm starts are considered as infeasible. We conclude that using (MC) as a base model helps us to eliminate some of the numerical problems.

8.3.4 Performance of the Compact Procedure

We present the performance of the procedure in Algorithm 5 in Chapter 7. In this procedure, we basically solve UB-P- and LB-P- in parallel on two different machines. We send the solutions to the main controller. We change the objective functions to c and continue to solve the problem in the remaining time. In the main controller, if the conditions in Theorem 13 is satisfied then the solution is optimal and the process is stopped. Otherwise, we wait for the results coming from the parallel processes and announce the first result the solver returns.

We use two different time limits for the parallel procedures: 25 seconds and 550 seconds.

8.3.4.1 When the Time Limit is 25 seconds

When we solve the model (MC-WPRB) by the procedure we described above, in 21% of the instances we find the optimal solution in 6 seconds by Theorem 13. In 61% of the instances, the models (UB-P-WPRB) and (LB-P-WPRB) yield the same solution. Moreover, solution of the model (UB-P-WPRB), $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ results in higher function c value than the solution of the model (LB-P-WPRB), $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, in 7% of the instances, i.e., $c(\bar{\mathbf{x}}, \bar{\mathbf{y}}) > c(\underline{\mathbf{x}}, \underline{\mathbf{y}})$. Solution of the model (LB-P-WPRB) has higher function c value than solution of the model (UB-P-WPRB) in 10% of the instances. Similarly, if we solve the model (MC-WPAB) by the procedure in 6 seconds on average we show that 21% of the instances are solved to optimality by Theorem 13. The remaining instances have the following shares: the models (UB-P-WPAB) and (LB-P-WPAB) yield the same solution in 61% of the instances, the model (UB-P-WPAB) provide better solutions than the model (LB-P-WPAB) in 8% of the instances, and the best solution of the model (LB-P-WPAB) has better value than the best solution of the

model (UB-P-WPAB) in 9% of the instances.

The results of the procedure are displayed in Table 8.9. UB // LB + MC-WPRB represents the results when the procedure is applied for the model (MC-WPRB) and UB // LB + MC-WPAB stands for the procedure we are using to solve the model (MC-WPAB). When we compare the results in Table 8.6 and Table 8.9, the procedure applied to the model (MC-WPRB) decreases the average solution time and average relative gap values and the instances solved to optimality is almost the same for. On the other hand, the procedure improves all performance metrics for the model (MC-WPAB).

Table 8.9: Performance of the procedure when we allocate 25 seconds to the heuristics

	OS	TLS	TTL	RG
	(%)	(%)	(secs.)	(%)
UB // LB + MC-WPRB	88.49	11.51	42	0.002
UB // LB + MC-WPAB	87.12	12.88	38	0.004

In Table 8.10, we present the number of instances that the models (MC-WPRB) and (MC-WPAB), and the procedures UB//LB+MC-WPRB and UB//LB+MC-WPAB are OS, TLS and NS. For 299 instances where the model (MC-WPRB) and the procedure UB//LB+MC-WPRB are OS, the procedure UB//LB+MC-WPRB is significantly faster than the cold stated (MC-WPRB) on the average where 95% confidence interval is given by (45 ± 10) seconds. In 39 instances both the model (MC-WPRB) and the procedure UB//LB+MC-WPRB hit the time limit with at least one feasible solution. For those instances, the average relative gap of the procedure UB//LB+MC-WPRB is significantly less than the average relative gap of the model (MC-WPRB), where the 95% confidence interval is $(0.040\% \pm 0.029\%)$. For 292 instances where the model (MC-WPAB) and the procedure UB//LB+MC-WPAB are OS, the procedure

is significantly faster than the cold started model on the average where 95% confidence interval is (33 ± 9) seconds. In 42 instances both the model (MC-WPAB) and the procedure UB//LB+MC-WPAB hit the time limit with at least one feasible solution. For those instances, the average relative gap of the procedure UB//LB+MC-WPAB is significantly less than the average relative gap of the model (MC-WPAB), where the 95% confidence interval is $(0.011\% \pm 0.010\%)$.

Table 8.10: Comparison of cold started models and procedure in Algorithm 5 when we allocate 25 seconds to the heuristics

(a) (MC-WPRB) vs (UB//LB+MC-WPRB)				(b) (MC-WPAB) vs (UB//LB+MC-WPAB)			
		UB//LB+MC-WPRB				UB//LB+MC-WPAB	
		OS	TLS			OS	TLS
(MC-WPRB)	OS	299	1	(MC-WPAB)	OS	292	2
	TLS	24	39		TLS	26	42
	NS	-	2		NS	1	2

We investigate the instances where the initial solution is not optimal by using Theorem 13. We focus on the improvements from the best solution of the approximate problems. Table 8.11 demonstrates the quality of the initial solutions. We divide the instances into two: (1) “improved” shows the ones where objective value arising from the warm starts improve, (2) in “not improved” instances the objective function do not improve. In improved instances the additional time needed to find the optimal solution is more than 200 seconds on average. On the other hand, in not improved instances the additional time to find the optimum is less than 60 seconds on average.

Table 8.11: Quality of the initial solutions

		OS (%)	TLS (%)	Additional Time To Find The Optimum (secs)	Surplus Improvement For OS (TL)	Surplus Improvement For TLS (TL)
(MC-WPRB)	Improved	0.7	6.9	268	13,027	1,110
	Not Improved	84.7	7.6	51	-	-
(MC-WPAB)	Improved	2	5	202	4,665	10,934
	Not Improved	81	12	38	-	-

There are 19 instances that Algorithm 2 returns 15,000 TL more surplus than the procedure presented on Algorithm 5. In addition to that, by Theorem 13 we already find the optimal solution for 13 instances where those instances are not optimal in Table 8.9. The reason for this difference arises from the fact that the approximate models can find better results than the model we solve in Algorithm 5 between 25 seconds and 600 seconds. Hence, instead of allocating 25 seconds to the approximate models, we need to solve these models in 550 seconds and apply the procedure defined in Algorithm 5.

