SIMPLE CHAOTIC SYSTEMS

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ABSTRACT

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Jerk dynamics is third-order explicit autonomous differential equations, and an important tool to understand the properties of functionally very simple but nonlinear three-dimensional dynamical systems that can exhibit chaotic long time behavior. The subclass Newtonian jerky dynamics is a jerky dynamics that can be derived from one-dimensional Newton equation and constitute simple extensions of the well-known oscillator dynamics to the third-order differential equations that can be non-steady and non-periodic bounded long time dynamics. In this thesis, an algebraically simple system, the Genesio system is recast into a jerky dynamics and it is showed that its jerk function is a Newtonian jerky. Moreover, the global dynamical properties of the Genesio system are investigated as a jerky dynamics by numerical examinations.

Keywords: Chaos, Jerky dynamics, Newtonian jerky dynamics, Algebraically simple systems, Genesio system, Dynamical properties.

ÖZET

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Jerk dinamiği, üçüncü mertebeden açık otonom olan adi diferansiyel denklemlerdir ve fonksiyonel olarak çok basit olmasına rağmen üç boyutlu lineer olmayan dinamik sistemlerin kaotik özelliklerinin anlaşılmasında önemli bir araçtır. Newtonyen jerk dinamiği alt sınıfı, bir boyutlu Newton denkleminden elde edilen bir jerk dinamiğidir ve iyi bilinen salınıcı dinamiğinin uzun süre sınırlı ancak düzensiz ve periyodik olmayan davranışa sahip üçüncü mertebeden diferansiyel denklemlere genişletilmesini teşkil eder. Bu tezde cebirsel olarak basit bir yapıya sahip olan Genesio sistem bir jerk dinamiği olarak düzenlenmiştir ve elde edilen jerk fonksiyonunun Newtonyen jerk yapısında olduğu gösterilmiştir. Ayrıca bu sistemin global dinamik özellikleri, sayısal yöntemlerle bir jerk dinamiği yapısında incelenmiştir.

Anahtar Kelimeler: Kaos, Jerk dinamiği, Newton jerk dinamiği, Cebirsel olarak basit sistemler, Dinamik özellikler

To my family and my supervisor

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CHAPTER 1

INTRODUCTION

Some aspects of chaos have been known for over a hundred years. Isaac Newton was said to get headaches thinking about 3-body problem (Sun, Moon, and Earth). In 1890, King Oscar II of Sweden announced a prize for anyone who could solve the n-body problem and hence demonstrate stability of the solar system. This prize was awarded to Jules Henri Poincaré who show that even the 3-body problem has no solution, [1, 2]. He went on to deduce many of the properties of chaotic systems including the sensitive dependence on initial conditions. With the successes of linear models in the sciences and the lack of powerful computers, the work of these early nonlinear dynamists went largely unnoticed and undeveloped for many decades.

Following the seminal studies of Lorenz [3] and Rössler [4, 5] the investigation of the time evolution of dynamical systems in physics, mathematics and many other fields of science has been preferentially focused on the exploration and understanding of aperiodic or chaotic behavior within the last four decades. Although there is no commonly accepted [3] definition of chaotic behavior, it is widely accepted that, (i) a non-periodic bounded long-time evolution of a time-continuous dynamical system leading to a strange attractor in its phase space, and (ii) a sensitive dependence on the initial conditions being characterized by the appearance of at least one positive Lyapunov exponent are signatures of a chaotic dynamics. Despite the extensive work on chaos there are still many unsolved issues. One of them is: What are the necessary and sufficient requirements in a time-continuous, autonomous dynamical system to exhibit chaos?

The necessary requirements are well-known. By virtue of the Poincaré-Bendixon Theorem [6–10], chaotic behavior in a time-continuous, autonomous dynamical system requires at least three-dimensional phase-space and at least one nonlinearity as chaos is a typical nonlinear phenomenon. However, it is not definitely known even for the simplest case of three-dimensional dynamical systems as to what degree of nonlinearity is sufficient for the existence of chaos. Hence the minimal functional form of three-dimensional chaotic vector fields is not known and opens a largely unexplored field during the last decade.

In recent years, a number of workers [11–23] have attempt to find three-dimensional systems which are functionally as simple as possible but nevertheless chaotic. Using extensive computer search, Sprott [11] found 19 distinct chaotic models (one of them is conservative and the remaining are dissipative, referred as models A to S) with three-dimensional vector fields that consists of five terms including two nonlinearities or six terms with one quadratic nonlinearity. With these 19 chaotic systems, Sprott further concluded that out of 19 possible chaotic systems, some of the cases may not be distinct since it requires a tedious calculation to test the algebraic equivalence and further study is required for investigating the algebraic simplicity these systems. Subsequently, Hoover [12] pointed out that the only conservative model (model A) found by Sprott is an already known special case of Nosé-Hoover thermostat dynamical system, which exhibits time reversible Hamiltonian chaos [13]. Except these the other models B to S were apparently unknown. On the other hand, in 1996, Göttlieb [14] reported that Sprott's model A can be recast into an explicit third-order form

$$\dddot{x} = J(x, \dot{x}, \ddot{x}),$$

which he called "jerk function". Because it involves a third derivative of x, which in a mechanical system is a rate of change of acceleration, it is called jerk [15]. Since it is known that any explicit ordinary differential equation can be recast in the form of a system of coupled first-order differential equations, the converse does not hold in general. Even if one may reduce the dynamical system to a jerk form for each of the phase space variables, the resulting differential equation may look quite different, *i.e.*, there may be different possible jerk forms of a single dynamical system. With this, Göttlieb [14] asked a provocative question “What is the simplest jerk function that gives chaos?”. By following the study of Göttlieb, Linz [16] reported that the original Rössler model, Lorenz model and Sprott’s model R can be reduced to a jerk form and further showed that the jerk form of Rössler and Lorenz models are rather complicated and are not suitable candidates for the Göttlieb’s simplest jerk function. However, the jerk form of Sprott model R demonstrates the existence of a much simpler form of jerk function that exhibits chaos. Getting inspired with the Linz’s response to Göttlieb’s question, Sprott [17, 18] again employed an extensive search but this time for the jerk function (not for the three-dimensional vector fields) containing minimum number of terms and at most a quadratic function of $\ddot{x}$, $\dot{x}$ and x. As a result, Sprott found a variety of cases including two having a total of three terms with two quadratic nonlinearity and a particular case having three terms with a single quadratic nonlinearity

$$\dddot{x} = -A\ddot{x} + \dot{x}^2 + x$$

Sprott also showed that the dynamical system representation of the above system is simpler than any previous work. Sprott also reported that all the system discovered by him share a common route to chaos.

By following Linz’s study [16], Eichorn *et al* [19] used the method of Gröbner bases and showed that fifteen of Sprott’s chaotic flows [14] can be recast into a jerk form. They also showed that these fifteen models, the Rössler toroidal model [5] and Sprott’s minimal chaotic flow [17] can be arranged into seven classes (referred as JD_1

and JD_7) of jerky dynamics as a hierarchy of quadratic jerk equations with increasingly many terms. Such a classification provides simple means to compare the functional complexity of different systems and also demonstrate the equivalence of cases not otherwise apparent (which was left unresolved by Sprott [12]). In a subsequent study, Eichorn *et al* [20] examined the simple cases of JD_1 and JD_2 in more detail and identified the regions of parameter space over which they exhibit chaos.

Besides the work on the jerk functions including the polynomial nonlinearities, some authors [21–23] have considered the jerk function having piecewise nonlinearities. In such a study, Linz and Sprott [21] have made an extensive search for the algebraically simplest dissipative chaotic flow with absolute-value nonlinearity and discovered the case

$$\dddot{x} = -A\ddot{x} + \dot{x} + |x| - 1,$$

which shows the chaotic behavior. For $A = 0.6$, it is also shown that this system exhibits period-doubling route to chaos and resembles the quadratic jerk function.

It is evident from the discussion given above that the jerk dynamics is an interesting subclass of dynamical systems that can exhibit many major features of the regular and chaotic behaviors, hence the jerk dynamical systems can be considered as a powerful example to demonstrate a variety of dynamical behaviors in simple dynamical systems.

The subclass Newtonian jerky dynamics is a jerky dynamics that can be derived from one-dimensional Newtonian equation $m\ddot{x} = F$ by taking its derivative with respect to time. Newtonian jerky dynamics constitute simple extensions of the well-known oscillator dynamics to the third-order differential equations that can have non-steady and non-periodic bounded long-time dynamics. So, Newtonian jerky dynamics seem to be an interesting special case of three-dimensional dynamical systems that apparently already have many ingredients that lead to chaotic and non-chaotic behavior. Because they are natural generalizations of the dynamics of linear and nonlinear oscillators, they allow at least some intuitive and analytical insights into their dynamics.

In this thesis, we study the Genesio system [24] as a jerky dynamics. As a one of the simple dynamical system, the Genesio system has a single quadratic nonlinearity. We show that the Genesio system can be recast into a jerky dynamics and the resulting jerk function fits into the class JD_2. Its jerk function consists of four terms with one quadratic nonlinearity. It is also rooted in Newton's equation, that is, it is a Newtonian jerky dynamics. Moreover, we investigate the global dynamics of the Genesio system and show that it shares the common route to chaos as systems in the class JD_2.

CHAPTER 2

JERKY DYNAMICS

Jerky dynamics [25] is a simple form of differential equations with an explicit time dependence. Some one-dimensional jerky systems can exhibit chaos. Indeed, there are several simple examples of jerky dynamics that show chaotic behavior [11, 16, 17]. Jerky dynamics can also be found in several examples of nonmechanical disciplines of physics. The first example of a jerky chaotic third-order differential equation can be found in [26]. This example is a simple oscillator model of thermal convection. In fact, jerky dynamics are conceptually simpler than dynamical systems, and they show all major features of three-dimensional vector fields. In particular, they can be considered as the natural generalization of oscillator dynamics, and hence they might serve as a useful tool to obtain further insight into nonchaotic and chaotic behavior including the routes to chaos.

2.1 Basics

Generally speaking, an autonomous dynamical system is specified by a set of n coupled first-order, ordinary differential equations that are not explicitly dependent on time t. In particular, three-dimensional dynamical systems are specified by

$$\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x}) \tag{2.1}$$

where $\mathbf{x} = (x, y, z)^{\mathsf{T}}$ denotes a point in a three-dimensional phase space $\Gamma \subseteq \mathbb{R}^3$,

$\mathbf{V}(\mathbf{x})$, in general, the nonlinear vector field of the dynamical system. Specifying the initial conditions $\mathbf{x}(t=0)=\mathbf{x_0},\ \mathbf{x}(t)$ represent the orbit or trajectory of the dynamical systems (2.1) in the phase space. It is also a well-known fact [27] that any autonomous nth order ODE that is given in an explicit form can be recast into an n-dimensional dynamical system. In particular, third-order explicit ODEs

$$\dddot{x} = J(x,\dot{x},\ddot{x}) \tag{2.2}$$

can be immediately transformed into a dynamical system (2.1) by introducing, for example,

$$\dot{x} = v, \quad \dot{v} = a, \quad \dot{a} = J(x,v,a) \tag{2.3}$$

The first and second components of this system are algebraically very simple (in fact are the simplest ones for an effectively three-dimensional dynamical system) and are identical for all jerky dynamics. The third component consists of the, in general, nonlinear jerk function $J(x,v,a)$. After transforming a three-dimensional dynamical system to a jerky dynamics, solely the number and types of terms appearing in the jerk function reflect the complexity of the linear and nonlinear couplings between the different variables of the original dynamical system. In this sense, one can understand the result that different dynamical systems with the same total of terms and nonlinearities can lead to jerky dynamics with different number of terms and nonlinearities [19]. Therefore, the functional complexity of different dynamical systems can be compared more directly and more clearly if these systems are recast as jerky dynamics. From this point of view, functionally elementary jerky dynamics, that exhibit chaos, also constitute the simplest chaotic three-dimensional dynamical systems. The contrary, however, is generally not true.

Motivated by the mechanical interpretation of Eq. (2.2) as evolution equation for the rate of change of the acceleration or the jerk, a third-order ODE of the form (2.2) that is (i) autonomous and (ii) explicit is called a jerky dynamics. Obviously, jerky dynamics (2.2) are a restricted class of all third-order ODEs and also of all third-order dynamical systems (2.1). As a necessary, but not sufficient requirement for a nontrivial jerky dynamics that is not just the derivative of a second-order explicit autonomous ODE, the jerk function must depend explicitly on x. Under certain constraints, in particular, when the acceleration $\ddot{x}$ enters only linearly into the jerky dynamics, it can also be interpreted as the derivative of a one-dimensional Newtonian equation with a memory term that depends on the dynamical history of the motion [16].

