

DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**DEVELOPMENT OF A WAVELET BASED
TECHNIQUE FOR VIBRO-ACOUSTIC ANALYSIS
OF CONNECTED STRUCTURES**



by
Murat KARA

November, 2019
İZMİR

DEVELOPMENT OF A WAVELET BASED TECHNIQUE FOR VIBRO-ACOUSTIC ANALYSIS OF CONNECTED STRUCTURES

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Program**

**by
Murat KARA**

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İZMİR**

Ph. D. THESIS EXAMINATION RESULT FORM

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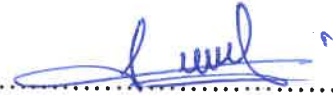
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Murat KARA

DEVELOPMENT OF A WAVELET BASED TECHNIQUE FOR VIBRO-ACOUSTIC ANALYSIS OF CONNECTED STRUCTURES

ABSTRACT

Discrete Singular Convolution (DSC) method is proposed as a wavelet based technique having global methods' accuracy with local methods' flexibility. In recent years, researches on the development of the method are still proceeding. Although performed improvements increase the use of the method, there are still unsolved problems via the DSC, i.e. implementation of impedance boundary condition, connecting non-stepped structures, combining the DSC with a proper uncertainty analysis tool to analyse uncertain structures. This doctorate thesis mainly deals with the development of a DSC algorithm to analyse uncertain connected structures.

The original studies in this thesis may be grouped in three categories, i) solution of individual structures with impedance boundary conditions, ii) analysis of connected structures, iii) uncertainty analysis of individual / connected structures. For the first category, bars and acoustic ducts having impedance boundary are selected as test structures. For the second category, algorithms to connect individual structures are proposed and tested on three different systems, i.e., i) acoustic transmission lines, ii) connected beams having different connection orientations, iii) stiffened plate constructed by connecting a beam and a plate. Polynomial Chaos Expansion (PCE) is combined with the DSC method for the third category.

All results are validated with any of analytical solutions (if applicable), numerical methods and reference studies from the literature. It is proper to say that, the proposed methodologies present unique advantages in the vibration / acoustic analysis of structures.

Keywords: Discrete Singular Convolution, Taylor series expansion, Polynomial Chaos Expansion, connected structures, uncertainty analysis

BAĞLI YAPILARIN VİBRO-AKUSTİK ANALİZİ İÇİN DALGACIK TABANLI BİR TEKNİK GELİŞTİRİLMESİ

ÖZ

Ayrık Tekil Konvolüsyonu (ATK) yöntemi global yöntemlerin hassasiyetine ve lokal yöntemlerin esnekliğine sahip dalgacık tabanlı bir yöntem olarak geliştirilmiştir. Son yıllarda, yöntemin geliştirilmesine yönelik araştırmalar devam etmektedir. Yapılan geliştirmelerin yöntemin kullanımını yaygınlaştırmasına rağmen, empedans sınır koşulunun uygulanması, basamak olmayan yapıların bağlanması, belirsiz yapıların analizi için ATK ile uygun bir belirsizlik analiz yönteminin birleştirilmesi gibi problemler ATK ile hala çözülememektedir. Bu doktora tezi temel olarak belirsiz bağlı yapıları analiz etmek için ATK algoritması geliştirilmesi ile ilgilenir.

Bu tezdeki orijinal çalışmalar, üç kategoride gruplandırılabilir, i) empedans sınır koşullu tekil yapıların çözümü, ii) bağlı yapıların analizi, iii) tekil / bağlı yapıların belirsizlik analizi. İlk kategori için, empedans sınır koşullu çubuk ve akustik kanallar test yapıları olarak seçilmiştir. İkinci kategori için, tekil yapıları bağlamak amacıyla algoritmalar önerilmiş ve üç ayrı sistem üzerinde test edilmiştir, i) akustik iletim hatları, ii) farklı bağlantı oryantasyonlarına sahip bağlı çubuklar, iii) bir çubuk ile bir plakanın bağlanması ile oluşturulan güçlendirilmiş plaka. Üçüncü kategori için ise Çok terimli Kaos Açılımı (ÇKA), ATK ile bir araya getirilmiştir.

Tüm sonuçlar analitik çözümler (eğer var ise), sayısal yöntemler ve literatürdeki referans çalışmalar ile karşılaştırılmıştır. Önerilen metodolojilerin yapıların titreşim / akustik analizlerinde eşsiz avantajlar sunduğunu söylemek uygundur.

Anahtar Kelimeler: Ayrık Tekil Konvolüsyonu, Taylor serisi açılımı, Çok terimli Kaos Açılımı, bağlı yapılar, belirsizlik analizi

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CHAPTER ONE

INTRODUCTION AND LITERATURE REVIEW

1.1 Literature Review

Vibro-acoustic analysis is one of the most important issues in design of aerospace, marine and automotive structures. It includes the determination of vibration and acoustic characteristics of mechanical systems in order to control their dynamic behaviours considering safety, durability and stability. The analysis is generally carried out via three different manners; as i) analytical, ii) numerical and iii) experimental. Table 1.1 presents a simple classification for some common methodologies used in vibro-acoustic analysis.

Table 1.1 General classifications of methodologies considered in vibro-acoustics analyses (FEM: Finite Element Method, BEM: Boundary Element Method, FDM: Finite Difference Method, DSC: Discrete singular convolution, MC: Monte Carlo Analysis, PCE: Polynomial Chaos Expansion, SMA: Statistical Moments Approach, FORM, SORM: First and Second Order Reliability Methods)

Analytical Methods	Numerical Methods				Experimental Methods
	Local			Global	
	Deterministic	Statistical			
		Probabilistic	Non-Probabilistic		
Exact Solutions of Partial Differential Equations (PDEs)	FEM, BEM, FDM, DSC	FEM-Monte Carlo, PCE, SMA, FORM, SORM, SEA	Anti-optimization, Fuzzy, Interval Analysis	Galerkin, Rayleigh-Ritz, DSC	Modal Analysis, Impact Hammer Test, Frequency Sweeping Method, Power Injection Method

Analytical evaluations are generally possible only for some waveguides (longitudinal, torsional, bending etc.) of academic structures such as strings, beams, plates, shells etc. with simple boundary conditions. They are mostly used for either

for the validation study of numerical methods before analysing complex structures or simplified models of exact structures for example; a plane wing may be modelled as a cantilever beam.

For relatively more complex structures (with complex boundaries, arbitrary shaped or built-up), numerical and experimental techniques should be considered for more realistic analyses. An appropriate experimental setup may not be feasible for large structures since they are generally expensive to construct, and test reliability is strongly dependent on the expertise of the test performer as well as the test environment. Mostly, reliability of the tests is supplied with experimental repeatability (is running the same test for several times) of the results. In repeatable tests, the results must be in a high confidence level, preferably, 90% or higher. Therefore, experimental analysis is also time consuming when the required time for evaluating the results is compared to the numerical methods.

Due to the mentioned disadvantages of analytical and experimental methods, numerical approaches are much more preferred among researchers. Numerical methods may be grouped in several types. The most suitable one is to group them as global and local methods as given in Table 1.1. Galerkin approach (Rao, 2011b), Rayleigh-Ritz approach (Rao, 2011a) are the most known global methods. In these techniques, a set of functions are assumed as basis of the response, and coefficients of these basis functions are estimated by some mathematical manipulations. Global methods are highly accurate in spectral (frequency dependent) calculations compared to local methods, but they are less accurate for complex geometries such as arbitrary shaped structures. Approximate deterministic techniques such as finite element and boundary element methods (FEM, BEM) are two of the most widely used local techniques. These techniques are available to estimate local response. However, they are limited to be used in low frequency range since higher frequencies are computationally expensive and their theory may fail at these frequencies.

In numerical analysis, the statistical methods must also be used in the analysis due to the uncertainty phenomena. Although uncertainty will be explained in the

following pages in detail, it is helpful to give a brief description in order to express importance of the statistical methods. Uncertainty is evaluating different characteristics from sample to sample of the products produced in the same production line. Uncertainty in the structures is critical especially in high frequency vibro-acoustics of aerospace engineering products.

At this point, it is useful to define the frequency ranges, which are abstractly constructed in vibro-acoustic analyses:

a) Low Frequency Region: Vibro-acoustic response is dominated by resonant peaks. The response is highly sensitive to the boundary conditions. Deterministic techniques such as FEM and BEM are reliably used. Le Bot et. al. (2010) presents typical values of maximum frequency for a finite element model of order of several millions degrees of freedom as given in Table 1.2. It is also stated by Marburg (2008) and Langer et. al. (2017) that, six or higher number of elements per wavelength are required in FEM for the evaluation of a consistent result.

Table 1.2 Typical values of maximum frequency for a finite element model of order of several millions degrees of freedom (Le Bot et al., 2010)

Structures	Maximum Frequency
Automotive	500 Hz
Aircraft	50 Hz
Large Ship	10 Hz
Launcher, Building	1 Hz

b) High Frequency Region: The vibration response is smooth due to modal overlapping which is defined as a measure of overlapping modes in a certain frequency band. Boundary conditions do not affect the response much. The response is sensitive to variations in material and local changes. Statistical and energetic methods are used in predictions. In this range, Statistical Energy Analysis (Lyon & DeJong, 1995), Power / Energy Flow Method (Fredö, 1997; Vlahopoulos & Zhao, 2001), Wave Intensity Technique (Langley, 1992), Energy Finite Element Method (EFEM) (Zhang, Vlahopoulos, & Kuangcheng, 2005) are mostly used techniques.

c) Mid Frequency Region: It is a transition region including the characteristics of both low and high frequency ranges. Several hybrid methods combining low and high frequency techniques in different manners are developed (De Rosa & Franco, 2010; Ji, Mace, & Pinnington, 2006; Reynders, Langley, Dijckmans, & Vermeir, 2014; Seçgin, Kara, & Ozankan, 2016; Vanmaele, Vandepitte, & Desmet, 2007; Vlahopoulos & Zhao, 2001). Beside that, there are also studies to extend SEA to lower frequencies. Various authors have decided to adopt an hybrid approach by mixing the FEM with SEA (Cotoni, Shorter, & Langley, 2007; Langley & Cordioli, 2009; Seçgin et al., 2016; Shorter & Langley, 2005). The finite element part of the implementation is used to compute the low frequency response while the SEA part tackles its high frequency counterpart. However, a real gap still exists between the methods in terms of frequency applicability. The component mode synthesis is used by Cotoni et. al. (2008) to reduce the computational cost of the finite element problem while bringing some benefits to the SEA approach. However, researches still continue to develop more efficient techniques for mid frequency region. Seçgin and his colleagues introduced a modal impedance technique for mid and high frequency vibration analysis for connected systems (Seçgin, Kara, & Güler, 2019; Seçgin et al., 2016; Seçgin, Kara, & Ozankan, 2019). Guyader and his co-workers (2001; 2012) proposed Statistical Modal Energy Distribution Analysis (SMEDA) method to determine energy densities of coupled subsystems. Although the boundary condition implementation of subsystems via SMEDA and the modal impedance technique (Seçgin, Kara, & Güler, 2019; Seçgin et al., 2016; Seçgin, Kara, & Ozankan, 2019) seems similar, there are some main differences between the methodologies in the calculation of coupling loss factor (CLF), input power and energies of subsystems. Briefly, SMEDA computes CLFs regarding on modal approach of SEA. However, modal impedance technique sticks with the wave approach of SEA, thus uses modal data to compute impedances required in the wave approach. A useful comparison between most commonly used numerical approaches given by Seçgin (2008) is presented here in Table 1.3.

Table 1.3 Specifications and capabilities of conventional vibro-acoustic methods (Seçgin, 2008)

Criterion	FEM*, BEM*	SEA*	EFA (EFEM*)	RTM*
Principle	-Energy minimization principle	-Energy (Power) flow balance principle -Statistical modeling principle	-Vibration conduction principle -Energy flow balance principle	-Directional energy flow balance principle
Element modelling	-Using finite and boundary elements	-Using modal or geometrical macro elements	-Using finite and boundary elements	-Circular ray tubes
Requirements	-Sufficient number of small elements -Sufficient HD and CPU	-High modal density -High modal overlap count -Accurate determination of CLF and DLF	-High modal density -High modal overlap count -Sufficient HD and CPU	-High modal density -High modal overlap count -Specular boundaries
Excitation frequency range	-Low frequency	-High frequency	-High frequency	-High frequency
Frequency bandwidth	-Discrete or narrow bandwidth	-Wide bandwidth	-Wide bandwidth	-Wide bandwidth
Response characteristics	-Discrete (time, space, frequency)	-Average (time, space, frequency)	-Discrete (space) -Average (time, frequency)	-Discrete (directional, space) -Average (time, frequency)
Loss of local information	-Low	-High	-Low	-High
Strong fluid-structure interaction	√	-	√	?
Response parameters	-Displacement -Velocity -Acceleration	-Modal energy	-Modal energy	-Modal energy
Prediction of modal behavior	√	-	-	-
Applications	-Simple -Complex structures	-Simple -Complex structures	- Simple -Complex structures	-Simple structures
Numerical algorithm	-Meshing -Standard equation solver	-Sub-structuring -Standard equation solver	-Thermal FEM algorithm -Standard equation solver	-Efficient ray tracing algorithm
Computation cost	-Depending on the number of elements	-Low	-Depending on the number of elements	-High

*FEM: Finite element method, BEM: Boundary element method, SEA: Statistical energy analysis, EFEM: Energy finite element analysis, RTM: Ray tracing method

Apart from those methods, a wavelet based method called as Discrete Singular Convolution (DSC) introduced by Wei (1999) proved itself as a powerful tool in

vibration analysis. Since the thesis takes the DSC on board as a main approach, it is sensible here to discuss the DSC and its main applications related to the thesis. The DSC is based on wavelets and distribution theory. Before introducing the DSC, it is useful to give a brief information on the wavelets and its advantages. As it is known, wavelet is a waveform that has zero mean and limited duration. From the mathematical point of view, any function can be generated by dilation and translation of wavelets. A disadvantage of the Fourier analysis is that frequency information can be extracted for the complete duration of signal of $f(t)$. Since the Fourier transform of a signal $F(\omega)$ is calculated over all time, the information it provides arises from an average over the whole length of the signal. If there is a local oscillation in time history, this will be seen from Fourier transform, but its location on time axis will be lost. This disadvantage is overcome by wavelet analysis, which it provides an alternative way of breaking a signal into its parts. Today, the wavelets are widely used in various areas including dynamic analysis, speech and image processing, etc.

The DSC showed its accuracy in vibro-acoustics. This method has been successfully used for solving several different kinds of partial differential equations; as in quantum mechanics (Wei, 1999, 2000a, 2000b), non-linear differential equations (Feng & Wei, 2002; Wei, 2000a), vibrations (Wei, 2001c, 2001a, 2001b; Wei, Zhao, & Xiang, 2001, 2002b; S. Zhao, Wei, & Xiang, 2005; Y. B. Zhao & Wei, 2002), flow analysis of incompressible fluids (Wan, Patnaik, & Wei, 2002; Wei, 2001a) and solution of Helmholtz equation with high wavenumbers (Bao, Wei, & Zhao, 2004). Civalek and co-workers have also made great contribution to the development of the DSC especially for the vibration analysis of laminated plates (Baltacıoğlu, Akgöz, & Civalek, 2010; Baltacıoğlu, Civalek, Akgöz, & Demir, 2011; Civalek, 2006, 2007, 2013; Gürses, Civalek, Korkmaz, & Ersoy, 2009) and of nano-structures (Civalek, 2017; Gürses, Akgöz, & Civalek, 2012; Mercan & Civalek, 2016). There are also important studies on the application of the DSC to wave propagation problems (Li, Wang, Lu, Zhang, & Li, 2012; Wang, Xu, & Xu, 2010), transient analysis (Wan et al., 2002; Wang et al., 2010; Wei, 2001a), free vibration analysis of symmetrical laminated composite plates (Seçgin & Sarigül, 2008) and high frequency vibration analyses (Seçgin & Sarigül, 2009; Wei, Zhao, & Xiang,

2002a). Beside that, Shokrollahi et. al. (2014) investigated the natural frequencies of non-uniform bars those have different combinations of cross-sections via the DSC. Civalek et. al. (2008b, 2008a, 2009; 2010; 2009) applied the DSC to more complex non-uniform structures i.e. shell and membranes. However, the DSC suffers from providing accuracy in the modelling of complex boundary conditions when the classical boundary condition implementation is applied.

The classical boundary condition procedure uses a form of linear function to relate ghost and structural points to satisfy Dirichlet and Neumann boundary conditions, which arise in the DSC algorithm as symmetric and anti-symmetric extensions. This procedure successfully works for the simply-supported (pinned) and fixed boundaries of the bending beams/plates at which the displacement function is zero. However, it fails in multiple boundary conditions such as free edges and Robin boundary, which holds a coupled condition that the derivative of a function depends on the function itself. Impedance boundary in vibro-acoustic is one of the well-known types of Robin boundary.

In handling free edges, Zhao et. al. (2005) introduced an iteratively matched boundary method for the treatment of free edged beams using the DSC. In this method, an iterative way with finite difference approximations is employed as such that only the number of ghost points is handled as much as the number of boundary conditions at each stage. This approach yields accurate predictions of the first few modes in vibration analysis; however, it is not much accurate in higher modes and not so straightforward for the applications. Recently, Wang et. al. (2017; 2010; 2017; 2011) have introduced another implementation procedure for free boundaries based on Taylor series expansion (TSE). The approach determines the natural frequencies of thin beams and thin plates even in higher modes successfully. These studies ignited a spark for other researchers to analyse the structures with complex boundary conditions those are unsolvable with classical boundary condition implementation procedure (Gibigaye, Yabi, & Degan, 2018; Kara & Seçgin, 2019). In this manner, Gibigaye et. al. (2018) used DSC-TSE method for the free vibration analysis of plates restrained on elastic Vlasov soil type foundation. Kara and Seçgin (2019)

expand the DSC-TSE method by solving one dimensional bar and acoustic duct with impedance boundary problem.

Up to this point, the studies on the DSC are on individual structures. However, a numerical method must also solve not only individual structures (bar, duct, beam, plate, etc...) but also complex structures (connected individual structures). Firstly, Duan and Wang (2013, 2014) proposed a DSC based method to analyse the step beams and plates by setting a n th order polynomial at each side of step. However, this method is not easy to employ. Later, the DSC method with Taylor Series Expansion (called as DSC-TSE approach) is also implemented on static, buckling and free vibration analysis of step beams (Wang & Duan, 2014) and circular thin plates with stepped thickness (Duan, Wang, & Jin, 2014).

In vibro-acoustics, it is also important to state here a phenomenon called as “uncertainty”. In engineering sense, “uncertainty” is generally defined as uncontrollable variability in the dynamic response properties and it is unavoidable due to reasons such as production tolerances, manufacturing defects, environmental factors, etc. Even if each sample of uncertain structure shows small variability from the desired property, samples of uncertain structure may show large-scale variation in dynamic response (Fahy, 1994). Fahy (1994) exposed the uncertainty by measuring structure-borne sound of driver position in 57 samples of Isuzu pick-up trucks (shown in Figure 1.1) which are produced in the same production line.

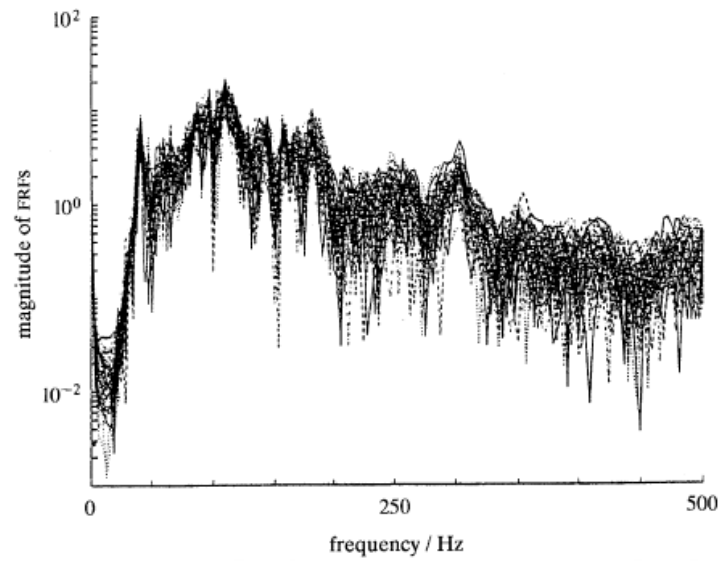


Figure 1.1 Uncertain acoustic response of pickup trucks (Fahy, 1994)

For reliable analysis, engineers or designers have to consider these uncertainties in all stage of the production. In this regard, dynamics of uncertainty propagation should be well modeled and thus an accurate prediction should be performed via one of the proper probabilistic or non-probabilistic approaches as presented in Table 1.1.

Many researches exist in literature for uncertainty propagation problems. It is more appropriate to categorize these studies based on their methodologies. Uncertainty propagation methods can be collected in two packages; probabilistic and non-probabilistic approaches, mainly.

Non-probabilistic methods such as interval analysis (Moore, 1979), anti-optimization technique (Elishakoff, Haftka, & Fang, 1994), fuzzy set theory (Hanss, 2013; Kumar & Schuhmacher, 2005) etc. are used in case where the statistical properties of uncertain parameters are not known but the limits of them are known. In such methods, only the limits of the uncertain response variable can be obtained.

Statistics and distribution of uncertain parameters are known in probabilistic methods such as Monte Carlo simulation (Evans & Swartz, 2000; Rubinstein & Kroese, 2016), first and second order reliability methods (Choi & Yoo, 2012; Hohenbichler & Rackwitz, 1989; Keane & Price, 1997), numerical integration based

methods (D. H. Evans, 1972; Lee & Kwak, 2006; Rahman & Xu, 2004; Seo & Kwak, 2002) and spectral methods (Ghanem & Spanos, 2003; Lucor, Su, & Karniadakis, 2004; Sepahvand, 2017; Sepahvand, Marburg, & Hardtke, 2007, 2010). In those techniques, uncertain parameters are modelled or sampled in accordance with their statistical properties, and response variable statistics are obtained by an appropriate mathematical tool. Nevertheless, numerous experiments need to be performed in order to evaluate statistics of the uncertain parameter, successfully. Among these methods, Monte Carlo (MC) based simulations are much straight forward for general application of vibro-acoustic systems, therefore they are most commonly used. However, due to high computation time and memory usage, and due to the fact that it does not include stochastic evaluation in response parameters, MC based simulations are still in question for researchers.

The spectral methods like Karhunen-Loeve (KL) expansion (Ghanem & Spanos, 2003) or Polynomial Chaos Expansion (PCE) (Lucor et al., 2004; Sepahvand, 2017; Sepahvand et al., 2007, 2010), which may also be called as functional methods (because they are able to formulate the uncertain response), are widely used in uncertainty analyses. It should be noted that, these methods are similar in basis since the first order PCE expansion is called as KL expansion. In PCE technique, distributions of uncertain variables are modelled using appropriate polynomial and desired response is calculated again in terms of a polynomial with constant coefficients. Selection of inappropriate distribution-polynomial type couple leads to high computational time since the higher order terms of the polynomial do not converge to zero. An advantage of this technique is applicability using with a numerical method i.e., finite element method (Dammak, Koubaa, El Hami, Walha, & Haddar, 2019; Ghanem & Spanos, 2003), finite difference method (Sepahvand et al., 2007) or differential equations (Sepahvand et al., 2010).

At this point, it is useful to quote the studies on uncertain structures those are handled via the DSC together with an uncertainty analysis tool. Seçgin et. al. (2013; 2012; 2018) combined the DSC with uncertainty analysing methods. He and his colleagues performed Monte Carlo simulations by using the DSC for a thin isotropic

plate and laminated composite plates in order to estimate the statistical bounds of vibration via extreme value model (Seçgin, 2013; Seçgin et al., 2012). In another study, Seçgin and Kara (2018) developed a closed form solution methodology called as statistical moment analysis, for the analysis of uncertain thin beams. In that study, the DSC presents a unique advantage since the characteristic matrix obtained via the DSC is independent from physical and mechanical properties of the considered structure. Then, the same approach is also extended for the analysis of thin isotropic and composite plates (Seçgin & Kara, 2019b, 2019a). However, there is only one study in literature on uncertain beams handled via combination of DSC and PCE (Kara & Seçgin, 2018) which is an output of this thesis.

1.2 Objective of the Thesis

In the scope of the literature review, it is mainly aimed to develop a technique having the following capabilities in this thesis;

- Global methods' accuracy with local methods' flexibility
- Applicable to connected structures
- Handling uncertainties without Monte-Carlo simulations.

As explained in Section 1.1, there are many numerical solution techniques on the calculation of the vibro-acoustic response. In the scope of Section 1.1, Table 1.4 has been prepared for more close comparison of a local method (finite element method), a global method (Galerkin approach) and a local and global method (discrete singular convolution). In this regard, following remarks inferred from Table 1.4 answer the question **“Why the DSC and PCE have been selected as numerical approaches to be improved to achieve the thesis objectives?”**;

Table 1.4 Comparison of some numerical local and global methods

Property	Finite Element Method (Local)	Galerkin Approach (Global)	Discrete Singular Convolution (Local+Global)
Reliable Frequency interval	lower frequencies	depends on number of basis function	full spectrum
Estimation of local information	+	-	+
Working on connected and coupled structures	+	-	limited
Accuracy	depends on number of elements	close to analytical results (nearly exact)	close to analytical results (nearly exact)
Estimation of uncertain structures	Monte Carlo simulation or different techniques	Monte Carlo simulation	Monte Carlo simulation or different techniques
Computation time	depends on number of elements	lower than finite element method	lower than finite element method

1. The DSC may consider wider frequency range when comparing with other methods.
2. The DSC provides local response as FEM does.
3. DSC application of connected structures is limited to stepped structures. Extending the application area of this technique for any connected structures such as L-type jointed beams, beam-plate structures, duct acoustic systems with high accuracy would attract a lot of attention for two basic reasons;
 - a. As it is known, Galerkin method cannot be applied for complex structures, since it uses mathematical model in minimizing the residuals of the basis functions which is impractical to apply.
 - b. Accurate FE model needs to have very small elements. In this case, the computational costs expand excessively. Therefore, the DSC may be superior to FEM.
4. Monte Carlo (MC) simulation is not efficient to analyse stochastic systems due to the limitations of computational costs. Polynomial Chaos Expansion (PCE) for stochastic analysis is more efficient (Sepahvand et al., 2010).

To sum up, at the end of the thesis, it is aimed to improve the DSC method to have capability of solving connected (relatively complex) structures having the uncertainty modelled via Polynomial Chaos Expansion (PCE).

1.3 Thesis Organization

This thesis consists of nine chapters including introduction and conclusions excluding Appendices.

Chapter 1: A comprehensive literature review and objective of the Thesis are given.

Chapter 2: Wave equations of some structural and acoustic waveguides are given briefly. Beside that, the boundary condition equations and the analytical natural frequencies are also presented.

Chapter 3: The philosophy of the DSC method, its grid scheme and boundary condition implementation procedures are expressed in detail.

Chapter 4: Application of the DSC method for the analysis of some waveguides are given. In this regard, longitudinal vibrations of bars with different boundary conditions including mechanical impedance boundary, sound propagation in ducts, bending vibrations of thin beams and plates are demonstrated.

Chapter 5: The DSC algorithm for connected structures, which is based on the continuity equations at the connection point, is expressed on the analysis of transmission lines.

Chapter 6: The numerical analysis of the individual continuous systems via the DSC method is presented. In this context, free and forced vibration / acoustic analyses of bars / ducts, free vibration analyses of inhomogeneous bending beams and free vibration analysis of bending plates are performed.

Chapter 7: The DSC methodology for the connected structures is applied on several vibration and acoustic structures. In this regard, considered structures are grouped on three main topics, i.e., i) acoustic transmission lines, ii) connected beams having I-, L- and T-type connection orientations, iii) stiffened plate that is constructed by connecting a beam and a plate.

Chapter 8: In this Chapter, analyses of uncertain structures are performed via DSC-PCE method. In this regard, firstly, an introduction to PCE is given. Then, the procedure of the PCE for the implementation of DSC is presented. Finally, numerical analyses of uncertain inhomogeneous individual beam and uncertain L-type connected beams are performed.

Chapter 9: This chapter gives a short review and underlines the outcomes of this Thesis.

Appendix 1: Stiffness matrices according to boundary condition implementation procedure are expressed.

Appendix 2: The procedure of two load method is given in order to determine the acoustic performance parameters of a muffler/insulation material.

Appendix 3: The PCE application for analytical solution of uncertain differential equations is presented.

Appendix 4: This section includes the list of symbols.

CHAPTER TWO

WAVE EQUATIONS OF STRUCTURAL AND ACOUSTIC WAVEGUIDES

2.1 Introduction

In engineering, a complex vibro-acoustic structure consists of continuous structural or acoustic elements such as string, duct, bar, beam, membrane, plate, etc... In analyses, each structural or acoustic element can be modelled via waveguides. Table 2.1 represents some of waveguides of the structural and acoustic elements. The wave motion of these elements can be expressed in terms of partial differential equations (PDEs). This chapter is devoted to PDEs and common boundary conditions of some waveguides.

Table 2.1 Waveguides of some structural elements

Structural element	Waveguide
Bar	Longitudinal
Acoustic duct	Longitudinal
Rod	Torsional
Beam	Bending
Membrane	Bending
Plate	Bending, Inplane

2.2 Longitudinal Vibration of Bars

In Figure 2.1a, an elastic bar with the length of l is shown. The PDE of non-uniform bars are expressed by (Rao, 2011a),

$$\frac{\partial}{\partial x} \left[E(x)S(x) \frac{\partial u(x,t)}{\partial x} \right] - \rho(x)S(x) \frac{\partial^2 u(x,t)}{\partial t^2} = -f(x,t) . \quad (2.1)$$

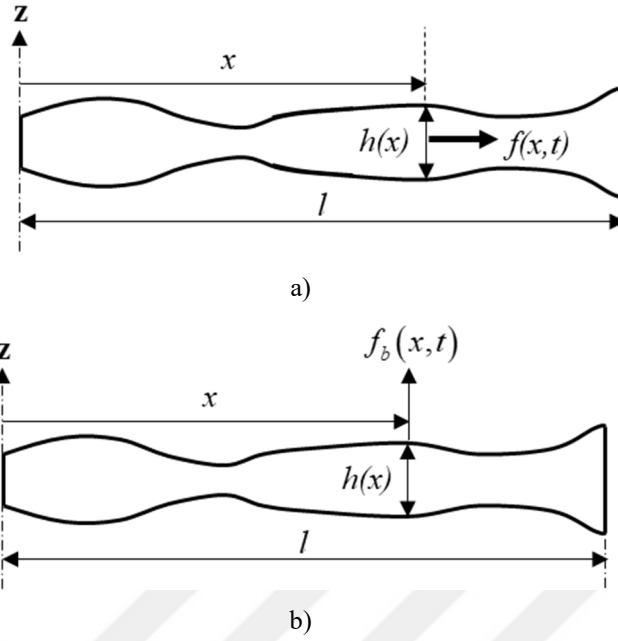


Figure 2.1 Schematic representation of a non-uniform bar/beam a) bar, b) beam

Here, $S(x)$ is the cross-sectional area, $E(x)$ is elasticity (Young) modulus, $\rho(x)$ is density, $f(x,t)$ represents the external force, $h(x)$ is thickness and $u(x,t)$ is the space and time dependent axial displacement.

For the uniform bar, the equation of motion (EoM) reduces to,

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c_L^2} \frac{\partial^2 u(x,t)}{\partial t^2} = -\frac{f(x,t)}{ES}, \quad (2.2)$$

where $c_L = \sqrt{E/\rho}$ is the longitudinal wave speed. Assuming harmonic time variant vibration displacement solution, $u(x,t) = U(x)\exp(j\omega_n t)$ by noting that $j = \sqrt{-1}$, homogenous form of Eq. (2.2) can be written as (Rao, 2011a);

$$\frac{d^2 U(x)}{dx^2} + \frac{\omega_n^2}{c_L^2} U(x) = 0, \quad (2.3)$$

where, ω_n is the natural angular frequency (r/s) of the bar. The solution of Eq. (2.3) depends on the boundary conditions of bars. The common boundary conditions (BCs) of bars are expressed as follows (Rao, 2011a):

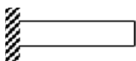
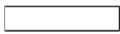
$$\text{Fixed end: } U(x) = 0, \quad (a)$$

$$\text{Free end: } ES \frac{dU(x)}{dx} = 0, \quad (b) \quad (2.4)$$

$$\text{Mechanical impedance (MI) end: } ES \frac{dU(x)}{dx} = a_1 Z_m U(x). \quad (c)$$

Here, $Z_m = k_s + j\omega_n c - m\omega_n^2$ is the mechanical impedance, k_s , c and m are the stiffness, damping element and mass at the boundary and a_1 is equal to 1 at $x=0$, whereas $a_1 = -1$ at $x=l$. Analytical natural frequencies and mode shapes can be found in Table 2.2 for fixed and free boundary condition combinations.

Table 2.2 Analytical natural frequencies and mode shapes of vibrating bar (Rao, 2011a)

BC	Schematic of BC	Natural frequency (ω_n)	Mode shape	n
Fixed-Fixed		$n\pi c_L / l$	$\sin\left(\frac{n\pi x}{l}\right)$	$1, 2, 3, \dots$
Fixed-Free		$n\pi c_L / 2l$	$\sin\left(\frac{n\pi x}{2l}\right)$	$1, 3, 5, \dots$
Free-Free		$n\pi c_L / l$	$\cos\left(\frac{n\pi x}{l}\right)$	$0, 1, 2, \dots$

2.3 Sound Propagation in Acoustic Ducts

Consider the duct shown in Figure 2.2. Sound propagates in a duct with the same intensity along the duct up to a certain frequency. That is a cut-off frequency of the plane wave condition. To provide plane wave condition, the wavelength of sound wave must be much larger than the diameter of the duct. The cut-off frequency is defined as (Munjial, 1987),

$$f < \frac{1.84}{\pi d} c_s, \quad (2.5)$$

where, c_s is the speed of sound, d is the diameter of duct and f is the frequency of propagating sound ($\omega/2\pi$) in Hz. Inhomogeneous sound pressure wave equation can be given as (Nefske & Sung, 2006);

$$\frac{\partial^2 p(x,t)}{\partial x^2} - \frac{1}{c_s^2} \frac{\partial^2 p(x,t)}{\partial t^2} = q(x,t). \quad (2.6)$$

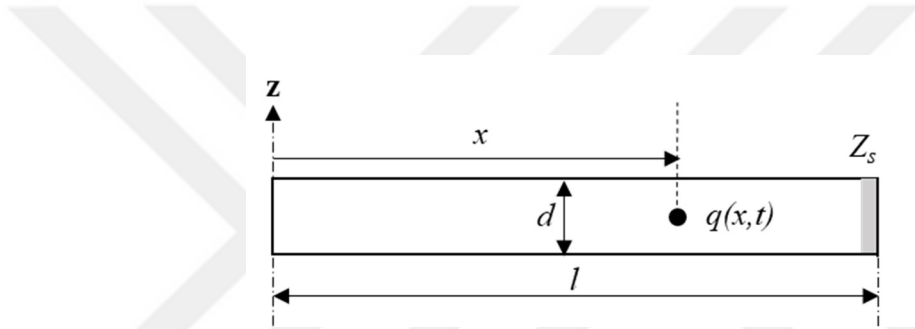


Figure 2.2 Schematic representation of a duct with hard wall at $x=0$ and hard wall with acoustic impedance boundary at $x=l$

Here, $p(x,t) = P(x)\exp(j\omega t)$ is the time-harmonic sound pressure in field x , ω is the circular frequency of sound wave, $q(x,t)$ is the acoustic source creating sound pressure per unit area ($\text{Pa}\cdot\text{m}^{-2}$) and it may be calculated for a monopole point source as follows:

$$q = \frac{j\omega\rho_s Q}{V}, \quad (2.7)$$

where, V represents the volume of the duct, Q represents the volume velocity of the source and ρ_s represents the air density. Homogenous form of Eq. (2.6) reduces to,

$$\frac{d^2 P(x)}{dx^2} + \frac{\omega_n^2}{c_s^2} P(x) = 0. \quad (2.8)$$

The acoustic ducts may have several different boundary conditions, such as acoustic hard boundary, acoustic soft boundary, flexible wall boundary, hard / flexible wall with acoustic impedance boundary, etc... Here one can find equations of acoustic hard, acoustic soft (also known as open to atmosphere) and acoustic hard with acoustic impedance boundaries:

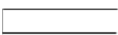
$$\text{Soft BC:} \quad P(x) = 0, \quad (a)$$

$$\text{Hard BC:} \quad \frac{dP(x)}{dx} = 0, \quad (b) \quad (2.9)$$

$$\text{Hard with impedance BC:} \quad \frac{dP(x)}{dx} + a_1 Z_a P(x) = 0. \quad (c)$$

Here, $Z_a = j\omega\rho_s/Z_s$, (Z_s is the surface acoustic impedance of a material applied to the boundary) and a_1 is equal to 1 at $x=0$, whereas $a_1 = -1$ at $x=l$. Analytical natural frequencies and mode shapes of ducts for the acoustic hard (H) and the acoustic soft (S) boundary condition combinations are given in Table 2.3.

Table 2.3 Analytical natural frequencies and mode shapes of acoustic duct (Nefske & Sung, 2006)

BC	Schematic of BC	Natural frequency (ω_n)	Mode shape	n
H-H		$nc_s/2l$	$\cos\left(\frac{n\pi x}{l}\right)$	$0, 1, 2, \dots$
H-S		$nc_s/4l$	$\cos\left(\frac{n\pi x}{2l}\right)$	$1, 3, 5, \dots$
S-S		$nc_s/2l$	$\sin\left(\frac{n\pi x}{l}\right)$	$1, 2, 3, \dots$

2.4 Bending Vibrations of Thin Beams

Before introducing the wave equation, it is useful to define the meaning of “thin structures”. Here, the term “thin” implies that thickness of the structure is very small compared with its characteristic length. With this assumption, some mechanical effects such as shear deformation and rotary inertias are neglected in the computations. This assumption implies that the points on the axial position remains in the same straight line that is perpendicular to the rotational axis when it is deformed.

Partial differential equation of a non-uniform “thin” beam shown in Figure 2.1b subjected to a harmonic excitation with $f_b = F_0 \exp(j\omega t)$ is given as (Rao, 2011a):

$$\frac{\partial^2}{\partial x^2} \left(E(x) I_y(x) \frac{\partial^2 w(x,t)}{\partial x^2} \right) + \rho(x) S(x) \frac{\partial^2 w(x,t)}{\partial t^2} = F_0 \delta(x - x_0) \exp(j\omega t). \quad (2.10)$$

For uniform beams, Eq. (2.10) reduces to,

$$\frac{\partial^4 w(x,t)}{\partial x^4} + \frac{1}{c_b^2} \frac{\partial^2 w(x,t)}{\partial t^2} = \frac{F_0}{EI_y} \delta(x - x_0). \quad (2.11)$$

Here, w is the bending displacement in z -direction, $I_y(x)$ is the space dependent moment of inertia about y -axis, f_b is the dynamic force with F_0 amplitude, $\delta(\cdot)$ is the dirac delta function and $c_b = \sqrt{EI_y / \rho S}$ is bending wave speed. Assuming free vibration of the beam with $w(x,t) = W(x) \exp(j\omega_n t)$, Eq. (2.10) can be written as;

$$\frac{d^4 W(x)}{dx^4} - \frac{\omega_n^2}{c_b^2} W(x) = 0. \quad (2.12)$$

The boundary conditions are mathematically expressed as follows (Rao, 2011a):

$$\text{Pinned (P) BC:} \quad W = 0, \quad EI_y \frac{d^2 W}{dx^2} = 0. \quad (2.13)$$

$$\text{Clamped (C) BC:} \quad W = 0, \quad \frac{dW}{dx} = 0. \quad (2.14)$$

$$\text{Free (F) BC:} \quad EI_y \frac{d^2 W}{dx^2} = 0, \quad EI_y \frac{d^3 W}{dx^3} = 0. \quad (2.15)$$

Beside these boundary conditions, the beam may have impedance at its boundary. In this case, boundary condition equations for translational or rotational impedance can be written as (Rao, 2011a):

$$EI_y \frac{d^2 W}{dx^2} = 0, \quad EI_y \frac{d^3 W}{dx^3} = a_1 [k_s + j\omega_n c - m\omega_n^2] W, \quad (2.16)$$

$$EI_y \frac{d^2 W}{dx^2} = a_1 \left[k_{st} \frac{dW}{dx} + j\omega_n c_t \frac{dW}{dx} - \omega_n^2 I_0 \frac{dW}{dx} \right], \quad EI_y \frac{d^3 W}{dx^3} = 0, \quad (2.17)$$

respectively. Note that, $a_1 = -1$ at $x=0$ and $a_1 = 1$ at $x=l$ in Eq. (2.16) whereas $a_1 = 1$ at $x=0$ and $a_1 = -1$ at $x=l$ in Eq. (2.17) (Rao, 2011a). Analytical natural frequencies of thin beams are calculated as follows,

$$\omega_n = \lambda_n^2 \sqrt{\frac{EI_y}{\rho S l^4}}, \quad (2.18)$$

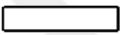
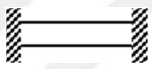
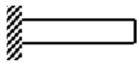
where λ_n is natural frequency parameter and its values are tabulated in Table 2.4 for different boundary conditions.

2.5 Bending Vibrations of Thin Plates

The differential equation of a bending vibration of a thin plate subjected to a harmonic force can be given as (Leissa, 1969):

$$D \left(\frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = f_b \delta(x - x_0) \delta(y - y_0). \quad (2.19)$$

Table 2.4 Analytical natural frequency parameters and mode shapes of thin beam (Rao, 2011a)

BC	Schematic of BC	Mode shape	First four λ_n	Other λ_n $n=4, 5, 6, \dots$
P-P		$\sin(\lambda_n x)$	π 2π 3π 4π	πn
F-F		$\sin(\lambda_n x) + \sinh(\lambda_n x)$ $+ \alpha_n (\cos(\lambda_n x) + \cosh(\lambda_n x))$ where $\alpha_n = \frac{\sin(\lambda_n l) - \sinh(\lambda_n l)}{\cosh(\lambda_n l) - \cos(\lambda_n l)}$	0 4.730041 7.853205 10.995608	$\pi n - \frac{\pi}{2}$
C-C		$\sinh(\lambda_n x) - \sin(\lambda_n x)$ $+ \alpha_n (\cosh(\lambda_n x) - \cos(\lambda_n x))$ where $\alpha_n = \frac{\sin(\lambda_n l) - \sinh(\lambda_n l)}{\cosh(\lambda_n l) - \cos(\lambda_n l)}$	4.730041 7.853205 10.995608 14.137165	$\pi n + \frac{\pi}{2}$
C-F		$\sin(\lambda_n x) - \sinh(\lambda_n x)$ $- \alpha_n (\cos(\lambda_n x) - \cosh(\lambda_n x))$ where $\alpha_n = \frac{\sin(\lambda_n l) + \sinh(\lambda_n l)}{\cosh(\lambda_n l) + \cos(\lambda_n l)}$	1.875104 4.694091 7.854757 10.995541	$\pi n - \frac{\pi}{2}$

Here, $D = Eh^3 / 12(1 - \nu^2)$ is bending rigidity, h is the thickness of the plate and ν is Poisson's ratio. For harmonic free vibration, Eq. (2.19) can be rewritten in terms of free vibration displacement $w(x, y, t) = W(x, y) \exp(j\omega_n t)$ as;

$$\frac{Eh^2}{12\rho(1 - \nu^2)} \left(\frac{\partial^4 w(x, y)}{\partial x^4} + 2 \frac{\partial^4 w(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y)}{\partial y^4} \right) - \omega_n^2 w(x, y) = 0. \quad (2.20)$$

The boundary conditions can be given for the clamped (C) edges (Leissa, 1969),

$$W(x, y) = \frac{\partial W(x, y)}{\partial x} = 0, \text{ at } x = \{0, a\}, \quad (2.21)$$

$$W(x, y) = \frac{\partial W(x, y)}{\partial y} = 0, \text{ at } y = \{0, b\}, \quad (2.22)$$

for pinned (P) edges (Leissa, 1969),

$$W(x, y) = D \left[\frac{\partial^2 W(x, y)}{\partial x^2} + \nu \frac{\partial^2 W(x, y)}{\partial y^2} \right] = 0, \text{ at } x = \{0, a\}, \quad (2.23)$$

$$W(x, y) = D \left[\frac{\partial^2 W(x, y)}{\partial y^2} + \nu \frac{\partial^2 W(x, y)}{\partial x^2} \right] = 0, \text{ at } y = \{0, b\}, \quad (2.24)$$

and for free (F) edges at $x = \{0, a\}$ (Leissa, 1969),

$$D \left[\frac{\partial^2 W(x, y)}{\partial x^2} + \nu \frac{\partial^2 W(x, y)}{\partial y^2} \right] = D \left[\frac{\partial^3 W(x, y)}{\partial x^3} + (2 - \nu) \frac{\partial^3 W(x, y)}{\partial x \partial y^2} \right] = 0, \quad (2.25)$$

and at $y = \{0, b\}$,

$$D \left[\frac{\partial^2 W(x, y)}{\partial y^2} + \nu \frac{\partial^2 W(x, y)}{\partial x^2} \right] = D \left[\frac{\partial^3 W(x, y)}{\partial y^3} + (2 - \nu) \frac{\partial^3 W(x, y)}{\partial y \partial x^2} \right] = 0, \quad (2.26)$$

and following boundary condition equation should also be satisfied at a free corner (Wang & Yuan, 2017):

$$2D(1 - \nu) \frac{\partial^2 W(x, y)}{\partial x \partial y} = 0. \quad (2.27)$$

Here, a and b represents the length of plate on x - and y - direction, respectively. For fully simply supported (pinned) plates, analytical natural frequencies can be obtained by (Whitney, 1987),

$$\omega_n = \sqrt{\frac{D}{\rho h} \left(\left(\frac{i_1 \pi}{a} \right)^4 + 2 \left(\frac{i_1 \pi}{a} \right)^2 \left(\frac{i_2 \pi}{b} \right)^2 + \left(\frac{i_2 \pi}{b} \right)^4 \right)}, \quad (2.28)$$

where, $i_1 = 1, 2, 3, \dots$ and $i_2 = 1, 2, 3, \dots$ Non-dimensional natural frequency parameters may be calculated by (Leissa, 1969),

$$\lambda = \omega_n a^2 \sqrt{\rho / D}. \quad (2.29)$$

CHAPTER THREE

DISCRETE SINGULAR CONVOLUTION (DSC) METHOD

3.1 Introduction

Discrete Singular Convolution (DSC) was introduced by Wei in 1999 (Wei, 1999). In early 2000s, the DSC has been applied to various problems with complex geometry and boundary conditions. It proved its accuracy in solving several physics and engineering problems modelled with partial differential equations. The method discretizes a structure in three domains; structural domain and left and right ghost (auxiliary) domains. The wavelet scaling functions are utilized to relate a discretized point with the others, so a convolution kernel required in the DSC. The main difficulty of the application of the DSC method is to relate points in ghost domain (ghost points) in terms of the points in the structural domain (structural points). This implementation requires an appropriate boundary condition algorithm.

In this chapter, the theory of the DSC and its boundary condition implementation procedures are presented in detail.

3.2 Theory of the DSC

The theory of the DSC relies on the theory of the distributions and wavelets. Let T and $\eta(t)$ be a distribution and element of the space of test functions. In this case, a singular convolution is defined as (Wei, 1999; Wei et al., 2002b),

$$G(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)dx, \quad (3.1)$$

where, $T(t-x)$ is a singular kernel. Higher order singular kernel of delta type is generally used for the numerical solution of partial differential equations (PDEs):

$$T(x) = \delta^{(n)}(x), \quad n=1,2,\dots \quad (3.2)$$

Since the kernels are singular, a sequence of approximation (T_α) of distribution T can be constructed for easy computation:

$$\lim_{\alpha \rightarrow \alpha_0} T_\alpha(x) \rightarrow T(x). \quad (3.3)$$

Here, α_0 is a generalized limit. Since continuous functions are modelled as discretized points in the computers, Eq. (3.1) can be expressed as,

$$G_\alpha(x) = \sum_k T_\alpha(x - x_k) g(x_k), \quad (3.4)$$

and it may now be called as “*discrete singular convolution*”. Here, G_α is approximation of real function G and $\{x_k\}$ is discretized point vector. Note that, the real test function η is replaced with test function g .

The limit operation in Eq. (3.3) and the domain discretization procedures in Eq. (3.4) reveal an advantage and a disadvantage of the DSC method. If the delta kernel would be modelled by computers, the exact solution of PDEs in the frequency spectrum was obtained since the Fourier transform of the delta kernel equals to unit function. However, the use of discrete functions, which present properties of the delta kernel in the limit, will clearly decrease the accuracy of the solution in frequency spectrum. On the contrary, the use of discrete functions instead of delta kernels leads to estimate local solution of PDE. Thankfully, these aspects make the DSC method known as a local spectral method.