8.3.4.2 When the Time Limit is 550 seconds

When we solve the model (MC-WPRB) by the procedure we described above, in 30% of the instances we find the optimal solution by Theorem 13 in 33 seconds on average. Similarly, when we solve the model (MC-WPAB), in 30% of the instances are optimal in 37 seconds.

The results of the procedure are displayed in Table 8.9, from which we observe that allocating 550 seconds for the heuristics results in better performance than allocating 25 seconds.

Table 8.12: Performance of the procedure when we allocate 550 seconds for the heuristics

	OS	TLS	TTL	RG
	(%)	(%)	(secs.)	(%)
UB // LB + MC-WPRB	93.42	6.58	41	0.001
UB // LB + MC-WPAB	93.15	6.85	39	0.005

In Table 8.13, we present the number of instances that the models (MC-WPRB) and (MC-WPAB), and the procedures UB//LB+MC-WPRB and UB//LB+MC-WPAB are OS, TLS and NS when we allocate 550 seconds for the heuristics. For 298 instances the model (MC-WPRB) and the procedure UB//LB+MC-WPRB are OS. For those instances, the procedure runs significantly faster than the procedure on the average, where the 95% confidence interval is (46 ± 10.49) seconds. For 20 instances both the model (MC-WPRB) and the procedure UB//LB+MC-WPRB hit the time limit with at least one feasible solution. For those instances, the average relative gap of the procedure is significantly less than the average relative gap of the model where 95% confidence interval is $(0.059\% \pm 0.051\%)$. For 276 instances where the model (MC-WPAB) and the procedure UB//LB+MC-WPAB are OS, the procedure stops significantly faster than the model where the 95% confidence interval is (28 ± 13.3) seconds. In 6 instances both the model (MC-WPAB) and the procedure UB//LB+MC-WPAB hit the time limit with at least one feasible solution.

We also investigate the impact of increasing the relative gap. For this aim, we only focus on the instances where the optimal solution could not be found in Table 8.12. We group the instances which have a relative gap less than 1E-7, 1E-6, 1E-5, and 1E-4. We present the new optimality percentages in Table 8.14.

Table 8.13: Comparison of cold started models and procedure in Algorithm 5 when we allocate 550 seconds for the heuristics

(a) (MC-WPRB) vs (UB//LB+MC-WPRB)				(b) (MC-WPAB) vs (UB//LB+MC-WPAB)			
		UB//LB+MC-WPRB				UB//LB+MC-WPAB	
		OS	TLS			OS	TLS
(MC-WPRB)	OS	298	2	(MC-WPAB)	OS	276	18
	TLS	43	20		TLS	62	6
	NS	-	2		NS	3	-

Table 8.14: Change in the optimality percentages with the relative gap values

		1E-7	1E-6	1E-5	1E-4
(MC-WPRB)	# TLS	24	21	11	0
	OS %	93.42	94.2	97	100
(MC-WPAB)	# TLS	25	25	18	9
	OS %	93.15	93.15	95.07	99.45

According to Table 8.14, if the relative gap is set to 1E-4, then all instances are optimally solved for the model (MC-WPRB). Similarly, 99.45% of the instances are optimal when the relative gap is 1E-4 for the model (MC-WPAB). However, we sacrifice up to 99,999 TL ($\bar{9} \times 10^8 \times 10^{-4}$) surplus value with this relative gap.

We also conduct experiments on the strong cuts we defined in Chapter 6. They improve the gaps on average. However, they are not statistically significant, therefore, we do not include the results in this study.

8.3.5 Performance of the Decomposition Procedure

We present the performance of the decomposition procedure where we use the cuts defined in Theorems 14 and 27. Since we do not use w variables in the decomposition procedure, there is no significant improvement when we use MC as a base model or SDB as a base model. We present the results where MC is used as a base model.

Table 8.15 presents the results when we use the cut defined Theorem 14 and 27 to solve the models (MC-WPRB) and (MC-WPAB), respectively.

Table 8.15: Performance of decomposition procedures with exact cuts

	OS	TLS	NS	TTL	RG
	(%)	(%)	(%)	(secs.)	(%)
(MC-WPRB)	40.3	59.7	-	37	0.003
(MC-WPAB)	72.6	26.6	0.8	56	0.003

Although the number of instances solved to optimality is worse than the Procedure defined in Figure 5, relative gap values are not as high as the values defined in the procedure.

Average time and the number of optimal instances are not high with the exact cuts. To solve the problem in a short time, we test the heuristic cuts defined in Lemma 28 and 29. Table 8.16 presents the results, where the average number of nodes discovered is denoted by NN. To find the instances solved to optimality, we count the instances where the solution of the heuristic cut is optimal for the original problem. Average elapsed time is the average run time for 365 instances. Average relative gap shows the relative distance between the best objective value obtained by the heuristic cut and Algorithm 5. We only calculate the relative gap value for the instances which are not solved to optimality. The corresponding absolute gap value is around 12,000 TL.

Table 8.16: Performance of decomposition procedures with heuristic cuts

	OS	TTL	RG	NN
	(%)	(secs.)	(%)	
(MC-WPRB)	49.3	39	0.002	1479
(MC-WPAB)	70.7	26	0.002	771

In both of the heuristic and exact cuts, the cuts for the model (MC-WPRB) has lower optimality ratios than the ones for the model (MC-WPAB). We suspect that the reason for this phenomena arises from the data set. To support this argument, we will use a random data-set, and check whether we observe such a trend or not in the next chapter.

