DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOME RESULTS ON THE GEOMETRY OF BANACH SPACES

by

Abdulbaki AŞUR

September, 2022

İZMİR

SOME RESULTS ON THE GEOMETRY OF BANACH SPACES

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We have read the thesis entitled "**SOME RESULTS ON THE GEOMETRY OF BANACH SPACES**" completed by **ABDULBAKİ AŞUR** under supervision of **PROF. DR. SELÇUK DEMİR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Abdulbaki AŞUR

SOME RESULTS ON THE GEOMETRY OF BANACH SPACES

ABSTRACT

The goal of this study is to get some geometric conclusions from Banach spaces and find some useful applications. In this study, we start by giving some definitions about convexity and measure theory. The Brunn-Minkowski theorem, one of the most significant in convex geometry and functional analysis, is next explored, along with one of its applications, isoperimetric inequality.

Second, we prove F. John's theorem, which describes ellipsoids of maximal volume in symmetric convex bodies. The proof which is given is a relatively recent one due to Schuster & Gruber (2005).

Finally, we give some basic results in Banach Space theory which are related somehow to the geometric ideas given in John's theorem by using the ideas in Dvoretzky & Rogers (1950).

Keywords: Banach space, ellipsoid, John's theorem, symmetric convex body, unconditionally convergent series

BANACH UZAYLARININ GEOMETRİSİ ÜZERİNE BAZI SONUÇLAR

ÖZ

Bu çalışmanın amacı, Banach uzaylarının bazı geometrik sonuçlarını elde etmek ve bazı faydalı uygulamalar bulmaktır. Çalışmaya, konvekslik ve ölçüm teorisi hakkında bazı tanımlar vererek başlıyoruz. Daha sonra konveks geometri ve fonksiyonel analizin en önemli teoremlerinden biri olan Brunn-Minkowski teoremi ve bu teoremin uygulamalarından biri olan İzoperimetrik eşitsizlik tartışılmıştır.

İkinci olarak, simetrik dışbükey cisimlerde maksimum hacimli elipsoidleri karakterize eden F. John teoremini kanıtlıyoruz. Daha yeni ve anlaşılır olması sebebiyle Schuster & Gruber (2005)'teki kanıttan yararlanıldı.

Son olarak, Dvoretzky & Rogers (1950)'deki fikirleri kullanarak John'un teoreminde verilen geometrik fikirlerle bir şekilde ilişkili Banach Uzayı teorisinde bazı temel sonuçlar veriyoruz.

Anahtar kelimeler: Banach uzayı, elipsoid, John teoremi, koşulsuz yakınsak seri, simetrik konveks cisim

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CHAPTER 1
INTRODUCTION

1.1 Background

1. The l_p norm $||.||_p$ is described for $p \in [1, \infty)$ by
$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p},$$
where $x \in \mathbb{R}^n$ and $||x||_\infty = \max\{|x_i| \ : \ i = 1, 2, \ldots, n\}$.

2. A subset C of $\mathbb{R}^n$ is said to be *convex* if, for every x and y in C, the line segment that combines x and y is included in C. In other words, if $x, y \in C$,
$$tx + (1-t)y \in C$$
for every $t \in [0, 1]$.

3. If a convex set C is compact and its interior is not empty, then we say that C is a *convex body*. An important example for convex body is the unit Euclidean ball
$$B_2^n = \{x \in \mathbb{R}^n \mid ||x||_2 \leq 1\}.$$

4. A *box* B in $\mathbb{R}^n$ is defined by
$$B = \prod_{i=1}^{n} [a_i, b_i]$$
where $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$. Note that every open set in $\mathbb{R}^n$ can be written as a countable union of almost disjoint boxes.

5. The *Lebesgue measure m* is the unique translation invariant Borel measure such that
$$m\left(\prod_{j=1}^{k} [a_j, b_j] \right) = \prod_{j=1}^{k} (b_j - a_j).$$
Let E be a *measurable* set in $\mathbb{R}^n$. Then
$$m(E) = \sup\{m(K) \mid K \subseteq E, \ and \ K \ is \ compact\}$$
$$= \inf\{m(U) \mid E \subseteq U, \ and \ U \ is \ open\}$$

CHAPTER 2
CONVEXITY IN FUNCTIONAL ANALYSIS

2.1 Minkowski sum of sets

Definition 2.1.1. *In $\mathbb{R}^n$, the Minkowski sum of two (non-empty) sets, A and B, is defined to be the set $A+B$ that can be obtained by adding each element in A to each element in B. That is,*

$$A+B=\{a+b\ :\ a\in A,\ b\in B\}.$$

If c is a nonzero real number and K is a set in $\mathbb{R}^n$, then we may define

$$cK=\{cx\ :\ x\in K\}.$$

and K is called symmetric if $-K=K$

Note that if A and B are two convex bodies in $\mathbb{R}^n$, so is $A+B$.

2.2 The Brunn-Minkowski inequality

Theorem 2.2.1. *Let $A,B\subseteq\mathbb{R}^n$ be two measurable sets. If $A+B$ is measurable, then*

$$m(A+B)^{\frac{1}{n}}\geq m(A)^{\frac{1}{n}}+m(B)^{\frac{1}{n}}. \tag{2.1}$$

The equality holds when A and B are homothetic (A is a translation or an expansion of B) (Pak (2010)).

Proof. First, assume that A and B are boxes in $\mathbb{R}^n$ with side lengths $a_1,...,a_n$ and $b_1,...,b_n$, respectively. Then (2.1) becomes

$$\left(\prod_{i=1}^{n}(a_i+b_i)\right)^{\frac{1}{n}}\geq\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}+\left(\prod_{i=1}^{n}b_i\right)^{\frac{1}{n}}.$$

Observe that if $\lambda_1,\lambda_2,...,\lambda_n>0$, then

$$\left(\prod_{i=1}^{n}(\lambda_i a_i+\lambda_i b_i)\right)^{\frac{1}{n}}\geq\left(\prod_{i=1}^{n}\lambda_i a_i\right)^{\frac{1}{n}}+\left(\prod_{i=1}^{n}\lambda_i b_i\right)^{\frac{1}{n}}.$$

Replacing a_i and b_i with $\lambda_i a_i$ and $\lambda_i b_i$ does not change the truth of the inequality. Set $\lambda_i = \frac{1}{a_i+b_i}$. It is enough to prove that if $a_i + b_i = 1$ for each i, then

$$1 \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}}$$

By AM-GM inequality

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} b_i \geq \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}}.$$

The sum of the above inequalities gives the required result. Therefore the inequality holds for any two boxes.

Next, suppose A and B are finite union of disjoint boxes in $\mathbb{R}^n$. Describe the fol lowing subsets of $\mathbb{R}^n$:

$$A^+ = A \cap \{x \in \mathbb{R}^n \mid x_j \geq 0\}, \quad A^- = A \cap \{x \in \mathbb{R}^n \mid x_j \leq 0\},$$

$$B^+ = B \cap \{x \in \mathbb{R}^n \mid x_j \geq 0\}, \quad B^- = B \cap \{x \in \mathbb{R}^n \mid x_j \leq 0\}.$$

for some j. Translate A and B so that the following conditions are met:

1. A has some pair of boxes separated by the hyperplane

   $$\{x \in \mathbb{R}^n \mid x_j = 0\}.$$

   Namely, there is a box that is completely in the halfspace

   $$\{x \in \mathbb{R}^n \mid x_j \geq 0\}$$

   and there is another box which is in its complement halfspace (see figure 2.1).