The following classes of three-dimensional dynamical systems

$$\dot{x} = V_1(x,y,z) \tag{2.4a}$$

$$\dot{y} = V_2(x,y,z) \tag{2.4b}$$

$$\dot{z} = V_3(x,y,z) \tag{2.4c}$$

can be recast into an equivalent jerky dynamics (2.2) and a systematic transformation method can be determined to obtain Eq. (2.2) from Eqs. (2.4) if it exists at all. We call a jerky dynamics in the variable x, Eq. (2.2), equivalent to the dynamical system (2.1) or (2.4) if, for the same initial conditions, the signals $x(t)$ generated by Eqs. (2.2) and (2.4) are identical.

Definition 1. *(i) A jerky dynamics is dynamically equivalent to a dynamical system if, for the same initial conditions, both generate the same signal $x(t)$. (ii) A jerky dynamics is topologically equivalent to a dynamical system if, in additionally, any trajectory of the dynamical system corresponds to exactly one trajectory of the jerky dynamics if interpreted as a dynamical system, and vice versa.*

Definition 1 tells us that dynamical equivalence requires the existence of a unique and well-defined jerk function $J(x,\dot{x},\ddot{x})$. For topological equivalence there must also exist a globally invertible, (at least) continuous transformation between the dynamical system (2.4) and the jerky dynamics (2.2). If, moreover, this transformation is a diffeomorphism, *i.e.*, invertible and differentiable, the dynamical system (2.4) and the jerky dynamics (2.2) should be called *diffeomorphically equivalent*. Obviously, topological or diffeomorphic equivalence implies dynamical equivalence. The converse, however, is not true.

2.2 Transformable Dynamical Systems

As already mentioned, any jerky dynamics (2.2) can be rewritten in the form of a dynamical system

$$\dot{\mathbf{u}} = \mathbf{W}(\mathbf{u}) \tag{2.5}$$

by introducing $\mathbf{u} = (x,v,a)^\mathsf{T}$ and $\mathbf{W}(\mathbf{u}) = [v,a,J(x,v,a)]^\mathsf{T}$. If there is a jerky dynamics (2.2) or, equivalently, a dynamical system (2.5) for the system (2.4), then there must also be a transformation $\mathbf{T} = (T_1,T_2,T_3)^\mathsf{T} : \mathbf{x} \to \mathbf{u}$ of variables

$$\mathbf{u} = \mathbf{T}(\mathbf{x}) \tag{2.6}$$

that converts the original dynamical system (2.4) to the dynamical system (2.5). From Eqs. (2.4), (2.5), (2.6) we can read off the components of $\mathbf{T}$,

$$T_1(\mathbf{x}) = x, \tag{2.7a}$$

$$T_2(\mathbf{x}) = V_1(\mathbf{x}), \tag{2.7b}$$

$$T_3(\mathbf{x}) = \dot{\mathbf{x}} \cdot \nabla V_1(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \cdot \nabla V_1(\mathbf{x}) \tag{2.7c}$$

with $\nabla = (\partial_x, \partial_y, \partial_z)^{\mathsf{T}}$. Obviously, the transformation $\mathbf{T}$ depends on the structure of the vector field $\mathbf{V}(\mathbf{x})$ of the original dynamical system (2.4). It can be calculated for any arbitrary vector field. In order to find an equivalent jerky dynamics from $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$, however, there must also be a uniquely determined inverse transformation $\mathbf{T}^{-1} = (T_1^{-1}, T_2^{-1}, T_3^{-1})^{\mathsf{T}} : \mathbf{u} \to \mathbf{x}$ given by

$$\mathbf{x} = \mathbf{T}^{-1}(\mathbf{u}) \tag{2.8}$$

such that $\mathbf{T}^{-1}$ maps the systems (2.5) onto (2.4). By virtue of Eqs. (2.7), the condition of invertibility of $\mathbf{T}$ is effectively a constraint on the general form of the vector field $\mathbf{V}(\mathbf{x})$, and, therefore, defines the dynamical systems that possess an equivalent jerky dynamics.

In the following, it proves convenient to distinguish explicitly between the linear and nonlinear parts of the vector field. Therefore, we write it in the general form

$$\mathbf{V}(\mathbf{x}) = \mathbf{c} + \mathsf{B}\mathbf{x} + \mathbf{n}(\mathbf{x}) \tag{2.9}$$

where $\mathbf{c} \in \mathbb{R}^3$ is a vector of constants, $\mathsf{B} \in \mathbb{R}^{3\times 3}$ a matrix with constant coefficients, $b_{ij} \quad (i, j = 1, 2, 3)$ and $\mathbf{n}(\mathbf{x}) = [n_1(\mathbf{x}), n_2(\mathbf{x}), n_3(\mathbf{x})]^{\mathsf{T}}$ a three-dimensional vector of solely nonlinear functions in x, y, z that are at least twice differentiable and do not contain additive constants. Then the following holds.

Theorem 2.1. *Any dynamical system of the functional form*

$$\dot{x} = c_1 + b_{11}x + b_{12}y + b_{13}z + n_1(x) \tag{2.10a}$$

$$\dot{y} = c_2 + b_{21}x + b_{22}y + b_{23}z + n_2(\mathbf{x}) \tag{2.10b}$$

$$\dot{z} = c_3 + b_{31}x + b_{32}y + b_{33}z + n_3(\mathbf{x}) \tag{2.10c}$$

with $n_i \quad (i = 1,2,3)$ being nonlinear functions of the indicated arguments can be reduced to a jerky dynamics, $\dddot{x} = J(x, \dot{x}, \ddot{x})$, if the conditions

$$b_{12} n_2(\mathbf{x}) + b_{13} n_3(\mathbf{x}) = f(x, b_{12} y + b_{13} z) \tag{2.11a}$$

with f being an arbitrary function of the indicated arguments and

$$b_{12}^2 b_{23} - b_{13}^2 b_{32} + b_{12} b_{13} (b_{33} - b_{22}) \neq 0 \tag{2.11b}$$

hold.

Proof. *The demonstration of the theorem requires three steps : the calculation of the transformation* $\mathbf{T}$ *and its inverse* $\mathbf{T}^{-1}$ *and, finally, the derivation of the jerky dynamics. The transformation* $\mathbf{T}$ *can immediately be obtained from the dynamical system (2.10) by virtue of Eqs. (2.7),*

$$T_1(\mathbf{x}) = x \tag{2.12a}$$

$$T_2(\mathbf{x}) = c_1 + \mathbf{b}^1 \cdot \mathbf{x} + n_1(x) \tag{2.12b}$$

$$\begin{aligned} T_3(\mathbf{x}) = {} & \mathbf{c} \cdot \mathbf{b}^1 + \mathbf{b}^1 \cdot \mathbf{b}_1 x + \mathbf{b}^1 \cdot \mathbf{b}_2 y + \mathbf{b}^1 \cdot \mathbf{b}_3 z + \\ & + \mathbf{b}^1 \cdot \mathbf{n}(\mathbf{x}) + [c_1 + \mathbf{b}^1 \cdot \mathbf{x} + n_1(x)] \partial_x n_1(x) \end{aligned} \tag{2.12c}$$

For convenience, we have introduced in Eqs. (2.12) the notation

$$\mathbf{b}^i = (b_{i1}, b_{i2}, b_{i3})^T \qquad (i = 1,2,3) \tag{2.13a}$$

for the row vectors and

$$\mathbf{b}_j = (b_{1j}, b_{2j}, b_{3j})^T \qquad (j = 1, 2, 3)$$

for the column vectors of the matrix $B = (b_{ij})$ introduced in Eq. (2.9). The dot denotes the scalar product.

To calculate the inverse transformation $\mathbf{T}^{-1}$, we have to solve Eqs. (2.12) with respect to x,y, and z. Since $x = T_1(\mathbf{x})$ and Eq. (2.12a) hold and, therefore, T_1 maps x only onto itself, solely the second and third components of $\mathbf{T}$ and the variables y and z need to be considered while x can be handled like a simple parameter. To solve Eqs. (2.12b) and (2.12c) with respect to y and z, both variables should enter only linearly into T_2 and T_3. In T_3, however, nonlinear terms are present that contain y and z. Since we want to determine y and z as functions of $\mathbf{u}$, this does not cause a problem if these nonlinearities can be replaced by terms of $\mathbf{u}$. This explains why one has to demand the condition (2.11a) since in this case the part $\mathbf{b^1} \cdot \mathbf{n}(\mathbf{x})$ of Eq. (2.12c) can be written as $b_{11}n_1(x) + f(x, b_{12}y + b_{13}z)$. Using Eq. (2.12b) and $v = T_2(\mathbf{x})$, the second argument of f can be substituted by an expression that solely depends on x and v. Next, again using $v = T_2(\mathbf{x})$ and Eq. (2.12b) we can rewrite the second line of Eq. (2.12c) as $v\partial_x n_1(x)$. As a consequence, T_3 depends linearly on y and z. Taking into account Eq. (2.6), the second and third component of the transformation $\mathbf{T}$, (2.12b) and (2.12c), can therefore be written as

$$\begin{pmatrix} v \\ a \end{pmatrix} = \begin{pmatrix} r(x) \\ s(x,v) \end{pmatrix} + M \begin{pmatrix} y \\ z \end{pmatrix} \tag{2.14}$$

with the abbreviations

$$r(x) = c_1 + b_{11}x + n_1(x), \tag{2.15a}$$

$$s(x,v) = \mathbf{c} \cdot \mathbf{b^1} + \mathbf{b^1} \cdot \mathbf{b_1}x + v\partial_x n_1(x) + b_{11}n_1(x) + f(x, v - r(x)) \tag{2.15b}$$

and the matrix

$$M = \begin{pmatrix} b_{12} & b_{12} \\ \mathbf{b^1} \cdot \mathbf{b_2} & \mathbf{b^1} \cdot \mathbf{b_3}. \end{pmatrix} \tag{2.16}$$

Therefore, the problem of calculating $\mathbf{T}^{-1}$ *is reduced to a simple matrix inversion. The condition that is necessary for invertibility of* M *is given by* $det M = b_{12}^2 b_{23} - b_{13}^2 b_{32} + b_{12}b_{13}(b_{33} - b_{22}) \neq 0$. *This is exactly the condition (2.11b). Since we require that Eq. (2.11b) holds, it follows that the inverse of* M *and,therefore, also the inverse transformation* $\mathbf{T}^{-1}$ *exist. Consequently, from Eq.(2.14) one can finally calculate* $\mathbf{T}^{-1}$ *by additionally taking into account Eq. (2.8). The result reads*

$$T_1^{-1}(\mathbf{u}) = x, \tag{2.17a}$$

$$\begin{aligned} T_2^{-1}(\mathbf{u}) = \{&\mathbf{c} \cdot \mathbf{b^1} b_{13} - c_1 \mathbf{b^1} \cdot \mathbf{b_3} + (b_{12}A_{32} - b_{13}A_{22})x \\ &- \left[\mathbf{b^1} \cdot \mathbf{b_3} + b_{13}\partial_x n_1(x)\right] v - b_{13}a - (b_{12}b_{23} \\ &+ b_{13}b_{33})n_1(x) + b_{13}f(x, v - r(x))\}/det M, \end{aligned} \tag{2.17b}$$

$$\begin{aligned} T_3^{-1}(\mathbf{u}) = \{&c_1 \mathbf{b^1} \cdot \mathbf{b_2} - \mathbf{c} \cdot \mathbf{b^1} b_{12} + (b_{12}A_{33} - b_{13}A_{23})x \\ &+ \left[\mathbf{b^1} \cdot \mathbf{b_2} + b_{12}\partial_x n_1(x)\right] v + b_{12}a + (b_{12}b_{22} \\ &+ b_{13}b_{32})n_1(x) - b_{12}f(x, v - r(x))\}/det M, \end{aligned} \tag{2.17c}$$

where A_{ij} *denotes the cofactor to the element* b_{ij} *of the matrix* B.