Regularized Shannon delta sequence kernel (RSDK) is the most used delta sequence kernel in vibration problems, which is given as (Wei et al., 2002b):

$$T_\alpha(x) = \frac{\sin(\alpha x)}{\pi x} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad (3.5)$$

where, the first term and second term in the multiplication are known as the Shannon father wavelet and Gaussian regularizer, respectively. $\sigma = r\Delta$ represents the regularization parameter, Δ shows the grid spacing (distance between discretization points) and r is the regularization coefficient which can be selected by trial and error to confirm accuracy. One can refer to Seçgin and Sarigül (Seçgin & Sarigül, 2009) for detailed selection of regularization parameter, r . For vibration problems, sampling parameter (α) can be chosen according to the Nyquist frequency as π/Δ . In the discretized form, Eq. (3.5) is written by,

$$\delta_{\Delta,\sigma}^{(n)}(x - x_k) = \left(\frac{d}{dx}\right)^{(n)} \left(\frac{\sin\left(\frac{\pi}{\Delta}(x - x_k)\right)}{\frac{\pi}{\Delta}(x - x_k)} \exp\left(\frac{-(x - x_k)^2}{2\sigma^2}\right) \right). \quad (3.6)$$

Here, n indicates the order of the derivative, subscript k represents an individual point on any domain. Substituting Eq. (3.6) into Eq. (3.4) leads to,

$$g^{(n)}(x) \approx \sum_{k=-\infty}^{\infty} \left(\frac{d}{dx}\right)^{(n)} \left(\frac{\sin\left(\frac{\pi}{\Delta}(x - x_k)\right)}{\frac{\pi}{\Delta}(x - x_k)} \exp\left(\frac{-(x - x_k)^2}{2\sigma^2}\right) \right) g(x_k). \quad (3.7)$$

Eq. (3.7) enables to estimate derivative of a function by using series sum of multiplication of delta kernel and value of the function itself. However, modelling infinite series sum is not meaningful for the numerical computations; therefore, it must be truncated to a finite value of M :

$$g^{(n)}(x_m) \approx \sum_{k=-M}^M \delta_{\Delta,\sigma}^{(n)}(x - x_k) \Big|_{x=x_m} g(x_k). \quad (3.8)$$

As it is observed from Eq. (3.8), the derivative of the function is calculated by the function values at $(2M+1)$ points (M discrete points at left and right of the considered point, each), therefore M is known as the computational bandwidth. It is clear that, series of sum in Eq. (3.8) requires the function values at non-structural points such as, $g_{-M}, \dots, g_{-2}, g_{-1}, g_N, g_{N+1}, \dots, g_{N+M-1}$ which are called as ghost points. Figure 3.1 presents a 1D uniform discretization scheme for the DSC method. For an effective algorithm, three different indices; $i=0, 1, 2, \dots, N-1$, $k=-M, -M+1, \dots, 0, \dots, M-1, M$ and $j=-M, \dots, 0, \dots, N-1, \dots, N+M-1$ may be determined with the condition that $N \geq M+1$.

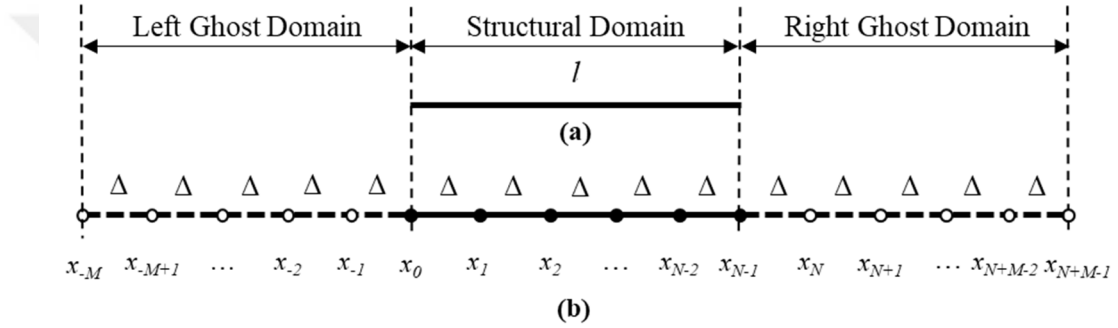


Figure 3.1 DSC discretization scheme for 1-D structures (a) a 1D structure with length l , b) Computational domain for the structure (Seçgin & Sarigül, 2008)

For the discretization shown in Figure 3.1, following uniform grid spacing may be defined as;

$$k\Delta = x_0 - x_k = x_1 - x_{k+1} = \dots = x_{N-1} - x_{N-1+k} . \quad (3.9)$$

Since the RSDK is a translationally invariant function, the function values can be written to form as,

$$\{C_{-M}^{(n)}, \dots, C_0^{(n)}, \dots, C_M^{(n)}\} = \{\delta_{\Delta, \sigma}^{(n)}(-M\Delta), \dots, \delta_{\Delta, \sigma}^{(n)}(0), \dots, \delta_{\Delta, \sigma}^{(n)}(M\Delta)\} . \quad (3.10)$$

Finally, Eq. (3.8) is expressed by using Eq. (3.10) together with the introduced indices as,

$$g^{(n)}(x_i) \approx \sum_{k=-M}^M C_k^{(n)} g(x_{i+k}), \quad (3.11)$$

or simply,

$$g_i^{(n)} \approx \sum_{k=-M}^M C_k^{(n)} g_{i+k} . \quad (3.12)$$

The first, second, third and fourth order derivatives of the RSDK are respectively given below (Seçgin, 2008; Wei, 2001b):

$$\delta_{\alpha,\sigma}^{(1)}(x_d) = \left(\frac{\cos(\alpha x_d)}{x_d} - \frac{\sin(\alpha x_d)}{\alpha x_d^2} - \frac{\sin(\alpha x_d)}{\alpha \sigma^2} \right) \cdot \exp\left(\frac{-x_d^2}{2\sigma^2}\right), \quad (3.13)$$

$$\delta_{\alpha,\sigma}^{(2)}(x_d) = \left\{ \begin{array}{l} -\left(\frac{\alpha \sin(\alpha x_d)}{x_d} + 2 \frac{\cos(\alpha x_d)}{x_d^2} \right) \\ -2 \left(\frac{\cos(\alpha x_d)}{\sigma^2} - \frac{\sin(\alpha x_d)}{\alpha x_d^3} \right) \\ + \left(\frac{\sin(\alpha x_d)}{\alpha x_d \sigma^2} + \frac{\sin(\alpha x_d)}{\alpha \sigma^4} x_d \right) \end{array} \right\} \cdot \exp\left(\frac{-x_d^2}{2\sigma^2}\right), \quad (3.14)$$

$$\delta_{\alpha,\sigma}^{(3)}(x_d) = \left\{ \begin{array}{l} -\left(\frac{\alpha^2 \cos(\alpha x_d)}{x_d} + 3 \frac{\alpha \sin(\alpha x_d)}{x_d^2} \right) - \left(\frac{x_d^2 \sin(\alpha x_d)}{\alpha \sigma^6} \right) \\ + 3 \left(\frac{\alpha \sin(\alpha x_d)}{\sigma^2} + 2 \frac{\cos(\alpha x_d)}{x_d^3} + \frac{\cos(\alpha x_d)}{x_d \sigma^2} \right) \\ + 3 \left(\frac{x_d \cos(\alpha x_d)}{\sigma^4} - 2 \frac{\sin(\alpha x_d)}{\alpha x_d^4} - \frac{\sin(\alpha x_d)}{\alpha x_d^2 \sigma^2} \right) \end{array} \right\} \cdot \exp\left(\frac{-x_d^2}{2\sigma^2}\right), \quad (3.15)$$

$$\delta_{\alpha,\sigma}^{(4)}(x_d) = \left\{ \begin{aligned} & \left(4 \frac{\alpha^2 \cos(\alpha x_d)}{x_d^2} + \frac{\alpha^3 \sin(\alpha x_d)}{x_d} \right) \\ & + 4 \left(\frac{\alpha^2 \cos(\alpha x_d)}{\sigma^2} - 3 \frac{\alpha \sin(\alpha x_d)}{x_d^3} \right) \\ & - 6 \left(\frac{\alpha \sin(\alpha x_d)}{x_d \sigma^2} + \frac{\alpha x_d \sin(\alpha x_d)}{\sigma^4} \right) \\ & - 12 \left(2 \frac{\cos(\alpha x_d)}{x_d^4} + \frac{\cos(\alpha x_d)}{x_d^2 \sigma^4} \right) \\ & - 4 \left(\frac{x_d^2 \cos(\alpha x_d)}{\sigma^6} - 6 \frac{\sin(\alpha x_d)}{\alpha x_d^5} \right) \\ & + 3 \left(4 \frac{\sin(\alpha x_d)}{\alpha x_d^3 \sigma^2} + \frac{\sin(\alpha x_d)}{\alpha x_d \sigma^4} \right) \\ & - \left(2 \frac{x_d \sin(\alpha x_d)}{\alpha \sigma^6} - \frac{x_d^3 \sin(\alpha x_d)}{\alpha \sigma^8} \right) \end{aligned} \right\} \cdot \exp\left(\frac{-x_d^2}{2\sigma^2}\right) \quad (3.16)$$

Here, $x_d = x - x_k$ and note that, $\delta_{\alpha,\sigma}^{(1)}(0) = 0$, $\delta_{\alpha,\sigma}^{(2)}(0) = -1/\sigma^2 - \pi^2/(3\Delta^2)$,
 $\delta_{\alpha,\sigma}^{(3)}(0) = 0$ and $\delta_{\alpha,\sigma}^{(4)}(0) = \left\{ 3 + 2(\pi^2/\Delta^2)\sigma^2 + (\pi^4/(5\Delta^4))\sigma^4 \right\} / \Delta^4$.

3.3 Boundary Condition Implementation Procedures

Here, the boundary condition implementation procedures are outlined for bending beams. Let us write n th order derivative of bending displacement at the boundaries ($x=0$ and $x=l$) by using Eq. (3.12):

$$W_0^{(n)} \approx \sum_{k=-M}^M C_k^{(n)} W_k = C_{-M}^{(n)} W_{-M} + \dots + C_{-1}^{(n)} W_{-1} + C_0^{(n)} W_0 + C_1^{(n)} W_1 + \dots + C_M^{(n)} W_M, \quad (3.17)$$

$$W_{N-1}^{(n)} \approx \sum_{k=-M}^M C_k^{(n)} W_{N-1+k} = C_{-M}^{(n)} W_{N-1-M} + \dots + C_M^{(n)} W_{N-1+M}. \quad (3.18)$$

In Eqs. (3.17)-(3.18), the main problem to overcome is to accurately define the function values of ghost points ($\{W_{-M}, W_{-M+1}, \dots, W_{-1}, W_N, W_{N+1}, \dots, W_{N+M-1}\}$). This must

be handled by constructing a relation between ghost and structural points with regard to the boundary condition at both ends.

In literature, two main approaches are developed to model the boundary conditions in the DSC: i) classical implementation, ii) Taylor series expansion implementation. The former one is quite straightforward in the implementation of some generic boundaries. However, it fails in free and complex boundary conditions. The issues on that problem are discussed in the following sections in detail.

3.3.1 Classical Implementation Procedure

This procedure is successfully used in simply-supported and fixed boundaries for beams, plates and shells, and fixed and free boundaries for bars, ducts. However, it fails for some complex boundaries such as free boundary of bending beams and Robin type boundaries.

In the classical implementation, a linear relation between structural and ghost points are constructed as follows with noting that $k = 1, 2, \dots, M$ (Wei (2001c));

$$\text{for left end,} \quad W_{-k} - W_0 = A_k (W_k - W_0), \quad (\text{a})$$

$$\text{for right end,} \quad W_{N-1+k} - W_{N-1} = A_{N-1+k} (W_{N-1-k} - W_{N-1}). \quad (\text{b}) \quad (3.19)$$

Here, A_k and A_{N-1+k} are the coefficients which must be determined to satisfy left and right boundaries. Reorganizing Eq. (3.19) yields,

$$W_{-k} = A_k W_k + (1 - A_k) W_0, \quad (\text{a})$$

$$W_{N-1+k} = A_{N-1+k} W_{N-1-k} + (1 - A_{N-1+k}) W_{N-1}. \quad (\text{b}) \quad (3.20)$$

Then the function values of ghost points can be written in terms of structural points by using Eq. (3.20) with proper A_k and A_{N-1+k} coefficients.

To apply the classical procedure for the bending beam, one can write the zeroth, first, the second and the third derivative of a displacement function W at the left boundary by using the DSC definition in Eq. (3.17) together with using Eq. (3.20a), respectively:

$$W_0^{(0)} = \sum_{k=-M}^M C_k^{(0)} W_k = \left(C_0^{(0)} + \sum_{k=1}^M C_k^{(0)} (1 - A_k) \right) W_0 + \sum_{k=1}^M C_k^{(0)} (1 + A_k) W_k, \quad (3.21)$$

$$W_0^{(1)} = \sum_{k=-M}^M C_k^{(1)} W_k = \left(C_0^{(1)} - \sum_{k=1}^M C_k^{(1)} (1 - A_k) \right) W_0 + \sum_{k=1}^M C_k^{(1)} (1 - A_k) W_k, \quad (3.22)$$

$$W_0^{(2)} = \sum_{k=-M}^M C_k^{(2)} W_k = \left(C_0^{(2)} - \sum_{k=1}^M C_k^{(2)} (1 - A_k) \right) W_0 + \sum_{k=1}^M C_k^{(2)} (1 + A_k) W_k, \quad (3.23)$$

and

$$W_0^{(3)} = \sum_{k=-M}^M C_k^{(3)} W_k = \left(C_0^{(3)} - \sum_{k=1}^M C_k^{(3)} (1 - A_k) \right) W_0 + \sum_{k=1}^M C_k^{(3)} (1 - A_k) W_k. \quad (3.24)$$

Note that, $C_k^{(0)} = C_{-k}^{(0)}$, $C_k^{(1)} = -C_{-k}^{(1)}$, $C_k^{(2)} = C_{-k}^{(2)}$ and $C_k^{(3)} = -C_{-k}^{(3)}$ for RSDK.

For pinned boundary (Eq. (2.13)), Eq. (3.23) is satisfied by choosing $A_k = -1 \forall k$ with $w_0 = 0$ (anti-symmetric extension), whereas Eq. (3.22) is satisfied by choosing $A_k = +1 \forall k$ with $w_0 = 0$ (symmetric extension) for fixed boundary (Eq. (2.14)). However, for free boundary (Eq. (2.15)), it becomes evident from Eqs. (3.23) and (3.24) that no A_k exists to satisfy those equations. Note that the same procedure can also be applied and the same coefficients may be obtained for the right boundary.

3.3.2 Taylor Series Expansion Implementation Procedure

Since the classical approach is not capable of modelling free boundary conditions of bending beams/plates alternative approaches are required. In this manner, Wang

et. al. (Wang, 2017; Wang & Xu, 2010; Wang & Yuan, 2017; Xu & Wang, 2011) used Taylor series expansion (TSE) to solve free boundary condition of bending structures. In this procedure, the displacement function W can be written by using Taylor series with backward and forward expansion for the left boundary:

$$W_{-k} = W_0 + \sum_{s=1}^{\infty} \frac{(-1)^s (k\Delta)^s}{s!} W_0^{(s)}, \quad (3.25)$$

$$W_k = W_0 + \sum_{s=1}^{\infty} \frac{(k\Delta)^s}{s!} W_0^{(s)}, \quad (3.26)$$

and for the right boundary:

$$W_{N-1+k} = W_{N-1} + \sum_{s=1}^{\infty} \frac{(k\Delta)^s}{s!} W_{N-1}^{(s)}, \quad (3.27)$$

$$W_{N-1-k} = W_{N-1} + \sum_{s=1}^{\infty} \frac{(-1)^s (k\Delta)^s}{s!} W_{N-1}^{(s)}, \quad (3.28)$$

respectively, and where s is an integer. There are two approaches in this procedure, which will be called here as DSC-TSE 1 method and DSC-TSE 2 method, respectively. In fact, these two methods are in the same basis to relate structural and ghost points, but mathematical operations are different, i.e., subtraction or summation.

3.3.2.1 DSC-TSE 1 Procedure

In this procedure, one writes the displacement function of the ghost points at left and at right in terms of the structure points by subtracting Eqs. (3.25) from (3.26) and Eqs. (3.27) from (3.28);

$$\begin{aligned}
W_{-k} &\approx W_k - 2k\Delta W_0^{(1)} - \frac{(k\Delta)^3}{3} W_0^{(3)} - \frac{(k\Delta)^5}{60} W_0^{(5)} - \dots \\
&\approx W_k - 2 \sum_{s=1}^{S_T} (k\Delta)^{2s-1} \frac{W_0^{(2s-1)}}{(2s-1)!},
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
W_{N-1+k} &\approx W_{N-1+k} + 2k\Delta W_{N-1}^{(1)} + \frac{(k\Delta)^3}{3} W_{N-1}^{(3)} + \frac{(k\Delta)^5}{60} W_{N-1}^{(5)} + \dots \\
&\approx W_{N-1+k} + 2 \sum_{s=1}^{S_T} (k\Delta)^{2s-1} \frac{W_{N-1}^{(2s-1)}}{(2s-1)!}.
\end{aligned} \tag{3.30}$$

respectively. Equations (3.29) and (3.30) can be limited to a finite value S_T . This value is selected as proper to the number of boundary conditions equations. For example, there are two boundary condition equations for bending vibration of beams at each boundary, therefore $S_T=2$. $S_T=1$ for ducts since there is only one boundary equation at one end. Now, Eqs. (3.29) and (3.30) can easily be adapted to the DSC equations to get rid of the ghost points.

3.3.2.2 DSC-TSE 2 Procedure

Similar to the DSC-TSE 1 procedure, one can relate structural and ghost points by summing Eqs. (3.25) and (3.26) and Eqs. (3.27) and (3.28) for the left and right boundaries;

$$\begin{aligned}
W_{-k} &\approx -W_k + 2W_0 + (k\Delta)^2 W_0^{(2)} + \frac{1}{12} (k\Delta)^4 W_0^{(4)} + \dots \\
&\approx -W_k + 2W_0 + 2 \sum_{s=1}^{S_T} (k\Delta)^{2s} \frac{W_0^{(2s)}}{(2s)!},
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
W_{N-1+k} &\approx -W_{N-1-k} + 2W_{N-1} + (k\Delta)^2 W_{N-1}^{(2)} + \frac{1}{12} (k\Delta)^4 W_{N-1}^{(4)} \\
&\approx -W_{N-1-k} + 2W_{N-1} + 2 \sum_{s=1}^{S_T} (k\Delta)^{2s} \frac{W_{N-1}^{(2s)}}{(2s)!},
\end{aligned} \tag{3.32}$$

respectively.

It is seen from Eqs. (3.29) and (3.30) or Eqs. (3.31) and (3.32) that, applying TSE in the DSC equation brings $2S_T$ additional degrees of freedom (DoFs) at the boundaries (derivative of the displacement function at boundaries). Note that, the highest order of the derivative term in DSC-TSE 2 is larger than the DSC-TSE 1.



CHAPTER FOUR

MODELLING WAVEGUIDES VIA THE DSC

4.1 Introduction

In this chapter, modelling procedures of some waveguides via the DSC method are introduced. Despite the procedures are in the same basis for different waveguides, there are minor differences in modelling. In this regard, modelled waveguides are given by corresponding subsection numbers below:

- Longitudinal vibrations of bars and sound propagation in ducts (Section 4.2),
- Bending vibrations of beams (Section 4.3),
- Bending vibrations of plates (Section 4.4).

4.2 Modelling Longitudinal Waveguides in 1-D Structures

One can model the longitudinal vibrations of bars or sound propagation in ducts with the following steps:

1. The discretization procedure is applied as presented in Figure 3.1. Then, n th order derivative ($n=2$ for the longitudinal waveguides) of axial displacement for bars (of sound pressure for ducts) at each structural point on x -space, can be written by using Eq. (3.12) as follows:

$$U_0^{(n)} \approx \sum_{k=-M}^M C_k^{(n)} U_k = C_{-M}^{(n)} U_{-M} + \dots + C_{-1}^{(n)} U_{-1} + C_0^{(n)} U_0 + C_1^{(n)} U_1 + \dots + C_M^{(n)} U_M, \quad (4.1)$$

$$U_1^{(n)} \approx \sum_{k=-M}^M C_k^{(n)} U_{1+k} = C_{-M}^{(n)} U_{-M+1} + \dots + C_M^{(n)} U_{M+1}, \quad (4.2)$$

$$U_2^{(n)} \approx \sum_{k=-M}^M C_k^{(n)} U_{2+k} = C_{-M}^{(n)} U_{-M+2} + \dots + C_M^{(n)} U_{M+2}, \quad (4.3)$$

$\vdots$

$$U_{N-1}^{(n)} \approx \sum_{k=-M}^M C_k^{(n)} U_{N-1+k} = C_{-M}^{(n)} U_{N-1-M} + \dots + C_M^{(n)} U_{N-1+M}. \quad (4.4)$$

Eqs. (4.1)-(4.4) can be represented in a matrix form for $N=N_x$ as follows:

$$\begin{Bmatrix} U_0^{(n)} \\ U_1^{(n)} \\ \vdots \\ U_{N-2}^{(n)} \\ U_{N-1}^{(n)} \end{Bmatrix}_{N \times 1} = \mathbf{C}_{N \times (N+2M)} \begin{Bmatrix} U_{-M} \\ \vdots \\ U_{-1} \\ U_0 \\ U_1 \\ \vdots \\ U_{N-1} \\ \vdots \\ U_{N+M-1} \end{Bmatrix}_{(N+2M) \times 1}, \quad (4.5)$$

where,

$$\mathbf{C} = \begin{bmatrix} C_{-M}^{(n)} & \dots & \dots & C_0^{(n)} & \dots & \dots & C_M^{(n)} & 0 & \dots & \dots & 0 \\ 0 & C_{-M}^{(n)} & \dots & \dots & C_0^{(n)} & \dots & \dots & C_M^{(n)} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & C_0^{(n)} & \ddots & \ddots & C_M^{(n)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & C_{-M}^{(n)} & \dots & \dots & C_0^{(n)} & \ddots & \ddots & C_M^{(n)} \end{bmatrix}. \quad (4.6)$$

Note that, the notation of given equations in this section is written for longitudinal vibration of bars, however, one can also obtain equations of 1D acoustic problems by replacing U , c_L , Z_m/ES and f/ES with P , c_s , Z_a and q , respectively.

2. Eqs. (3.20) with $A_k = -1$ is used for fixed boundary and $A_k = 1$ is used for the free boundary condition. However, it should be noted that, Eqs. (3.20) with $A_k = -1$ is used for acoustic soft boundary and $A_k = 1$ is used for the acoustic hard boundary condition for the duct acoustic problem. Beside that, Eqs. (3.29)-(3.30) or Eqs. (3.31)-(3.32) with $S=I$ are used for the impedance boundaries. However, DSC-TSE 1 approach has much more straightforward in the impedance boundary condition implementation. One can write the following equation by substituting derivative of displacement in Eq. (2.4c) into Eqs. (3.29) and (3.30) for the left and right impedance boundary condition as:

$$U_{-k} = U_k + 2k\Delta \frac{Z_m}{ES} U_0, \quad (4.7)$$

$$U_{N-1+k} = U_{N-1+k} - 2k\Delta \frac{Z_m}{ES} U_{N-1}. \quad (4.8)$$

Eqs. (4.7)-(4.8) construct a direct relation between ghost and structural points.

3. Eq. (2.3) becomes as follows:

$$(\mathbf{T}^L + \mathbf{T}^S + \mathbf{T}^R) \mathbf{u} + \frac{\omega_n^2}{c^2} \mathbf{I} \mathbf{u} = \{\mathbf{0}\} \quad (4.9)$$

Note that $\mathbf{I}$ is unit matrix, $\mathbf{u}_{N \times 1}^{(2)} = (\mathbf{T}^L + \mathbf{T}^S + \mathbf{T}^R)_{N \times N} \mathbf{u}_{N \times 1}$, $\mathbf{u} = \{U_0, U_1, \dots, U_{N-1}\}^T$, $\mathbf{T}^L$, $\mathbf{T}^S$ and $\mathbf{T}^R$ are the matrices including kernel information for the left-ghost, structural and right-ghost points, respectively. The information of the (i,j)th element of these matrices can be found directly as follows:

$$T_{(i,j)}^S = \begin{cases} C_{j-i}^{(n)} & \text{for } M - |j-i| \geq 0 \\ 0 & \text{for others} \end{cases}, \quad (4.10)$$

Note that, $n=2$ in Eq. (4.10). For impedance boundary at the left end,

$$T_{(i,j)}^L = \begin{cases} C_{2-j-i}^{(2)}, & \text{for } i+j \leq M+2 \text{ and } j \neq 1 \\ -2 \frac{Z_m}{ES} \sum_{k=i}^M C_{-k}^{(2)} (k-i+1) \Delta, & \text{for } i \leq M \text{ and } j=1 \\ 0, & \text{others} \end{cases}, \quad (4.11)$$

For impedance boundary at the right end,

$$\mathbf{T}_{(i,j)}^R = \begin{cases} C_{2(M+1)-i-j}^{(2)}, & \text{for } i+j \geq M+2 \text{ and } j \neq M+1 \\ -2 \frac{Z_m}{ES} \sum_{k=M-i+2}^M C_k^{(2)} (k-M+i-1) \Delta, & \text{for } i \geq 2 \text{ and } j = M+1 \\ 0, & \text{others} \end{cases} \quad (4.12)$$

4. Eq. (4.9) is then solved via a standard eigensolver to obtain natural frequencies. One can also attain the vibration response to a harmonic excitation $f = F(x) \exp(j\omega t)$ with ω angular frequency of excitation by,

$$\mathbf{u} = \left(\mathbf{T}^L + \mathbf{T}^M + \mathbf{T}^R + \frac{\omega^2}{c_L^2} \mathbf{I} \right)^{-1} \frac{\mathbf{f}}{ES}. \quad (4.13)$$

Here, $\mathbf{f}$ is the forcing vector discretized in a DSC spatial domain. Note that, forcing vector must be enlarged with the number of discretization points in the computations.

4.3 Modelling of Thin Beams

In Section 3.3.1, it is clearly shown that, classical implementation procedure is not capable of modelling the free boundary conditions; therefore, TSE must be used for this boundary condition. Since the modelling procedures of the beams and the bars are very similar, only the differences will be explained herein:

1. Wave equation of bending beams is written by using Eq. (3.12) in a discretized domain. After this step, Eq. (4.5) is obtained for $n=4$.
2. Then, as expressed in Section 3.3.1, Eqs. (3.20) with $A_k = 1$ is used for fixed (clamped) boundary and $A_k = -1$ is used for the simply-supported boundary condition. Beside that, Eqs. (3.29)-(3.30) or Eqs. (3.31)-(3.32) are used for free edges. Therefore, n th derivative of the displacement can be modeled as:

$$\mathbf{w}_{N \times 1}^{(n)} = (\mathbf{T}^S + \mathbf{T}^L + \mathbf{T}^R) \bar{\mathbf{w}} = \mathbf{D}^{(n)} \bar{\mathbf{w}}. \quad (4.14)$$

Here, $\mathbf{w}^{(n)}$ is the vector of derivatives of displacement at structural points, $\mathbf{T}^L$, $\mathbf{T}^S$ and $\mathbf{T}^R$ matrices have N rows but number of columns are determined with respect to DoFs after implementing the boundary condition. Vector $\bar{\mathbf{w}}$ includes elements of DoFs after implementing the boundary condition, i.e., $\bar{\mathbf{w}} = \{W_0, W_1, W_2, \dots, W_{N-1}\}^T$ for classical boundary condition implementation procedure, $\bar{\mathbf{w}} = \{W_0, W_1, W_2, \dots, W_{N-1}, W_0^{(1)}, W_0^{(3)}, W_{N-1}^{(1)}, W_{N-1}^{(3)}\}^T$ for DSC-TSE 1 and $\bar{\mathbf{w}} = \{W_0, W_1, W_2, \dots, W_{N-1}, W_0^{(2)}, W_0^{(4)}, W_{N-1}^{(2)}, W_{N-1}^{(4)}\}^T$ for DSC-TSE 2. Note that, dimension of matrix $\mathbf{D}^{(n)} = \mathbf{T}^S + \mathbf{T}^L + \mathbf{T}^R$ enlarges from $N \times N$ by two columns for each free edge, i.e., dimension of $\mathbf{D}^{(n)}$ for fixed-free beam is $N \times (N+2)$. The information of the (i,j)th element of matrix $\mathbf{T}^S$ can be found directly by Eq. (4.10) for $n=4$. However, elements of the matrices $\mathbf{T}^L$ and $\mathbf{T}^R$ are different according to the boundary condition implementation procedure. For the classical boundary condition implementation procedure, they are obtained by,

$$\mathbf{T}_{(i,j)}^L = \begin{cases} C_{2-j-i}^{(n)} A_{j-1} & \text{for } i+j \leq M+2 \text{ and } j \neq 1 \\ \sum_{k=i}^M C_{-k}^{(n)} (1 - A_{k-i+1}) & \text{for } i=1, 2, \dots, M \text{ and } j=1, \\ 0 & \text{others} \end{cases} \quad (4.15)$$

$$\mathbf{T}_{(i,j)}^R = \begin{cases} C_{2N-i-j}^{(n)} A_{N-1-k} & \text{for } i+j \geq 2N-M+1 \text{ and } j \neq N \\ \sum_{k=N+1-i}^M C_k^{(n)} (1 - A_{k-N+1+i}) & \text{for } i=2, \dots, M+1 \text{ and } j=N \\ 0 & \text{others} \end{cases} \quad (4.16)$$

Similarly, they may be calculated for the DSC-TSE 1 approach as;

$$T_{(i,j)}^L = \begin{cases} C_{2-j-i}^{(n)}, & \text{for } i+j \leq M+2 \text{ and } 2 \leq j \leq M+1 \\ -2 \sum_{k=i}^M C_{-k}^{(n)} (k-i+1) \Delta, & \text{for } j = N+1 \text{ and } i < M+1 \\ -\frac{1}{3} \sum_{k=i}^M C_{-k}^{(n)} ((k-i+1) \Delta)^3, & \text{for } j = N+2 \text{ and } i < M+1 \\ 0, & \text{others} \end{cases}, \quad (4.17)$$

$$T_{(i,j)}^R = \begin{cases} C_{2N-i-j}^{(n)}, & \text{for } i+j \geq 2N-M \text{ and } j < N \\ 2 \sum_{k=N+1-i}^M C_k^{(n)} (k-M+i-1) \Delta, & \text{for } j = N+3 \text{ and } i > 1 \\ \frac{1}{3} \sum_{k=N+1-i}^M C_k^{(n)} ((k-M+i-1) \Delta)^3, & \text{for } j = N+4 \text{ and } i > 1 \\ 0, & \text{others} \end{cases}, \quad (4.18)$$

and for DSC-TSE 2 approach,

$$T_{(i,j)}^L = \begin{cases} -C_{2-j-i}^{(n)}, & \text{for } i+j \leq M+2 \text{ and } 2 \leq j \leq M+1 \\ 2 \sum_{k=i}^M C_{-k}^{(n)}, & \text{for } j = 1 \text{ and } i \leq M+1 \\ \sum_{k=i}^M C_{-k}^{(n)} ((k-i+1) \Delta)^2, & \text{for } j = N+1 \text{ and } i < M+1 \\ \frac{1}{12} \sum_{k=i}^M C_{-k}^{(n)} ((k-i+1) \Delta)^4, & \text{for } j = N+2 \text{ and } i < M+1 \\ 0, & \text{others} \end{cases}, \quad (4.19)$$

$$T_{(i,j)}^R = \begin{cases} -C_{2N-i-j}^{(n)}, & \text{for } i+j \geq 2N-M \text{ and } j < N \\ 2 \sum_{k=N+1-i}^M C_k^{(n)}, & \text{for } j = N \text{ and } i > 1 \\ \sum_{k=N+1-i}^M C_k^{(n)} ((k-M+i-1) \Delta)^2, & \text{for } j = N+3 \text{ and } i > 1 \\ \frac{1}{12} \sum_{k=N+1-i}^M C_k^{(n)} ((k-M+i-1) \Delta)^4, & \text{for } j = N+4 \text{ and } i > 1 \\ 0, & \text{others} \end{cases}. \quad (4.20)$$

One can refer to Appendix 1 to see the form of the matrices for $N=4$ and $M=3$.

Let us rename the matrix $\mathbf{D}^{(n)}$ as “ $\mathbf{D}_x^{(4)}$ ” since the beam is assumed to be laid in x -space and order of the differentiation (n) equals to 4 for the beam.

3. Next, Eq. (2.12) can be modelled as;

$$\mathbf{D}_x^{(4)} \bar{\mathbf{w}} = \frac{\omega_n^2}{c_b^2} \mathbf{I} \mathbf{w}. \quad (4.21)$$

Here, vector $\bar{\mathbf{w}}$ includes structural points and their derivative terms at the boundary, whereas $\mathbf{w}$ includes only structural points of the beam. Dimensions of matrices and vectors in Eq. (4.21) vary according to the boundary conditions of the beam. Dimension of unit matrix $\mathbf{I}$ is $N \times N$ and of vector $\mathbf{w}$ is $N \times 1$ whereas dimensions of the matrix $\mathbf{D}_x^{(4)}$ and vector $\bar{\mathbf{w}}$ are given in Table 4.1. Matrix $\mathbf{D}_x^{(4)}$ is an augmented matrix that consists of two matrices; the matrix including coefficients of structural points $\mathbf{D}_s$, and the matrix including coefficients of derivative terms $\mathbf{D}_d$:

$$\mathbf{D}_x^{(4)} = [\mathbf{D}_s, \mathbf{D}_d]. \quad (4.22)$$

Note that, dimension of the matrix $\mathbf{D}_s$ is $N \times N$ and remaining part of the matrix $\mathbf{D}_x^{(4)}$ equals to $\mathbf{D}_d$.

In Table 4.1, “C”, “P” and “F” represent clamped, pinned and free boundary conditions, respectively. Notation “FC” represents that beam is free at $x=0$ and clamped at $x=l$.

Table 4.1 Dimensions of matrices and vectors according to the boundary conditions of beam

BCs	Dimension of $\mathbf{D}_x^{(4)}$ matrix	Dimension of $\bar{\mathbf{w}}$ vector
Both ends are fixed or pinned.		
CC / PP / CP / PC	$N \times N$	$N \times 1$
One end is free and the other is fixed or pinned.		
FC / FP	$N \times (N+2)$	$(N+2) \times 1$
CF / PF	$N \times (N+2)$	$(N+2) \times 1$
Both edges are free.		
FF	$N \times (N+4)$	$(N+4) \times 1$

4. In order to calculate the natural frequencies of the beam, derivative terms at the boundary should be written in terms of structural points. Thus, matrix $\mathbf{D}_x^{(4)}$ in Eq. (4.21) is then written as a square matrix. This is achieved by applying boundary conditions. Assuming the beam is free at $x=0$ and $x=l$, free boundary condition equations (Eq. (2.15)) is written in terms of the DSC and they can be written in the following matrix form:

$$\mathbf{B}_{BC} \bar{\mathbf{w}} = \{0\}. \quad (4.23)$$

where, $\mathbf{B}_{BC} = [\mathbf{C}_{coef}, \mathbf{B}]$, $\mathbf{C}_{coef}$ and $\mathbf{B}$ are the matrices those include coefficients of structural and derivative points in $\mathbf{B}_{BC}$, respectively. One can calculate derivatives of displacement vector ($\mathbf{w}_d = \{W_0^{(1)}, W_0^{(3)}, W_{N-1}^{(1)}, W_{N-1}^{(3)}\}^T$ for Eqs. (3.29) and (3.30) or $\mathbf{w}_d = \{W_0^{(2)}, W_0^{(4)}, W_{N-1}^{(2)}, W_{N-1}^{(4)}\}^T$ for Eqs. (3.31) and (3.32)) in terms of the displacements, $\mathbf{w}_{Nx1}$ by rearranging Eq. (4.23) as follows:

$$(\mathbf{C}_{coef})_{4 \times N_x} \mathbf{w}_{Nx1} = -\mathbf{B}_{4 \times 4} (\mathbf{w}_d)_{4 \times 1}, \quad (4.24)$$

Eq. (4.24) leads to,

$$\mathbf{w}_d = -\mathbf{B}^{-1}\mathbf{C}_{\text{Coef}}\mathbf{w}. \quad (4.25)$$

5. Then, left hand side of Eq. (4.21) can be rewritten by $(\mathbf{D}_s - \mathbf{D}_d\mathbf{B}^{-1}\mathbf{C}_{\text{Coef}})\mathbf{w}$, and Eq. (4.21) can be written as follows:

$$\left[\mathbf{D}_s - \mathbf{D}_d\mathbf{B}^{-1}\mathbf{C}_{\text{Coef}} - \frac{\omega_n^2}{c^2}\mathbf{I} \right] \mathbf{w} = 0. \quad (4.26)$$

6. Next, zero displacement rows and columns are deleted in Eq. (4.26) for the clamped and pinned boundary conditions (if exists). After that, natural frequencies of the beam can be calculated by solving the eigenvalues of Eq. (4.26).

4.4 Modelling of Thin Plates

In the DSC method, plates can be modelled by following steps:

1. One of the dimensions (i.e. x -dimension) of the plate is modelled such that it is a beam with the same length of plate's x -dimension.
2. Eq. (3.12) is written for $n=2$ and $n=4$ since equation of motion of the plate requires second and forth order derivatives. Then, appropriate boundary condition implementation procedure is performed as expressed in the previous section. At the end of this step, second and forth order derivatives of one dimension of the plate are calculated in terms of the DSC. Let us call these two matrices as $\mathbf{D}_x^{(2)}$ and $\mathbf{D}_x^{(4)}$, respectively.
3. Steps 1 and 2 are performed for the other dimension (y dimension) of the plate. Assume that $\mathbf{D}_y^{(2)}$ and $\mathbf{D}_y^{(4)}$ matrices are calculated at the end of this step.
4. Then, the equation of motion of the plate (Eq. (2.20)) may be written by distributing the calculated matrices via tensor product as follows:

$$\left(\mathbf{D}_x^{(4)} \otimes \mathbf{I}_y + 2\mathbf{D}_x^{(2)} \otimes \mathbf{D}_y^{(2)} + \mathbf{I}_x \otimes \mathbf{D}_y^{(4)} \right) \mathbf{w}_{xy} - \frac{12\rho(1-\nu^2)}{Eh^2} \omega_n^2 (\mathbf{I}_x \otimes \mathbf{I}_y) \mathbf{w}_{xy} = 0, \quad (4.27)$$

or simply,

$$\overline{\mathbf{D}}\mathbf{w}_{xy} = \frac{12\rho(1-\nu^2)}{Eh^2}\omega_n^2(\mathbf{I}_x \otimes \mathbf{I}_y)\mathbf{w}_{xy}. \quad (4.28)$$

Here, $\otimes$ represents tensor product, vector $\overline{\mathbf{w}}_{xy}$ includes structural points and derivative terms whereas $\mathbf{w}_{xy}$ includes only structural points of the plate. The dimensions of matrices and vectors vary according to the boundary conditions of the plate. The dimension of matrix $\mathbf{I}_x \otimes \mathbf{I}_y$ is $N_x \times N_y$ and the vector $\mathbf{w}_{xy}$ is $(N_x \cdot N_y) \times 1$, whereas dimensions of the matrix $\overline{\mathbf{D}}$ and vector $\overline{\mathbf{w}}_{xy}$ are given in Table 4.2. Note that, $\overline{\mathbf{D}} = [\mathbf{D}_s, \mathbf{D}_d]$ and $\mathbf{D}_s$ and $\mathbf{D}_d$ are the matrices including coefficients of structural and derivative points, respectively.

In Table 4.2, “C” represents clamped boundary condition but it should be noted that the same dimension of matrices and vectors are valid for the pinned (simply-supported) boundary condition. In Figure 4.1, a schematic representation of plate with “FCPF” boundary is presented from top view. Notation “FCPF” represents that plate is free at $x=0$, clamped at $y=0$, pinned at $x=l_x$ and free at $y=l_y$.

Table 4.2 Dimensions of matrices and vectors according to the boundary conditions of plate

BCs	Dimension of $\bar{\mathbf{D}}$ matrix	Dimension of $\bar{\mathbf{w}}_{xy}$ vector
All edges are fixed.		
CCCC	$(N_x \cdot N_y) \times (N_x \cdot N_y)$	$(N_x \cdot N_y) \times 1$
One edge is free and the others are fixed.		
FCCC	$(N_x \cdot N_y) \times ((N_x+2) \cdot N_y)$	$((N_x+2) \cdot N_y) \times 1$
CCFC	$(N_x \cdot N_y) \times ((N_x+2) \cdot N_y)$	$((N_x+2) \cdot N_y) \times 1$
CFCC	$(N_x \cdot N_y) \times (N_x \cdot (N_y+2))$	$(N_x \cdot (N_y+2)) \times 1$
CCCF	$(N_x \cdot N_y) \times (N_x \cdot (N_y+2))$	$(N_x \cdot (N_y+2)) \times 1$
Any two edges are free and the others are fixed.		
FCFC	$(N_x+4) \times N_y$	$((N_x+4) \cdot N_y) \times 1$
CFCF	$N_x \times (N_y+4)$	$(N_x \cdot (N_y+4)) \times 1$
FFCC	$(N_x+2) \times (N_y+2)$	$((N_x+2) \cdot (N_y+2)) \times 1$
FCCF	$(N_x+2) \times (N_y+2)$	$((N_x+2) \cdot (N_y+2)) \times 1$
CFFC	$(N_x+2) \times (N_y+2)$	$((N_x+2) \cdot (N_y+2)) \times 1$
CCFF	$(N_x+2) \times (N_y+2)$	$((N_x+2) \cdot (N_y+2)) \times 1$
One edge is fixed and the others are free.		
CFFF	$(N_x+2) \times (N_y+4)$	$((N_x+2) \cdot (N_y+4)) \times 1$
FFCF	$(N_x+2) \times (N_y+4)$	$((N_x+2) \cdot (N_y+4)) \times 1$
FCFF	$(N_x+4) \times (N_y+2)$	$((N_x+4) \cdot (N_y+2)) \times 1$
FFFC	$(N_x+4) \times (N_y+2)$	$((N_x+4) \cdot (N_y+2)) \times 1$
All edges are free.		
FFFF	$(N_x+4) \times (N_y+4)$	$((N_x+4) \cdot (N_y+4)) \times 1$

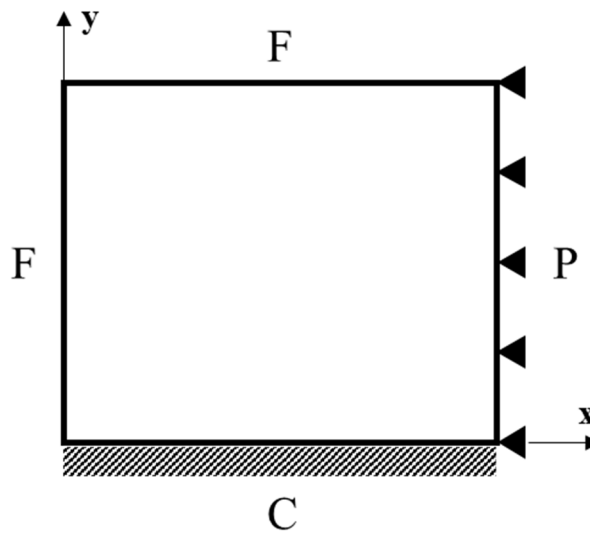


Figure 4.1 Schematic representation of “FCPF” plate

For the sake of simplicity, the remaining procedure is outlined for FCCC plate.

For this boundary condition combination, the vector $\{\overline{\mathbf{w}_{xy}}\}_{((N_x+2)N_y) \times 1}$ have

$\{W_{00}, W_{01}, \dots, W_{0N_y-1}, W_{10}, \dots, W_{01}^{(1)}, W_{02}^{(1)}, \dots, W_{0N_y-1}^{(1)}, W_{01}^{(3)}, W_{02}^{(3)}, \dots, W_{0N_y-1}^{(3)}\}^T$ elements for

DSC-TSE 1 approach. Similarly, $\{\overline{\mathbf{w}_{xy}}\}_{((N_x+2)N_y) \times 1}$ includes second and forth

order derivatives in DSC-TSE 2 approach instead of first and third order derivatives. Note that, number of derivative terms is $2N_y$ for this structure and it can be determined for other boundary condition cases.

5. In order to calculate the natural frequencies of the plate, the derivative terms should be written in terms of the structural points. It is performed with the same procedure that is given for the beam. Note that, the boundary condition equation must also be distributed via tensor product. Since the plate is free at $x=0$, Eq. (2.25) is written for each structural point in y-direction in terms of the DSC as follows:

$$\left[\mathbf{c}_{x0}^{(2)} \otimes \mathbf{D}_y^{(0)} + \nu (\mathbf{c}_{x0}^{(0)} \otimes \mathbf{D}_y^{(2)}) \right]_{N_y \times ((N_x+2)N_y)} \overline{\mathbf{w}_{xy}} = \{0\}, \quad (4.29)$$

and

$$\left[\mathbf{c}_{x0}^{(3)} \otimes \mathbf{D}_y^{(0)} + (2-\nu) (\mathbf{c}_{x0}^{(1)} \otimes \mathbf{D}_y^{(2)}) \right]_{N_y \times ((N_x+2)N_y)} \overline{\mathbf{w}_{xy}} = \{0\}. \quad (4.30)$$

Here, $\mathbf{c}_{x0}^{(2)}$ is vector form of Eq. (3.12) with $i=0$ and $n=2$ after applying TSE (Eq (3.29) or Eq. (3.31)). For the other boundary condition cases, i.e., FFFF case, Eqs. (2.25) and (2.26) are written for each edge in the form of Eqs. (4.29) and (4.30).

6. Eqs. (4.29) and (4.30) lead to $2N_y$ additional equations. Therefore, these equations can now be rearranged as a linear set of equations:

$$\mathbf{B}_{\text{BC}} \overline{\mathbf{w}_{\text{xy}}} = [\mathbf{C}_{\text{Coef}}]_{2N_y, x 2N_x, N_y} \{\mathbf{w}_{\text{xy}}\}_{N_x, N_y, x1} + [\mathbf{B}]_{2N_y, x 2N_y} \{\mathbf{w}_{\text{d}}\}_{2N_y, x1} = \{0\}, \quad (4.31)$$

where, $\mathbf{B}_{\text{BC}} = [\mathbf{C}_{\text{Coef}}, \mathbf{B}]$. One can calculate the derivatives of the displacement vector ($\{\mathbf{w}_{\text{d}}\}_{2N_y, x1}$) in terms of the displacements, $\{\mathbf{w}_{\text{xy}}\}_{N_x, N_y, x1}$ by using Eq. (4.25).

7. Eq. (4.28) can be written as follows:

$$\left(\mathbf{D}_{\text{S}} - \mathbf{D}_{\text{D}} \mathbf{B}^{-1} \mathbf{C}_{\text{Coef}} - \frac{12\rho(1-\nu^2)}{Eh^2} \omega_n^2 (\mathbf{I}_x \otimes \mathbf{I}_y) \right) \mathbf{w}_{\text{xy}} = 0. \quad (4.32)$$

8. Zero displacement rows and columns are deleted in Eq. (4.32) for the clamped and the pinned boundary conditions. Next, the natural frequencies of the plate can be calculated by solving the eigenvalues.

CHAPTER FIVE

THE DSC ALGORITHM FOR CONNECTED STRUCTURES

5.1 Introduction

Up to this point, the DSC method is expressed for some waveguides; however, the main research subject is applying the DSC for connected structures in this thesis. In this manner, there are some studies in the literature for vibration analysis of connected structures via the DSC (Duan & Wang, 2013, 2014; Duan et al., 2014; Wang & Duan, 2014), but they are limited to the stepped vibratory structures. In this thesis, the DSC-TSE method is implemented for novel structures. Here, “DSC-TSE” represents the DSC analysis of connected structures by using Taylor Series Expansion in modelling the continuity equation relations of the structures. Although the DSC-TSE is expressed here for the acoustic analysis of transmission lines, it may also be implemented for different connected structures. The details on the modelling of other structures will be given in Chapter 7.

5.2 The DSC for the Transmission Lines

The algorithm is expressed on a transmission line composed of J duct elements (units) with different diameters (d_j) and lengths (L_j) as shown in Figure 5.1. In order to express a general case, boundary conditions are shown for surface impedances of

$$Z_{a,1} \text{ and } Z_{a,2} \text{ at } x = 0 \text{ and } x = \sum_{j=1}^J L_j .$$

For the shown system, common boundary conditions at the ends of the entire structure must be satisfied. Besides, the boundary conditions of the common ends of any two ducts (such as, internal boundary at $x = \sum L_j$ where $j=1,2,3,\dots,J-1$) can be written by using continuity equations:

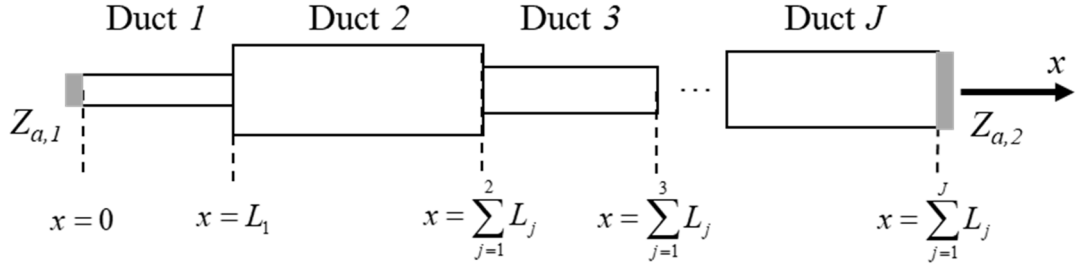


Figure 5.1 Schematic view of considered structure

$${}^jP_{N_j-1} - {}^{j+1}P_0 = 0, \quad (5.1)$$

$$S_j {}^jP_{N_j-1}^{(1)} - S_{j+1} {}^{j+1}P_0^{(1)} = 0. \quad (5.2)$$

Here, S_j is the cross-sectional area of duct j and P_i is acoustic pressure at discretized point of i .