8.4 Summary

In this chapter, we analyze the performance of our models and solution methodologies with the annual real data taken from the Turkish DAM operator. We focus on optimality percentage, solution time for the optimal solutions, and gaps for the instances where the solver stops due to time limit. We show that, MC is a good base model because it reduces the solution times and increases the optimality ratios for the both problem variants. Since there are instances where there is no solution by only solving the models (MC-WPRB) and (MC-WPAB), we apply the Algorithm 2. When the objective function is linear, the models are solved to optimality in almost all instances in 19 seconds on average. By using Algorithm 2 we can guarantee the optimal solution of the models (MC-WPRB) and (MC-WPAB) in 30% of the instances. We can find the optimal solution for 90.2% and 91.5% of the instances for the models (MC-WPRB) and (MC-WPAB), respectively.

These optimality ratios are calculated by assuming that we use two machines to solve each problem. We need a compact design we focus on adding a warm start to

the problems. We allocate 25 seconds for the approximate models and add the best of their results as a warm start and solve the main problem. Although we could not achieve to have an optimality ratio more than 90% by adding warm start, this method decreases the average solution times and gaps.

We also test the performance of Algorithm 5. We investigate the performance under 25 and 550 seconds time allocation for the approximate models. As a result, an optimal solution of the models (MC-WPRB) and (MC-WPAB) is found in 93.42% and 93.15% of the instances in approximately 40 seconds on average.

As a final step, we also test the decomposition procedure we discussed in Chapter 7. We solve 40.3% of the instances to optimality with the cut defined in Theorem 14 in 37 seconds with 0.003% relative gap. Similarly 72.6% of the instances are optimally solved with the cut defined in Theorem 27 in 56 seconds on average with 0.003% relative gap. These relative gap values are lower than the relative gap values when we solve the models (MC-WPAB) or (MC-WPRB).

As a result our procedure defined in Algorithm 5 performs better than the decomposition procedure when we use Turkish DAM data. In the next chapter, we will perform a scenario analysis. We present under which conditions our procedure still performs better than the decomposition procedure.

Chapter 9

SCENARIO ANALYSIS

In the previous chapter, we show that our proposed procedures perform well on real data-sets. However, we do not know how the performance of these procedures changes when the number of orders change. In this chapter we make scenario analysis and compare the performance of Algorithm 5 and heuristic decomposition procedure. As we show in the previous Chapter, Algorithm 5 is the best way to solve the problem when there is real data. However, the models we solve in Algorithm 5 have lots of constraints and binary variables. On the other hand, the model we solve with the heuristic decomposition procedure has a smaller size. Hence, it may be a good alternative solution method when the problem size increases.

In this chapter, we will seek the answers of the following questions:

- We observed in the previous chapter that Algorithm 5 exhibits good performance on real data. Under which conditions does it still show a good performance?
- If Algorithm 5 does not perform well, can we find better solutions with the heuristic decomposition procedure?
- We observed that the proposed decomposition procedure exhibits better performance on the model (MC-WPAB) in comparison with the model (MC-WPRB). Does the performance difference arise from the specific input data?

To answer these questions, we are going to create some data sets and analyze the impact of some parameters on the solution time and the quality of the solution. First, we present the parameters we will change. Then, we explain the details of the data

preparation. Afterwards, we describe the experiments and present the corresponding results.

9.1 Parameters

We will use the number of periods, block orders, linked block orders and flexible orders as a parameter.

Number of periods and segments ($|T|$): In real cases, the number of periods is generally 24. However, there is a “half hour day-ahead 15:30 auction” in the UK where the number of periods is 48. We will choose the number of periods from the set $\{24, 48, 96\}$. Most of the binary variables arise from the variables w in the mathematical models used in Algorithm 5. The number of these variables is related to the number of segments in the total curve in each period. To capture the impact of increasing the number of segments, we will increase the number of segments while we increase the number of periods.

Number of block orders ($|B|$): In Europe, the number of block and flexible orders is more than 4500 (N-Side, 2016). In the literature, there are some studies which conduct experiments with approximately 600 block orders per day (Martin et al., 2014; Madani and Van Vyve, 2015). Therefore, we will prepare 3 different data sets where the daily number of block orders is 150, 300 and 600, in each data set.

Number of flexible orders ($|F|$): We will investigate the cases where the number of flexible bids is 0, 29 and 59 per day.

Number of linked block orders ($|B^c|$): Although Martin et al. (2014) conduct experiments with linked block orders, there is no information about the average number of linked block order per day. We only know that they used the real data from 2009. We pick the ratio of the number of linked block orders to the number of block orders as 0, 20%, and 40%.

9.2 Data

We randomly generate 30 instances for each choice of parameters. We use the market clearing prices and the corresponding supply and demand curves published on Turkish DAM operator's website. We use the real curves instead of generating the hourly orders from scratch. Prices of the block and flexible orders are generated by the market clearing prices. We downloaded the aggregated supply and demand curves published on Turkish DAM operator's website for the days between 01 March 2018 and 31 March 2018 (31 days). These curves also includes matching quantities for block and flexible orders. Matching quantities of block and flexible orders are also public. We subtract the quantity arising from matched block and flexible orders from these curves and obtain the submitted aggregated supply and demand curves for each hour. Then, we randomly select one day from these 31 days and add the submitted aggregated supply and demand curves of that day to an instance. We repeat this procedure to create each instance.

In order to investigate the impact of the number of periods and segments, we randomly select $|T|/24$ days from the market data. We take their submitted aggregated curves and create new aggregated curves such a way that the same periods of the different days have consecutive periods in our new generated data.