The general form of the jerky dynamics corresponding to the dynamical system (2.10) can be obtained by the following procedure. The derivative of the third compo-

nent of Eq. (2.6) with respect to time reads

$$\dot{a} = \dddot{x} = \mathbf{V}(\mathbf{x}) \cdot \nabla T_3(\mathbf{x}). \tag{2.18}$$

Since $\dot{a} = \dddot{x} = J(\mathbf{u})$ holds, Eq. (2.18) yields the jerk function $J(\mathbf{u})$. The expression (2.18), however, still depends on y and z, so that we must insert the inverses $y = T_2^{-1}(\mathbf{u})$ and $z = T_3^{-1}(\mathbf{u})$ to obtain J as a function of $\mathbf{u}$. A straightforward, but somewhat tedious calculation then leads to the final result for the jerky dynamics. Using the matrix $A = (A_{ij})$ (with the cofactor A_{ij} of B) and accordingly to Eqs. (2.13) defined row and column vectors $\mathbf{A^i}$ and $\mathbf{A_j}$, it reads

$$J(\mathbf{u}) = g(x,v)a + h(x,v)v + k(\mathbf{u}) \tag{2.19}$$

with

$$g(x,v) = trB + \partial_x n_1(x) + f'[x, v - r(x)], \tag{2.20a}$$

$$h(x,v) = -trA - (b_{22} + b_{33})\partial_x n_1(x) + \partial_x f(x, v - r(x)) + v\partial_x^2 n_1(x) + [b_{11} + \partial_x n_1(x)] f'(x, v - r(x)), \tag{2.20b}$$

$$k(\mathbf{u}) = \mathbf{c} \cdot \mathbf{A_1} + \mathbf{b^1} \cdot \mathbf{A^1} x + \mathbf{A_1} \cdot \mathbf{n}(x, T_2^{-1}(\mathbf{u}), T_3^{-1}(\mathbf{u})), \tag{2.20c}$$

and $\partial_x f$ denoting the derivative with respect to x only of the first argument of f, f' the derivative with respect to the second argument of f and tr the trace of a matrix. If the nonlinear functions n_2 or n_3 in $k(\mathbf{u})$ depend on y or z, one has to insert the inverse transformations $y = T_2^{-1}(\mathbf{u})$ and $z = T_3^{-1}(\mathbf{u})$, Eqs. (2.17b) and (2.17c). This completes the proof. ■

As we see from Theorem (2.1), the transformations used have the restriction that the variable in the scalar differentiable equation is the same as one of the state variables of the three-dimensional system. A consequence is that a linear transformation is sometimes not possible; the resulting transformation is nonlinear. We now show that by removing this restriction, these models can be transformed to jerky dynamics via an affine transformation.

Definition 2. *Let B be an $n \times n$ matrix and $\mathbf{b}$ be an n by 1 vector. The pair $(B, \mathbf{b})$ is controllable if the matrix*

$$K = K(B, \mathbf{b}) = (\mathbf{b} | B\mathbf{b} | B^2\mathbf{b} | \ldots | B^{n-1}\mathbf{b})$$

is nonsingular. The matrix K is called the controllability matrix.

Theorem 2.2. *If the nonlinear part of the system (2.9) has the form, $n(\mathbf{x}) = \mathbf{b} f(\mathbf{x})$ where $\mathbf{b}$ is n by 1 vector and f is a real-valued function and if $(B, \mathbf{b})$ is controllable then the system*

$$\mathbf{x} = \mathbf{c} + B\mathbf{x} + \mathbf{b} f(\mathbf{x}) \tag{2.21}$$

is topologically conjugate to a scalar ordinary differential equation via an affine transformation.

Proof. *It is a well-known result in linear systems theory that there exists a nonsingular matrix T such that*

$$TAT^{-1} = \begin{pmatrix} 0 & 1 & & \\ \ddots & \ddots & & \\ & & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-1}, \end{pmatrix}$$

$$T\mathbf{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1, \end{pmatrix}$$

provided $(A, \mathbf{b})$ *is controllable and the characteristic polynomial of* A *is* $\lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_0$. *In particular,* T *can be found using the following equation :*

$$T^{-1} = K \begin{pmatrix} p_1 & p_2 & \ldots & p_{n-1} & 1 \\ p_2 & p_3 & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ p_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Let $T\mathbf{c} = (t_1, t_2, \ldots, t_n)^T$. *Define the vector* v *as* $v = (0, t_1, \ldots, t_{n-1})^T$. *Applying the state transformation* $\omega = Tx + v$, *we obtain the transformed system:* $\dot{\omega} = TAT^{-1}\omega + T\mathbf{b}f(T^{-1}(\omega - v)) + T\mathbf{c} - TAT^{-1}v$. *Since* $T\mathbf{c} - TAT^{-1}v$ *is of the form* $(0, \ldots, 0, \sigma)^T$, *the transformed system can be rewritten as a scalar ordinary differential equation.* ■

2.3 Minimal Chaotic Flows

The nonlinear dynamical systems A to S found by Sprott and a toroidal Rössler system are minimal dynamical systems that can show chaotic behavior for some parameter range where minimal is understood in an algebraic sense. Models A to E have only five terms with two quadratic nonlinearities and models F to S and TR have six terms with one quadratic nonlinearity. Moreover, Sprott also has found dynamical systems with five terms and only one quadratic nonlinearity that are chaotic in a certain parameter range [16, 17]. These models are already given in form of a jerky dynamics. Zhang and Heidel [28] have shown that three-dimensional dissipative quadratic systems with less than five terms cannot exhibit chaotic behavior.

Using the results of Sec. 2.2, we infer that the models A to C and E do not belong to the class of dynamical systems (2.10). On the other hand, we can analytically calculate all existing equivalent jerky dynamics for each of the systems D and F to S and TR.

The resulting jerky dynamics as well as the corresponding transformations $\mathbf{T}$ and their inverses $\mathbf{T}^{-1}$ are given in Table (5.1). In [19], the method of comprehensive Gröbner bases has been used to verify whether there are additionally equivalent jerky dynamics that are not contained in the class (2.10), and, simultaneously to check the analytical results. It turns out that all existing jerky dynamics are of the type being described by Eqs. (2.10).

The entry "none" for the models A, B, C and E means that there are no equivalent jerky dynamics for these systems in the sense that the jerk function is a nonsingular and polynomial expression. Therefore, only one of the models with two nonlinearities, namely, system D, can be converted into an equivalent jerky dynamics. This is mainly due to the fact that two of the total of five terms contained in these models are nonlinear. This leads, in general, to a lack of sufficiently many linear terms that are necessary for the existence of the inverse transformation. On the other hand, any system with six terms and only one nonlinearity (F to S and TR) can be recast into at least one jerky dynamics. The models F, I, and L possess even two.

From Table (5.1) one can see that most of the transformations are linear (models D, F (the jerky dynamics in x), G, I, L, M, O, Q, R, S, and TR), solely one contains a cubic term (model P) and the remaining are quadratic. In contrast to that, all derived jerky dynamics have only one quadratic nonlinearities. Moreover, it is interesting that not all of the six possible combinations of x, $\dot{x}$, $\ddot{x}$ appear in the models. Terms like $\dot{x}\ddot{x}$ and $\ddot{x}^2$ are missing. The absence of the $\ddot{x}^2$ term means that all the models are even Newtonian jerky [20].

2.3.1 Dynamical Systems with two or three jerky dynamics

To obtain dynamical systems of the class (2.10) with two simultaneously existing jerky dynamics, *e.g.*, in x and y, one has to restrict the nonlinear function $n_2(\mathbf{x})$ such that it is only function of y, $n_2(\mathbf{x}) = n_2(y)$. This follows directly from Eqs. (2.10). In addition to the conditions (2.11),

$$b_{12}n_2(y) + b_{13}n_3(\mathbf{x}) = f_1(x, b_{12}y + b_{13}z) \tag{2.22a}$$

$$b_{12}^2 b_{23} - b_{13}^2 b_{32} + b_{12}b_{13}(b_{33} - b_{22}) \neq 0 \tag{2.22b}$$

that ensure the existence of the jerky dynamics in x, there are also corresponding contraints for the jerky dynamics in y that read explicitly

$$b_{21}n_1(x) + b_{23}n_3(\mathbf{x}) = f_2(y, b_{21}x + b_{23}z) \tag{2.23a}$$

$$b_{23}^2 b_{31} - b_{21}^2 b_{13} + b_{21}b_{23}(b_{11} - b_{33}) \neq 0 \tag{2.23b}$$

where f_1 and f_2 are functions of the indicated arguments. Any dynamical system of functional form (2.10) with $n_2(\mathbf{x}) = n_2(y)$ that fulfills the conditions (2.22a-b) and (2.23a-b) can be recast into an equivalent jerky dynamics in its variables x and y. For simultaneously existing jerky dynamics in two other variables one has to take into account permutations of variables and indices, respectively.

For dynamical systems that possess simultaneously three jerky dynamics, further constraints apply. Clearly, $n_3(\mathbf{x}) = n_3(z)$ must hold. Furthermore, in addition to Eqs. (2.22a-b) and (2.23a-b) there is a third condition reading explicitly

$$b_{31}n_1(x) + b_{32}n_2(y) = f_3(z, b_{31}x + b_{32}y) \tag{2.24a}$$

$$b_{31}^2 b_{12} - b_{32}^2 b_{21} + b_{31}b_{32}(b_{22} - b_{11}) \neq 0 \tag{2.24b}$$

Combining the conditions (2.22a-b), (2.23a-b), and (2.24a-b), one obtains two distinct dynamical systems that simultaneously possess three equivalent jerky dynamics, namely, first

$$\dot{x} = c_1 + b_{11}x + b_{12}y + b_{13}z + n_1(x) \tag{2.25a}$$

$$\dot{y} = c_2 + b_{21}x + b_{22}y \tag{2.25b}$$

$$\dot{z} = c_3 + b_{31}x + b_{33}z \tag{2.25c}$$

with the constraints

$$b_{12} \neq 0, \quad b_{13} \neq 0, \quad b_{21} \neq 0, \quad b_{31} \neq 0 \tag{2.26}$$

and second

$$\dot{x} = c_1 + b_{11}x + b_{12}y + n_1(x) \tag{2.27a}$$

$$\dot{y} = c_2 + b_{22}y + b_{23}z + n_2(y) \tag{2.27b}$$

$$\dot{z} = c_3 + b_{31}x + b_{33}z + n_3(z) \tag{2.27c}$$

with the constraints

$$b_{12} \neq 0, \quad b_{23} \neq 0, \quad b_{31} \neq 0 \tag{2.28}$$

In general, for both systems (2.25) and (2.27) one should also consider permutations of variables and indices, respectively. Therefore, there are dynamical systems with nonlinearities in each component of the vector field that possess three equivalent jerky dynamics. Even if a three-dimensional dynamical system can be transformed to a jerky dynamics in each of its variables x, y, and z, the resulting three scalar differential equations are, at least in general, not of the same functional form.

2.3.2 Relations between the jerky dynamics

Comparing the jerky dynamics of Table (5.1), one observes that some of them look rather similar as far as the functional form of the jerk functions is concerned. Consider *e.g.*, the jerky dynamics of system J and the jerky dynamics in y of model I. Both consist of the same terms; they only differ with respect to their coefficients. Also the jerky dynamics of model L (in x) and of N are of the same functional form apart from an additional nonzero constant. The system F and H possess even identical jerky dynamics. Furthermore, there are three models (F, I, and L) that possess two equivalent jerky dynamics that contain similar terms. These observations raise the question of whether there are relations between different (but similar) jerky dynamics, or if there are even invertible transformations that map onto each other. To discuss this point in more detail, we consider a transformation $\mathbf{T}_J : (x, \dot{x}, \ddot{x}) \to (\xi, \dot{\xi}, \ddot{\xi})$ with

$$\xi = T_{J,1}(x, \dot{x}, \ddot{x}) \tag{2.29a}$$

$$\dot{\xi} = T_{J,2}(x, \dot{x}, \ddot{x}) \tag{2.29b}$$

$$\ddot{\xi} = T_{J,3}(x, \dot{x}, \ddot{x}) \tag{2.29c}$$

that maps a jerky dynamics $\dddot{x} = J_x(x, \dot{x}, \ddot{x})$ onto another jerky dynamics $\dddot{\xi} = J_\xi(\xi, \dot{\xi}, \ddot{\xi})$. Such a transformation is completely determined by its first component $T_{J,1}$ and the jerk function J_x. The second and third components $T_{J,2}$ and $T_{J,3}$ are only derivatives of $T_{J,1}$ with respect to time and the jerk function J_x must be used to substitute $\dddot{x}$ terms that appear after each derivation. Therefore, the transformation (2.29a-c) contains only one independent component that can be chosen as $\xi = T_{J,1}(x, \dot{x}, \ddot{x})$. This property and the condition of invertibility strongly restrict the class of possible transformations (2.29a-c) between different jerky dynamics. Therefore, it is only possible for very special cases to convert similar jerky dynamics to each other by invertible transformations.