By using common boundary conditions and the continuity equations, the DSC algorithm is applied by the following procedure:

1. Wave equation (Eq. (2.8)) of each individual duct is modelled in a classical DSC discretized domain.
2. If the transmission line has a soft or hard end at $x=0$ or $x=\sum_{j=1}^J L_j$, Eqs. (3.20)

are used with $A_k = -1$ for soft end, $A_k = 1$ for hard end in order to relate ghost and structural points. For the other cases including connection to another duct case, Eqs. (3.29) and (3.30) or Eqs. (3.31) and (3.32) with $S_T = 1$ is applied. For those cases, each duct unit will have two degrees of freedom (DOFs) at their own boundaries; i.e, one for the pressure, another for the derivative of the pressure as shown in Figure 5.2. Note that, DoFs after implementation TSE 2 is presented in Figure 5.2. However, similar procedure may be applied for TSE 1. Therefore, there will be $2J$ additional DOFs for the considered structure shown in Figure 5.1

$({}^1P_0^{(1)}, {}^1P_{N_1-1}^{(1)}, {}^2P_0^{(1)}, {}^2P_{N_2-1}^{(1)}, \dots, {}^{J-1}P_0^{(1)}, {}^{J-1}P_{N_{J-1}-1}^{(1)}, {}^JP_0^{(1)}, {}^JP_{N_J-1}^{(1)})$ for the DSC-TSE 1 or

$({}^1P_0^{(2)}, {}^1P_{N_1-1}^{(2)}, {}^2P_0^{(2)}, {}^2P_{N_2-1}^{(2)}, \dots, {}^{J-1}P_0^{(2)}, {}^{J-1}P_{N_{J-1}-1}^{(2)}, {}^JP_0^{(2)}, {}^JP_{N_J-1}^{(2)})$ for the DSC-TSE 2

where pre-subscript of pressure shows duct number) and those can be handled by $2(J-1)$ additional continuity equations given in Eqs. (5.1)-(5.2) and two boundary condition equations. Note that, the continuity equations may be gained with a large value in order to model the pressure continuity strictly. Beside that, the first order derivatives of the pressures for the duct j and $j+1$ must be written in terms of DSC in modelling Eq. (5.2).

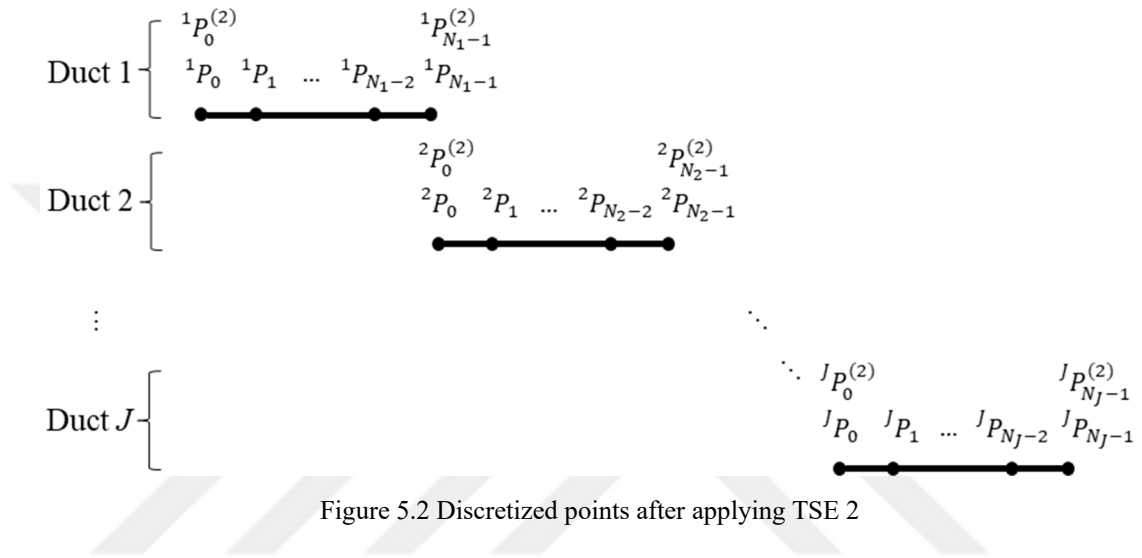


Figure 5.2 Discretized points after applying TSE 2

3. Steps 1-2 lead $\sum_{j=1}^J N_j + 2J$ independent equations (where N_j represents the discretization numbers of duct j) and they can be written as,

$$(\mathbf{K} + \mathbf{M})\mathbf{p} = \{\mathbf{0}\} . \quad (5.3)$$

Here, $\mathbf{K}$ is the global stiffness matrix, $\mathbf{M}$ is the global mass matrix and it is a function of natural frequency (ω_n) and

$\mathbf{p} = \{^1P_0, ^1P_1, \dots, ^1P_{N_1-1}, ^1P_0^{(2)}, ^1P_{N_2-1}^{(2)}, ^2P_0, \dots, ^J P_{N_J-1}, ^J P_0^{(2)}, ^J P_{N_J-1}^{(2)}\}$ for the DSC-TSE 2.

Note that, mass matrix is a function of the natural frequency. Global stiffness and mass matrices can be constructed by performing the following two steps:

- Place the local stiffness and mass matrices of each duct unit diagonally,

- Place the continuity and boundary condition equations to last rows.

After these steps, following equations may be written:

$$\mathbf{K} = \begin{bmatrix} (\mathbf{D}_1^{(2)})_{N_1 \times (N_1+2)} & 0 & \dots & \dots & 0 \\ 0 & (\mathbf{D}_2^{(2)})_{N_2 \times (N_2+2)} & 0 & 0 & \vdots \\ \vdots & 0 & (\mathbf{D}_3^{(2)})_{N_3 \times (N_3+2)} & 0 & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & \dots & \dots & 0 & (\mathbf{D}_J^{(2)})_{N_J \times (N_J+2)} \\ & & (\mathbf{C}_K)_{2J \times \sum_{j=1}^J N_j+2} & & \end{bmatrix}, \quad (5.4)$$

$$\mathbf{M} = \begin{bmatrix} \frac{\omega_n^2}{c_s^2} \mathbf{I}_{N_1 \times (N_1+2)} & 0 & \dots & \dots & 0 \\ 0 & \frac{\omega_n^2}{c_s^2} \mathbf{I}_{N_2 \times (N_2+2)} & 0 & 0 & \vdots \\ \vdots & 0 & \frac{\omega_n^2}{c_s^2} \mathbf{I}_{N_3 \times (N_3+2)} & 0 & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{\omega_n^2}{c_s^2} \mathbf{I}_{N_J \times (N_J+2)} \\ & & (\mathbf{C}_M)_{2J \times \sum_{j=1}^J N_j+2} & & \end{bmatrix}. \quad (5.5)$$

Here, $\mathbf{I}$ represents the unit matrix, $\mathbf{C}_K$ and $\mathbf{C}_M$ are matrices obtained from the continuity and boundary condition equations. The matrix $\mathbf{C}_M$ includes the coefficient terms of the natural frequency in continuity and boundary condition equations whereas $\mathbf{C}_K$ includes the remaining coefficient terms in those equations.

$\mathbf{D}_j^{(2)}$ is the local stiffness matrix of duct j and

$$\mathbf{D}_j^{(2)} = (\mathbf{T}^S + \mathbf{T}^L + \mathbf{T}^R)_{N_j \times (N_j+2)}. \quad (5.6)$$

The elements of $\mathbf{T}^S$ matrix are given in Eqs. (4.10) whereas $\mathbf{T}^L$ and $\mathbf{T}^R$ are given in Eqs. (4.17)-(4.18) for DSC-TSE 1 and given in Eqs. (4.19)-(4.20) for DSC-TSE 2 approach.

One can also obtain the acoustic response to a harmonic excitation $q(x, t) = Q(x) \exp(j\omega t)$ with ω angular frequency of excitation by,

$$\{\mathbf{p}\} = (\mathbf{K} + \mathbf{M}(\omega))^{-1} \{\mathbf{q}\}. \quad (5.7)$$

Here, $\{\mathbf{q}\}$ is the excitation vector discretized in a DSC spatial domain. Note that, excitation vector must be enlarged with discretization number of corresponding duct as in forced vibration of bending beams.

CHAPTER SIX

NUMERICAL ANALYSIS OF INDIVIDUAL CONTINUOUS SYSTEMS VIA THE DSC

6.1 Introduction

In this Chapter, vibration and acoustic analysis of single (individual) continuous structures with different boundary conditions are performed via the DSC method. In the analyses, after a convergence study is carried out for different boundary conditions, the results are compared with analytical and/or finite element method (FEM). Analysed structures are given below together with the Section numbers:

- Free and forced vibration analyses of bars are presented in Section 6.2.
- Free and forced acoustic analyses of ducts are presented in Section 6.3.
- Free vibration analyses of inhomogeneous bending beams are presented in Section 6.4.
- Free vibration analysis of bending plates is presented in Section 6.5.

In the computations, impedance and free boundaries are treated by using Taylor series expansion implementation whereas the others are handled via classical implementation procedure.

6.2 Free and Forced Vibration Analysis of a Bar

Longitudinal vibration of a circular aluminium bar for which the respective properties are given in Table 6.1 is considered. For this structure, first ten natural frequencies and frequency response function between 0-10000 Hz subjected to a point excitation are investigated for different boundary conditions. Note that internal damping of the bar is assumed to be negligible in the study.

Table 6.1 Properties of the bar

Property (Unit)	Value
Elasticity Modulus (GPa)	70
Poisson's ratio	0.3
Density (kg/m ³)	2700
Cross-section (m ²)	$0.1^2 \pi$
Length (m)	1

6.2.1 Free Vibration Analysis of the Bar

Natural frequencies in Hz ($f_n = \omega_n / 2\pi$) of the bar are determined by using the DSC under different boundary conditions of the left and right ends; i) fixed-fixed, ii) fixed-free, iii) free-free, iv) impedance-fixed, v) impedance-free and vi) impedance-impedance. Among those boundary conditions, convergence study is performed for fixed-fixed, fixed-free and free-free boundaries with $N=11, 21$ and 31 discretization points, the computational bandwidth $M=N-1$ and the regularization coefficient $r=N/10+1$. For those BCs, $Error\%$ obtained by $(f_{n,DSC} - f_{an}) / f_{an} \times 100$ for natural frequencies is plotted in Figure 6.1. Note that, analytical natural frequencies are obtained by using Table 2.2.

It is seen from Figure 6.1 that, we are able to estimate the first ten natural frequencies with $Error\%$ under the value of 1% with $N=21$. Then, regularization coefficient of the Regularized Shannon Delta Kernel (RSDK) is optimized for the first and tenth mode of the fixed-fixed bar for $N=21$. In this regard, the regularization coefficient is varied between 1.9 and 6 and $Error\%$ with respect to different regularization coefficients are presented in Figure 6.2 for the first and the tenth mode.

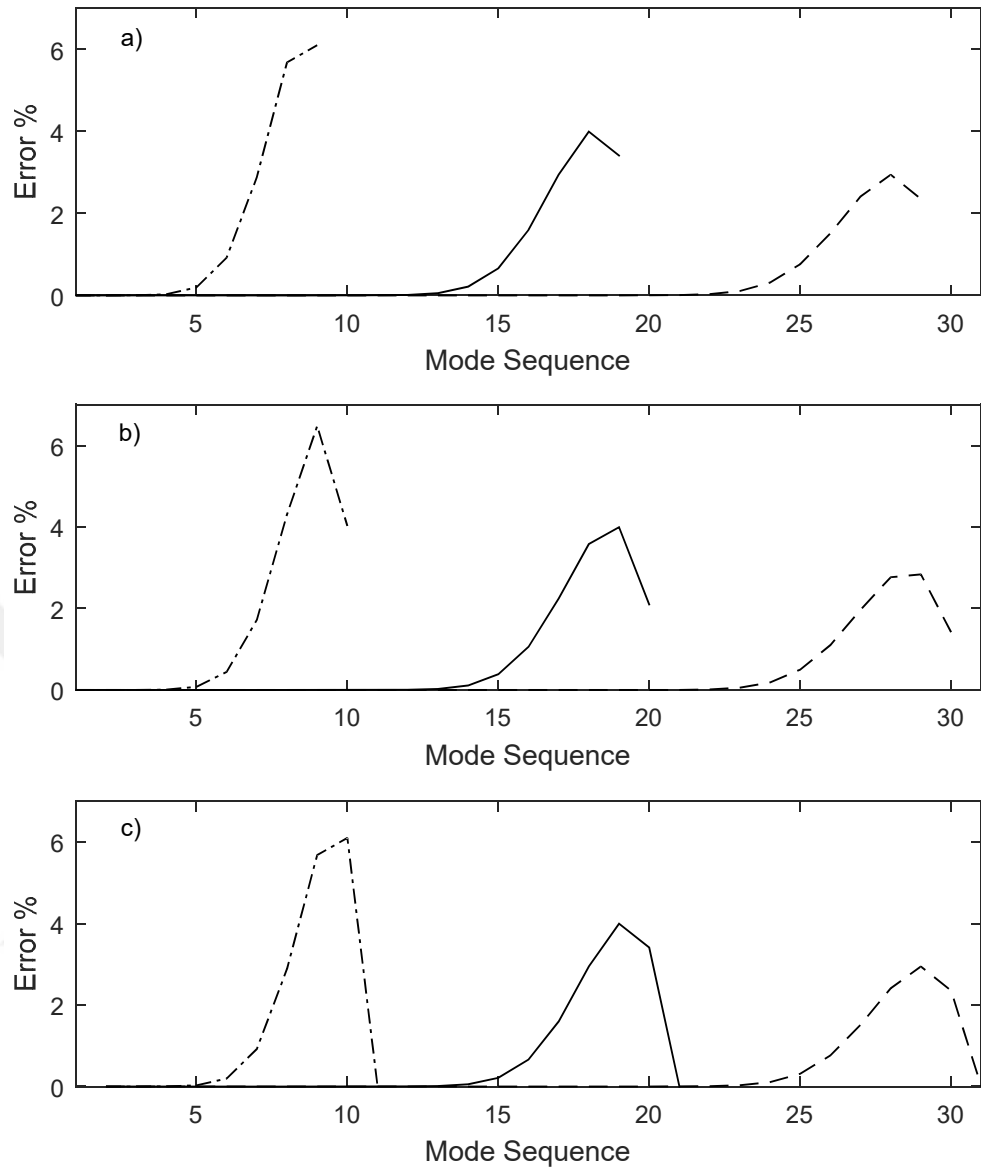


Figure 6.1 Relative errors in the natural frequencies of the bar obtained via the DSC a) fixed-fixed boundary, b) free-fixed boundary, c) free-free boundary (dash dotted line: $N=11$, solid line: $N=21$, dashed line: $N=31$)

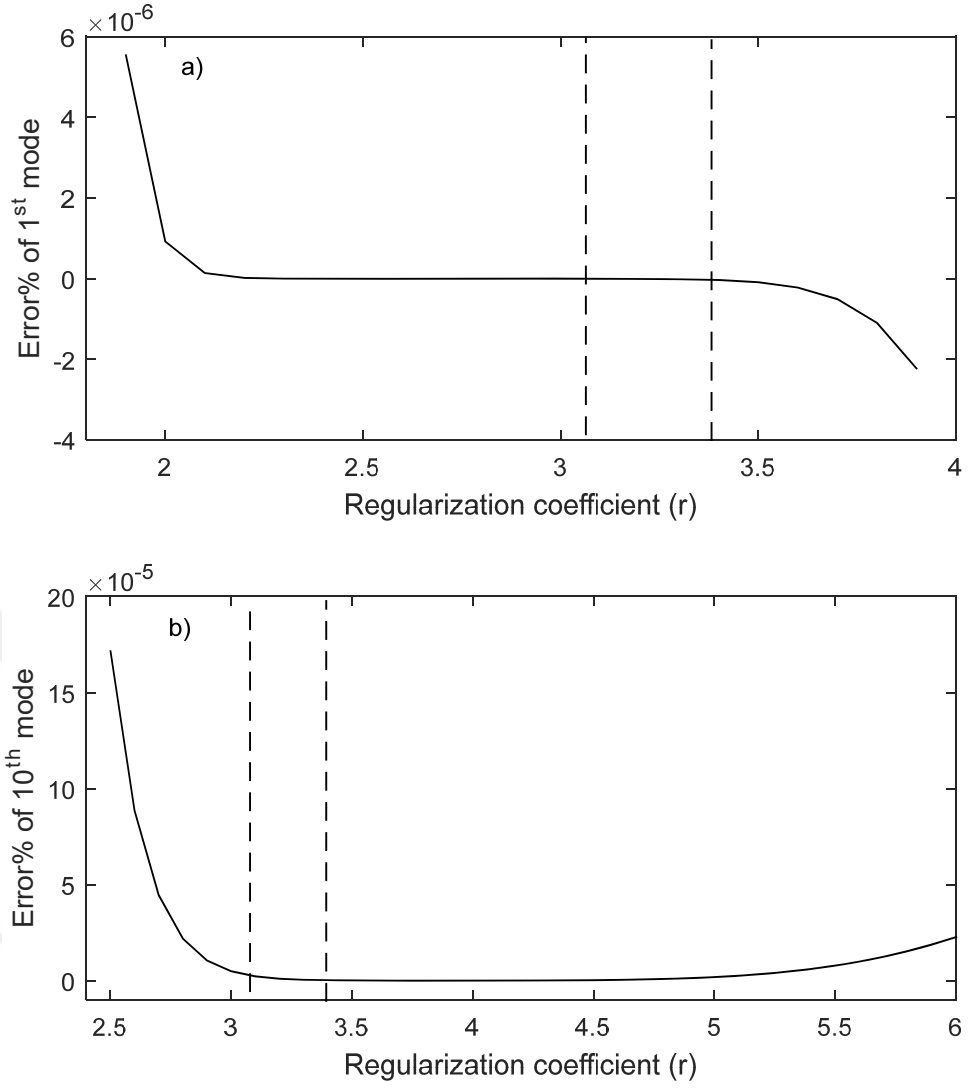


Figure 6.2 Relative errors in the natural frequencies of the bar for different regularization coefficients
a) Error % of the first mode, b) Error % of the tenth mode

The lowest *Error%*'s are computed with $r=2.7$ and $r=3.8$ for the first and the tenth natural frequencies, respectively. The computed natural frequencies for these regularization coefficients are tabulated in Table 6.2. Although, selecting one of the values found for optimum r for any of the frequencies yields worse results for the other natural frequency. However, a small region of r varying between 3.1-3.4 (interval shown in Figure 6.2 with dashed lines) is found for optimum accuracy for all computed natural frequencies. Therefore, regularization coefficient (r) is selected as 3.1 in the analyses.

Table 6.2 Natural frequencies of the fixed-fixed bar with $N=21$ for $r=2.7$ and $r=3.8$ (Hz)

Mode Consequence	Analytical	DSC	
		$r=2.7$	$r=3.8$
1	2545.875	2545.875	2545.873
2	5091.751	5091.751	5091.752
3	7637.626	7637.626	7637.625
4	10183.502	10183.502	10183.502
5	12729.377	12729.377	12729.376
6	15275.252	15275.253	15275.253
7	17821.128	17821.131	17821.127
8	20367.003	20367.031	20367.003
9	22912.878	22913.070	22912.878
10	25458.754	25459.886	25458.754

After determining the optimum number of discretization and the value of regularization coefficient, natural frequencies are computed for several different boundary condition cases. Natural frequencies obtained by the DSC are then compared with those obtained by either analytical derivation (in case they are available) or by finite element method (FEM) or both as given in Tables 6.3-6.5. FEM calculations are performed by ANSYS. The bar is modelled by using two-node Beam44 element type, the damper-spring is modelled by Combin14 element whereas Mass21 element is used for point mass. Beam44 and Mass21 elements are capable of modelling six degrees of freedom (DOF) at each node but in this study all DOFs are restricted except the displacement DOF on x -direction to model the considered bar. For mechanical impedance (MI) boundary conditions, two different sets of mass and stiffness are selected as follows:

Set 1) Masses and stiffness at the left and right ends are identical ($m_0 = m_l = 1$ kg, $k_{s,0} = k_{s,l} = 10^6$ Nm⁻¹) (Table 6.4).

Set 2) Masses and stiffness are not identical at the left and right boundary ($m_0 = 5$, $m_l = 15$ kg; $k_{s,0} = 10^6$ Nm⁻¹, $k_{s,l} = 0.5 \times 10^6$ Nm⁻¹) (Table 6.5).

Table 6.3 Natural frequencies of the bar under fixed and free end BC combinations (Hz)

Mode Seq.	fixed-fixed								free-fixed			free-free		
	Analy.	DSC			FEM				Analy.	DSC	FEM	Analy.	DSC	FEM
		N=11	N=21	N=31	N=20	N=40	N=60	N=120					N=21	N=120
1	2545.88	2545.86	2545.88	2545.88	2548.49	2546.53	2546.17	2545.95	1272.94	1272.94	1272.95	0.00	0.00	0.00
2	5091.75	5091.77	5091.75	5091.75	5112.71	5096.99	5094.08	5092.33	3818.81	3818.81	3819.06	2545.88	2545.88	2545.95
3	7637.63	7637.85	7637.63	7637.63	7708.48	7655.31	7645.48	7639.59	6364.69	6364.69	6365.82	5091.75	5091.75	5092.33
4	10183.50	10186.38	10183.50	10183.50	10351.73	10225.43	10202.12	10188.16	8910.56	8910.56	8913.68	7637.63	7637.63	7639.59
5	12729.38	12754.08	12729.38	12729.38	13058.55	12811.32	12765.76	12738.47	11456.44	11456.44	11463.07	10183.50	10183.50	10188.16
6	15275.25	15415.76	15275.25	15275.25	15844.94	15416.95	15338.14	15290.96	14002.31	14002.31	14014.41	12729.38	12729.38	12738.47
7	17821.13	18335.18	17821.13	17821.13	18726.50	18046.33	17921.04	17846.08	16548.19	16548.19	16568.16	15275.25	15275.25	15290.96
8	20367.00	21524.39	20367.00	20367.00	21717.66	20703.47	20516.21	20404.25	19094.07	19094.07	19124.75	17821.13	17821.13	17846.08
9	22912.88	24311.24	22912.88	22912.88	24830.43	23392.38	23125.43	22965.92	21639.94	21639.94	21684.62	20367.00	20367.00	20404.25
10	25458.75	-	25458.81	25458.75	28072.29	26117.09	25750.48	25531.52	24185.82	24185.83	24248.20	22912.88	22912.88	22965.92

Table 6.4 Natural frequencies of the bar under mechanical impedance, fixed, and free end BC combinations for the Set 1 (Hz)

Mode Sequence	MI-fixed B.C.							MI-free B.C.			MI-MI B.C.		
	DSC			FEM	Deviation %			DSC	FEM	Deviation	DSC	FEM	Deviation
	N=11	N=21	N=31	N=120				N=21	N=120	%	N=21	N=120	%
1	1258.31	1258.33	1258.34	1258.34	0.00	0.00	0.00	17.18	17.19	0.04	24.15	24.16	0.04
2	3773.65	3774.24	3774.35	3774.64	0.03	0.01	0.01	2516.28	2516.39	0.00	2487.38	2487.51	0.01
3	6287.14	6289.87	6290.37	6291.79	0.07	0.03	0.02	5032.12	5033.09	0.02	4973.88	4975.21	0.03
4	8798.03	8804.84	8806.23	8810.19	0.14	0.06	0.04	7547.45	7550.81	0.04	7459.44	7464.05	0.06
5	11311.39	11318.92	11321.84	11330.30	0.17	0.10	0.07	10062.01	10070.00	0.08	9943.51	9954.53	0.11
6	13862.82	13831.91	13837.15	13852.61	-0.07	0.15	0.11	12575.56	12591.15	0.12	12425.67	12447.15	0.17
7	16568.34	16343.65	16352.13	16377.60	-1.16	0.21	0.16	15087.94	15114.74	0.18	14905.58	14942.43	0.25
8	19560.34	18854.09	18866.77	18905.73	-3.46	0.27	0.21	17599.04	17641.24	0.24	17383.03	17440.90	0.33
9	22582.41	21363.30	21381.07	21437.49	-5.34	0.35	0.26	20108.84	20171.13	0.31	19858.00	19943.05	0.43
10	24757.71	23871.51	23895.07	23973.35	-3.27	0.42	0.33	22617.50	22704.88	0.38	22330.73	22449.39	0.53

Table 6.5 Natural frequencies of the bar under mechanical impedance, fixed, and free end BC combinations for the Set 2 (Hz)

Mode Sequence	MI-fixed B.C.			MI-free B.C.			MI-MI B.C.		
	DSC N=21	FEM N=120	Deviation %	DSC N=21	FEM N=120	Deviation %	DSC N=21	FEM N=120	Deviation %
1	1202.45	1202.44	0.00	16.79	16.80	0.04	19.04	19.04	0.03
2	3609.98	3610.79	0.02	2405.38	2405.57	0.01	2078.77	2079.00	0.01
3	6026.58	6030.43	0.06	4816.87	4818.87	0.04	4240.78	4243.64	0.07
4	8455.95	8465.58	0.11	7239.52	7245.91	0.09	6496.22	6504.52	0.13
5	10899.73	10917.77	0.17	9676.00	9689.54	0.14	8821.39	8837.52	0.18
6	13357.99	13386.55	0.21	12127.09	12150.16	0.19	11195.57	11221.21	0.23
7	15829.88	15870.55	0.26	14592.30	14626.75	0.24	13605.72	13641.74	0.26
8	18314.23	18368.13	0.29	17070.57	17117.74	0.28	16043.85	16090.21	0.29
9	20809.92	20877.79	0.33	19560.72	19621.53	0.31	18504.84	18560.65	0.30
10	23316.09	23398.31	0.35	22061.74	22136.76	0.34	20985.44	21048.99	0.30

Followings can be inferred from Tables 6.3-6.5:

- All computed results are consistent with each other.
- In Table 6.3, for fixed-fixed boundary condition (BC), the DSC results converge to analytical results as the number of discretization points increase as expected. For fixed-fixed, free-fixed, free-free BCs, the DSC results are the same up to two digits of accuracy with the analytical ones for $N=21$ except mode 10.
- The accuracy of the finite element method for the fixed-fixed BC enhances with the increasing number of elements in higher order modes, as expected. In other BC cases, the results of only $N=120$ are presented since its deviations from the analytical results in fixed-fixed BC case are lower than 1%.
- In Tables 6.4-6.5, deviations of the natural frequencies between the DSC and the FEM are also obtained via $Deviation\% = (f_{n,FEM} - f_{n,DSC}) / f_{n,FEM} \times 100$. It is seen from Table 6.4, the deviations belonging to 21 and 31 discretization points are lower than 1%.
- It is seen that in all impedance boundary cases, when the other end is not fixed, the first mode is the mode of single degree of freedom system dominated by spring and total mass of the bar (i.e., bar mass+ point mass at the boundary).
- Consequently, it is seen that even for a very small number of discretization points in the DSC, the proposed methodology is quite effective and accurate when compared with the analytical and FEM results.

6.2.2 Frequency Response Function of the Bar

Next, the frequency response functions of the bar are determined for the same boundary condition cases considered in the free vibration analysis. The force is applied at $x=0.2$ m from the left boundary with the amplitude of 1 N, and frequency response functions (FRFs) are measured from the excitation points, i.e., driving point FRFs. Calculated FRFs are compared with those of FEM for fixed-fixed, fixed-free, free-free boundary conditions in Figure 6.3 and for the other boundary condition cases in Figure 6.4. Note that N is selected as 21 in DSC computations whereas 41 elements are used for FEM. This is due to FEM having lower than 1% error for the considered range (1-10000 Hz) in natural frequency calculations. Besides that,

damping constants of mechanical impedances used in FRFs are selected in accordance with $c = 0.1\sqrt{k_s m}$.

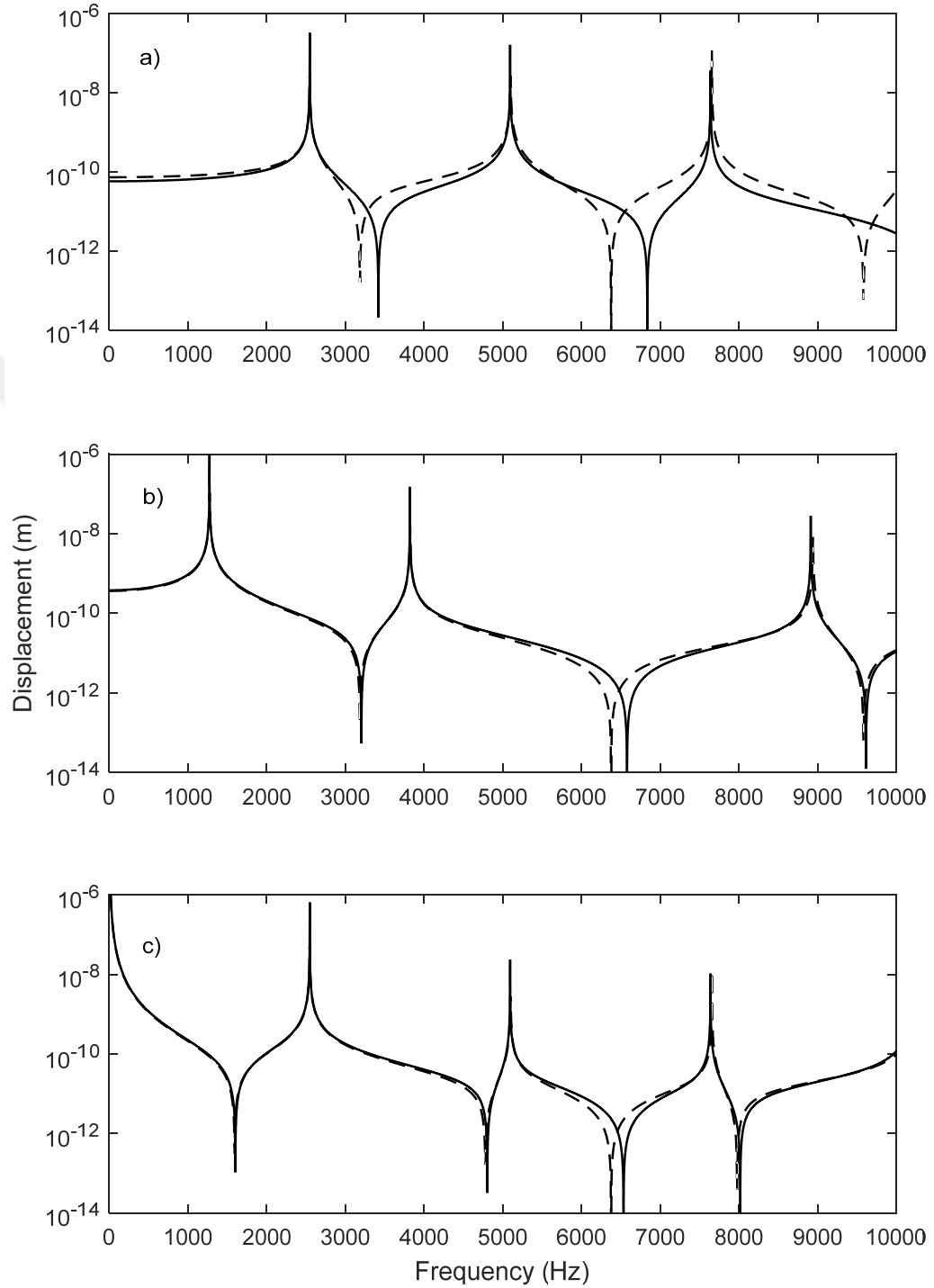


Figure 6.3 Frequency response function of the bar a) fixed-fixed bar, b) free-fixed bar, c) free-free bar (solid line: DSC; dash line: FEM)

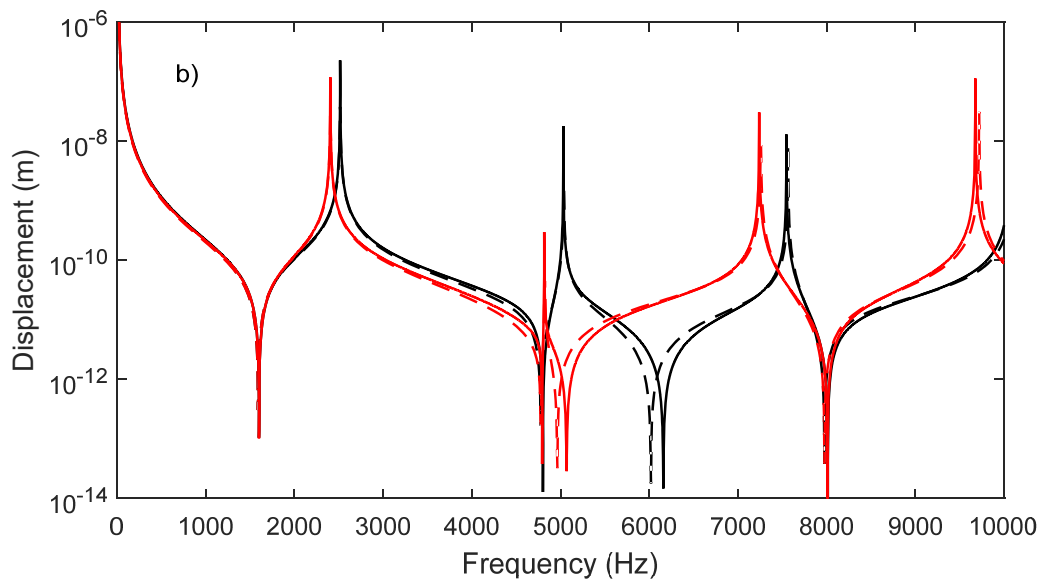
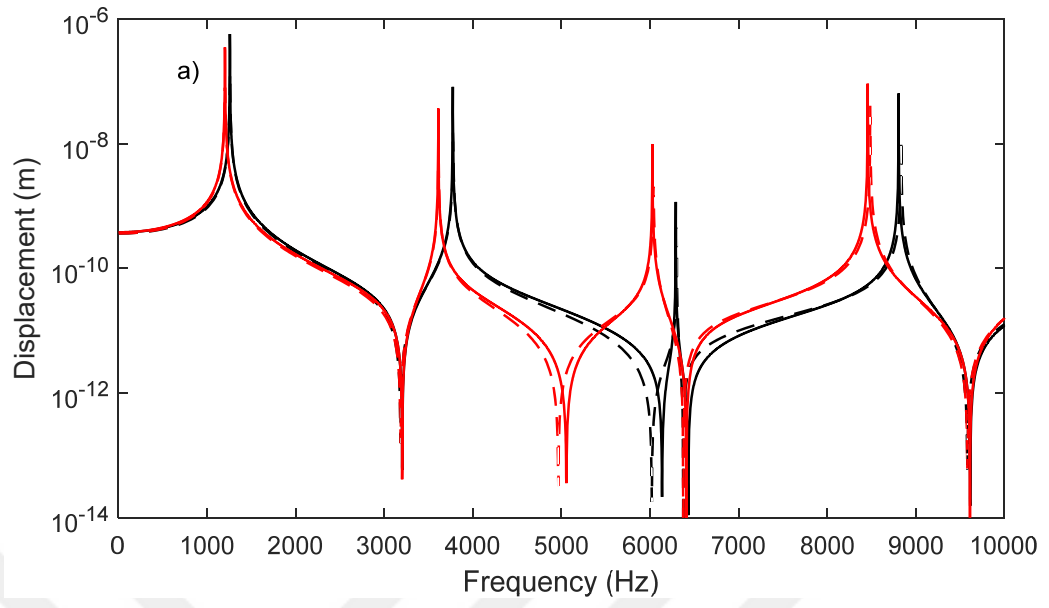


Figure 6.4 Frequency response function of the bar a) MI-fixed bar, b) MI-free bar, c) MI-MI bar (black/red solid line: DSC; black/red dash line: FEM; black: mass-damper-spring set 1; red mass-damper-spring set 2)

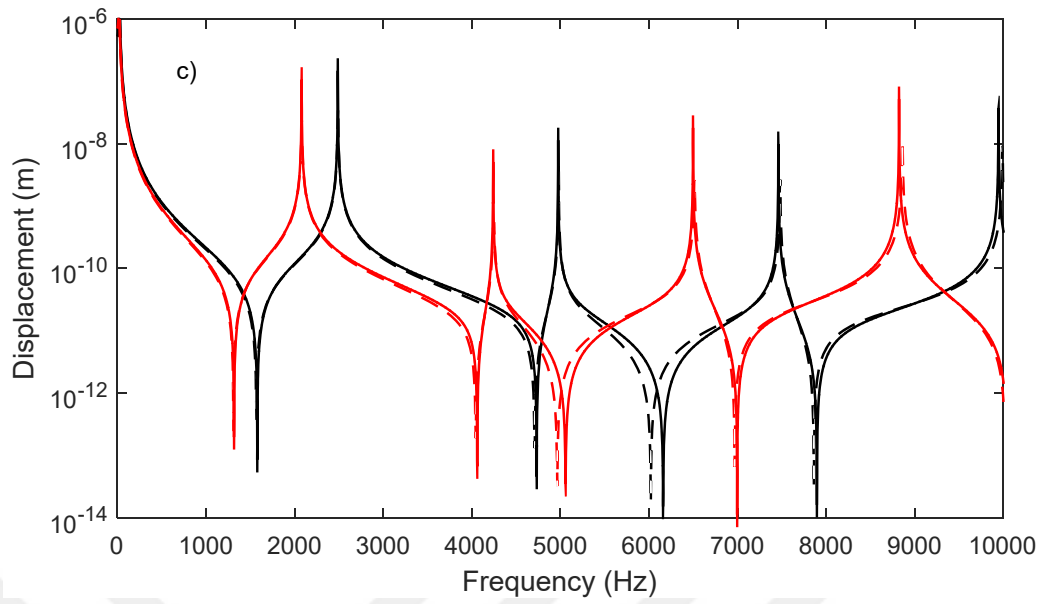


Figure 6.4 continues

Following conclusions can be drawn from Figures 6.3-6.4:

1. DSC results are very consistent with FEM computations including the mechanical impedance boundary conditions.
2. There are small deviations in anti-resonance frequency estimations. These are probably caused by numerical errors in either DSC or FEM or in both of them.
3. The proposed method is capable of estimating FRF of the bar with mechanical impedance boundary for different sets of impedances.

6.3 Free and Forced Acoustic Analysis of a Duct

6.3.1 Natural Frequencies of the Duct

In the analysis of ducts, the regularization parameter is selected as $r=3.1$ and the discretization number is selected as $N=21$ as selected for longitudinally vibrating bar since their equations of the motion are similar. Natural frequencies of the duct with the length of 1 m and diameter of 10 cm are calculated to demonstrate the accuracy of the DSC. Three boundary conditions are selected as test cases, i.e, hard-hard, soft-

hard and soft-soft. Calculated natural frequencies are compared with analytical ones in Table 6.6. *Error %* is also presented in Table 6.6.

Table 6.6 Natural frequencies (Hz) of the duct ($N=21$)

Mode seq.	Soft-Soft			Hard-Hard			Soft-Hard		
	DSC	Analy.	Error %	DSC	Analy.	Error %	DSC	Analy.	Error %
1	171.50	171.50	-8.36E-08	0.01	0.00	Undfn.	85.75	85.75	-2.38E-08
2	343.00	343.00	2.03E-08	171.50	171.50	-4.41E-08	257.25	257.25	7.94E-09
3	514.50	514.50	-8.62E-09	343.00	343.00	1.05E-08	428.75	428.75	-4.72E-09
4	686.00	686.00	4.53E-09	514.50	514.50	-4.23E-09	600.25	600.25	3.34E-09
5	857.50	857.50	-2.28E-09	686.00	686.00	2.06E-09	771.75	771.75	-2.48E-09
6	1029.00	1029.00	8.81E-09	857.50	857.50	-7.03E-10	943.25	943.25	3.63E-09
7	1200.50	1200.50	1.29E-07	1029.00	1029.00	7.71E-09	1114.75	1114.75	2.97E-08
8	1372.00	1372.00	1.92E-06	1200.50	1200.50	1.30E-07	1286.25	1286.25	5.13E-07
9	1543.50	1543.50	2.31E-05	1372.00	1372.00	1.92E-06	1457.75	1457.75	6.83E-06
10	1715.00	1715.00	2.24E-04	1543.50	1543.50	2.31E-05	1629.25	1629.25	7.38E-05

It is seen from Table 6.6 that, natural frequencies obtained via the DSC are consistent with the analytical ones even for very low number of discretization.

6.3.2 Application of the DSC to the Sound Propagation in Ducts with Impedance Boundary

Here, sound propagation in a circular duct with impedance boundaries is handled via the DSC. The duct is excited by a point monopole sound source at $x=0.2$ m. In the study, the volume velocity of the source is selected as $Q=10^{-5}$ m³s⁻¹. Methodology is tested by the calculation of sound pressure distribution along tube axis and by the frequency response of sound pressure at driving point. All results are compared with finite element solutions.

6.3.2.1 Sound Pressure Distribution in the Duct with Several Constant Impedance Boundaries

Although the surface acoustic impedance (Z_a) is mostly frequency dependent, the values of surface impedances are selected to be constant with respect to frequency in these analyses. Here six different boundary condition sets are constructed by using several surface impedance values corresponding to several impedance types such as

hard, soft, fully anechoic and insulation material etc. Sound pressure distribution is tested for two different frequencies; 500 Hz and 1000 Hz. In Table 6.7, boundary condition sets and frequencies corresponding to the figure numbers are presented. Note that N is selected as 41 in the DSC computations in order to present a good spatial distribution and similarly FEM results are extracted from 41 equally spaced spatial nodes.

Table 6.7 Boundary condition sets and frequencies corresponding to the figure numbers

Sets	$Z_{a,0}$ ($\text{kgm}^{-2}\text{s}^{-1}$)	$Z_{a,L}$ ($\text{kgm}^{-2}\text{s}^{-1}$)	500 Hz	1000 Hz
Set 1	10^{10} (Hard)	10^{10} (Hard)	Fig. 6.5a	Fig. 6.6a
Set 2	10^{-4} (Soft)	10^{-4} (Soft)	Fig. 6.5b	Fig. 6.6b
Set 3	420.17 (Fully anechoic)	420.17 (Fully anechoic)	Fig. 6.5c	Fig. 6.6c
Set 4	10^{-4} (Soft)	10^{10} (Hard)	Fig. 6.5d	Fig. 6.6d
Set 5	300+750j (Impedance 1)	10^{10} (Hard)	Fig. 6.5e	Fig. 6.6e
Set 6	300+750j (Impedance 1)	750+5000j (Impedance 2)	Fig. 6.5f	Fig. 6.6f

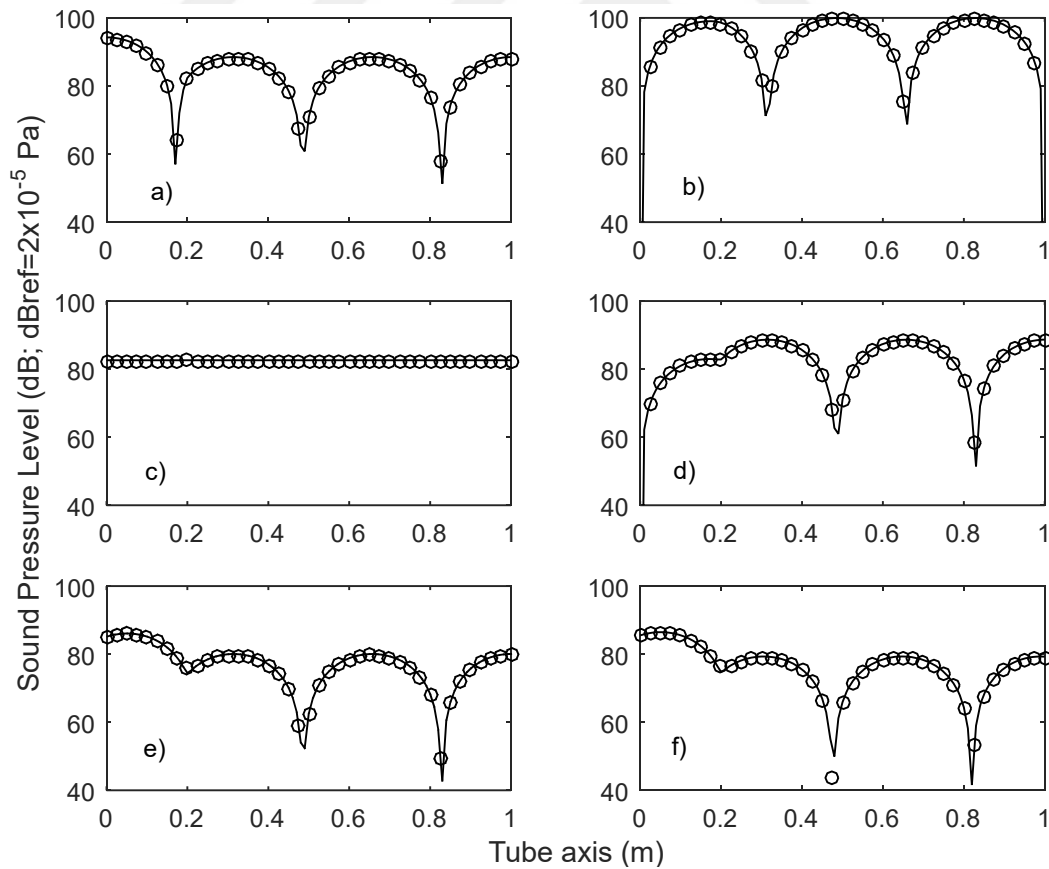


Figure 6.5 Sound pressure level distribution along the tube axis for different boundary condition sets at 500 Hz a) Set 1, b) Set 2, c) Set 3, d) Set 4, e) Set 5, f) Set 6 (solid line: DSC; 'o': FEM)

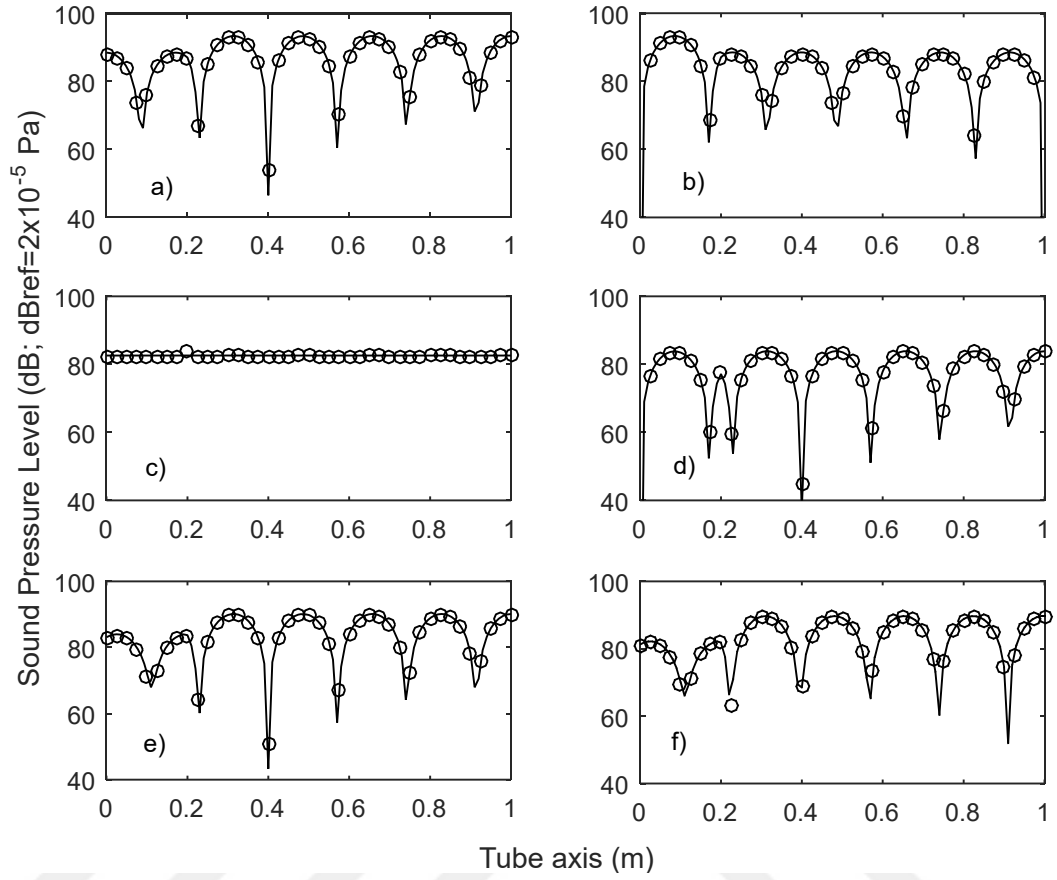


Figure 6.6 Sound pressure level distribution along the tube axis for different boundary condition sets at 1000 Hz a) Set 1, b) Set 2, c) Set 3, d) Set 4, e) Set 5, f) Set 6 (solid line: DSC; 'o': FEM)

It is seen from Figures 6.5 and 6.6 that, the calculated results are consistent with the finite element solutions for all impedance sets. This implies that the proposed methodology can be reliably used for the impedance boundary conditions in acoustics as well.

