SOME RESULTS ON CONTROLLABILITY AND OBSERVABILITY OF DISCRETE MODELS

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ABSTRACT

SOME RESULTS ON CONTROLLABILITY AND OBSERVABILITY OF DISCRETE MODELS

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In this thesis, we obtain novel results on controllability and observability of some discrete models. The thesis consantrates on two main parts. In the first part, some topological results are obtained and investigated by associating their connection in discrete control systems. In the second part, a novel sampling pattern is constructed and added to the literature related to preserving the property of controllability/observability of sampled-data control systems.

Firstly, we start with examining some special type of sequences in which each candidate is a polynomial with its coefficients are of integers taken from a discrete set. Here, we contribute facts on behaviour of elements of some constructed sets related to the context, some of which are general version of those defined in the works [17, 18]. As one application of the theory, we propose a one dimensional controllability problem of a convex polyhedra with finite number of controls acting on the system and prove that the reachable set of the discrete model is dense in the set of real numbers under the suitable finite control set.

Secondly, we design a new nonequidistant sampling pattern such that the property of controllability and observability of the linear continuous time systems is preserved during the procedure of the sample and zero-order hold operation. Moreover, we provide one application in the concept of controllability of the pendulum in two dimensional setting and obtain corresponding digital controls which make the corresponding sampled-data system controllable.

Keywords: Linear sample-data systems, Kronecker's theorem, non-equidistant sampling, observability, controllability of finite dimensional systems.

ÖZ

AYRIK MODELLERİN KONTROL EDİLEBİLİRLİĞİ VE GÖZLENEBİLİRLİĞİ ÜZERİNE BAZI SONUÇLAR

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Bu tezde, bazı ayrık sistemlerin kontrol edilebilirliği ve gözlenebilirliği üzerine yeni sonuçlar elde ettik. Bu tez, iki ana bölümden oluşmaktadır. Birinci bölümde, bazı özel kümelerin topolojileri üzerine yeni sonuçlar elde edildi ve bu sonuçların literatürden bazı ayrık kontrol yapıları ile ilişkisi incelendi. İkinci bölümde ise, sürekli sistemlerin ayrık yapılarında, sistemin kontrol edilebilirliğini yada gözlenebilirliğini koruyan yeni bir zaman örneklemesi yada zaman çizergesi/düzeni önerildi.

Başlangıç olarak, biz polinom tarzı bazı özel dizilerin oluşturduğu kümelerin topolojileri üzerinde yoğunlaştık. Bu bağlamda, literatürde mevcut olan bazı kümelere genel bir bakış açısı geliştirildi ve yeni sonuçlar elde edildi. Bu sunulan araştırmaya bir uygulama olarak, bir boyutlu dışbükey çokyüzlünün sonlu bir kümeden alınan kontroller vasıtasıyla, kontrol edilebilirliği incelendi ve ulaşabileceği noktaların yoğunluğu araştırıldı.

Tezin ikinci kısımında, biz yeni bir zaman çizergesi oluşturduk öyle ki sürekli sistemlerden ayrık sistemlere geçildiğinde, sürekli yapıların sahip oldukları kontrol edilebilirlik ve gözlenebilirlik özeliklerinin bu bağlamda oluşan ayrık yapılarda korunmasını sağlaması hedef alındı. Sunulan araştırmaya bir uygulama olarak, teori bazında oluşturulan bazı dijital kontroller vasıtasıyla iki boyutlu sarkaç probleminin kontrol edilebilirliği incelendi.

Anahtar Kelimeler: Doğrusal ayrık veri sistemleri, Kronecker teoremi, eşit olmayan zaman düzeni, sonlu sistemlerin kontrol edilebilirliği, gözlenebilirlik.

This thesis is dedicated to the memory of my father

Prof. Mehmet Hamidoğlu Tagiyev

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CHAPTER 1

INTRODUCTION

Recently, there has been an increasing tendency toward building a connection between continuous or dynamical real life processes with their corresponding discrete models in many fields of mathematics; control theory, numerical analysis, number theory, theories emerged from computer science and industry, etc [2, 6, 11, 30, 32, 33, 35, 44, 49, 54]. Roughly speaking, the difference of continuous and discrete processes could be considered as the difference of a box of matchsticks and a piece of cable or wire: in the former each matchsticks has its own nature which can be handled discretely, while in the latter points on a wire are continuously spread out from one end to the other. Here, integers are useful tools to deal with the first, but not with the second where namely real numbers play their parts. For given an arbitrary real number, it is possible to reduce or enlarge its size by multiplying a fixed real number to reach the actual length of wire. Hence, the length of wire is a continuous variable just any other quantities like mass, time, volume, density, speed and so on. Most of the time, in discrete models, one to one correspondence is built between integers and objects taken from the countable set under consideration whereas in continuous models, their dimensions are measured through analysis.

Although, many notions in either of the branches could be meaningless when carried to the other, interesting and challenging questions and corresponding answers come forward when an interaction between the two can be designed [1, 8, 11, 33, 42, 44]. For example, we can immediately design jumping problem of a grasshopper on the road as grasshopper is stepping forward or backward with countable many moves where the jump size is equally separated. Here, the problem may be appeared to be discrete problem whenever the jump size is taken as an integer. In that problem, the grasshopper is controlled by an integer which can be considered as one control variable which makes the reachability set of grasshopper not dense on the road, i.e., it misses uncountable many points on the road. However, the problem may be well suited to continuous case by considering, instead of one control variable, at least two controls in such a way that they are rationally independent, i.e., the ratio of one jump size to another is irrational number. In this case, one can predict that the grasshopper could reach nearly all points on the road which means that the limiting case of those points provides the whole road. The idea behind this methodology comes from Kronecker's density theorem ([10,12,29,38]) where the connection between discrete and continuous case could be built intuitively.

In real life processes, a continuous model which is adaptable to dijital platforms always plays a crucial role and many methods and discretization techniques have been developed for considering those problems in a discrete fashion. One of the main purposes in discretization processes is building an efficient and time saving environments with the aim of protecting properties of continuos case such as controllability, observability or stability of the system after the transfer. Hence, designing a proper time schedule becomes challenging to create a better ground for efficiency of the discretization procedure most of the time. As it is discussed in the jumping problem of a grasshopper, we may intuitively connect both cases with Kronecker's density theorem by some means.

In the following sections, we mention about the problems under consideration and motivations behind them. Here, our main body of the thesis consists of two parts. In the first part, topological aspects of some special sequences are examined and their relation with the controllability problem of some discrete linear control models is discussed. In the second part, we discuss about building a better and efficient sampling time pattern for controllable or observable linear sampled-data systems. Here, we consantrated on samplings for which controllability and observability of the linear continuous time systems is preserved during the procedure of discretization. Moreover, in both parts, we provide the novelity of the work and its contributions to control theory.

1.1 On Behaviour of Some Special Sequences

1.1.1 Motivation and Problem Definition

Much attention has recently been attracted to the topological structure of sets of polynomials with bounded integer coefficients with applications in analysis and theory of numbers [4, 23, 24]. Among these sets, the one that holds its importance in this context, is the following spectrum

$$X_m(\lambda) = \Big\{c_0 + c_1\lambda + \cdots + c_k\lambda^k : c_i \in \{0, 1, \cdots, m\}, k \in \mathbb{N}\Big\}$$

which has been introduced by Erdös et al. in [18] for the case $1 < \lambda < 2$ and $m = 1$. The structure of the set $X_m(\lambda)$ was examined and results were obtained about the behaviour of differences between consecutive terms of the set for cases of all transcendental and Pisot numbers (see also [17]). Recall that a Pisot number is an algebraic integer which is greater than one and all of its conjugates have modulus less than one. There are huge literature for the application of Pisot numbers in theory of numbers, we refer to [4, 24, 25]. Recent progress was made in [23] where answers were given for perspectives and open problem raised by Erdös et al. in [18]. In [23], the following set of polynomials was considered

$$Y_m(\lambda) = \Big\{c_0 + c_1\lambda + \cdots + c_k\lambda^k : c_i \in \{0, \pm1, \pm2, \cdots, \pm m\}, k \in \mathbb{N}\Big\}$$

for real number $\lambda > 1$, and $m \neq 0$ integer. It was shown that $Y_m(\lambda)$ is dense in $\mathbb{R}$ if and only if $\lambda < m + 1$ and λ is not a Pisot number. It should be noted that $Y_m(\lambda)$ is not dense in $\mathbb{R}$ whenever λ is a Pisot number (see [25]), or $\lambda \geq m + 1$ (see [19]). In addition to that, $Y_m(\lambda)$ has a finite accumulation point in $\mathbb{R}$ if and only if $\lambda < m + 1$ and λ is not a Pisot number (see [4]).

In this part, we study the topological structure of the set $X_m(\lambda)$ for the case $m = \lfloor\lambda\rfloor$, i.e., integer part of λ and $\lambda > 1$. Here, we analyse the behaviour of differences of consecutive terms of the set and obtain related results in the general setting. Moreover, the work is extended to the new class of $X := X_{\lfloor\lambda\rfloor}(\lambda)$ as the set of pairwise combination of terms from the set $X_m(\lambda)$ with a transcendental number $\gamma \in (0, 1)$ in the following sense

$$Y := \{\alpha_i + \gamma\alpha_j | \ \alpha_i, \alpha_j \in X \ \text{ for } i, j \in \mathbb{N}\}.$$

Here, we study the behaviour of elements of the set Y and prove our main result that the gap between each consecutive terms of the set tends to zero. As an application of the main result, we concentrate on the controllability problem of a convex polyhedra rolling in $\mathbb{R}$. More precisely, the problem is a discrete model of the form

$$x_{n+1} = \lambda x_n + \eta, \quad x_0 = 0, \tag{1.1}$$

where $\lambda > 1$ and η is ranging from finite control set. Our motivation comes from here that each discrete points of the system can be represented as a polynomial in λ with bounded coefficients emerged from the finite set. In literature, the problem is related to robotics and we refer to the papers [9, 13, 14] for analysing controllability property of the system under finite control set. Those works are answering the problem whether it is possible to design a finite control set in such a way that the reachable set of a convex polyhedra with those controls acting on the system is dense in the space. Several techniques applied to the papers [30, 49] to obtain denseness criteria of a vehicle's reachable set in some certain spaces and to the paper [35] for a model of a robot's finger in the framework of the theory of expansions in non-integer bases where the density of its reachability set was studied.

1.1.2 Contributions and Novelties

As a contribution, we obtain novel results related to behaviour of elements of the set $X_m(\lambda)$ for the case $m = \lfloor \lambda \rfloor$ and any $\lambda > 1$. The results gathered in this field are generalized version of some results obtained in the works [17, 18]. Moreover, a new set Y is introduced in the literature and behaviour of differences of consecutive terms of the set is investigated. Proof techniques applied in this section are adopted from the papers [17, 18] and without utilizing Pisot number analysis, topological properties of the sets X and Y are derived and finally, it is concluded that differences of successive terms of Y approach to zero. Hence, the paper develops a different point of view in regard to the recent progress of the work [23].

As an application of the theory, we consider a problem of controllability in robotics [9, 13, 14]. More precisely, we analyse the interaction in between displacements of a convex polyhedra by means of finite controls in one dimensional space. The movement of a convex polyhedra for each step can be described as the first order recurrence relation. The work answers density property of the reachable set of the system for $\lambda > \mathcal{A} := \frac{1+\sqrt{5}}{2}$. Hence, this result opens a new window for the use of the topological properties of the set Y in discrete linear control models like (1.1).

1.2 On Construction of Sampling Schedule in Sampled-Data Systems

1.2.1 Motivation and Problem Definition

In many forms of real life processes, a continuous time model is implemented in a digital fasion on a computer. Since the computer is viewed as a finite state machine, it is clearly incapable of recording infinitely many data evaluated in a given continous time interval. In this context, discretiation of continuous systems or dynamic processes by means of time or state sampling techniques step forward to overcome this factor to allow the computer to calculate the necessary data in finitely many steps. The concept of observability/controllability of control systems takes an important role in this regard. More precisely, most interesting results follow from carrying those properties to discrete systems that are emerged from real life problems. In literature, we refer to [36] for general knowledge about control systems with controllability, observability and stability analysis; [46] for controllability and observability of discrete processes.

Recently, there has been growing interest in preserving controllability and observability after discretization with sampling patterns. In this regard, we refer to the papers [5–7] for preserving the property of observability after discretization of continuous time system under equidistant sampling. In [7], observability of nonlinear models were examined for compact manifolds and under the assumption of infinitesimal observability, observability property is carried out almost everywhere. Later proved in [6] that the assertion still holds by removing the previous infinitesimal observability assumption which is difficult to check and adding analyticity property of continuous system.

In this section, we consider the problem of controllability and observability of sampled-data systems by means of the sample and hold technique. Roughly speaking, this technique depends on the fact that the inputs and outputs of the system are evaluated at times $0, T, 2T, \cdots$ (T as sampling time) and the control stays fixed in $[iT, (i+1)T)$ for $i \in \mathbb{N}$. Here, the sampling times are equally separeted with sampling period T, i.e., the sampling has an equidistant property. Equidistant sampling and a zero-order hold were studied and understood completely in the literature [31, 41].

Once the observable/controllable continuous time system is discretized by use of the sample and hold operation, it is possible to lose the property of observability/controllability during the process [31]. It is thus of interest to design an appropriate sampling which preserves that property after the discretization. For example, consider the signal $y(t) = \sin(2\pi\tau t)$ for the second order observable system with arbitrary parameter $\tau \in \mathbb{R}$. If we let the time is sampled by $t_i = iT$ for $i \in \mathbb{N}$, then $y(t_i) = 0$ for all $i \in \mathbb{N}$, whenever the parameter τ hits the value $1/T$, which means the discretized system is not observable. Hence for that example, observability of the system is not carried out by equidistant sampling defined above. Fortunately, that problem can be fixed if one more intermediate sampling time T_1 with the property that $T/T_1 \in \mathbb{R}/\mathbb{Q}$ is added. More precisely, we observe the system at times iT_1 when it is not observable at times iT for $i \in \mathbb{N}$ and conversely. However, the above strategy does not applied to higher order systems. As an example, consider the following signal output of the sixth order observable system with parameter ρ

$$y(t) = \sin(2\pi(T + T_1)\rho t) - \sin(2\pi T_1 \rho t) - \sin(2\pi T \rho t)$$

It is easy to check that $y(iT) = 0$ and $y(iT_1) = 0$ for all $i \in \mathbb{N}$, whenever $\rho = 1/T_1 T$, i.e., discrete system is not observable under any sampling times T and T_1. This observation motivates us to construct nonequidistant sampling in such a way that observability/controllability of the continous time system can be carried out by the discretization. A simple nonequidistant periodic sampling is created in [33] and proves that observability/controllability of the continuous time system is preserved during discretization with the periodic sampling pattern and zero-order hold operation [16, 27].

In the context of sampled-data systems with nonequidistant/variable time samplings, we refer to the papers, [2, 3, 32, 51] for designing observers related to certain classes of non-linear systems with both sampled and delayed measurements and building an appropriate upper bound for the sampling schedule ensuring exponential convergence of the observer system error. In [21], the observer design problem is addressed for a class of continuous-time dynamical systems with uncertain sampled measurements and it is proven that under suitable chosen upper bound for the diameter of sampling partition, the observation error converges to zero exponentially. Moreover, the paper [42] provides a sufficient condition on the sampling time which allows to preserve the observability property of continuous dynamical systems with discrete-time output measurements.

In this part, we consider the following continuous-time problem

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + Bu(t), \quad t > 0 \\ y(t) = Cx(t), \quad t > 0 \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^l$ and $u(t) \in \mathbb{R}^m$ for each fixed $t \in \mathbb{R}^+$. Here, we can see that $x(t)$ is the state of the system, $u(t)$ control input, $y(t)$ is signal output.