However, different jerky dynamics that belong to the same dynamical systems can always be transformed to each other by an invertible transformation. This follows from the fact that the jerky dynamics themselves are obtained from the dynamical system via invertible transformations. For illustrations, assume that we have a dynamical system that possesses two equivalent jerky dynamics (like the models F, I , and L) for its variables, say, x and y. Suppose that the jerky dynamics are calculated from the dynamical system by the invertible transformations $\mathbf{T}_x : (x,y,z) \to (x,\dot{x},\ddot{x})$ and $\mathbf{T}_y : (x,y,z) \to (y,\dot{y},\ddot{y})$. Then, we can immediately write down the transformation $\mathbf{T}_J$ that maps the jerky dynamics in x to the one in y as a combination of $\mathbf{T}_x^{-1}$ and $\mathbf{T}_y$, $\mathbf{T}_J = \mathbf{T}_y \circ \mathbf{T}_x^{-1}$. Moreover, since we have $y = T_{J,1}(x,\dot{x},\ddot{x})$ and also $y = T_{x,2}^{-1}(x,\dot{x},\ddot{x})$, one can immediately read off the characteristic first component of $\mathbf{T}_J$ from $\mathbf{T}_x^{-1}$. Analogous arguments are valid for the inverse $\mathbf{T}_J^{-1}$.

2.3.3 Classification of simple chaotic jerky dynamics

To discuss the relations of jerky dynamics that belong to different dynamical systems, we consider the linear transformation

$$\xi = k(x+c), \quad \dot{\xi} = k\dot{x}, \quad \ddot{\xi} = k\ddot{x} \tag{2.30}$$

(with $k, c \in R,$ and $k \neq 0$) that moves the origin and simultaneously rescales variables. This transformation is the only one that (i) converts a jerky dynamics (for a real variable x) into another equivalent jerky dynamics (for a new dynamical variable ξ), (ii) is invertible and, (iii) independent of the specific form of the jerk function $J_x(x,\dot{x},\ddot{x})$. In general Eq. (2.30) does not convert the different jerky dynamics of Table (5.1) to each other, but transforms them to seven basic classes of jerky dynamics that differ by the type and the number of terms appearing in the corresponding seven jerky functions. Transformations between these different classes have not been found. In Table

(5.2) the basic classes (denoted by JD_1 to JD_7) are listed as well as the models that belong to each class. Moreover the concrete realization of the first component of the transformation (2.30) for each model is given.

The jerky dynamics that belong to the same basic class are not necessarily identical. Their jerky functions do consist of the same functional form, but, in general, with different parameters or combinations of parameters as coefficients. Using the values of the parameters α, β, and γ for which Sprott found chaotic behavior [11], one can easily determine values of the coefficients k_i of the basic jerky dynamics JD_1 to JD_7 that lead to a chaotic dynamics. These values of the coefficients are also shown in Table (5.2).

It is interesting that the transformations that convert the models of Table (5.1) to the, in general, algebraically simpler basic classes of jerky dynamics generate more complicated dynamical systems with a larger total number of terms on the right-hand side of the basic jerky dynamics of Table (5.2) varies from four to seven and the number of nonlinear terms from one to four although all the corresponding minimal dynamical systems (except from model D) have the same number of terms and nonlinearities. Therefore, simplicity of the dynamical systems does, at least in general, not correspond to simplicity of the equivalent jerky dynamics.

2.3.4 Conditions that exclude chaos

Looking at the functional form of a jerky dynamics $\dddot{x} = J(x, \dot{x}, \ddot{x})$, it is highly nontrivial to decide whether it can have chaotic solutions for some parameter ranges or not. On a pragmatic level, chaotic dynamics means that the long time evolution of the underlying system is (i) bounded, *i.e.*, $|x(t)| < \infty$ for all t, and (ii) neither a fixed point nor a periodic or quasiperiodic solution.

For some subclasses of jerky dynamics, however, one can derive a simple criterion under what circumstances aperiodic or chaotic solutions cannot appear. Consider the following integro-differential equation

$$\ddot{x} + \Omega(x,\dot{x}) = \int^t f(x(\tau),x'(\tau),x''(\tau))d\tau \tag{2.31}$$

with Ω and f being differentiable functions with respect to their arguments and the prime denoting the derivative with respect to τ. Taking the time derivative of Eq. (2.31), one obtains a jerky dynamics

$$\dddot{x} + p(x,\dot{x})\ddot{x} + q(x,\dot{x},\ddot{x}) = 0 \tag{2.32}$$

with

$$p(x,\dot{x}) = \partial_{\dot{x}}\Omega(x,\dot{x}) \tag{2.33a}$$

$$q(x,\dot{x},\ddot{x}) = \dot{x}\partial_x\Omega(x,\dot{x}) - f(x,\dot{x},\ddot{x}) \tag{2.33b}$$

In turn, any jerky dynamics $\dddot{x} = J(x,\dot{x},\ddot{x})$ can be recast in the functional form (2.32). Moreover, if $p(x,\dot{x})$ and $q(x,\dot{x},\ddot{x})$ are integrable functions with respect to their arguments, it can also be written in form of Eq. (2.31). Then the following holds.

Theorem 2.3. *Any jerky dynamics (2.32) with integrable $p(x,\dot{x})$ and $q(x,\dot{x},\ddot{x})$ cannot show chaotic behavior if $f(x,\dot{x},\ddot{x})$ is either a positive or a negative semidefinite function for all x, $\dot{x}$ and $\ddot{x}$.*

Proof. *To demonstrate the statement, we first write Eq. (2.31) as*

$$\ddot{x} + \Omega(x,\dot{x}) = h \tag{2.34a}$$

$$\dot{h} = f(x,\dot{x},\ddot{x}) \tag{2.34b}$$

The condition that $f(x,\dot{x},\ddot{x})$ is positive (or negative) semidefinite for all x, $\dot{x}$ and $\ddot{x}$ and, *therefore, also for all t, implies that* $\dot{h} \geq 0$ $\left[or\ \dot{h} \leq 0\right]$ *holds for all t. Consequently,* $h(t)$ *is a monotonically increasing (or decreasing) function of t. In the long-time limit, the modulus of* $h(t)$ *can only attain zero, a finite nonzero constant or infinity.*

If $\lim_{t\to\infty} |h(t)| = C < \infty$ *holds, the time evolution of Eq. (2.34a) reduces to an effectively second-order dynamics*

$$\ddot{x} + \Omega(x, \dot{x}) = \pm C, \tag{2.35}$$

in the long-time limit $\lim t \to \infty$*. By virtue of the Poincaré-Bendixson theorem, the time evolution of Eq. (2.35) can only approach a fixed point (including infinity) or be periodic.*

If $\lim_{t\to\infty} |h(t)| = \infty$ *holds, the time evolution of Eq. (2.34a) eventually escapes to infinity. Fixed points and bounded solutions cannot be attained, since the left-hand side of Eq. (2.34a) also has to diverge. Therefore, the proof is complete.* ■

Two remarks are in order. (i) The theorem generalizes a previously presented theorem in [11] in two respects. (a) It does not require the boundedness of $\ddot{x} + \Omega(x, \dot{x}) = 0$. (b) It is not restricted to Newtonian jerky dynamics, *i.e.*, f is also allowed to depend on $\ddot{x}$. (ii) Under the conditions stated in the theorem, not only chaotic solutions are excluded, but also quasiperiodic and even period-doubling solutions cannot exist in the long-time limit.

For the seven basic classes of jerky dynamics JD_1 to JD_7 listed in Table (5.2), it will be shown that chaotic dynamics is excluded for some ranges of the parameters k_i.

According to the above discussion, a jerky dynamics can, in general, be written as an integro-differential equation

$$\ddot{\xi} + \Omega(\xi, \dot{\xi}) = \int^t f(\xi(\tau), \xi'(\tau), \xi''(\tau)) d\tau \tag{2.36}$$

where the prime denotes the derivative with respect to τ. It can be proven that the jerky dynamics that underlines, Eq. (2.36) cannot exhibit irregular dynamics if the $f(\xi,\xi',\xi'')$ of the memory term is either positive semidefinite or negative semidefinite for all ξ,ξ' and ξ''.

Since all the models JD_1 to JD_2 do not contain $\ddot{x}$ term, the integrand f of Eq. (2.36) does not depend on ξ'', *i.e.*, $f = f(\xi,\xi')$. For the model JD_1 the function f reads

$$f(\xi,\xi') = k_2\xi + k_3 \tag{2.37}$$

and is therefore also independent of ξ'. Taking into account the above statement, we can immediately infer that chaotic dynamics cannot appear for $k_2 = 0$. For k_3 no condition can be given; in particular, $k_3 = 0$ is not excluded. In fact, Sprott's simplest dissipative chaotic jerky dynamics *SJ* (cf Table (5.2)) serves as example for a JD_1 model with $k_3 = 0$ that can exhibit chaotic dynamics.

In this context, it is interesting that the jerk model

$$\dddot{\xi} = k_1\ddot{\xi} + k_2\dot{\xi} + \xi\dot{\xi} + k_3 \tag{2.38}$$

cannot show irregular behavior at all. This also follows from the Theorem 2.3. Eq. (2.38) is very similar to the basic model JD_1; the term linear in ξ has only been substituted by a term linear in $\dot{\xi}$.

For the remaining basic models JD_2 to JD_7 the structure of the jerky dynamics has been chosen such that the integrand of the memory term is of the functional form

$$f(\xi,\dot{\xi}) = A\xi^2 + B\xi'^2 + C \tag{2.39}$$

The parameters A, B, and C for each model are determined by the coefficients of the nonlinear terms ξ^2, $\dot{\xi}^2$, and $\xi\ddot{\xi}$ and by the additive constant of the underlying jerky dynamics. If the corresponding terms are not present A, B, and/or C are zero. From the specific form (2.39) of $f(\xi,\xi')$ we obtain conditions on the (relative) signs of the coefficients A, B, and C that exclude chaotic behavior of the corresponding jerky dynamics. In particular, irregular dynamics cannot appear if either $A \geq 0,\ B \geq 0,\ C \geq 0$ or $A \leq 0,\ B \leq 0,\ C \leq 0$ hold simultaneously. For the jerky dynamics JD_2, *e.g.*, we have $A = 1,\ B = 0,\ C = k_3$ and we can infer that for $k_3 > 0$ chaotic dynamics cannot appear. This condition translates into conditions for the parameters α and β of each model that is contained in the class JD_2. For the model M, *e.g.*, we obtain $-\alpha - \frac{1}{4}\beta^2 > 0$. Similar considerations hold for the other basic classes.

CHAPTER 3

NEWTONIAN JERKY DYNAMICS

A Newtonian jerky dynamics, [16, 18], can be defined as the subclass of all jerky dynamics that can be derived from one-dimensional Newtonian equation $m\ddot{x} = F$ by taking its derivative with respect to time. Therefore a Newtonian jerky dynamics has an immediate mechanical interpretation as the one-dimensional motion of a point particle of mass m under the influence of a generally complicated force F that can depend on the instantaneous position and velocity, the time and, in general, also on the position and velocity history. Rescaling the force F by the (finite, nonzero) mass m, $F/m \to F$, for convenience, one obtains for the time derivative of the Newtonian equation

$$\dddot{x} = \dot{F} = \dot{x}\partial_x F + \ddot{x}\partial_{\dot{x}} F + \partial_t F \tag{3.1}$$

Independent of the specific form of the force F, one infers two properties [16,18]. First, a Newtonian jerky dynamics can only contain the acceleration $\ddot{x}$ linearly, although in general with a prefactor that can depend on the position x and the velocity $\dot{x}$. This follows from the fact that the force F cannot contain dependencies on the acceleration $\ddot{x}$. Second, only forces that (i) are independent of the time t or (ii) depend linearly on the time via an additive term Ct with C being a constant, can lead to a Newtonian jerky dynamics, Eq. (3.1). This is because the jerk function $J(x, \dot{x}, \ddot{x})$ should not be explicitly dependent on the time.

For the specific form of the force F, one can quite generally think of the following options.

Case(a). The force F is only function of the instantaneous position and velocity of the particle and of the time, $F = F_{inst} = f_0(x,\dot{x}) + Ct + D$ with C and D being constants. This case has been discussed by Sprott [18]. Since the jerk function J is assumed to be autonomous, its functional form must be given by $J(x,\dot{x},\ddot{x}) = \dot{x}\partial_x f_0(x,\dot{x}) + \ddot{x}\partial_{\dot{x}} f_0(x,\dot{x}) + C$. In particular, an additive term that depends solely on the position x does not appear in the jerk function [16], [18]. An extensive search by Sprott [18] for such jerk dynamics that have only quadratic nonlinearities did not show any indication for the appearance of irregular solutions.