6.3.2.2 Frequency Response Function of the Duct with Frequency Dependent Impedance Boundary

In this step of the study, the frequency response of sound pressure of the duct with glasswool attached to both ends subjected to a monopole excitation is obtained. Frequency-dependent surface acoustic impedance of the glasswool is experimentally determined by impedance tube measurement according to the procedure given in

“ASTM E1050 – 12” (ASTM E1050 - 12, Standard test method for impedance and absorption of acoustical materials using a tube, two microphones and a digital frequency analysis system, 2012). A schematic representation of the experimental setup is shown in Figure 6.7. The procedure for the experiment to determine surface acoustic impedance is briefly outlined below (ASTM E1050 - 12, Standard test method for impedance and absorption of acoustical materials using a tube, two microphones and a digital frequency analysis system, 2012):

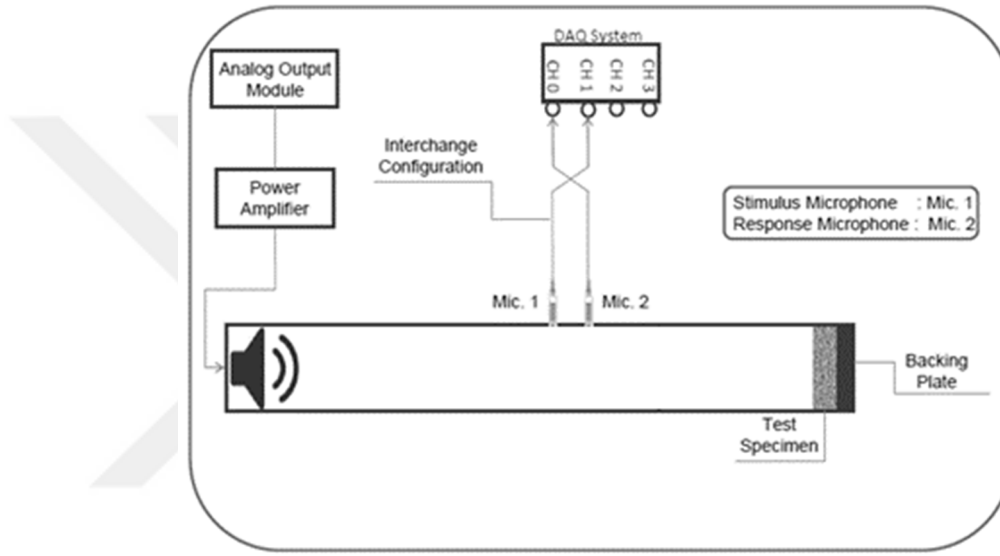


Figure 6.7 Schematic experimental setup for the impedance calculation of glasswool (Kara & Seçgin, 2019)

- A circular sample of the glasswool is placed to the one end of the impedance tube over a backing plate. Then, a loudspeaker is driven by a white noise from the other end of the tube.
- Complex frequency response function $H_{m,12} = P_{m,2}/P_{m,1}$ is determined by using the measured data from the microphones. Here $P_{m,i}$ denotes the sound pressure measured from i th microphone.
- Then, microphones are interchanged to disregard measurement errors, and $H_{m,21} = P_{m,1}/P_{m,2}$ is determined.

- After that, average of complex frequency response $H_a = \sqrt{H_{m,12} \cdot H_{m,21}}$ is obtained, and reflection coefficient of the glasswool is calculated by the following formula:

$$R = |R| \exp(j\phi_R) = \frac{H_a - \exp(-jk_w s_m)}{\exp(jk_w s_m) - H_a} \exp(j2k_w(L + s_m)). \quad (6.1)$$

Here, s_m is the distance between microphones, L is the distance between the sample and microphone 2 and k_w is the wavenumber. Now, one can determine the surface impedance of the material by,

$$Z_s = (\rho_s c_s) \frac{1 + R}{1 - R}. \quad (6.2)$$

According to the given procedure above, the real and the imaginary parts of the measured impedance are presented in Figure 6.8.

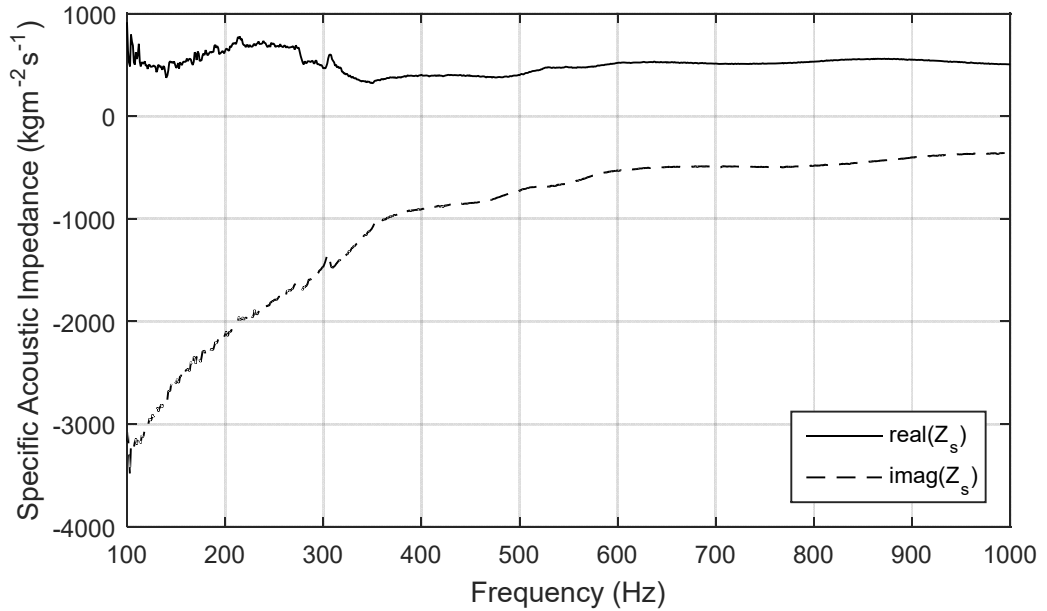
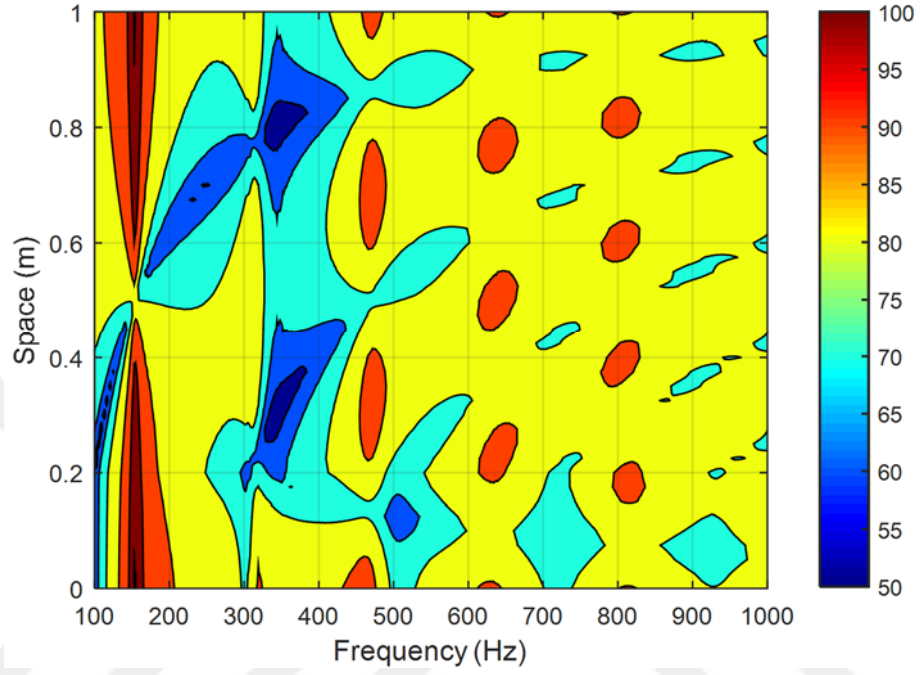


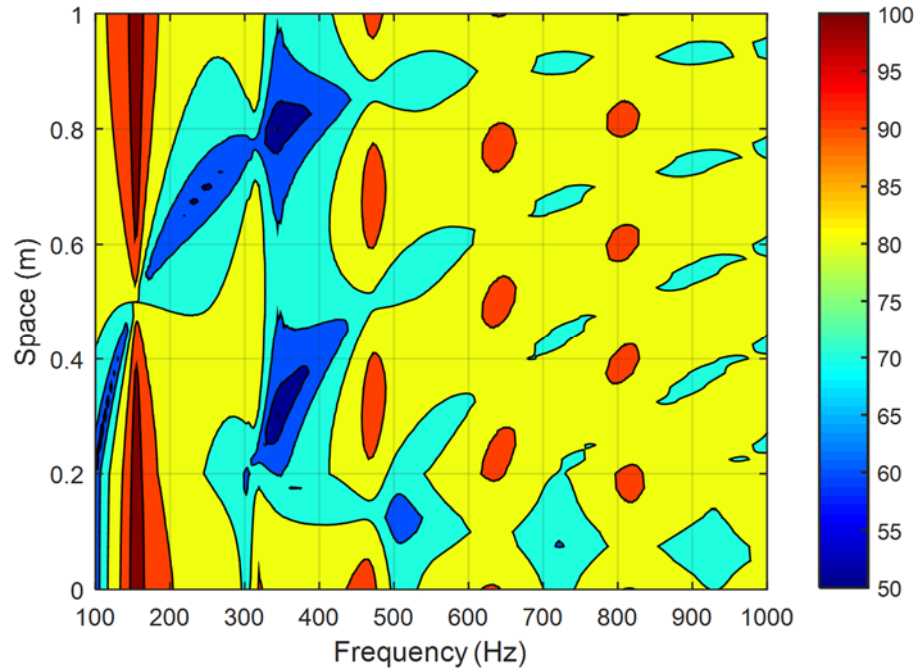
Figure 6.8 Measured real and imaginary parts of surface acoustic impedance (Z_s) of glasswool

The measured impedances are then fed to the DSC and FEM to obtain the response of the duct. In this manner, the sound pressure level with respect to

frequency and space is presented in Figure 6.9. Beside that, the frequency responses of the sound pressure for several spatial points; $x=0.1$ m, $x=0.2$ m (driving point), $x=0.4$ m, $x=0.5$ m, $x=0.75$ m are also presented in Figure 6.10 to compare the DSC and FEM results, easily.



a)



b)

Figure 6.9 Sound pressure level (dB) response of the duct with respect to frequency and space, a) DSC, b) FEM

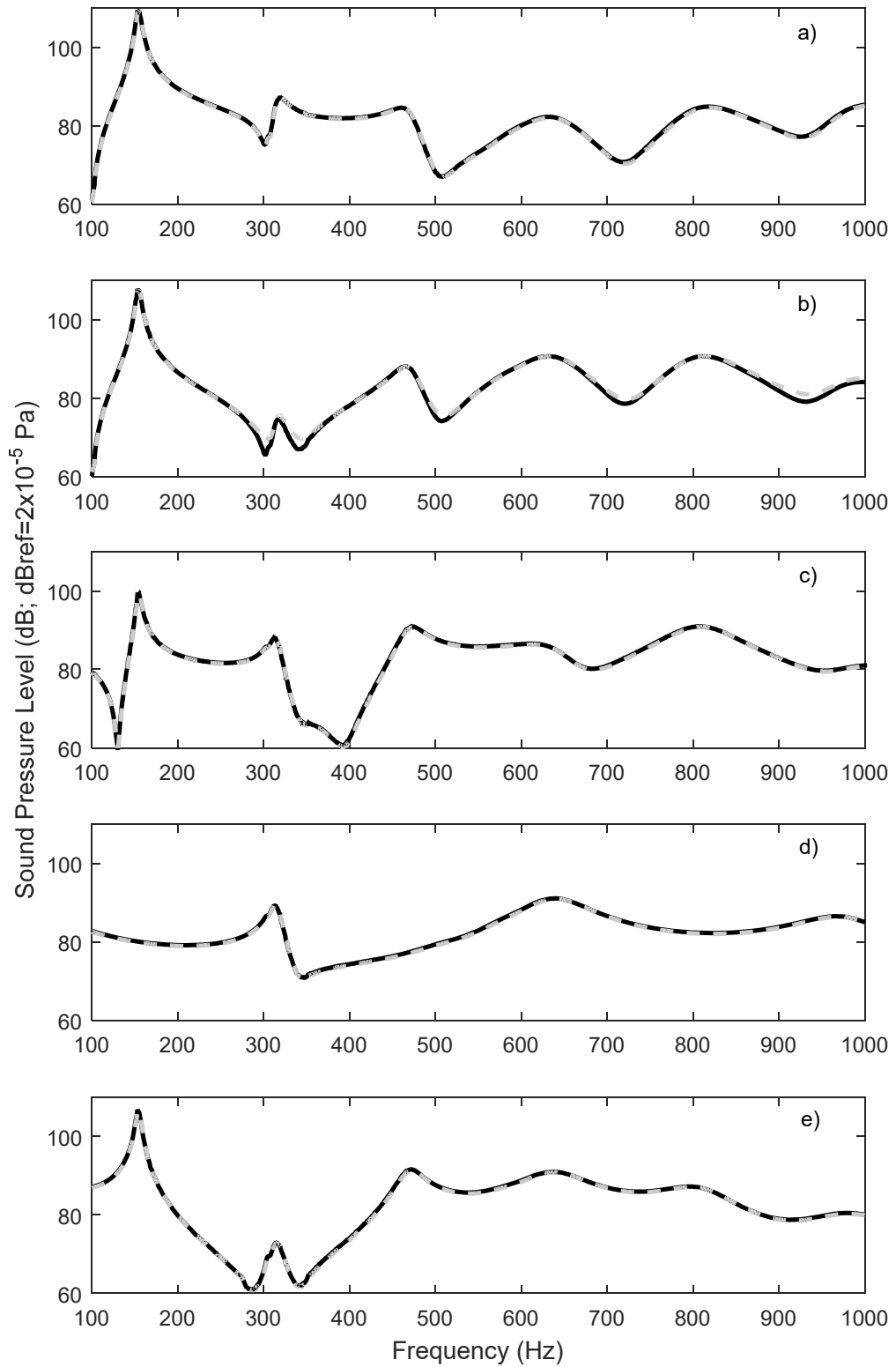


Figure 6.10 Frequency response function of the duct at a) $x=0.1$ m, b) $x=0.2$ m (driving point), c) $x=0.4$ m, d) $x=0.5$ m, e) $x=0.75$ m (black solid line: DSC, gray dash line: finite element)

It is seen from Figure 6.9 that, the results show the similar behaviour in all frequencies and axial positions. However, the amplitudes of sound pressure level are not clearly monitored in this Figure. Therefore, it is useful to compare the DSC and FEM for certain axial positions between 100-1000 Hz. In this regard, it may be observed from Figure 6.10 that, the amplitudes of the DSC results perfectly match with the ones of FEM.

Results presented in Section 6.2 and 6.3 imply that, the DSC-TSE 1 is reliably applicable for systems having complex boundary conditions such as Robin boundary conditions, in free and forced vibration and acoustic problems.

6.4 Free Vibration Analysis of a Bending Thin Beam

In this section, the natural frequencies of uniform and inhomogeneous (non-uniform) thin beams are determined by DSC-TSE, and the DSC results are compared with FEM results.

6.4.1 Free Vibration Analysis of a Uniform Beam

The natural frequency parameters of a uniform beam are determined by using the DSC under various boundary conditions; i) clamped-clamped (C-C), ii) pinned–pinned (P-P), iii) free-free (F-F) and iv) clamped-free (C-F). Convergence study is performed for all boundary conditions with different number of discretization points. Note that, DSC-TSE 1 and DSC-TSE 2 approaches are applied for free boundary conditions and the classical implementation procedure is applied for pinned and clamped ends. In the analysis, 20 vibration modes are considered with $M=N-1$ and $r=N/10+1$. Computed natural frequency parameters (λ in Eq. (2.18)) are tabulated in Tables 6.8-6.9. *Error%* of natural frequencies is plotted in Figures 6.11-6.12. Note that, analytical natural frequencies are obtained by using Table 2.4.

Table 6.8 Natural frequency parameters of P-P and C-C beams (λ)

Mode Seq.	P-P				C-C			
	Analy.	DSC			Analy.	DSC		
		N=21	N=31	N=41		N=21	N=31	N=41
1	3.142	3.142	3.142	3.142	4.730	4.730	4.730	4.730
2	6.283	6.283	6.283	6.283	7.853	7.854	7.853	7.853
3	9.425	9.425	9.425	9.425	10.996	10.998	10.996	10.996
4	12.566	12.566	12.566	12.566	14.137	14.143	14.139	14.138
5	15.708	15.708	15.708	15.708	17.279	17.289	17.282	17.280
6	18.850	18.850	18.850	18.850	20.420	20.438	20.425	20.422
7	21.991	21.991	21.991	21.991	23.562	23.589	23.569	23.565
8	25.133	25.133	25.133	25.133	26.704	26.743	26.714	26.708
9	28.274	28.274	28.274	28.274	29.845	29.901	29.860	29.851
10	31.416	31.421	31.416	31.416	32.987	33.066	33.006	32.995
11	34.558	34.560	34.558	34.558	36.128	36.245	36.154	36.139
12	37.699	37.712	37.699	37.699	39.270	39.462	39.303	39.283
13	40.841	40.899	40.841	40.841	42.412	42.776	42.453	42.428
14	43.982	44.189	43.982	43.982	45.553	46.294	45.605	45.574
15	47.124	47.703	47.124	47.124	48.695	50.100	48.759	48.720
16	50.265	51.530	50.265	50.265	51.836	54.083	51.914	51.867
17	53.407	55.537	53.407	53.407	54.978	57.809	55.072	55.015
18	56.549	59.226	56.549	56.549	58.119	60.682	58.233	58.163
19	59.690	61.869	59.691	59.690	61.261	62.331	61.399	61.313
20	62.832		62.833	62.832	64.403		64.574	64.463

Table 6.9 Natural frequency parameters of F-F and C-F beams (λ)

Mode Seq.	F-F							C-F						
	Analy.	DSC-TSE 1			DSC-TSE 2			Analy.	DSC-TSE 1			DSC-TSE 2		
		N=21	N=31	N=41	N=21	N=31	N=41		N=21	N=31	N=41	N=21	N=31	N=41
1	0.000	0.093	0.030	0.017	0.093	0.030	0.016	1.875	1.875	1.875	1.875	1.875	1.875	1.875
2	4.730	4.721	4.726	4.728	4.730	4.730	4.730	4.694	4.690	4.692	4.693	4.694	4.694	4.694
3	7.853	7.830	7.842	7.847	7.853	7.853	7.853	7.855	7.843	7.849	7.852	7.855	7.855	7.855
4	10.996	10.951	10.975	10.984	10.996	10.996	10.996	10.996	10.974	10.985	10.990	10.997	10.996	10.996
5	14.137	14.066	14.103	14.118	14.138	14.137	14.137	14.137	14.104	14.121	14.128	14.140	14.138	14.137
6	17.279	17.177	17.229	17.250	17.280	17.279	17.279	17.279	17.233	17.255	17.265	17.285	17.280	17.279
7	20.420	20.284	20.353	20.381	20.423	20.421	20.420	20.420	20.361	20.389	20.402	20.430	20.423	20.421
8	23.562	23.390	23.475	23.510	23.568	23.563	23.562	23.562	23.489	23.522	23.538	23.578	23.566	23.563
9	26.704	26.496	26.594	26.638	26.715	26.705	26.704	26.704	26.618	26.654	26.673	26.729	26.709	26.706
10	29.845	29.604	29.713	29.765	29.865	29.847	29.846	29.845	29.751	29.786	29.808	29.883	29.853	29.848
11	32.987	32.718	32.830	32.891	33.020	32.990	32.988	32.987	32.891	32.918	32.943	33.043	32.998	32.991
12	36.128	35.846	35.946	36.016	36.186	36.134	36.130	36.128	36.044	36.050	36.077	36.216	36.144	36.134
13	39.270	39.013	39.062	39.140	39.376	39.278	39.272	39.270	39.236	39.182	39.211	39.419	39.291	39.278
14	42.412	42.271	42.179	42.263	42.616	42.423	42.414	42.412	42.522	42.315	42.345	42.695	42.438	42.421
15	45.553	45.710	45.296	45.386	45.960	45.570	45.557	45.553	45.999	45.450	45.479	46.121	45.587	45.566
16	48.695	49.403	48.415	48.508	49.467	48.718	48.700	48.695	49.746	48.586	48.614	49.770	48.738	48.710
17	51.836	53.263	51.536	51.630	53.140	51.867	51.843	51.836	53.667	51.724	51.748	53.596	51.891	51.855
18	54.978	56.964	54.660	54.752	56.787	55.019	54.987	54.978	57.386	54.865	54.883	57.298	55.046	55.001
19	58.119	60.051	57.790	57.875	59.948	58.174	58.132	58.119	60.379	58.010	58.019	60.335	58.204	58.148
20	61.261	62.107	60.926	60.998	62.084	61.333	61.277	61.261	62.232	61.161	61.155	62.223	61.366	61.295

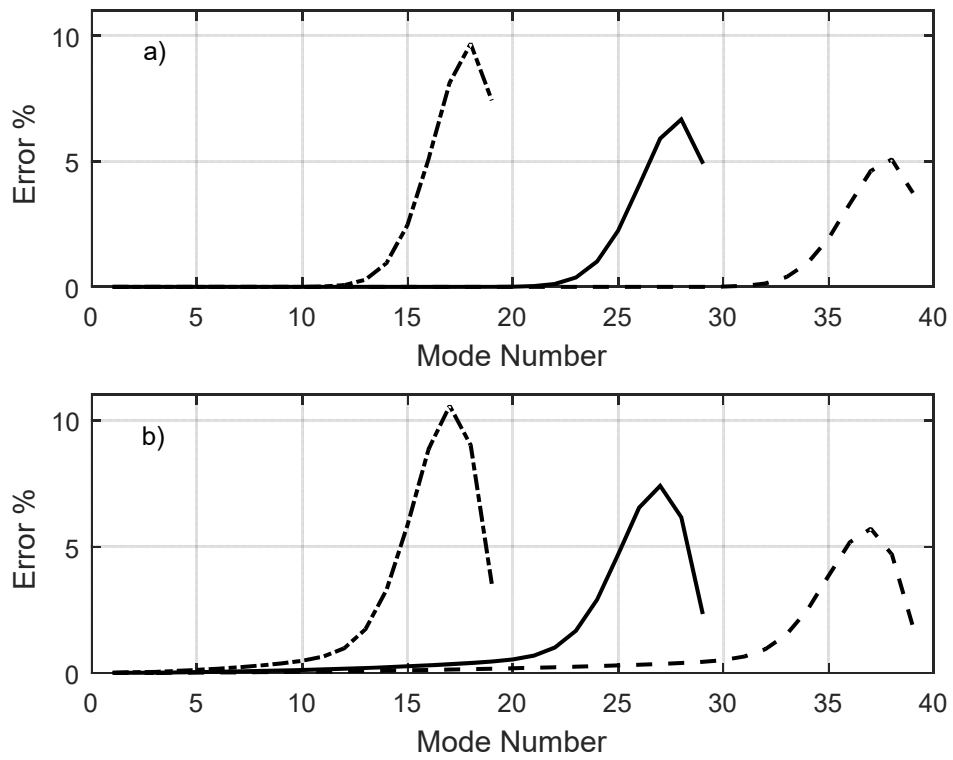


Figure 6.11 Error curve of beam a) P-P, b) C-C (dash dotted line: $N=21$, solid line: $N=31$, dashed line: $N=41$)

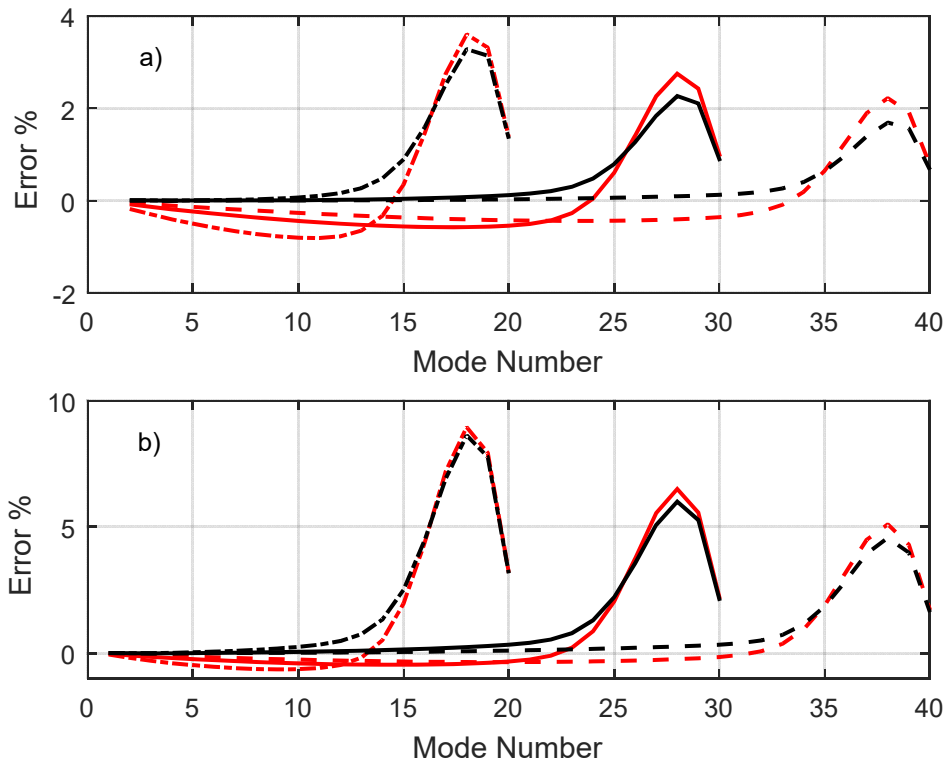


Figure 6.12 Error curves of beam with a) F-F, b) C-F (dash dotted line: $N=21$, solid line: $N=31$, dashed line: $N=41$, red: DSC-TSE1, black: DSC-TSE2)

Followings can be observed from Table 6.7, Table 6.8, Figure 6.11 and Figure 6.12:

- Predicted natural frequency parameters are consistent with analytical results for all BCs. Besides, the best results are obtained for pinned-pinned BC.
- The better results are computed when the discretization number is increased. $N=31$ may be selected for the further analyses when 1% is selected as the maximum allowable *Error%* in the computations.
- DSC-TSE 1 and DSC-TSE 2 approaches calculate a residual natural frequency parameter around zero as presented in Table 6.9.
- DSC-TSE 2 approach estimates the natural frequency parameters better than DSC-TSE 1 approach. This is due to the fact that, DSC-TSE 2 takes into account higher order derivatives than DSC-TSE 1.
- *Error%* is gradually increases in the higher order modes in the computations for all considered BCs. However, natural frequency parameters are estimated as lower than the analytical ones for the lower order modes for the DSC-TSE 1, i.e., an underestimating phenomenon.

After determining the number of discretization points as $N=31$, it is useful to optimize the regularization coefficient for the first mode for the beam with C-C, C-F and F-F boundaries. In this regard, regularization coefficient varied between 1.9-6.1 with the step of 0.1. The optimum regularization coefficient is obtained as 4.5 for C-C boundary, 4.1 (already used in computations) for C-F boundary and 4.9 for F-F boundary. Note that, regularization coefficient is optimized for DSC-TSE 2 approach in free boundary cases. The natural frequency parameters, which are presented in Tables 6.8 and 6.9, are updated in accordance with the optimized regularization coefficient for C-C and F-F boundaries in Table 6.10. Mode shapes of the beam for the considered boundary conditions are presented in Figures 6.13-6.16.

Table 6.10 Natural frequency parameters of the beam for the optimized regularization parameter (λ)

Mode Seq.	C-C			F-F		
	Analy.	r=4.1	r=4.5	Analy.	r=4.1	r=4.9
1	4.730	4.730	4.730	0.000	0.654	0.030
2	7.853	7.853	7.853	4.730	4.726	4.730
3	10.996	10.996	10.996	7.853	7.842	7.853
4	14.137	14.139	14.139	10.996	10.975	10.996
5	17.279	17.282	17.282	14.137	14.103	14.137
6	20.420	20.425	20.425	17.279	17.229	17.279
7	23.562	23.569	23.569	20.420	20.353	20.421
8	26.704	26.714	26.714	23.562	23.474	23.563
9	29.845	29.860	29.859	26.704	26.594	26.705
10	32.987	33.006	33.006	29.845	29.712	29.847
11	36.128	36.154	36.154	32.987	32.829	32.990
12	39.270	39.303	39.302	36.128	35.945	36.134
13	42.412	42.453	42.453	39.270	39.061	39.278
14	45.553	45.605	45.604	42.412	42.176	42.423
15	48.695	48.759	48.758	45.553	45.293	45.570
16	51.836	51.914	51.913	48.695	48.411	48.718
17	54.978	55.072	55.070	51.836	51.530	51.867
18	58.119	58.233	58.230	54.978	54.653	55.019
19	61.261	61.399	61.393	58.119	57.779	58.174
20	64.403	64.574	64.561	61.261	60.910	61.333

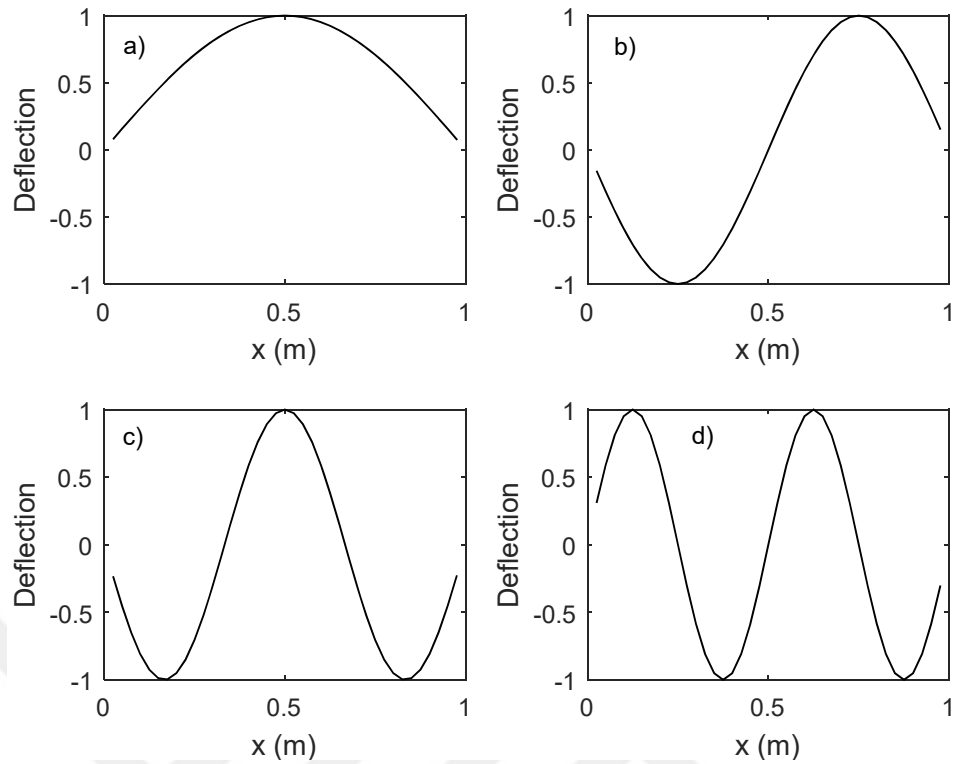


Figure 6.13 Mode shapes of P-P beam a) first mode, b) second mode, c) third mode, d) fourth mode

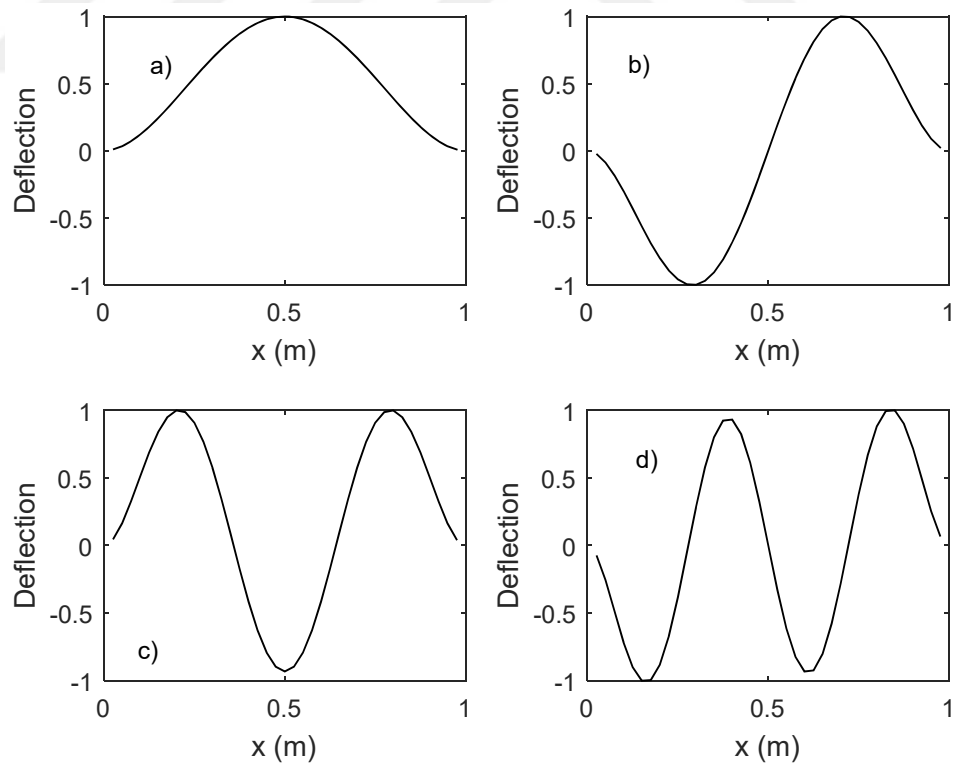


Figure 6.14 Mode shapes of C-C beam a) first mode, b) second mode, c) third mode, d) fourth mode

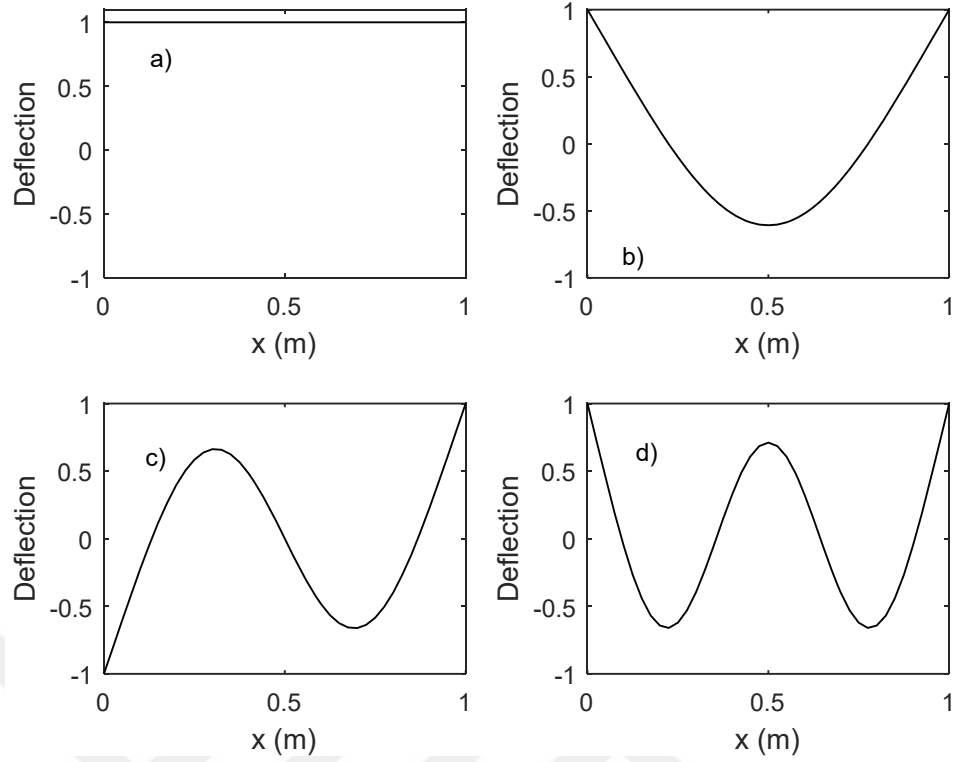


Figure 6.15 Mode shapes of F-F beam a) first mode, b) second mode, c) third mode, d) fourth mode

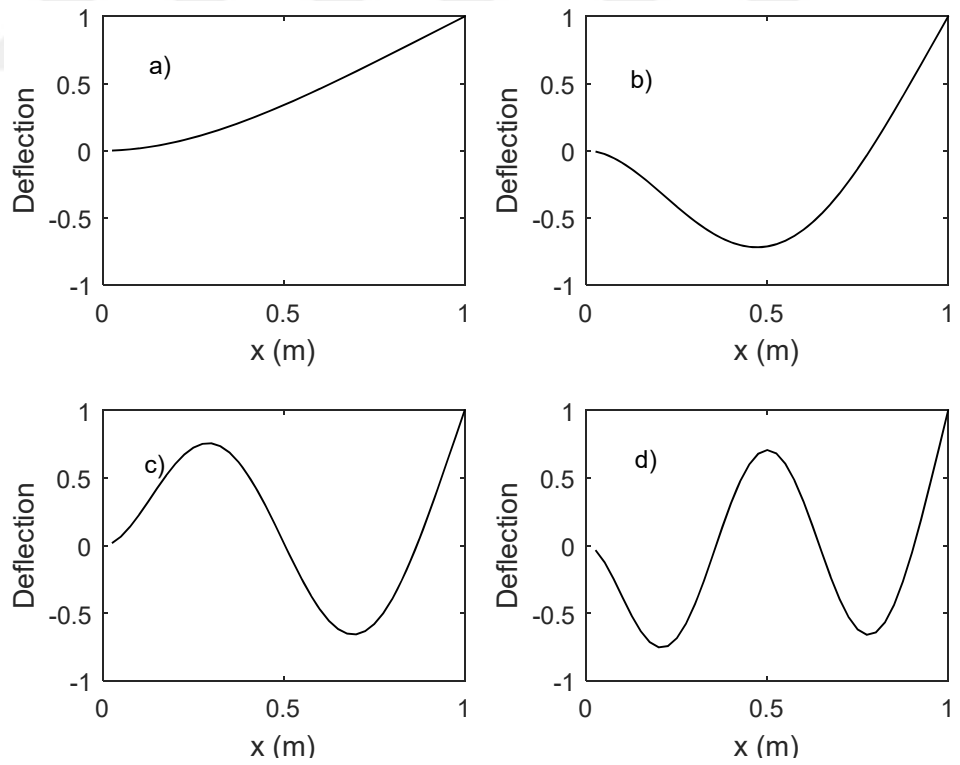


Figure 6.16 Mode shapes of C-F beam a) first mode, b) second mode, c) third mode, d) fourth mode

It is seen from Table 6.10 that, the results evaluated for optimized regularization parameter converge better than the ones of in Tables 6.8 and 6.9. Beside that, mode shapes show the expected characteristics of the boundary conditions.

6.4.2 Free Vibration Analysis of an Inhomogeneous Beam

6.4.2.1 The DSC Modelling Procedure of an Inhomogeneous Beam

Assume that the beam considered in Chapter 2 has constant width b , has thickness with the form of $h(x) = h(1 + \beta x)^2$ (where $\beta = (\sqrt{h_l/h} - 1)/l$, h and h_l are the thickness at $x=0$ and $x=l$, respectively,) has Young modulus in the form of $E(x) = E(1 + \gamma(x))$ (where E is spatial mean of Young modulus and $\gamma(x)$ denotes the space dependency). In this context, Eq. (2.10) can be rewritten for harmonic free vibration response ($w(x, t) = W(x) \exp(j\omega_n t)$) as follows:

$$\frac{\partial^2}{\partial x^2} \left[E(1 + \gamma(x)) \frac{h^3(1 + \beta x)^6}{12} \frac{\partial^2 W(x)}{\partial x^2} \right] - \omega_n^2 \rho(x) h(1 + \beta x)^2 W(x) = 0. \quad (6.3)$$

Eq. (6.3) can be expanded with noting that $r_\rho(x) = 1/\rho(x)$ as specific volume as;

$$\begin{aligned} & Eh^2 r_\rho(x) \left[\frac{1}{12} (1 + \gamma(x)) (1 + \beta x)^4 W^{(4)}(x) + \right. \\ & + \left((1 + \gamma(x)) \beta (1 + \beta x)^3 + \frac{1}{6} \gamma^{(1)}(x) (1 + \beta x)^4 \right) W^{(3)}(x) + \\ & + \left(\frac{1}{12} \gamma^{(2)}(x) (1 + \beta x)^4 + \gamma^{(1)}(x) \beta (1 + \beta x)^3 \right) W^{(2)}(x) + \\ & \left. \left(+ \frac{5}{2} (1 + \gamma(x)) \beta^2 (1 + \beta x)^2 \right) W^{(2)}(x, \vartheta) \right] - \lambda_n W(x) = 0. \end{aligned} \quad (6.4)$$

Here, $\lambda_n = \omega_n^2$ is the eigenvalue of the structure. One can follow the following steps in order to model a non-uniform thin beam via DSC:

1. The considered beam is discretized as shown in Figure 3.1. Eq. (6.4) is written for each structural point.
2. Eq. (3.12) is substituted for the derivative terms of equations obtained in Step 1.
3. Proper boundary condition implementation procedure is performed to get rid of the points in ghost domains.
4. Next, obtained set of equations (N independent equations) can be written in the following form:

$$(Eh^2r_\rho \mathbf{D}(\mathbf{x}) - \lambda_n \mathbf{I}) \mathbf{w}(\mathbf{x}) = 0, \quad (6.5)$$

where, $\mathbf{D}(\mathbf{x})$ is the dynamic stiffness matrix, $\mathbf{x} = \{X_0, X_1, \dots, X_{N-1}\}^T$ represents the vector for the discretized structural points and $\mathbf{w} = \{W_0, W_1, \dots, W_{N-1}\}^T$ is the corresponding bending displacement vector to the structural points. Note that, the specific volume is taken as uniform in Eq. (6.5). To show the content of the dynamic stiffness matrix, i th row of the matrix which belongs to $(i-1)$ th structural point is given as:

$$\begin{aligned} \{\mathbf{D}(\mathbf{x})\}_{1 \times N} = & \frac{1}{12}(1 + \gamma(X_{i-1}))(1 + \beta X_{i-1})^4 \{\mathbf{D}^{(4)}\} + \\ & + \left((1 + \gamma(X_{i-1}))\beta(1 + \beta X_{i-1})^3 + \frac{1}{6}\gamma^{(1)}(X_{i-1})(1 + \beta X_{i-1})^4 \right) \{\mathbf{D}^{(3)}\} + \\ & + \left(\frac{1}{12}\gamma^{(2)}(X_{i-1})(1 + \beta X_{i-1})^4 + \gamma^{(1)}(X_{i-1})\beta(1 + \beta X_{i-1})^3 \right) \{\mathbf{D}^{(2)}\} \\ & + \left(\frac{5}{2}(1 + \gamma(X_{i-1}))\beta^2(1 + \beta X_{i-1})^2 \right) \{\mathbf{D}^{(2)}\}. \end{aligned} \quad (6.6)$$

Here, $\{\mathbf{D}^{(n)}\}$ represents i th row of characteristic DSC matrix of differentiation order of n .

6.4.2.2 Numerical Studies for an Inhomogeneous Beam

In this part of the study, the natural frequencies of a thin, non-uniform beam with pinned boundary condition are computed via DSC. The mechanical and the physical properties of the beam are shown in Table 6.11. Here the following studies are performed:

Table 6.11 Mechanical and physical properties of the beam

Property (<i>Unit</i>)	Value
Young modulus (N/mm^2)	32.5
Density (kg/m^3)	2500
Thickness at $x=0$ (m)	0.25
Thickness at $x=l$ (m)	0.175
Width (m)	0.3
Length (m)	15

- Firstly, uniform thin beam is considered and first ten natural frequencies are obtained via the DSC and finite element method (FEM) with different numbers of elements ($N=20, 40, 120$ and 230). Since the natural frequency parameters are computed by the DSC method (Table 6.8), analyses are not again performed with different number of discretization numbers. However, the results of the DSC with $N=31$ are compared with the analytical solutions and FEM in Table 6.12.
- Next, free vibration analysis of the non-uniform beam (the beam with variable geometry and material properties in space) is performed. In this regard, the natural frequencies are determined for i) the beam with non-uniform thickness, ii) the beam with non-uniform Young modulus, iii) the beam with non-uniform thickness and Young modulus. Note that, $\gamma(x)$ is selected as in the form of $\gamma(x) = \gamma_0 \cos(2\pi x/l) + 0.5\gamma_0 \sin(4\pi x/l)$ where $\gamma_0 = 0.02$. Note that, non-uniformity of the thickness and Young modulus are presented together with those of uniform case in Figure 6.17. Computed natural frequencies are given in Table 6.13.

Table 6.12 Natural frequencies of the uniform beam (r/s)

Mode Seq.	Analy.	DSC N=31	FEM			
			N=20	N=60	N=120	N=230
1	11.414	11.414	11.414	11.414	11.414	11.414
2	45.656	45.656	45.656	45.656	45.656	45.656
3	102.726	102.726	102.730	102.726	102.726	102.726
4	182.624	182.624	182.644	182.624	182.624	182.624
5	285.350	285.350	285.424	285.351	285.350	285.350
6	410.904	410.904	411.124	410.907	410.905	410.904
7	559.287	559.287	559.835	559.294	559.287	559.287
8	730.497	730.497	731.707	730.512	730.498	730.497
9	924.535	924.535	926.960	924.566	924.537	924.535
10	1141.401	1141.401	1145.906	1141.460	1141.405	1141.401

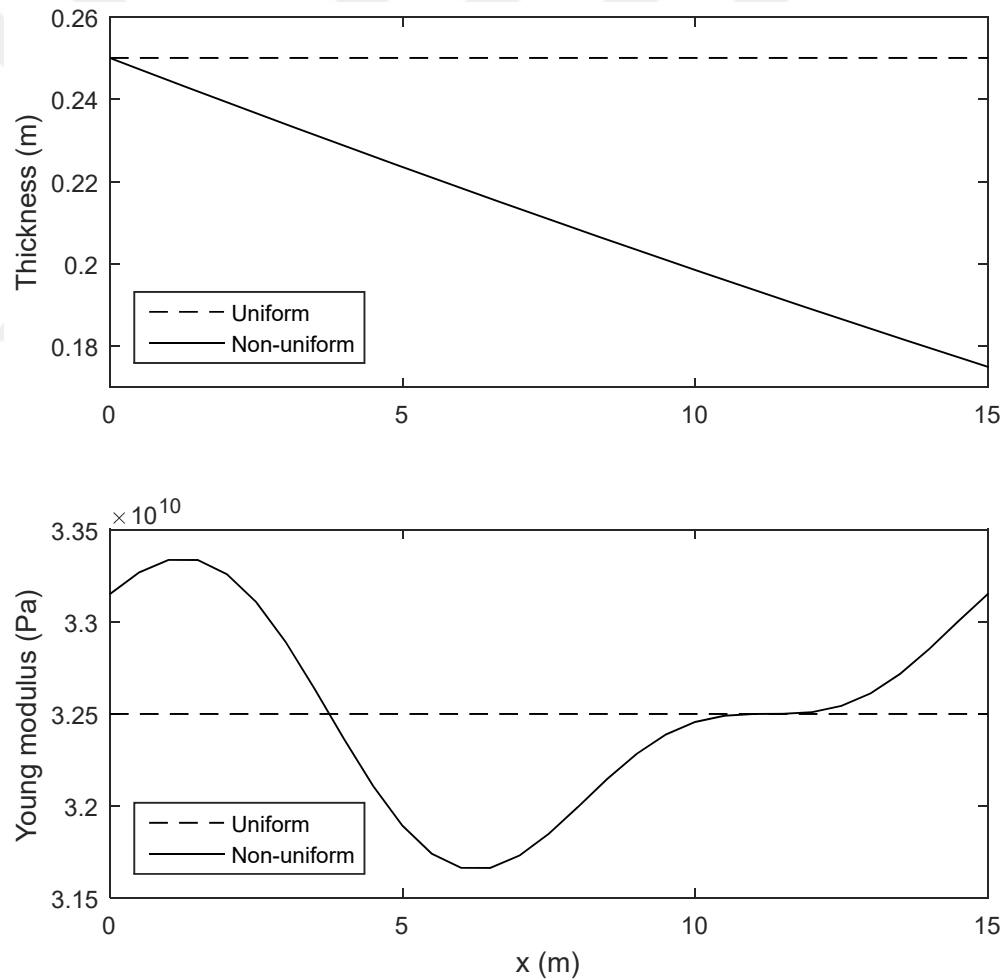


Figure 6.17 Non-uniform properties with respect to beam axis

Table 6.13 Natural frequencies of the non-uniform beam (r/s)

Mode Seq.	Non-uniform thickness			Non-uniform Young modulus		Non-uniform thickness and Young modulus	
	DSC <i>N</i> =31	FEM <i>N</i> =230	(Tan, Wang, & Jiao, 2016)	DSC <i>N</i> =31	FEM <i>N</i> =230	DSC <i>N</i> =31	FEM <i>N</i> =230
1	9.573	9.535	9.535	10.861	11.356	9.143	9.489
2	38.669	38.353	38.353	45.631	45.652	38.444	38.320
3	86.644	86.254	86.201	102.727	102.719	86.774	86.259
4	153.716	153.292		182.618	182.611	153.786	153.283
5	239.916	239.474		285.337	285.329	239.960	239.453
6	345.253	344.805		410.883	410.873	345.277	344.768
7	469.733	469.283		559.254	559.244	469.736	469.230
8	613.357	612.911		730.452	730.441	613.339	612.838
9	776.127	775.688		924.475	924.464	776.085	775.593
10	958.043	957.614		1141.325	1141.313	957.977	957.496

Followings can be inferred from Tables 6.12-6.13:

1. It is seen from Table 6.12 that, the DSC results have perfect accuracy. Besides, FEM is capable of estimating the natural frequencies up to 3 digits accuracy with $N=230$ elements. However, discretization number of $N=31$ is enough for the same accuracy for the DSC method. Therefore, $N=31$ for DSC and $N=230$ for FEM are selected in the following analyses.
2. For the beam with non-uniform thickness, the results of Tan et. al. (2016) is also used for comparison. In that study, an exact approach for a thin beam with the same mechanical and physical properties is considered. Results of Tan et. al. (2016) are very consistent with FEM results. Therefore, FEM results may be taken as reference for the other non-uniform cases.
3. For the non-uniform structures, the DSC and FEM results are very consistent with each other (Table 6.13).
4. To sum up, it can be said that the DSC method is very successful in the analysis of non-uniform bending beams.