Discretization of the system (1.2) with sampling $\{t_i\}_{i\in\mathbb{N}}$ by adopting zero-order hold [39] with $t_0 = t_k$ and $t = t_{k+1}$, gives us the following discrete form

$$\begin{cases} x_{i+1} = e^{A(t_{i+1}-t_i)}x_i + \displaystyle\int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-\tau)}Bd\tau \; u_i, \\ y_i = Cx_i, \end{cases} \tag{1.3}$$

where $x_i := x(t_i)$, $y_i := y(t_i)$, and u_i represents control input which is constant along the interval $[t_i, t_{i+1})$.

The problem is to design a time sampling $\{t_i\}_{i\in\mathbb{N}}$ in such a way that the property of controllability/observability of the continuous system (1.2) can be preserved during the process of discretization, i.e., the discrete system (1.3) becomes controllable/observable.

1.2.2 Contributions and Novelties

In this work, we contribute a novel sampling schedule in the context of nonequidistant/uncertain samplings and with its involvement in preserving observability/controllability of linear time invariant sampled-data systems. Moreover, we show that the number of steps required for observability/controllability of the sampled-data system is finite and beside that, step size, necessary to the transfer for two dimensional sampled-data system is sharpened when compared with the paper [33]. In addition, by using this sampling pattern in linear sampled-data systems, we see that it is feasible to make the upper diameter of the sampling sequence as small as possible which provides a better ground for the exponential convergence of observation error and stability analysis of sampled-data systems (see related problems from e.g., [3, 44, 51]).

1.3 The Outline of the Thesis

The thesis consists of four chapters and each of which is discussed below in detail.

Chapter 2 is devoted to some basic notions, definitions, techniques and theorems related to control theory. Those facts are gathered from different control books and research articles [22, 50]. Here, we define the concept of controllability and observability for both continuous and discrete case in finite dimensional linear systems. Moreover, we discuss necessary and sufficient conditions for systems being controllable or observable. In addition, we provide zero-order-hold discretization technique for sampled-data systems and remarks related to it.

Chapter 3 considers some topological results related to polynomial type sequences with bounded integer coefficients and their connection with controllability problem of a convex polyhdra with a finite control set. In section 3.1, we give preliminaries about some basic concepts, theorem and lemma related to the chapter. In this part, we mention about Kronecker's density theorem [29] and provide one important application which would be used throughout the thesis. Moreover, we consider certain subsets of X and Y, namely sets of all finite sum of even powers of λ terms in X and Y respectively. In addition, γ is defined as a transcendental number which is bounded by constants depending on λ for later purposes. In section 3.2, we provide the main results of the paper. Finally, in section 3.3, the theory is applied to the controllability problem of the system (1.1) with a finite set of controls in one dimensional case. In addition, the case $\lambda = 1$, is considered separately and proven that reachable set of the system is dense in $\mathbb{R}$ as a consequence of Kronecker's theorem.

Chapter 4 consantrates on designing time sampling patterns to preserve observability and controllability of linear sample-data systems. In section 4.1, we state the problem under consideration and adopt zero-order hold for the system. In this part, we define the concept of controllability and observability both for continuous time and sampled-data systems in a long time horizon. In section 4.2, we introduce a novel sampling pattern which consists of two parts, each of which constructed separately in the beginning and extended periodically by using the division algorithm. In section 4.3, we prove the main results of the paper. In addition, under the novel sampling, step size which guarantees the property of observability/controllability of the sampled-data system becomes less for two dimensional sampled-data systems when compared with sampling schedule proposed in the paper [33]. In section 4.4, we analyse the upper diameter of the sampling pattern constructed in section 4.3 and observe that the upper bound can be flexible in a way that makes it possible to decrease or increase the maximum gap of successive terms depending on the schedule. Finally, we note that the sampling pattern could be utilized for the problem of state observation of a flexible joint robotic arm proposed in [44]. In section 4.5, we consider the controllability problem of continuous time model of the pendulum in two dimensional setting, and build our corresponding controls which makes the sampled-data system controllable with sampling pattern introduced in section 4.2. Lastly, we conclude the chapter by pointing out two types of sampling and their involvements in the results obtained in section 4.3.

Chapter 5 concludes the thesis by pointing out some perspectives and future works related to the context.

CHAPTER 2

SOME BASIC FACTS IN CONTROL THEORY

Control theory is presently one of the most interdisciplinary areas of research with its most modern applications. In other words, control theory has been a discipline where many mathematical ideas and methods have designed to provide an environment which consists of plentiful crossing points of engineering and mathematics. There are huge literature in this context, we refer to [8, 15, 22, 36, 55, 56]. Throughout this chapter, we introduce some of basic mathematical concepts with related examples in control theory.

2.1 Controllability

Controlling a system can be considered as an action, i.e., to manage things in order to guarantee that the system behaves as desired. In this section, we define the concept of controllability in finite dimensional linear systems [22, 47, 55, 56].

Consider the following finite dimensional system

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + Bu(t), \quad t \in (0,T) \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $x_0 \in \mathbb{R}^n$. Here, the functions $x : [0,T] \to \mathbb{R}^n$ and $u : [0,T] \to \mathbb{R}^m$ are defined as the state and the control respectively. Of course, we assume that $m \leq n$ and the main goal is to control the system with minimum number of controls, i.e., minimize m.

Definition 2.1.1. *The system (2.1) is (exact) controllable in time $T > 0$ if for any initial and final data x_0 and x_1 in $\mathbb{R}^n$ it is possible to find a control $u \in L^2(0,T;\mathbb{R}^m)$*

such that the solution of (2.1) satisfies $x(T) = x_1$.

Now define the set of reachable states or reachability set as

$$\mathcal{R}(T, x_0) = \Big\{x(T) \in \mathbb{R}^n : \ x \text{ solution of (2.1) with } u \in L^2(0,T)^m\Big\}.$$

Here, we give another definition of controllability

Definition 2.1.2. *The system (2.1) is (exact) controllable in time* $T > 0$ *if* $\mathcal{R}(T, x_0) = \mathbb{R}^n$ *for any* $x_0 \in \mathbb{R}^n$.

In this context, we provide two examples. In the first example, we consider the system with two components in which only one component can be controlled by a given control function, the other component acts freely. This situation is different in the second example where the both of them interacts with one another by means of the control.

Example 1.

$$\begin{cases} x_1'(t) = 2x_1(t), & t \in (0,T) \\ x_2'(t) = x_2(t) + u(t), & t \in (0,T). \end{cases} \tag{2.2}$$

where $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Obviously, the above system (2.2) is not controllable since the first component $x_1(t)$ acts freely without any influence of the control $u(t)$.

Example 2. Consider the following harmonic oscillator

$$x''(t) + \alpha x(t) = u(t), \quad t \in (0,T). \tag{2.3}$$

where $\alpha \in \mathbb{R}$. Here, its corresponding finite dimensional system would be the following

$$\begin{cases} x_1'(t) = x_2(t), & t \in (0,T) \\ x_2'(t) = u(t) - \alpha x_1(t), & t \in (0,T). \end{cases} \tag{2.4}$$

where $A = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here, unlike Example 1, now the control has an influence on both components. Through this analysis, we cannot say immediately that the system (2.4) is not controllable. For invsetigating whether the system is controllable or not, we provide Kalman Rank Condition [8] for necessary and sufficent condition for controllability.

Theorem 2.1.3 (Kalman Rank Condition)**.** *The system (2.1) is (exact) controllable in some time $T > 0$ if and only if*

$$rank\Big[B, AB, A^2B, \cdots, A^{n-1}B\Big] = n. \tag{2.5}$$

Hence, we immediately have the following result

Corollary 2.1.4. *If the system (2.1) is controllable in some time $T > 0$, then it is controllable in any time.*

Therefore, by using this condition, we can easily have that the rank condition (2.5) in the first example is 1, i.e., the system (2.2) is not controllable. On the other hand, it can be easily checked that the rank condition (2.5) in the second example is full, i.e., the system (2.4) is controllable.

2.2 Observability

In this part, we consider the following finite dimensional system [8],

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \quad t > t_0 \\ y(t) = Cx(t), \quad t > t_0 \\ x(t_0) = x_0, \end{cases} \tag{2.6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^k$ and $u(t) \in \mathbb{R}^m$ for each fixed $t \in \mathbb{R}^+$. Here, $x(t)$ represents the state of the system, $u(t)$ represents the control input, $y(t)$ represents the signal output.

Definition 2.2.1. *The system (2.6) is (exact) observable if for any initial time t_0 and any initial data x_0, there exists a finite time T such that the knowledge of $u(t)$ and $y(t)$ for $t \in [t_0, T]$ suffices to determine x_0 uniquely.*

Here, the role of $y(t)$ is important to determine the initial data uniquely, i.e, there would be no loss of generality in supposing that the control is identically zero. Let us consider the following two examples about observable and unobservable systems.

Example 1. Consider the system described by

$$\begin{cases} x_1'(t) = x_1(t) - u(t), \\ x_2'(t) = -x_2(t) + u(t), \\ y(t) = x_1(t). \end{cases} \tag{2.7}$$

Here, it can be easily checked that the system satisfies the condition of Theorem (2.1.3), i.e., the system is controllable. On the other hand, from the first part of the equation, it can be seen that $x_1(t)$ which is $y(t)$ can be determined uniquely by $u(t)$ and $x_1(t_0)$. Hence, it is impossible to determine $x_2(t_0)$ by means of the output which makes the system not observable.

Example 2. Consider the system described by

$$\begin{cases} x_1'(t) = x_1(t) - u(t), \\ x_2'(t) = -x_2(t) + u(t), \\ y(t) = x_1(t) + x_2(t). \end{cases} \tag{2.8}$$

Here, one can see that both state components are involved in the system's output $y(t)$ which affects the behaviour of the system. Through the analysis before, we cannot say immediately that the system is not observable. Practical check for the answer of the question whether the system is observable or not, we provide the following rank condition [8] to build the necessary and sufficient environment for observability.

Theorem 2.2.2. *The system (2.6) is (exact) observable if and only if*

$$rank\Big[C, CA, CA^2, \cdots, CA^{n-1}\Big]^T = n. \tag{2.9}$$

By using Theorem 2.2.2, we see that the rank of the system (2.7) in the first example is one which makes it unobservable. On the other hand, it can be checked that the rank of the system (2.8) in the second example is full which provides the observability of (2.8).

2.3 Sampled Data Systems

Most models of real life processes are presented as continuous time models, usually in form of differential equations. For analysing those problems in digital platform via computers, we introduce the concept of sampling [26, 39]. Here, we consider a finite dimensional ordinary differential equation

$$x'(t) = f(x(t), u(t)) \tag{2.10}$$

with vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, state function $x : \mathbb{R} \to \mathbb{R}^n$ and control function $u : \mathbb{R} \to \mathbb{R}^m$. For corresponding discrete version of (2.10), we consider the following discrete system

$$x^+ = f(x, u) \tag{2.11}$$

where $f : X \times U \to X$ is a transition function assigning the state x^+ at the next time instant from the state value $x \in X$ and the control value $u \in U$ where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$.

The idea of sampling consists of defining a discrete time system (2.11) such that the trajectories of the discrete system and the continuous system (2.10) match at the sampling times $t_0 < t_1 < t_2 < \cdots < t_k$, namely,

$$x_c(t_i, t_0, x_0, u) = x_d(i, x_0, u) \quad i = 0, 1, 2, .., k, \tag{2.12}$$

where x_c solution of (2.10) and x_d solution of (2.11) with initial condition $x(t_0) = x_0$. Here, we define the sampling pattern as equidistant sampling if distances between each consecutive terms are equal to each other, i.e., $t_n = nT$ where $T > 0$ is the sampling period.

Example 1. Consider the following discrete model for the total distance covered by the vehicle

$$x^+ = x + u \tag{2.13}$$

where x^+ next position and x current position of the vehicle, in addition u is the control which manages the distance covered by the vehicle, i.e., for values $u < 0$, it goes backwards while for $u > 0$, it proceeds forwards.

Example 2. Consider the following continuous time control system

$$x'(t) = u(t) \tag{2.14}$$

The solution of (2.14) is

$$x_c(t, 0, x_0, u) = x_0 + \int_0^t u(\tau) d\tau. \tag{2.15}$$

Let $X = \mathbb{R}$ and $U = L^\infty([0, T], \mathbb{R})$, then corresponding discrete version of (2.14) would be of the following form

$$x^+ = x + \int_0^T u(\tau) d\tau. \tag{2.16}$$

If we fix the control function as $u(t) = u$ where u is a scalar in $\mathbb{R}$, then (2.16) becomes

$$x^+ = x + Tu. \tag{2.17}$$

Here, for letting $T = 1$, we obtain the discrete model (2.13).

2.3.1 Zero Order Hold

Discretization of continuous time system with higher order of derivatives by means of sample and hold techniques always causes problems such as complex circuitry, higher costs, large number of delayed pulses, serious stability problems and so on. Hence, in power approximation of a given function $x(t)$, use of only the first term in that to estimate the given function during the time interval $kT \leq t < (k+1)T$ for $k \in \mathbb{N}$ becomes more effective and powerful which is known as zero-order hold. It holds the value of $x(kT)$ for $kT \leq t < (k+1)T$ until the next sample $x((k+1)T)$ arrives. As it is noticed in this technique, the sampling pattern is taken as an equidistant manner with sampling period T. Moreover, a common assumption is that the continuous-time input signal, i.e., the control of the system is generated by a zero-order hold stays fixed in $[iT, (i+1)T)$ for $i \in \mathbb{N}$, which is called digital control in the literature [39].

To understand the concept better, let us apply this method to the following finite dimensional control system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{2.18}$$

where A is the state matrix, B is the input matrix, C is the output matrix, and D is the feedthrough matrix. Moreover, $x(t)$ is the state, $y(t)$ is the output signal, $u(t)$ is the control input for $t \in \mathbb{R}^+$.

The solution of the system (2.18) with initial data $x(t_0) = x_0$ and its output signal would be the following

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.19}$$

and

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2.20}$$

Now, take next time t_1, then our solution with initial condition $x(t_1)$ would be

$$x(t) = e^{A(t-t_1)}x(t_1) + \int_{t_1}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.21}$$

Moreover, in zero-order-hold, we assume that the control stays fixed in $[kT, (k+1)T)$ for $k \in \mathbb{N}$, i.e.,

$$u(t) = u(kT), \qquad \forall t \in [kT, (k+1)T).$$

Repeating the procedure done in (2.21), we have the following

$$x(t) = e^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.22}$$

Hence, letting $t = (k+1)T$ in (2.22) and doing some substitutions, we obtain the following

$$x((k+1)T) = e^{AT}x(kT) + \int_{0}^{T} e^{A\xi}Bu(kT)d\xi \tag{2.23}$$

where $\xi = (k+1)T - \tau$.

Therefore, discrete time description of the continuous time plant (2.18) by means of zero and hold would be

$$\begin{cases} x(k+1) = \hat{A}x(k) + \hat{B}u(k), \\ y(k+1) = Cx(k) + Du(k), \end{cases} \tag{2.24}$$

where

$$\hat{A} = e^{AT} \qquad \text{and} \qquad \hat{B} = \int_{0}^{T} e^{A\xi}Bd\xi. \tag{2.25}$$

The integral part in (2.25) can be solved explicitly whenever the state matrix A is non-singular, i.e., $\det A \neq 0$. More precisely, we can find the following

$$\hat{B} = A^{-1}(e^{AT} - I)B$$

where I is the identity matrix having the same dimension with the state matrix.

CHAPTER 3

ON BEHAVIOUR OF SOME SPECIAL SEQUENCES

3.1 Preliminaries

In this section, we provide some definitions and principles together with theorems. Firstly, we start with definition of algebraic and transcendental number.

Definition 3.1.1. *An algebraic number is any real number which is a root of a non-zero polynomial in one variable with rational coefficients. If a number is not algebraic then it is called a transcendental number.*

Pigeonhole Principle: for $n+1$ different real numbers $x_0, x_1, \cdots, x_n$ in closed interval $[0,1]$. Then, one can say that there exists two numbers x_i, x_j with $i \neq j$ satisfying

$$|x_i - x_j| \leq \frac{1}{n}.$$

Now, we give Kronecker's Theorem which relies on estimating any real number by means of given irrational number (see [29], [34]).