We generate a block order b as follows:

- **Number of periods:** We generate N_b with a uniform distribution between 3 and the maximum number of periods, i.e. $N_b \sim U[3, |T|]$.
- **Start period:** We generate this number with uniform distribution between 1 and the maximum number of periods minus the number of periods in order not to exceed the allowed period span. i.e. $U[1, |T| - N_b + 1]$.
- **Price:** We generate P_b by using Normal distribution with mean μ and variance σ^2 . We calculate μ by taking the average of the market clearing prices where

the block order b is valid. We calculate σ by the three sigma rule which assumes the 99.7% of values lie between ± 3 standard deviation of the mean. In our case, the price values are between the maximum price limit denoted by P_{max} and the minimum price limit denoted by P_{min} . Therefore, $\sigma = \min \left\{ \frac{\mu - P_{min}}{3}, \frac{P_{max} - \mu}{3} \right\}$. With probability less than 0.003 the price may be less than P_{min} or higher than P_{max} , in those cases we set the price to the closest price limit. By this method, the price values may have a precision of more than 2 digits. In that case, we round it to 2 digits of precision.

- **Quantity:** Q_b is generated by uniform distribution between -6000 and 6000. By this technique we may obtain a sell block or a buy block order with probability 0.5.

For the cases where a link relation exists, while generating block orders with probability p we randomly assign a linked block order (parent) to the current one. We set p as 0.2 and 0.4. We pay attention not to form a family with more than 3 generations. Also, each order has no more than three children and there is not any cyclic relation. The expected number of linked orders is 30 and 60 when we generate 150 block orders with probability 0.2 and 0.4, respectively.

The duration of the flexible orders is one period and their time window is the number of periods. We randomly generate the price and quantity of a flexible order, where P_f follows a Normal distribution with mean μ and variance σ^2 . We calculate μ by taking the maximum of the market clearing prices where the flexible order f is valid. We calculate σ by the three sigma rule described above. Since there is only sales flexible orders in Turkish DAM, Q_f is generated by a uniform distribution between -6000 and 0.

The remaining parametric values (e.g. price limits, number of generations in a block order family) are equal to the values that Turkish market operator uses.

9.3 Tests and Performance Measures

We use a similar test environment and solver parameters presented in the previous section.

We run the procedure outlined in Algorithm 5 by setting 550 seconds for the heuristic time limit and 600 seconds for the overall algorithms.

We investigate the performance of Algorithm 5 in terms of (1) percentage of instances that an optimal solution is found (OS), (2) percentage of the instances that a feasible solution is found but we could not prove its optimality due to time limit (TLS), (3) Average time for the instances which are solved to optimality (TTL), (4) average relative gap for the instances where a feasible solution is found but its optimality could not be proven within the given time limit (RG) and (5) average number of nodes discovered (NN).

We use a *paired t-test* to show whether two models or two solution methodologies perform significantly different from each other or not with respect to the measures objective function value, elapsed time, and the number of nodes discovered while solving the problem. For each performance metric pm , we build the hypothesis $H_{model-type, pm}^{pm}$, where *model-type* is either the models (MC-WPRB) or (MC-WPAB). We define the hypothesis as:

$$H^0(pm, model - type) : \mu(M_{model-type}^{pm}) - \mu(H_{model-type}^{pm}) = 0, \quad (9.1)$$

$$H^1(pm, model - type) : \mu(M_{model-type}^{pm}) - \mu(H_{model-type}^{pm}) \neq 0, \quad (9.2)$$

where M represents the procedure defined in Algorithm 5 and H stands for the heuristic cuts. The performance metric pm can be *obj*, *time*, or *node*, which corresponds to the objective function, elapsed time, and the average number of nodes discovered while solving the problem, respectively. We set the significance level to 5%. We only apply t-test to the instances where the solution times of the models (MC-WPRB) or (MC-WPAB) hit the time limit. If the number of data points is less than 30, we do not perform the t-test due to the low number of observations.

9.4 Results

In this section, we first present the results for different number of periods. Then, we focus on the impact of the number of block orders and flexible orders. Finally, we show how the performance of Algorithm 5 is affected by the number of linked block orders.

9.4.1 Impact of the Number of Periods

We generate 90 instances in total where in each instance $|B|$ is 150, B^c and $|F|$ are 0. For each set of 30 instances $|T|$ is 24, 48 or 96, respectively.

We show the results of Algorithm 5 with different $|T|$ values when we are solving the model (MC-WPRB) in Table 9.1 and the model (MC-WPAB) in Table 9.2. When the number of period limit increases, the percentage of cases solved to optimality decreases from 97% to 7% and the average solution time in each optimal case increases. The average relative gap values for the aborted cases increase. Therefore, the procedure deteriorates in all performance metrics when the number of periods increases. Since the average number of nodes discovered while solving the problem decreases, the average time spent on each node increases. To illustrate, the solver spends most of its time to solve the node problem when $|T|$ is 96 and the model (MC-WPAB) is solved.

Table 9.1: Impact of the number of periods: Results when the model (MC-WPRB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
$ T $	(%)	(%)	(secs.)	(%)	
24	97	3	93	0.000 ¹	521
48	50	50	203	0.003	371
96	13	87	554	2.748	48

Table 9.2: Impact of the number of periods: Results when the model (MC-WPAB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
$ T $	(%)	(%)	(secs.)	(%)	
24	97	3	100	0.003	822
48	77	23	236	3.627	303
96	7	93	584	3.837	1

In Table 9.3, we report the paired t-test results where the mean of differences is $\bar{X}$, lower confidence limit is (LCL), and the upper confidence limit (UCL).