2. It holds that

   $$\frac{m(A^+)}{m(A)} = \frac{m(B^+)}{m(B)}, \qquad \frac{m(A^-)}{m(A)} = \frac{m(B^-)}{m(B)}.$$

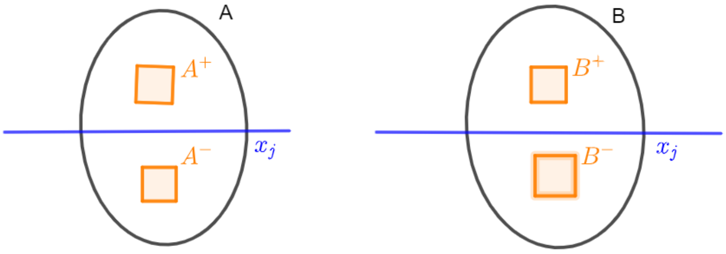

Figure 2.1 A^+, B^+, A^- and B^- as described in the condition 1.

Observe that moving A or B also moves $A+B$, so everything we say about the tra nslated sets is true for the original ones. Now we use induction on the total number of boxes in $A \cup B$. Since A^+ and A^- are subsets of A, we know that $A^+ \cup B^+$ and $A^- \cup B^-$ have fewer boxes than $A \cup B$. Therefore, (2.1) is true for them by the induction hypothesis. Moreover $A^+ + B^+$ and $A^- + B^-$ are disjoint because their signs change in the x_j-coordinate (Jonathan Kelner (2009)). Hence, we have

$$\begin{aligned}
m(A+B) &\geq m(A^+ + B^+) + m(A^- + B^-) \\
&\geq \left(m(A^+)^{\frac{1}{n}} + m(B^+)^{\frac{1}{n}}\right)^n + \left(m(A^-)^{\frac{1}{n}} + m(B^-)^{\frac{1}{n}}\right)^n \\
&= m(A^+)\left(1 + \frac{m(B^+)^{\frac{1}{n}}}{m(A^+)^{\frac{1}{n}}}\right)^n + m(A^-)\left(1 + \frac{m(B^-)^{\frac{1}{n}}}{m(A^-)^{\frac{1}{n}}}\right)^n \\
&= \left(m(A^+) + m(A^-)\right)\left(1 + \left(\frac{m(B)}{m(A)}\right)^{\frac{1}{n}}\right)^n \\
&= \left(m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}\right)^n .
\end{aligned}$$

Taking nth root of both sides completes the proof for elementary sets.

The previous argument shows that (2.1) is true for every elementary set (disjoint union of boxes). Let U be an open subset of $\mathbb{R}^n$ with finite measure ($m(U) < \infty$). If $B_1, B_2, \ldots$ are almost disjoint boxes such that

$$U = \bigcup_{k=1}^{\infty} B_k,$$

then

$$m(U) = \sum_{k=1}^{\infty} m(B_k) < \infty.$$

This means that for every $\epsilon > 0$, there exists a natural number N such that if $B_1, B_2, ..., B_N \subset U$, then

$$m(U) < \sum_{k=1}^{N} m(B_k) + \epsilon.$$

Next, suppose that A and B are open sets of finite measures. Let $\epsilon > 0$. Then there exist elementary sets $A_\epsilon \subset A$ and $B_\epsilon \subset B$ such that

$$m(A) \leq m(A_\epsilon) + \epsilon, \qquad m(B) \leq m(B_\epsilon) + \epsilon.$$

Notice that $A_\epsilon + B_\epsilon \subset A + B$. Then

$$m(A+B)^{\frac{1}{n}} \geq m(A_\epsilon + B_\epsilon)^{\frac{1}{n}} \geq m(A_\epsilon)^{\frac{1}{n}} + m(B_\epsilon)^{\frac{1}{n}}.$$

Letting $\epsilon \to 0$, we obtain that $A_\epsilon \to A$, $B_\epsilon \to B$ and

$$m(A+B)^{\frac{1}{n}} \geq m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}.$$

Therefore, (2.1) is true for open sets with finite measures.

Now suppose that A and B are compact sets. Let $\epsilon > 0$. Define

$$A_\epsilon = \{x \in \mathbb{R}^n \mid d(x,A) < \epsilon\}, \qquad B_\epsilon = \{x \in \mathbb{R}^n \mid d(x,B) < \epsilon\}.$$

A_ϵ and B_ϵ are bounded and open sets containing the sets A and B, respectively. By the previous argument

$$m(A_\epsilon + B_\epsilon)^{\frac{1}{n}} \geq m(A_\epsilon)^{\frac{1}{n}} + m(B_\epsilon)^{\frac{1}{n}} \geq m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}$$

As $\epsilon \to 0$ we get

$$m(A+B)^{\frac{1}{n}} \geq m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}.$$

So (2.1) is true for compact sets. If A and B are measurable sets, then

$$\begin{aligned} m(A) &= \sup\{m(K) \mid K \subseteq A, \textit{ and } K \textit{ is compact}\} \\ &= \inf\{m(U) \mid A \subseteq U, \textit{ and } U \textit{ is open}\} \end{aligned}$$

and by repeating the previous arguments, we have

$$m(A+B)^{\frac{1}{n}} \geq m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}.$$

□

2.3 Isoperimetric problem

Now, we show how the Brunn-Minkowski inequality works by using it to prove an important theorem in geometry. Consider the following problem:

Problem 1. In $\mathbb{R}^n$, find a body of given volume that has the smallest surface area.

From now on, we write $vol(\cdot)$ instead of $m(\cdot)$ to denote the Lebesgue measure or volume. Next, define the surface area of a body:

Definition 2.3.1. *Let K be a body in $\mathbb{R}^n$ with n-dimensional volume $vol(K)$ and define a set*

$$K_\epsilon = \{x \in \mathbb{R}^n \mid d(x,K) < \epsilon\} = \bigcup_{x \in K} B(x,\epsilon).$$

Then the surface area of K is given by

$$\alpha(K) = \limsup_{\epsilon \to 0^+} \frac{vol(K_\epsilon) - vol(K)}{\epsilon}.$$

The Isoperimetric inequality will give the answer to our problem.

Corollary 2.3.1 (Isoperimetric inequality (Ball (1997)))**.** *Let K be a convex body in $\mathbb{R}^n$ with $vol(K) > 0$ and let B be a closed Euclidean ball with $vol(B) = vol(K)$. Then for every $\epsilon > 0$, we have*

$$vol(K_\epsilon) \geq vol(B_\epsilon)$$

and

$$\alpha(K) \geq \alpha(B).$$

Proof. If $r > 0$, then $B = rB_2^n$. If we choose $r = \left(\frac{vol(K)}{vol(B_2^n)}\right)^{\frac{1}{n}}$, then

$$vol(B) = vol(rB_2^n) = r^n vol(B_2^n) = \frac{vol(K)}{vol(B_2^n)} vol(B_2^n) = vol(K).$$

We have $B_\epsilon = rB_2^n + \epsilon B_2^n = (r+\epsilon)B_2^n$ and $K_\epsilon = K + \epsilon B_2^n$, and by the Brunn-Minkowski inequality, we get

$$\begin{aligned} vol(K_\epsilon)^{\frac{1}{n}} &= vol(K + \epsilon B_2^n)^{\frac{1}{n}} \\ &\geq vol(K)^{\frac{1}{n}} + vol(\epsilon B_2^n)^{\frac{1}{n}} \\ &= vol(rB_2^n)^{\frac{1}{n}} + \epsilon vol(B_2^n)^{\frac{1}{n}} \\ &= (r+\epsilon) vol(B_2^n)^{\frac{1}{n}} = vol((r+\epsilon)B_2^n)^{\frac{1}{n}} = vol(B_\epsilon)^{\frac{1}{n}}. \end{aligned}$$

□

As a solution to the problem, a direct consequence of the Isoperimetric inequality is that a body of a fixed volume with the smallest surface area is the Euclidean ball.