Case(b). The force F contains in addition to case(a) an additive memory term, $F = F_{inst} + M$, with

$$M = \int^t f(x(\tau), \frac{dx}{d\tau}(\tau))d\tau = \int^t f(x,\dot{x})d\tau \tag{3.2}$$

depending, quite generally, on the positional and velocity history of the motion.

The total time derivative of the memory term is determined by

$$\dot{M} = \dot{x}\partial_x M + \ddot{x}\partial_{\dot{x}} M + \partial_t M = f(x(t),\dot{x}(t)) \tag{3.3}$$

since $\partial_x M = 0 = \partial_{\dot{x}} M$ holds. Therefore, the jerk function for case (b) is given by

$$J(x,\dot{x},\ddot{x}) = \dot{x}\partial_x f_0(x,\dot{x}) + \ddot{x}\partial_{\dot{x}} f_0(x,\dot{x}) + f(x,\dot{x}) + C \tag{3.4}$$

and can contain an additive term that solely depends on x. This offers a chance for finding chaotic solutions.

To summarize, Newtonian jerky dynamics constitute simple extensions of the well-known oscillator dynamics to third-order differential equations that can have non-steady and non-periodic bounded long-time dynamics.

3.1 Normal Form of a Newtonian Jerky Dynamics

Starting from a general ansatz of the force F, we have determined its corresponding jerk function in Eq. (3.4). Here we ask the opposite question: What is the general functional form of a jerky dynamics that is still Newtonian jerky? It will be shown that basically any well-behaved jerky dynamics that depends linearly on $\ddot{x}$ is Newtonian jerky.

Theorem 3.1. *Any jerky dynamics of the functional form*

$$\dddot{x} + p(x,\dot{x})\ddot{x} + q(x,\dot{x}) = 0 \tag{3.5}$$

with p and q being differentiable and integrable functions of their arguments x and $\dot{x}$, is Newtonian jerky.

Proof. *We have to show that Eq. (3.5) can be reduced to a Newtonian equation with a force F being of the form stated in case (b). Comparing Eq. (3.5) with Eq. (3.4), one directly identifies that $p(x,\dot{x}) = -\partial_{\dot{x}} f_0 = -\partial_{\dot{x}} F$ must hold. Next, we have to split $q(x,\dot{x})$ into two parts, one being the result of the derivative with respect to x of the "instantaneous" part of the force and the other coming from the memory term. To determine the first part denoted by $h(x,\dot{x})$ in the following, one can take advantage of the fact that Schwarz's theorem, $\partial_x \partial_{\dot{x}} F = \partial_{\dot{x}} \partial_x F$, must hold if the force F exists at all. As a result, one infers that $\partial_x p(x,\dot{x}) = \partial_{\dot{x}} h(x,\dot{x})$. Next, we have to distinguish between two cases.*

(i) $\partial_x p(x,\dot{x}) = 0$: In this case, p is either zero, a nonzero constant, or a function that solely depends on $\dot{x}$, $p = p(\dot{x})$. As a consequence, h can only be a either zero, a nonzero constant, or a function that solely depends on x, $h = h(x)$. Then, Eq. (3.5) can be rewritten in the form

$$\dddot{x} + p(\dot{x})\ddot{x} + h(x)\dot{x} + [q(x,\dot{x}) - h(x)\dot{x}] = 0, \tag{3.6}$$

from which one immediately reads off its corresponding Newtonian equation,

$$\ddot{x} = F = -\int^{\dot{x}} p(\dot{x})d\dot{x} - \int^{x} h(x)dx$$

$$-\int^{t} [q(x,\dot{x}) - h(x)\dot{x}]\, d\tau. \tag{3.7}$$

Note that the function $h(x)$ is basically arbitrary. It only has to be differentiable and integrable with respect to x.

(ii) $\partial_x p(x,\dot{x}) \neq 0$: In this case, $p(x,\dot{x})$ depends explicitly on x. From Schwarz's theorem, it follows that $h(x,\dot{x}) = \int^{\dot{x}} \partial_x p(x,\dot{x})d\dot{x}$ must hold. Therefore, Eq. (3.5) can be rewritten in the form

$$\dddot{x} + p(x,\dot{x})\ddot{x} + \left[\int^{\dot{x}} \partial_x p(x,\dot{x})d\dot{x}\right]\dot{x} + \left[q(x,\dot{x}) - \dot{x}\int^{\dot{x}} \partial_x p(x,\dot{x})\dot{x}\right] = 0 \tag{3.8}$$

from which its corresponding Newtonian equation,

$$\ddot{x} = F = -\int^{\dot{x}} p(x,\dot{x})d\dot{x} - \int^{t}\left[q(x,\dot{x}) - \dot{x}\int^{\dot{x}} \partial_x p(x,\dot{x})d\dot{x}\right] d\tau, \tag{3.9}$$

directly follows. Equations (3.7) and (3.9) constitute the desired expressions of the force that lead to a Newtonian jerky dynamics.Therefore, the proof is complete. ■

Several remarks are in order. (i) Including dependencies on the velocity in the memory term of the force F leads to a much wider class of Newtonian jerky dynamics in comparison to a memory term that only depends on the positional history [16]. (ii) An interesting point in the proof of the Theorem 3.1 is the ambiguity of the functional form of the force if $p(x,\dot{x})$ in Eq. (3.5) is independent of

x. This might appear somewhat strange, but can be easily explained by inspection of Eq. (3.4). For the special case in which p is independent of x, the instantaneous part of the force is given by $F_{inst} = f_1(\dot{x}) + f_2(x) + Ct$ and the Newtonian jerky dynamics by $\dddot{x} = C + \ddot{x}\partial_{\dot{x}} f_1(\dot{x}) + \dot{x}\partial_x f_2(x) + f(x,\dot{x}) = 0$. Then, an additional term $h(x)\dot{x} = \partial_t \int^t h(x)\dot{x} d\tau$ in the jerk function can be absorbed in either $\dot{x}\partial_x f_2(x)$ arising from the instantaneous part of the force F, or $f(x,\dot{x})$ coming from the memory term. (iii) In particular, Theorem 3.1 holds for the case that p and q are polynomial expressions in x and $\dot{x}$. (iv) If p in Eq. (3.5) is zero or a nonzero constant, then the simplest corresponding Newtonian equation [where $h(x)$ has been chosen to be zero] is determined by $\ddot{x} + p\dot{x} + \int^t q(x(\tau), \dot{\tau}) d\tau = 0$.

3.2 Are some Newtonian Jerky Dynamics Gradient Systems?

In the qualitative theory of dynamical systems [27,29] gradient systems play an interesting role. For these systems, one can rule out the existence of oscillatory solutions just by considering their vector fields. In particular, a dynamical system $\dot{\mathbf{x}} = \mathbf{W}(\mathbf{x})$ with $\mathbf{x} = (x, v, a)$ being a point of the three-dimensional phase space, is a gradient system if its vector field $\mathbf{W}(\mathbf{x})$ results from the gradient of a scalar potential, $\mathbf{W}(\mathbf{x}) = -\nabla\Phi(x)$. Using Eq. (3.5), we can write a Newtonian jerky dynamics as a dynamical system reading explicitly

$$\dot{x} = v, \quad \dot{v} = a, \quad \dot{a} = J(x, v, a) = -p(x, v)a - q(x, v) \tag{3.10}$$

. Unfortunately, however, the following holds.

Theorem 3.2. *Newtonian jerky dynamics are not gradient systems.*

Proof. *(by contradiction). Suppose that a scalar potential $\Phi(x, v, a)$ that reproduces the dynamical system (3.10) exists. Then, it must fulfill the three conditions,*

$$\partial_x \Phi = -v, \quad \partial_v \Phi = -a, \quad \partial_a \Phi = p(x,v)a + q(x,v). \tag{3.11}$$

Integrating the last equation of (3.11), yields the functional form of the scalar potential

$$\Phi(x,v,a) = \frac{1}{2} p(x,v)a^2 + q(x,v)a + \phi(x,v) \tag{3.12}$$

with $\phi(x,v)$ being a function of x and v only. Taking the derivative of (3.12) with respect to x and using $\partial_x \Phi = -v$ from Eq. (3.11), one infers that $\partial_x p(x,v) = 0 = \partial_x q(x,v), \partial_x \phi(x,v) = -v$. Therefore, $\phi(x,v) = -xv +$ arbitrary function of v must hold, and p and q can only be functions of v. Similarly, taking the derivative of (3.12) with respect to v and using $\partial_v \Phi = -a$ from Eq.(3.11), one finds that $\partial_v p(x,v) = 0 = \partial_v \phi(x,v)$ and $\partial_v q(x,v) = -1$. This implies that p and ϕ can only depend on x and $q(x,v) = -v$. Combining the results for the derivatives of Φ with respect to x and v, one infers that there is no scalar potential Φ since no functional form of $\phi(x,v)$ can satisfy the above conditions. ■

As a consequence, there is no elementary criterion that excludes periodic solutions in some classes of Newtonian jerky dynamics. This, however, does not imply that Newtonian jerky dynamics with a globally attracting fixed point cannot exist.

3.3 Nonchaotic Newtonian Jerky Dynamics

As soon as at least one quadratic nonlinearity is present in a Newtonian jerky dynamics, there seems to be some chance [16–18] to find, at least for some parameter ranges, chaotic dynamics in the sense that the long-time solutions are bounded, but not fixed points, periodic or quasiperiodic solutions. It is, however, a highly nontrivial open problem if a given functional form of a jerky dynamics can really show chaotic dynamics.

Here, this problem will be discussed from the opposite point view and determined some subclasses of nonlinear Newtonian jerky dynamics that cannot show chaotic behavior. Trivially, any Newtonian jerky dynamics that does not explicitly depend on x, $\dddot{x} = J(\dot{x}, \ddot{x})$, can be reduced to an oscillator dynamics by introducing $y = \dot{x}$.

Even if the jerky dynamics depends explicitly on x, there are classes of jerky functions that cannot lead to a chaotic dynamics. In particular, all Newtonian jerky dynamics with a conserved quantity are effectively second-order systems. By virtue of the Poincaré-Bendixson theorem, they cannot show irregular dynamics.

Another class of non-chaotic Newtonian jerky dynamics can be obtained by the following consideration. We can always split the term $q(x,\dot{x})$ in a Newtonian jerky dynamics, Eq. (3.5), into two terms according to

$$q(x,\dot{x}) = r(x,\dot{x})\dot{x} - s(x,\dot{x}), \tag{3.13}$$

where r and s are functions of the indicated arguments and $r(x,\dot{x})$ fulfills the Schwarz condition $\partial_x p(x,\dot{x}) = \partial_{\dot{x}} r(x,\dot{x})$. As a consequence, any Newtonian jerky dynamics can be rewritten as

$$\dddot{x} + p(x,\dot{x})\ddot{x} + r(x,\dot{x})\dot{x} = s(x,\dot{x}) \tag{3.14}$$

or, equivalently,

$$\frac{d}{dt}\left[\ddot{x} + \int^{\dot{x}} p(x,\dot{x})d\dot{x}\right] = \frac{d}{dt}\int^{t} s(x,\dot{x})d\tau. \tag{3.15}$$

Direct integration of Eq. (3.15) yields

$$\ddot{x} + \int^{\dot{x}} p(x,\dot{x})d\dot{x} = \int^{t} s(x,\dot{x})d\tau. \tag{3.16}$$

This shows most clearly that a Newtonian jerky dynamics can be interpreted as an oscillator [the left-hand side of (3.16)] coupled to an internal driving mechanism or feedback [the right-hand side of (3.16)] that is an integral over the history of its motion. This fact has some consequences for the possible dynamics of Newtonian jerky systems.