¹0.00064%

Table 9.3: Impact of the number of periods: Paired student t-test results when the models are solved with the exact and heuristic algorithms

model-Type	PM	$\bar{X}$	$p\text{-value}$	LCL	UCL
MC-WPRB	obj	243,047	0.0010	104,837	381,258
	time	578	0	562	594
	node	149	0.2658	-118	415
MC-WPAB	obj	51,552	0.2602	-39,892	142,996
	time	576	0	557	594
	node	-23	0.8139	-224	177

We report the results of *paired t-test* when we solve the models (MC-WPRB) and (MC-WPAB) with Algorithm 5 and the heuristic cut defined in Lemma 28. In 95% confidence interval, the heuristic cut is 578 seconds faster than Algorithm 5 on average when we solve the model (MC-WPRB). In 95% confidence interval, we fail to reject the hypothesis that the sample mean of the objective value Algorithm 5 and the heuristic cut defined in Lemma 29 in Table 9.3 are the same for the model (MC-WPAB). For the same model, Algorithm 5 is 576 seconds slower than the heuristic cut on average. We fail to reject the hypothesis that the sample mean of the number of nodes discovered in Algorithm 5 is equal to the sample mean of the number of nodes discovered using the heuristic cut for both of the model variants.

The average time spent on each node of Algorithm 5 is higher than the average time spent on each node of the heuristic time. This is an expected result because, increasing the period size by increasing the total number of segments gives rise to a larger number of binary variables in the models used in Algorithm 5.

Although the average objective value of Algorithm 5 is higher than the average objective value of the heuristic cut, in more than 50% of the cases, the heuristic cut and Algorithm 5 compute the same optimal value. Since the results for Algorithm 5

do not dominate the results of the heuristic cut in the model (MC-WPAB), we can also use the heuristic algorithm as an alternative solution method when the number of periods increases.

We do not observe that the heuristic cut defined in Lemma 29 has an advantage on the heuristic cut defined in Lemma 28 for the current data sets in terms of optimality percentages.

9.4.2 Impact of the Number of Block Orders

We generate 60 additional instances where in each instance $|T|$ is 24, $|B^c|$ and $|F|$ are 0. $|B|$ in each set of 30 instances is 300 and 600, respectively.

We report the results of Algorithm 5 in Tables 9.4 and 9.5 for the models (MC-WPRB) and (MC-WPAB), respectively. The number of instances solved to optimality decreases and the elapsed time to solve the cases to optimality and the relative gap of the aborted cases increase for both problem variants.

Table 9.4: Impact of the number of block orders: Results when the model (MC-WPRB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
$ B $	(%)	(%)	(secs.)	(%)	
150	97	3	93	0.000 ²	521
300	63	37	189	0.004	1429
600	23	67	292	7.939	815

²0.00064%

Table 9.5: Impact of the number of block orders: Results when the model (MC-WPAB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
$ B $	(%)	(%)	(secs.)	(%)	
150	97	3	100	0.003	822
300	67	33	259	0.012	1,805
600	27	73	277	4.253	1,633

We fail to reject that the objective difference of the two solution methodologies are the same. The average run time exhibits a similar behavior to the previous cases: The average running time of the heuristic cut is better than the average running time of Algorithm 5. The number of nodes explored by Algorithm 5 is less than the number of nodes explored by the decomposition with the heuristic cut. The reason for this difference is that the initial solution procedure in Algorithm 5 finds a solution in 550 seconds in most of the cases. When we change the objective to the quadratic one, the solver cannot explore nodes further. According to the t-test results we can use the heuristic decomposition as an alternative to the models, where we find an initial solution.

Table 9.6: Impact of the number of block orders: Paired student t-test results when the models are solved with the exact and heuristic algorithms

model-Type	PM	$\bar{X}$	$p\text{-value}$	LCL	UCL
MC-WPRB	obj	-39,109,788	0.08464	-83,855,589	5,636,013
	time	366	0	289	442
	node	-5,913	0	-8,000	-3,826
MC-WPAB	obj	-36,732	0.7224	-247,156	173,691
	time	224	0	141	328
	node	-6,5167	0	-8,350	-4,681

9.4.3 Impact of the Number of Flexible Orders

We generate 60 additional instances, where 30 of them have 29 flexible orders and the remaining cases have 59 flexible orders. For all of the cases, $|B|$ is set to 150, and $|T|$ is 24. There is no link relation in these data sets.

We present the results of Algorithm 5 in Tables 9.7 and 9.8 to solve the models (MC-WPRB) and (MC-WPAB), respectively. The percentage of cases solved to optimality decreases from 97% to 57% when the number of flexible orders increases. However, absolute gap values of these models are not higher than 10,000 TL, which is a tolerable result. The average solution time in each optimal case increases when the number of flexible orders increases. The reason for this increase is that the solver investigates more number of nodes on average when the number of flexible orders increases.

Table 9.7: Impact of the Number of Flexible Orders: Results when the model (MC-WPRB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
$ F $	(%)	(%)	(secs.)	(%)	
0	97	3	93	0.000 ³	521
29	93	7	122	0.002	6,306
59	80	20	281	0.000 ⁴	71,396

Table 9.8: Impact of the Number of Flexible Orders: Results when the model (MC-WPAB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
$ F $	(%)	(%)	(secs.)	(%)	
0	97	3	100	0.003	822
29	63	37	80	0.001	3,914
59	57	43	171	0.000 ⁵	25,071

Although, the procedure deteriorates in all performance metrics when the number of flexible orders increases, the number of instances where the procedure hits the time limit is not sufficiently large, i.e., greater than or equal to 30. Hence, we do not conduct paired t-tests.

We observe that, the average time to solve the model (MC-WPAB) with the

⁴0.00064%

⁴0.00098%

⁵0.00077%

heuristic decomposition is higher than that to solve the model (MC-WPRB). However, the decomposition method for solving the model (MC-WPAB) gives closer results to the objective value of Algorithm 5 than the decomposition method for solving the model (MC-WPRB).

9.4.4 Impact of the Number of Linked Orders

We also generate 60 additional instances, where 30 of them have the link probability p equal to 0.2 and the remaining cases are generated with $p = 0.4$. For all of the cases, $|B|$ is set as 150, and $|T|$ is 24, and $|F|$ is 59.