CHAPTER 3
ELLIPSOIDS AND JOHN'S THEOREM

3.1 Polarity

Definition 3.1.1. *The polar of a set $K \subseteq \mathbb{R}^n$ is defined by*

$$K^\circ := \{x \in \mathbb{R}^n \ : \ \langle x, y \rangle \leq 1 \quad \forall y \in K\}.$$

Now, we give some properties of polarity:

- If K is a symmetric convex body, then $(K^\circ)^\circ = K$, (bipolar theorem).
- $(B_1^n)^\circ = B_\infty^n$, $(B_2^n)^\circ = B_2^n$ and $(B_\infty^n)^\circ = B_1^n$.
- If $K \subseteq L$, then $L^\circ \subseteq K^\circ$.
- If $T \in GL_n(\mathbb{R})$ is an invertible linear map, then $(T(K))^\circ = (T^*)^{-1}(K^\circ)$, where T^* is the transpose (or adjoint) of T. In particlar, for any nonzero real number α, we have $(\alpha K)^\circ = \frac{1}{\alpha} K^\circ$.

3.2 Banach-Mazur distance and ellipsoids

If $X_1 = (\mathbb{R}^n, \|.\|)$ is a Banach space, the closed unit ball B_{X_1} of X_1 is a symmetric body. Suppose that $T \in GL_n(\mathbb{R})$,

$$X_1 = (\mathbb{R}^n, \|.\|), \qquad X_2 = (\mathbb{R}^n, \|T(.)\|)$$

and B_{X_1} and B_{X_2} are the closed unit balls of X_1 and X_2, respectively. Then the spaces X_1 and X_2 are isometric and $B_{X_2} = T^{-1}(B_{X_1})$. Studying n-dimensional Banach spaces is equivalent to studying symmetric bodies in $\mathbb{R}^n$ up to the action of $GL_n(\mathbb{R})$.

Definition 3.2.1 (Banach-Mazur distance)**.** *If X and Y are two n-dimensional Banach spaces, we may define their Banach-Mazur distance as*

$$d_{BM}(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| \ : \ T : X \to Y \textit{ is linear bijection}\}$$

where $\|T\| = \sup_{x\in B_X} \|T(x)\|$, *and* B_X *is the unit ball of* X. *The operator norm of* T *can also be written by* $\|T\| = \inf\{\lambda > 0 \ : \ T(B_X) \subseteq \lambda B_Y\}$.

For simplicity we write $d(X,Y)$ instead of $d_{BM}(X,Y)$. If K and L are two symmetric bodies in $\mathbb{R}^n$, then their Banach-Mazur distance is

$$d(K,L) = \inf\left\{\frac{b}{a} \ : \ aK \subseteq T(L) \subseteq bK,\ a,b > 0, \text{ and } T \in GL_n(\mathbb{R})\right\}.$$

Some basic properties of d are as follows:

- Symmetry: $d(K,L) = d(L,K)$ since $aK \subseteq T(L) \subseteq bK$ is equivalent to $\frac{1}{b}L \subseteq T^{-1}(K) \subseteq \frac{1}{a}L$.
- Invariance under polarity: $d(K,L) = d(K^\circ, L^\circ)$ because $aK \subseteq T(L) \subseteq bK$ is equivalent to $\frac{1}{b}K^\circ \subseteq (T^*)^{-1}(L^\circ) \subseteq \frac{1}{a}K^\circ$.
- $d(K,P) \le d(K,L)\cdot d(L,P)$.
- d(K,L)=1 if and only if $\exists T \in GL_n(\mathbb{R})$ such that $T(K) = L$.

We denote BM_n by the set of symmetric convex bodies in $\mathbb{R}^n$ up to the equivalence relation (Guillaume Aubrun (2020)).

$$K \sim L \iff \exists T \in GL_n(\mathbb{R}) \ : \ T(K) = L.$$

The space $(BM_n, \log d)$ is a metric space.

Definition 3.2.2 (Ellipsoid)**.** *An ellipsoid* $\mathcal{E} \subseteq \mathbb{R}^n$ *is a body that has the form*

$$\mathcal{E} = T(B_2^n),$$

where $T \in GL_n(\mathbb{R})$.

Now, we define the following sets:

$$M_n^+ = \{A \in M_n(\mathbb{R}) \ : \ A = A^T, \text{ and } \langle Ax, x\rangle \ge 0,\ \forall x \in \mathbb{R}^n\}$$

$$M_n^{++} = \{A \in M_n(\mathbb{R}) \ : \ A = A^T, \text{ and } \langle Ax, x\rangle > 0,\ \forall x \in \mathbb{R}^n\}$$

where $M_n(\mathbb{R})$ denotes the set of $n\times n$ matrices with real entries.

Proposition 3.2.1. *For $\mathcal{E} \subseteq \mathbb{R}^n$, the following are equivalent:*

1. *$\mathcal{E}$ is an ellipsoid.*
2. *$\exists A \in M_n^{++}$ such that $\mathcal{E} = A(B_2^n)$.*
3. *There is an orthonormal basis (f_i) of $\mathbb{R}^n$, and positive numbers (α_i), such that*
$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} \frac{\langle x, f_i\rangle^2}{\alpha_i^2} \leq 1 \right\}.$$
4. *$\mathcal{E}$ is the closed unit ball with respect to some norm coming from an inner product.*

Proof. (1. $\iff$ 2.): Recall the polar decomposition that any $T \in GL_n(\mathbb{R})$ can be written as $T = AO$ for some $O \in O(n)$ (the set of$n\times n$ orthogonal matrix) and $A \in M_n^{++}$.Then we have

$$\mathcal{E} = T(B_2^n) = A(O(B_2^n)) = A(B_2^n).$$

(2. $\implies$ 3.): By the spectral theorem, if $A \in M_n^{++}$, then there exists an orthonormal b asis (f_i) of $\mathbb{R}^n$ such that $A(f_i) = \alpha_i f_i$ for $\alpha_i > 0$. If $x \in \mathbb{R}^n$, we have

$$A(x) = \sum_{i=1}^{n} \alpha_i \langle x, f_i\rangle f_i$$

and

$$A^{-1}(x) = \sum_{i=1}^{n} \frac{\langle x, f_i\rangle}{\alpha_i} f_i.$$

It follows that

$$\begin{aligned} x \in \mathcal{E} &\iff x \in A(B_2^n) \\ &\iff A^{-1}(x) \in B_2^n \\ &\iff \|A^{-1}(x)\|_2 = \sum_{i=1}^{n} \frac{\langle x, f_i\rangle^2}{\alpha_i^2} \leq 1. \end{aligned}$$

(3. $\implies$ 4.): Define

$$Q(x,y) = \sum_{i=1}^{n} \frac{\langle x, f_i\rangle\langle y, f_i\rangle}{\alpha_i^2}.$$

Q is an inner product on $\mathbb{R}^n$ and

$$||x||_Q = Q(x,x) \leq 1.$$

So $\mathcal{E}$ is the closed unit ball with respect to the norm $||.||_Q$, that is,

$$\mathcal{E} = B_Q.$$

(4. $\implies$ 2.): If Q is an inner product, then there is an $A \in M_n^{++}$ such that

$$Q(x,x) = \langle x, A(x) \rangle$$

for every $x \in \mathbb{R}^n$. Since 3. and 4. are true, we have

$$\begin{aligned} Q(x,x) \leq 1 &\iff \langle x, A(x) \rangle \leq 1 \\ &\iff || \sqrt{A}(x)||^2 \leq 1 \\ &\iff x \in A^{-1/2}(B_2^n) \end{aligned}$$

Replacing $A^{-1/2}$ with A completes the proof. □

3.3 John's theorem

The following definition will be used for the rest of this study.

Definition 3.3.1. *Let K be a symmetric convex body in $\mathbb{R}^n$. An ellipsoid $\mathcal{E}$ of maximal volume inside K is called the John ellipsoid of K, and denoted by $\mathcal{E}_J(K)$. If $\mathcal{E}_J(K) = B_2^n$, we say that K is in the John position.*

John's theorem claims that such ellipsoid is unique and gives a characterization of a body being in John position.