Theorem 3.3. *If (i) the oscillator on the left-hand side of Eq. (3.15), $\ddot{x}+\int^{\dot{x}} p(x,\dot{x})d\dot{x} = 0$, possesses only bounded solutions and (ii) the integrand of memory term, $s(x,\dot{x})$, on the right-hand side of Eq. (3.15) is either positive semidefinite or negative semidefinite for all x and $\dot{x}$, then the Newtonian jerky dynamics (3.14) cannot show chaotic behavior.*

Proof. *To demonstrate the statement, one first infers by virtue of the Poincaré-Bendixson theorem [27] that (3.16) can only have fixed points or periodic solutions if the right-hand side of (3.16) is identical to zero. This happens if $s(x,\dot{x})$ is zero for all x and $\dot{x}$. Next we have to distinguish two cases. (i) $s(x,\dot{x})$ is a positive or negative constant for all x and $\dot{x}$. Then, the right-hand side of (3.16) increases or decreases linearly in time and the modulus of solution $x(t)$ of (3.16) diverges to infinity. (ii) In the general case, $s(x,\dot{x})$ can depends on x and/or $\dot{x}$ and satisfies the condition $s(x,\dot{x}) \geq 0$ or $s(x,\dot{x}) \leq 0$. Then, $s(x,\dot{x})$ can either approach zero, a positive or negative constant, oscillate with solely positive or negative values of s, or go to plus or minus infinity in the long time limit. In any case, the modulus of the time integral over $s(x,\dot{x})$ monotonically increases either to a constant or diverges to infinity. This implies that are possible long time solutions can only be either fixed points, periodic solutions, or divergences to infinity, but not irregular dynamics.* ■

Two remarks are in order. (i) As a consequence of the Theorem 3.3, a simple Newtonian jerky dynamics which possesses, as only nonlinearity, a term being quadratic in x,

$$\dddot{x}+A\ddot{x}+B\dot{x}\pm Cx^2 = 0, \tag{3.17}$$

with A, B and C positive semidefinite constants, cannot show chaotic dynamics. On the other hand, a Newtonian jerky dynamics of the functional form

$$\dddot{x} + A\ddot{x} - B\dot{x}^2 + Cx = 0, \tag{3.18}$$

can have chaotic solutions for some parameters since Sprott's minimal dissipative model $(B = 1$ and $C = 1)$ is a special case of (3.18). (ii) The left-hand side of (3.16) is not restricted to harmonic oscillators. It implies to any nonlinear oscillator with bounded solutions.

The theorem partly generalizes previous ideas and statements by Zhang and Heidel [28] and Sprott [18].

CHAPTER 4

GENESIO SYSTEM

In this chapter, we show that the Genesio system is one of the minimal chaotic jerky model and is also a Newtonian jerky, that is it can be derived from one-dimensional Newton equation. Furthermore, we investigate the global dynamical properties of the Genesio system, and show that the route to chaos is determined by a period-doubling cascade, that is initiated by a Hopf bifurcation.

4.1 System Description

The Genesio system, which was proposed by Genesio and Tesi [24], is described by the following simple three-dimensional autonomous system with only one quadratic nonlinear term:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -ax - by - cz + x^2 \end{aligned} \tag{4.1}$$

where a, b, c are real parameters. The state space plot for Eq. (4.1) is shown in Fig. 4.1.

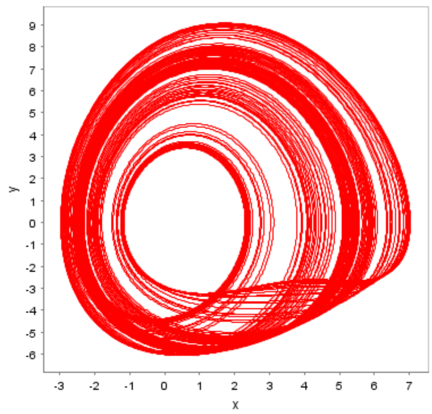


Figure 4.1: Chaotic attractor of the Genesio system for $a = 6.3$, $b = 2.92$, $c = 1.2$ and $x_0 = -2.0$, $y_0 = 0.5$, $z_0 = 2.0$

4.2 Genesio System as Jerky Dynamics

Theorem 4.1. *The Genesio system can be recast into a jerky dynamics, and the resulting form belongs to class JD_2.*

Proof. *Since the system (4.1) has the functional form in Eq. (2.10) and fulfills the conditions (2.11) in Sec. 2.1, it can be recast into a jerky dynamics in x. Rewriting the Eqs. (4.1) into a jerky dynamics in x leads to*

$$\dddot{x} = -c\ddot{x} - b\dot{x} - ax + x^2 \tag{4.2}$$

where $\dot{x} = \frac{dx}{dt}$. Using the linear and invertible transformation

$$\xi = x - \frac{a}{2}, \quad \dot{\xi} = \dot{x}, \quad \ddot{\xi} = \ddot{x}$$

in Eq. (4.2) we get equivalent jerky dynamics for a new dynamical variable ξ as

$$\dddot{\xi} = -c\ddot{\xi} - b\dot{\xi} + \xi^2 - \frac{a^2}{4}. \tag{4.3}$$

Comparing with Table (5.2), one can see that the resulting jerky form of the Genesio system belongs to the class JD_2 *with*

$$k_1 = -c, \quad k_2 = -b, \quad \textit{and} \quad k_3 = -\frac{a^2}{4}$$

coefficients. ■

Theorem 4.2. *Genesio system has no equivalent jerky dynamics in the variables y and z.*

Proof. *We can write the Genesio system (4.1) as*

$$\dot{\mathbf{x}} = \mathbf{c} + B\mathbf{x} + \mathbf{n}(\mathbf{x}) \tag{4.4}$$

where

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix}, \quad \mathbf{n}(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ x^2 \end{pmatrix}.$$

From the discussion given in Sec. 2.3.1, we know that for a simultaneous existence of a jerky dynamics in y and/or z, the first necessity

$$n_2(\mathbf{x}) = n_2(y) \quad \textit{and/or} \quad n_3(\mathbf{x}) = n_3(z) \tag{4.5}$$

must hold. Because of the corresponding nonlinear terms

$$n_2(\mathbf{x}) = 0 \quad \textit{and/or} \quad n_3(\mathbf{x}) = x^2 \neq n_3(z),$$

in Eq. 2.14 do not depend on y and z, respectively. Also the condition (2.23a) yields

$$f_2(y,z) = x^2,$$

which is meaningless. Therefore, the Genesio system cannot have equivalent jerky dynamics in y and/or z. ■

We note that the Genesio system has no equivalent jerky dynamics with the systems belonging to class JD_2

Theorem 4.3. *The Genesio system is a Newtonian jerky dynamics.*

Proof. *Eq. (4.3) can be written as*

$$\dddot{x} + p(x,\dot{x})\ddot{x} + q(x,\dot{x}) = 0 \tag{4.6}$$

where $p(x,\dot{x}) = c$ *and* $q(x,\dot{x}) = b\dot{x} - x^2 + \frac{a^2}{4}$*. Since both p and q are differential and integrable functions of their arguments x and* $\dot{x}$, *it is Newtonian jerky by Theorem 3.1.*
■

An immediate consequence of the Theorem 4.3 can be given as follows:

Corollary 4.1. *The Genesio system is not a gradient system.*

Proof. *By Theorem 4.3,we have the Genesio system is Newtonian jerky. So, Theorem 3.2 in Sec. 3.2 implies that it is not a gradient system.* ■

Theorem 4.4. *The Genesio system*

$$\dddot{x} + c\ddot{x} + b\dot{x} - x^2 + \frac{a^2}{4} = 0 \tag{4.7}$$

exhibits chaotic solutions for some parameter ranges.

Proof (1). *We write the Eq. (4.7) as*

$$\dddot{x} + c\ddot{x} + b\dot{x} = x^2 - \frac{a^2}{4}. \tag{4.8}$$

Integrating Eq.(4.8) with respect to t we get

$$\ddot{x} + c\dot{x} + bx = \int^t \left[x(\tau)^2 - \frac{a^2}{4} \right] d\tau. \tag{4.9}$$

The memory term $x^2 - \frac{a^2}{4}$ changes sign as x varies, that is, it is neither positive semi-definite nor negative semi-definite for all x. Thus the Genesio system can have chaotic solutions for some parameter ranges. ■

Proof (2). *From Eq. (4.6) we have*

$$q(x,\dot{x}) = b\dot{x} - x^2 + \frac{a^2}{4}$$

If we write

$$q(x,\dot{x}) = r(x,\dot{x})\dot{x} - s(x,\dot{x})$$

where $r(x,\dot{x}) = b$ and $s(x,\dot{x}) = x^2 - \frac{a^2}{4}$, the since $s(x,\dot{x})$ is neither positive semi-definite or negative semi-definite for all a and $\dot{x}$, by Theorem 3.3, the Genesio system can exhibit chaotic solutions for some parameter ranges.

4.3 Dynamical Properties of the Genesio System

In this section, we will investigate the global dynamical properties of the Genesio system and we will show that it shares the common route to chaos as systems in the class JD_2.

4.3.1 Analytical and Numerical Tools

We split the investigation of the Genesio system as a jerky dynamics into two parts. First, we use analytic methods to confine the regions of the parameter spaces where potentially interesting dynamics occur, *i.e.*, where the dynamics is divergent. These analytical methods are the stability analysis of fixed points, the Hopf bifurcation and the time evolution of the volume in phase space. Secondly, we use a numerical approach to detect different types of long-time dynamics and their dependence on the parameter that enter into the jerk system in question. These numerical studies are based on the concept of Lyapunov exponents.

(1) Analytical Methods

(i) Stability analysis of fixed points and Hopf Bifurcation:

Given jerky dynamics

$$\dddot{x} + c\ddot{x} + b\dot{x} - x^2 + \frac{a^2}{4} = 0 \tag{4.10}$$

the equilibria can be found by assuming that it has a fixed point $x^* \in \mathbb{R}$, $\dot{x}^* = 0$, $\ddot{x}^* = 0$, which leads to $J(x^*, 0, 0) = x^2 - \frac{a^2}{4} = 0$, or, $x^* = \pm\frac{a}{2}$. So, there are two equilibria: $E_+(\frac{a}{2}, 0, 0)$ and $E_-(-\frac{a}{2}, 0, 0)$. Linearizing the Eq. (4.10) about the equilibrium E_+ provides one real and a pair of complex conjugate eigenvalues along with the following characteristic equation

$$f(\lambda) = \lambda^3 + c\lambda^2 + b\lambda - a = 0 \tag{4.11}$$

and linearizing the Eq. (4.10) about the other equilibrium E_- yields the following characteristic equation

$$f(\lambda) = \lambda^3 + c\lambda^2 + b\lambda + a = 0. \tag{4.12}$$

According to Routh-Hurwitz criterion, the equilibrium E_+ is stable (*i.e.*, the real part of all roots $\lambda_i \quad (i = 1,2,3)$ of Eq. (4.11) are negative) only if the conditions

$$a < 0, \qquad b > 0, \qquad c > 0, \qquad bc + a > 0 \tag{4.13}$$

are fulfilled. For $bc = -a \quad (a < 0,\ b,c > 0)$, the fixed point becomes unstable and the two complex roots of (4.11) cross the imaginary axes, while the third root remains real and negative. Therefore, at $bc = -a$ a stable limit cycle arises via a Hopf-bifurcation, [30].

The equilibrium E_- has the same stability characterization. If

$$a > 0, \qquad b > 0, \qquad c > 0, \qquad bc - a > 0 \tag{4.14}$$

then Eq. (4.12) satisfies the Routh-Hurwitz rule and at $bc = a$ a stable limit cycle arises via a Hopf-bifurcation.

(ii) Time evolution of phase-space volume:

The volume contraction rate of a (three-dimensional) dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is given by $\wedge = \nabla \cdot \mathbf{F}$, and the volume contraction rate of a jerky dynamics is $\wedge = \partial_{\ddot{x}} J(x, \dot{x}, \ddot{x})$. So, the volume contraction rate of the Eq. (4.10) is $\wedge = \partial_{\ddot{x}}(-c\ddot{x} - b\dot{x} + x^2 - \frac{a^2}{4}) = -c$, *i.e.*, $\frac{1}{V}\frac{dV}{dt} = -c$, which can be solved to yield $V(t) = V(0)e^{-ct}$. Since c is positive, the jerky dynamics Eq. (4.10) is dissipative with solutions for $t \to \infty$ that contract at an exponential rate $-c$ onto an attractor of zero volume that may be an equilibrium point, a limit cycle, or a strange attractor. When $c = 0$, $\wedge$ is constant and the phase space volume conserved and the dynamical systems is conservative. When c is negative, $\wedge$ is positive and the volume expands exponentially fast and there are only unstable fixed points. Therefore, the dynamics diverges for $t \to \infty$ if the initial value does not lie exactly at such unstable set.

The equation (4.10) has three free parameters a, b, and c and the position of equilibria $E_+(\frac{a}{2},0,0)$ and $E_-(-\frac{a}{2},0,0)$ depend on the parameter a.