The results are presented in Tables 9.9 and 9.10. The number of block orders is fixed. Relating a block order to another decreases the number of possible feasible solutions because we eliminate solution $y_{b_m} = 0$ and $y_b = 0$ where b_m represents the parent order or block order b . Therefore, increasing the probability of the link relation increases OS up to 77% when we solve the model (MC-WPAB). However, in the formulation of the model (MC-WPRB) we add as many constraints for each linked block order as the number of paths that the block order has. Thus, when we solve the model (MC-WPRB), the number of optimal instances decreases with higher p values.

Table 9.9: Impact of the Number of Linked Block Orders: Results when the model (MC-WPRB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
p	(%)	(%)	(secs.)	(%)	
0	80	20	281	0.000 ⁶	71,396
0.2	80	20	119	0.001	7,350
0.4	67	33	189	0.001	10,594

Table 9.10: Impact of the Number of Linked Block Orders: Results when the model (MC-WPAB) is solved via Algorithm 5

	OS	TLS	TTL	RG	NN
p	(%)	(%)	(secs.)	(%)	
0	57	43	171	0.000 ⁷	25,071
0.2	63	37	193	0.001	16,393
0.4	77	23	139	0.003	7,920

In the model (MC-WPRB), there is no sufficient number of cases that hit the time limit, therefore, we do conduct a t-test to evaluate the performance of the heuristic cut. Moreover, the absolute gaps of the model (MC-WPAB) is no more than 10,000 TL. Therefore, we do not also use paired t-test for this model type. We observe the similar results that we observe in Section 9.4.3 about the performance of the heuristic cut.

9.5 Summary

In this chapter, we investigate the impact of the parameter changes on the performance of Algorithm 5. We consider an increment in the number of periods, number of block, flexible and linked block orders.

Algorithm 5 is the most promising solution methodology on the real data sets. However, the performance of Algorithm 5 in terms of percentage of instances solved to optimality and solution times deteriorates when the number of periods and the number of block and flexible orders increase. Increasing the link relation among the block orders simplifies the model (MC-WPAB) because the number of feasible solutions decreases

⁶0.00098%

⁷0.00077%

when we add link order condition. However, it decreases OS of the model (MC-WPRB) because the number of constraints increases since ξ_b increases.

We also focus on the performance of the heuristic cuts. We can use the heuristic cuts as an alternative initial solution methodology when the number of block orders increases. However, for the other cases, our initial solution methodology results in better objective function values. We observed in Chapter 8 that solving the model (MC-WPAB) with the decomposition procedure results in better OS than solving the model (MC-WPRB) with the same procedure. We could not observe such a trend with the random examples. Hence, we think that the reason for this difference is caused by the bidding behaviour of the Turkish day-ahead market participants.

Chapter 10

DISCUSSION AND CONCLUSION

In this study, we consider the problem in European day-ahead electricity markets when there are piece-wise hourly, block, linked block and flexible orders. We investigate the mathematical properties of the optimal solution of the classical problem of the power exchanges, which we call supply demand balance problem. In this problem, total surplus is maximized subject to supply demand balance in each period and the basic requirements of the orders such as all-or-nothing conditions of block and flexible orders, and link condition of block orders. To the best of our knowledge, this is the first study that focuses on the optimality conditions of the supply demand balance problem with piece-wise linear hourly, linked block and flexible orders.

Using piece-wise linear hourly orders results in an objective function with quadratic terms. We introduce a new market clearing model instead of the supply demand balance problem with a fewer number of decision variables. The proposed market clearing model brings a new interpretation of the surplus maximization problem.

By leveraging the optimality conditions, we propose a new IP pricing mechanism with additional constraints that support Walrasian equilibrium when there exist piece-wise hourly, block, linked block and flexible orders with additional uplift. This pricing mechanism is the complement of the mechanism embraced in the European day-ahead markets.

We investigate the two problem variants arising from the supply demand balance problem: The proposed variant and the one used in Nord Pool Spot, where the order types are similar with the ones we focus on this study. Since the supply demand balance problem is the basis for both the model of Nord Pool Spot and our proposed

model, we use the market clearing model in these formulations. We propose new strong valid cuts for the resulting problems.

Winner determination is an important feature in combinatorial auction designs. Therefore, solving the market clearing problems to optimality in a limited amount of time is our primal aim. We propose two different solution methodologies: Bender's decomposition and approximations. We conduct computational experiments to assess the performance of our models using the data procured from the Turkish DAM operator. We also perform scenario analysis to determine how robust our solution methodologies are with respect to changes in the problem parameters.

According to the results, decreasing the number quadratic terms improves the solution times and optimality rates. We solve 93% of the cases in approximately 40 seconds on average by using Algorithm 5 when the time limit for the approximate problems is 550 seconds. The exact decomposition procedures do not perform well on the real data set but the heuristic decomposition computes near optimal solutions in a short time.

We show in the scenario analysis that when the number of periods, block orders or flexible orders increase, the percentage of the optimal cases decreases in Algorithm 5. On the other hand, the heuristic cuts become a viable option for the exact solution method in terms of time and objective function when the number of block orders increases.

10.1 Managerial and Practical Insights

A market policy maker desires a competitive market design where the market clearing prices signal the true investment costs. It is important to have surplus maximizing combinatorial auction design by preserving fairness and truth telling. Fairness and truth telling are captured in the market clearing problem in European power exchanges. However, both Turkish and European day-ahead markets suffer from the equilibrium, which is important to keep the market working. We suggest the policy

makers to implement our proposed IP pricing variant to prevent the market failure. Truth telling problem can be eliminated by a strict market surveillance system.

Participants may have linear marginal costs. Allowing only piece-wise hourly orders forces the participants to bid a piece-wise linear curve which is almost a step function. We observe such a trend in the hourly order curves published on the Turkish day-ahead market. This gives rise to both technical problems when solving the market clearing problem and it creates undesired matching for the participants. Allowing mixed hourly order curves eliminates this problem.