Theorem 3.1.2. *If ξ is irrational, then the set of points $(\{n\xi\} : n \in \mathbb{N})$ is dense in the interval $(0,1)$, where $\{\cdot\}$ represents a fractional part of the number.*

There are several approaches and considerations of the theorem, done by Estermann in [20], by Bohr in [10], and a different geometric perspective was made by Lettenmeyer in [38]. As an application of Theorem 3.1.2, we provide the following useful result.

Lemma 3.1.3. *Let ξ and η be rationally independent positive real numbers, i.e., $\frac{\xi}{\eta}$ is an irrational number. Then, the set*

$$K := \{\, j\eta + i\xi \mid i, j \in \mathbb{Z}\}, \quad \textit{is dense in } \mathbb{R}.$$

The proof of Lemma 3.1.3 is straightforward. Namely, for given ξ and η be rationally independent numbers, we know that $\frac{\xi}{\eta} \in \mathbb{I}$. Then for any $\epsilon > 0$, we have from Theorem 3.1.2 that for any real number $\tau \in \mathbb{R}$,

$$|\{\kappa\} - N - M\frac{\xi}{\eta}| < \frac{\epsilon}{\eta},$$

where $\kappa = \dfrac{\tau}{\eta}$ and for some $M, N \in \mathbb{Z}$. Here, $\{\cdot\}$ represents a fractional part of the number. Hence, we conclude that

$$\Big|\tau - \Big(\big(\lfloor\kappa\rfloor + N\big)\eta + M\xi\Big)\Big| < \epsilon. \tag{3.1}$$

Observe that both $\lfloor\kappa\rfloor + N$ and M are integers in (3.1).

Let $\lambda > 1$ be a real number and for $n \in \mathbb{N}$, we consider the unique $(\lfloor\lambda\rfloor+1)$ expansion of n as the following way

$$n = c_0 + c_1(\lfloor\lambda\rfloor + 1) + c_2(\lfloor\lambda\rfloor + 1)^2 + \cdots + c_k(\lfloor\lambda\rfloor + 1)^k$$

where $c_i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$, and set the sequence of the following form

$$\alpha_n = c_0 + c_1\lambda + \cdots + c_k\lambda^k. \tag{3.2}$$

Define the set X as all collection of (3.2) and let the sequence $\{\beta_i\}$ be its increasing rearrangement. Let $\gamma \in (0, 1)$ be a transcendental number such that $\dfrac{\lambda}{\gamma} \in \mathbb{I}$ and satisfying the following inequality

$$\max\left\{\frac{1}{\lambda^2}, \frac{\lambda - 1}{\lambda\lfloor\lambda\rfloor}\right\} < \gamma < \frac{1}{\lambda}. \tag{3.3}$$

Define X_e as the set of all finite sum of even powers of λ terms in X. More precisely,

$$X_e := \Big\{\sum_{i=0}^{k} c_i\lambda^{2i} \mid k \in \mathbb{N}, c_i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\Big\}. \tag{3.4}$$

Now, define the following set

$$Y_e := \{\alpha_i + \gamma\alpha_j | \ \alpha_i, \alpha_j \in X_e \text{ for } i, j \in \mathbb{N}\}. \tag{3.5}$$

Let $\{\zeta_i\}$ and $\{\tau_i\}$ be the increasing rearrangement of Y_e and Y respectively, i.e.,

$$\zeta_0 < \zeta_1 < \zeta_2 \cdots \quad \text{and} \quad \tau_0 < \tau_1 < \tau_2 \cdots$$

Observe that $\zeta_0 = \tau_0 = 0$, $\zeta_1 = \tau_1 = \gamma$ and $\zeta_2 = \tau_2 = 1$. Moreover, define the following limits

$$\mathcal{I}(\lambda) = \liminf(\zeta_{i+1} - \zeta_i) \quad \text{and} \quad \mathcal{S}(\lambda) = \limsup(\tau_{i+1} - \tau_i). \tag{3.6}$$

Throughout this chapter, it is assumed that $\mathcal{A} := \frac{1+\sqrt{5}}{2}$.

Lemma 3.1.4. *For any $k \in \mathbb{N}$, the following inequalities*

$$\gamma\lambda^{2k+2} < 1 + \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i}, \tag{3.7}$$

and

$$\lambda^{2k+2} < 1 + \gamma \lfloor\lambda\rfloor \lambda^{2k+2} + \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i}, \tag{3.8}$$

hold.

Proof. Firstly, we prove the inequality (3.7). By using (3.3),

$$\begin{aligned} 1 + \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i} &> 1 + \Big(\lfloor\lambda\rfloor + \frac{\lambda-1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} \\ &= 1 + \Big(1 + \lfloor\lambda\rfloor - \frac{1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} \\ &> 1 + \Big(\lambda - \frac{1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} = 1 + \lambda^{2k+1} - \frac{1}{\lambda} > \gamma\lambda^{2k+2}. \end{aligned}$$

Namely, we obtain the first inequality. For the second one, we use (3.3), i.e.,

$$\begin{aligned} 1 + \gamma \lfloor\lambda\rfloor \lambda^{2k+2} + \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i} &> 1 + \frac{\lambda-1}{\lambda}\lambda^{2k+2} + \Big(\lfloor\lambda\rfloor + \frac{\lambda-1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} \\ &> 1 + \lambda^{2k+2} - \lambda^{2k+1} + \Big(\lambda - \frac{1}{\lambda}\Big) \sum_{i=0}^{k} \lambda^{2i} > \lambda^{2k+2}. \end{aligned}$$

Hence, we have the second inequality. □

Lemma 3.1.5. *For $\lambda > \mathcal{A}$, the following inequality*

$$\lambda^{2n+2} > \lfloor\lambda\rfloor \sum_{i=0}^{n} \lambda^{2i}$$

holds for all $n \in \mathbb{N}$.

Proof. Since $\lambda > \mathcal{A}$, we have that $\lambda^2 - \lambda - 1 > 0$. Here, we would prove that for any $n \in \mathbb{N}$,

$$\lambda^{2n+1} > \sum_{i=0}^{n} \lambda^{2i} \tag{3.9}$$

holds by induction. At first,

$$\lambda^2 - \lambda - 1 > 0 \iff \lambda^3 - \lambda^2 - \lambda > 0 \implies \lambda^3 - \lambda^2 - 1 > 0.$$

Hence, we have that $\lambda^3 > \lambda^2 + 1$, i.e., it is true for $n = 1$. Assume that (3.9) is true for n. Here, we observe that

$$1 + \lambda^2 + \cdots + \lambda^{2n} + \lambda^{2n+2} < \lambda^{2n+1} + \lambda^{2n+2} < \lambda^{2n+3},$$

which implies the case for $n + 1$. Therefore, we have (3.9) by induction. As a result,

$$\lambda^{2n+1} > \sum_{i=0}^{n} \lambda^{2i} \implies \lambda^{2n+2} > \lfloor \lambda \rfloor \sum_{i=0}^{n} \lambda^{2i},$$

which completes the proof. □

3.2 Main Results

Theorem 3.2.1. *For each $n \in \mathbb{N}$, the following inequality*

$$\beta_{n+1} - \beta_n \leq 1$$

holds.

Proof. To prove the theorem, we use induction by n. Here, for case $n = 1$, recall that $\beta_0 = 0$ and $\beta_1 = 1$, i.e., $\beta_1 - \beta_0 = 1$ which is true.

For induction hypothesis, assume that it is true for case n, i.e., $\beta_{i+1} - \beta_i \leq 1$ for $i = 1, 2, \cdots, n$. We prove that it is true for the case $n + 1$, i.e., $\beta_{n+2} - \beta_{n+1} \leq 1$.

Let $\beta_{n+1} = \xi_0 + \xi_1\lambda + \cdots + \xi_s\lambda^s$, for $\xi_i \in \{0, 1, \cdots, \lfloor \lambda \rfloor\}$. Here, ξ_0 is either $\lfloor \lambda \rfloor$ or different from $\lfloor \lambda \rfloor$.

If we have the case $\xi_0 \neq \lfloor \lambda \rfloor$, then ξ_0 would be in $\{0, 1, \cdots, \lfloor \lambda \rfloor - 1\}$, i.e., $\beta_{n+1} + 1 \in X$. This implies that $\beta_{n+1} \leq \beta_{n+2} \leq \beta_{n+1} + 1$ which means $\beta_{n+2} - \beta_{n+1} \leq 1$.

Now, assume that $\xi_0 = \lfloor\lambda\rfloor$. Let k be the largest integer such that $\xi_0 = \xi_1 = \cdots = \xi_k = \lfloor\lambda\rfloor$. Then, $\xi_{k+1} \in \{0, 1, \cdots, \lfloor\lambda\rfloor - 1\}$, i.e.,

$$\beta_{n+1} = \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k) + \xi_{k+1}\lambda^{k+1} + \sum_{i=k+2}^{s} \xi_i \lambda^i. \tag{3.10}$$

Notice from (3.10) that $\beta_{n+1} + \lambda^{k+1} \in X$. Hence, if one can have $c_0, c_1, ..c_k \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$ such that

$$\lfloor\lambda\rfloor (1+\lambda+\cdots+\lambda^k) < c_0+c_1\lambda+\cdots+c_k\lambda^k+\lambda^{k+1} \leq 1+\lfloor\lambda\rfloor (1+\lambda+\cdots+\lambda^k), \tag{3.11}$$

holds, then one can have the following

$$\beta_{n+2} \leq c_0 + c_1\lambda + \cdots + c_k\lambda^k + \lambda^{k+1} + \sum_{i=k+1}^{s} \xi_i \lambda^i \leq 1 + \beta_{n+1}$$

which proves the case $n + 1$. Therefore, it suffices to prove the existence of such $\{c_i\}_{i=0}^k$ satisfying (3.11).

Note firstly that $(\lfloor\lambda\rfloor + 1 - \lambda)(1 + \lambda + \cdots + \lambda^k) > 0$ which implies

$$\lambda^{k+1} < 1 + \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k). \tag{3.12}$$

From (3.12), if we have $\lambda^{k+1} > \lfloor\lambda\rfloor (1+\lambda+\cdots+\lambda^k)$, then we obtain (3.11) by simply choosing $c_0 = \cdots = c_k = 0$. Otherwise, assume that $\lambda^{k+1} \leq \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k)$. Let $\beta_\eta = \lfloor\lambda\rfloor (1 + \lambda + \cdots + \lambda^k)$ which is a term in X. Here, $\eta \leq n + 1$ and the next term would be of the form

$$\beta_{\eta+1} = c_0 + c_1\lambda + \cdots + c_k\lambda^k + \lambda^{k+1},$$

for some $c_0, c_1, ..c_k \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$. Since there would be no term of X in between β_η and $\beta_{\eta+1}$, we conclude that

$$X \cap (\beta_\eta - \lambda^{k+1}, \beta_{\eta+1} - \lambda^{k+1}) = \emptyset. \tag{3.13}$$

Observe that $c_0 + c_1\lambda + \cdots + c_k\lambda^k \in X$ and $\beta_\eta - \lambda^{k+1} + 1 < \beta_\eta$. Therefore, by using the induction hypothesis together with (3.13), one can have the following

$$\beta_\eta - \lambda^{k+1} < c_0 + c_1\lambda + \cdots + c_k\lambda^k \leq \beta_\eta - \lambda^{k+1} + 1 \tag{3.14}$$

which concludes the proof of (3.11). $\square$

Theorem 3.2.2. *For each $n \in \mathbb{N}$, the following inequality*

$$\zeta_{n+1} - \zeta_n \leq 1$$

holds.

Proof. We apply proof by induction. Here, for case $n = 1$ recall that $\zeta_0 = 0$ and $\zeta_1 = \gamma$, i.e., $\zeta_1 - \zeta_0 = \gamma < 1$ which is true.

For induction hypothesis, assume that it is true for case n, i.e., $\zeta_{i+1} - \zeta_i \leq 1$ for $i = 1, 2, \cdots, n$. We prove that it is true for the case $n+1$, i.e., $\zeta_{n+2} - \zeta_{n+1} \leq 1$. Let $\zeta_{n+1} = \xi_0 + \xi_1\lambda^2 + \cdots + \xi_s\lambda^{2s}$, where $\xi_i \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$. As it is examined in Theorem 3.2.1, either $\xi_0 \neq (1+\gamma)\lfloor\lambda\rfloor$ or $\xi_0 = (1+\gamma)\lfloor\lambda\rfloor$.

For the first case, one can say either $\zeta_{n+1} + 1 \in Y_e$ or $\zeta_{n+1} + \gamma \in Y_e$ or both of them holds. Hence, we have either

$$\zeta_{n+2} \leq \zeta_{n+1} + 1 \quad \text{or} \quad \zeta_{n+2} \leq \zeta_{n+1} + \gamma,$$

which proves the case for $n+1$.

For the second case, we assume that $\xi_0 = (1+\gamma)\lfloor\lambda\rfloor$. Let k be the largest integer such that $\xi_0 = \xi_1 = \cdots = \xi_k = (1+\gamma)\lfloor\lambda\rfloor$. More precisely,

$$\zeta_{n+1} = \lfloor\lambda\rfloor \left(1+\gamma\right) \sum_{i=0}^{k} \lambda^{2i} + \xi_{k+1}\lambda^{2k+2} + \sum_{i=k+2}^{s} \xi_i\lambda^{2i}.$$

Here, we examine two cases separately.

Case 1. Let us assume that $\xi_{k+1} = i + \lfloor\lambda\rfloor\gamma$, where $i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$. It can be seen from assumption on k that $i \neq \lfloor\lambda\rfloor$, which implies $\zeta_{n+1} + \lambda^{2k+2} \in Y_e$. More precisely,

$$\zeta_{n+1} - \lfloor\lambda\rfloor \left(1+\gamma\right) \sum_{i=0}^{k} \lambda^{2i} - \lfloor\lambda\rfloor \gamma\lambda^{2k+2} + \lambda^{2k+2} \in Y_e. \tag{3.15}$$

If we have

$$\lambda^{2k+2} > \lfloor\lambda\rfloor \gamma\lambda^{2k+2} + \lfloor\lambda\rfloor \left(1+\gamma\right) \sum_{i=0}^{k} \lambda^{2i}, \tag{3.16}$$

then, from Lemma 3.1.4, we conclude that

$$\zeta_{n+2} \leq \zeta_{n+1} - \lfloor\lambda\rfloor \left(1+\gamma\right) \sum_{i=0}^{k} \lambda^{2i} - \lfloor\lambda\rfloor \gamma\lambda^{2k+2} + \lambda^{2k+2} < 1 + \zeta_{n+1}.$$

However, if (3.16) is not satisfied, i.e.,

$$\lambda^{2k+2} \leq \lfloor\lambda\rfloor \gamma\lambda^{2k+2} + \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i},$$

then define $\zeta_\nu = \lfloor\lambda\rfloor \gamma\lambda^{2k+2} + \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i}$ which is an element of Y_e, and of course $\nu \leq n+1$. The next term is $\zeta_{\nu+1}$, and it would be in the following form

$$\zeta_{\nu+1} = c_0 + c_1\lambda^2 + \cdots + c_k\lambda^{2k} + \lambda^{2k+2}$$

for some $c_m \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$ and $m = 0, 1, \cdots, k$. Consider the open interval

$$I_\nu = (\zeta_\nu - \lambda^{2k+2}, \zeta_{\nu+1} - \lambda^{2k+2})$$

for some $c_i \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$. Note that

$$(\zeta_\nu, \zeta_{\nu+1}) \cap Y_e = \emptyset. \tag{3.17}$$

Moreover, (3.3) and (3.17) imply $I_\nu \cap Y_e = \emptyset$. Since, $\zeta_\nu - \lambda^{2k+2} + 1 < \zeta_\nu$, we apply induction hypothesis that

$$\zeta_\nu - \lambda^{2k+2} < c_0 + c_1\lambda^2 + \cdots + c_k\lambda^{2k} \leq \zeta_\nu - \lambda^{2k+2} + 1$$

which concludes

$$\zeta_{n+2} \leq \zeta_{n+1} - \zeta_\nu + c_0 + c_1\lambda^2 + \cdots + c_k\lambda^{2k} + \lambda^{2k+2} \leq 1 + \zeta_{n+1}.$$