6.5 Free Vibration Analysis of Bending Plates

According to the procedure given in Section 4.4, the natural frequency parameters of a thin plate are determined and compared with some of the results available in literature. The mechanical properties and geometry of the plate are given in Table 6.14. The results are obtained for several different boundary conditions with the combination of free (F), pinned (P), clamped (C) boundaries and they are presented in Tables 6.15-6.19. Note that, results are compared with several methods. Mode shapes of PPPP, FCCC, FPPP, FFPP and FFFF plates are also given in Figures 6.18-6.22, respectively.

Table 6.14 Mechanical properties and dimensions of the plate

Property [Unit]	Value
Elasticity(E) [GPa]	70
Poisson's ratio (ν)	0.3
Density (ρ) [kg/m^3]	2700
Dimensions [$m \times m \times m$]	1 x 1 x 1e-3

Table 6.15 Natural frequencies parameters (λ in Eq. (2.29)) of fully simply-supported thin plate ($N_x \times N_y = 15 \times 15$)

Mode Seq.	Analytical	DSC
1	19.739	19.741
2	49.348	49.348
3	49.348	49.348
4	78.957	78.956
5	98.696	98.696
6	98.696	98.696
7	128.304	128.304
8	128.304	128.304
9	167.783	167.784
10	167.783	167.784

Table 6.16 Natural frequency parameters of FCCC and FPPP plates (λ in Eq. (2.29))

B.C.	FCCC						FPPP					
Mode Seq.	DSC-TSE 1	DSC-TSE 2	GDQ (Yu, Xiang, & Wei, 2009)	MIB (Yu et al., 2009)	FEM	Analy. Approx. (Leissa, 1973)	DSC-TSE 1	DSC-TSE 2	GDQ (Yu et al., 2009)	MIB (Yu et al., 2009)	FEM	Analy. Approx. (Leissa, 1973)
	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$		$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	
1	23.840	23.953	24.025	23.987	24.204	24.020	11.664	11.683	11.685	11.685	11.734	11.685
2	39.784	39.973	40.147	40.194	40.321	40.039	27.669	27.749	27.756	27.756	27.887	27.756
3	62.942	63.258	63.494	63.325	65.525	63.493	41.066	41.176	41.197	41.193	42.047	41.197
4	76.267	76.682	76.845	76.913	78.512	76.761	58.810	59.023	59.066	59.064	59.889	59.066
5	80.110	80.527	80.901	80.809	82.577	80.713	61.563	61.845	61.861	61.859	63.012	61.861

Table 6.17 Natural frequency parameters of FFPP and FFPC plates (λ in Eq. (2.29))

B.C.	FFPP						FFPC					
Mode Seq.	DSC-TSE 1	DSC-TSE 2	GDQ (Yu et al., 2009)	MIB (Yu et al., 2009)	FEM	Analy. Approx. (Leissa, 1973)	DSC-TSE 1	DSC-TSE 2	GDQ (Yu et al., 2009)	MIB (Yu et al., 2009)	FEM	Analy. Approx. (Leissa, 1973)
	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$		$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	
1	3.353	3.375	3.549	3.483	3.369	3.369	5.327	5.364	5.780	5.746	5.354	5.364
2	17.224	17.311	17.316	17.321	17.393	17.407	18.936	19.072	20.703	19.467	19.165	19.171
3	19.163	19.298	17.662	19.623	19.411	19.367	24.500	24.683	20.926	24.553	24.908	24.768
4	37.867	38.183	36.576	38.647	38.440	38.291	42.642	43.048	40.296	43.597	43.416	43.191
5	50.583	50.996	51.039	51.089	52.133	51.324	52.209	52.666	52.255	53.701	53.864	53.000

Table 6.18 Natural frequency parameters of FFCC and FFFP plates (λ in Eq. (2.29))

B.C.	FFCC						FFFP					
Mode Seq.	DSC-TSE 1	DSC-TSE 2	GDQ (Yu et al., 2009)	MIB (Yu et al., 2009)	FEM	Analy. Approx. (Leissa, 1973)	DSC-TSE 1	DSC-TSE 2	GDQ Ref. (Yu et al., 2009)	MIB Ref. (Yu et al., 2009)	FEM	Analy. Approx. (Leissa, 1973)
	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$		$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$	
1	6.878	6.939	7.873	7.149	6.925	6.942	6.593	6.657	5.161	6.866	6.648	6.648
2	23.695	23.906	23.615	24.323	24.095	24.034	14.832	14.900	14.725	14.919	14.993	15.023
3	26.391	26.601	23.873	26.249	26.851	26.681	25.083	25.370	23.082	25.934	25.499	25.492
4	47.079	47.592	44.587	47.865	48.068	47.785	25.725	25.997	24.156	26.347	26.160	26.126
5	62.118	62.704	62.730	62.719	64.513	63.039	47.942	48.403	46.296	48.598	49.123	48.711

Table 6.19 Natural frequency parameters of FFFF plate (λ in Eq. (2.29))

B.C.	FFFF										
Mode Seq.	DSC-TSE 1	DSC-TSE 1	DSC-TSE 1	DSC-TSE 2	DSC-LK (Wang & Xu, 2010)	DSC-LK Ref. (Wang & Yuan, 2017)	GDQ (Yu et al., 2009)	MIB (Yu et al., 2009)	FEM	Analy. Approx. (Narita & Leissa, 1986)	Analy. Approx. (Leissa, 1973)
	$N_x=N_y=15$	$N_x=N_y=22$	$N_x=N_y=33$	$N_x=N_y=33$	$N_x=N_y=33$	$N_x=N_y=22$	$N_x=N_y=15$	$N_x=N_y=15$	$N_x=N_y=15$		
1	13.280	13.364	13.419	13.517	13.419	13.468	10.303	14.571	13.493	13.468	13.489
2	19.430	19.510	19.559	19.596	19.559	19.595	19.596	19.599	19.701	19.596	19.789
3	24.031	24.143	24.213	24.287	24.213	24.271	22.146	24.406	24.473	24.271	24.432
4	34.178	34.465	34.647	34.850	34.649	34.799	30.026	35.488	35.000	34.801	35.024
5	34.178	34.465	34.647	34.850	34.649	34.799	30.803	35.761	35.000	34.801	35.024

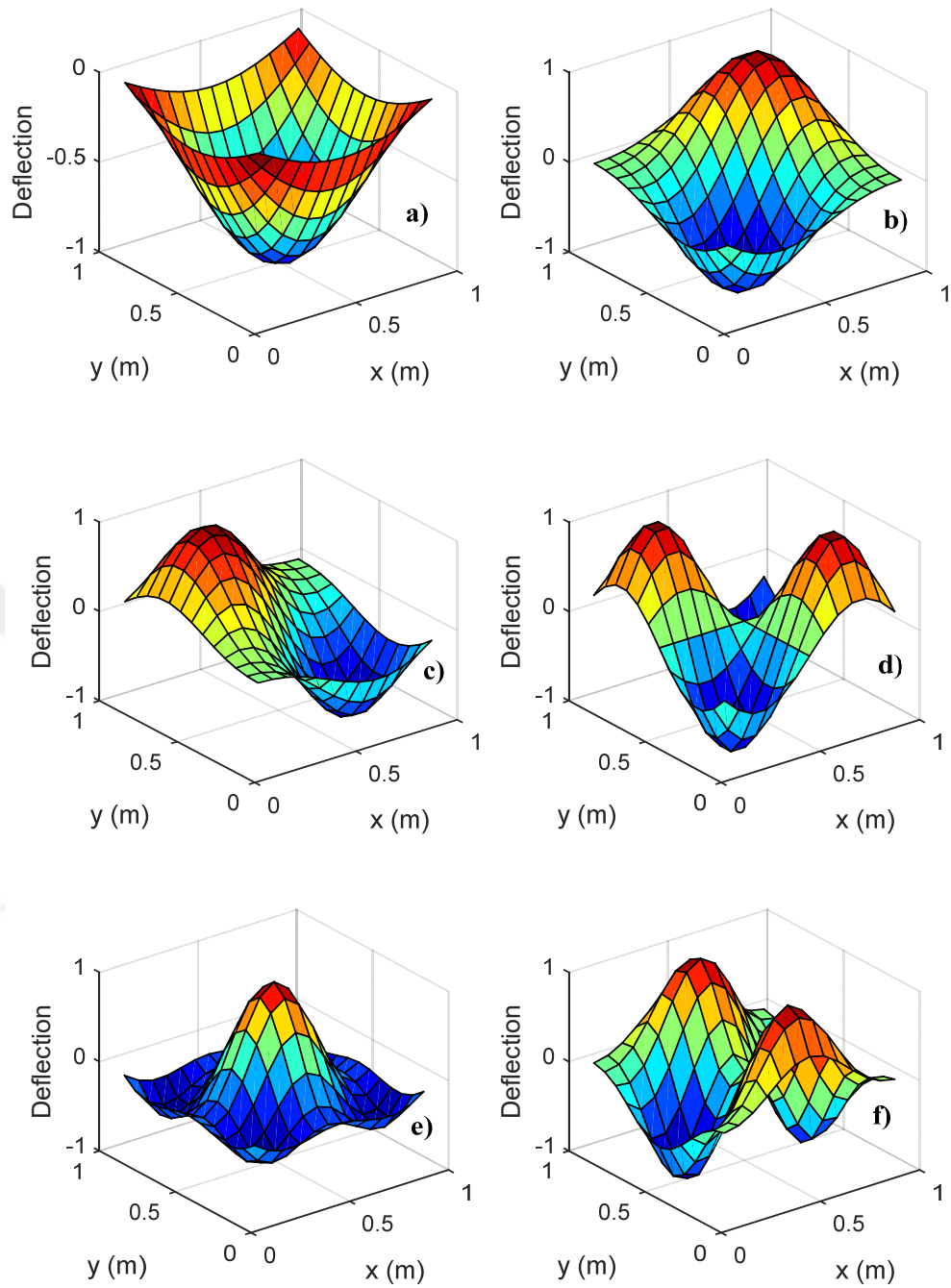


Figure 6.18 Mode shapes of PPPP plate a) mode sequence (MS) number: 1, b) MS number: 2, c) MS number: 3, d) MS number: 4, e) MS number: 5, f) MS number: 6

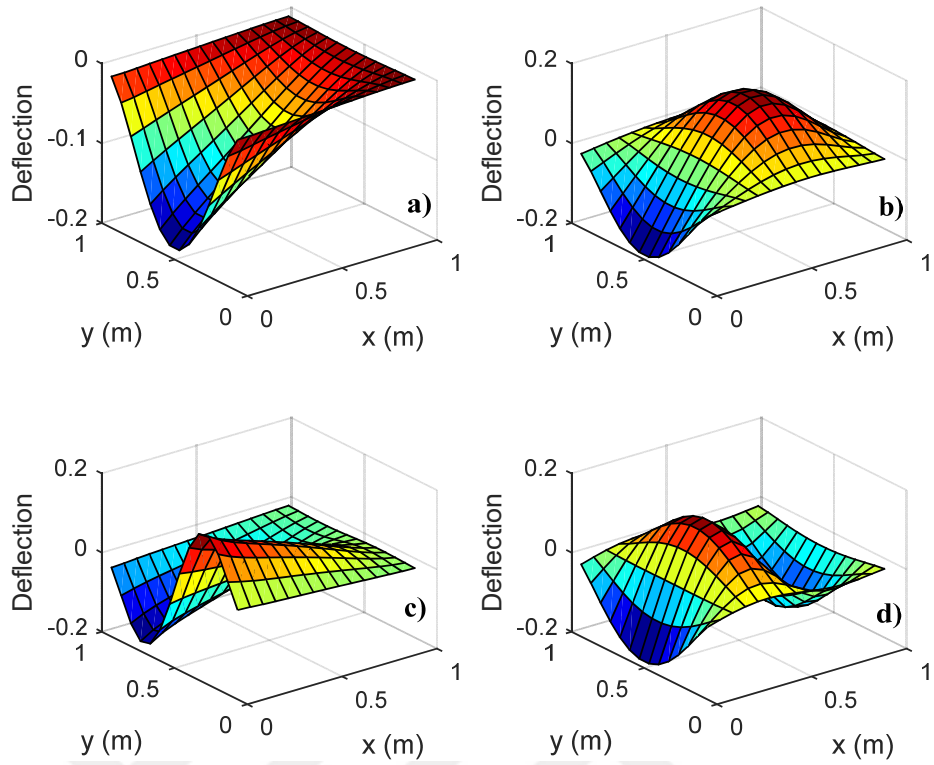


Figure 6.19 Mode shapes of FCCC plate a) MS number: 1, b) MS number: 2, c) MS number: 3, d) MS number: 4

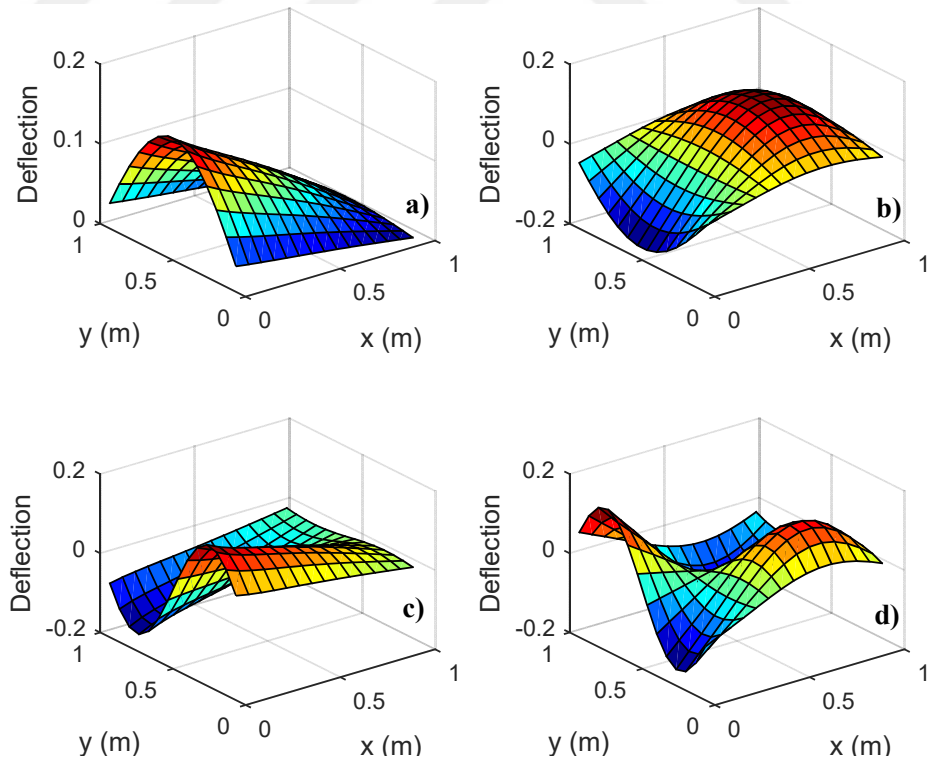


Figure 6.20 Mode shapes of FPPP plate a) MS number: 1, b) MS number: 2, c) MS number: 3, d) MS number: 4

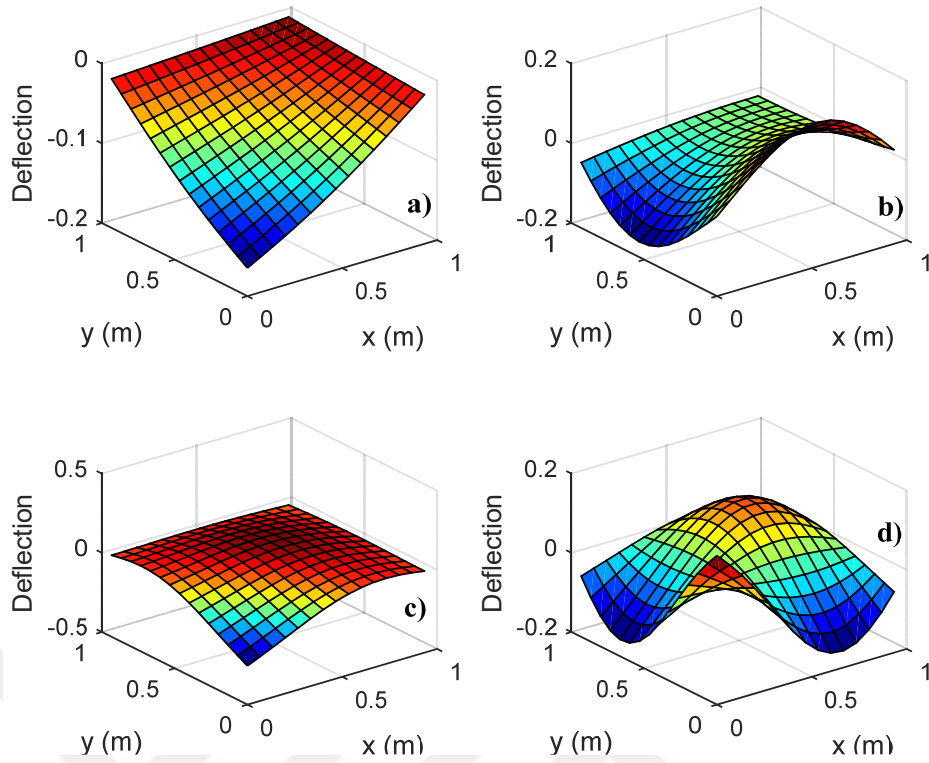


Figure 6.21 Mode shapes of FFPP plate a) MS number: 1, b) MS number: 2, c) MS number: 3, d) MS number: 4

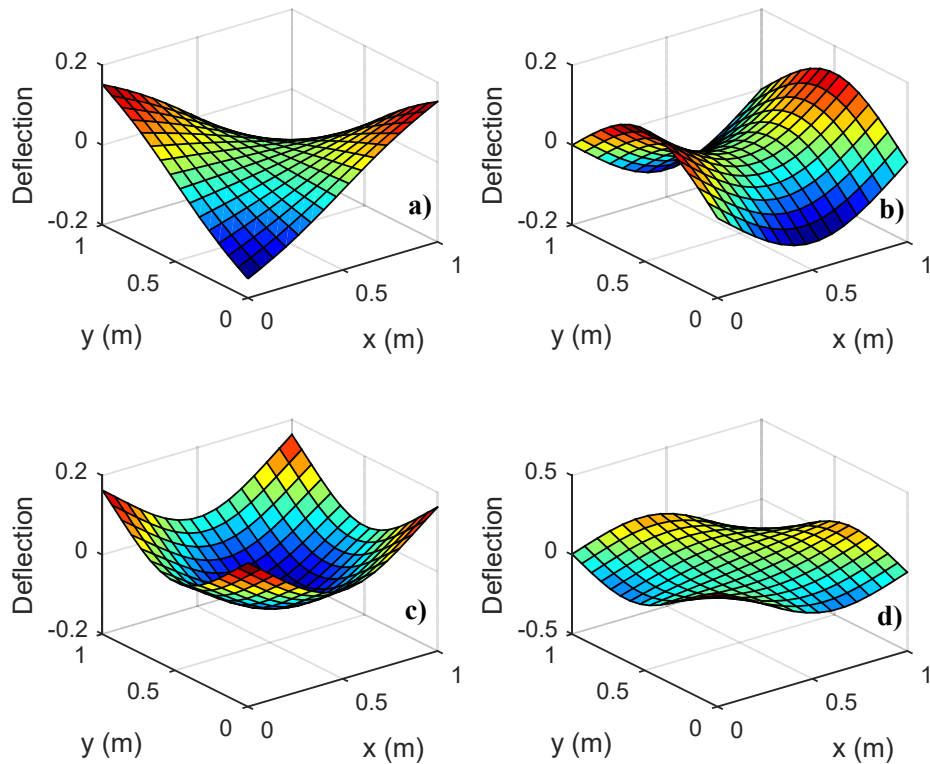


Figure 6.22 Mode shapes of FFFF plate a) MS number: 1, b) MS number: 2, c) MS number: 3, d) MS number: 4

In Tables 6.15-6.19, GDQ represents results of General Differential Quadrature method, MIB is matched interface and boundary method, FEM is finite element method, DSC-LK is DSC analysis by using Lagrange's delta sequence kernel. In the finite element analysis, ANSYS Shell 181 elements are used which is four node element and are capable of solving plate problems. Followings can be inferred from the tables:

1. For the fully simply-supported plate, symmetric modes are calculated successfully.
2. DSC results are very consistent with the results used in the comparisons.
3. DSC-TSE 2 presents better accuracy compared to DSC-TSE 1.
4. DSC-TSE 2 results are much more consistent with analytical results than FEM is.
5. MIB method is an accurate approach in lower modes but accuracy in the higher order modes is in question.
6. In the scope of these interpretations, DSC-TSE 2 approach can be used reliably in free edged plates.

Figures 6.18-6.22 present mode shapes, which are successfully obtained via the DSC method.

CHAPTER SEVEN

NUMERICAL STUDIES ON CONNECTED STRUCTURES VIA THE DSC

7.1 Introduction

In this Section, the DSC methodology is applied for the connected structures on several vibration and acoustic systems. In this regard, the considered systems are grouped on three main topics, i.e., i) acoustic transmission lines, ii) connected beams having I-, L- and T- type connection orientation, iii) a beam connected to a plate called as “stiffened plate”. Different capabilities of the DSC are tested by considering these systems, and they may be listed as follows:

- In the analysis of the acoustic transmission lines, the natural frequencies, the mode shapes and the power transmission coefficient which is an important parameter for analysis of mufflers, is determined by combining the DSC with two-load method (which is an experimental method used in acoustics). Here, combining the DSC-two load method to estimate the power transmission coefficient is performed for the first time in literature.
- In the analysis of I-, L- and T-type connected beams, the analysis of structures having mode conversion is considered via the DSC method for the first time in literature.
- Finally, natural frequencies of a complex structure are determined by using DSC method. In this context, a plate connected to a beam, which is called as “stiffened plate” is also considered for the first time in literature.

7.2 Numerical Analysis of Acoustic Transmission Lines

In this study, natural frequencies, mode shapes and power transmission coefficients of transmission lines are determined. In this regard, the proposed DSC-TSE algorithm (in Chapter 5) is applied to transmission lines with two, three and five duct elements.

7.2.1 Natural Frequencies and Mode Shapes of a Uniform Duct

Before analysing transmission lines, firstly, natural frequencies and mode shapes of a uniform duct with the length of 0.3 m and the diameter of 0.01 m are determined by using conventional DSC and the proposed DSC-TSE algorithm. Here, DSC-TSE states the DSC algorithm, which is developed for connected structures using TSE 2. In this manner, this structure is modelled with three different geometrical set:

- I. Modelling a single duct by using conventional DSC ($l=0.3\ m$),
- II. Modelling a transmission line with two duct elements having with the same diameter but different lengths by using DSC-TSE ($l_1=0.05\ m$, $l_2=0.25\ m$; $l=l_1+l_2$),
- III. Modelling the structure with identical two ducts by using DSC-TSE ($l_1=l_2=0.15\ m$; $l=l_1+l_2$).

The analyses are performed with different grid spacing (0.03 m , 0.015 m and 0.01 m for the first and third geometrical sets and 0.0125 m , 0.01 m and 0.005 m for the second geometrical set). The results are compared with the ones of finite element (FE) and analytical solutions. FE analyses are also performed with the element sizes of 0.0125 m , 0.01 m and 0.005 m . The results of acoustic hard (H) – acoustic hard (H) and acoustic hard (H) – acoustic soft (S) boundary conditions are given with *Error%* in Tables 7.1 and 7.2, respectively. Note that, M is selected as $N-1$ in the computations. Some of the mode shapes calculated by using DSC-TSE are compared with analytical ones in Figure 7.1.

Table 7.1 Natural frequencies of H-H uniform duct (Hz)

Mode	Analy.	conventional DSC ($l_1=30e-2$)			DSC-TSE ($l_1=5e-2$; $l_2=25e-2$)			DSC-TSE ($l_1=15e-2$; $l_2=15e-2$)			FEM ($l_1=30e-2$)		
		$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$	$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$
1	0.00	1.15	0.02	0.00	142.13	0.00	4.38	0.00	3.58	0.00	0.00	0.00	0.00
2	571.67	571.67	571.67	571.67	602.07	563.05	571.69	573.12	571.67	571.67	572.00	572.00	572.00
3	1143.33	1143.34	1143.33	1143.33	1148.27	1142.11	1143.34	1152.27	1143.57	1143.40	1144.00	1144.00	1144.00
4	1715.00	1715.05	1715.00	1715.00	1713.29	1715.72	1715.00	1719.73	1715.00	1715.00	1716.00	1716.00	1716.00
5	2286.67	2287.31	2286.67	2286.67	2284.44	2287.99	2286.68	2333.49	2288.72	2287.21	2288.10	2288.10	2288.00
6	2858.33	2863.88	2858.33	2858.33	2856.88	2861.08	2858.43	2926.75	2858.36	2858.33	2860.40	2860.20	2860.00
7	3430.00	3461.55	3430.00	3430.00	3429.91	3435.26	3430.22	3657.75	3437.39	3431.85	3432.90	3432.40	3432.00
8	4001.67	4117.10	4001.67	4001.67	4004.20	4008.48	4001.93	4311.46	4002.05	4001.67	4005.90	4004.80	4004.10
9	4573.33	4833.22	4573.33	4573.33	4579.94	4578.63	4573.47	5083.45	4593.50	4577.79	4579.60	4577.50	4576.10
10	5145.00	5459.00	5145.00	5145.00	5158.08	5147.40	5145.00	5522.12	5148.98	5145.00	5154.50	5150.70	5148.20
Error %													
Mode		conventional DSC ($l_1=30e-2$)			DSC-TSE ($l_1=5e-2$; $l_2=25e-2$)			DSC-TSE ($l_1=15e-2$; $l_2=15e-2$)			FEM ($l_1=30e-2$)		
		$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$	$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$
1	-	undfn.	undfn.	undfn.	undfn.	undfn.	undfn.	undfn.	undfn.	undfn.	undfn.	undfn.	undfn.
2	-	0.00	0.00	0.00	5.32	-1.51	0.00	0.25	0.00	0.00	0.06	0.06	0.06
3	-	0.00	0.00	0.00	0.43	-0.11	0.00	0.78	0.02	0.01	0.06	0.06	0.06
4	-	0.00	0.00	0.00	-0.10	0.04	0.00	0.28	0.00	0.00	0.06	0.06	0.06
5	-	0.03	0.00	0.00	-0.10	0.06	0.00	2.05	0.09	0.02	0.06	0.06	0.06
6	-	0.19	0.00	0.00	-0.05	0.10	0.00	2.39	0.00	0.00	0.07	0.07	0.06
7	-	0.92	0.00	0.00	0.00	0.15	0.01	6.64	0.22	0.05	0.08	0.07	0.06
8	-	2.88	0.00	0.00	0.06	0.17	0.01	7.74	0.01	0.00	0.11	0.08	0.06
9	-	5.68	0.00	0.00	0.14	0.12	0.00	11.15	0.44	0.10	0.14	0.09	0.06
10	-	6.10	0.00	0.00	0.25	0.05	0.00	7.33	0.08	0.00	0.18	0.11	0.06

Table 7.2 Natural frequencies of H-S uniform duct (Hz)

Mode	Analy.	conventional DSC ($l_1=30e-2$)			DSC-TSE ($l_1=5e-2$; $l_2=25e-2$)			DSC-TSE ($l_1=15e-2$; $l_2=15e-2$)			FEM ($l_1=30e-2$)		
		$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$	$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$
1	285.83	285.83	285.83	285.83	349.74	262.93	285.90	281.33	285.85	285.83	286.00	286.00	286.00
2	857.50	857.50	857.50	857.50	871.35	853.78	857.51	862.10	857.54	857.51	858.01	858.01	858.01
3	1429.17	1429.18	1429.17	1429.17	1429.50	1429.26	1429.17	1436.82	1429.41	1429.23	1430.00	1430.00	1430.00
4	2000.83	2001.03	2000.83	2000.83	1998.52	2001.86	2000.83	2020.54	2001.52	2001.01	2002.10	2002.00	2002.00
5	2572.50	2574.49	2572.50	2572.50	2570.61	2574.36	2572.55	2629.14	2573.99	2572.89	2574.20	2574.10	2574.00
6	3144.17	3158.12	3144.17	3144.17	3143.28	3148.12	3144.33	3280.28	3146.97	3144.88	3146.60	3146.30	3146.00
7	3715.83	3779.74	3715.83	3715.83	3716.88	3722.17	3716.10	3983.84	3720.73	3717.01	3719.30	3718.60	3718.10
8	4287.50	4472.32	4287.50	4287.50	4291.87	4293.96	4287.72	4706.05	4295.74	4289.32	4292.70	4291.10	4290.10
9	4859.17	5174.17	4859.17	4859.17	4868.57	4862.87	4859.21	5337.22	4873.03	4861.84	4866.90	4864.10	4862.20
10	5430.83	5649.53	5430.84	5430.83	5448.90	5433.05	5430.90	5728.80	5454.76	5434.61	5442.40	5437.50	5434.30
Error %													
Mode		conventional DSC ($l_1=30e-2$)			DSC-TSE ($l_1=5e-2$; $l_2=25e-2$)			DSC-TSE ($l_1=15e-2$; $l_2=15e-2$)			FEM ($l_1=30e-2$)		
		$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$	$\Delta=0.03$	$\Delta=0.015$	$\Delta=0.01$	$\Delta=0.0125$	$\Delta=0.01$	$\Delta=0.005$
1	-	0.00	0.00	0.00	22.36	-8.01	0.02	-1.58	0.01	0.00	0.06	0.06	0.06
2	-	0.00	0.00	0.00	1.61	-0.43	0.00	0.54	0.01	0.00	0.06	0.06	0.06
3	-	0.00	0.00	0.00	0.02	0.01	0.00	0.54	0.02	0.00	0.06	0.06	0.06
4	-	0.01	0.00	0.00	-0.12	0.05	0.00	0.98	0.03	0.01	0.06	0.06	0.06
5	-	0.08	0.00	0.00	-0.07	0.07	0.00	2.20	0.06	0.02	0.07	0.06	0.06
6	-	0.44	0.00	0.00	-0.03	0.13	0.01	4.33	0.09	0.02	0.08	0.07	0.06
7	-	1.72	0.00	0.00	0.03	0.17	0.01	7.21	0.13	0.03	0.09	0.07	0.06
8	-	4.31	0.00	0.00	0.10	0.15	0.01	9.76	0.19	0.04	0.12	0.08	0.06
9	-	6.48	0.00	0.00	0.19	0.08	0.00	9.84	0.29	0.06	0.16	0.10	0.06
10	-	4.03	0.00	0.00	0.33	0.04	0.00	5.49	0.44	0.07	0.21	0.12	0.06

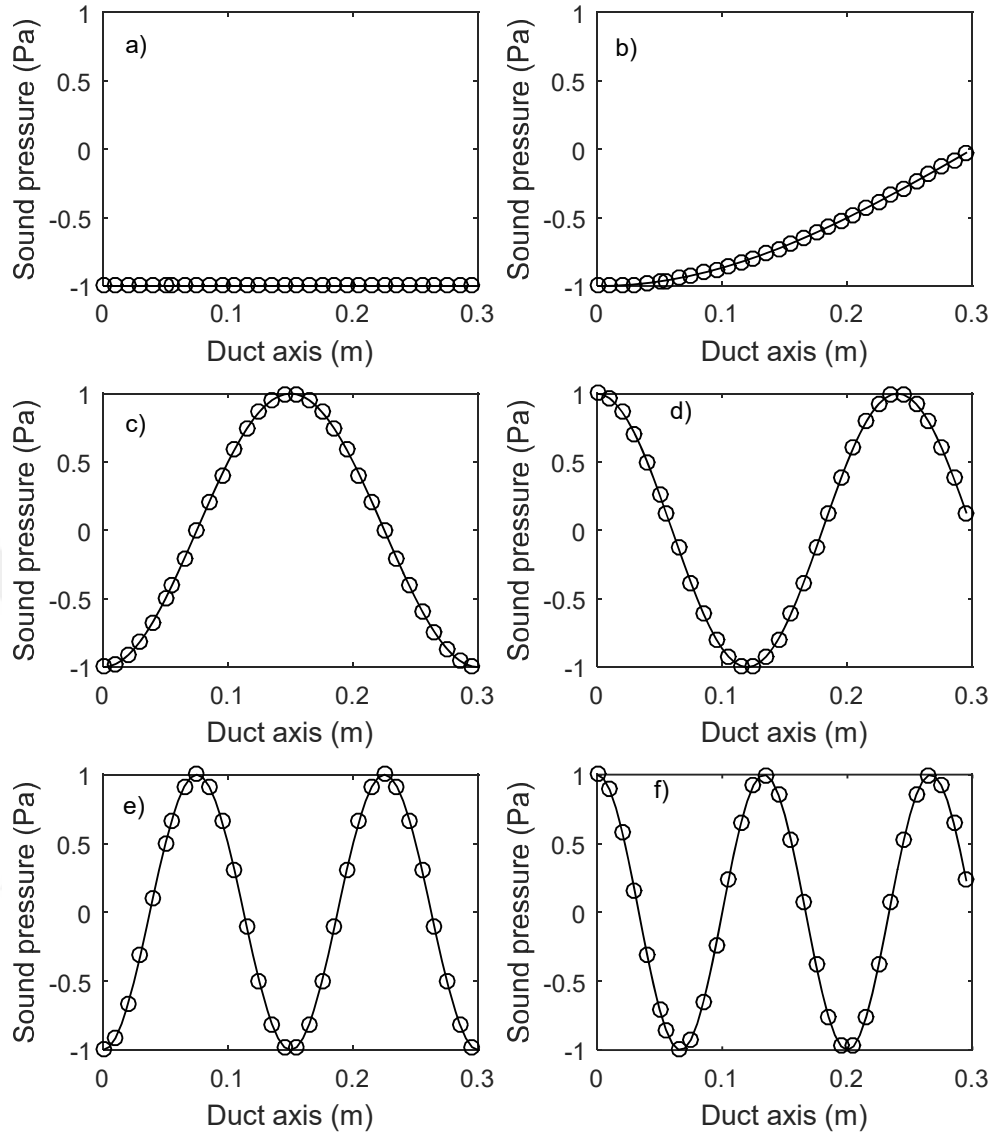


Figure 7.1 Normalized mode shapes of uniform duct a) first mode of H-H duct, b) first mode of H-S duct, c) third mode of H-H duct, d) third mode of H-S duct, e) fifth mode of H-H duct, f) fifth mode of H-S duct (solid line: DSC, 'o': analytical)

Followings can be inferred from Tables 7.1-7.2 and Figure 7.1:

1. DSC-TSE is capable of estimating the natural frequencies and the mode shapes of the uniform duct for different boundary conditions. However, the natural frequencies are much more consistent with analytical ones when the lengths of the connected ducts are close to each other since bandwidth M is increasing even similar grid spacing are used in modelling the ducts in those cases.

2. Rate of convergences of the DSC-TSE is not as high as the ones of the conventional DSC.
3. *Error %* of FEM does not reduce below 0.06 % when numbers of elements are increased. However, proposed DSC may have lower error values than FEM with decreasing the grid spacing. In the further analyses of acoustic ducts, grid spacing and length of finite element is selected as 0.005 *m*.

7.2.2 Natural Frequencies and Transmission Coefficients of Transmission Lines

After determining the grid spacing for the DSC and element size for FEM, analyses of transmission lines with two, three and five duct units as schematically shown in Figure 5.1 are performed. In the analyses, physical properties of ducts are tabulated in Table 7.3.

Table 7.3 Physical properties of ducts

Duct No (<i>j</i>)	Length (<i>l_j</i>) [m]	Diameter (<i>d_j</i>) [m]
1, 3, 5	0.05	0.01
2, 4	0.25	0.01-0.05

The natural frequencies of transmission lines with two, three and five duct elements for different diameter ratios are computed with the DSC-TSE and FEM. Results are compared in Tables 7.4-7.9.

Table 7.4 Natural frequencies of H-H transmission line with two duct elements for different diameter ratios (Hz)

Mode Seq.	$d_2/d_1=2$			$d_2/d_1=3$			$d_2/d_1=4$		
	DSC	FEM	Dev.	DSC	FEM	Dev.	DSC	FEM	Dev.
	$\Delta=5e-3$	$\Delta=5e-3$	%	$\Delta=5e-3$	$\Delta=5e-3$	%	$\Delta=5e-3$	$\Delta=5e-3$	%
1	2.34	0.00	undfn.	1.58	0.00	undfn.	1.19	0.00	undfn.
2	649.42	649.80	-0.06	668.98	669.37	-0.06	676.28	676.67	-0.06
3	1259.80	1260.50	-0.06	1311.08	1311.80	-0.05	1334.78	1335.60	-0.06
4	1715.00	1716.00	-0.06	1715.00	1716.00	-0.06	1715.00	1716.00	-0.06
5	2170.23	2171.50	-0.06	2118.95	2120.20	-0.06	2095.25	2096.50	-0.06
6	2780.66	2782.20	-0.06	2761.09	2762.70	-0.06	2753.79	2755.40	-0.06
7	3430.15	3432.00	-0.05	3430.14	3432.00	-0.05	3430.13	3432.00	-0.05
8	4079.66	4081.90	-0.05	4099.21	4101.40	-0.05	4106.50	4108.80	-0.06
9	4690.04	4692.70	-0.06	4741.37	4744.00	-0.06	4765.09	4767.70	-0.05
10	5145.00	5148.20	-0.06	5145.01	5148.20	-0.06	5145.01	5148.20	-0.06

Table 7.5 Natural frequencies of H-S transmission line with two duct elements for different diameter ratios (Hz)

Mode	$d_2/d_1=2$			$d_2/d_1=3$			$d_2/d_1=4$		
	DSC	FEM	Dev.	DSC	FEM	Dev.	DSC	FEM	Dev.
	$\Delta=5e-3$	$\Delta=5e-3$	%	$\Delta=5e-3$	$\Delta=5e-3$	%	$\Delta=5e-3$	$\Delta=5e-3$	%
1	326.24	326.41	-0.05	335.32	335.51	-0.06	338.63	338.83	-0.06
2	964.47	965.03	-0.06	997.76	998.34	-0.06	1010.90	1011.50	-0.06
3	1512.56	1513.40	-0.06	1566.27	1567.20	-0.06	1599.14	1600.10	-0.06
4	1917.45	1918.60	-0.06	1863.74	1864.80	-0.06	1830.87	1831.90	-0.06
5	2465.59	2467.00	-0.06	2432.29	2433.70	-0.06	2419.15	2420.50	-0.06
6	3103.89	3105.60	-0.05	3094.79	3096.50	-0.06	3091.47	3093.20	-0.06
7	3756.42	3758.50	-0.06	3765.49	3767.60	-0.06	3768.80	3770.90	-0.06
8	4394.74	4397.10	-0.05	4428.04	4430.40	-0.05	4441.18	4443.60	-0.05
9	4942.67	4945.60	-0.06	4996.42	4999.40	-0.06	5029.31	5032.30	-0.06
10	5347.60	5350.80	-0.06	5293.93	5297.10	-0.06	5261.07	5264.20	-0.06

Table 7.6 Natural frequencies of H-H transmission line with three duct elements for different diameter ratios (Hz)

Mode	$d_2/d_1=2$			$d_2/d_1=3$			$d_2/d_1=4$			$d_2/d_1=5$		
	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %
1	3.23	0.00	undfn.	2.21	0.00	undfn.	1.68	0.00	undfn.	1.35	0.00	undfn.
2	617.32	617.66	-0.06	653.01	653.39	-0.06	666.91	667.29	-0.06	673.62	674.01	-0.06
3	1180.45	1181.10	-0.06	1263.67	1264.40	-0.06	1303.61	1304.40	-0.06	1325.34	1326.10	-0.06
4	1577.17	1578.10	-0.06	1632.70	1633.70	-0.06	1661.71	1662.70	-0.06	1678.15	1679.10	-0.06
5	1852.84	1853.90	-0.06	1797.30	1798.40	-0.06	1768.29	1769.30	-0.06	1751.84	1752.90	-0.06
6	2249.62	2250.90	-0.06	2166.39	2167.60	-0.06	2126.46	2127.70	-0.06	2104.72	2105.90	-0.06
7	2812.85	2814.40	-0.05	2777.14	2778.60	-0.05	2763.23	2764.70	-0.05	2756.52	2758.00	-0.05
8	3430.29	3432.00	-0.05	3430.27	3432.00	-0.05	3430.26	3432.00	-0.05	3430.26	3432.00	-0.05
9	4047.76	4049.70	-0.05	4083.45	4085.50	-0.05	4097.34	4099.40	-0.05	4104.05	4106.10	-0.05
10	4610.90	4613.20	-0.05	4694.20	4696.50	-0.05	4734.18	4736.50	-0.05	4755.94	4758.20	-0.05

Table 7.7 Natural frequencies of H-S transmission line with three duct elements for different diameter ratios (Hz)

Mode	$d_2/d_1=2$			$d_2/d_1=3$			$d_2/d_1=4$			$d_2/d_1=5$		
	DSC $\Delta=5e-3$	FEM $\Delta=5e-3$	Dev. %	DSC $\Delta=5e-3$	FEM $\Delta=5e-3$	Dev. %	DSC $\Delta=5e-3$	FEM $\Delta=5e-3$	Dev. %	DSC $\Delta=5e-3$	FEM $\Delta=5e-3$	Dev. %
1	196.02	196.11	-0.05	146.85	146.93	-0.05	115.08	115.14	-0.05	94.01	94.06	-0.05
2	713.60	714.01	-0.06	700.19	700.60	-0.06	694.38	694.78	-0.06	691.48	691.88	-0.06
3	1278.38	1279.10	-0.06	1319.02	1319.80	-0.06	1339.21	1340.00	-0.06	1350.01	1350.80	-0.06
4	1715.01	1716.00	-0.06	1715.00	1716.00	-0.06	1715.00	1716.00	-0.06	1715.00	1716.00	-0.06
5	2151.69	2152.90	-0.06	2111.05	2112.20	-0.05	2090.85	2092.00	-0.05	2080.05	2081.20	-0.06
6	2716.53	2718.00	-0.05	2729.94	2731.40	-0.05	2735.76	2737.20	-0.05	2738.66	2740.10	-0.05
7	3234.11	3235.90	-0.06	3283.26	3285.10	-0.06	3315.03	3316.90	-0.06	3336.10	3338.00	-0.06
8	3626.15	3628.20	-0.06	3576.98	3579.00	-0.06	3545.21	3547.20	-0.06	3524.14	3526.10	-0.06
9	4144.03	4146.10	-0.05	4130.64	4132.70	-0.05	4124.81	4126.90	-0.05	4121.91	4124.00	-0.05
10	4708.92	4711.30	-0.05	4749.62	4751.90	-0.05	4769.84	4772.10	-0.05	4780.65	4782.90	-0.05

Table 7.8 Natural frequencies of H-H transmission line with five duct elements for different diameter ratios (Hz)

Mode	$d_2/d_1=2$			$d_2/d_1=3$			$d_2/d_1=4$			$d_2/d_1=5$		
	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %	DSC $\Lambda=5e-3$	FEM $\Lambda=5e-3$	Dev. %
1	2.83	0.00	undfn.	1.93	0.00	undfn.	1.46	0.00	undfn.	1.17	0.00	undfn.
2	240.05	240.17	-0.05	192.16	192.26	-0.05	155.30	155.39	-0.06	128.90	128.97	-0.06
3	634.21	634.57	-0.06	661.64	662.03	-0.06	672.02	672.41	-0.06	676.97	677.37	-0.06
4	770.42	770.87	-0.06	733.98	734.41	-0.06	715.25	715.66	-0.06	705.39	705.80	-0.06
5	1231.77	1232.50	-0.06	1296.90	1297.60	-0.05	1326.19	1327.00	-0.06	1341.44	1342.20	-0.06
6	1320.27	1321.00	-0.06	1340.34	1341.10	-0.06	1352.02	1352.80	-0.06	1358.51	1359.30	-0.06
7	1685.99	1687.00	-0.06	1706.43	1707.40	-0.06	1711.76	1712.80	-0.06	1713.55	1714.60	-0.06
8	1744.02	1745.00	-0.06	1723.58	1724.60	-0.06	1718.24	1719.30	-0.06	1716.45	1717.50	-0.06
9	2109.78	2111.00	-0.06	2089.73	2090.90	-0.06	2078.05	2079.20	-0.06	2071.56	2072.70	-0.06
10	2198.30	2199.50	-0.05	2133.17	2134.40	-0.06	2103.88	2105.10	-0.06	2088.63	2089.80	-0.06

Table 7.9 Natural frequencies of H-S transmission line with five duct elements for different diameter ratios (Hz)

Mode	d₂/d₁=2			d₂/d₁=3			d₂/d₁=4			d₂/d₁=5		
	DSC Δ=5e-3	FEM Δ=5e-3	Dev. %	DSC Δ=5e-3	FEM Δ=5e-3	Dev. %	DSC Δ=5e-3	FEM Δ=5e-3	Dev. %	DSC Δ=5e-3	FEM Δ=5e-3	Dev. %
1	110.58	110.61	-0.03	86.12	86.15	-0.04	68.84	68.87	-0.04	56.85	56.87	-0.04
2	313.32	313.50	-0.06	235.05	235.18	-0.06	184.77	184.88	-0.06	151.27	151.36	-0.06
3	672.31	672.70	-0.06	679.24	679.64	-0.06	682.08	682.47	-0.06	683.45	683.85	-0.06
4	826.70	827.18	-0.06	762.11	762.55	-0.06	732.01	732.44	-0.06	716.45	716.87	-0.06
5	1262.91	1263.60	-0.05	1312.00	1312.80	-0.06	1335.23	1336.00	-0.06	1347.45	1348.20	-0.06
6	1414.55	1415.40	-0.06	1393.30	1394.10	-0.06	1384.43	1385.20	-0.06	1380.09	1380.90	-0.06
7	1715.01	1716.00	-0.06	1715.00	1716.00	-0.06	1715.00	1716.00	-0.06	1715.00	1716.00	-0.06
8	2015.52	2016.70	-0.06	2036.77	2037.90	-0.06	2045.64	2046.80	-0.06	2049.98	2051.10	-0.05
9	2167.16	2168.40	-0.06	2118.07	2119.30	-0.06	2094.84	2096.00	-0.06	2082.62	2083.80	-0.06
10	2603.40	2604.80	-0.05	2668.00	2669.50	-0.06	2698.11	2699.60	-0.06	2713.67	2715.20	-0.06

It may be inferred from Tables 7.4-7.9 that, the proposed algorithm is capable of estimating natural frequencies for the transmission lines with different boundary conditions and diameter ratios. Besides, it should be noted that, *Error%*'s are around 0.06% which were the ones of FEM in convergence study for the selected element sizes.

Next, the power transmission coefficients (t_c) are obtained for transmission lines with the three and five duct elements for the considered diameter ratios via the DSC-TSE. The results are compared with FEM and the transfer matrix method (TMM) in Figures 7.2-7.7. Note that, power transmission coefficients are calculated up to 4000 Hz with two load method (ASTM E2611-17, Standard test method for normal incidence determination of porous material acoustical properties based on the transfer matrix method, 2017) which is standardized with ASTM E2611-17. In the DSC and FEM computations and hard and anechoic ends ($Z_s=420.18 \text{ kgm}^{-2}\text{s}^{-1}$) are selected as the loads. One can refer to Appendix 2 for the details of the two-load method.