10.2 Future Directions

Although electricity storage is not an economically viable option with today's technologies, it may be a cheaper and cost-effective in the future. At one point, the day-ahead markets may be abolished. However, between the transformation period, integrating the storage capacities into the day-ahead markets may be necessary.

As a future work, integration of the storage technologies into the day-ahead markets can be studied. Particularly, we suggest a new arbitrage order type. In this order type, the clearing algorithm decides to buy the electricity in lower prices and sell it at the higher prices with a quantity at most at the capacity of the battery.

It is important to have strategyproof mechanism in day-ahead market design, where there is no incentive for the market participants to hide or lie about their private information. Another future direction can be finding an alternative objective function in addition to total surplus to achieve strategyproofness and pareto optimality. For this aim, fairness can be an alternative objective function to investigate.



Appendices

Appendix A

CHANGED CONSTRAINTS IN THE MODELS (MC-WPRB) AND (MC-WPAB) WHEN SDB IS USED AS A BASE MODEL

In this section, we present the constraints that change when we use SDB as a base model to formulate the supply demand balance problem variants.

- Objective function would be:

$$\begin{aligned} & \sum_{t \in T, i \in I_t} \left(\delta_{i,t} P_{k,t}^D \Delta Q_{i,t}^D - \frac{\delta_{i,t}^2 \Delta P_{k,t}^D \Delta Q_{i,t}^D}{2} - \sigma_{i,t} P_{i,t}^S \Delta Q_{i,t}^S - \frac{\sigma_{i,t}^2 \Delta P_{k,t}^S \Delta Q_{i,t}^S}{2} \right) \\ & + \sum_{t \in T} \left(\delta_t^c P_{1,t}^D \Delta Q_{1,t}^D - \sigma_t^c P_{1,t}^S \Delta Q_{1,t}^S \right) + \sum_{b \in B} y_b P_b Q_{b,t} + \sum_{f \in F, t \in T} y_{ft} P_f Q_f \end{aligned}$$

- Constraint defined by (6.26) and (6.68) is replaced by the following inequality.

$$\sum_{i \in I_t} \delta_{i,t} \Delta Q_{i,t}^D - \sigma_{i,t} \Delta Q_{i,t}^S + \delta_t^c \Delta Q_{1,t}^D - \sigma_t^c \Delta Q_{1,t}^S \sum_{b \in B} y_b Q_{b,t} + \sum_{f \in F} y_{ft} Q_f = 0 \quad \forall t \in T.$$

- (6.29)-(6.31) and (6.71)-(6.73) are replaced by the following inequality.

$$\omega_{0,t} \leq \delta_t^c \leq 1, \forall t \in T$$

$$\omega_{i,t} \leq \delta_{i,t} \leq \omega_{(i-1)t}, \forall i \in I_t, t \in T$$

$$\omega_{K_t,t} = 1,$$

$$\omega_{0,t}^S \leq \sigma_t^c \leq 1, \forall t \in T$$

$$\omega_{i,t}^S \leq \sigma_{i,t} \leq \omega_{(i-1)t}^S, \forall i \in I_t, t \in T$$

$$\omega_{K_t,t}^S = 1,$$

$$\omega_{i,t} \in \{0, 1\} \forall i = \{0, \dots, K_t\}, t \in T.$$

- (6.32) and (6.74) are replaced by the following inequalities:

$$p_t = P_{1,t}^D - \sum_{i \in I_t} \Delta P_{i,t}^D \delta_{i,t}, \forall t \in T$$

$$p_t = P_{1,t}^S + \sum_{i \in I_t} \Delta P_{i,t} \sigma_{i,t}$$

We also provide the valid cuts which are depend on w variables when we use SDB as a base model for problem variants which we call the model (SDB-WPRB) and the model (SDB-WPAB).

Let k be a block or flexible order with price P_k which is valid in period set T_k . For all $t \in T_k$ define the set of segments I_{kt} for a sales order k as $I_{kt} = \{i \in I_t | P_{i,t} \leq P_k < P_{(i+1),t}\}$ and for a purchase order b , $I_{kt} = \{i \in I_t | P_{(i+1),t} < P_k \leq P_{i,t}\}$.

Note that, there is a strict order in price levels for both demand and supply segments except for the curtailable demand and supply segments. Therefore, $I_{kt} = \emptyset$ when $P_k = P_{K_t,t}^D$ and $bf \in B^S \cup F^S$ or $P_k = P_{K_t,t}$ and $bf \in B^D \cup F^D$. For the other cases, there exists exactly one segment that covers P_k . Hence, the cardinality of set I_{kt} is at most one for all $t \in T_k$, $bf \in B$ or $bf \in F$.

Theorem 16. *Let k be a block or flexible order with price P_k which is valid in period set T_k where $|T_k| = N_k$. Then, the following is a strong valid inequality for model (SDB-WPRB).*

$$\sum_{t \in T_b, i \in I_{bt}} \omega_{i,t} \leq (N_b - 1) + y_b, \quad \forall b \in B \setminus \{B^c \cup B^m\}. \quad (\text{A.1})$$

$$\sum_{b_l \in \phi_b, t \in T_{b_l}, i \in I_{b_l t}} \omega_{i,t} \leq \sum_{b_l \in \phi_b} N_{b_l} + y_b - 1, \quad \forall b \in B^m \setminus B^c, \quad (\text{A.2})$$

$$\sum_{b_l \in \phi_b, t \in T_{b_l}, i \in I_{b_l t}} \omega_{i,t} \leq \sum_{b_l \in \phi_b} N_{b_l} - y_{b_m} + y_b, \quad \forall b \in B^c, b \in \Theta_{b_m}. \quad (\text{A.3})$$

$$\sum_{t \in T} y_{ft} \geq \omega_{i,t}, \quad \forall i \in I_{ft}, t \in T, f \in F. \quad (\text{A.4})$$

Let k be a block or flexible order with price P_k . For all $t \in T_k$ define the set of segments I_{kt} for a sales order k as $I_{kt} = \{i \in I_t^S | P_{i,t}^L < P_k \leq P_{i,t}^H\}$ and for a purchase order b , $I_{kt} = \{i \in I_t^D | P_{i,t}^L \leq P_k < P_{i,t}^H\}$. Recall that there is a strict order in price levels for both demand and supply segments except for the first ones which are represented with 1 and $n_t^D + 2$. Hence, $\{1, n_t^D\} \notin I_{kt}$ for all $t \in T$ and $bf \in B \cup F$.