Case 2. In this case, we suppose that $\xi_{k+1} = i + j\gamma$, where $i \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$ and $j \in \{0, 1, \cdots, (\lfloor\lambda\rfloor - 1)\}$. Therefore, $\zeta_{n+1} + \gamma\lambda^{2k+2} \in Y_e$. Namely,

$$\zeta_{n+1} - \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i} + \gamma\lambda^{2k+2} \in Y_e. \tag{3.18}$$

If we have

$$\gamma\lambda^{2k+2} > \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i}, \tag{3.19}$$

then from Lemma (3.1.4), we conclude that

$$\xi_{k+2} \leq \xi_{k+1} - \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i} + \gamma\lambda^{2k+2} < 1 + \xi_{k+1}.$$

However, if (3.16) is not satisfied, i.e.,

$$\gamma\lambda^{2k+2} \leq \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i}.$$

Let us define $\zeta_\mu = \lfloor\lambda\rfloor \Big(1+\gamma\Big) \sum_{i=0}^{k} \lambda^{2i}$ which is an element of Y_e, and clearly $\mu \leq n+1$. By using (3.3), the next term would be in the following form

$$\zeta_{\mu+1} = \ell_0 + \ell_1\lambda^2 + \cdots + \ell_k\lambda^{2k} + \gamma\lambda^{2k+2}$$

for some $\ell_m \in \{i + j\gamma \mid i, j \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}\}$ and $m = 0, 1, \cdots, k$. Define the following open interval

$$I_\mu = (\zeta_\mu - \gamma\lambda^{2k+2}, \zeta_{\mu+1} - \gamma\lambda^{2k+2}).$$

From $(\zeta_\mu, \zeta_{\mu+1}) \cap Y_e = \emptyset$ and (3.3), we have $I_\mu \cap Y_e = \emptyset$. As it is discussed previously, by utilizing induction hypothesis, we obtain

$$\zeta_\mu - \gamma\lambda^{2k+2} < \ell_0 + \ell_1\lambda^2 + \cdots + \ell_k\lambda^{2k} \leq \zeta_\mu - \gamma\lambda^{2k+2} + 1$$

which concludes

$$\zeta_{n+2} \leq \zeta_{n+1} - \zeta_\mu + \ell_0 + \ell_1\lambda^2 + \cdots + \ell_k\lambda^{2k} + \gamma\lambda^{2k+2} \leq 1 + \zeta_{n+1}.$$

Hence, for both possible cases, we prove that the hypothesis is true for the case $n+1$. As a result, we obtain that $\zeta_{n+1} - \zeta_n \leq 1$ for $n \in \mathbb{N}$. $\square$

Theorem 3.2.3. *For $\lambda > \mathcal{A}$, the following*

$$\mathcal{I}(\lambda) = 0,$$

satisfies.

Proof. For $n \in \mathbb{N}$, we consider the following close interval of the form

$$\mathcal{J}_n := \Big[0, (1+\gamma)\lfloor\lambda\rfloor(1 + \lambda^2 + \cdots + \lambda^{2n})\Big]. \tag{3.20}$$

Coefficients of each powers of λ can be any value from $\{0, 1, \cdots, \lfloor\lambda\rfloor\}$, i.e., there are $\lfloor\lambda\rfloor + 1$ different possible values. Hence, the set $\mathcal{J}_n \cap Y_e$ contains $(\lfloor\lambda\rfloor + 1)^{2n+2}$ number of ζ_i's. Here, we consider two cases: whether λ is a transcendental or an algebraic number.

Case 1. Assume that λ is a transcendental number. Then, there are $(\lfloor\lambda\rfloor + 1)^{2n+2}$ different elements of the set $\mathcal{J}_n \cap Y_e$. This comes from the fact that $\dfrac{\lambda}{\gamma} \in \mathbb{I}$ and a transcendental number cannot be expressed as a root of polynomial with integer coefficients. In addition to that, for given any $\epsilon > 0$, it is possible to find a large $m \in \mathbb{N}$ such that

$$(1+\gamma)\lfloor\lambda\rfloor(1+\lambda^2+\cdots+\lambda^{2m}) < \epsilon\big((1+\lfloor\lambda\rfloor)^{2m+2}-1\big). \tag{3.21}$$

Here, we obtain that $(\lfloor\lambda\rfloor+1)^{2m+2}-1$ number of intervals with length ϵ cover $\mathcal{J}_m$ and there are $(\lfloor\lambda\rfloor + 1)^{2m+2}$ number of different ζ_i's in $\mathcal{J}_m$. Therefore, from pigeonhole principle, one interval contains at least two different values from $\mathcal{J}_m$, i.e., this is true for arbitrary small ϵ, implies $\mathcal{I}(\lambda) = 0$.

Case 2. Let λ be an algebraic number. In this part, we prove that for any $k \in \mathbb{N}$, λ cannot be the root of following polynomial of even powers

$$P(x) := c_0 + c_1x^2 + \cdots + c_kx^{2k}$$

for $c_i \in \{0, \pm1, \cdots, \pm\lfloor\lambda\rfloor\}$ whenever $\lambda > \mathcal{A}$.

Assume the contrary, i.e., if such $k \in \mathbb{N}$ exists, then we have that $c_k\lambda^{2k} = -c_{k-1}\lambda^{2k-2} - \cdots - c_0$. Suppose without loss of generality that $c_k > 0$, by using Lemma 3.1.5, we obtain

$$c_k\lambda^{2k} \geq \lambda^{2k} > \lfloor\lambda\rfloor \sum_{i=0}^{k-1}\lambda^{2i} \geq -c_{k-1}\lambda^{2k-2} - \cdots - c_0$$

which contradicts with λ being the root of $P(x)$. This observation shows us that there would be no common terms of Y_e in $\mathcal{J}_n$. More precisely, there would be $(\lfloor\lambda\rfloor+1)^{2n+2}$ terms which cannot coincide in the set $\mathcal{J}_n \cap Y_e$. As it is discussed before, by applying same procedure, we obtain that $\mathcal{I}(\lambda) = 0$. $\square$

Theorem 3.2.4. *The following*

$$\liminf_{k\to\infty}(\beta_{k+1} - \beta_k) = 0,$$

satisfies for all transcendental $\lambda > 1$.

The proof of Theorem 3.2.4 is very similar to the proof of Theorem 3.2.3, so it is omitted.

Lemma 3.2.5. *For $\lambda > \mathcal{A}$ and given any $\epsilon > 0$, one can find a subsequence $\{z_n\}$ of ζ_n such that the following two conditions are satisfied*

(i) $\sum_{i\in I} z_i \in Y_e$ *where* $I \subset \mathbb{N}$.

(ii) $\epsilon < z_{2k+1} - z_{2k} < (1 + \lfloor\lambda\rfloor)^2\epsilon$ *for* $k \in \mathbb{N}$.

Proof. We give a proof by induction. Firstly, we prove that both cases hold for $n = 1$, i.e., find $z_1, z_0 \in Y_e$ such that both conditions satisfy. From Theorem 3.2.3, we have that $\mathcal{I}(\lambda) = 0$ which means for an arbitrary $\epsilon > 0$, it is possible to find $k \in \mathbb{N}$ such that $0 < \zeta_{k+1} - \zeta_k < \epsilon$. More precisely, we have

$$0 < \beta_{s_{k+1}} - \beta_{s_k} + \gamma(\beta_{r_{k+1}} - \beta_{r_k}) < \epsilon$$

where $\zeta_{k+1} = \beta_{s_{k+1}} + \gamma\beta_{r_{k+1}}$ and $\zeta_k = \beta_{s_k} + \gamma\beta_{r_k}$.

If $\beta_{s_{k+1}}$ and β_{s_k} share the same term λ^{2n} for some $n \in \mathbb{N}$, with coefficient $\nu_{k+1}, \nu_k \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$ respectively. Without loss of generality, assume that $\nu_{k+1} \leq \nu_k$, then we establish new terms as

$$\hat{\beta}_{s_{k+1}} = \beta_{s_{k+1}} - \nu_{k+1}\lambda^{2n} \quad \text{and} \quad \hat{\beta}_{s_k} = \beta_{s_k} - \nu_{k+1}\lambda^{2n}. \tag{3.22}$$

It should be noted that there may be more terms λ^n such that both $\beta_{s_{k+1}}$ and β_{s_k} share, but doing the same process as it is done in (3.22), we can still have that $\hat{\beta}_{s_{k+1}} + \hat{\beta}_{s_k} \in X_e$ which means both terms share no common terms λ^n. Same observation is valid for $\beta_{r_{k+1}}$ and β_{r_k}, i.e., one can have $\hat{\beta}_{r_{k+1}}$ and $\hat{\beta}_{r_k}$ such that $\hat{\beta}_{r_{k+1}} - \hat{\beta}_{r_k} = \beta_{r_{k+1}} - \beta_{r_k}$ and $\hat{\beta}_{r_{k+1}} + \hat{\beta}_{r_k} \in X_e$. Hence, our new constructed terms would be the following

$$\hat{\zeta}_{k+1} = \hat{\beta}_{s_{k+1}} + \gamma\hat{\beta}_{r_{k+1}} \quad \text{and} \quad \hat{\zeta}_k = \hat{\beta}_{s_k} + \gamma\hat{\beta}_{r_k}$$

satisfying both $\hat{\zeta}_{k+1} + \hat{\zeta}_k \in Y_e$ and $0 < \hat{\zeta}_{k+1} - \hat{\zeta}_k < \epsilon$. Let $m \in \mathbb{N}$ be the least even number satisfying the following relation

$$\lambda^m(\hat{\zeta}_{k+1} - \hat{\zeta}_k) < \epsilon < \lambda^{m+2}(\hat{\zeta}_{k+1} - \hat{\zeta}_k). \tag{3.23}$$

Multiplying (3.23) by λ^2, gives the following

$$\epsilon < \lambda^{m+2}(\hat{\zeta}_{k+1} - \hat{\zeta}_k) < \lambda^2\epsilon < (\lfloor\lambda\rfloor + 1)^2\epsilon.$$

Now let $z_1 = \lambda^{m+2}\hat{\zeta}_{k+1}$ and $z_0 = \lambda^{m+2}\hat{\zeta}_k$, then we establish the first two terms satisfying both conditions. Now, let us assume that it is true for the case n, i.e., there exist $z_0, z_1, \cdots, z_{2n-1}$ such that both conditions (i) and (ii) hold. We prove that the case is true for $n+1$, i.e., we construct z_{2k} and z_{2k+1} in such a way that these together with $z_0, z_1, \cdots, z_{2n-1}$ satisfy (i) and (ii). Let p be the highest even power of λ in z_i for $i = 0, 1, \cdots, 2n-1$. Here, one can find $\eta \in \mathbb{N}$ such that

$$0 < \zeta_{\eta+1} - \zeta_\eta < \frac{\epsilon}{\lambda^p},$$

where $\zeta_{\eta+1} = \beta_{u_{\eta+1}} + \gamma\beta_{w_{\eta+1}}$ and $\zeta_\eta = \beta_{u_\eta} + \gamma\beta_{w_\eta}$. If $\beta_{u_{\eta+1}}$ and β_{u_η} share the same term λ^{2n} for some $n \in \mathbb{N}$, with coefficients $\nu_\eta, \nu_{\eta+1} \in \{0, 1, \cdots, \lfloor\lambda\rfloor\}$ respectively and satisfying $\nu_{\eta+1} \le \nu_\eta$, then simply choosing

$$\hat{\beta}_{u_{\eta+1}} = \beta_{u_{\eta+1}} - \nu_{\eta+1}\lambda^{2n} \quad \text{and} \quad \hat{\beta}_{u_\eta} = \beta_{u_\eta} - \nu_{\eta+1}\lambda^{2n},$$

we have that $\hat{\beta}_{u_{\eta+1}} + \hat{\beta}_{u_\eta} \in X_e$. Similarly, one can build $\hat{\beta}_{w_{\eta+1}}$ and $\hat{\beta}_{w_\eta}$ such that $\beta_{w_{\eta+1}} - \beta_{w_\eta} = \hat{\beta}_{w_{\eta+1}} - \hat{\beta}_{w_\eta}$ and $\hat{\beta}_{w_{\eta+1}} + \hat{\beta}_{w_\eta} \in X_e$. Hence, we construct the following terms

$$\hat{\zeta}_{\eta+1} = \hat{\beta}_{u_{\eta+1}} + \gamma\hat{\beta}_{w_{\eta+1}} \quad \text{and} \quad \hat{\zeta}_\eta = \hat{\beta}_{u_\eta} + \gamma\hat{\beta}_{w_\eta}$$

which satisfy $\hat{\zeta}_{\eta+1} + \hat{\zeta}_\eta \in Y_e$ and $0 < \hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta < \frac{\epsilon}{\lambda^p}$. Let $\ell \in \mathbb{N}$ be an even number such that

$$\lambda^\ell(\hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta) < \frac{\epsilon}{\lambda^p} < \lambda^{\ell+2}(\hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta) \implies \frac{\epsilon}{\lambda^p} < \lambda^{\ell+2}(\hat{\zeta}_{\eta+1} - \hat{\zeta}_\eta) < (\lfloor\lambda\rfloor + 1)^2\frac{\epsilon}{\lambda^p}.$$

Now, we choose our next candidates as $z_{2n+1} = \lambda^{p+\ell+2}\hat{\zeta}_{\eta+1}$ and $z_{2n} = \lambda^{p+\ell+2}\hat{\zeta}_\eta$, which together with $z_0, z_1, \cdots, z_{2n-1}$ satisfy both conditions (i) and (ii). Hence, the proof is completed by induction. □

Lemma 3.2.6. *For $\lambda > \mathcal{A}$ and given any $\epsilon > 0$, it is possible to find a finite subsequence $\{\rho_i\}_{i=0}^k$ of $\{\zeta_n\}$ such that the following two conditions hold*

(i) $\rho_0 < \rho_1 < \cdots < \rho_k$ *and* $\rho_k - \rho_0 > \lambda$.

(ii) $\rho_i - \rho_{i-1} < (1 + \lfloor\lambda\rfloor)^2\epsilon$ *for* $i = 1, \cdots, k$.

Proof. Let $\epsilon > 0$ given and $k > \dfrac{\lambda}{\epsilon}$. Now, we consider the sequence $\{z_i\}_{i=0}^{2k-1}$ defined in Lemma 3.2.5 and adopt the following construction for $\{\rho_i\}_{i=0}^{k}$

$$\rho_0 = \sum_{i=0}^{k-1} z_{2i} \qquad \text{and} \qquad \rho_i = \rho_{i-1} + z_{2i-1} - z_{2i-2} \;\text{ for }\; i = 1, 2, \cdots, k. \tag{3.24}$$

From first condition of Lemma 3.2.5, we obtain that $\rho_i \in Y_e$ for $i = 0, .., k$. Moreover, from the second condition of Lemma 3.2.5, we have both cases (i) and (ii). $\square$

Theorem 3.2.7. *The following*

$$\mathcal{S}(\lambda) = 0,$$

holds for $\lambda > \mathcal{A}$.