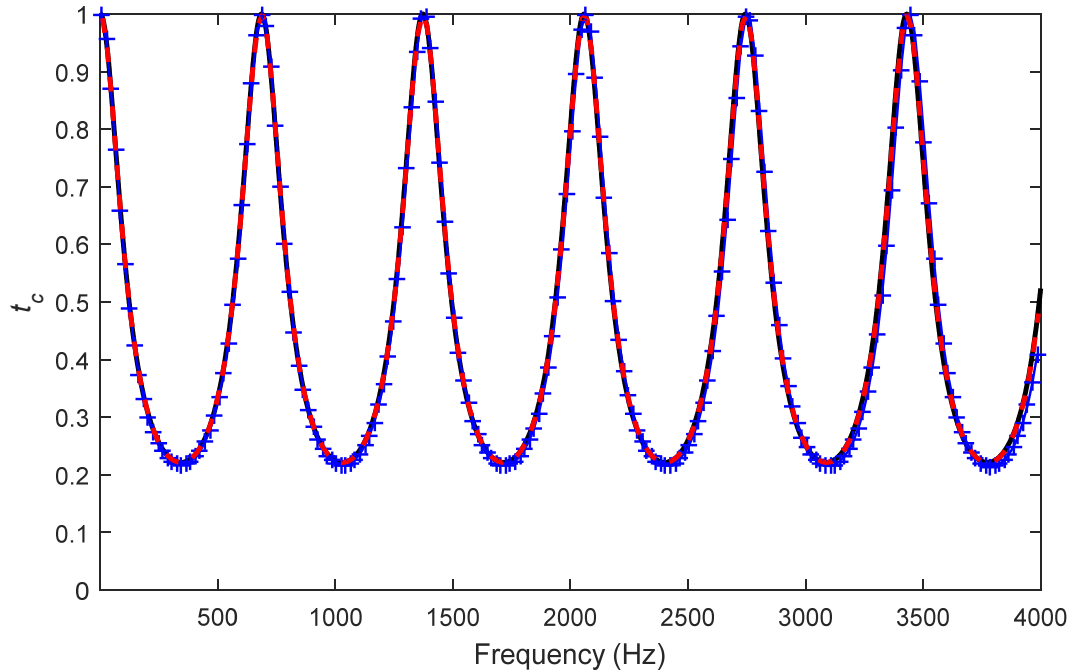


Figure 7.2 Power transmission coefficient for the transmission line with three duct elements for $d_2/d_1=2$ (Black: DSC, blue +: FEM, red: TMM)

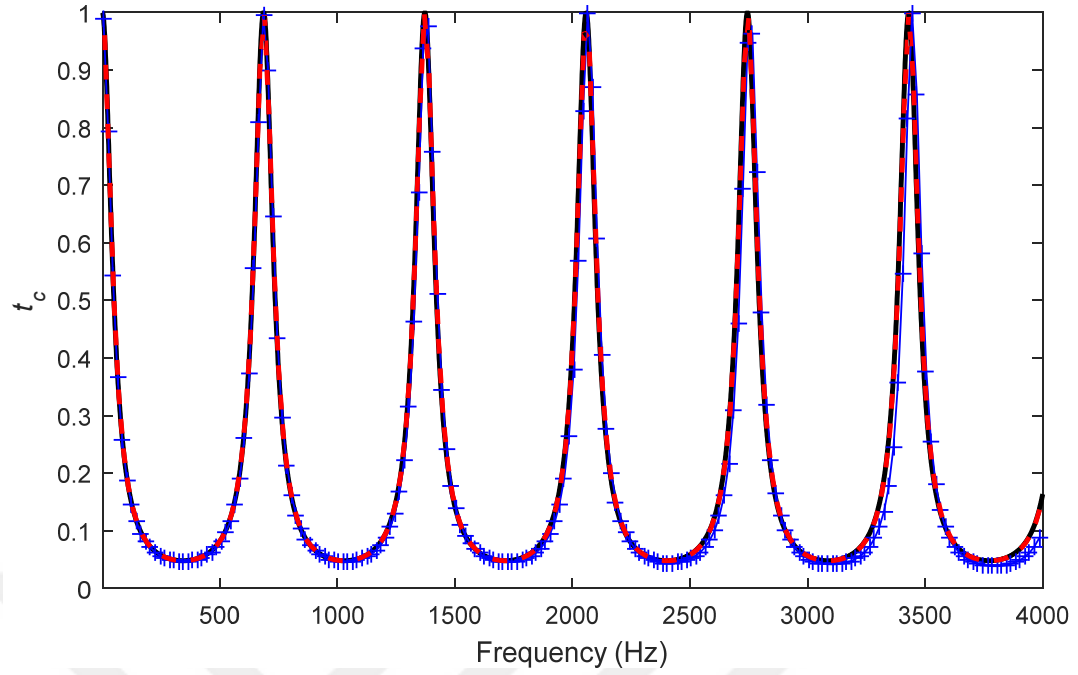


Figure 7.3 Power transmission coefficient for the transmission line with three duct elements for $d_2/d_1=3$ (Black: DSC, blue +: FEM, red: TMM)

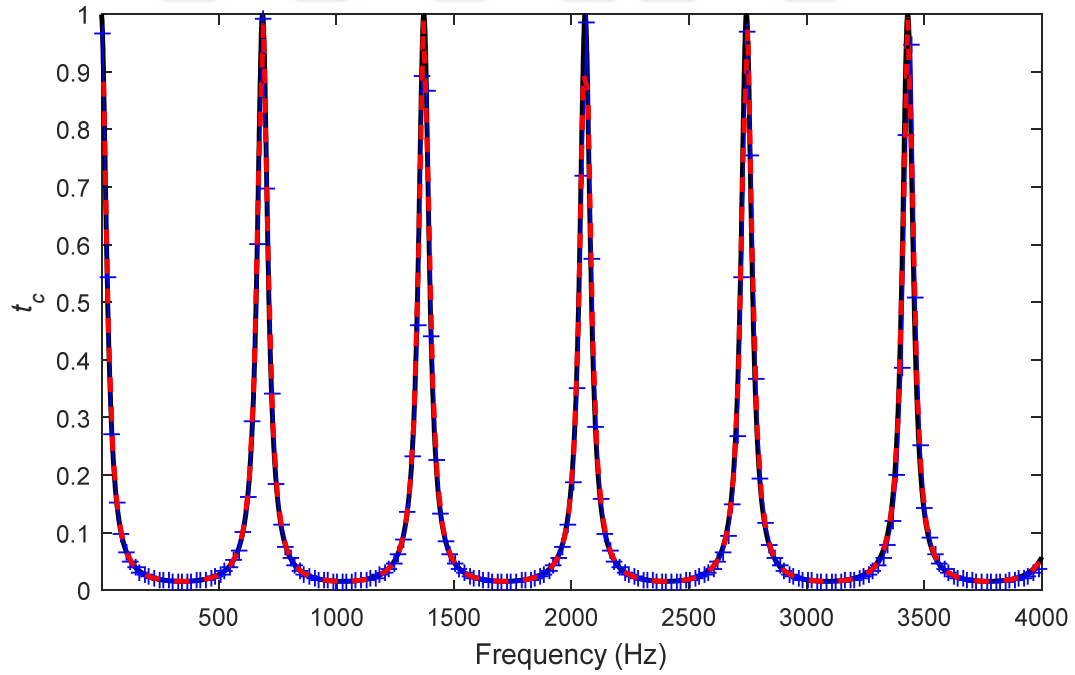


Figure 7.4 Power transmission coefficient for the transmission line with three duct elements for $d_2/d_1=4$ (Black: DSC, blue +: FEM, red: TMM)

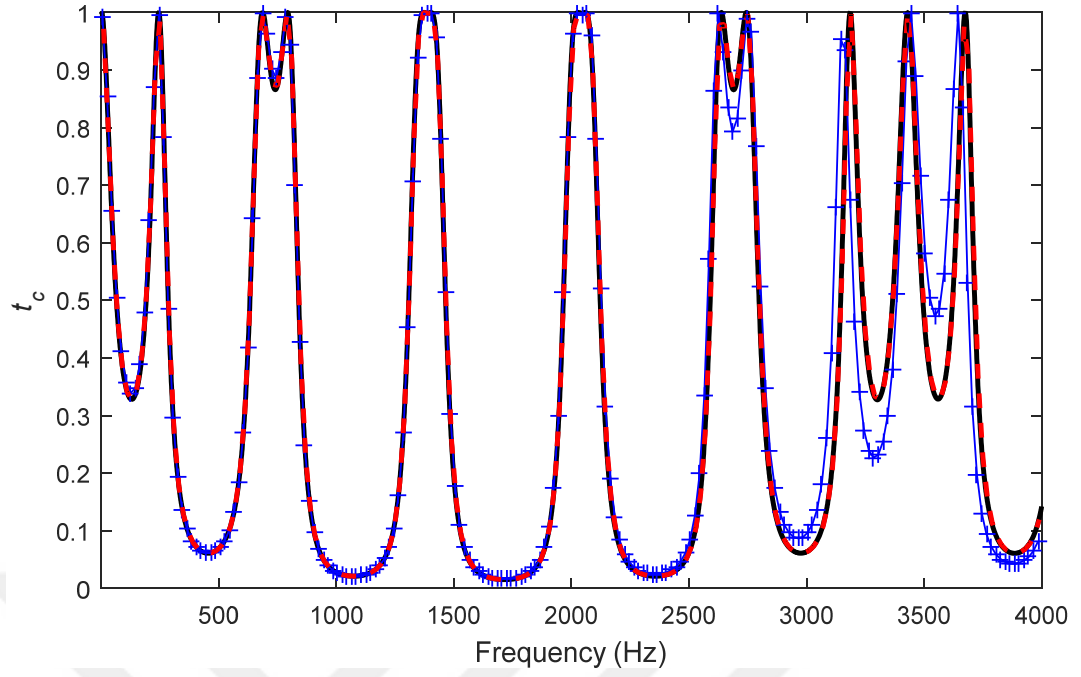


Figure 7.5 Power transmission coefficient for the transmission line with five duct elements for $d_2/d_1=2$ (Black: DSC, blue +: FEM, red: TMM)

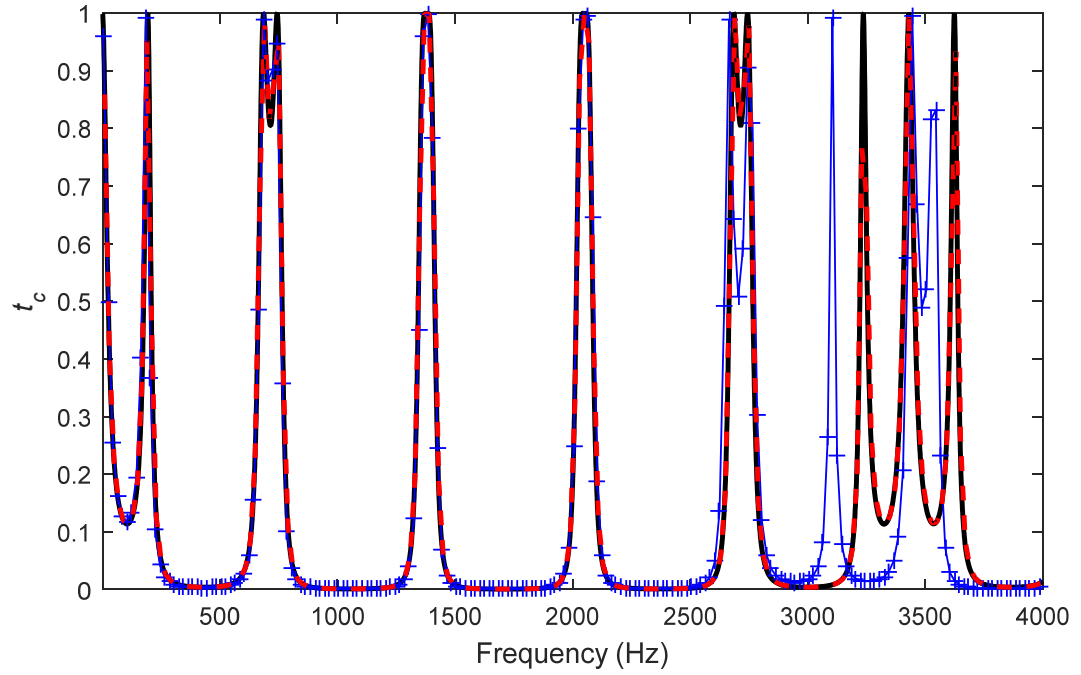


Figure 7.6 Power transmission coefficient for the transmission line with five duct elements for $d_2/d_1=3$ (Black: DSC, blue +: FEM, red: TMM)

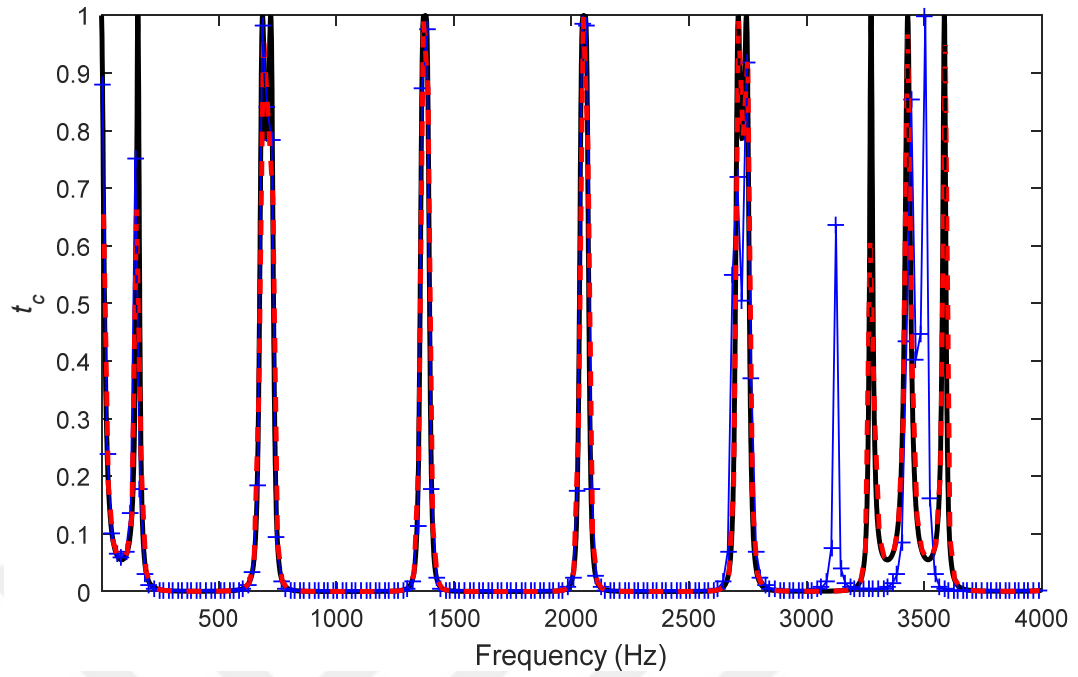


Figure 7.7 Power transmission coefficient for the transmission line with five duct elements for $d_2/d_1=4$ (Black: DSC, blue +: FEM, red: TMM)

Followings can be inferred from Figures 7.2-7.7:

- The DSC-TSE method is capable of estimating the power transmission coefficient when it is used with two-load method. Note that, there are some deviations in higher frequencies from finite element method. This may be due to the fact that FEM uses 3D elements in power transmission coefficient computations. However, perfect match is seen between TMM and the DSC method.
- In general, these results show that, proposed algorithm is successful in handling transmission line acoustic systems.
- Increasing the diameter ratio leads lower transmission coefficient with wider frequency range as expected.
- For the transmission line with three duct units, lowest transmission coefficients are observed at the frequency of $c_s/4l_2$.
- For the transmission line with five duct units, Fabry-Perot resonance frequency (first peak in curves) is observed in lower frequencies when the diameter ratio increases.

- “Bragg scattering” phenomena occurs at the frequency of $f_{Bragg} = c_s / (2\Delta l)$ if the periodic transmission lines are considered (Cai, Wang, Chu, Liu, & Xu, 2017). For the five-duct-unit, Bragg scattering frequency is observed at $f_{Bragg} = 3430$ Hz where $\Delta l = 0.05$ m. This is the starting frequency of the Bragg band.
- The proposed method is also capable of estimating the transmission coefficient of periodic structures (transmission line with five duct element). Therefore, the method may be used to analyse acoustic metamaterials if the other acoustic structures (such as, resonators, inline cavities, mufflers with extended inlet/outlet ducts, etc...) are validated for the DSC with numerical and/or analytical methods.

7.3 Numerical Analyses of Connected Beams

After validating the proposed method on acoustic transmission lines, it is aimed to test the methodology on more complex structures. In this regard, three different structures are considered i.e., I-, L- and T-type connected beams as shown in Figure 7.8. Following considerations on modelling should be noted;

- Considering bending waveguides, only the bending moment transmission is considered in L-type connected beams. However, the bending force transmission is also considered in modelling of I-type connected beams.
- For longitudinal and bending waveguides, the bending force in a beam is converted to the axial force in the other connected structure and vice versa for L-type connected beams. This phenomenon is known as “mode conversion”. However, there is no mode conversion for I-type connected structure.
- For T-type connected structure, all of these transmission types applies.

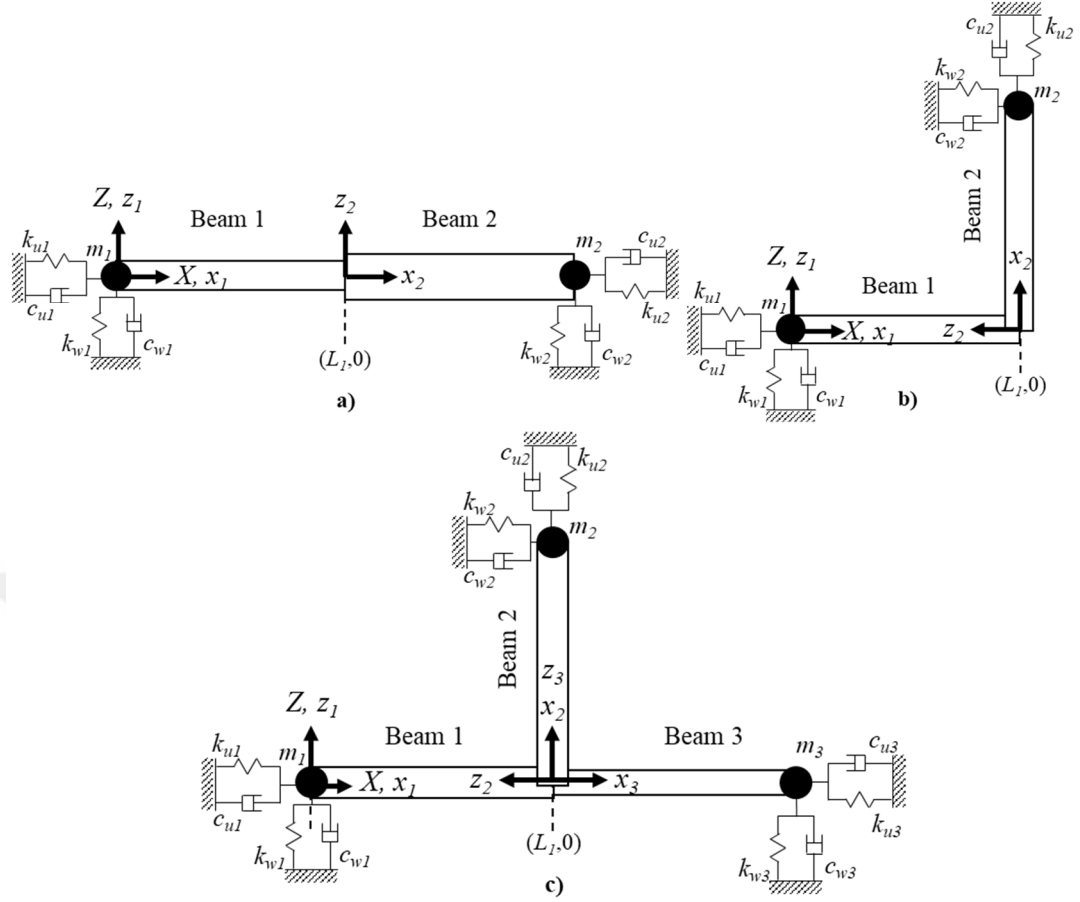


Figure 7.8 Schematic representations of analysed structures a) I-type, b) L-type, c) T-type

One can construct the structures (shown in Figure 7.8) via modelling the longitudinal and bending vibrations of each beam via the DSC-TSE as given in Chapter 5. However, TSE is applied for $S_T = 1$ for the longitudinal vibrations whereas $S_T = 2$ for bending vibrations. In this regard, there will be three additional DoFs (second and forth order derivatives of bending displacement and second order derivative of axial displacement) at each boundary apart from the displacements themselves. Therefore, there are 12 additional DoFs for I- and L-type connected beams, whereas 18 additional DoFs are valid for T-type connected beams. However, these additional DoFs are handled with boundary and continuity equations. There are two BC equations for bending vibration whereas there is only one BC equation for longitudinal vibration of each beam. Thus, they lead six and nine BC equations for I- and L-types, and T-type, respectively. Apart from the BC equations, one can write the continuity equations at $x=X_0$ for I-type as (Cremer, Heckl, & Petersson, 2005),

$$\text{Displacement cont. in } x\text{-axis:} \quad {}^1U_{N_1-1} - {}^2U_0 = 0, \quad (7.1)$$

$$\text{Displacement cont. in } z\text{-axis:} \quad {}^1W_{N_1-1} - {}^2W_0 = 0, \quad (7.2)$$

$$\text{Slope cont. about } y\text{-axis:} \quad {}^1W_{N_1-1}^{(1)} - {}^2W_0^{(1)} = 0, \quad (7.3)$$

$$\text{Moment cont. about } y\text{-axis:} \quad E_1 I_1 {}^1W_{N_1-1}^{(2)} - E_2 I_2 {}^2W_0^{(2)} = 0, \quad (7.4)$$

$$\text{Force cont. in } x\text{-axis:} \quad E_1 S_1 {}^1U_{N_1-1}^{(1)} - E_2 S_2 {}^2U_0^{(1)} = 0, \quad (7.5)$$

$$\text{Force cont. in } z\text{-axis:} \quad E_1 I_1 {}^1W_{N_1-1}^{(3)} - E_2 I_2 {}^2W_0^{(3)} = 0, \quad (7.6)$$

whereas they may be written for L-type connected beam as (Cremer et al., 2005),

$$\text{Displacement cont. in } x\text{-axis:} \quad {}^1U_{N_1-1} + {}^2W_0 = 0, \quad (7.7)$$

$$\text{Displacement cont. in } z\text{-axis:} \quad {}^1W_{N_1-1} - {}^2U_0 = 0, \quad (7.8)$$

$$\text{Slope cont. about } y\text{-axis:} \quad {}^1W_{N_1-1}^{(1)} - {}^2W_0^{(1)} = 0, \quad (7.9)$$

$$\text{Moment cont. about } y\text{-axis:} \quad E_1 I_1 {}^1W_{N_1-1}^{(2)} - E_2 I_2 {}^2W_0^{(2)} = 0, \quad (7.10)$$

$$\text{Force cont. in } x\text{-axis:} \quad E_1 S_1 {}^1U_{N_1-1}^{(1)} + E_2 I_2 {}^2W_0^{(3)} = 0, \quad (7.11)$$

$$\text{Force cont. in } z\text{-axis:} \quad E_1 I_1 {}^1W_{N_1-1}^{(3)} - E_2 S_2 {}^2U_0^{(1)} = 0. \quad (7.12)$$

Similarly, one can write the continuity equations for T-type connected beams as follows (Cremer et al., 2005):

$$\text{Displacement cont. in } x\text{-axis:} \quad {}^1U_{N_1-1} + {}^2W_0 = 0, \quad (7.13)$$

$$2^{\text{nd}} \text{ displacement cont. in } x\text{-axis:} \quad {}^1U_{N_1-1} + {}^3U_0 = 0, \quad (7.14)$$

$$\text{Displacement cont. in } z\text{-axis:} \quad {}^1W_{N_1-1} - {}^2U_0 = 0, \quad (7.15)$$

$$2^{\text{nd}} \text{ displacement cont. in } z\text{-axis:} \quad {}^1W_{N_1-1} - {}^3W_0 = 0, \quad (7.16)$$

$$\text{Slope cont. about } y\text{-axis:} \quad {}^1W_{N_1-1}^{(1)} - {}^2W_0^{(1)} = 0, \quad (7.17)$$

$$2^{\text{nd}} \text{ slope cont. about } y\text{-axis:} \quad {}^1W_{N_1-1}^{(1)} - {}^3W_0^{(1)} = 0, \quad (7.18)$$

$$\text{Moment cont. about } y\text{-axis:} \quad E_1 I_1 {}^1 W_{N_1-1}^{(2)} - E_2 I_2 {}^2 W_0^{(2)} - E_3 I_3 {}^3 W_0^{(2)} = 0, \quad (7.19)$$

$$\text{Force cont. in } x\text{-axis:} \quad E_1 S_1 {}^1 U_{N_1-1}^{(1)} + E_2 I_2 {}^2 W_0^{(3)} - E_3 S_3 {}^3 U_0^{(1)} = 0, \quad (7.20)$$

$$\text{Force cont. in } z\text{-axis:} \quad E_1 I_1 {}^1 W_{N_1-1}^{(3)} - E_2 S_2 {}^2 U_0^{(1)} - E_3 I_3 {}^3 W_0^{(3)} = 0. \quad (7.21)$$

7.3.1 Natural Frequency Analysis of Connected Beams

Natural frequencies of these structures are determined with the DSC-TSE. The mechanical and the physical properties of the analysed beams are given in Table 7.10. Different boundary conditions including translational mechanical impedance (MI) boundary are considered in the analyses. In Table 7.11, table numbers of the calculated natural frequencies corresponding to the boundary conditions are given. Note that, mass and stiffness at the boundaries are selected as identical and $m=4$ kg and $k_s=20000 \text{ Nm}^{-1}$, respectively. In the analyses, 30 modes are considered and the results are compared with FEM for all structures. However, analytical and conventional DSC solutions are also obtained for I-type connected beam except MI-MI boundary condition. In order to perform a convergence study, different numbers of discretization for the DSC and number of elements for FEM are considered in the analysis of I-type connected beam. Note that, regularization parameter is also optimized for this structure. After determining optimum number of discretization, number of element and regularization parameter, analyses of L- and T-type beams are performed and results are presented in Tables 7.12-7.17 as shown in Table 7.11.

Table 7.10 Mechanical and physical properties of beams

Property (Unit)	Value
Elasticity (GPa)	70
Poisson's ratio	0.3
Density (kg/m^3)	2700
Length (m)	0.5
Width (m)	10e-3
Thickness (m)	1e-3

Table 7.11 Table numbers corresponding to the boundary conditions of structures

BC Structure	P-P or P-P-P	F-F	C-F	MI-MI or MI-MI-MI
I-Type	Table 7.12	Table 7.13	Table 7.14	Table 7.15
L-Type	Table 7.16	-	-	Table 7.16
T-Type	Table 7.17	-	-	Table 7.17

Table 7.12 Natural frequencies of I-type connected beam with P-P boundaries (Hz)

Mode Seq.	Analytical	Conventional DSC		DSC-TSE						FEM					
		N=41 (r=5.1)	Error %	N=21x2 (r=3.3)	Error %	N=31x2 (r=4.6)	Error %	N=41x2 (r=5.4)	Error %	N=100x2	Error %	N=200x2	Error %	N=400x2	Error %
1	2.3089	2.3089	0.0000	2.3089	0.0001	2.3089	0.0001	2.3089	0.0001	2.3089	0.0020	2.3089	0.0020	2.3089	0.0020
2	9.2354	9.2354	0.0000	9.2354	-0.0005	9.2354	-0.0001	9.2354	0.0000	9.2364	0.0107	9.2356	0.0020	9.2354	-0.0002
3	20.7797	20.7797	0.0000	20.7809	0.0059	20.7800	0.0016	20.7798	0.0006	20.7848	0.0246	20.7807	0.0049	20.7797	0.0001
4	36.9417	36.9417	0.0000	36.9417	0.0000	36.9417	0.0000	36.9417	0.0000	36.9577	0.0434	36.9449	0.0088	36.9417	0.0001
5	57.7213	57.7213	0.0000	57.7305	0.0158	57.7239	0.0044	57.7224	0.0018	57.7605	0.0678	57.7293	0.0138	57.7215	0.0003
6	83.1187	83.1187	0.0000	83.1187	0.0000	83.1187	0.0000	83.1187	0.0000	83.1999	0.0977	83.1352	0.0198	83.1190	0.0003
7	113.1338	113.1338	0.0000	113.1689	0.0310	113.1435	0.0086	113.1378	0.0036	113.2842	0.1329	113.1643	0.0269	113.1343	0.0004
8	147.7666	147.7666	0.0000	147.7666	0.0000	147.7666	0.0000	147.7666	0.0000	148.0232	0.1736	147.8186	0.0352	147.7675	0.0006
9	187.0171	187.0171	0.0000	187.1146	0.0521	187.0438	0.0143	187.0282	0.0059	187.4283	0.2199	187.1004	0.0445	187.0185	0.0007
10	230.8854	230.8854	0.0000	230.8854	0.0000	230.8854	0.0000	230.8854	0.0000	231.5123	0.2715	231.0122	0.0549	230.8875	0.0009
11	279.3713	279.3713	0.0000	279.5939	0.0797	279.4313	0.0215	279.3960	0.0089	280.2895	0.3287	279.5570	0.0665	279.3743	0.0011
12	332.4749	332.4749	0.0000	332.4749	0.0000	332.4749	0.0000	332.4749	0.0000	333.7760	0.3913	332.7380	0.0791	332.4793	0.0013
13	390.1963	390.1963	0.0000	390.6435	0.1146	390.3146	0.0303	390.2448	0.0124	391.9891	0.4595	390.5587	0.0929	390.2022	0.0015
14	452.5353	452.5353	0.0000	452.5353	0.0000	452.5353	0.0000	452.5353	0.0000	454.9480	0.5332	453.0228	0.1077	452.5434	0.0018
15	519.4921	519.4921	0.0000	520.3140	0.1582	519.7046	0.0409	519.5787	0.0167	522.6733	0.6124	520.1345	0.1237	519.5027	0.0020
16	591.0665	591.0665	0.0000	591.0665	0.0000	591.0665	0.0000	591.0665	0.0000	595.1871	0.6971	591.8983	0.1407	591.0803	0.0023
17	667.2587	667.2587	0.0000	668.6760	0.2124	667.6146	0.0533	667.4028	0.0216	672.5133	0.7875	668.3188	0.1589	667.2762	0.0026
18	748.0686	748.0686	0.0000	748.0689	0.0000	748.0686	0.0000	748.0686	0.0000	754.6773	0.8834	749.4012	0.1781	748.0906	0.0029
19	833.4961	833.4961	0.0000	835.8324	0.2803	834.0610	0.0678	833.7231	0.0272	841.7062	0.9850	835.1507	0.1985	833.5235	0.0033
20	923.5414	923.5414	0.0000	923.5455	0.0004	923.5414	0.0000	923.5414	0.0000	933.6286	1.0922	925.5730	0.2200	923.5750	0.0036
21	1018.2044	1018.2044	0.0000	1021.9530	0.3682	1019.0636	0.0844	1018.5466	0.0336	1030.4748	1.2051	1020.6741	0.2426	1018.2453	0.0040
22	1117.4851	1117.4851	0.0000	1117.5259	0.0036	1117.4851	0.0000	1117.4851	0.0000	1132.2769	1.3237	1120.4602	0.2662	1117.5344	0.0044
23	1221.3835	1221.3835	0.0000	1227.4010	0.4927	1222.6465	0.1034	1221.8814	0.0408	1239.0685	1.4479	1224.9381	0.2910	1221.4424	0.0048
24	1329.8997	1329.8997	0.0000	1330.2205	0.0241	1329.8997	0.0000	1329.8997	0.0000	1350.8851	1.5780	1334.1144	0.3169	1329.9694	0.0052
25	1443.0335	1443.0335	0.0000	1453.2121	0.7054	1444.8389	0.1251	1443.7370	0.0488	1467.7639	1.7138	1447.9965	0.3439	1443.1157	0.0057
26	1560.7850	1560.7852	0.0000	1562.7420	0.1254	1560.7850	0.0000	1560.7850	0.0000	1589.7437	1.8554	1566.5919	0.3720	1560.8812	0.0062
27	1683.1543	1683.1554	0.0001	1702.5618	1.1530	1685.6764	0.1498	1684.1244	0.0576	1716.8651	2.0028	1689.9084	0.4013	1683.2662	0.0067
28	1810.1412	1810.1488	0.0004	1819.3368	0.5080	1810.1412	0.0000	1810.1412	0.0000	1849.1707	2.1562	1817.9542	0.4316	1810.2708	0.0072
29	1941.7459	1941.7896	0.0023	1983.5754	2.1542	1945.2029	0.1780	1943.0559	0.0675	1986.7048	2.3154	1950.7377	0.4631	1941.8950	0.0077
30	2077.9682	2078.1841	0.0104	2111.0148	1.5903	2077.9682	0.0000	2077.9682	0.0000	2129.5134	2.4806	2088.2677	0.4957	2078.1391	0.0082

Table 7.13 Natural frequencies of I-type connected beam with F-F boundaries (Hz)

Mode Seq.	Analy.	Conventional DSC		DSC-TSE						FEM					
		N=41 (r=5.1)	Error %	N=21x2 (r=3.3)	Error %	N=31x2 (r=4.6)	Error %	N=41x2 (r=5.4)	Error %	N=100x2	Error %	N=200x2	Error %	N=400x2	Error %
1	0.0000	0.0000	undfn.	0.0001	undfn.	0.0001	undfn.	0.0001	undfn.	0.0001	undfn.	0.0000	undfn.	0.0000	undfn.
2	0.0000	0.0001	undfn.	0.4078	undfn.	0.0832	undfn.	0.0131	undfn.	0.0001	undfn.	0.0001	undfn.	0.0000	undfn.
3	5.2338	5.2339	0.0017	5.2340	0.0035	5.2339	0.0021	5.2339	0.0019	5.2341	0.0053	5.2339	0.0015	5.2339	0.0015
4	14.4267	14.4275	0.0052	14.4275	0.0054	14.4275	0.0052	14.4275	0.0052	14.4293	0.0178	14.4278	0.0074	14.4274	0.0047
5	28.2856	28.2836	-0.0071	28.2859	0.0008	28.2842	-0.0050	28.2839	-0.0062	28.2912	0.0196	28.2851	-0.0019	28.2836	-0.0072
6	46.7543	46.7544	0.0002	46.7544	0.0003	46.7543	0.0000	46.7543	0.0000	46.7761	0.0467	46.7585	0.0090	46.7541	-0.0004
7	69.8428	69.8431	0.0004	69.8565	0.0196	69.8466	0.0053	69.8444	0.0022	69.8931	0.0720	69.8527	0.0141	69.8426	-0.0003
8	97.5491	97.5498	0.0008	97.5500	0.0009	97.5492	0.0001	97.5491	0.0000	97.6492	0.1027	97.5688	0.0202	97.5487	-0.0004
9	129.8730	129.8747	0.0013	129.9218	0.0376	129.8861	0.0101	129.8784	0.0041	130.0533	0.1388	129.9087	0.0275	129.8726	-0.0003
10	166.8147	166.8183	0.0022	166.8191	0.0027	166.8152	0.0003	166.8148	0.0001	167.1157	0.1805	166.8744	0.0358	166.8142	-0.0003
11	208.3740	208.3809	0.0033	208.5052	0.0630	208.4083	0.0164	208.3880	0.0067	208.8481	0.2275	208.4684	0.0453	208.3736	-0.0002
12	254.5511	254.5636	0.0049	254.5665	0.0060	254.5529	0.0007	254.5515	0.0002	255.2639	0.2800	254.6931	0.0558	254.5507	-0.0002
13	305.3459	305.3671	0.0070	305.6426	0.0972	305.4211	0.0246	305.3762	0.0099	306.3781	0.3380	305.5518	0.0674	305.3457	-0.0001
14	360.7584	360.7930	0.0096	360.8014	0.0119	360.7633	0.0014	360.7595	0.0003	362.2070	0.4016	361.0476	0.0802	360.7585	0.0000
15	420.7886	420.8428	0.0129	421.3869	0.1422	420.9351	0.0348	420.8469	0.0139	422.7686	0.4706	421.1841	0.0940	420.7892	0.0002
16	485.4365	485.5188	0.0170	485.5399	0.0213	485.4483	0.0024	485.4392	0.0006	488.0825	0.5451	485.9653	0.1089	485.4380	0.0003
17	554.7021	554.8234	0.0219	555.8154	0.2007	554.9640	0.0472	554.8052	0.0186	558.1698	0.6252	555.3954	0.1250	554.7047	0.0005
18	628.5854	628.7599	0.0278	628.8084	0.0355	628.6108	0.0040	628.5913	0.0009	633.0531	0.7108	629.4788	0.1421	628.5895	0.0007
19	707.0864	707.3318	0.0347	709.0418	0.2765	707.5256	0.0621	707.2571	0.0241	712.7569	0.8020	708.2204	0.1604	707.0924	0.0008
20	790.2051	790.5434	0.0428	790.6492	0.0562	790.2549	0.0063	790.2168	0.0015	797.3069	0.8987	791.6254	0.1797	790.2136	0.0011
21	877.9416	878.3998	0.0522	881.2438	0.3761	878.6422	0.0798	878.2100	0.0306	886.7308	1.0011	879.6991	0.2002	877.9530	0.0013
22	970.2957	970.9066	0.0630	971.1396	0.0870	970.3865	0.0094	970.3171	0.0022	981.0577	1.1091	972.4473	0.2217	970.3108	0.0016
23	1067.2676	1068.0702	0.0752	1072.7389	0.5126	1068.3416	0.1006	1067.6729	0.0380	1080.3185	1.2228	1069.8761	0.2444	1067.2871	0.0018
24	1168.8571	1169.8979	0.0890	1170.4658	0.1376	1169.0135	0.0134	1168.8941	0.0032	1184.5455	1.3422	1171.9918	0.2682	1168.8819	0.0021
25	1275.0644	1276.3977	0.1046	1284.2246	0.7184	1276.6587	0.1250	1275.6564	0.0464	1293.7732	1.4673	1278.8011	0.2931	1275.0954	0.0024
26	1385.8894	1387.5784	0.1219	1389.2260	0.2408	1386.1462	0.0185	1385.9502	0.0044	1408.0373	1.5981	1390.3110	0.3190	1385.9276	0.0028
27	1501.3320	1503.4501	0.1411	1517.4597	1.0742	1503.6363	0.1535	1502.1732	0.0560	1527.3754	1.7347	1506.5288	0.3461	1501.3787	0.0031
28	1621.3924	1624.0249	0.1624	1629.4069	0.4943	1621.7979	0.0250	1621.4887	0.0059	1651.8270	1.8771	1627.4621	0.3743	1621.4488	0.0035
29	1746.0705	1749.3197	0.1861	1776.6378	1.7506	1749.3275	0.1865	1747.2381	0.0669	1781.4333	2.0253	1753.1188	0.4037	1746.1380	0.0039
30	1875.3663	1879.3653	0.2132	1896.3987	1.1215	1875.9850	0.0330	1875.5134	0.0078	1916.2371	2.1793	1883.5071	0.4341	1875.4464	0.0043

Table 7.14 Natural frequencies of I-type connected beam with C-F boundaries (Hz)

Mode Seq.	Analy.	Conventional DSC		DSC-TSE						FEM					
		N=41 (r=5.1)	Error %	N=21x2 (r=2.7)	Error %	N=31x2 (r=3.6)	Error %	N=41x2 (r=4.8)	Error %	N=100x2	Error %	N=200x2	Error %	N=400x2	Error %
1	0.8224	0.8225	0.0111	0.8225	0.0111	0.8225	0.0111	0.8225	0.0111	0.8225	0.0085	0.8225	0.0085	0.8225	0.0085
2	5.1545	5.1547	0.0044	5.1548	0.0060	5.1547	0.0044	5.1547	0.0041	5.1549	0.0086	5.1547	0.0047	5.1546	0.0028
3	14.4341	14.4334	-0.0048	14.4334	-0.0044	14.4332	-0.0057	14.4332	-0.0060	14.4353	0.0085	14.4336	-0.0033	14.4331	-0.0068
4	28.2856	28.2841	-0.0056	28.2866	0.0033	28.2841	-0.0053	28.2836	-0.0071	28.2918	0.0218	28.2849	-0.0026	28.2832	-0.0086
5	46.7543	46.7564	0.0046	46.7570	0.0058	46.7550	0.0014	46.7546	0.0006	46.7781	0.0509	46.7589	0.0099	46.7540	-0.0006
6	69.8428	69.8476	0.0068	69.8632	0.0291	69.8481	0.0076	69.8449	0.0030	69.8967	0.0771	69.8533	0.0150	69.8424	-0.0006
7	97.5491	97.5585	0.0097	97.5610	0.0123	97.5520	0.0030	97.5502	0.0012	97.6552	0.1088	97.5698	0.0213	97.5484	-0.0007
8	129.8730	129.8899	0.0130	129.9449	0.0554	129.8914	0.0142	129.8804	0.0057	130.0625	0.1459	129.9102	0.0286	129.8721	-0.0007
9	166.8147	166.8429	0.0169	166.8508	0.0217	166.8233	0.0052	166.8181	0.0020	167.1290	0.1884	166.8766	0.0371	166.8135	-0.0007
10	208.3740	208.4188	0.0215	208.5647	0.0915	208.4219	0.0230	208.3932	0.0092	208.8668	0.2365	208.4713	0.0467	208.3726	-0.0007
11	254.5511	254.6189	0.0266	254.6388	0.0345	254.5716	0.0080	254.5591	0.0031	255.2892	0.2900	254.6972	0.0574	254.5494	-0.0007
12	305.3459	305.4451	0.0325	305.7711	0.1392	305.4499	0.0341	305.3872	0.0135	306.4114	0.3490	305.5571	0.0692	305.3440	-0.0006
13	360.7584	360.8994	0.0391	360.9438	0.0514	360.8002	0.0116	360.7745	0.0045	362.2498	0.4134	361.0544	0.0821	360.7564	-0.0005
14	420.7886	420.9840	0.0465	421.6359	0.2014	420.9890	0.0476	420.8677	0.0188	422.8226	0.4834	421.1927	0.0960	420.7866	-0.0005
15	485.4365	485.7018	0.0547	485.7958	0.0740	485.5138	0.0159	485.4661	0.0061	488.1496	0.5589	485.9759	0.1111	485.4346	-0.0004
16	554.7021	555.0556	0.0637	556.2718	0.2830	555.0563	0.0639	554.8408	0.0250	558.2519	0.6400	555.4083	0.1273	554.7006	-0.0003
17	628.5854	629.0491	0.0738	629.2506	0.1058	628.7182	0.0211	628.6357	0.0080	633.1525	0.7266	629.4944	0.1446	628.5846	-0.0001
18	707.0864	707.6861	0.0848	709.8801	0.3951	707.6731	0.0830	707.3141	0.0322	712.8757	0.8188	708.2390	0.1630	707.0865	0.0000
19	790.2051	790.9711	0.0969	791.4460	0.1570	790.4204	0.0272	790.2858	0.0102	797.4478	0.9166	791.6474	0.1825	790.2066	0.0002
20	877.9416	878.9092	0.1102	882.8884	0.5635	878.8657	0.1053	878.2965	0.0404	886.8962	1.0200	879.7248	0.2031	877.9449	0.0004
21	970.2957	971.5059	0.1247	972.7879	0.2568	970.6293	0.0344	970.4193	0.0127	981.2505	1.1290	972.4772	0.2248	970.3014	0.0006
22	1067.2676	1068.7679	0.1406	1076.3165	0.8479	1068.6664	0.1311	1067.7985	0.0497	1080.5417	1.2437	1069.9107	0.2477	1067.2762	0.0008
23	1168.8571	1170.7021	0.1578	1174.4752	0.4806	1169.3558	0.0427	1169.0395	0.0156	1184.8023	1.3642	1172.0314	0.2716	1168.8695	0.0011
24	1272.9377	1272.9377	0.0000	1272.9387	0.0001	1272.9380	0.0000	1275.8324	0.2274	1272.9410	0.0003	1272.9385	0.0001	1272.9379	0.0000
25	1275.0644	1277.3169	0.1767	1292.5529	1.3716	1277.1148	0.1608	1386.1503	8.7122	1294.0668	1.4903	1278.8463	0.2966	1275.0812	0.0013
26	1385.8894	1388.6217	0.1972	1399.6029	0.9895	1386.6131	0.0522	1502.4123	8.4078	1408.3712	1.6222	1390.3623	0.3227	1385.9116	0.0016
27	1501.3320	1504.6277	0.2195	1536.4472	2.3389	1504.2598	0.1950	1621.7563	8.0212	1527.7535	1.7599	1506.5867	0.3500	1501.3606	0.0019
28	1621.3924	1625.3491	0.2440	1654.5690	2.0462	1622.4175	0.0632	1747.5546	7.7811	1652.2531	1.9033	1627.5271	0.3784	1621.4285	0.0022
29	1746.0705	1750.8070	0.2713	1815.5569	3.9796	1750.1629	0.2344	1875.8626	7.4334	1781.9116	2.0527	1753.1915	0.4078	1746.1152	0.0026
30	1875.3663	1881.0429	0.3027	1948.8886	3.9204	1876.7893	0.0759	2011.2782	7.2472	1916.7719	2.2079	1883.5881	0.4384	1875.4210	0.0029

Table 7.15 Natural frequencies of I-type connected beam with MI-MI boundaries (Hz)

Mode Seq.	DSC-TSE			FEM	Dev. %		
	N=21x2 (r=3.2)	N=31x2 (r=4.4)	N=41x2 (r=5.4)		N=21x2 (r=3.2)	N=31x2 (r=4.4)	N=41x2 (r=5.4)
1	2.3090	2.3090	2.3090	2.3087	0.0131	0.0128	0.0127
2	9.2419	9.2419	9.2419	9.2289	0.1412	0.1414	0.1414
3	11.2486	11.2486	11.2486	11.2350	0.1209	0.1209	0.1209
4	11.2576	11.2576	11.2576	11.2503	0.0650	0.0650	0.0650
5	11.2730	11.2730	11.2730	11.2593	0.1214	0.1216	0.1216
6	20.7764	20.7755	20.7753	20.7842	-0.0376	-0.0417	-0.0426
7	36.9382	36.9382	36.9382	36.9452	-0.0190	-0.0190	-0.0190
8	57.7271	57.7206	57.7191	57.7248	0.0041	-0.0073	-0.0099
9	83.1155	83.1155	83.1155	83.1222	-0.0080	-0.0080	-0.0080
10	93.5350	93.5350	93.5350	94.7742	-1.3075	-1.3076	-1.3076
11	113.1659	113.1404	113.1347	113.1375	0.0251	0.0025	-0.0025
12	147.7635	147.7635	147.7635	147.7707	-0.0049	-0.0049	-0.0049
13	187.1122	187.0407	187.0250	187.0217	0.0484	0.0102	0.0018
14	230.8822	230.8822	230.8822	230.8906	-0.0036	-0.0036	-0.0036
15	279.5928	279.4284	279.3929	279.3775	0.0771	0.0182	0.0055
16	332.4718	332.4718	332.4718	332.4824	-0.0032	-0.0032	-0.0032
17	390.6450	390.3120	390.2416	390.2054	0.1127	0.0273	0.0093
18	452.5321	452.5321	452.5321	452.5465	-0.0032	-0.0032	-0.0032
19	520.3206	519.7024	519.5756	519.5058	0.1568	0.0378	0.0134
20	591.0634	591.0634	591.0634	591.0834	-0.0034	-0.0034	-0.0034
21	668.6925	667.6131	667.3997	667.2794	0.2118	0.0500	0.0180
22	748.0662	748.0654	748.0654	748.0937	-0.0037	-0.0038	-0.0038
23	835.8702	834.0605	833.7199	833.5267	0.2812	0.0640	0.0232
24	923.5475	923.5383	923.5383	923.5782	-0.0033	-0.0043	-0.0043
25	1022.0467	1019.0647	1018.5434	1018.2484	0.3730	0.0802	0.0290
26	1117.5622	1117.4820	1117.4820	1117.5375	0.0022	-0.0050	-0.0050
27	1227.6721	1222.6500	1221.8782	1221.4455	0.5098	0.0986	0.0354
28	1330.4477	1329.8965	1329.8965	1329.9726	0.0357	-0.0057	-0.0057
29	1454.0660	1444.8459	1443.7339	1443.1189	0.7586	0.1197	0.0426
30	1563.7598	1560.7819	1560.7819	1560.8844	0.1842	-0.0066	-0.0066

Table 7.16 Natural frequencies of L-type connected beams with P-P and MI-MI boundaries (Hz)

Mode Seq.	P-P			MI-MI		
	DSC-TSE	FEM	Dev. %	DSC-TSE	FEM	Dev. %
	N=31x2 (r=4.6)	N=400x2		N=31x2 (r=4.4)	N=400x2	
1	9.2354	9.2354	0.0003	9.2355	9.2224	0.1418
2	14.4276	14.4273	0.0024	11.2514	11.2165	0.3114
3	36.9421	36.9412	0.0026	11.2590	11.237	0.1958
4	46.7559	46.7524	0.0074	11.2590	11.2485	0.0934
5	83.1212	83.1165	0.0056	11.2755	11.2626	0.1144
6	97.5559	97.5413	0.0150	14.4362	14.4507	-0.1001
7	147.7744	147.7597	0.0100	36.9422	36.9482	-0.0164
8	166.8347	166.7929	0.0251	46.7592	46.7624	-0.0068
9	230.9045	230.8683	0.0157	83.1212	83.123	-0.0022
10	254.5982	254.5016	0.0379	97.5591	97.551	0.0083
11	332.5148	332.4393	0.0227	147.7745	147.7661	0.0057
12	360.8540	360.6597	0.0539	166.8380	166.8025	0.0213
13	452.6101	452.4683	0.0313	230.9045	230.8747	0.0129
14	485.6125	485.2575	0.0732	254.6016	254.5112	0.0355
15	591.1965	590.9497	0.0418	332.5150	332.4457	0.0208
16	628.8878	628.2814	0.0965	360.8578	360.6695	0.0522
17	748.2826	747.875	0.0545	452.6104	452.4749	0.0299
18	790.6995	789.7119	0.1251	485.6171	485.2677	0.0720
19	923.8812	923.2317	0.0704	591.1969	590.9565	0.0407
20	971.0775	969.5186	0.1608	628.8938	628.292	0.0958
21	1118.0130	1116.999	0.0908	748.2834	747.8822	0.0536
22	1170.0709	1167.651	0.2073	790.7081	789.7232	0.1247
23	1330.7144	1329.138	0.1186	923.8825	923.2394	0.0697
24	1387.7694	1384.012	0.2715	971.0906	969.5312	0.1608
25	1562.0599	1559.567	0.1598	1118.0151	1117.007	0.0902
26	1624.3634	1618.391	0.3690	1170.0916	1167.665	0.2078
27	1812.2325	1808.083	0.2295	1330.7180	1329.148	0.1182
28	1880.3519	1870.214	0.5421	1387.8032	1384.03	0.2726
29	2081.8550	2073.958	0.3808	1562.0662	1559.58	0.1594
30	2157.6308	2137.156	0.9580	1624.4205	1618.418	0.3709

Table 7.17 Natural frequencies of T-type connected beams with P-P-P and MI-MI-MI boundaries (Hz)

Mode Seq.	P-P-P			MI-MI-MI		
	DSC-TSE	FEM	Dev. %	DSC-TSE	FEM	Dev. %
	N=31x2 (r=4.6)	N=400x2		N=31x2 (r=4.4)	N=400x2	
1	9.2354	9.2354	0.0001	9.2409	9.2278	0.1416
2	14.4276	14.4271	0.0031	11.2492	11.1997	0.4420
3	14.4278	14.4274	0.0028	11.2507	11.2288	0.1954
4	36.9417	36.9416	0.0004	11.2507	11.2475	0.0288
5	46.7548	46.7508	0.0086	11.2647	11.2491	0.1387
6	46.7574	46.7534	0.0086	11.2647	11.2597	0.0445
7	83.1191	83.1186	0.0006	14.4253	14.4403	-0.1037
8	97.5514	97.5346	0.0172	14.4524	14.4663	-0.0958
9	97.5627	97.5459	0.0172	36.9388	36.9457	-0.0186
10	147.7679	147.7662	0.0012	46.7537	46.7590	-0.0114
11	166.8215	166.773	0.0291	46.7675	46.7675	0.0000
12	166.8546	166.8062	0.0290	83.1165	83.1223	-0.0070
13	230.8885	230.8843	0.0018	93.5350	94.7742	-1.3076
14	254.5671	254.4548	0.0441	97.5503	97.5506	-0.0003
15	254.6447	254.5326	0.0440	97.5724	97.5512	0.0217
16	332.4816	332.4726	0.0027	147.7653	147.7699	-0.0031
17	360.7910	360.5648	0.0627	166.8204	166.7890	0.0188
18	360.9482	360.7228	0.0625	166.8643	166.8115	0.0316
19	452.5478	452.5308	0.0037	230.8859	230.8880	-0.0009
20	485.4967	485.0826	0.0854	254.5661	254.4710	0.0374
21	485.7853	485.3735	0.0848	254.6545	254.5380	0.0458
22	591.0881	591.0585	0.0050	332.4789	332.4763	0.0008
23	628.6890	627.9801	0.1129	360.7901	360.5814	0.0579
24	629.1833	628.4806	0.1118	360.9583	360.7281	0.0638
25	748.1041	748.0546	0.0066	452.5452	452.5346	0.0023
26	790.3746	789.2165	0.1467	485.4959	485.0998	0.0817
27	791.1806	790.0379	0.1446	485.7960	485.3790	0.0859
28	923.5979	923.5176	0.0087	591.0856	591.0622	0.0040
29	970.5633	968.7281	0.1894	628.6886	627.9983	0.1099
30	971.8348	970.0361	0.1854	629.1950	628.4862	0.1128

Followings can be inferred from Tables 7.12-7.17:

1. Proposed method is capable of estimating the natural frequencies of the uniform beam by modelling the beam as two separate beams (Tables 7.12-7.14). Different numbers of discretization points in the proposed algorithm and element sizes in the FEM computations are considered. It is seen that, all of the results are consistent with the analytical solutions. However, $N=31 \times 2$ (which means each beam is discretized with 31 structural points) for the DSC is capable of estimating the natural frequencies with the error lower than 0.5%. Therefore, $N=31 \times 2$ is selected for the further analysis.