Theorem 17. *Let k be a block or flexible order with price P_k which is valid in period set T_k . Then, the following are valid inequalities for model (SDB-WPAB).*

$$\sum_{t \in T_b, i \in I_{bt}} \omega_{(i-1)t} \geq y_b, \quad \forall b \in B \setminus B^m. \quad (\text{A.5})$$

$$\sum_{b_l \in \phi_b, t \in T_{b_l}, i \in I_{b_l t}} \omega_{(i-1)t} \geq y_b, \quad \forall b \in B^m. \quad (\text{A.6})$$

$$\omega_{(i-1)t} \geq y_{ft}, \quad \forall i \in I_{ft}, \quad f \in F, \quad t \in T. \quad (\text{A.7})$$

Appendix B

SUPPLEMENTARY RESULTS

In this section, we first show the average daily model sizes when we use the real data-set. We present average number of binary variables (**BV**), average number of non-negative continuous variables (**CV**), average number of linear constraints (**LC**) and average number of quadratic constraints (**QC**).

	BV	CV	LC	QC
(MC)	244	9,377	9,430	-
(SDB-WPRB)	16,041	15,845	31,983	-
(MC-WPRB)	9,597	9,401	19,095	-
(MC-WPRB-Q)	244	18,706	19,295	1
(SDB-WPAB)	16,041	15,870	32,013	-
(MC-WPAB)	9,597	9,426	19,125	-
(MC-WPAB-Q)	244	19,002	19,051	1

Table B.1: Size of the models when the real data is used

We also randomly select 10 instances generated in Chapter 9 and we run the model (MC-WPRB). We present the relative gap values in 20, 40, ..., 600 seconds in Figure B.1. For these instances, the relative gap values can be 14% in 60 seconds but they reach 0.00% in 240 seconds.

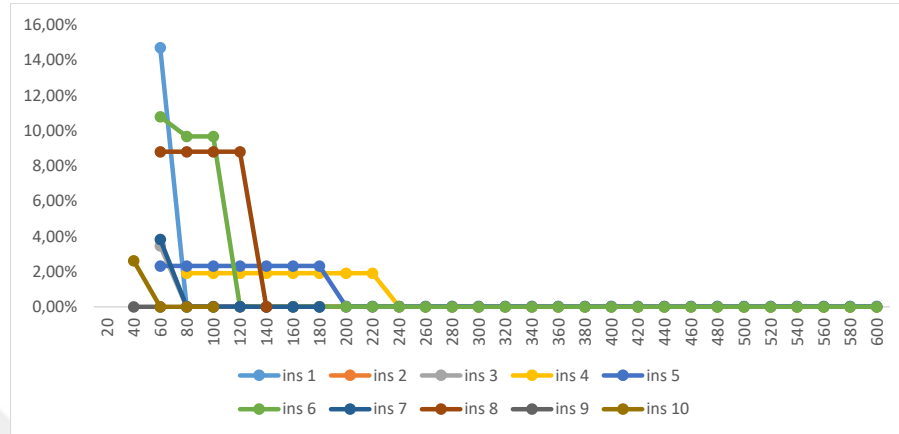


Figure B.1: Relative gap values for 10 randomly selected instances

We set the limits of y axis to 0.03% and present the relative gap values in Figure B.2. In 5 instances solver hits the time limit (ins 1,2,4,5, and 6). For ins 3, 7, 8 and 10, the relative gap for the first integer feasible solution found is more than 1%, but the solver can find the optimal solution at most 240 seconds. For ins 9, the first integer feasible solution found has small relative gap and the solver can find the optimal solution in 60 seconds.

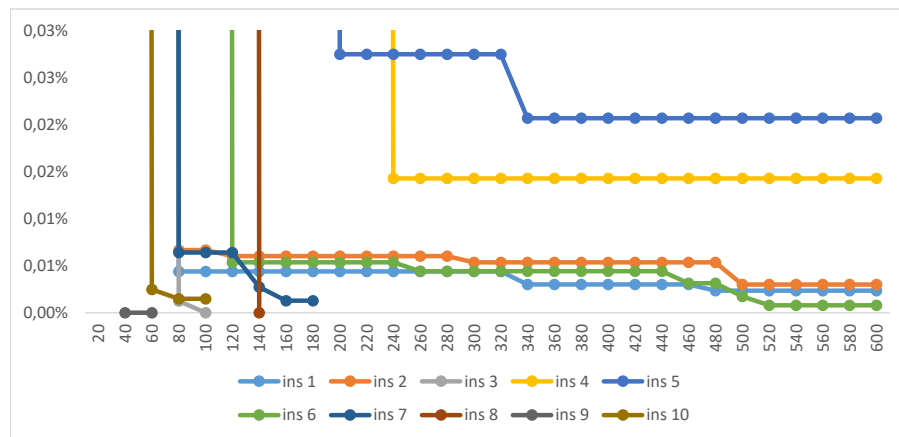


Figure B.2: Relative gap values for 10 randomly selected instances

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