Proof. For given an arbitrary $\epsilon > 0$, construct the finite subsequence $\{\rho_i\}_{i=0}^{k}$ of $\{\zeta_n\}$ defined in Lemma 3.2.6. Namely, we have

$$\rho_0 < \rho_1 < \cdots < \rho_k \;\text{ with }\; \rho_i - \rho_{i-1} < (1 + \lfloor\lambda\rfloor)^2\epsilon \;\text{ and }\; \rho_k - \rho_0 > \lambda.$$

Define the set of the following form

$$Y_o := \lambda Y_e = \{\lambda\zeta_n\}_{n\in\mathbb{N}}.$$

which is the collection of all odd powers of λ, and so, $Y_o \cap Y_e = \emptyset$. Moreover, from Theorem 3.2.2, it can be concluded that for any interval of length λ contains at least one element from the set Y_o. Now, we consider the following open intervals

$$I_\tau := (\tau, \tau + (1 + \lfloor\lambda\rfloor)^2\epsilon) \subset (0, \infty). \tag{3.25}$$

for $\tau > \rho_0 + \lambda$, and show that $I_\tau \cap Y \neq \emptyset$. By using above argument, one can find at least one element

$$\nu \in [\tau - \rho_0 - \lambda, \tau - \rho_0] \cap Y_o. \tag{3.26}$$

Here, ν consists of only odd powers and from (3.26), we have that

$$\nu + \rho_0 \leq \tau \leq \nu + \rho_k,$$

which means the set I_τ would contain at least one number of the form $\nu + \rho_i$ for some $i = 0, 1, \cdots, k$. Since, $\nu + \rho_i \in Y$ for all i, we have $I_\tau \cap Y \neq \emptyset$.

As a result, we obtain that

$$\mathcal{S}(\lambda) \leq (1 + \lfloor \lambda \rfloor)^2 \epsilon,$$

for arbitrary $\epsilon > 0$. Hence, we have $\mathcal{S}(\lambda) = 0$. □

Since $\{\tau_i\}$ is the increasing rearrangement of the set Y, this implies that $\tau_{i+1} - \tau_i \geq 0$ for every $i \in \mathbb{N}$. As a result, by using Theorem 3.2.7, we obtain the following result

Corollary 3.2.8. *The following*

$$\lim_{k \to \infty} (\tau_{k+1} - \tau_k) = 0,$$

holds for $\lambda > \mathcal{A}$.

3.3 Controllability of Discrete Models Under a Finite Control Set

In this part, we consider the discrete model (1.1) with its controls taking from the finite set, say $\mathcal{I}$. Here, the aim is to build a finite control set $\mathcal{I}$ in such a way that the robot can reach nearly all points in the space. Let us design the following two sets

$$\mathcal{C}_\lambda = \{0, \pm 1, \pm 2, \cdots, \pm \lfloor \lambda \rfloor\} \text{ and } \mathcal{C}_{\lambda,\gamma} = \{i + j\gamma \mid i, j \in \mathcal{C}_\lambda\} \tag{3.27}$$

and reachable set for the model (1.1) with control set $\mathcal{I}$

$$\mathcal{R}(0, \mathcal{I}) = \Big\{ \sum_{i=0}^{n} c_i \lambda^i \Big| c_i \in \mathcal{I}, n \in \mathbb{N} \Big\}. \tag{3.28}$$

Firstly, let us consider the case $\lambda = 1$ for the model (1.1). If η is ranging in the set $\mathcal{C}_1$, then the reachable set $\mathcal{R}(0, \mathcal{C}_1)$ of (1.1) would be the set of integers which is not dense in $\mathbb{R}$. As a result of Lemma 3.1.3, for $\eta \in \mathcal{I} = \{\pm p, \pm q\}$ where p, q are rationally independent numbers, we have that the reachable set $\mathcal{R}(0, \mathcal{I})$ of (1.1) is dense in $\mathbb{R}$. Therefore, for density, the control set can be designed as $\mathcal{I} = \{\pm p, \pm q\}$ for the case $\lambda = 1$.

For case $\lambda > 1$, we provide the following result for density.

Theorem 3.3.1. *The reachable set* $\mathcal{R}(0, \mathcal{C}_{\lambda,\gamma})$ *of (1.1) is dense in* $\mathbb{R}$ *for* $\lambda > \mathcal{A}$.

Proof. Let $\upsilon \in \mathbb{R}$ be a real number and by applying division algorithm, we would have its λ expansion as

$$\upsilon = \sum_{i=0}^{s} c_i \lambda^i + r_\upsilon, \quad \text{where } c_i \in \mathcal{C}_\lambda \text{ and } 0 \leq r_\upsilon < 1.$$

For $r_\upsilon = 0$, we have that $\upsilon \in \mathcal{R}(0, \mathcal{C}_{\lambda,\gamma})$. Let us assume that $0 < r_\upsilon < 1$.

From Theorem 3.2.7, for $\epsilon = \dfrac{r_\upsilon}{\lambda^p} > 0$, where p is a large integer satisfying $p > s$, it is possible to find $k \in \mathbb{N}$ such that

$$\tau_k - \tau_{k-1} \approx \epsilon \qquad \text{for} \quad \tau_k, \tau_{k-1} \in Y.$$

As a result,

$$\upsilon = \sum_{i=0}^{s} c_i \lambda^i + r_\upsilon \approx \sum_{i=0}^{s} c_i \lambda^i + \lambda^p \tau_k - \lambda^p \tau_{k-1}.$$

where coefficients of right hand side belong to the set $\mathcal{C}_{\lambda,\gamma}$, which is an element of $\mathcal{R}(0, \mathcal{C}_{\lambda,\gamma})$. Since, υ is arbitrary here, we conclude that the reachable set $\mathcal{R}(0, \mathcal{C}_{\lambda,\gamma})$ of (1.1) is dense in $\mathbb{R}$. $\square$

CHAPTER 4

ON CONSTRUCTION OF SAMPLING SCHEDULE IN SAMPLED-DATA SYSTEMS

4.1 Preliminaries

Here, we consider the problem controllability and observability of the system (1.2) in infinite time horizon as follows [8, 48]

Definition 4.1.1 (Conrollability)**.** *The continuous time system (1.2) is said to be controllable if for any $t_0 \geq 0$, initial state $x(t_0) = x_0$ and final state x_1, there exists a finite time $t_1 > t_0$ and a control $u(t)$ for $t_0 \leq t \leq t_1$ such that $x(t_1) = x_1$.*

Definition 4.1.2 (Observability)**.** *The continuous time system (1.2) is said to be observable if for any $t_0 \geq 0$ and initial state $x(t_0) = x_0$ there exists a finite time $t_1 > t_0$ such that the knowledge of $u(t)$ and $y(t)$ for $t_0 \leq t \leq t_1$ suffices to determine x_0 uniquely.*

Note that the solution of (1.2) is of the form

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{4.1}$$

where $t_0 = 0$ for our case. By applying zero-order hold (2.3.1) with sampling schedule $\{t_i\}_{i=0}^{\infty}$ to the system (1.2), we obtain the following discrete system

$$\begin{cases} x_{i+1} = e^{A(t_{i+1}-t_i)}x_i + \displaystyle\int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-\tau)}Bd\tau\; u_i, \\ y_i = Cx_i, \end{cases} \tag{4.2}$$

where $x_i := x(t_i)$, $y_i := y(t_i)$, and u_i represents control input which is constant along the interval $[t_i, t_{i+1})$. In addition, we define the notion of (exact) observability and (exact) controllability of the system (1.3) in the following sense [33].

Definition 4.1.3. *The discrete system (1.3) is said to be observable if there exists a positive integer* ν *(number of steps) such that any* t_s *initial time, the state* x_s *is uniquely determined from* $\{y_s, y_{s+1}, \cdots, y_{s+\nu}\}$ *and* $\{u_s, u_{s+1}, \cdots, u_{s+\nu-1}\}$*.*

Definition 4.1.4. *The discrete system (1.3) is said to be controllable, if there exists a positive integer* ν *(number of steps) such that for every* t_s *initial time, and given any initial and final states* x_0 *and* x^1 *respectively, there exists control sequence* $\{u_s, u_{s+1}, \cdots, u_{s+\nu-1}\}$ *which transfers* $x_{s+\nu}$ *to the final state* x^1*, i.e.,* $x_s = x_0$ *and* $x_{s+\nu} = x^1$*.*

Moreover, we provide Cayley Hamilton Theorem and its extended version [50].

Theorem 4.1.5. *(Cayley Hamilton Theorem). Let* A *be* $n \times n$ *matrix, then*

$$A^n + c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots + c_1 A + c_0 I_n = 0,$$

where c_i *some real numbers for* $i = 0, 1, \cdots, n-1$*.*

Now, we give its extended version in our case [37].

Lemma 4.1.6. *Let* A *be* $n \times n$ *invertible matrix, then for any* $i, j \in \mathbb{Z}$*, one can have*

$$A^j = c_0 A^i + c_1 A^{i+1} + \cdots + c_{n-2}A^{i+n-2} + c_{n-1}A^{i+n-1}$$

where c_i *some real numbers for* $i = 0, 1, \cdots, n-1$*.*

Proof. Let us fix $i, j \in \mathbb{Z}$. By using Theorem 4.1.5, one can have

$$A^{j-i} = c_0 I_n + c_1 A + \cdots + c_{n-2}A^{n-2} + c_{n-1}A^{n-1}, \tag{4.3}$$

where c_i some real numbers for $i = 0, 1, \cdots, n-1$. Hence, by multiplying (4.3) with A^i, we get the desired result. $\square$

4.2 Construction of Sampling Pattern

Now, we propose a strategy to build a non-equidistant sampling which consists of two parts. For each part, we define an algorithm and design our sampling pattern accordingly. To start with, we consider any irrational number $\xi \in \mathbb{R}^+$ satisfying the following relation

$$\eta\xi < 1 < (\eta+1)\xi \quad \text{and} \quad \eta \geq n \quad \text{for } \eta \in \mathbb{N}. \tag{4.4}$$

In (4.4), n represents the dimension of state space of (1.2). Let us give the first and second part of the construction.

First part: Define S_k as the sum of first k positive integers, i.e.,

$$S_k = 1 + 2 + \cdots + k \quad \text{and} \quad S_0 = 0.$$

Here, we build the sampling as the following way

$$t_m = (k - (m - S_k))\xi + \frac{(m - S_k)}{\eta} \quad \text{where} \quad S_k \leq m < S_{k+1}. \tag{4.5}$$

According to (4.5), the first term of the time sequence is $t_0 = 0$, second term is $t_1 = \xi$, third term is $t_2 = \frac{1}{\eta}$ and so on. Note that the construction is well defined, i.e., for $t_s = t_p$ where $s, p \in Z_0^+$ implies $s = p$.

With the algorithm defined in (4.5), we pick the first sampling times until the index $m = S_n + n$, namely there would be $\frac{n(n+3)}{2}$ numbers in the first part, i.e., only sampling times: $t_0, t_1, t_2, \cdots, t_{S_n+n}$. Moreover, we observe that the sampling pattern is in ascending order due to the relation between ξ and η given in (4.4).

Second part: Define the following construction

$$t_j^i = (n-j)\xi + (i+j)\frac{1}{\eta} \quad \text{for} \quad 2 \leq i+j \leq n-1 \quad \text{and} \quad i, j \in \mathbb{Z}^+. \tag{4.6}$$

Here, we note that there are $\frac{(n-2)(n-1)}{2}$ number of sampling times and the construction (4.6) is well-defined. Moreover, for any $i, j \in \mathbb{Z}^+$, $t_s^i < t_p^i$ and $t_j^s < t_j^p$ for $s < p$, i.e., the construction (4.6) are in ascending order due to the inequality (4.4). In addition, by using the inequality (4.4), one can have the following inequalities

$$t_{n-k-1}^k < t_1^{k+1} \quad \text{for} \quad k = 1, 2, \cdots, n-3. \tag{4.7}$$

By utilizing (4.7), we see that the second part of the sampling pattern can be ordered in an ascending manner as

$$t_1^1, t_2^1, \cdots, t_{n-2}^1, \;\; t_1^2, t_2^2, \cdots, t_{n-3}^2, \;\; \cdots \;\; , t_1^k, t_2^k, \cdots \;\; , t_{n-k-1}^k, \;\; \cdots, \;\; t_1^{n-2}.$$

Here, the only problem in this part is to implement these sampling times in a sampled-data system. However, the construction (4.6) can be simply adjusted to only one variable in the following sense

$$t_j^i = t_{S_{n+1}-S_{i-1}+(i-1)(n-1)+j-1}, \tag{4.8}$$

which would be a continuation of the time sequence constructed in the first part.

As a result, we pick $t_0, t_1, \cdots, t_{S_n+n}$ sampling times from first part and continue with $t_{S_{n+1}}$, $t_{S_{n+1}+1}$, $\cdots$, t_{n^2+1} sampling times from second part. Let S denotes the set of all n^2+2 points, i.e.,

$$S := \{t_0, t_1, \cdots, t_{n^2+1}\}. \tag{4.9}$$

Next, we extend the set S periodically without interrupting the nature of these n^2+2 points as the following way:

$$t_m = \gamma t_{n^2+1} + t_k \text{ where } m = \gamma(n^2+1) + k. \tag{4.10}$$

Observe that for any $s \in \mathbb{N}$, one can obtain the following

$$t_{s+n^2+1} = t_s + t_{n^2+1}. \tag{4.11}$$

The fact (4.11) comes from the division algorithm: let $s = \gamma_1(n^2+1) + k_1$, then $s + n^2 + 1 = (\gamma_1+1)(n^2+1) + k_1$. By using the construction (4.10), we have

$$t_{s+n^2+1} = (\gamma_1+1)t_{n^2+1} + t_{k_1} = (\gamma_1 t_{n^2+1} + t_{k_1}) + t_{n^2+1} = t_s + t_{n^2+1}.$$

To understand the above construction better, we provide two and three dimensional version of the presented sampling pattern.

<u>*Two dimensional construction:*</u>

$$\begin{array}{llllll}
1^{st}\, row & \rightarrow & 0 & \xi & 2\xi & \cdots \\
2^{nd}\, row & \rightarrow & \frac{1}{\eta} & \xi + \frac{1}{\eta} & \cdots & \\
3^{rd}\, row & \rightarrow & \frac{2}{\eta} & \cdots & & \\
\vdots & & & & &
\end{array}$$

In two dimensional construction, all the entries are taken from first part setting. Namely, $t_0 = 0$, $t_1 = \xi$, $t_2 = \frac{1}{\eta}$, $t_3 = 2\xi$, $t_4 = \xi + \frac{1}{\eta}$ and finally $t_5 = \frac{2}{\eta}$. Then, we extend the setting by adopting division algorithm defined in (4.10).

<u>*Three dimensional construction:*</u>

$$
\begin{array}{lccccc}
1^{st}\ row & \to & 0 & \xi & 2\xi & 3\xi & \cdots \\
2^{nd}\ row & \to & \frac{1}{\eta} & \xi + \frac{1}{\eta} & 2\xi + \frac{1}{\eta} & \cdots & \\
3^{rd}\ row & \to & \frac{2}{\eta} & \xi + \frac{2}{\eta} & 2\xi + \frac{2}{\eta} & \cdots & \\
4^{th}\ row & \to & \frac{3}{\eta} & \cdots & & & \\
\vdots & & & & & &
\end{array}
$$

In the above setting, the enteries $t_0 = 0$, $t_1 = \xi$, $t_2 = \frac{1}{\eta}$, $t_3 = 2\xi$, $t_4 = \xi + \frac{1}{\eta}$, $t_5 = \frac{2}{\eta}$, $t_6 = 3\xi$, $t_7 = 2\xi + \frac{1}{\eta}$, $t_8 = \xi + \frac{2}{\eta}$, and finally $t_9 = \frac{3}{\eta}$ are taken from first part construction, but the entry, $t_{10} = 2\xi + \frac{2}{\eta}$ is taken from second part construction. As it is mentioned in two dimensional setting, we extend the set $\{t_0, t_1, \cdots, t_{10}\}$ periodically by utilizing division algorithm with mod 10.

Now, we are ready to discretize the continuous system (1.2) by means of sample and zero-order hold with above constructed pattern $\{t_i\}_{i=0}^{\infty}$. In the next section, we consider the discrete problem under sampling pattern $\{t_i\}_{i=0}^{\infty}$ and prove that if its continuous version is observable or controllable, then same properties would be carried out in the discrete case.