Theorem 3.3.1 (John's theorem (Schuster & Gruber (2005)))**.** *If $K \subseteq \mathbb{R}^n$ is a symmetric convex body, then there exists a unique ellipsoid $\mathcal{E}_J(K)$ of maximal volume* *in K. Moreover $\mathcal{E}_J(K) = B_2^n$ if and only if $B_2^n \subseteq K$, and there exist $x_1, x_2, \ldots, x_m \in \partial K \cap \partial B_2^n$ and positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that*

$$\sum_{i=1}^{m} \lambda_i = 1 \qquad \textit{and} \qquad \frac{Id}{n} = \sum_{i=1}^{m} \lambda_i \cdot x_i x_i^T,$$

that is,

$$\frac{Id}{n} \in conv\{xx^T \ : \ x \in \partial K \cap \partial B_2^n\}$$

Intitively, if $B_2^n \subseteq K$ but without enough contact point, then there is a way to construct another ellipsoid inside K with a larger volume (John (1948)). For every symmetric convex body $K \subseteq \mathbb{R}^n$, there is $T \in GL_n(\mathbb{R})$ such that $T(K)$ is in the John position.

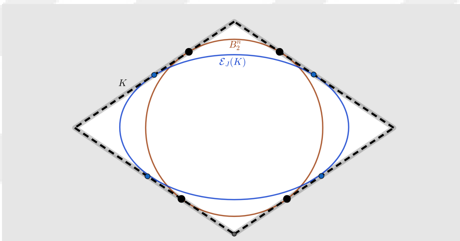

Figure 3.1 An ellipsoid (circle) without enough contact points can grow.

Proof of the Theorem 3.3.1. Let K be a symmetric convex body in $\mathbb{R}^n$. First, we must show the existence of the ellipsoid of maximal volume. We note that if $\mathcal{E} = T(B_2^n)$, then $vol(\mathcal{E}) = |\det(T)| vol(B_2^n)$. The set

$$\mathcal{F} = \{T \in M_n(\mathbb{R}) \ : \ T(B_2^n) \subseteq K\}$$

is compact (see Appendix A-1) and therefore the continous function $|\det(\cdot)|$ reaches its maximum. So there exists an ellipsoid of maximal volume.

We can now prove the uniqueness part. Suppose $\mathcal{E}_1$, $\mathcal{E}_2$ are two different ellipsoids of the same volume in K. We may write

$$\mathcal{E}_1 = T_1(B_2^n), \qquad \mathcal{E}_2 = T_2(B_2^n)$$

for $T_1, T_2 \in M_n^{++}$ with $T_1 \neq T_2$. However, we have $\det(T_1) = \det(T_2)$. Since the function $\log\det$ is stricly concave on M_n^{++}, (see Appendix A-2) we get

$$\det\left(\frac{T_1+T_2}{2}\right) > \sqrt{\det(T_1)\det(T_2)}.$$

Put $T = \frac{T_1+T_2}{2}$ and $\mathcal{E} = \frac{\mathcal{E}_1+\mathcal{E}_2}{2}$. We have

$$\mathcal{E} = T(B_2^n) = \frac{T_1(B_2^n)+T_2(B_2^n)}{2} = \frac{\mathcal{E}_1+\mathcal{E}_2}{2} \subseteq K.$$

If $\det(T_1) = \det(T_2)$, then $\sqrt{\det(T_1)\det(T_2)} = |\det(T_1)|$, and so

$$|\det(T)| = \left|\det\left(\frac{T_1+T_2}{2}\right)\right| > |\det(T_1)|.$$

Therefore,

$$vol(\mathcal{E}) = |\det(T)|vol(B_2^n) > |\det(T_1)|vol(B_2^n) = vol(\mathcal{E}_1).$$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two different ellipsoids with the same volume, then there is another ellipsoid $\mathcal{E}$ of greater volume inside K.

Now, we prove the characterization. Suppose $B_2^n \subseteq K$ and there exist $x_1, x_2, \ldots, x_m \in \partial K \cap \partial B_2^n$ and positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$\sum_{i=1}^{m} \lambda_i = 1 \qquad \text{and} \qquad \frac{Id}{n} = \sum_{i=1}^{m} \lambda_i \cdot x_i x_i^T.$$

Let $\mathcal{E} \subseteq K$ be an ellipsoid. Then

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{\langle x, f_i \rangle^2}{\alpha_i^2} \leq 1 \right\}$$

for some orthonormal basis $\{f_1, f_2, \ldots, f_n\}$ and positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $i \leq m$ and consider $x_i \in \partial K \cap \partial B_2^n$. K is contained in the set

$$\{x \in \mathbb{R}^n \ : \ \langle x, x_i \rangle \leq 1\}.$$

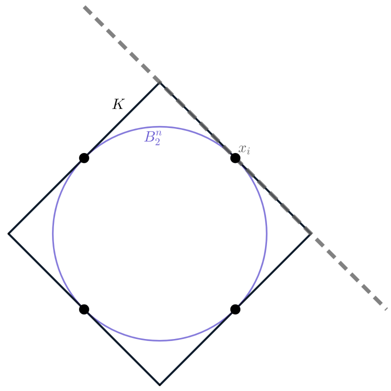

Figure 3.2 K is contained on one side.

Then $\mathcal{E} \subseteq K \subseteq \{x \in \mathbb{R}^n \ : \ \langle x, x_i \rangle \leq 1\}$. Then we have $\langle p, x_i \rangle \leq 1$ for every $p \in \mathcal{E}$. So $x_i \in \mathcal{E}^\circ$. Therefore

$$\sum_{j=1}^{n} \alpha_j^2 \cdot \langle x_i, f_j \rangle^2 \leq 1, \qquad \forall i \leq m$$

since $(\alpha K)^\circ = \frac{1}{\alpha} K^\circ$ for $\alpha > 0$. So

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j^2 \cdot \langle x_i, f_j \rangle^2 \leq 1 &\implies \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} \alpha_j^2 \cdot \langle x_i, f_j \rangle^2 \leq 1 \\ &\implies \sum_{j=1}^{n} \alpha_j^2 \sum_{i=1}^{m} \lambda_i \cdot \langle x_i, f_j \rangle^2 \leq 1 \end{aligned} \tag{3.1}$$

by hypothesis

$$Id = n \sum_{i=1}^{m} \lambda_i \cdot x_i x_i^T$$

we have

$$\begin{aligned} f_j^T \cdot Id \cdot f_j &= n \sum_{i=1}^{m} \lambda_i \cdot f_j^T x_i x_i^T f_j \\ &= n \sum_{i=1}^{m} \lambda_i \cdot \langle x_i, f_j \rangle^2. \end{aligned}$$

By hypothesis, we have $f_j^T \cdot Id \cdot f_j = 1$ which implies

$$1 = n \sum_{i=1}^{m} \lambda_i \cdot \langle x_i, f_j \rangle^2.$$

Therefore, by (3.1)

$$\sum_{j=1}^{n} \alpha_j^2 \leq n.$$

By the AM-GM inequality, we obtain

$$\prod_{j=1}^{n} \alpha_j \leq 1.$$

Since

$$vol(\mathcal{E}) = \prod_{j=1}^{n} \alpha_j \cdot vol(B_2^n),$$

we get

$$vol(\mathcal{E}) \leq vol(B_2^n).$$

Thus, K is in John position.