2. Results with $N=200 \times 2$ elements in FEM are capable of estimating the natural frequencies with the error below 0.5%. This proves that the proposed algorithm is superior to FEM for this problem. However, since the results of FEM is used as a reference for the further computations, the number of the elements is selected as $N=400 \times 2$ which has the error below 0.01%. It should also be noted that FEM estimates some of the natural frequencies lower than the ones of analytical solutions. This is due to the fact that, the element in FEM method uses Timoshenko beam theory whose natural frequencies are under estimated than those of Euler beam theory.
3. For MI-MI boundary of I-type connected beams, three additional natural frequencies around 11 Hz are obtained. These are natural frequencies dominated by springs and total mass of the structure. Similar natural frequencies are also obtained for L- and T-type connected structures.
4. For the L- and T-type structures, natural frequencies obtained via DSC are consistent with the ones of FEM. However, it is seen that, higher deviations are observed especially in the higher order modes as expected.
5. Some of the natural frequencies belong to I-type beam are also observed in the ones of L- and T-type connected beams. These are modes of the individual beams which are used to construct I-type structure and can be called as “local modes”.
6. Results show that, the constructed relations for all structures lead correct modelling of considered structures. It should be noted that, these relations include mode transmission type (such as, moment transmission for all structures) and mode conversion equations (such as, longitudinal force transmitted as bending force or vice versa for L- and T-type structures).
7. In general, these results show that, proposed algorithm is successful in handling the connected beams with different orientation.

7.4 Numerical Analysis of a Stiffened Plate

The proposed DSC-TSE methodology has some difficulties on the modelling of connected beam-plate structures. The method is not so straightforward to model the structures connected via internal points, because continuity equations at internal

points are more complex. Therefore, another approach is proposed in the thesis and it is named here as DSC-c method. Here, DSC-c denotes DSC method for connected structures. The method is applied to a “stiffened plate” problem. This approach separately models the beam and plate in the DSC manner with appropriate boundary condition and solves simultaneously in a global DSC matrix.

Let us consider the schematic structure shown in Figure 7.9. For this connected structure, differential equations of the beam and the plate must be satisfied among the discretization points on connection line, i.e., $x=0.5$ m line, whereas only the differential equation of plate must be satisfied for the other discretized points. Therefore, equation of motion of the structure may be written as follows:

$$D \left(\frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + EI_x N_x \frac{\partial^4 w(x, y, t)}{\partial y^4} + \rho \left(h_p \frac{\partial^2 w(x, y, t)}{\partial t^2} + S N_x \frac{\partial^2 w(x, y, t)}{\partial t^2} \right) = 0, \text{ for } x=0.5, (7.22)$$

$$D \left(\frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + \rho h_p \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0, \text{ others. (7.23)}$$

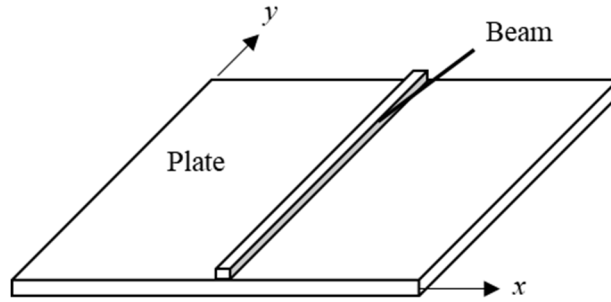


Figure 7.9 Schematic representation of a stiffened plate

The DSC modelling procedure of this structure is outlined below:

- The DSC matrix of the beam and the plate are evaluated by Eq. (4.26) and Eq. (4.32), respectively as they are two separate structures. In order to model the beam in two-dimensional domain, equation of motion of the beam must be enlarged

with the discretization number on y -direction as shown in Eq. (7.23). Note that, the beam and y -dimension of the plate must be discretized with the same number of discretization points ($N_b=N_y$ where N_b and N_y represent the discretization number of beam and plate on y -axis, respectively).

- Next, the DSC matrix of the beam must be accordingly placed in ones of the plate. It is performed by distributing the structural points of the beam on the plate as follows:

$$\mathbf{D}_{BP} = \mathbf{D}_{Dist} \otimes \mathbf{D}_B, \quad (7.24)$$

Here, $\mathbf{D}_{BP}$ represents matrix of the beam that is distributed on the plate, $\mathbf{D}_{Dist}$ is the distribution matrix and its dimension is arranged in accordance with the number of structural points of the plate on x -dimension, i.e., $(N_x \times N_x)$ and $\mathbf{D}_B$ is the left hand side of Eq. (4.26):

$$\mathbf{D}_B = \mathbf{D}_S - \mathbf{D}_D \mathbf{B}^{-1} \mathbf{C}_{Coef} - \frac{\omega_n^2}{c^2} \mathbf{I}. \quad (7.25)$$

According to the position of the beam, the elements of the distribution matrix are arranged. Since the beam lays on $x=0.5$ m line which is the mid-position in the discretization on x -space, the element at $(N_x + 1)/2$ th row and column must be one whereas the others are zero:

$$\mathbf{D}_{Dist} = \begin{bmatrix} 0 & \dots & \dots & 0 & \dots & \dots & 0 \\ 0 & \ddots & \dots & \vdots & \dots & \ddots & 0 \\ \vdots & \vdots & 0 & 0 & 0 & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & 0 & 0 & \vdots & \vdots \\ 0 & \ddots & \dots & \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \dots & \dots & 0 \end{bmatrix}_{N_x \times N_x}. \quad (7.26)$$

One can shift the position of the beam by rearranging the distribution matrix.

- The global matrix of the stiffened plate is evaluated by summing up Eq. (4.32) and Eq. (7.24). Then, natural frequencies of the structure can be obtained by solving eigenvalue problem of global matrix.

7.4.1 Free Vibration Analysis of a Stiffened Plate

The proposed methodology is applied to structure shown in Figure 7.9. The boundary condition of the system is assumed to be as fully simply-supported. The physical and the mechanical properties of the plate are presented in Table 6.14. One can also refer to Table 6.1 for the mechanical properties of the beam with length of 1 m and the width of 0.001 m.

As a result, the first twenty natural frequencies obtained via DSC are compared with the ones of FEM for different thickness of the beam in Table 7.18. The first six mode shapes of the structure having the beam thickness of 0.2 m are evaluated with the DSC and FEM and presented in Table 7.19 together with an individual plate with simply-supported boundary condition. Beside that, the corresponding mode shapes of the beam are also presented in Table 7.20. Note that, the discretization number and regularization coefficient in each spatial axis are selected as $N=N_x=N_y=31$ and $r=4.9$ for the DSC, respectively, $N=200$ elements in each spatial axis are selected for FEM analysis. The number of elements in FEM is selected after performing a convergence study. In the convergence study, a fully simply-supported plate is considered and number of elements having *Error%* lower than 0.01% from analytical results are selected in these analyses.

Table 7.18 Natural frequencies of the stiffened plate (Hz)

Mode Seq.	h=0.001 m			h=0.005 m			h=0.01 m			h=0.02 m		
	DSC-c	FEM	Dev. %	DSC-c	FEM	Dev. %	DSC-c	FEM	Dev. %	DSC-c	FEM	Dev. %
1	4.8368	4.8419	-0.1050	4.9551	4.9761	-0.4224	5.7904	5.8181	-0.4766	9.6683	9.7015	-0.3421
2	12.0965	12.1038	-0.0606	12.1017	12.1149	-0.1091	12.1017	12.1289	-0.2244	12.1017	12.1559	-0.4460
3	12.1017	12.1097	-0.0662	12.8901	12.9276	-0.2904	16.6767	16.6765	0.0011	19.3627	19.4942	-0.6746
4	19.3627	19.3671	-0.0228	19.3627	19.3947	-0.1650	19.3627	19.4289	-0.3408	21.8755	21.8472	0.1293
5	24.1787	24.2089	-0.1249	24.1082	24.2363	-0.5287	24.1813	24.4311	-1.0223	25.5285	26.0782	-2.1080
6	24.1968	24.2265	-0.1226	26.0710	26.1284	-0.2197	31.3657	31.3413	0.0778	31.4644	31.6448	-0.5702
7	31.4348	31.4722	-0.1190	31.4644	31.5114	-0.1492	31.4644	31.5579	-0.2964	33.9618	33.9494	0.0367
8	31.4644	31.4735	-0.0290	31.6512	31.8123	-0.5064	34.0279	34.3475	-0.9305	41.1457	41.2260	-0.1947
9	41.1375	41.1644	-0.0654	41.1457	41.1775	-0.0772	41.1457	41.1939	-0.1170	47.5891	47.8517	-0.5489
10	41.1457	41.1961	-0.1223	44.2225	44.2987	-0.1721	48.4067	48.5235	-0.2407	48.4067	48.6197	-0.4380
11	43.5314	43.5835	-0.1196	44.6361	44.8483	-0.4731	48.4067	48.5270	-0.2479	48.4067	48.6336	-0.4665
12	48.4067	48.4274	-0.0427	48.4067	48.4709	-0.1324	49.1454	49.1319	0.0276	50.4915	50.4936	-0.0042
13	48.4067	48.4274	-0.0427	48.4067	48.4717	-0.1340	52.9406	53.1951	-0.4785	60.5084	60.8962	-0.6368
14	60.4693	60.5335	-0.1061	60.5084	60.6128	-0.1722	60.5084	60.7108	-0.3334	62.3659	63.6680	-2.0451
15	60.5084	60.5477	-0.0649	62.6219	62.9821	-0.5720	62.3902	63.0566	-1.0569	66.0243	65.9861	0.0579
16	62.8641	62.9716	-0.1707	63.1696	63.4438	-0.4321	70.1897	70.3370	-0.2093	70.1897	70.4363	-0.3500
17	62.9184	63.0243	-0.1680	67.1059	67.2030	-0.1444	70.7194	71.1409	-0.5926	71.9278	71.9541	-0.0366
18	70.1188	70.2312	-0.1600	69.9877	70.2815	-0.4180	71.1267	71.4994	-0.5213	77.4508	77.9766	-0.6744
19	70.1897	70.2350	-0.0644	70.1897	70.3828	-0.2743	76.1113	76.1700	-0.0771	79.9626	81.8665	-2.3257
20	77.4508	77.4860	-0.0455	77.4508	77.5955	-0.1865	77.4508	77.7289	-0.3578	83.4640	83.4300	0.0408

Table 7.19 Mode shapes of stiffened plate for $h=0.02$ m and single plate

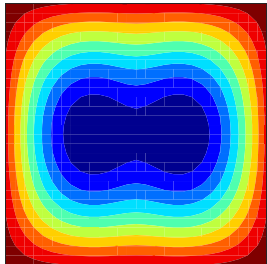
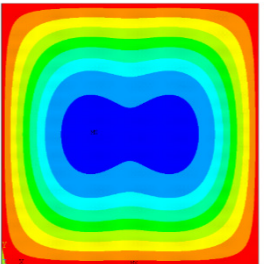
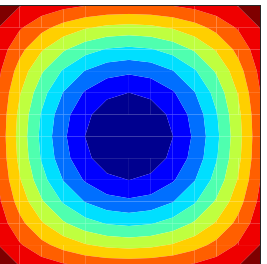
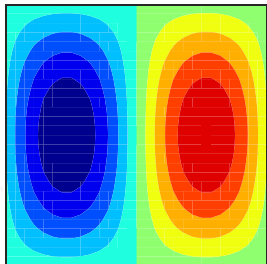
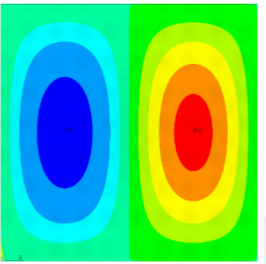
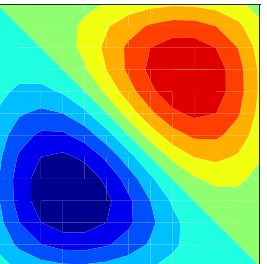
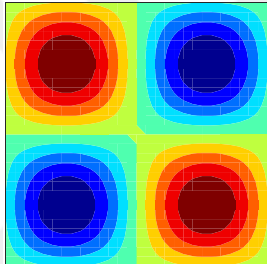
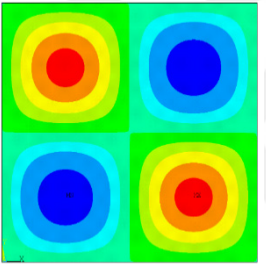
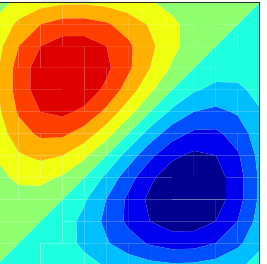
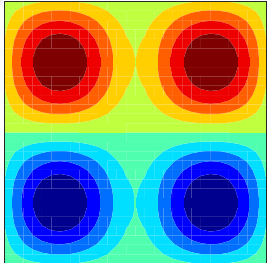
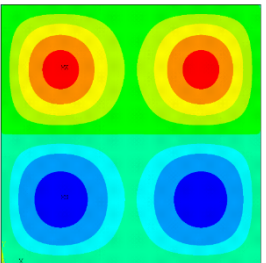
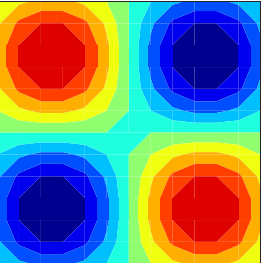
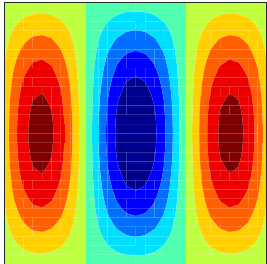
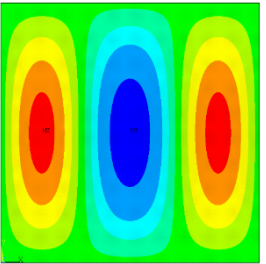
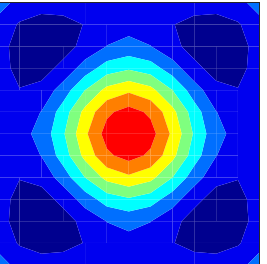
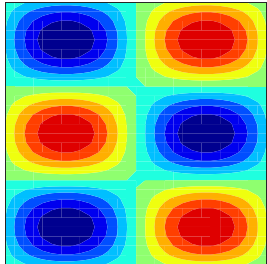
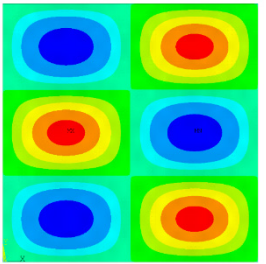
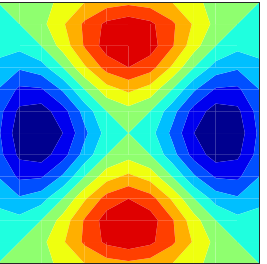
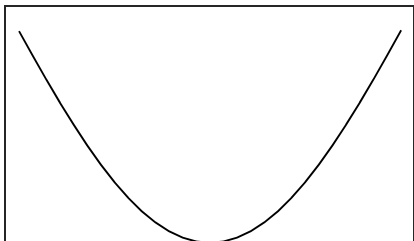
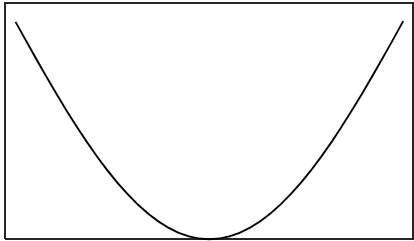
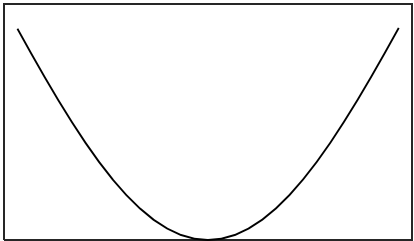
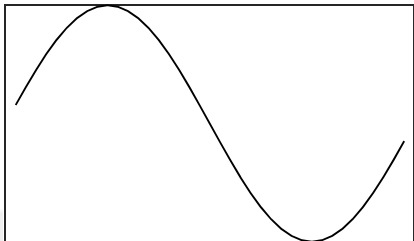
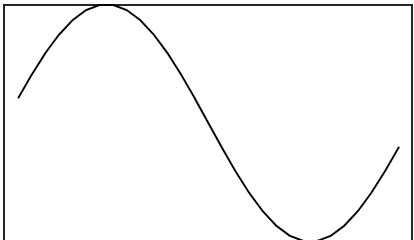
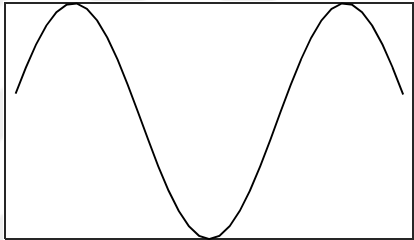
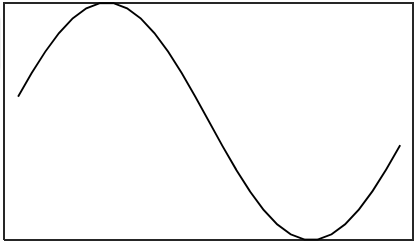
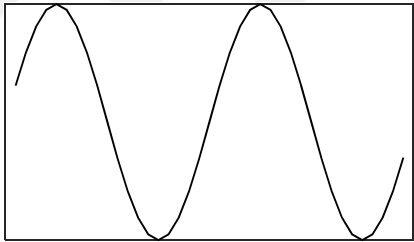
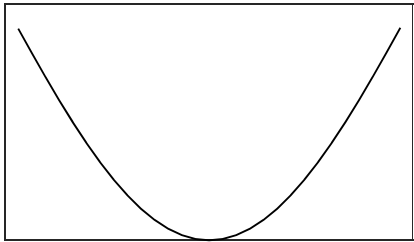
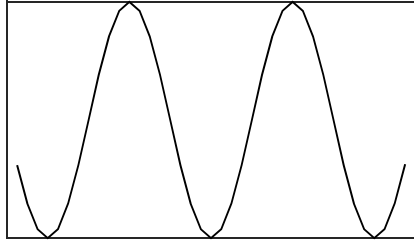
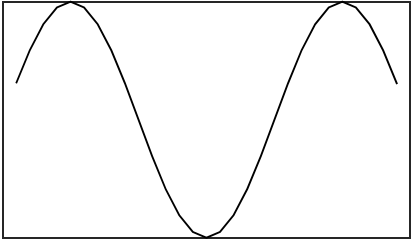
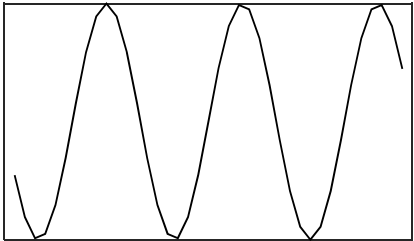
Mode Seq.	DSC-c for line connected stif. plate	FEM for line connected stif. plate	DSC for simply supported plate
1			
2			
3			
4			
5			
6			

Table 7.20 Mode shapes at $x=0.5$ m of the stiffened plate and simply-supported beam

Mode Seq.	DSC-c for line connected stif. plate	DSC for simply-supported beam
1		
2		
3		
4		
5		
6		

Followings can be inferred from results:

- The results show that, the DSC-c method is able to determine the vibration characteristics of stiffened plates.
- The deviation of the DSC-c method from FEM increases as the thickness of the beam increases. This may be due to the fact that, the shear effects in the DSC are neglected. Therefore, the model may be improved to consider this effect for further comparisons.
- Mode shapes of DSC-c and FEM for the structure are consistent with each other.
- When the mode shapes of the fully simply-supported beam and plate are compared with stiffened plate, it can be observed that the vibration characteristics of individual beam and plate completely change.
- The mode shapes of the simply-supported beam are changed by connecting the beam to plate.
- The effect of connecting beam is clearly observed on the first mode of the stiffened plate by reduced vibration displacement at $x=0.5$ m line.
- For the second and third mode shapes of the stiffened plate, the symmetric mode behaviours evaluated in second and third modes of simply-supported plate are not observed anymore.
- The fourth mode of the simply-supported plate is evaluated as the third mode of the stiffened plate.

CHAPTER EIGHT

POLYNOMIAL CHAOS EXPANSION FOR UNCERTAINTY ANALYSIS

8.1 Introduction

Up to this point, the DSC methodology and its applications are considered for deterministic structures. However, as explained in Chapter 1, uncertainty in structures is an important phenomena and it must be considered in the analysis starting from the design stage. In this regard, Polynomial Chaos Expansion (PCE), which is a spectral analysis tool, is considered for the uncertainty analysis. In the numerical analysis, the DSC is used as a partial differential equation solver whereas the PCE is for uncertainty tool. Here, the integrated methodology will be called as DSC-PCE throughout the thesis. In the applications included in the thesis, two different uncertain structures are considered;

1. Uncertain inhomogeneous individual beam,
2. Uncertain L-type connected beams.

Before starting to analyse the structures, firstly, an introduction to PCE is given in Sections 8.2 and 8.3. Then, the procedure of the PCE implemented to the DSC is given in Section 8.4. Finally, numerical results of considered structures are given in Section 8.5.

8.2 Polynomial Chaos Expansion (PCE)

Polynomial Chaos Expansion (PCE) is known as a functional uncertainty analysis tool because it able to formulate the uncertain response as expressed in Chapter 1. According to Polynomial Chaos Expansion (PCE), any uncertain variable Y can be expressed as the sum of orthogonal polynomials (Ghanem & Spanos, 2003; Sepahvand et al., 2010):

$$\begin{aligned}
Y(\xi) &= y_0 \psi_0 + \sum_{i_1=1}^{\infty} y_{i_1} \psi_1(\xi_{i_1}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} y_{i_1 i_2} \psi_2(\xi_{i_1}, \xi_{i_2}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} y_{i_1 i_2 i_3} \psi_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) + \dots \\
&= \sum_{i=0}^{\infty} y_i \psi_i(\xi)
\end{aligned} \tag{8.1}$$

Here, $\xi = \{\xi_1, \xi_2, \dots, \xi_n\}$ is the vector containing stochastic parameters, y_i is deterministic coefficient of orthogonal polynomial basis of $\psi_i(\cdot)$. Since polynomial terms are orthogonal to each other, the scalar product of any two of the polynomial basis satisfies:

$$\langle \psi_i \cdot \psi_j \rangle = \begin{cases} 0, & \text{for } i \neq j \\ \langle \psi_i^2 \rangle, & \text{for } i = j \end{cases} \tag{8.2}$$

Here, $\langle \rangle$ shows the inner product. In PCE, unknown deterministic coefficients of polynomials can be determined by using Galerkin projection (Ghanem & Spanos, 2003) which may be described as calculating average of the considered equation and basis polynomial function multiplication:

$$y_i = \frac{1}{\langle \psi_i^2 \rangle} \int_{\Omega_1} \dots \int_{\Omega_n} \langle Y, \psi_k(\xi) \rangle d\mu_n(\xi_n) \dots d\mu_1(\xi_1). \tag{8.3}$$

Note that, $d\mu(\xi)$ is the probability density function of random space (Ω) of uncertain parameter (ξ) and k is an integer starting from zero. If all random variables are uncorrelated, this leads to the following,

$$d\mu(\xi) = \mu_1(\xi_1) \mu_2(\xi_2) \dots \mu_n(\xi_n) d\mu_1 d\mu_2 \dots d\mu_n. \tag{8.4}$$

Here, $\mu_i(\xi_i)$ is the probability density function. One can have the following equation by substituting Eq. (8.4) into Eq. (8.3):

$$y_i = \frac{1}{\langle \psi_i^2 \rangle} \int_{\Omega_1} \int_{\Omega_2} \dots \int_{\Omega_n} Y \psi_k(\xi) \mu_1(\xi_1) \mu_2(\xi_2) \dots \mu_n(\xi_n) d\xi_1 d\xi_2 \dots d\xi_n . \quad (8.5)$$

Several types of polynomial basis functions (ψ_i) such as Laguerre polynomial, Jacobi polynomial, Legendre polynomial, etc. can be used in PCE, but Hermite polynomials are the most suitable one for normal distribution (Ghanem & Spanos, 2003):

$$H_i(\xi) = (-1)^i \exp\left(\frac{1}{2}\xi^T \xi\right) \frac{\partial^i}{\partial \xi_{i_1} \partial \xi_{i_2} \dots \partial \xi_{i_n}} \exp\left(-\frac{1}{2}\xi^T \xi\right). \quad (8.6)$$

As shown in Eq. (8.1), PC expansion of an uncertain variable can be extended up to infinity. In the application of PC expansion, it makes sense to expand the uncertain variable up to a finite number. Organizing Eq. (8.1) for the finite series:

$$Y = \sum_{i=0}^{N_{PC}} y_i \psi_i(\xi), \quad (8.7)$$

where,

$$N_{PC} = \frac{(n_u + p_u)!}{n_u! p_u!} - 1. \quad (8.8)$$

Here, p_u shows the order of polynomial, n_u shows number of uncertain parameter (ξ_i) to define uncertain variable (Y). For $n_u=2$ and $p_u=3$ and using Hermitian polynomials for the orthogonal polynomial, Eq. (8.1) can be expressed as:

$$\begin{aligned} Y = & y_0 H_0 + \sum_{i_1=1}^2 y_{i_1} H_1(\xi_{i_1}) + \sum_{i_1=1}^2 \sum_{i_2=1}^{i_1} y_{i_1 i_2} H_2(\xi_{i_1}, \xi_{i_2}) \\ & + \sum_{i_1=1}^2 \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} y_{i_1 i_2 i_3} H_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) \end{aligned} \quad (a)$$

$$\begin{aligned}
Y = & y_0 H_0 + \underbrace{y_1 H_1(\xi_1)}_{i_1=1} + \underbrace{y_2 H_1(\xi_2)}_{i_1=2} + \underbrace{y_{11} H_2(\xi_1, \xi_1)}_{i_1=i_2=1} + \underbrace{y_{21} H_2(\xi_2, \xi_1)}_{i_1=2, i_2=1} + \\
& \underbrace{y_{22} H_2(\xi_2, \xi_2)}_{i_1=i_2=2} + \underbrace{y_{111} H_3(\xi_1, \xi_1, \xi_1)}_{i_1=i_2=i_3=1} + \underbrace{y_{211} H_3(\xi_2, \xi_1, \xi_1)}_{i_1=2, i_2=i_3=1} + \\
& \underbrace{y_{221} H_3(\xi_2, \xi_2, \xi_1)}_{i_1=i_2=2, i_3=1} + \underbrace{y_{222} H_3(\xi_2, \xi_2, \xi_2)}_{i_1=i_2=i_3=2}
\end{aligned} \quad , \quad (b) \quad (8.9)$$

where,

$$H_0 = 1, \quad (a)$$

$$H_1(\xi_1) = (-1)^1 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial}{\partial \xi_1} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = \xi_1, \quad (b)$$

$$H_1(\xi_2) = \xi_2, \quad (c)$$

$$H_2(\xi_1, \xi_1) = (-1)^2 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial^2}{\partial \xi_1 \partial \xi_1} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = \xi_1^2 - 1, \quad (d)$$

$$H_2(\xi_2, \xi_2) = (-1)^2 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial^2}{\partial \xi_2 \partial \xi_2} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = \xi_2^2 - 1, \quad (e)$$

$$H_2(\xi_2, \xi_1) = (-1)^2 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial^2}{\partial \xi_2 \partial \xi_1} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = \xi_1 \xi_2, \quad (f) \quad (8.10)$$

$$H_3(\xi_1, \xi_1, \xi_1) = (-1)^3 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial^3}{\partial \xi_1 \partial \xi_1 \partial \xi_1} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = \xi_1^3 - 3\xi_1, \quad (g)$$

$$H_3(\xi_2, \xi_2, \xi_2) = (-1)^3 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial^3}{\partial \xi_2 \partial \xi_2 \partial \xi_2} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = \xi_2^3 - 3\xi_2, \quad (h)$$

$$H_3(\xi_2, \xi_1, \xi_1) = (-1)^3 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial^3}{\partial \xi_2 \partial \xi_1 \partial \xi_1} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = (\xi_1^2 - 1)\xi_2, \quad (i)$$

$$H_3(\xi_2, \xi_2, \xi_1) = (-1)^3 e^{\frac{1}{2}(\xi_1^2 + \xi_2^2)} \frac{\partial^3}{\partial \xi_2 \partial \xi_2 \partial \xi_1} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = (\xi_2^2 - 1)\xi_1. \quad (j)$$

Note that, one can refer to Ghanem and Spanos (2003) for different order of polynomials and dimensions.

8.3 Representation of an Uncertain Variable via Optimal PCE

There is an orthogonal optimal polynomial basis for each distribution type in PCE, i.e. Hermitian polynomials for Gaussian distribution, Laguerre polynomials for Gamma distribution, Jacobi polynomials for Beta distribution and Legendre polynomials for Uniform distributions. In these polynomial types, weighting functions coincide with the distribution type and if these basis polynomials are used for the corresponding distribution types, PC expansion of the uncertain parameters is optimal representation. In this case, minimum number of coefficients of polynomial is enough to represent the uncertain variable. If appropriate polynomial basis-distribution type is not used for the uncertain variable, a space transformation is required to derive non-optimal PCE (Sepahvand et al., 2010). In non-optimal cases, there are more coefficients needed to express the uncertain parameter as a polynomial.

If the uncertain parameter ξ is defined in the interval of $\xi \in [a, b]$ and the uncertain variable Y is defined as $Y \in [c, d]$, then a simple shifting needs to be performed to derive uncertain variable in terms of uncertain parameter ξ :

$$\int_c^Y \mu_1(\tau) d\tau = \int_a^\xi \mu_2(\tau) d\tau \Rightarrow Y = f(\xi) . \quad (8.11)$$

An example may be given to model normal distribution via Hermitian polynomials in order to demonstrate this procedure. Assuming the uncertain parameter ξ has a normal distribution with the statistical properties $\{\varepsilon_\xi, \chi_\xi\} = \{0, 1\}$ (ε and χ represent the mean and standard deviation) whereas uncertain Young modulus of a structure, E has $\{\varepsilon_E, \chi_E\} = \{68 \times 10^9, 0.05 \times 68 \times 10^9\}$. Eq. (8.11) is applied as in the following:

$$\int_{-\infty}^E \frac{1}{\sqrt{2\pi}\chi_E} \exp\left(\frac{-(\tau - \varepsilon_E)^2}{2\chi_E^2}\right) d\tau = \int_{-\infty}^{\xi} \frac{1}{\sqrt{2\pi}\chi_{\xi}} \exp\left(\frac{-(\tau - \varepsilon_{\xi})^2}{2\chi_{\xi}^2}\right) d\tau. \quad (8.12)$$

and,

$$0.5 + 0.5\operatorname{erf}\left[\sqrt{2}\left(\frac{E - \varepsilon_E}{2\chi_E}\right)\right] = 0.5 + 0.5\operatorname{erf}\left[\sqrt{2}\left(\frac{\xi - \varepsilon_{\xi}}{2\chi_{\xi}}\right)\right]. \quad (8.13)$$

Eq. (8.13) leads to,

$$E = \left(\frac{\xi - \varepsilon_{\xi}}{\chi_{\xi}}\right)\chi_E + \varepsilon_E = 34 \times 10^8 \xi + 68 \times 10^9. \quad (8.14)$$

Eq. (8.14) show that, only two terms of Hermite polynomials are enough to model an uncertain normal distributed variable in the optimal PCE.

8.4 DSC-PCE Method for Uncertainty Analysis of Structures

PCE is an uncertainty tool that may be applied with either analytical or numerical methods to solve uncertain differential equations (Sepahvand et al., 2010). One can refer to Appendix 3 for the PCE application to analytical differential equations. In this section, uncertainty analysis of structures via PCE is introduced with a numerical method, i.e, DSC method. In this regard, the methodology is expressed on the evaluation of distribution of natural frequencies of uncertain inhomogeneous thin beams.

Consider the inhomogeneous beam given in Section 6.4.2 as uncertain. For that uncertain beam, Eq. (6.5) can be rearranged with stochastic parameters $\vartheta = \{\vartheta_1, \vartheta_2, \vartheta_3\}$ ($\vartheta_1, \vartheta_2, \vartheta_3$ are the uncertain parameters affecting elasticity modulus, thickness and density, respectively) as follows;

$$\left(E(\vartheta_1)h^2(\vartheta_2)r_\rho(\vartheta_3)\mathbf{D}(\mathbf{x})-\lambda_n(\vartheta)\mathbf{I}\right)\mathbf{w}(\mathbf{x},\vartheta)=0. \quad (8.15)$$

As it may be inferred from Eq. (8.15) that, dynamic stiffness matrix is independent from uncertain structure. It gives unique advantage in uncertainty modelling.

As expressed in previous section, a first order polynomial is sufficient in PCE representation of the normal distribution with Hermite polynomials. In this case, stochastic parameters may be described as $\vartheta_i = \{\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,n}\}$. It is assumed that, all uncertain variables depend on single uncertain variable ($\vartheta_i = \xi_{i,1}$) in this study. So, uncertain variables can be written as follows:

$$E(\vartheta_1) = \sum_{i=0}^1 E_i H_i(\xi_{1,1}) = E_0 + E_1 \xi_{1,1} = \begin{Bmatrix} E_0 & E_1 \end{Bmatrix} \begin{Bmatrix} 1 \\ \xi_{1,1} \end{Bmatrix} \quad (8.16)$$

$$h^2(\vartheta_2) = \sum_{j=0}^1 h_j^2 H_j(\xi_{2,1}) = h_0^2 + h_1^2 \xi_{2,1} = \begin{Bmatrix} h_0^2 & h_1^2 \end{Bmatrix} \begin{Bmatrix} 1 \\ \xi_{2,1} \end{Bmatrix} \quad (8.17)$$

$$r_\rho(\vartheta_3) = \sum_{k=0}^1 r_{\rho k} H_k(\xi_{3,1}) = r_{\rho 0} + r_{\rho 1} \xi_{3,1} = \begin{Bmatrix} r_{\rho 0} & r_{\rho 1} \end{Bmatrix} \begin{Bmatrix} 1 \\ \xi_{3,1} \end{Bmatrix} \quad (8.18)$$

Note that, E_0 , h_0 and $r_{\rho 0}$ are means of elasticity, thickness at $x=0$ and specific volume, respectively, whereas, E_1 , h_1 and $r_{\rho 1}$ are corresponding standard deviations of elasticity, thickness at $x=0$ and specific volume. Polynomial terms of the corresponding eigenvalues are written by using tensor product of Eqs. (8.16)-(8.18):

$$\psi_t(\xi_{1,1}, \xi_{2,1}, \xi_{3,1}) = \begin{Bmatrix} 1 \\ \xi_{1,1} \end{Bmatrix} \otimes \begin{Bmatrix} 1 \\ \xi_{2,1} \end{Bmatrix} \otimes \begin{Bmatrix} 1 \\ \xi_{3,1} \end{Bmatrix} \quad (8.19)$$

The dimension of the vector $\psi_t(\xi_{1,1}, \xi_{2,1}, \xi_{3,1})$ is 8×1 . In PCE, any variable is written as a series sum starting from zero. Therefore, upper limit of the series sum for

the uncertain eigenvalues is obtained as $N_\lambda = 2^3 - 1 = 7$. In this context, uncertain eigenvalues of the considered beam can be described as follows:

$$\begin{aligned}\lambda_n &= \sum_{t=0}^{N_\lambda} \lambda_{n,t} \psi_t(\xi_{1,1}, \xi_{2,1}, \xi_{3,1}) \\ &= \lambda_{n,0} + \lambda_{n,1} \xi_{1,1} + \lambda_{n,3} \xi_{2,1} + \lambda_{n,4} \xi_{3,1} + \lambda_{n,4} \xi_{1,1} \xi_{2,1} \\ &\quad + \lambda_{n,5} \xi_{1,1} \xi_{3,1} + \lambda_{n,6} \xi_{2,1} \xi_{3,1} + \lambda_{n,7} \xi_{1,1} \xi_{2,1} \xi_{3,1} .\end{aligned}\quad (8.20)$$

Substituting Eqs. (8.16)-(8.18) and (8.20) into Eq. (8.15) leads to:

$$\left(\sum_{k=0}^1 \sum_{j=0}^1 \sum_{i=0}^1 E_i H_i(\xi_{1,1}) h_j^2 H_j(\xi_{2,1}) r_{\rho k} H_k(\xi_{3,1}) [\mathbf{D}(\mathbf{X})] - \sum_{t=0}^{N_\lambda} \lambda_{n,t} \psi_t(\xi_{1,1}, \xi_{2,1}, \xi_{3,1}) [\mathbf{I}] \right) \{ \mathbf{W}(\mathbf{X}, \vartheta) \} = 0 . \quad (8.21)$$

Let us multiply Eq. (8.21) with u th term of the orthogonal basis of ψ_t and apply the Galerkin projection:

$$\left(\sum_{k=0}^1 \sum_{j=0}^1 \sum_{i=0}^1 E_i h_j^2 r_{\rho k} e_{ijk u} [\mathbf{D}(\mathbf{X})] - \lambda_{n,u} e_u^2 [\mathbf{I}] \right) \{ \mathbf{W}(\mathbf{X}, \vartheta) \} = 0 \quad (8.22)$$

Here, $k=0,1,\dots,N_\lambda$, and

$$\begin{aligned}e_{ijk u} &= \int_{\Omega_3} \int_{\Omega_2} \int_{\Omega_1} H_i(\xi_{1,1}) H_j(\xi_{2,1}) H_k(\xi_{3,1}) \psi_u(\xi_{1,1}, \xi_{2,1}, \xi_{3,1}) \mu_1(\xi_{1,1}) \mu_2(\xi_{2,1}) \mu_3(\xi_{3,1}) d\xi_{1,1} d\xi_{2,1} d\xi_{3,1} , \\ e_u^2 &= \int_{\Omega_3} \int_{\Omega_2} \int_{\Omega_1} [\psi_u(\xi_{1,1}, \xi_{2,1}, \xi_{3,1})]^2 \mu_1(\xi_{1,1}) \mu_2(\xi_{2,1}) \mu_3(\xi_{3,1}) d\xi_{1,1} d\xi_{2,1} d\xi_{3,1} .\end{aligned}$$

One can calculate $\lambda_{n,u}$ by changing u starting from zero up to N_λ in Eq. (8.22) by using any eigenvalue solver.

8.5 Numerical Uncertainty Analysis of Different Structures via DSC-PCE

The proposed methodology is applied to two different structures:

1. Uncertain inhomogeneous individual beam is considered to apply the methodology for an individual waveguide in Section 8.5.1.
2. Uncertain L-type connected structure is considered as uncertain connected structure in Section 8.5.2.

8.5.1 DSC-PCE Method for Bending Beams

The effects of uncertain geometrical and material property on natural frequencies of the beam with the mechanical and the physical properties given in Table 6.11 are examined by using DSC and PC expansion (DSC-PCE) together in order to show the applicability of proposed methodology. In this regard, Young modulus and specific volume are selected as uncertain to simulate the uncertain material property whereas thickness is for uncertain geometrical property. In the analysis, standard deviations of uncertain variables are selected to be as 5% of their mean values, which are given in Table 6.11. Several uncertainty cases are considered for uniform and non-uniform beam as tabulated in Table 8.1. In order to estimate the uncertain natural frequencies via DSC-PCE, firstly, coefficients given in Eq. (8.20) are determined by using Eq. (8.22). Then, the uncertain parameters are created with 10000 samples according to the Normal distribution and fed to Eq. (8.20) to determine the uncertain eigenvalues. Next, samples of uncertain natural frequencies are determined via $\omega_n = \sqrt{\lambda_n}$ and normalized probability distributions of natural frequencies are obtained. In the presentation of the results, firstly, distributions of natural frequencies of C-F beam are given in Figure 8.1 for modes 1, 2, 10 and 20. Beside that, the mean and the standard deviation of natural frequencies for the first 20 modes are given in Table 8.2. The results are compared with Monte Carlo (MC) simulation. Then, normalized probability distribution functions of the first and tenth modes are compared with MC simulations for the Cases 2-5 in Figures 8.2-8.5. Note that, 10000 samples of uncertain variables are also used in MC simulations. Since the DSC method is

verified with analytical and FEM solutions for the deterministic beam in Section 6.4, here only uncertainty analyses are verified with Monte-Carlo simulation.

Table 8.1 Uncertainty cases

Boundary Condition	Uncertainty	Uniform	Non-uniform		
			$h(x)$	$E(x)$	$h(x), E(x)$
C-F	$h(\xi_{2,1})$	Case 1			
P-P	$E(\xi_{1,1}), h(\xi_{2,1}), r(\xi_{3,1})$	Case 2	Case 3	Case 4	Case 5

It is seen from Figures 8.1-8.5 and Table 8.2 that results of the DSC-PCE are consistent with MC simulation for the considered modes of all uncertain cases. However, small deviations of the results are due to difference in sampling parameters. Beside that, the elapsed time for the PCE is half of MC simulation for each case. Results show that, DSC-PCE method can reliably be used for uncertainty analysis of inhomogeneous beams.