Remark 4.2.1. *For $n = 2$, the construction (4.10) becomes*

$$t_m = \gamma t_5 + t_k \ \ where \ \ m = 5\gamma + k, \ \ for \ \ k = 0, 1, 2, 3, 4.$$

Hence, one can say that

$$\gamma t_5 \le t_m < (\gamma + 1)t_5 \implies \frac{2}{\eta}\left\lfloor \frac{m}{5} \right\rfloor < t_m < \frac{2}{\eta}\Big(\left\lfloor \frac{m}{5} \right\rfloor + 1\Big) \implies t_m < \left\lfloor \frac{m}{5} \right\rfloor + 1.$$

Last inequality comes from the fact that $\eta \ge 2$. In general, it can be proved that

$$t_m < (n-1)\left(\left\lfloor \frac{m}{n^2+1} \right\rfloor + 1\right).$$

Hence, each term in the sampling pattern is bounded from above by a number which depends on its index and dimension of the space.

Remark 4.2.2. *From fact (4.11), one can understand that the constructed sampling pattern $\{t_i\}_{i=0}^{\infty}$ acts periodically which makes the corresponding sampled-data system periodically time-varying, i.e., all its properties become uniform in time. In the next section, this motivates us to consider, without loss of generality, only the case $t_0 = 0$, instead of any arbitrary initial data t_k.*

4.3 Discretization Without Loss of Observability/Controllability

In this section, we show that the property of observability/controllability of continuous time system (1.2) can be carried over sampled-data system (1.3) by sampling pattern (4.9). To start with, we provide the first main theorem of the paper.

Theorem 4.3.1 (Observability)**.** *Assume that the system (1.2) is discretized with the sampling pattern $\{t_i\}_{i=0}^{\infty}$ defined in (4.10). Then, the observability of the system (1.2) implies the observability of the system (1.3).*

Proof. First of all, assume that the system (1.2) is observable. Here, we proceed proof by contradiction. Namely, we assume the contrary of the statement, i.e., the system (1.3) is not observable. As it is mentioned in Remark 4.2.2., it suffices to consider only the case $t_0 = 0$. Hence, contrary of the statement means the following set of equations

$$y_k = Ce^{At_k}x_0, \quad \text{for any } t_k \in \{t_i\}_{i=0}^{\infty},$$

has no unique solution $x(0) = x_0 \in \mathbb{R}^n$. More precisely, there would be no $\nu \in \mathbb{N}$ (step size) such that x_0 is uniquely determined by $\{y_0, y_1, \cdots, y_\nu\}$. Here, without loss of generality, it can be assumed that control inputs act on the discrete system (1.3) are identically zero. Therefore, one can find a non-zero vector $\psi \in \mathbb{R}^n$ such that

$$Ce^{At_k}\psi = 0 \qquad \text{for } k \in \mathbb{N}. \tag{4.12}$$

By using (4.12), we show that for given any $i, j \in \mathbb{Z}$, the following equality holds

$$Ce^{A(\frac{j}{\eta}+i\xi)}\psi = 0. \tag{4.13}$$

Note firstly that $t_{S_{j+1}-1} = \frac{j}{\eta} \in S$ for $j = 0, 1, \cdots, n-1$ and the matrix $M := e^{A\frac{1}{\eta}}$ is non-singular then it follows from Lemma 4.1.6 that any power of M can be written as a linear combination of $M^0, M^1, \cdots, M^{n-1}$. From (4.12), we see that for given any $j \in \mathbb{Z}$

$$Ce^{A\frac{j}{\eta}}\psi = CM^j\psi = C\sum_{v=0}^{n-1}\alpha_v M^v\psi = C\sum_{v=0}^{n-1}\alpha_v e^{A\frac{v}{\eta}}\psi = 0,$$

for some $\{\alpha_v\}_{v=0}^{n-1} \in \mathbb{R}$. Similarly, from (4.12) and the fact that $t_{S_i} = i\xi \in S$ for $i = 0, 1, \cdots, n-1$, one can obtain $Ce^{Ai\xi}\psi = 0$ for $i \in \mathbb{Z}$ from Lemma 4.1.6.

Hence from above observations we have that for any $i, j \in \mathbb{Z}$

$$\begin{aligned} Ce^{A(\frac{j}{\eta}+i\xi)}\psi &= C\sum_{v=0}^{n-1}\alpha_v e^{A\frac{v}{\eta}}\sum_{v=0}^{n-1}\beta_w e^{Aw\xi}\psi \\ &= \sum_{v,w=0}^{n-1}\alpha_v\beta_w Ce^{A(\frac{v}{\eta}+w\xi)}\psi = 0, \end{aligned} \tag{4.14}$$

for some $\{\alpha_v\}_{v=0}^{n-1}$ and $\{\beta_v\}_{w=0}^{n-1} \in \mathbb{R}$. Last equality of (4.14) comes from the fact that $\frac{v}{\eta} + w\xi \in S$ for $v, w = 0, 1, \cdots, n-1$. Hence, we conclude from Lemma 3.1.3 and analyticity of exponential function that

$$Ce^{At}\psi = 0 \quad \text{for all } t \geq 0. \tag{4.15}$$

This fact together with Cayley Hamilton Theorem implies

$$rank\Big[C, CA, CA^2, \cdots, CA^{n-1}\Big]^T \neq n.$$

Namely, the system (1.2) is not observable which contradicts with the assumption. Hence, we prove that the system (1.3) is observable. $\square$

Now, we state the second main theorem of the paper in which the controllability of continuous time system (1.2) is preserved during the discretization with sampling pattern (4.9).

Theorem 4.3.2 (Controllability)**.** *Assume that the system (1.2) is discretized with the sampling pattern $\{t_i\}_{i=0}^{\infty}$ given in (4.10). Then, the controllability of the system (1.2) implies the controllability of the system (1.3).*

Proof. From Definition 4.1.4, controllability of the sampled-data system (1.3) is defined that for given any initial time t_k, and $x_0, x_1 \in \mathbb{R}^n$ arbitrary initial and final states respectively, there exists a positive integer $\nu \in \mathbb{Z}_0^+$ and sequence of controls $\{u_k, u_{k+1}, \cdots, u_{\nu+k-1}\}$ that derives $x_{\nu+k}$ to x_1, i.e., $x_{\nu+k} = x_1$.

As it is discussed in Remark 4.2.2, discretization with time pattern $\{t_i\}_{i=0}^{\infty}$ makes the system uniform in time, i.e., it suffices to prove the theorem for the case $t_0 = 0$.

Note that the state of the system (1.3) at $t = t_\nu$ is given by

$$x_\nu = e^{A(t_\nu - t_0)} x_0 + \sum_{k=0}^{\nu-1} \int_{t_k}^{t_{k+1}} e^{A(t_\nu - \zeta)} B d\zeta \, u_k,$$

which means

$$e^{-At_\nu} x_\nu - e^{-At_0} x_0 = \sum_{k=0}^{\nu-1} \int_{t_k}^{t_{k+1}} e^{-A\zeta} B d\zeta \, u_k. \tag{4.16}$$

For controllability, x_ν and x_0 need to be arbitrary vectors in $\mathbb{R}^n$. Hence, the left hand side of (4.16) can be any vector in $\mathbb{R}^n$.

The right side of (4.16), on the other hand, becomes a certain set of vectors built by control sequence $\{u_0, u_1, \cdots, u_{\nu-1}\}$ which depends on ν. Clearly, number of these vectors increases whenever ν grows and they span a subspace of $\mathbb{R}^n$ for each ν. If for some finite ν, those vectors span the entire space $\mathbb{R}^n$, then sampled-data system (1.3) becomes controllable.

On the other hand, if (1.3) is not controllable, then the span of these vectors from right hand side of (4.16) becomes a proper subset of $\mathbb{R}^n$ for all $\nu \in \mathbb{N}$.

This implies that one can find a vector $\psi \in \mathbb{R}^n$ such that

$$\int_{t_k}^{t_{k+1}} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } t_k \in \{t_i\}_{i=0}^{\infty} \tag{4.17}$$

where ψ^T is transpose of ψ. To prove the theorem, we assume the contrary of the statement and at the end, we obtain a contradiction. Hence, the equality (4.17) is valid, and clearly one can have the following similar situation

$$\int_{0}^{t_k} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } t_k \in \{t_i\}_{i=0}^{\infty}.$$

First observation is that $t_{S_i} = i\xi \in S$ for $i = 1, \cdots, n$, which gives

$$\int_{0}^{i\xi} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } i = 1, 2, \cdots, n. \tag{4.18}$$

By using (4.18), we show that

$$\int_{k\xi}^{(k+1)\xi} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{holds for each } k \in \mathbb{Z}. \tag{4.19}$$

To prove the above identity, we apply Lemma 4.1.6 to non singular matrix $K := e^{-A\xi}$, i.e., any power of K can be represented as a linear combination of $K^0, K^1, \cdots, K^{n-1}$. Namely,

$$\begin{aligned}
\int_{k\xi}^{(k+1)\xi} \psi^T e^{-A\zeta} B d\zeta &= \int_0^{\xi} \psi^T e^{-Ak\xi} e^{-A\zeta} B d\zeta \\
&= \int_0^{\xi} \psi^T \sum_{v=0}^{n-1} \alpha_v e^{-Av\xi} e^{-A\zeta} B d\zeta \\
&= \sum_{v=0}^{n-1} \alpha_v \int_{v\xi}^{(v+1)\xi} \psi^T e^{-A\zeta} B d\zeta = 0
\end{aligned}$$

for some $\{\alpha_v\}_{v=0}^{n-1} \in \mathbb{R}$. Last part comes from the fact (4.18).

Second observation is that $t_{S_{j+1}-1} = \frac{j}{\eta} \in S$ for $j = 1, \cdots, n$ which means

$$\int_0^{\frac{j}{\eta}} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } j = 1, 2, \cdots, n. \tag{4.20}$$

From (4.20) , we obtain the following

$$\int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{holds for each } k \in \mathbb{Z}. \tag{4.21}$$

Again, we utilize Lemma 4.1.6 for non singular matrix $\hat{K} := e^{-A\frac{1}{\eta}}$, i.e.,

$$\begin{aligned}
\int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} \psi^T e^{-A\zeta} B d\zeta &= \int_0^{\frac{1}{\eta}} \psi^T e^{-A\frac{k}{\eta}} e^{-A\zeta} B d\zeta \\
&= \int_0^{\frac{1}{\eta}} \psi^T \sum_{v=0}^{n-1} \beta_v e^{-A\frac{v}{\eta}} e^{-A\zeta} B d\zeta \\
&= \sum_{v=0}^{n-1} \beta_v \int_{\frac{v}{\eta}}^{\frac{v+1}{\eta}} \psi^T e^{-A\zeta} B d\zeta = 0
\end{aligned}$$

for some $\{\beta_v\}_{v=0}^{n-1} \subset \mathbb{R}$. Now, by using (4.19) and (4.21), we prove the following

$$\int_0^{\frac{j}{\eta}+i\xi} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{holds for each } i, j \in \mathbb{Z}. \tag{4.22}$$

In other words,

$$\begin{aligned}
\int_0^{\frac{j}{\eta}+i\xi} \psi^T e^{-A\zeta} B d\zeta &= \int_{i\xi}^{\frac{j}{\eta}+i\xi} \psi^T e^{-A\zeta} B d\zeta \\
&= \int_0^{\frac{j}{\eta}} \psi^T e^{-Ai\xi} e^{-A\zeta} B d\zeta \\
&= \sum_{v=0}^{n-1} \gamma_v \Big\{ \int_{v\xi}^{\frac{j}{\eta}+v\xi} \psi^T e^{-A\zeta} B d\zeta + \int_{\frac{j}{\eta}}^{v\xi} \psi^T e^{-A\zeta} B d\zeta \Big\} \\
&= \sum_{v=0}^{n-1} \gamma_v \int_{\frac{j}{\eta}}^{\frac{j}{\eta}+v\xi} \psi^T e^{-A\zeta} B d\zeta = \sum_{v=0}^{n-1} \gamma_v \int_0^{v\xi} \psi^T e^{-A\frac{j}{\eta}} e^{-A\zeta} B d\zeta \\
&= \sum_{v=0}^{n-1} \gamma_v \int_0^{v\xi} \psi^T \sum_{w=0}^{n-1} \sigma_w e^{-A\frac{w}{\eta}} e^{-A\zeta} B d\zeta \\
&= \sum_{v,w=0}^{n-1} \gamma_v \sigma_w \int_0^{\frac{w}{\eta}+v\xi} \psi^T e^{-A\zeta} B d\zeta = 0,
\end{aligned}$$

for some coefficients $\{\gamma_v\}_{v=0}^{n-1}, \{\sigma_w\}_{w=0}^{n-1} \subset \mathbb{R}$ coming from Lemma 4.1.6. The last part of above equality results from the fact that $\frac{w}{\eta}+v\xi \in S$ for $w, v = 0, 1, 2, \cdots, n-1$.

As a result, from Lemma 3.1.3 and together with the result (4.22), we obtain the following

$$\int_0^t \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for any } t \in \mathbb{R}^+. \tag{4.23}$$

This means that $\psi^T e^{At} B \equiv 0$. This fact together with Cayley Hamilton Theorem implies

$$rank\Big[B, AB, A^2B, \cdots, A^{n-1}B\Big] \neq n,$$

i.e., contradicting with Kalman rank condition. As a result, controllability of (1.2) implies controllability of (1.3). □

Remark 4.3.3. *It is shown in Theorem 4.3.1 that for all $t_k \in S$ satisfying*

$$Ce^{At_k}\psi = 0 \quad \textit{implies} \quad Ce^{A(\frac{j}{\eta}+i\xi)}\psi = 0 \ \textit{for any } i, j \in \mathbb{Z}.$$

Similarly, in Theorem 4.3.2, for all $t_k \in S$ satisfying

$$\int_0^{t_k} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \textit{implies} \quad \int_0^{\frac{j}{\eta}+i\xi} \psi^T e^{-A\zeta} B d\zeta = 0 \ \textit{for any } i, j \in \mathbb{Z}.$$

From Theorem 4.3.1 and Theorem 4.3.2, it is understood that property of observability/controllability can be carried over sampled-data system by sampling pattern $\{t_i\}_{i=0}^{\infty}$. The following results show that the number of steps ν required for observability/controllability is bounded.

Corollary 4.3.4. *If the continuous system (1.2) is observable/controllable, then the discrete system (1.3) with sampling pattern $\{t_i\}_{i=0}^{\infty}$ is observable/controllable with $\nu = 2(n^2+1)$.*

Proof. Observability: The proof is by contradiction. Assume that for some $s \in \mathbb{N}$, the state x_s cannot uniquely be determined from the measurement interval $[t_s, t_{s+2(n^2+1)-1}]$, means that one can find a non-zero vector $\psi \in \mathbb{R}^n$ such that

$$Ce^{A(t_{s+i}-t_s)}\psi = 0 \qquad \text{for} \quad i = 0, 1, \cdots, 2(n^2+1)-1. \tag{4.24}$$

Let $s = \gamma(n^2+1)+k$, we have $t_s = \gamma t_{n^2+1} + t_k$ where $\gamma \in \mathbb{N}$, and $k \in \{0, 1, \cdots, n^2\}$. Here, we examine two cases:

Case 1: Assume that $k = 0$, then $t_{s+i} - t_s = t_i$ for $i = 0, 1, \cdots, n^2+1$. Hence, (4.24) becomes

$$Ce^{At_i}\psi = 0 \qquad \text{for} \quad i = 0, 1, \cdots, n^2+1. \tag{4.25}$$

Here, it follows from Theorem 4.3.1 and Remark 4.3.3, that (4.15) holds in this case.

Case 2: Assume that $k \neq 0$, then the following sampling times

$$(\gamma+1)t_{n^2+1} + t_i, \qquad \text{for} \quad i = 0, 1, \cdots, n^2+1, \tag{4.26}$$

satisfy the relation (4.24). In other words, sampling times given in (4.26) are those among $\{t_s, t_{s+1}, \cdots, t_{s+2(n^2+1)-1}\}$. By defining

$$\hat{C} = Ce^{A((\gamma+1)t_{n^2+1}-t_s)}$$

(4.24) becomes

$$\hat{C}e^{At_i}\psi = 0 \qquad \text{for} \quad i = 0, 1, \cdots, n^2+1.$$

Again, from Theorem 4.3.1 and Remark 4.3.3, we have (4.15).