Conversely, suppose that K is in the John position and assume the contrary that

$$\frac{Id}{n} \notin conv\{xx^T \ : \ x \in \partial K \cap \partial B_2^n\}.$$

Then by the Hahn-Banach theorem we can find a linear functional φ on $M_n^{sa}(\mathbb{R})$ such that for every $x \in \partial K \cap \partial B_2^n$,

$$\varphi\left(\frac{Id}{n}\right) < \varphi(xx^T).$$

There is a natural inner product on $M_n^{sa}(\mathbb{R})$: $(A, B) \rightarrow Tr(AB)$ for every $A, B \in M_n^{sa}(\mathbb{R})$. Therefore, by the Riesz Representation Theorem, there exists some $H \in M_n^{sa}(\mathbb{R})$ such that

$$\varphi(A) = Tr(AH)$$

for every $A \in M_n^{sa}(\mathbb{R})$. So

$$\varphi\left(\frac{Id}{n}\right) = Tr\left(\frac{Id}{n}H\right) = \frac{1}{n}Tr(H) < Tr(Hxx^T)$$

for every $x \in \partial K \cap \partial B_2^n$. Since $Tr(Hxx^T) = x^T Hx$, (see Appendix A-3) we have

$$\frac{1}{n}Tr(H) < \langle x, Hx\rangle$$

for every $x \in \partial K \cap \partial B_2^n$ ($|x| = \|x\|_2 = 1$). If we replace H by $H - \frac{1}{n}Tr(H)$, then we may assume $Tr(H) = 0$ (see Appendix A-4). For $\delta > 0$, let

$$\mathcal{E}_\delta = \{x \ : \ \langle x, (Id + \delta H)x\rangle \leq 1\},$$

and let $M = \max\{||\lambda||_\infty \ : \ \lambda \text{ is the eigenvalue of H}\}$.

$$\begin{aligned}\delta < \frac{1}{M} &\implies Id + \delta H \text{ is positive definite}\\ &\implies \mathcal{E}_\delta \text{ is an ellipsoid.}\end{aligned}$$

We claim that for δ small enough, we have $\mathcal{E}_\delta \subseteq K$. We compare the corresponding norms $||x||_{\mathcal{E}_\delta}$ and $||x||_K$.

$$||x||_K \leq 1 \quad \forall x \in \mathcal{E}_\delta \implies \mathcal{E}_\delta \subseteq K.$$

This can be computed as:

$$||x||_{\mathcal{E}_\delta} = \inf\{t \geq 0 \ : \ x \in t\mathcal{E}_\delta\} = \sqrt{\langle x, (Id + \delta H)x\rangle}.$$

It follows that

$$\begin{aligned}||x||^2_{\mathcal{E}_\delta} - ||x||^2_K &= \langle x, (Id + \delta H)x\rangle - ||x||^2_K\\ &= \underbrace{|x|^2 - ||x||^2_K}_{f(x)} + \delta \underbrace{\langle x, Hx\rangle}_{g(x)}.\end{aligned}$$

The continuous functions f and g satisfy the following properties:

1. $f \geq 0$ on ∂B^n_2,
2. $g > 0$ on the set $\{x \in \partial B^n_2 \ : \ f(x) = 0\}$, and $f + \delta g > 0$ on ∂B^n_2 for δ small enough (see Appendix A-5).

So there is an $\alpha > 0$ such that $f(x) + \delta g(x) \geq \alpha$ for every $x \in \partial B^n_2$. It follows that $(1+\varepsilon)\mathcal{E}_\delta \subseteq K$ for some $\varepsilon > 0$. To check this, let $x \in \mathcal{E}_\delta$. Then $||x||_{\mathcal{E}_\delta} \leq 1$ and

$$\begin{aligned}||x||^2_{\mathcal{E}_\delta} - ||x||^2_K \geq \alpha &\implies ||x||^2_K + \alpha \leq 1\\ &\implies ||x||^2_K \leq 1 - \alpha\\ &\implies \left\| \frac{1}{\sqrt{1-\alpha}} x \right\|^2_K \leq 1,\end{aligned}$$

so $\frac{1}{\sqrt{1-\alpha}}x \in K$, and since $1-\alpha > 0$, we have $\frac{1}{\sqrt{1-\alpha}} > 1$. So we can say

$$1+\varepsilon = \frac{1}{\sqrt{1-\alpha}}$$

for some $\varepsilon > 0$. Therefore

$$\|(1+\varepsilon)x\|_K^2 \leq 1,$$

so $(1+\varepsilon)x \in K$. Thus, $(1+\varepsilon)\mathcal{E}_\delta \subseteq K$.

Let μ_j be the eigenvalues of $Id + \delta H$. We have

$$\sum_{j=1}^{n} \mu_j = Tr(Id+\delta H) = n + \delta Tr(H) = n.$$

Let $T_\delta = Id + \delta H$. Then since $T_\delta \in M_n^{++}$

$$\det(T_\delta) = \prod_{j=1}^{n} \mu_j > 0,$$

and we have

$$\begin{aligned}
\mathcal{E}_\delta &= \{x \ : \ \langle x, (Id+\delta H)x\rangle \leq 1\} \\
&= \{x \ : \ \langle x, T_\delta x\rangle \leq 1\} \\
&= \{x \ : \ \|T_\delta^{1/2} x\|^2 \leq 1\} \\
&= T_\delta^{-1/2}(B_2^n).
\end{aligned}$$

Therefore,

$$\frac{vol(\mathcal{E}_\delta)}{vol(B_2^n)} = \frac{|\det(T_\delta^{-1/2})| vol(B_2^n)}{vol(B_2^n)} = \prod_{j=1}^{n} (\mu_j)^{-1/2}.$$

By the AM-GM inequality

$$\left(\prod_{j=1}^{n} \mu_j\right)^{1/n} \leq \frac{1}{n}\sum_{j=1}^{n} \mu_j = 1,$$

and therefore

$$\frac{vol(\mathcal{E}_\delta)}{vol(B_2^n)} \geq 1 \implies vol(\mathcal{E}_\delta) \geq vol(B_2^n).$$

Since $(1+\varepsilon)^n > 1$ for some $\varepsilon > 0$, we have

$$vol((1+\varepsilon)\mathcal{E}_\delta) = (1+\varepsilon)^n vol(\mathcal{E}_\delta) \geq (1+\varepsilon)^n vol(B_2^n) > vol(B_2^n).$$

Therefore, $vol((1+\varepsilon)\mathcal{E}_\delta) > vol(B_2^n)$. This contradicts to our hypothesis that $\mathcal{E}_J(K) = B_2^n$. $\square$

3.4 Distance estimations

Recall the Banach-Mazur distance between two symmetric convex body K, L in $\mathbb{R}^n$

$$d_{BM}(K,L) = \inf\left\{\frac{b}{a} \ : \ aK \subseteq T(L) \subseteq bK,\ a,b>0,\ \text{and } T \in GL_n(\mathbb{R})\right\}.$$

Some of the most interesting applications of John's theorem are the following corollaries:

Corollary 3.4.1. *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body. Then*

$$\mathcal{E}_J(K) \subseteq K \subseteq \sqrt{n}\mathcal{E}_J(K).$$

Proof. Suppose that K is a symmetric convex body in $\mathbb{R}^n$. Then there is a $T \in GL_n(\mathbb{R})$ such that $L = T(K)$ is in John position. Then we may assume that K is in John position. Suppose $\mathcal{E}_J(K) = B_2^n$. Then it is enough to show that

$$B_2^n \subseteq K \subseteq \sqrt{n}B_2^n.$$

So we have to show the inclusion $K \subseteq \sqrt{n}B_2^n$. If $\mathcal{E}_J(K) = B_2^n$ and $B_2^n \subseteq K$, then by John's theorem, there are contact points $x_i \in \partial K \cap \partial B_2^n$ and real numbers $\lambda_i > 0$ such that

$$\sum_{i=1}^n \lambda_i = 1, \quad \text{and} \quad \frac{Id}{n} = \sum_{i=1}^n \lambda_i \cdot x_i x_i^T.$$

Since $K \subseteq \{y \in \mathbb{R}^n \ : \ \langle y, x_i\rangle \le 1\}$. Then for every $x \in K$, we have $\langle x, x_i\rangle \le 1$. Therefore,

$$\begin{aligned}\frac{Id}{n} = \sum_{i=1}^n \lambda_i \cdot x_i x_i^T \implies x^T x &= n\sum_{i=1}^n \lambda_i \cdot x^T x_i x_i^T x \\ \implies |x|^2 &= n\sum_{i=1}^n \lambda_i \cdot \langle x, x_i\rangle^2 \le n\sum_{i=1}^n \lambda_i = n \\ \implies |x| &\le \sqrt{n},\end{aligned}$$

we have $x \in \sqrt{n}B_2^n$. So $K \subseteq \sqrt{n}B_2^n$. □

Corollary 3.4.2. *If $K \subseteq \mathbb{R}^n$ is a symmetric convex body, then*

$$d_{BM}(K, B_2^n) \leq \sqrt{n}.$$

Proof. Let K be a symmetric convex body in $\mathbb{R}^n$. Then we have:

$$\begin{aligned} d_{BM}(K, B_2^n) &= \inf\left\{\frac{b}{a} \ : \ aB_2^n \subseteq T(K) \subseteq bB_2^n,\ a,b > 0,\ \text{and}\ T \in GL_n(\mathbb{R})\right\} \\ &\leq \sqrt{n}. \end{aligned}$$

The last inequality comes from the previous corollary. □

Theorem 3.4.3. *Let BM_n denote the set of all symmetric convex bodies in $\mathbb{R}^n$. Then the metric space (BM_n, d_{BM}) is compact.*

CHAPTER 4
SERIES IN BANACH SPACES

4.1 Absolute and unconditional convergence

The following definition is essential for the rest of this section.