To get the parameter-independent equilibria we use the transformation

$$\bar{x} = \frac{2}{a}x, \qquad \bar{t} = ct \tag{4.15}$$

for x and t yielding the new quantities $\bar{x}$ and $\bar{t}$. With the substitution of Eq. (4.15), Eq. (4.10) becomes

$$\dddot{\bar{x}} = -\ddot{\bar{x}} - \frac{b}{c^2}\dot{\bar{x}} - \frac{-a}{2c^3}(\bar{x}^2 - 1)$$

and introducing new parameters $\alpha = \frac{b}{c^2}$, $\beta = -\frac{a}{2c^3}$ and then dropping the overbars we write

$$\dddot{x} = -\ddot{x} - \alpha\dot{x} - \beta(x^2 - 1) \tag{4.16}$$

Eq. (4.16) possesses the two stationary points $x^* = \pm 1$, $\dot{x}^* = 0$, $\ddot{x}^* = 0$. Analyzing their stability leads to the characteristic equation

$$\lambda^3 + \lambda^2 + \alpha\lambda \pm 2\beta = 0 \tag{4.17}$$

It follows that $x^* = 1$ is stable only for $\alpha > 0$, $\beta > 0$ and $\alpha > 2\beta$ and it becomes unstable at the line $\alpha = 2\beta$ $(\alpha, \beta > 0)$ via a Hopf-bifurcation. Similarly, $x^* = -1$ is stable only for $\alpha > 0$, $\beta < 0$ and $\alpha > -2\beta$ and becomes unstable at the line $\alpha = -2\beta$ $(\alpha > 0, \beta < 0)$ via a Hopf-bifurcation. These stability properties of the fixed points also reflect the symmetry of the Genesio system.

The Eq. (4.16) is invariant under $x \to -x$ and $\beta \to -\beta$. Therefore knowing the solution of $x(t)$ of Eq. (4.16) for a certain value of parameter β and certain initial values, the dynamics of the corresponding sign-inverted β and initial values is given by $-x(t)$. Due to this symmetry property of Eq. (4.16), we need only to consider initial values close to one of the two stationary points, *e.g.*, $x^* = -1$, $\dot{x}^* = 0$, $\ddot{x}^* = 0$. Then, the most interesting region of the (α, β)-parameter plane is the one for $\alpha < -2\beta$ and positive α.

(2) Numerical Methods

(i) Lyapunov exponents

The Lyapunov exponents of a dynamical system constitute a measure for the exponential divergence (or convergence) of initially nearby trajectories. The property of Lyapunov exponents, which is important for our problem is the following : the type of an attractor that is approached in the long-time limit is characterized by the spectrum of the corresponding Lyapunov exponents. For phase space dimension three, *e.g.*, an attracting fixed point possesses the Lyapunov spectrum $-,-,-$, an attracting limit cycle $0,-,-$, an attracting two-torus $0,0,-$ and a chaotic attractor $+,0,-$. Therefore to determine the long-time behavior of a jerky dynamics, *i.e.*, the type of the attractor, we will numerically compute the set of all Lyapunov exponents.

(ii) Lyapunov Dimension

Another important quantitative concept for the characterization of different dynamical long-time behavior, including (dissipative) chaos, is the dimension of an attractor. This concept is based on the metric properties of an attractor, *e.g.*, the Hausdorff dimension or on its invariant measure, leading to information dimension. Based on their dimensions, a simple classification of attractor is possible. For instance, a fixed point has dimension zero, a stable limit cycle has dimension one, a two-dimensional torus has dimension two, while the dimension of (dissipative) chaotic attractor take on the values that are non-integers. Lyapunov exponents can also be used to define a dimension like quantity of an attractor called Lyapunov dimension, which is defined as

$$D_L = m + \frac{\sum_{i=1}^{m} \lambda_i}{|\lambda_{m+1}|} \tag{4.18}$$

where $m < n$ is the largest index such that $\sum_{i=1}^{m} \lambda_i \geq 0$ is valid. Therefore, to show that the Eq. (4.11) has a (dissipative) chaotic attractor we will compute the Lyapunov dimension and we will plot its variation with the parameter β.

4.3.2 Numerical Results and Discussion

In this section we present the results of our investigation of the global dynamics of Genesio system in jerky form (4.16). Numerical computations are performed using Mathematica, Dynamics and iDMC softwares. The initial values are chosen close to the fixed point $x^* = -1$, $\dot{x}^* = 0$, $\ddot{x}^* = 0$.

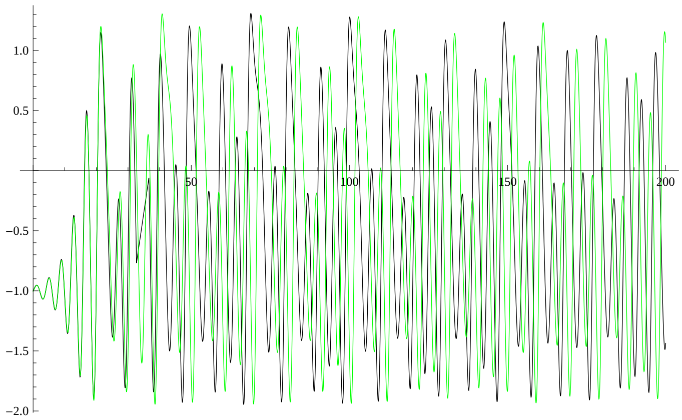


Figure 4.2: Numerical solutions of Eq.(4.16) in the case $\beta = -1.9$

In Fig. 4.2 we show some numerical solutions of Eq. (4.16) in case $\beta = -1.9$. The chaotic nature of these solutions is evident, not just because of their irregularity, but because of their extreme sensitivity to initial conditions. With an initial difference of just 1 part in 1000 the oscillation sequences are seen diverging as t becomes greater than about 30. Moreover, even if we reduce the discrepancy in the initial conditions by a factor of 100, to just 1 part in 100000, we only manage to keep the solutions together for a little longer, till t is about 40, after which they once again go their separate ways.

For the parameter regions $\alpha < 0$, or $\beta < 0$, no bounded solutions have been found. This suggests that the Genesio system does probably not possess at all a stable attractor in these regions. For the region $\alpha > 0$ and $\beta > 0$ the resulting Lyapunov spectra are shown in Fig. 4.3. The fixed point domain is followed by a large limit cycle region and a structured chaotic region. However the chaotic region is not present for $\alpha \leq 1.6$. At $\alpha = 1.6$ its boundary is formed by two tongues that reach into the limit cycle domain. For smaller parameter values one only finds some chaotic points at $\alpha = 0.75$ and $\beta = 0.71$. Moreover there are islands with bounded dynamics (limit cycles and strange attractors) located within the diverging region.

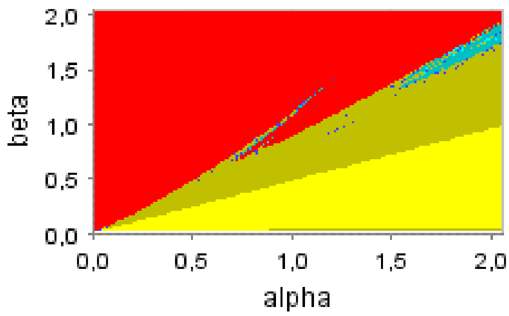
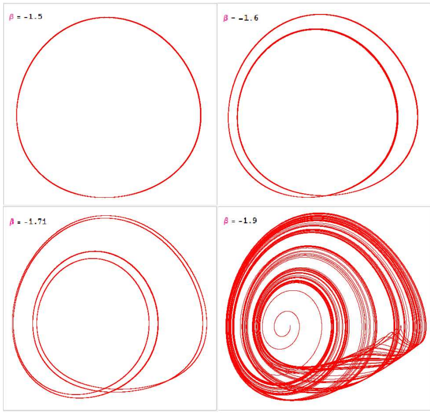


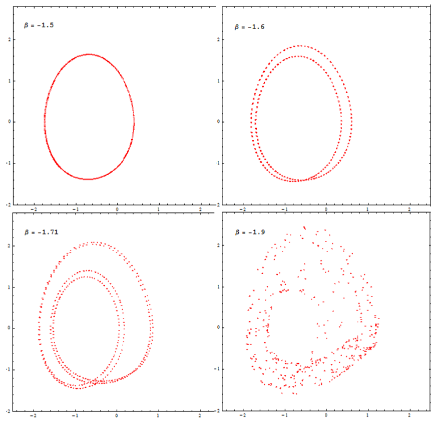
Figure 4.3: Lyapunov spectra for the Eq.(4.16), calculated for the initial values $x_0 = -1.005, y_0 = 0.05, z_0 = 0.05$. The different colors classify the long-time dynamical behavior: red= divergence, yellow= fixed point, green= limit cycle, light blue= strange attractor, blue= chaos.

To illustrate the corresponding dynamics in more detail, several Fiegenbaum diagrams for fixed α and increasing β are shown in Figs. 4.6- 4.11.

Fig. 4.4 shows two-dimensional projections of the system's attractor for different values of β (with $\alpha = 2.03$ held fixed). At $\beta = -1.5$ the attractor is a stable limit cycle. As β is decreased to -1.6, the limit cycle goes around twice before closing, and its period is approximately twice that of the original cycle. This is what period-doubling looks like in a continuous-time system. In fact, somewhere between $\beta = -1.5$ and -1.6, a period-doubling bifurcation of cycles must have occured. Another period-doubling bifurcation creates the four-loop cycle shown at $\beta = -1.71$. After an infinite cascade of further period-doublings, we obtain the strange attractor shown at $\beta = -1.9$.



(a) Period-doubling sequence



(b) Poincaré sections

Figure 4.4: Two-dimensional projections and corresponding Poincaré sections of the Eq. (4.16)'s attractor for different values of β

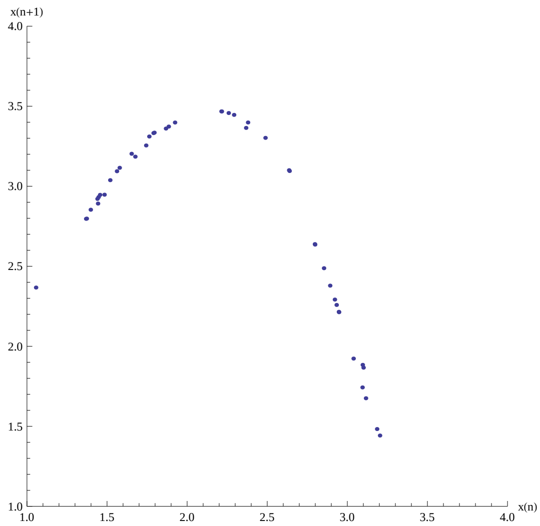


Figure 4.5: First return map

For a computed value of β, we record the successive local maxima of $x(t)$ for a trajectory on the strange attractor. Fig. 4.5 shows x_{n+1} vs. x_n, where x_n denotes the nth local maximum. The data points fall nearly on a one-dimensional curve. These one-dimensional maps are obtained to compare the different dynamics on a dynamical system. Such maps with parabola like maxima are well-known for the generation of the chaotic solution through period-doubling route and it gives us a clue the route to chaos in the jerk dynamical systems under consideration. In Fig. 4.5, the number of different maxima tells us the period of the attractor. For instance, at $\beta = -1.71$ the attractor is period-4 (Fig. 4.4), and hence there are four local maxima of $x(t)$. All of these points are graphed above $\beta = -1.71$ in Fig. 4.6. We proceed in this way for all values of β, thereby sweeping out the orbit diagram.

As stated in Sec. 4.3.1, Eq. (4.16) is invariant under $x(t) \to -x(t)$ and $\beta \to -\beta$. Due to the symmetry of this equation we choose initial values close to the equilibrium E_- that exhibits very interesting dynamics. We have computed all graphs for the initial values $x^*(0) = -1.005$, $\dot{x}^*(0) = 0.05$, $\ddot{x}^*(0) = 0.05$.

In Fig. 4.6(a), we have shown the bifurcation diagram, which is the plot of successive maxima of the long time evolution of $x(t)$ as a function of system parameter β for fixed value of the parameter $\alpha = 2.03$. By Fig. 4.6(a), it becomes clear that the jerk dynamical system Eq. (4.16) having quadratic nonlinearity shows the chaos, via a cascade of period-doubling bifurcations, as β is increased (and decreased). In Fig 4.6(b), the Lyapunov exponents as a function of the parameter β is for the same range as that of the bifurcation plot. In Fig. 4.6(c), the Lyapunov dimension is for the same range of β as in Fig. 4.6(a) and (b). It can be clearly seen that all the three frames have one-to-one correspondence with each other, *i.e.*, we observe that for the values of parameter β where bifurcation diagram shows the limit cycle solutions, the largest Lyapunov exponent (blue) is negative as well as the dimension of the attractor is two. However, for the values of the parameter β, where the bifurcation plot shows the existence of aperiodic behavior (chaotic), the largest Lyapunov exponent is positive as well as the dimension of the attractor is a non-integer 2.14 between two and three.