Therefore, for both cases, we get (4.15) which gives a contradiction. Hence, the statement is true for $\nu = 2(n^2+1)$.

Contrability: The proof is by contradiction. Suppose that for some $s \in \mathbb{N}$ and $x_0, x_1 \in \mathbb{R}^n$ such that there is no control sequence $\{u_s, u_{s+1}, \cdots, u_{s+2(n^2+1)-1}\}$ which transfers initial and final states $x_s = x_0$ and $x_{s+2(n^2+1)} = x_1$.

As it was mentioned in Theorem 4.3.2, contrary of the statement means the existence of a non-zero vector $\psi \in \mathbb{R}^n$ satisfying

$$\int_{t_{s+i}}^{t_{s+i+1}} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } i \in \{0, 1, \cdots, 2(n^2+1)-1\}, \tag{4.27}$$

where ψ^T is transpose of ψ. In this proof, we consider two cases:

Case 1: Assume that $k = 0$, then for each sampling times

$$\gamma t_{n^2+1} + t_i, \qquad \text{for} \quad i = 0, 1, \cdots, n^2+1, \tag{4.28}$$

which are from the set $\{t_s, t_{s+1}, \cdots, t_{s+2(n^2+1)-1}\}$, satisfies (4.27).

Let $\ell_0^T = \psi^T e^{-A\gamma t_{n^2+1}}$, then we have from (4.27) that

$$\int_0^{t_i} \ell_0^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } i \in \{0, 1, \cdots, n^2+1\}. \tag{4.29}$$

From Remark 4.3.3 and Theorem 4.3.2, we obtain

$$\int_0^{\frac{j}{\eta}+i\xi} \ell_0^T e^{-A\zeta} B d\zeta = 0 \quad \text{for any } i, j \in \mathbb{Z}. \tag{4.30}$$

Case 2: Assume that $k \neq 0$, then by using the fact that all sampling times defined in (4.26) are from the set $\{t_s, t_{s+1}, \cdots, t_{s+2(n^2+1)-1}\}$, that each of those satisfying (4.27), i.e.,

$$\int_0^{t_i} \ell_1^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } i \in \{0, 1, \cdots, n^2+1\}, \tag{4.31}$$

where $\ell_1^T = \psi^T e^{-A(\gamma+1)t_{n^2+1}}$. Similarly, we conclude that

$$\int_0^{\frac{j}{\eta}+i\xi} \ell_1^T e^{-A\zeta} B d\zeta = 0 \quad \text{for any } i, j \in \mathbb{Z}. \tag{4.32}$$

From both cases, we prove that

$$\int_0^{\frac{j}{\eta}+i\xi} \ell_k^T e^{-A\zeta} B d\zeta = 0, \quad k = 0, 1 \quad \text{and} \quad \text{for any } i, j \in \mathbb{Z}. \tag{4.33}$$

Since (4.33) is true for all $i, j \in \mathbb{Z}$, from Lemma 3.1.3, we conclude (4.23) which gives us a contradiction. $\square$

Remark 4.3.5. *In Corollary 4.3.4, we see that the number of steps which is necessary to transfer the property of controllable/observable from continuous system (1.2) to the sampled-data system (1.3) could be* $\nu = 2(n^2 + 1)$. *In addition, any number which is greater than the value* ν *satisfies the relation as well.*

Next observation would answer to the question, whether it is possible to find the number of steps ν for the transfer by providing a better estimate other than $\nu = 2(n^2 + 1)$. In this part, we examine the case $n = 2$ and find out that it is sufficient to guarantee the transfer if we have at least $\nu = 5$ instead of $\nu = 10$, as required in Corollary 4.3.4.

Corollary 4.3.6. *If the continuous system (1.2) is observable/controllable for the case* $n = 2$, *then the discrete system (1.3) with sampling pattern* $\{t_i\}_{i=0}^{\infty}$ *is observable/-controllable with number of step* $\nu = 5$.

Proof. To start with, we assume that the control system (1.2) is controllable for $n = 2$. Now, we follow proof by contradiction. Namely, suppose that for some $s \in \mathbb{N}$ and $x_0, x_1 \in \mathbb{R}^2$ such that there is no control sequence $\{u_s, u_{s+1}, \cdots, u_{s+4}\}$ which transfers the initial and final state as $x_s = x_0$ and $x_{s+5} = x_1$.

Observe that control input $u(t)$ is considered at points

$$\{t_s, t_{s+1}, t_{s+2}, t_{s+3}, t_{s+4}\}.$$

Let $s = 5\gamma + k$, where $\gamma \in \mathbb{N}$ and $k \in \{0, 1, \cdots, 4\}$. It comes from (4.10) that $t_s = \gamma t_5 + t_k$, and from the construction of sampling pattern for case $n = 2$, we see that the remainder part could be

$$t_0 = 0, \;\; t_1 = \xi, \;\; t_2 = \frac{1}{\eta}, \;\; t_3 = 2\xi, \;\; t_4 = \frac{1}{\eta} + \xi. \tag{4.34}$$

The solution of (1.3) at final time x_1 with initial data x_0 can be written as

$$e^{-At_{s+5}} x_1 = e^{-At_s} x_0 + \sum_{i=s}^{s+4} \int_{t_i}^{t_{i+1}} e^{-A\zeta} B d\zeta \; u_i. \tag{4.35}$$

As it is discussed in the proof of Theorem 4.3.2, opposite of the statement means one can find a non-zero vector $\psi \in \mathbb{R}^2$ such that

$$\int_{t_{s+i}}^{t_{s+i+1}} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for all } i \in \{0, 1, 2, 3, 4\}, \tag{4.36}$$

where ψ^T is transpose of ψ. Since, the sampling pattern is non-equidistant with period $n^2 + 1,$ which is five in our case, we practice case analysis. For this purpose, we consider five cases:

Case 1: Assume $k = 0$, i.e., the remainder $t_k = t_0 = 0$ and let $\phi_0 = \psi^T e^{-\gamma A t_5}$. Hence, (4.36) implies the following

$$\int_{0}^{t_i} \phi_0 e^{-A\zeta} B d\zeta = 0 \quad \text{for } i \in \{1, 2, 3, 4, 5\}. \tag{4.37}$$

By using Remark 4.3.3, we obtain the following

$$\int_{0}^{\frac{j}{\eta}+i\xi} \phi_0 e^{-A\zeta} B d\zeta = 0 \quad \text{for any } i, j \in \mathbb{Z}. \tag{4.38}$$

Case 2: Assume $k = 1$ i.e., $t_k = t_1 = \xi$ and let $\phi_1 = \psi^T e^{-A(\gamma t_5 + \xi)}$. Again, (4.36) implies the following

$$\int_{t_k - \xi}^{t_{k+1}-\xi} \phi_1 e^{-A\zeta} B d\zeta = 0 \quad \text{for } i \in \{1, 2, 3, 4, 5\}. \tag{4.39}$$

Observe that

$$t_1 - \xi = 0,\ t_2 - \xi = \frac{1}{\eta} - \xi,\ t_3 - \xi = \xi,\ t_4 - \xi = \frac{1}{\eta},\ t_5 - \xi = \frac{2}{\eta} - \xi,\ t_6 - \xi = \frac{2}{\eta}.$$

Moreover, from Lemma 4.1.6, we have that

$$\int_{0}^{\frac{1}{\eta}} \phi_1 e^{-A\zeta} B d\zeta = \int_{0}^{\frac{2}{\eta}} \phi_1 e^{-A\zeta} B d\zeta = 0 \implies \int_{0}^{\frac{i}{\eta}} \phi_1 e^{-A\zeta} B d\zeta = 0, \tag{4.40}$$

for any $i \in \mathbb{Z}$. By using (4.40) and Lemma 4.1.6, we obtain for any $i \in \mathbb{Z}$

$$\int_{\frac{i}{\eta}}^{\frac{i}{\eta}-\xi} \phi_1 e^{-A\zeta} B d\zeta = c_1 \int_{\frac{1}{\eta}}^{\frac{1}{\eta}-\xi} \phi_1 e^{-A\zeta} B d\zeta + c_2 \int_{\frac{2}{\eta}}^{\frac{2}{\eta}-\xi} \phi_1 e^{-A\zeta} B d\zeta = 0,$$

for some $c_1, c_2 \in \mathbb{R}$. Hence, for $i = 0$, we have that

$$\int_{0}^{-\xi} \phi_1 e^{-A\zeta} B d\zeta = 0. \tag{4.41}$$

Also, by using Lemma 4.1.6 and (4.41), one can easily obtain

$$\int_0^{-\xi} \phi_1 e^{-A\zeta} B d\zeta = \int_0^{\xi} \phi_1 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{i\xi} \phi_1 e^{-A\zeta} B d\zeta = 0, \tag{4.42}$$

for any $i \in \mathbb{Z}$. As a result, we have that

$$\int_0^{\frac{j}{\eta}+i\xi} \phi_1 e^{-A\zeta} B d\zeta = \sum_{v=-1}^{0} \sum_{w=0}^{1} r_v l_w \int_0^{\frac{w}{\eta}+v\xi} \phi_1 e^{-A\zeta} B d\zeta = 0, \tag{4.43}$$

for some r_v, l_w real numbers.

Case 3: Assume $k = 2$, i.e., $t_k = t_2 = \frac{1}{\eta}$ and let $\phi_2 = \psi^T e^{-A(\gamma t_5 + \frac{1}{\eta})}$. From (4.36), we have

$$\int_{t_k - \frac{1}{\eta}}^{t_{k+1} - \frac{1}{\eta}} \phi_2 e^{-A\zeta} B d\zeta = 0 \quad \text{for } i \in \{2, 3, 4, 5, 6\}. \tag{4.44}$$

Moreover,

$$t_2 - \frac{1}{\eta} = 0,\ t_3 - \frac{1}{\eta} = 2\xi - \frac{1}{\eta},\ t_4 - \frac{1}{\eta} = \xi,\ t_5 - \frac{1}{\eta} = \frac{1}{\eta},\ t_6 - \frac{1}{\eta} = \xi + \frac{1}{\eta},\ t_7 - \frac{1}{\eta} = \frac{2}{\eta}.$$

Observe that,

$$\int_0^{\frac{1}{\eta}} \phi_2 e^{-A\zeta} B d\zeta = \int_0^{\frac{2}{\eta}} \phi_2 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{\frac{i}{\eta}} \phi_2 e^{-A\zeta} B d\zeta = 0,$$

for any $i \in \mathbb{Z}$, then we have

$$\int_{\frac{i}{\eta}}^{\frac{i}{\eta}+\xi} \phi_2 e^{-A\zeta} B d\zeta = c_1 \int_0^{\xi} \phi_2 e^{-A\zeta} B d\zeta + c_2 \int_{\frac{1}{\eta}}^{\frac{1}{\eta}+\xi} \phi_2 e^{-A\zeta} B d\zeta = 0,$$

and by using above equality for $i = -1$, we obtain

$$\int_{i\xi}^{-\frac{1}{\eta}+i\xi} \phi_2 e^{-A\zeta} B d\zeta = b_1 \int_0^{-\frac{1}{\eta}} \phi_2 e^{-A\zeta} B d\zeta + b_2 \int_{\xi}^{-\frac{1}{\eta}+\xi} \phi_2 e^{-A\zeta} B d\zeta = 0,$$

where c_1, c_2, b_1, b_2 real numbers coming from Lemma 4.1.6. From the last part, when $i = 2$, we have

$$\int_{2\xi}^{-\frac{1}{\eta}+2\xi} \phi_2 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{2\xi} \phi_2 e^{-A\zeta} B d\zeta = 0.$$

Hence,

$$\int_0^{\xi} \phi_2 e^{-A\zeta} B d\zeta = \int_0^{2\xi} \phi_2 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{i\xi} \phi_2 e^{-A\zeta} B d\zeta = 0, \tag{4.45}$$

for any $i \in \mathbb{Z}$. In this case, for any $i, j \in \mathbb{Z}$

$$\int_0^{\frac{j}{\eta}+i\xi} \phi_2 e^{-A\zeta} B d\zeta = \sum_{v,w=0}^{1} \alpha_v \beta_w \int_0^{\frac{w}{\eta}+v\xi} \phi_2 e^{-A\zeta} B d\zeta = 0.$$

Case 4: Assume $k = 3$, i.e., $t_k = t_3 = 2\xi$ and let $\phi_3 = \psi^T e^{-A(\gamma t_5 + 2\xi)}$.

From (4.36), we obtain that

$$\int_{t_i-2\xi}^{t_{i+1}-2\xi} \phi_3 e^{-A\zeta} B d\zeta = 0 \quad \text{for } i \in \{3,4,5,6,7\}. \tag{4.46}$$

Observe that $t_k - 2\xi$ would be the following

$$0,\ \frac{1}{\eta} - \xi,\ \frac{2}{\eta} - 2\xi,\ \frac{2}{\eta} - \xi,\ \frac{3}{\eta} - 2\xi,\ \frac{2}{\eta},$$

for $k = 3,4,5,6,7,8$ respectively. From Lemma 4.1.6, for any $i \in \mathbb{Z}$

$$\int_{i(\frac{1}{\eta}-\xi)}^{\frac{1}{\eta}+i(\frac{1}{\eta}-\xi)} \phi_3 e^{-A\zeta} B d\zeta = c_1 \int_{\frac{1}{\eta}-\xi}^{\frac{2}{\eta}-\xi} \phi_3 e^{-A\zeta} B d\zeta + c_2 \int_{2(\frac{1}{\eta}-\xi)}^{\frac{3}{\eta}-2\xi} \phi_3 e^{-A\zeta} B d\zeta = 0,$$

where c_1, c_2 some real numbers. Let $i = 0$, then

$$\int_0^{\frac{1}{\eta}} \phi_3 e^{-A\zeta} B d\zeta = \int_0^{\frac{2}{\eta}} \phi_3 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{\frac{i}{\eta}} \phi_3 e^{-A\zeta} B d\zeta = 0,$$

and so

$$\int_0^{\frac{1}{\eta}} \phi_3 e^{-A\zeta} B d\zeta = 0 \implies \int_{\frac{1}{\eta}}^{\frac{1}{\eta}-\xi} \phi_3 e^{-A\zeta} B d\zeta = 0.$$

Moreover, by using the last identity, one can have for any $i \in \mathbb{Z}$, that

$$\int_{\frac{i}{\eta}}^{\frac{i}{\eta}-\xi} \phi_3 e^{-A\zeta} B d\zeta = b_1 \int_{\frac{1}{\eta}}^{\frac{1}{\eta}-\xi} \phi_3 e^{-A\zeta} B d\zeta + b_2 \int_{\frac{2}{\eta}}^{\frac{2}{\eta}-\xi} \phi_3 e^{-A\zeta} B d\zeta = 0,$$

for some b_1, b_2 real numbers. Again, using Lemma 4.1.6, we have for any $i \in \mathbb{Z}$

$$\int_{\frac{i}{\eta}}^{\frac{i}{\eta}-2\xi} \phi_3 e^{-A\zeta} B d\zeta = d_1 \int_{\frac{2}{\eta}}^{\frac{2}{\eta}-2\xi} \phi_3 e^{-A\zeta} B d\zeta + d_2 \int_{\frac{3}{\eta}}^{\frac{3}{\eta}-2\xi} \phi_3 e^{-A\zeta} B d\zeta = 0,$$

for some $d_1, d_2 \in \mathbb{R}$. Hence, for $i = 0$,

$$\int_0^{-2\xi} \phi_3 e^{-A\zeta} B d\zeta = \int_0^{-\xi} \phi_3 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{i\xi} \phi_3 e^{-A\zeta} B d\zeta = 0.$$

In this part, we obtain that

$$\int_0^{\frac{j}{\eta}+i\xi} \phi_3 e^{-A\zeta} B d\zeta = \sum_{v=-1}^{0} \sum_{w=0}^{1} r_v l_w \int_0^{\frac{w}{\eta}+v\xi} \phi_3 e^{-A\zeta} B d\zeta = 0. \tag{4.47}$$

Case 5: Assume $k = 4$, i.e., $t_k = t_4 = \xi + \frac{1}{\eta}$ and let $\phi_3 = \psi^T e^{-A(\gamma t_5 + \xi + \frac{1}{\eta})}$.