Definition 4.1.1. *Let B be a Banach space and let $||x||$ denote any norm of x for every $x \in B$. The series*

$$\sum_{n=1}^{\infty} x_n \tag{4.1}$$

1. *is called absolutely convergent if $\sum ||x_n|| < \infty$;*
2. *is called unconditionally convergent if for any permutation σ of $\mathbb{N}$, the series $\sum x_{\sigma(n)}$ still converges where the sequence $(x_{\sigma(n)})_{n=1}^{\infty}$ is any rearrangement of the sequence $(x_n)_{n=1}^{\infty}$.*

Due to the definition above, the result below is quite natural.

Lemma 4.1.1. *If x_n is a sequence of real numbers, then being unconditionally and absolutely convergent of the series $\sum x_n$ are equivalent.*

The following result is true for all Banach spaces and its proof comes from the real case.

Lemma 4.1.2. *Every absolutely convergent series in a Banach space is also unconditionally convergent.*

4.2 Banach spaces and symmetric convex bodies

Now, let's prove a lemma that is about finite dimensional Banach spaces and symm etric convex bodies (Day (1973)).

Lemma 4.2.1. *Let B be a Banach space with* $\dim(B) = n$*. Then there exist n points $x_1, x_2, \ldots, x_n$ in B with $\|x_i\| = 1$ such that for each $i = 1, 2, \ldots, n$ and all real $t_1, t_2, \ldots, t_i$ we have*

$$\left\| \sum_{j\le i} t_j x_j \right\| \le \left[1 + \left(\frac{i(i-1)}{n} \right)^{1/2} \right] \left(\sum_{j\le i} t_j^2 \right)^{1/2}.$$

Proof. Suppose that B is an n-dimensional Banach space. Let C be a closed unit ball of B, that is

$$C = \{x \in B \;:\; \|x\| \le 1\}.$$

By the John's theorem (Theorem 3.3.1), C contains a unique ellipsoid $\mathcal{E}$ of maximal volume. Let $\|x\|_C$ and $\|x\|_{\mathcal{E}}$ denote the norms of C and $\mathcal{E}$, respectively. Since all norms on a finite dimensional Banach spaces are equivalent, we may assume that $\|x\|_{\mathcal{E}}$ is the Euclidean norm ($\|x\|_{\mathcal{E}} = \|x\|_2$). For this reason, we may assume that C is in John position, i.e., $\mathcal{E} = B_2^n$.

We search for x_i such that $\|x_i\|_{\mathcal{E}} = \|x_i\|_C = 1$ and that x_i are approximately orthogonal. By induction on i, we shall find an orthonormal basis $u_1, u_2, \ldots, u_n$ of $(B, \|\cdot\|_2)$ and points $x_1, x_2, \ldots, x_n$ with $\|x_i\|_{\mathcal{E}} = \|x_i\|_C = 1$, $x_i = (x_{i1}, x_{i2}, \ldots, x_{ii}, 0, \ldots, 0)$ and

1. $x_i = \sum_{j\le i} a_{ij} u_j$ and $a_{ij} \ge 0$, and
2. $a_{i1}^2 + a_{i2}^2 + \cdots + a_{i(i-1)}^2 \le (i-1)/n$.

Let $u_1 = x_1$ be arbitrary with $\|x_1\|_{\mathcal{E}} = \|x_1\|_C = 1$. Suppose that we have found $x_1, x_2, \ldots, x_i$ and $u_1, u_2, \ldots, u_i$ ($1 \le i < n$) that satisfy the properties above. Extend $u_1, u_2, \ldots, u_i$ to an orthonormal basis with suitable $u_{i+1}, u_{i+2}, \ldots, u_n$. Let $\varepsilon > 0$. Consider the ellipsoid $\mathcal{E}_\varepsilon$ given by

$$(1+\varepsilon)^{n-i}(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_i^2) + (1+\varepsilon+\varepsilon^2)^{-i}(\alpha_{i+1}^2 + \cdots + \alpha_n^2) \le 1 \tag{4.2}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the components of a vector in $\mathcal{E}_\varepsilon$. Then

$$vol(\mathcal{E}_\varepsilon) > vol(\mathcal{E}).$$

So there exists a point $p_\varepsilon \in \mathcal{E}_\varepsilon \cap C$ with $p_\varepsilon \notin \mathcal{E}$ and $\|p_\varepsilon\|_C = 1$. Put $p_\varepsilon = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then

$$\sum_{j=1}^{n} \alpha_j^2 \geq 1. \tag{4.3}$$

Subtracting (4.2) from (4.3) yields

$$(1+\varepsilon)^{n-i}\left(\sum_{j=1}^{i} \alpha_j^2\right) + (1+\varepsilon+\varepsilon^2)^{-i}\left(\sum_{j=i+1}^{n} \alpha_j^2\right) - \left(\sum_{j=1}^{n} \alpha_j^2\right) \leq 0$$

and so

$$[(1+\varepsilon)^n - 1]\sum_{j=1}^{i} \alpha_j^2 + [(1+\varepsilon+\varepsilon^2)^{-i} - 1]\sum_{j=i+1}^{n} \alpha_j^2 \leq 0. \tag{4.4}$$

By compactness, there exists a subsequence p_{ε_k} converging to some x_{i+1} in $\partial C \cap \partial \mathcal{E}$ with $\|x_{i+1}\|_{\mathcal{E}} = \|x_{i+1}\|_C = 1$ where $\varepsilon_k \to 0$. If x_{i+1} has coordinates $a_1, a_2, \ldots, a_n$ where $a_m = 0$ for $i+1 < m \leq n$, we have (see Appendix A-8)

$$(n-i)\sum_{j=1}^{i} a_j^2 + (-i)\sum_{j=i+1}^{n} a_j^2 \leq 0$$

and then if $\|x_{i+1}\|^2 = \sum a_j^2 = 1$, then

$$(n-i)\sum_{j=1}^{i} a_j^2 - i a_{i+1}^2 \leq 0 \implies n\sum_{j=1}^{i} a_j^2 - i\underbrace{\left(\sum_{j=1}^{i} a_j^2 + a_{i+1}^2\right)}_{1} \leq 0$$

$$\implies \sum_{j \leq i} a_j^2 \leq \frac{i}{n}.$$

Now choose u_{i+1} orthogonal to $u_1, u_2, \ldots, u_i$ in the space $\langle u_1, u_2, \ldots, u_i, x_{i+1}\rangle$ and complete this sequence to a new orthonormal basis. This gives a new representation of x_{i+1} which satisfies the two conditions above. This induction process defines x_i and

u_i for every $i \le n$. $\|u_i\| = 1$ and the condition 2 above implies that

$$\begin{aligned}
\|x_i - u_i\|_{\mathcal{E}}^2 &= \left\| \sum_{j \le i} a_{ij} u_j - u_i \right\|_{\mathcal{E}}^2 \\
&= \left\| \sum_{j < i} a_{ij} u_j - u_i(1 - a_{ii}) \right\|_{\mathcal{E}}^2 \\
&\le \sum_{j<i} a_{ij}^2 + (1 - a_{ii})^2 \\
&\le \frac{i-1}{n} + \frac{i-1}{n} = \frac{2(i-1)}{n}.
\end{aligned}$$