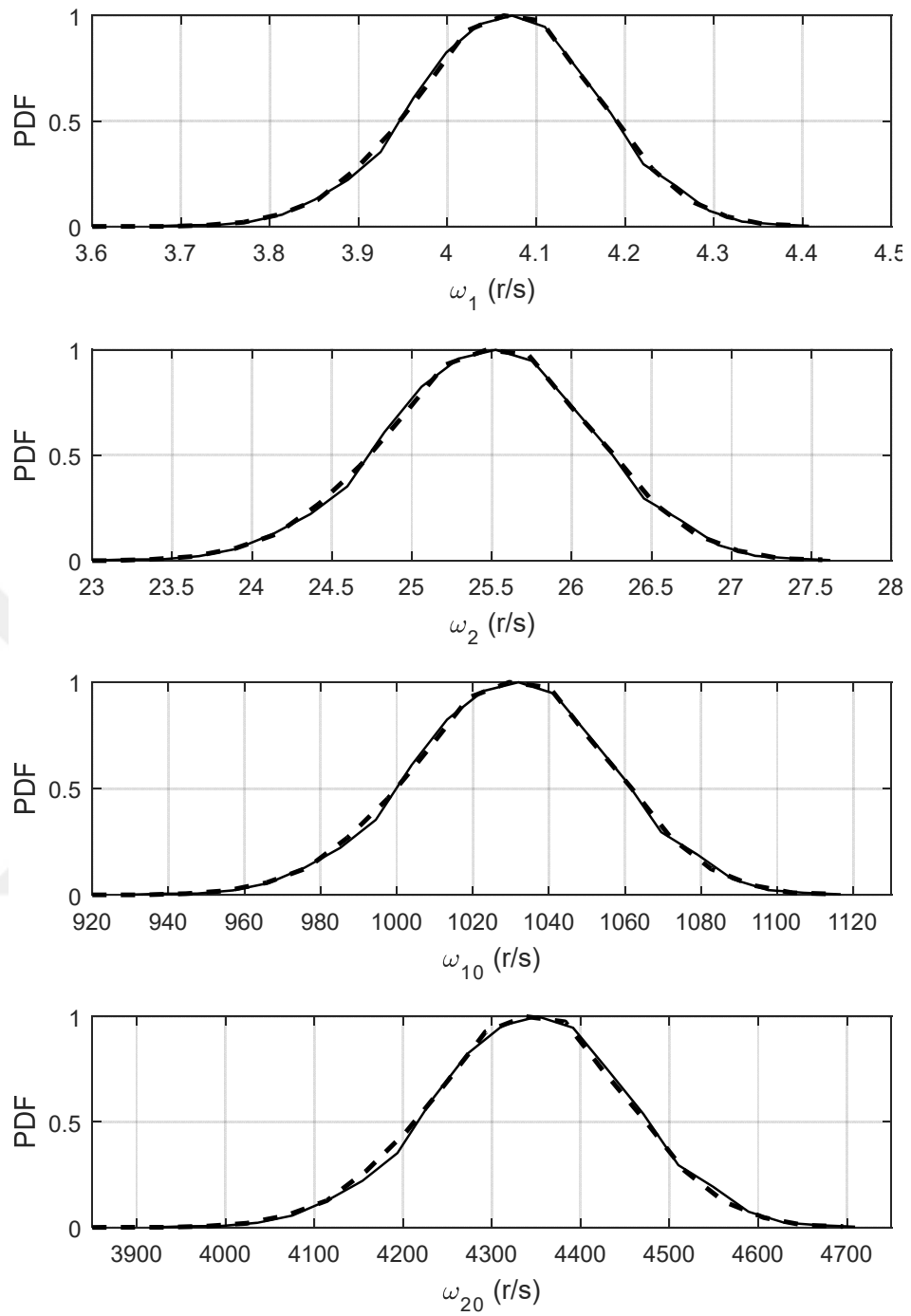


Figure 8.1 Normalized probability distribution of uncertain natural frequencies for Case 1 (solid line: PCE, dash line: Monte Carlo)

Table 8.2 Statistical properties of natural frequencies obtained via PCE and MC simulations (r/s) for Case 1

Mode seq.	PCE		MC		Deviation %	
	mean	std. dev.	mean	std. dev.	mean	std. dev.
1	4.066	0.102	4.064	0.102	0.042	-0.028
2	25.483	0.638	25.474	0.638	0.036	-0.034
3	71.354	1.786	71.334	1.787	0.028	-0.042
4	139.830	3.500	139.789	3.501	0.029	-0.041
5	231.160	5.786	231.062	5.787	0.042	-0.028
6	345.333	8.643	345.166	8.645	0.048	-0.022
7	482.359	12.073	482.092	12.074	0.055	-0.015
8	642.251	16.074	641.838	16.075	0.064	-0.006
9	825.021	20.649	824.406	20.648	0.075	0.004
10	1030.688	25.796	1029.794	25.792	0.087	0.017
11	1259.274	31.517	1258.003	31.508	0.101	0.031
12	1510.805	37.813	1509.033	37.795	0.117	0.047
13	1785.313	44.683	1782.885	44.654	0.136	0.066
14	2082.835	52.130	2079.557	52.084	0.158	0.087
15	2403.418	60.153	2399.049	60.086	0.182	0.112
16	2747.119	68.756	2741.363	68.660	0.210	0.140
17	3114.006	77.938	3106.498	77.805	0.242	0.171
18	3504.175	87.703	3494.454	87.522	0.278	0.208
19	3917.766	98.055	3905.230	97.810	0.321	0.251
20	4355.062	109.000	4338.828	108.670	0.374	0.304

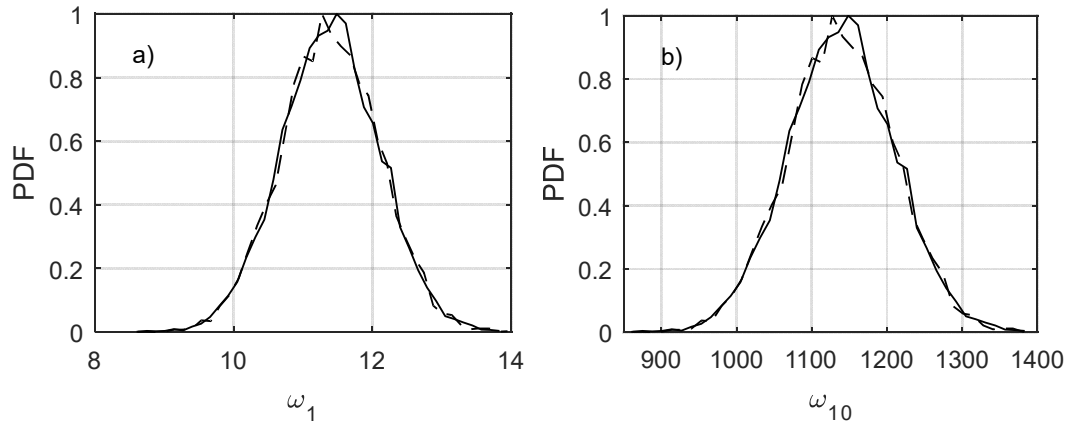


Figure 8.2 Normalized probability distribution of natural frequency (r/s) for uncertain Case 2, a) Mode 1, b) Mode 10 (solid line: PCE, dash line: Monte Carlo)

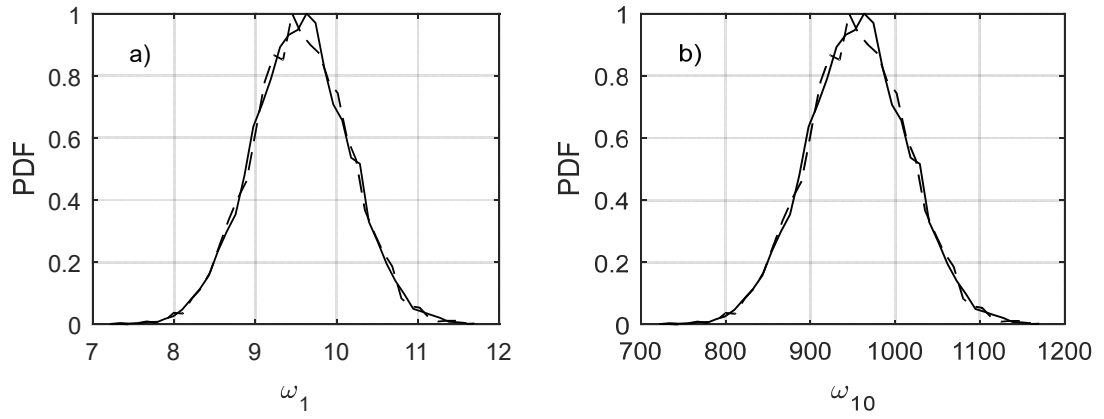


Figure 8.3 Normalized probability distribution of natural frequency (r/s) for uncertain Case 3, a) Mode 1, b) Mode 10 (solid line: PCE, dash line: Monte Carlo)

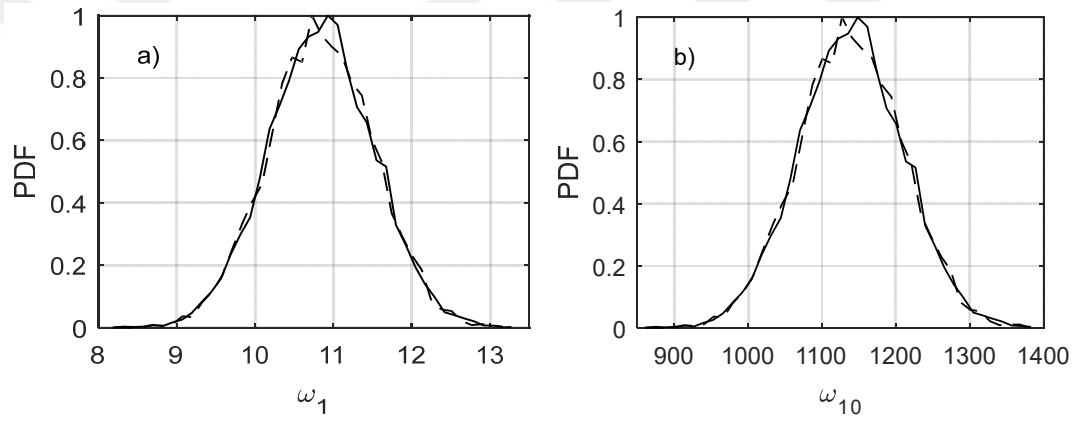


Figure 8.4 Normalized probability distribution of natural frequency (r/s) for uncertain Case 4, a) Mode 1, b) Mode 10 (solid line: PCE, dash line: Monte Carlo)

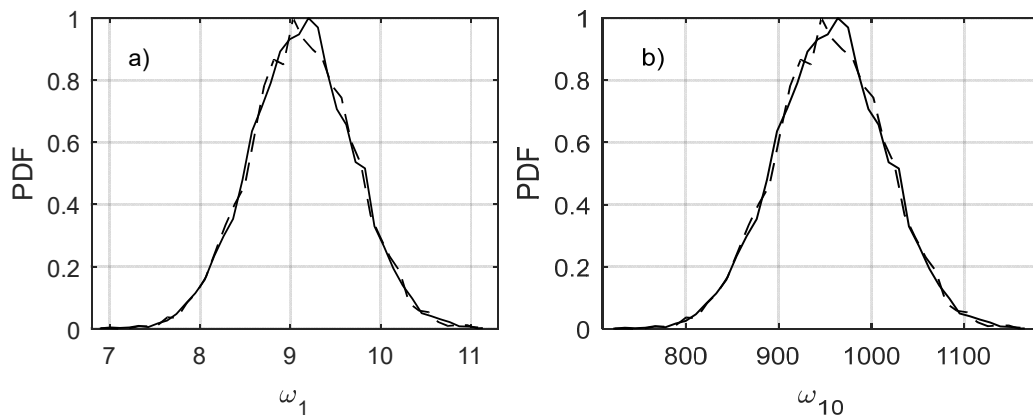


Figure 8.5 Normalized probability distribution of natural frequency (r/s) for uncertain Case 5, a) Mode 1, b) Mode 10 (solid line: PCE, dash line: Monte Carlo)

8.5.2 DSC-PCE for L-Type Connected Beams

In this section, the main goal is to present applicability of the proposed method for the connected uncertain structures. In this regard, L-type connected structure with P-P boundaries that is considered in Section 7.3, is assumed to be uncertain. Uncertainty is constructed by selecting the elasticity of the structure as stochastic parameter. In this regard, mean and standard deviation of the elasticity is selected as $\{\varepsilon_E, \chi_E\} = \{70, 0.05 \times 70\}$ GPa. The same procedure given in previous Section 8.4 is implemented to estimate probability distribution functions of natural frequencies. In the analysis, DSC parameters are selected as the same with the ones in Section 7.3. Monte Carlo simulation with 10000 samples are used for comparison. It should be noted that, the DSC method is used to solve the equation of motion of the structure for the PCE and Monte Carlo simulation. Distributions of the natural frequencies of mode 1, 2, 5, 10 are given in Figure 8.6. Besides, the mean and the standard deviation of the natural frequencies are tabulated in Table 8.3.

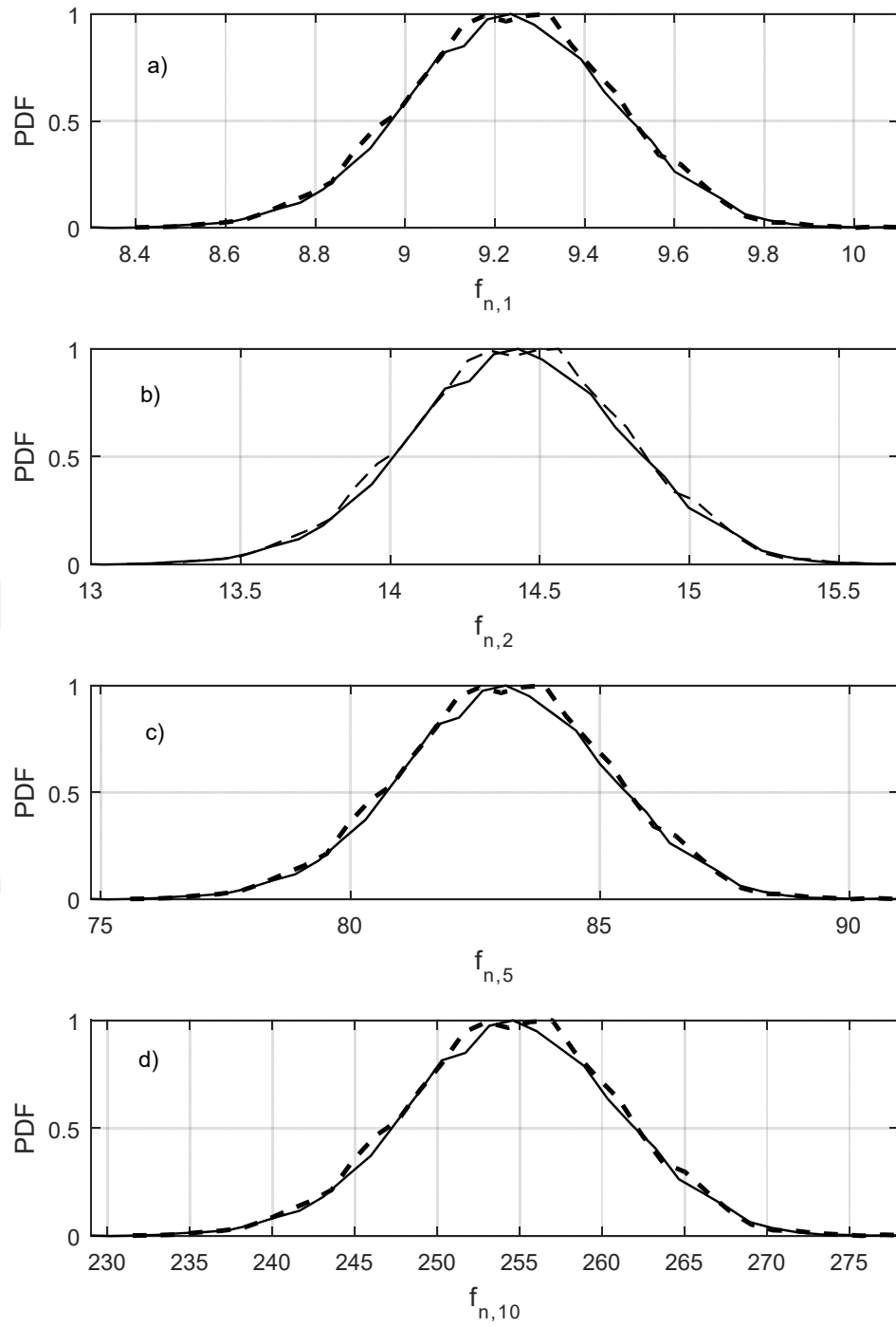


Figure 8.6 Normalized probability distribution of natural frequency (Hz) for uncertain L-type connected structure a) Mode 1, b) Mode 2, c) Mode 5, d) Mode 10 (solid line: PCE, dash line: Monte Carlo)

Table 8.3 Statistical properties of natural frequencies of L-type connected beams obtained via PCE and MC simulations (Hz)

Mode seq.	PCE		MC		Deviation %	
	mean	std. dev.	mean	std. dev.	mean	std. dev.
1	9.233	0.231	9.232	0.231	0.0094	0.0794
2	14.423	0.361	14.422	0.361	0.0094	0.0794
3	36.931	0.925	36.927	0.924	0.0094	0.0794
4	46.741	1.170	46.737	1.169	0.0094	0.0794
5	83.095	2.080	83.087	2.079	0.0094	0.0794
6	97.525	2.442	97.516	2.440	0.0094	0.0794
7	147.728	3.698	147.714	3.695	0.0094	0.0794
8	166.783	4.175	166.767	4.172	0.0094	0.0794
9	230.832	5.779	230.811	5.774	0.0094	0.0794
10	254.519	6.372	254.495	6.367	0.0094	0.0794
11	332.411	8.322	332.380	8.315	0.0094	0.0794
12	360.741	9.031	360.707	9.024	0.0094	0.0794
13	452.469	11.328	452.426	11.319	0.0094	0.0794
14	485.461	12.154	485.415	12.144	0.0094	0.0794
15	591.012	14.796	590.956	14.784	0.0094	0.0794
16	628.692	15.739	628.632	15.727	0.0094	0.0794
17	748.049	18.728	747.978	18.713	0.0094	0.0794
18	790.453	19.789	790.378	19.774	0.0094	0.0794
19	923.593	23.122	923.506	23.104	0.0094	0.0794
20	970.775	24.304	970.684	24.284	0.0094	0.0794
21	1117.664	27.981	1117.558	27.959	0.0094	0.0794
22	1169.707	29.284	1169.597	29.261	0.0094	0.0794
23	1330.299	33.304	1330.173	33.278	0.0094	0.0794
24	1387.339	34.732	1387.208	34.705	0.0094	0.0794
25	1561.572	39.094	1561.425	39.063	0.0094	0.0794
26	1623.861	40.654	1623.708	40.622	0.0094	0.0794
27	1811.667	45.356	1811.496	45.320	0.0094	0.0794
28	1879.773	47.061	1879.595	47.023	0.0094	0.0794
29	2081.205	52.104	2081.009	52.062	0.0094	0.0794
30	2156.972	54.000	2156.768	53.958	0.0094	0.0794

As shown from Figure 8.6 and Table 8.3 that, calculated results are consistent with each other. Beside that, computational time is 556 sec for Monte Carlo simulation whereas it is 1.5 sec for DSC-PCE. This proves that, Monte Carlo simulation is not a low cost methodology in the sense of computational cost when the considered structure is starting to become more complex.

CHAPTER NINE

CONCLUSION

9.1 Review of the Thesis

The main objective of this thesis is proposing a wavelet methodology for the analysis of uncertain connected structures. In this regard, Discrete Singular Convolution (DSC) method and Polynomial Chaos Expansion (PCE) are introduced as a partial differential equation solver and uncertainty modelling tool, respectively. The DSC is examined for implementing connected structures since it is highly accurate in vibration problems even for low numbers of discretization points whereas, the Polynomial Chaos Expansion (PCE) is investigated in this thesis since it presents a mathematical function in terms of uncertain parameters for the uncertain response variables.

Although the main aim of the thesis is analysing uncertain connected structures, a novel problem that had not been solved up to this thesis, is considered i.e., free and forced vibration and acoustic analysis of individual structures with impedance boundaries. Taylor series expansion (TSE) is used with the DSC for the solution of the problem. In this regard, a longitudinally vibrating bar and sound propagation in a duct are considered as numerical examples and natural frequencies, mode shapes and forced responses are calculated as results. The comparisons with finite element method (FEM) show that, the proposed methodology estimates the results with the similar accuracies for lower numbers of discretization in the DSC.

Secondly, two different algorithms for the connected structures are also demonstrated. The first one uses the continuity equations at the boundaries in order to relate the connected structures by using Taylor Series Expansion (TSE), called as DSC-TSE. The proposed DSC-TSE methodology has the capability of modelling the structures those are connected to each other at their boundaries. This algorithm is implemented to two different systems i.e., i) acoustic transmission lines, ii) connected beams having different connection orientations. In the analyses of these

structures, natural frequencies for different boundary conditions are mainly focused. However, transmission coefficients are successfully calculated by combining the DSC with two-load method for acoustic transmission lines. It is shown that, the results are consistent with other numerical methods and rate of convergences of the DSC-TSE is not as high as the ones of the conventional DSC when a uniform structure is considered but it is better than the ones of the FEM.

On the contrary, some difficulties in DSC-TSE are observed when the connection is not supplied at the boundary since continuity equations at internal points are more complex. Therefore, another approach is proposed in the thesis and it is named here as DSC-c method. The DSC-c relies on the simultaneous solution of all differential equations discretized in a global matrix. This approach separately models the beam and plate in the DSC manner with appropriate boundary condition and solves simultaneously in a global DSC matrix. The method is applied to a beam connected to a plate called as “stiffened plate” problem and free vibration analysis is performed. The results are verified with the ones of FEM for different thicknesses of the beam. The effect of the connected beam is clearly observed on the mode shapes of the plate by reduced vibration at the discretization line of the connected beam. Beside that, mode shapes of stiffened plate are compared with ones of a fully simply-supported plate and beam and change in mode shapes are also clearly observed.

Finally, the studies are completed by performing uncertainty analysis. In the uncertainty analysis, probability distribution of natural frequencies of a bending beam and L-type connected beams are determined with combined DSC-PCE method. In this methodology, DSC presents a unique advantage since the characteristic matrix is independent of the material properties. Therefore, once, the DSC matrix for the structure is obtained, uncertainty analysis may be performed via PCE. It is shown that, computational time is considerably reduced when Monte Carlo simulation is taken as reference.

9.2 Main Findings of the Studies in the Thesis

As expressed in previous section, there are several studies in this Thesis. Now it will be helpful to give the remarkable findings of the Thesis:

- Discrete Singular Convolution (DSC) method is highly convergent for several waveguides.
- It is observed from the comparisons with analytical solutions, the DSC estimates more accurate results than Finite Element Method (FEM) does, when the same number of discretization is used.
- Together with the successful implementation of free boundary condition for bending beams and plates, Taylor Series Expansion (TSE) with the DSC is also able to model the impedance boundary conditions of the structures or Robin Boundary condition equations.
- The DSC converges better to analytical results when the expansion order of the TSE is higher (DSC-TSE 2 approach).
- The DSC methodology may be applied for the analysis of inhomogeneous structures.
- DSC-TSE may easily be adopted for the analysis of the connected structures when the connection is supplied at the boundaries.
- The DSC-TSE is successfully applied on the analysis of periodic acoustic structures. It is also shown that, this approach may be used with two-load method in order to estimate transmission coefficient of acoustic structures.
- The DSC-TSE is able to estimate the natural frequencies of the structures having mode conversion.
- For the structures having internal connection points, DSC-c method is more straightforward since this method simultaneously solves the differential equations at the discretized points of connection.
- Since the dynamic stiffness matrix obtained via the DSC is independent from the mechanical and the physical properties of the structure, it is very easy and cheaper to combine the DSC method with the PCE in handling uncertainties.

9.3 Contributions of the Thesis to the Literature

By the thesis, following contributions to the literature are made;

- M. Kara and A. Seçgin (2019). Discrete singular convolution method for one-dimensional vibration and acoustics problems with impedance boundaries. Journal of Sound and Vibration, 446, 22-36. <https://doi.org/10.1016/j.jsv.2019.01.028>
- M. Kara and A. Seçgin (2018). İnce bir çubuğun belirsiz doğal frekanslarının çokterimli kaos açılımı ile matematiksel olarak modellenmesi. Dokuz Eylül Üniversitesi Fen ve Mühendislik Dergisi, 20, 60, 852-862. <https://doi.org/10.21205/deufmd.2018206067>
- M. Kara and A. Seçgin. Polynomial chaos expansion based discrete singular convolution method for the free vibration analysis of uncertain non-uniform beams, *in preparation*.
- M. Kara and A. Seçgin. Discrete singular convolution method for the acoustic analysis of transmission lines, *in preparation*.
- M. Kara and A. Seçgin. Vibration analysis of structures having mode conversion via discrete singular convolution method, *in preparation*.

Beside that, following conference papers are also outcomes of the Thesis:

- M. Kara and A. Seçgin (2019). Geometrik belirsizliğe sahip ince çubukların çok terimli kaos açılımı temelli ayırık tekil konvolüsyon yöntemi ile titreşim analizi. In the Proceeding book of “19. Ulusal Makina Teorisi ve Sempozyumu”, 30-37, 04-06 Eylül 2019, İskenderun, Türkiye
- M. Kara and A. Seçgin. Examining effect of uncertain material impedances on sound pressure distribution in acoustic impedance tubes via discrete singular convolution, *in preparation*.
- M. Kara and A. Seçgin. Discrete singular convolution method for Euler beams with mechanical impedance boundaries, *in preparation*.

9.4 Suggestions for Future Works

The performed studies in thesis ignited following works, which may be considered in future;

- The methodologies presented here may be extended to the high frequency analysis of connected composite plates with different orientations.
- Vibration energy transmission paths may be determined by present algorithms.
- The methodologies presented here may be applied to the acoustic-structure interaction problems.
- Since the solutions of all boundary condition cases are available, a new connection algorithm that is based on impedance of the structures may be developed.
- The DSC may be applied to high frequency vibrations of periodic structures by using Bloch-Floquet boundary conditions.
- The methodologies presented here may also be implemented for high frequency nonlinear vibration analysis.

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APPENDICES

APPENDIX-1: Stiffness Matrices

As expressed in Chapter 3, there are two different boundary condition implementation procedures in the DSC algorithm, i) classical approach ii) Taylor series expansion approach. According to the applied implementation procedure, derivatives of structural points are constructed by summing up three matrices, i.e., $\mathbf{T}^L$, $\mathbf{T}^S$ and $\mathbf{T}^R$. They are the matrices including kernel information for the left-ghost, structural and right-ghost points, respectively. In this appendix, these matrices are given for those procedures with $N=4$ and $M=3$. For the classical approach, $\mathbf{T}^L$, $\mathbf{T}^S$ and $\mathbf{T}^R$ are expressed as follows:

$$\begin{aligned}
 \begin{Bmatrix} W_0^{(n)} \\ W_1^{(n)} \\ W_2^{(n)} \\ W_3^{(n)} \end{Bmatrix} &= \begin{bmatrix} \sum_{k=1}^3 C_{-k}^{(n)} (1 - A_k) & C_{-1}^{(n)} A_k & C_{-2}^{(n)} A_k & C_{-3}^{(n)} A_k \\ \sum_{k=2}^3 C_{-k}^{(n)} (1 - A_k) & C_{-2}^{(n)} A_k & C_{-3}^{(n)} A_k & 0 \\ \sum_{k=3}^3 C_{-k}^{(n)} (1 - A_k) & C_{-3}^{(n)} A_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{Bmatrix} \\
 &+ \begin{bmatrix} C_0^{(n)} & C_1^{(n)} & C_2^{(n)} & C_3^{(n)} \\ C_{-1}^{(n)} & C_0^{(n)} & C_1^{(n)} & C_2^{(n)} \\ C_{-2}^{(n)} & C_{-1}^{(n)} & C_0^{(n)} & C_1^{(n)} \\ C_{-3}^{(n)} & C_{-2}^{(n)} & C_{-1}^{(n)} & C_0^{(n)} \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{Bmatrix} \\
 &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & C_3^{(n)} & \sum_{k=3}^3 C_k^{(n)} (1 - A_{N-1-k}) \\ 0 & C_3^{(n)} & C_2^{(n)} & \sum_{k=2}^3 C_k^{(n)} (1 - A_{N-1-k}) \\ C_3^{(n)} & C_2^{(n)} & C_1^{(n)} & \sum_{k=1}^3 C_k^{(n)} (1 - A_{N-1-k}) \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{Bmatrix}, \tag{A.1}
 \end{aligned}$$

respectively. Beside the classical approach, derivatives of structural points are written for DSC-TSE 1 approach as,

$$\begin{aligned}
 \begin{Bmatrix} W_0^{(n)} \\ W_1^{(n)} \\ W_2^{(n)} \\ W_3^{(n)} \end{Bmatrix} &= \begin{bmatrix} 0 & C_{-1}^{(n)} & C_{-2}^{(n)} & C_{-3}^{(n)} & -2\sum_{k=1}^3 C_{-k}^{(n)}(k\Delta) & -\frac{1}{3}\sum_{k=1}^3 C_{-k}^{(n)}(k\Delta)^3 & 0 & 0 \\ 0 & C_{-2}^{(n)} & C_{-3}^{(n)} & 0 & -2\sum_{k=2}^3 C_{-k}^{(n)}((k-1)\Delta) & -\frac{1}{3}\sum_{k=2}^3 C_{-k}^{(n)}((k-1)\Delta)^3 & 0 & 0 \\ 0 & C_{-3}^{(n)} & 0 & 0 & -2\sum_{k=3}^3 C_{-k}^{(n)}((k-2)\Delta) & -\frac{1}{3}\sum_{k=3}^3 C_{-k}^{(n)}((k-2)\Delta)^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_0^{(1)} \\ W_0^{(3)} \\ W_3^{(1)} \\ W_3^{(3)} \end{Bmatrix} \\
 &+ \begin{bmatrix} C_0^{(n)} & C_1^{(n)} & C_2^{(n)} & C_3^{(n)} & 0 & 0 & 0 & 0 \\ C_{-1}^{(n)} & C_0^{(n)} & C_1^{(n)} & C_2^{(n)} & 0 & 0 & 0 & 0 \\ C_{-2}^{(n)} & C_{-1}^{(n)} & C_0^{(n)} & C_1^{(n)} & 0 & 0 & 0 & 0 \\ C_{-3}^{(n)} & C_{-2}^{(n)} & C_{-1}^{(n)} & C_0^{(n)} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_0^{(1)} \\ W_0^{(3)} \\ W_3^{(1)} \\ W_3^{(3)} \end{Bmatrix} \\
 &+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3^{(n)} & 0 & 0 & 0 & 2\sum_{k=3}^3 C_k^{(n)}((k-2)\Delta) & \frac{1}{3}\sum_{k=3}^3 C_k^{(n)}((k-2)\Delta)^3 \\ 0 & C_3^{(n)} & C_2^{(n)} & 0 & 0 & 0 & 2\sum_{k=2}^3 C_k^{(n)}((k-1)\Delta) & \frac{1}{3}\sum_{k=2}^3 C_k^{(n)}((k-1)\Delta)^3 \\ C_3^{(n)} & C_2^{(n)} & C_1^{(n)} & 0 & 0 & 0 & 2\sum_{k=1}^3 C_k^{(n)}(k\Delta) & \frac{1}{3}\sum_{k=1}^3 C_k^{(n)}(k\Delta)^3 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_0^{(1)} \\ W_0^{(3)} \\ W_3^{(1)} \\ W_3^{(3)} \end{Bmatrix}, \quad (\text{A.2})
 \end{aligned}$$

whereas they are for DSC-TSE 2 approach as,

$$\begin{Bmatrix} W_0^{(n)} \\ W_1^{(n)} \\ W_2^{(n)} \\ W_3^{(n)} \end{Bmatrix} = \begin{bmatrix} 2\sum_{k=1}^3 C_{-k}^{(n)} & -C_{-1}^{(n)} & -C_{-2}^{(n)} & -C_{-3}^{(n)} & \sum_{k=1}^3 C_{-k}^{(n)} (k\Delta)^2 & \frac{1}{12}\sum_{k=1}^3 C_{-k}^{(n)} (k\Delta)^4 & 0 & 0 \\ 2\sum_{k=2}^3 C_{-k}^{(n)} & -C_{-2}^{(n)} & -C_{-3}^{(n)} & 0 & \sum_{k=2}^3 C_{-k}^{(n)} ((k-1)\Delta)^2 & \frac{1}{12}\sum_{k=2}^3 C_{-k}^{(n)} ((k-1)\Delta)^4 & 0 & 0 \\ 2\sum_{k=3}^3 C_{-k}^{(n)} & -C_{-3}^{(n)} & 0 & 0 & \sum_{k=3}^3 C_{-k}^{(n)} ((k-2)\Delta)^2 & \frac{1}{12}\sum_{k=3}^3 C_{-k}^{(n)} ((k-2)\Delta)^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_0^{(1)} \\ W_0^{(3)} \\ W_3^{(1)} \\ W_3^{(3)} \end{Bmatrix}$$

$$+ \begin{bmatrix} C_0^{(n)} & C_1^{(n)} & C_2^{(n)} & C_3^{(n)} & 0 & 0 & 0 & 0 \\ C_{-1}^{(n)} & C_0^{(n)} & C_1^{(n)} & C_2^{(n)} & 0 & 0 & 0 & 0 \\ C_{-2}^{(n)} & C_{-1}^{(n)} & C_0^{(n)} & C_1^{(n)} & 0 & 0 & 0 & 0 \\ C_{-3}^{(n)} & C_{-2}^{(n)} & C_{-1}^{(n)} & C_0^{(n)} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_0^{(1)} \\ W_0^{(3)} \\ W_3^{(1)} \\ W_3^{(3)} \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C_3^{(n)} & 2\sum_{k=3}^3 C_k^{(n)} & 0 & 0 & \sum_{k=3}^3 C_k^{(n)} ((k-2)\Delta)^2 & \frac{1}{12}\sum_{k=3}^3 C_k^{(n)} ((k-2)\Delta)^4 \\ 0 & -C_3^{(n)} & -C_2^{(n)} & 2\sum_{k=2}^3 C_k^{(n)} & 0 & 0 & \sum_{k=2}^3 C_k^{(n)} ((k-1)\Delta)^2 & \frac{1}{12}\sum_{k=2}^3 C_k^{(n)} ((k-1)\Delta)^4 \\ -C_3^{(n)} & -C_2^{(n)} & -C_1^{(n)} & 2\sum_{k=1}^3 C_k^{(n)} & 0 & 0 & \sum_{k=1}^3 C_k^{(n)} (k\Delta)^2 & \frac{1}{12}\sum_{k=1}^3 C_k^{(n)} (k\Delta)^4 \end{bmatrix} \begin{Bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_0^{(1)} \\ W_0^{(3)} \\ W_3^{(1)} \\ W_3^{(3)} \end{Bmatrix} \quad (\text{A.3})$$

APPENDIX-2: Two Load Method

For passive insulation materials and complex acoustical structures, analytical determination of transfer matrix elements is not an easy task. Therefore, experimental procedures i.e., two-load method (ASTM E2611-17, Standard test method for normal incidence determination of porous material acoustical properties based on the transfer matrix method, 2017; Lung & Doige, 1983) and two-source method (Munjal & Doige, 1990) are mostly used in those cases. According to the two-load method, transfer matrix elements of test specimen are obtained by measuring two complex pressure response functions ($H_{ij} = P_i/P_j$) at either side of considered structures as shown in Figure A.1 for each load. In the measurements, the tube must be ended with any two of hard end, soft end, anechoic end or a sound absorber material.

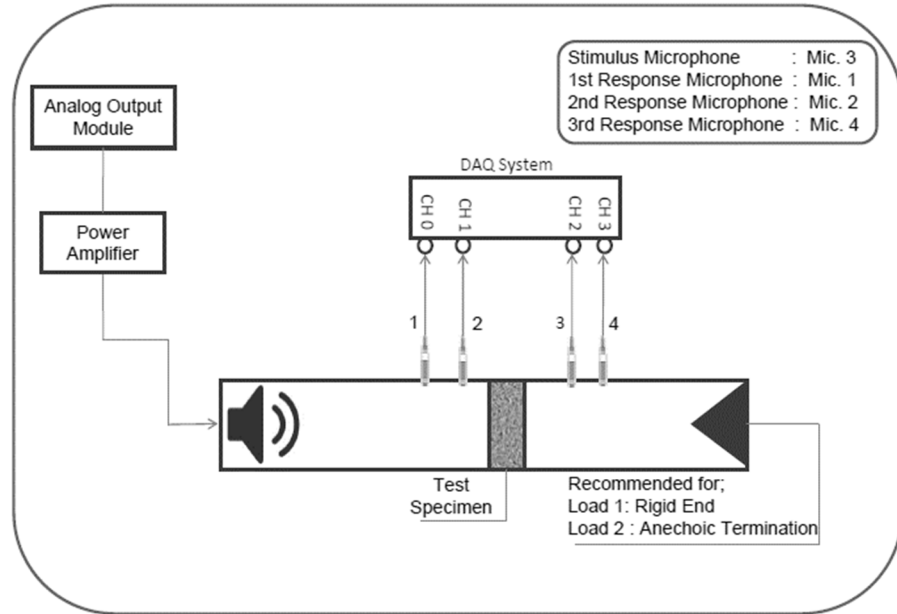


Figure A.1 Experimental setup for the impedance measurement

Transfer matrix elements may be calculated by determining sound pressures and particle velocities of test specimen at the inlet and outlet for each load as follows:

$$p_0 = A + B , \quad (\text{A.4})$$

$$p_d = Ce^{-jkd} + De^{jkd}, \quad (\text{A.5})$$

$$u_0 = (A - B) / \rho c, \quad (\text{A.6})$$

$$u_d = (Ce^{-jkd} - De^{jkd}) / \rho c. \quad (\text{A.7})$$

Here, d is the thickness of test specimen, k is wavenumber and A , B , C and D are forward and backwards travelling waves on either side of test specimen and can be calculated by using complex sound pressure response functions:

$$A = j \frac{H_{13}e^{-jkl_1} - H_{23}e^{-jk(l_1+s_1)}}{2\sin(ks_1)}, \quad (\text{A.8})$$

$$B = j \frac{H_{23}e^{jk(l_1+s_1)} - H_{13}e^{jkl_1}}{2\sin(ks_1)}, \quad (\text{A.9})$$

$$C = j \frac{H_{33}e^{jk(l_2+s_2)} - H_{43}e^{jkl_2}}{2\sin(ks_2)}, \quad (\text{A.10})$$

$$D = j \frac{H_{43}e^{-jkl_2} - H_{33}e^{-jk(l_2+s_2)}}{2\sin(ks_2)}. \quad (\text{A.11})$$

Here, l_1 and l_2 are distances of 2nd and 3rd microphones to the inlet of specimen, s_1 and s_2 are distance between microphones. One can determine transfer matrix elements by using Eqs. (A.4)-(A.7) as follows:

$$T_{(1,1)} = \frac{p_{0a}u_{db} - p_{0b}u_{da}}{p_{da}u_{db} - p_{db}u_{da}}, \quad (\text{A.12})$$

$$T_{(1,2)} = \frac{p_{0b}p_{da} - p_{0a}p_{db}}{p_{da}u_{db} - p_{db}u_{da}}, \quad (\text{A.13})$$

$$T_{(2,1)} = \frac{u_{0a}u_{db} - u_{0b}u_{da}}{p_{da}u_{db} - p_{db}u_{da}}, \quad (\text{A.14})$$

$$T_{(2,2)} = \frac{p_{da}u_{0b} - p_{db}u_{0a}}{p_{da}u_{db} - p_{db}u_{da}}. \quad (\text{A.15})$$

APPENDIX-3: PCE for First Order Ordinary Differential Equation

In this subsection, a first order stochastic differential equation solved by Sepahvand et. al. (2010) is considered. Calculated results are also compared with results of the same study. Consider the following ordinary stochastic differential equation with the uncertain parameter K ,

$$\frac{dY}{dt} = -K(\xi_1)Y, \quad Y = Y(t, \xi_1) \quad \text{and} \quad Y(0, \xi_1) = 1. \quad (\text{A.16})$$

Here, ξ_1 is the uncertain (random) variable having normal distribution with the statistical properties of $[\mu_\xi, \sigma_\xi] = [0, 1]$. Uncertain parameter K which has also the same statistical properties of random variable, is represented by the truncated PC expansion of,

$$K = \sum_{i=0}^{N_k} k_i \psi_i(\xi_1). \quad (\text{A.17})$$

From the Chapter 8, it is known that $N_k=1$ and $[k_0, k_1] = [0, 1]$ if the Hermitian polynomial is used as basis. So, uncertain variable has the form of $K = \xi_1$. Since K is uncertain, solution of the differential equation is also depends on uncertain variable ξ_1 . Solution of differential equation Y may be expressed as:

$$Y(t, \xi_1) = \sum_{j=0}^{N_y} y_j(t) \psi_j(\xi_1). \quad (\text{A.18})$$

Substituting Eqs. (A.17) and (A.18) into the Eq. (A.16) leads to,

$$\sum_{j=0}^{N_y} \frac{dy_j(t)}{dt} \psi_j(\xi_1) = - \sum_{i=0}^{N_k} k_i \psi_i(\xi_1) \sum_{j=0}^{N_y} y_j(t) \psi_j(\xi_1). \quad (\text{A.19})$$

Multiplying of both side of Eq. (A.19) by basis functions $\psi_m(\xi_1)$ which are the set of orthogonal polynomials span the random space. Using the orthogonality principle leads to,

$$\frac{dy_m(t)}{dt} = -\frac{1}{e_{mm}^2} \sum_{j=0}^{N_y} \sum_{i=0}^{N_k} k_i y_j(t) e_{ijm}, \quad m = 0, 1, 2, \dots, N_y. \quad (\text{A.20})$$

where,

$$e_{ijm} = \langle \psi_i \psi_j \psi_m \rangle = \int_{\Omega} \psi_i(\xi_1) \psi_j(\xi_1) \psi_m(\xi_1) p(\xi_1) d\xi_1, \quad (\text{A.21})$$

$$e_m^2 = \langle \psi_m^2 \rangle = \int_{\Omega} \psi_m(\xi_1) \psi_m(\xi_1) p(\xi_1) d\xi_1. \quad (\text{A.22})$$

Solution procedure of Eq. (A.20) is for $N_y=4$ as below:

The following equations are can be calculated for $m=0, 1, 2, 3, 4$.

$$\frac{dy_0}{dt} = -\frac{1}{e_{00}^2} \left[k_0 y_0(t) e_{000} + k_1 y_0(t) e_{100} + k_0 y_1(t) e_{010} + k_1 y_1(t) e_{110} + k_0 y_2(t) e_{020} + \dots \right] \quad (\text{A.22})$$

$$\frac{dy_1}{dt} = -\frac{1}{e_{11}^2} \left[k_0 y_0(t) e_{001} + k_1 y_0(t) e_{101} + k_0 y_1(t) e_{011} + k_1 y_1(t) e_{111} + k_0 y_2(t) e_{021} + \dots \right] \quad (\text{A.23})$$

$$\frac{dy_2}{dt} = -\frac{1}{e_{22}^2} \left[k_0 y_0(t) e_{002} + k_1 y_0(t) e_{102} + k_0 y_1(t) e_{012} + k_1 y_1(t) e_{112} + k_0 y_2(t) e_{022} + \dots \right] \quad (\text{A.24})$$

$$\frac{dy_3}{dt} = -\frac{1}{e_{33}^2} \left[k_0 y_0(t) e_{003} + k_1 y_0(t) e_{103} + k_0 y_1(t) e_{013} + k_1 y_1(t) e_{113} + k_0 y_2(t) e_{023} + \dots \right] \quad (\text{A.25})$$

$$\frac{dy_4}{dt} = -\frac{1}{e_{44}^2} \left[k_0 y_0(t) e_{004} + k_1 y_0(t) e_{104} + k_0 y_1(t) e_{014} + k_1 y_1(t) e_{114} + k_0 y_2(t) e_{024} + \dots \right] \quad (\text{A.26})$$

These equations forms ordinary differential equation system and can be written in a matrix form,

$$\begin{pmatrix} \frac{dy_0}{dt} \\ \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \\ \frac{dy_4}{dt} \end{pmatrix} = - \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ K_{51} & \dots & \dots & \dots & K_{55} \end{bmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \quad (\text{A.27})$$

where,

$$K_{pr} = \frac{\sum_{i=0}^{N_k} k_i e_{itm}}{e_{mm}^2} \quad m = p-1; t = r-1. \quad (\text{A.28})$$

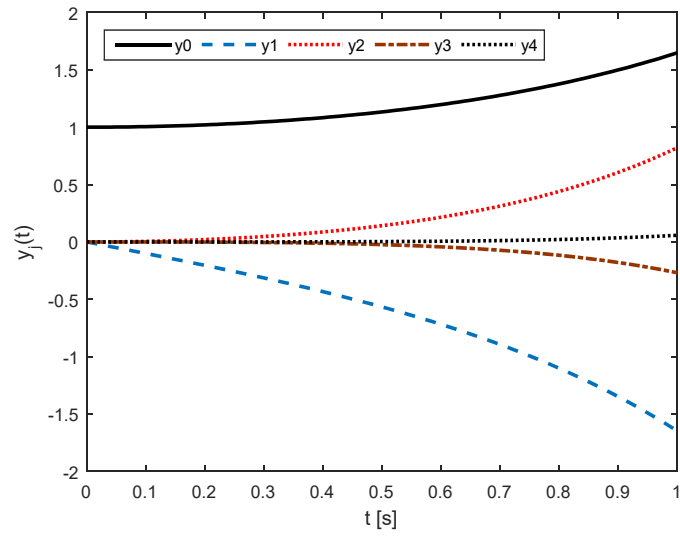
Beside them, initial condition of the differential equation given in Eq. (A.16) also needs to be projected onto $y_m(0)$. This is performed by;

$$Y(0, \xi_1) = \sum_{j=0}^{N_y} y_j(0) \psi_j(\xi_1) = 1 \quad (\text{A.29})$$

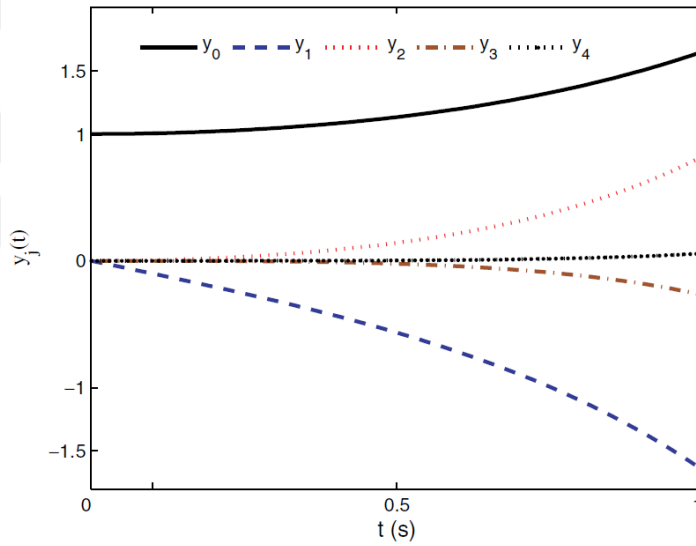
and

$$y_m(0) = \frac{\langle 1, \psi_m \rangle}{e_{mm}^2} \quad (\text{A.30})$$

Eq. (A.27) can be solved with the initial conditions calculated from Eq. (A.30) via Matlab. Calculated results are given in Fig. A.2a where results from Ref. (Sepahvand et al., 2010) are given in Fig. A.2b.



a)



b)

Figure A.2 a) Coefficients of PC expansion of Eq. (44) b) Results of Ref. (Sepahvand et al., 2010)

It is seen that, calculated coefficients of PC expansion are the same by Sepahvand et. al. (2010).

APPENDIX-4: List of Symbols

a_1	: a constant for the mechanical impedance boundary condition position
a	: plate length in x -direction
A_k	: boundary condition coefficients in DSC
b	: plate length in y -direction
$\mathbf{B}$	: matrix that includes coefficients of derivative points in $\mathbf{B}_{BC}$
$\mathbf{B}_{BC}$	: boundary condition equation matrix written in terms of DSC
c	: dashpot at the boundary
c_b	: bending wave speed
c_L	: longitudinal wave speed
c_s	: speed of sound
C	: a collective symbol for the kernel
$\mathbf{c}_{x0}^{(2)}$	: vector form of Eq. (3.12) with $i=0$ and $n=2$ after applying TSE
$\mathbf{C}_K$	: matrix obtained from the continuity and boundary condition equations
$\mathbf{C}_M$	: matrix obtained from the continuity and boundary condition equations
$\mathbf{C}_{Coef}$	: matrix that includes coefficients of structural points in $\mathbf{B}_{BC}$
d	: diameter of duct
$d\mu(\xi)$	: probability function of random space (Ω) of uncertain parameter
D	: bending rigidity
$\{\mathbf{D}^{(n)}\}$	: i th row of characteristic DSC matrix of differentiation order of n
$\mathbf{D}(\mathbf{x})$	: dynamic stiffness matrix
DSC	: Discrete Singular Convolution
DSC-TSE	: DSC applied with Taylor Series Expansion for connected structure
DSC-c	: simultaneous solution for connected structures via the DSC
E	: Elasticity (Young) modulus,
FEM	: Finite Element Method
f	: frequency in Hz

f_{an}	: analytical natural frequency in Hz
$f(x, t)$	: external longitudinal force
f_b	: external bending force
F_0	: amplitude of the harmonic excitation force
F	: force
$\mathbf{f}$	: forcing vector discretized in a DSC spatial domain
G	: a function
G_α	: approximation of real function
g	: a test function
h	: thickness
$H_{m,12}$	: complex frequency response function
H_a	: average of complex frequency response
H	: Hermitian polynomial
$I_y(x)$	: space dependent moment of inertia about y-axis
j	: complex number, $\sqrt{-1}$
j	: duct number in transmission line
J	: total numbers of duct elements in transmission lines
k_s	: stiffness at the boundary
k_w	: wavenumber
$\mathbf{K}$	: global stiffness matrix
l	: length of bar / beam
L	: distance between the sample and microphone 2
m	: mass at the boundary
M	: computational bandwidth
$\mathbf{M}$	: global mass matrix
n	: order of the derivative
n_u	: number of uncertain parameter to define uncertain variable
N	: number of discretization in DSC
N_{PC}	: expansion order of PCE

$q(x, t)$	: acoustic source creating sound pressure per unit area (Pa.m^{-2})
Q	: volume velocity of the source
$p(x, t)$	: time-harmonic sound pressure
p_u	: order of polynomial to define uncertain variable
$P(x)$	: space dependent sound pressure
$P_{m,i}$	: sound pressure measured from i th microphone
PCE	: Polynomial Chaos Expansion
R	: reflection coefficient
r	: regularization coefficient
r_p	: specific volume
RSDK	: Regularized Shannon Delta sequence Kernel
s	: derivative order of Taylor Series Expansion
s_m	: distance between microphones,
S	: the cross-sectional area,
S_T	: finite value of the order of TSE
T	: a distribution
$T(t - x)$	: a singular kernel
$\mathbf{T}^L$	: matrix including kernel information for the left-ghost points
$\mathbf{T}^S$	: matrix including kernel information for the structural points
$\mathbf{T}^R$	: matrix including kernel information for the right-ghost points
$u(x, t)$	: space and time dependent axial displacement
$U(x)$	: space dependent axial displacement
$\mathbf{u}$	: axial displacement vector discretized in a DSC spatial domain
V	: volume of the duct
y_i	: deterministic coefficient of orthogonal polynomial basis
Y	: an uncertain variable
$w(x, t)$	: time and space dependent bending displacement in z direction
$W(x)$	: space dependent bending displacement in z direction
$\mathbf{w}$	: bending displacement vector discretized in a DSC spatial domain

$\bar{\mathbf{w}}$	: vector that includes the elements of DoFs after implementing boundary condition
$\mathbf{w}_d$	: vector that contains derivatives of boundary displacements
$\{x_k\}, \mathbf{x}$	: discretized point vector
Z_a	: $j\omega\rho/Z_s$
Z_m	: mechanical impedance
Z_s	: acoustic impedance of material applied at the boundary
α	: sampling parameter
α_0	: a generalized limit
β	: space dependent parameter coefficient of thickness
$\gamma(x)$	: the space dependency of elasticity
$\delta^{(n)}(x)$	: higher order singular kernel of delta type
Δ	: grid spacing, distance between discretization points
ε	: mean of uncertain parameter
$\eta(t)$	: element of the space of test functions
λ_n	: natural frequency parameter (eigenvalue)
$\mu_i(\xi_i)$	: probability density function
ν	: Poisson's ratio
ξ_i	: i th stochastic / uncertain parameter
ξ	: vector containing stochastic parameters
$\rho(x)$	: space dependent density
ρ_s	: air density
σ	: regularization parameter
χ	: standard deviation of uncertain parameter
$\psi_i(\cdot)$	: orthogonal polynomial basis
ω	: excitation frequency in r/s
ω_n	: natural circular frequency in r/s

ω_{an} : analytical natural frequency in r/s

Ω : space of random variable

$\vartheta_1, \vartheta_2, \vartheta_3$: uncertain parameters affecting elasticity modulus, thickness and density

$\langle \rangle$: expected value