Using (4.36), we get that

$$\int_{t_i-(\xi+\frac{1}{\eta})}^{t_{i+1}-(\xi+\frac{1}{\eta})} \phi_4 e^{-A\zeta} B d\zeta = 0 \quad \text{for } i \in \{4,5,6,7,8\}. \tag{4.48}$$

See that $t_k - (\xi + \frac{1}{\eta})$ is of the following form

$$0, \ \frac{1}{\eta} - \xi, \ \frac{1}{\eta}, \ \frac{2}{\eta} - \xi, \ \frac{1}{\eta} + \xi, \ \frac{2}{\eta},$$

for $k = 4,5,6,7,8,9$ respectively. From Lemma 4.1.6, for any $i \in \mathbb{Z}$,

$$\int_{\frac{i}{\eta}}^{\frac{i}{\eta}-\xi} \phi_4 e^{-A\zeta} B d\zeta = b_1 \int_{\frac{1}{\eta}}^{\frac{1}{\eta}-\xi} \phi_4 e^{-A\zeta} B d\zeta + b_2 \int_{\frac{2}{\eta}}^{\frac{2}{\eta}-\xi} \phi_4 e^{-A\zeta} B d\zeta = 0,$$

where b_1, b_2 some real numbers and for $i = 0$, we have

$$\int_0^{-\xi} \phi_4 e^{-A\zeta} B d\zeta = 0 \implies \int_{-\xi}^{\frac{1}{\eta}-\xi} \phi_4 e^{-A\zeta} B d\zeta = 0.$$

By using last identity, we have

$$\int_{i\xi}^{\frac{1}{\eta}+i\xi} \phi_4 e^{-A\zeta} B d\zeta = b_1 \int_0^{\frac{1}{\eta}} \phi_4 e^{-A\zeta} B d\zeta + b_2 \int_{-\xi}^{\frac{1}{\eta}-\xi} \phi_4 e^{-A\zeta} B d\zeta = 0,$$

where b_1, b_2 real numbers coming from Lemma 4.1.6. For $i = 1$,

$$\int_{\xi}^{\frac{1}{\eta}+\xi} \phi_4 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{\xi} \phi_4 e^{-A\zeta} B d\zeta = 0.$$

Hence, we have for any $i \in \mathbb{Z}$,

$$\int_0^{-\xi} \phi_4 e^{-A\zeta} B d\zeta = \int_0^{\xi} \phi_4 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{i\xi} \phi_4 e^{-A\zeta} B d\zeta = 0,$$

and

$$\int_0^{\frac{1}{\eta}} \phi_4 e^{-A\zeta} B d\zeta = \int_0^{\frac{2}{\eta}} \phi_4 e^{-A\zeta} B d\zeta = 0 \implies \int_0^{\frac{i}{\eta}} \phi_4 e^{-A\zeta} B d\zeta = 0,$$

which both concludes that

$$\int_0^{\frac{j}{\eta}+i\xi} \phi_4 e^{-A\zeta} B d\zeta = \sum_{v=-1}^{0} \sum_{w=0}^{1} r_v l_w \int_0^{\frac{w}{\eta}+v\xi} \phi_4 e^{-A\zeta} B d\zeta = 0, \tag{4.49}$$

for some constants l_i, r_i.

Hence, for all possible five cases for the remainder part of t_s, we have the following

$$\int_0^{\frac{j}{\eta}+i\xi} \phi_k e^{-A\zeta} B d\zeta = 0 \quad \text{for any } i,j \in \mathbb{Z}, \ \text{ and } \ k = 0,1,2,3,4. \tag{4.50}$$

Since this is true for all $i, j \in \mathbb{Z}$, we can say that

$$\int_0^{\frac{j}{\eta}+i\xi} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for any } i, j \in \mathbb{Z}. \tag{4.51}$$

Hence, we conclude from Lemma 3.1.3 that

$$\int_0^{t} \psi^T e^{-A\zeta} B d\zeta = 0 \quad \text{for any } t \in \mathbb{R}^+, \tag{4.52}$$

means (1.2) is not controllable for the case $n = 2$ which gives us a contradiction. Therefore, there is no such $s \in \mathbb{N}$ violates the terms of statement.

For observability, the result can be derived from duality. Namely, we consider five cases and follow similar approaches as shown in Corollary 4.3.4. □

Remark 4.3.7. *From Corollary 4.3.6, we understand that for the transfer of observability/controllability from continous system (1.2) to the sampled-data system (1.3), it is sufficent to have only $\nu = 5$ number of steps in two dimensional control systems. The bound for step size ν is sharpened when compared with Corollary 4.3.4. Moreover, step size is decreased if it is compared with the simple periodic nonequidistant time sampling presented in the work [33] which should be at least $\nu = 6$ in two dimensional control systems.*

4.4 On Small/Large Upper Diameter of the Sampling Schedule

In this section, we analyse the upper diameter of the sampling pattern defined in (4.10) and see that it is possible to build a small or large upper bound for the gaps between consecutive terms of the time schedule, depending on the problem under consideration, by changing the role of our two candidates ξ and η satisfying the relation (4.4). Here, we note that designing controllers for sampled-data systems by using the emulation approach, it is necessary to have sufficiently small sampling periods for the pattern. For this design, it is significant to choice the sampling period which admits maximum allowance that guarantees stability of the system. But, in our case, we are dealing with sampled data system under the non-equidistant samplings, i.e., uncertain sampling schedules, distances between consecutive time samplings in the schedule are always changing. In this part, we design the upper bound for the constructed sampling pattern (4.10) and observe that the bound depends on the magnitude of ξ or η for the desired purposes. Now, we define the upper bound for the schedule in the following way (4.10):

$$T := \sup_{i\in\mathbb{N}}(t_{i+1} - t_i) \tag{4.53}$$

Since, the construction (4.10) is periodic with period length $n^2 + 1$, it is possible to rewrite (4.53) as

$$T = \max_{t_i\in S}(t_{i+1} - t_i) \tag{4.54}$$

It can be seen from section 4.2, through both parts of the construction together with (4.7) that possible gaps in between consecutive terms would be either

$$\frac{1}{\eta} - \xi \quad \text{or} \quad (n-k)\xi - (n-k-1)\frac{1}{\eta} \quad \text{for} \quad k = 0, 1, \cdots, n-1 \tag{4.55}$$

Hence, by using the relation (4.4), among the candidates from (4.55) the maximum value is attained when $k = n - 1$, which would be ξ. Therefore, the upper bound for the sampling schedule would be $T = \xi$. Moreover, we know from (4.4) that

$$\frac{1}{\eta+1} < \xi < \frac{1}{\eta} \quad \text{and} \quad \eta \geq n \tag{4.56}$$

which means ξ becomes close to zero for large values η. This provides us an advantage since the gap which depends on ξ could be too small depending on the choose of η.

Now, let us consider the reverse case. More precisely, we ask the following question: whether it is possible to design a sampling pattern like (4.10) in such a way that the gap between consecutive terms becomes as large as we want. The answer to this question would be affirmative. For this purpose, we roughly build the new set by switching the role of ξ and η in (4.10) with the following sense

$$\hat{S} := \{i\eta + j\frac{1}{\xi} \;\Big|\; i, j \in \mathbb{N},\ \eta\xi < 1 < (\eta+1)\xi \quad \text{with} \quad \eta \geq n\} \tag{4.57}$$

Of course, before using the set $\hat{S}$ for later purposes, it should be adapted to the construction parts defined in section 4.2 and its periodic extension. Moreover, the idea of Kronecker's theorem would be still valid for this framework which provides main results in section 4.3.

By using previous discussion, in this case, possible gaps between successive terms would be either

$$\frac{1}{\xi} - \eta \quad \text{or} \quad (n-k)\eta - (n-k-1)\frac{1}{\xi} \quad \text{for} \quad k = 0, 1, \cdots, n-1, \tag{4.58}$$

which provides us an upper bound for the new sampling pattern (4.57) as $T = \eta$ and its behaviour totally depends on ξ. In other words, the upper bound gets larger values whenever ξ gets closer to zero.

As a result, both construction of sampling patterns creates a flexible ground for the problem under consideration in the context of sampled-data systems. As an example, we may consider the problem of designing an observer which exponentially estimates the state of a flexible joint robotic arm proposed in [44] for $n = 4$ with upper diameter of sampling schedule given as 0.1. By adapting these assumptions, we let $\eta \geq 10$ which makes upper bound $T = \xi \leq 0.1$ in our case. Hence, for that design, it is feasible to use the sampling schedule proposed in (4.10).

4.5 Controllability of the Pendulum with Digital Controls

In this part, we provide one application of the research about the control problem related to the pendulum (see [22]).

Let us consider the following control problem:

$$\ddot{\phi}(t) - \phi(t) = u(t), \tag{4.59}$$

where $\phi(t)$ is angle of displacement of the pendulum and $u(t)$ is the control. The problem is deriving $(\phi, \dot{\phi})$ to desired state $(0, 0)$ for all small initial data without interrupting its nature. Here, we represent second order system (4.59) as a finite dimensional system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t > 0, \tag{4.60}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the state $x(t) = (\phi(t), \dot{\phi}(t)) \in \mathbb{R}^2$ for $t \in \mathbb{R}^+$. Observe that Kalman's rank condition satisfies for the system (4.60), namely, $rank[B : AB] = 2$, which is of full rank. Hence, the continuous time system (4.60) is controllable for any final time T.

Applying Theorem 4.3.2 to the system (4.60), and using Corollary 4.3.6, we can say that the corresponding sampled-data system is controllable with at least $\nu = 5$, number of steps or controls. Let us design our controls for controllability. But before that, we design our sampling (4.9) for the case $n = 2$:

$$t_m = \gamma\frac{2}{\eta} + t_k, \text{ where } m = 5\gamma + k \text{ for } k \in \{0, 1, 2, 3, 4\},$$

where t_k is of the form (4.34). Since (4.60) is controllable for any time T, the corresponding sampled-data system is controllable, i.e., for any x^0 initial and x^1 final data, there exists control sequence $\{u_s, u_{s+1}, u_{s+2}, u_{s+3}, u_{s+4}\}$ such that $x_s = x^0$ and $x_{s+5} = x^1$:

$$e^{-At_{s+5}}x^1 - e^{-At_s}x^0 = \sum_{i=s}^{s+4}\int_{t_i}^{t_{i+1}} e^{-A\zeta}Bd\zeta\, u_s, \tag{4.61}$$

where $u(t_i) = u_i \in \mathbb{R}$ for $i \in \{0, 1, 2, 3, 4\}$, each of which is constant along the interval $[t_i, t_{i+1})$.

It is shown in Corollary 4.3.6 that for any $s \in \mathbb{N}$,

$$span\{v_s, v_{s+1}, v_{s+2}, v_{s+3}, v_{s+4}\} = \mathbb{R}^2,$$

where

$$v_{s+i} = \int_{t_{s+i}}^{t_{s+i+1}} e^{-A\tau}Bd\tau \quad \text{ for } i = 0, 1, \cdots, 4. \tag{4.62}$$

Let us fix the initial data x^0. Among vectors in (4.62), one can find two linearly independent vectors with the least index numbers, say v_α and v_β where $\alpha, \beta \in \{s, s+1, \cdots, s+4\}$ with $\alpha > \beta$. Hence, any vector in $\mathbb{R}^2$ can be written as a linear combination of v_α and v_β. Namely,

$$-e^{-At_s}x^0 = c_1v_\alpha + c_2v_\beta \quad \text{for some} \quad c_1, c_2 \in \mathbb{R}.$$

Hence, we provide the following strategy for building controls:

$$u_\alpha = c_1, \quad u_\beta = c_2, \quad \text{and} \quad u_i = 0 \quad \text{for} \quad i \notin \{\alpha, \beta\}. \tag{4.63}$$

Under the control sequence (4.63), we have that

$$\begin{aligned} e^{-At_{s+5}}x_{s+5} - e^{-At_s}x^0 &= \sum_{i=s}^{s+4}\int_{t_i}^{t_{i+1}} e^{-A\tau}Bd\tau\, u_i \\ &= c_1v_\alpha + c_2v_\beta, \end{aligned} \tag{4.64}$$

which means that the final state x_{s+5} achieves the equilibrium, i.e., $x_{s+5} = 0$.

As a result, we conclude that the control sequence $(u_s, u_{s+1}, \cdots, u_{s+4})$ constructed in (4.63) preserves the controllability of the continuous time system (4.59) after the discretization.

4.6 Samplings Compared with the Literature

This section is devoted to two types of sampling schedule and their relations with Theorem 4.3.1 and Theorem 4.3.2.

Shifted sampling pattern: For any $\kappa \in \mathbb{R}^+$, consider the following sampling

$$\hat{t}_m = \kappa + t_m \quad \text{where} \quad t_m \in S. \tag{4.65}$$

The idea is to shift the sampling (4.9) by a constant κ. The construction (4.65) can be useful and more practical for the discretization of continuous system (1.2) with initial time $t_0 = \kappa$. Here, by following similar approaches performed in Section 4.3, it can be shown that Theorem 4.3.1 and Theorem 4.3.2 are both valid with shifted sampling (4.65).

Periodic nonequidistant sampling pattern: The following sampling is the special form of sampling pattern proposed in [33]

$$t_m = \frac{j}{\eta} + i\xi \quad \text{where } m = (n+1)j + i \quad \text{for } i = 0, 1, 2 \cdots, n \text{ and } m, i \in \mathbb{Z}_0^+. \tag{4.66}$$

Here, n is the dimension of the state space of (1.2), ξ is an irrational number and $\eta \in \mathbb{N}$ satisfying the inequality $\eta\xi < \frac{1}{n}$. Observe from (4.66) that the construction is non-equidistant and periodic, i.e., each period has a block of n successive time intervals of length ξ continued with one time interval with the length $\frac{1}{\eta} - n\xi$. It can be shown that Theorem 4.3.1 and Theorem 4.3.2 hold for (4.66). Moreover, the number of sampling times which is required to observe/control the sampled-data system is at least $\nu = n(n+1)$.

CHAPTER 5

CONCLUSIONS

In conclusion, problems related to controllability and observability of some discrete models are investigated and in this framework, novel results are derived when compared with the literature. More precisely, we design perspectives proposed in Chapter 3 and Chapter 4 respectively which are applied to problems gathered from control theory: controllability of the pendulum in two dimensional setting with digital controls [28], controllability problem of a convex polyhedra with finite number of controls in one dimensional setting [9, 14], the problem of state observation of a flexible joint robotic arm in four dimensional setting proposed in [44] with the constructed samplings. In addition, proposed sampling pattern could be used in problems related to building state observers [2, 3, 32, 51]. As a result, we believe that the work would provide fruitful environment in the context of controllability and observability of discrete systems for both readers and the author himself.

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PUBLICATIONS

International Journal Publications

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2. Hamidoğlu, A. "On general form of the Tanh method and its application to nonlinear partial differential equations", Numerical Algebra, Control and Optimization, 6(2), 175–181, (2016).

3. Hamidoğlu, A. "Null Controllability of Heat Equation with Switching Controls under Robin's Boundary Condition", Hacettepe Journal of Mathematics and Statistics, 45(2), 373-379, (2016).

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International Conference Publications

1. Hamidoğlu, A. "A Strategy for Finding Integer Almost Periods in the Theory of Almost Periodic Functions", Young Invited Speaker in 3rd Caucasian Mathematics Conference (CMC-III), Rostov-Don/Russia, (2019).

2. Hamidoğlu, A. "A New Strategy for Finding Switching Controls in Controllability Problems", The 6th International Conference on Controls and Applications (COIA 2018), 2, 135–137, Baku/Azerbaijan, (2018).

3. Hamidoğlu, A. "On Generalization of Tanh Method and Its Application to Nonlinear Partial Differential Equations", 28th European Conference on Operational Research, Poznan/Poland, (2016).