Since $\mathcal{E} \subseteq C$, we have $\|x\|_C \le \|x\|_{\mathcal{E}}$ for all x. Thus,

$$\begin{aligned}
\left\| \sum_{j \le i} t_j x_j \right\|_C &\le \left\| \sum_{j \le i} t_j u_j \right\|_C + \left\| \sum_{j \le i} t_j (x_j - u_j) \right\|_C \\
&\le \left\| \sum_{j \le i} t_j u_j \right\|_C + \sum_{j \le i} |t_j| \left(\frac{2(i-1)}{n} \right)^{1/2} \\
&\le \left(\sum_{j \le i} t_j^2 \right)^{1/2} + \left(\sum_{j \le i} t_j^2 \right)^{1/2} \left(\sum_{j \le i} \frac{2(i-1)}{n} \right)^{1/2} \\
&= \left[1 + \left(\frac{i(i-1)}{n} \right)^{1/2} \right] \left(\sum_{j \le i} t_j^2 \right)^{1/2}
\end{aligned}$$

where the last inequality comes from the Cauchy-Schwarz inequality. □

The following result is extremely straightforward.

Lemma 4.2.2. *If $n \ge i(i-1)$, then any n-dimensional Banach space B contains i unit vectors $x_1, x_2, \ldots, x_i$ such that*

$$\left\| \sum_{j \le i} t_j x_j \right\| \le 2 \left(\sum_{j \le i} t_j^2 \right)^{1/2}$$

for any real $t_1, t_2, \ldots, t_i$.

4.3 Dvoretzky-Rogers Theorem

Our main goal is giving a clear proof of the following result.

Theorem 4.3.1 (Dvoretzky-Rogers)**.** *Let B be an infinite-dimensional Banach space and let c_n be a sequence of positive terms which is square summable, that is, c_n satisfies the condition $\sum c_n^2 < \infty$. Then B contains an unconditionally convergent series $\sum_n x_n$ satisfying $\|x_n\| = c_n$ for every $n = 1, 2, \ldots$.*

If $c_n = 1/n$, the immediate consequence of Theorem 4.3.1 is as follows:

Corollary 4.3.2 (Dvoretzky & Rogers (1950))**.** *A Banach space B is infinite dimensional iff there exists an unconditionally convergent series which is not absolutely convergent. In other words, there is a one-to-one correspondence between unconditionally convergent series and absolutely convergent series in a Banach space B iff B is of finite dimension.*

Proof of Theorem 4.3.1. Suppose that $\sum c_n^2 < \infty$. Find an increasing sequence N_j such that, if $\sigma_j = (N_j, N_{j+1}]$, one has

$$\sum_j \left(\sum_{k \in \sigma_j} c_k^2 \right)^{1/2} < \infty.$$

In B, there exists a sequence of linearly independent, finite dimensional subspaces of sufficiently high dimensional such that we can find vectors $A_{N_j+1}, \ldots, A_{N_{j+1}}$ satisfying the conditions of lemma 4.2.2. Let $x_n = c_n A_n$. If σ is a finite set of integers,

$$\left\| \sum_{k \in \sigma} x_k \right\| \leq \sum_j \left\| \sum_{k \in \sigma \cap \sigma_j} x_k \right\| \leq \sum_j 2 \left(\sum_{k \in \sigma \cap \sigma_j} c_k^2 \right)^{1/2}.$$

This can be made arbitrarily small by keeping σ disjoint from enough of the initial blocks $\sigma_1, \sigma_2, \ldots$. This shows that $\sum x_n$ is unconditionally Cauchy (see Appendix A-7). Since B is a Banach space, then the series $\sum_n x_n$ is unconditionally convergent. □

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APPENDICES

A-1 If K is a symmetric convex body in $\mathbb{R}^n$, then the set

$$\mathcal{F} = \{T \in M_n(\mathbb{R}) \ : \ T(B_2^n) \subseteq K\}$$

is compact.

Proof. We have to prove that $\mathcal{F}$ is bounded and closed. Since $M_n(\mathbb{R})$ is finite-dimensional, all norms on $M_n(\mathbb{R})$ are equivalent. Then we can use any norm on $M_n(\mathbb{R})$ to show compactness.

$$\begin{aligned} K \text{ is symmetric convex body} &\implies K \text{ is compact} \\ &\implies \exists\, r > 0 \text{ such that } K \subseteq rB_2^n, \end{aligned}$$

and since $||T||_{op} = \sup_{||x||_2 \leq 1} ||T(x)|| = ||T|_{B_2^n}||_\infty$, we have:

$$\begin{aligned} T \in \mathcal{F} &\implies ||T||_{op} \leq r \\ &\implies \mathcal{F} \text{ is bounded.} \end{aligned}$$

Now, we show the closeness of $\mathcal{F}$. Let $(T_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ such that $T_m \to T$. It is enough to show that $T \in \mathcal{F}$. We have:

$$\begin{aligned} T_m \to T &\implies ||T_m||_{op} \to ||T||_{op} \\ &\implies ||T_m|_{B_2^n}||_\infty \to ||T|_{B_2^n}||_\infty \\ &\implies T_m \to T \text{ uniformly on } B_2^n. \end{aligned}$$

Suppose there exists $x \in B_2^n$ such that $T(x) \notin \mathcal{F}$, that is, $T(x) \notin K$. This means that there exists $r > 0$ such that $B(T(x), r) \cap K = \emptyset$;

$$\begin{aligned} T_m \to T \text{ uniformly on } B_2^n &\implies T_m \to T \text{ pointwise on } B_2^n \\ &\implies T_m(x) \to T(x) \\ &\implies \exists M > 0 \ : \ \forall m \geq M \quad ||T_m(x) - T(x)|| < r \\ &\implies T_m(x) \in B(T(x), r), \end{aligned}$$

and we know that

$$\begin{aligned} T_m \in \mathcal{F} &\implies T_m(B_2^n) \subseteq K \\ &\implies T_m(x) \in K. \end{aligned}$$

Thus $T_m(x) \in B(T(x), r) \cap K$, a contradiction. Therefore for any $x \in B_2^n$, we get

$$\begin{aligned} T(x) \in K &\implies T(B_2^n) \subseteq K \\ &\implies T \in \mathcal{F} \\ &\implies \mathcal{F} \text{ is closed.} \end{aligned}$$

□

A-2 The function log det is strictly concave on M_n^{++}.

Proof. Let $T_1, T_2 \in M_n^{++}$. Since $\log\det$ is continuous, it is enough to show the midpoint concavity, that is, we must show that

$$\log\det\left(\frac{T_1+T_2}{2}\right) > \frac{\log\det T_1 + \log\det T_2}{2} = \log\sqrt{\det(T_1)\det(T_2)}.$$

If we denote $A = T_1^{-1/2} T_2 T_1^{-1/2} \in M_n^{++}$, then $\det(A) = \det(T_2)/\det(T_1)$. Let λ_i be the eigenvalues of A. Since log is concave, we have

$$\begin{aligned} \log\det\left(\frac{Id+A}{2}\right) &= \log\prod_{i=1}^{n}\left(\frac{1+\lambda_i}{2}\right) \\ &= \sum_{i=1}^{n}\log\left(\frac{1+\lambda_i}{2}\right) \\ &\geq \sum_{i=1}^{n}\frac{\log 1 + \log\lambda_i}{2} \\ &= \frac{1}{2}\sum_{i=1}^{n}\log\lambda_i \\ &= \frac{1}{2}\log\prod_{i=1}^{n}\lambda_i = \frac{1}{2}\log\det(A) = \log\sqrt{\frac{\det(T_2)}{\det(T_1)}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \det\left(\frac{T_1+T_2}{2}\right) &= \det(T_1)\det\left(\frac{Id+A}{2}\right) \\ &\geq \det(T_1)\log\sqrt{\frac{\det(T_2)}{\det(T_1)}} = \sqrt{\det(T_1)\det(T_2)}, \end{aligned}$$

and since log is strictly concave, there is equality if and only if $\lambda_i = 1$ for every i, that is, $T_1 = T_2$. If $T_1 \neq T_2$, then

$$\log\det\left(\frac{T_1+T_2}{2}\right) > \log\sqrt{\det(T_1)\det(T_2)},$$

and we obtain

$$\det\left(\frac{T_1+T_2}{2}\right) > \sqrt{\det(T_1)\det(T_2)}.$$

□

A-3 If $H \in M_n^{sa}$, that is $H^T = H$, and x is n-vector, then

$$Tr(Hxx^T) = x^T Hx.$$

Proof. If $H \in M_n^{sa}$ and x is n-vector, then the product Hx is also n-vector. Put $y = Hx$. Then:

$$\begin{aligned} Tr(Hxx^T) &= Tr(yx^T) \\ &= \sum_{i=1}^{n} y_i x_i \\ &= \langle y, x\rangle \\ &= y^T x = (Hx)^T x = x^T Hx. \end{aligned}$$

□

A-4 Suppose

$$\frac{1}{n}Tr(H) < \langle x, Hx\rangle$$

for some $H \in M_n^{sa}(\mathbb{R})$. If $G = H - \frac{1}{n}Tr(H)Id$, then

$$Tr(G) = 0 \quad \text{and} \quad \langle x, Gx\rangle > 0$$

for every $x \in \partial K \cap \partial B_2^n$.

Proof. Put $G = H - \frac{1}{n}Tr(H)Id$, then

$$\begin{aligned} Tr(G) &= Tr\left(H - \frac{1}{n}Tr(H)Id\right) \\ &= Tr(H) - \frac{1}{n}Tr(H)Tr(Id) \\ &= Tr(H) - \frac{1}{n}Tr(H)n \\ &= 0 \end{aligned}$$

and if $x \in \partial K \cap \partial B_2^n$, then

$$\begin{aligned}\langle x, Gx\rangle &= \left\langle x, \left(H - \frac{1}{n}Tr(H)Id\right)x\right\rangle \\ &= \left\langle x, Hx - \frac{1}{n}Tr(H)x\right\rangle \\ &= \langle x, Hx\rangle - \frac{1}{n}Tr(H)\underbrace{\langle x, x\rangle}_{1} \\ &= \langle x, Hx\rangle - \frac{1}{n}Tr(H) > 0.\end{aligned}$$

Therefore, we have

$$0 = \frac{1}{n}Tr(G) < \langle x, Gx\rangle,$$

for every $x \in \partial K \cap \partial B_2^n$. □

A-5 If $f(x) = |x|^2 - \|x\|_K^2$ and $g(x) = \langle x, Hx\rangle$, then $f \geq 0$ on ∂B_2^n, $g > 0$ on the set $\{x \in \partial B_2^n \ : \ f(x) = 0\}$, and $f + \delta g > 0$ on ∂B_2^n for δ small enough.

Proof. First, we show $f \geq 0$ on ∂B_2^n. We have

$$B_2^n \subseteq K = \{x \ : \ \|x\|_K \leq 1\},$$

and therefore

$$\begin{aligned}x \in \partial B_2^n &\implies |x| = 1 \text{ and } \|x\|_K \leq 1 \\ &\implies |x|^2 - \|x\|_K^2 = 1 - \|x\|_K^2 \geq 0. \\ &\implies f(x) \geq 0.\end{aligned}$$

Second, we show that $g > 0$ on the set $\{x \in \partial B_2^n \ : \ f(x) = 0\}$.

$$\begin{aligned}x \in \partial K \cap \partial B_2^n &\implies f(x) = |x|^2 - \|x\|_K^2 = 0 \\ &\implies g(x) = \langle x, Hx\rangle > \frac{1}{n}Tr(H) = 0 \\ &\implies g > 0 \text{ on } \{x \in \partial B_2^n \ : \ f(x) = 0\}.\end{aligned}$$

Finally we prove that $f + \delta g > 0$ on ∂B_2^n. Define a set

$$A := \{x \ : \ g(x) \leq 0\}.$$

Then A is compact since it is a closed subset of the compact set ∂B_2^n. By the above arguments, $g(x) > 0$, if $f(x) = 0$. Then for every $x \in A$, we have $f(x) \neq 0$. So $f > 0$ on A.

Let $m = \min\{f(x) \ : \ x \in A\}$. Then $m > 0$. So

$$\begin{aligned} \delta < \frac{m}{\|g\|_\infty} &\implies -\delta g(x) \leq \delta \|g\|_\infty < m \leq f(x) \quad \forall x \in A \\ &\implies f + \delta g > 0 \text{ on } \partial B_2^n. \end{aligned}$$

□

A-6 Suppose that (4.4) holds. Then we have

$$(n-i)\sum_{j=1}^{i} a_j^2 - i\sum_{j=i+1}^{n} a_j^2 \leq 0.$$

Proof. Suppose that

$$[(1+\varepsilon)^{n-i} - 1]\sum_{j=1}^{i} \alpha_j^2 + [(1+\varepsilon+\varepsilon^2)^{-i} - 1]\sum_{j=i+1}^{n} \alpha_j^2 \leq 0$$

holds where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the coordinates of p_ε By compactness, there is some x_{i+1} with $\|x_{i+1}\|_{\mathcal{E}} = \|x_{i+1}\|_C = 1$ such that $p_{\varepsilon_k} \to x_{i+1}$ where $\varepsilon_k \to 0$. If x_{i+1} has coordinates $a_1, a_2, \ldots, a_n$ where $a_m = 0$ for $i+1 < m \leq n$, then we have $\alpha_{j_k} \to a_j$ and so

$$[(1+\varepsilon_k)^{n-i} - 1]\sum_{j=1}^{i} \alpha_{j_k}^2 + [(1+\varepsilon_k+\varepsilon_k^2)^{-i} - 1]\sum_{j=i+1}^{n} \alpha_{j_k}^2 \leq 0.$$

If we divide each side by ε_k, we get

$$\frac{(1+\varepsilon_k)^{n-i} - 1}{\varepsilon_k}\sum_{j=1}^{i} \alpha_{j_k}^2 + \frac{(1+\varepsilon_k+\varepsilon_k^2)^{-i} - 1}{\varepsilon_k}\sum_{j=i+1}^{n} \alpha_{j_k}^2 \leq 0.$$

Since $\varepsilon_k \to 0$ and $\alpha_{j_k} \to a_j$, by the L'Hospital rule, we have

$$\lim_{k\to\infty}\left(\frac{(1+\varepsilon_k)^{n-i} - 1}{\varepsilon_k}\sum_{j=1}^{i} \alpha_{j_k}^2\right) = (n-i)\sum_{j=1}^{i} a_j^2,$$

and

$$\lim_{k\to\infty}\left(\frac{(1+\varepsilon_k+\varepsilon_k^2)^{-i} - 1}{\varepsilon_k}\sum_{j=i+1}^{n} \alpha_{j_k}^2\right) = (-i)\sum_{j=i+1}^{n} a_j^2.$$

Thus, the result follows. □

A-7 A series $\sum x_n$ converges unconditionally to x, if for every $\varepsilon > 0$, there exists a finite subset F of $\mathbb{N}$ such that for every $G \supseteq F$ finite, we have

$$\left\| \sum_{n \in G} x_n - x \right\| < \varepsilon.$$

A series $\sum x_n$ is unconditionally Cauchy if for every $\varepsilon > 0$, there exists a finite subset F of $\mathbb{N}$ such that for every $G \subseteq \mathbb{N}$ finite and $F \cap G = \emptyset$, we have

$$\left\| \sum_{n \in G} x_n - x \right\| < \varepsilon.$$

If a space B is complete, then unconditionally Cauchy series are unconditionally convergent.