In Fig 4.7, we show the bifurcation diagram for only positive values of β with in detail to investigate the route to chaos. The route from fixed point dynamics over limit cycles to chaos is determined by the well-known period-doubling cascade and initiated by a Hopf bifurcation at the line $\alpha = -2\beta$. Therefore the limit cycle domain, that follows the fixed point region $(0.68 < \beta < 1)$, consists of periodic attractors with periods $2^n \quad (n \in \mathbf{N})$, $0.68 < \beta < 1.74$. Short after the period-32 at $\beta = 1.74$, can be seen in Fig. 4.7(b), the various curves in the figure seem to expand explosively and merge together to produce an area of almost solid red. This region consists of an infinite series of period-doubling bifurcations. This behavior is indicative of the onset

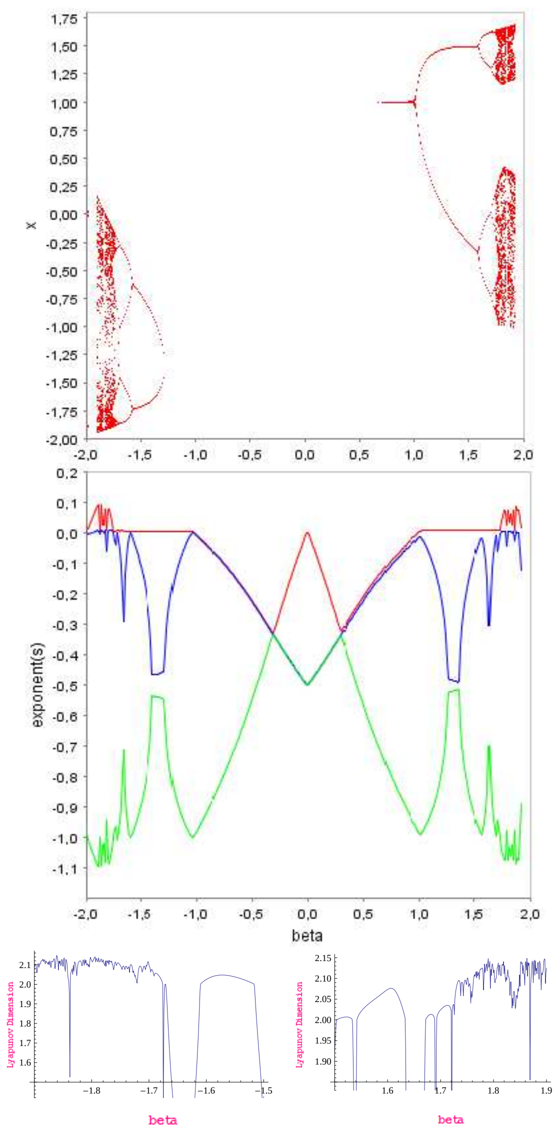
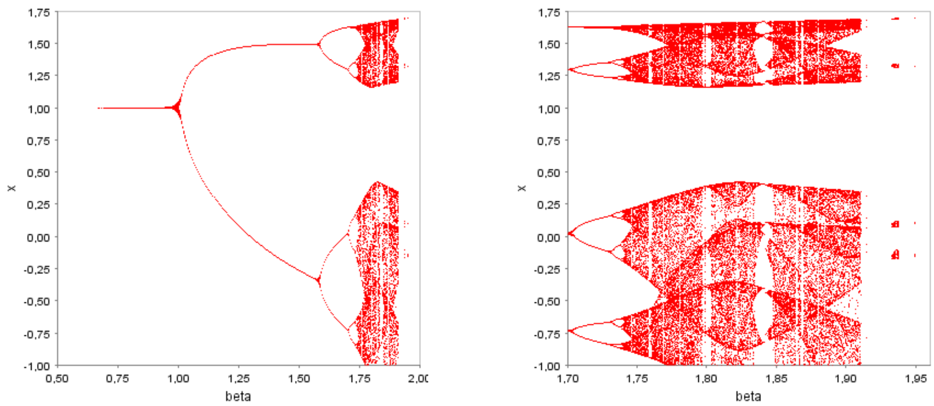


Figure 4.6: (a) The bifurcation diagram showing period-doubling route to chaos, (b) the Lyapunov exponents, (c) the Lyapunov dimension of the attractor.

of chaos. It is also seen that the chaotic region contains many narrow windows, which are called limit cycle windows, for example about $\beta = 1.8$, in which chaos reverts to



(a) $\beta > 0$ (b) Detailed view for $\beta > 1.70$

Figure 4.7: The bifurcation diagram with fixed $\alpha = 2.03$ and increasing positive β values.

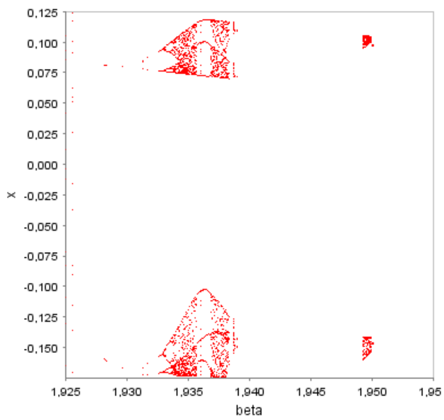
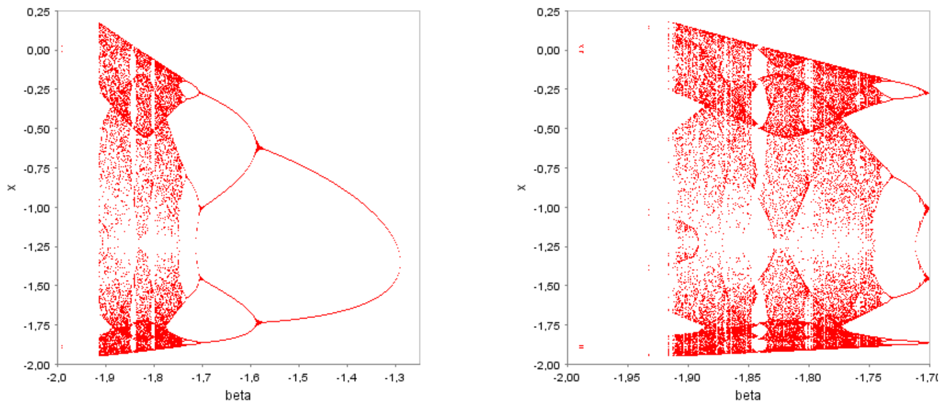


Figure 4.8: An enlargement of the island of bounded dynamics in the region of diverging dynamics seen in Fig. 4.7.

limit cycle for a short interval in β. As β is further increased, the limit cycle windows break down, and eventually disappears. Thus, limit cycle solutions suddenly are changed into chaotic solutions again. This aperiodic behavior is seen until the parameter β is about 1.91. Moreover, there is also four islands of bounded dynamics located in the diverging region after $\beta = 1.91$. These tiny islands consist of chaotic attractors that, with increasing β, bifurcate backwards by period-halfing to a limit cycle of period-2, as can be seen from Fig. 4.8.

Fig. 4.9, for only negative values of β, shows the similar dynamics discussed above.



(a) $\beta < 0$ (b) Detailed view for $\beta < -1.70$

Figure 4.9: The bifurcation diagram with fixed $\alpha = 2.03$ and decreasing negative β values.

Further investigations on different dynamics for other fixed $\alpha > 0$ values, are showed as follows.

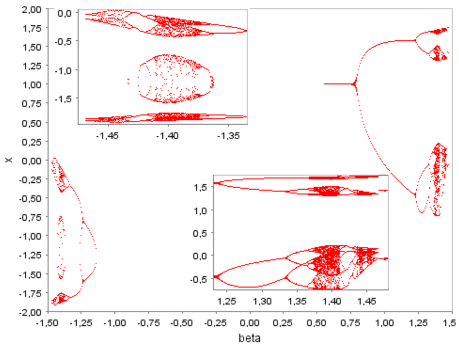


Figure 4.10: The bifurcation diagram with $\alpha = 1.6$.

With a fixed value of $\alpha = 1.6$ an increasing β $\alpha \leq -2\beta$, one finds two period-doubling bifurcation scenarios to chaos with subsequent reverse bifurcations to a limit cycle of period 2^n $(n \in \mathbf{N})$ in Fig. 4.10. Parts of the last limit cycle range, however, consist of a new stable attractor.

In Fig. 4.11, α is chosen to be 0.75 and is, therefore, from the region where only a few chaotic points are present and the area with bounded dynamics looks frayed. It shows only fragments of a complete Feigenbaum (bifurcation) diagram. This is again due to changing boundaries of the attracted region in phase space upon changing system parameters. This dependence of basin of attraction on system parameters can be overcome by choosing appropriate initial values that are still close to the fixed point $x^* = -1$, $\dot{x}^* = 0$, $\ddot{x}^* = 0$.

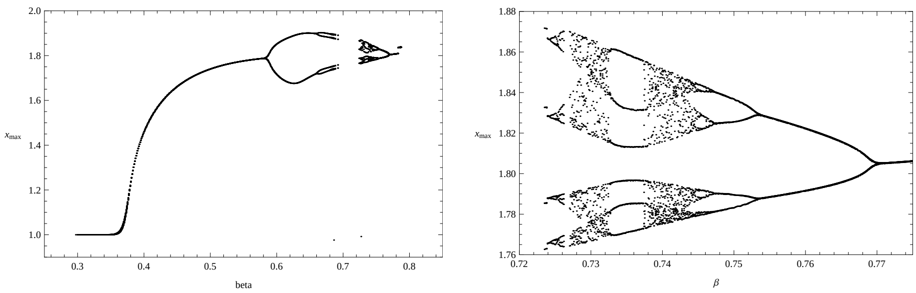


Figure 4.11: (a) The bifurcation diagram with $\alpha = 0.75$. (b) Detailed view of (a), showing forward bifurcations to chaos and backward bifurcations that lead to a period-two limit cycle.

CHAPTER 5

DISCUSSION

In this thesis, we showed that Genesio system is one of the functionally simplest polynomial class of jerky dynamics, JD_2 and it is a Newtonian jerky dynamics. We also investigated some aspects of the dynamical properties of the Genesio system and we have found for this system there are not only few parameters but wide ranges of parameter values that lead to chaotic long-time dynamics. Moreover, also large parameter regions are present where the long-time attractor of the dynamics consists of stable limit cycles. The route to chaos is determined by a period-doubling cascade (for appropriately varied system parameters) that is initiated by a Hopf bifurcation. Varying the initial conditions, we also able to detect several coexisting stable attractors. Therefore, despite its functional simplicity, the Genesio system shows a rich diversity of dynamical behavior.

Table 5.1: Minimal Models and Their Jerky Dynamics.

Model	*Equations*	*Jerky Dynamics*	*Transformations*	*Inverse Transformations*
A	$\dot{x}=y$ $\dot{y}=-x+yz$ $\dot{z}=\alpha-y^2$	none	none	none
B	$\dot{x}=yz$ $\dot{y}=x-y$ $\dot{z}=\alpha-xy$	none	none	none
C	$\dot{x}=yz$ $\dot{y}=x-y$ $\dot{z}=\alpha-x^2$	none	none	none
D	$\dot{x}=-y$ $\dot{y}=x+z$ $\dot{z}=xz+\alpha y^2$	$\dddot{x}=x\ddot{x}-\dot{x}-\alpha\dot{x}^2+x^2$	$\dot{x}=-y$ $\ddot{x}=-x-z$	$y=-\dot{x}$ $z=-x-\ddot{x}$
E	$\dot{x}=yz$ $\dot{y}=x^2-y$ $\dot{z}=\alpha-\beta x$	none	none	none
F	$\dot{x}=y+z$ $\dot{y}=-x+\alpha y$ $\dot{z}=x^2-z$	$\dddot{x}=(\alpha-1)\ddot{x}+(\alpha-1)\dot{x}$ $-x-\alpha x^2+2x\dot{x}$	$\dot{x}=y+z$ $\ddot{x}=-x+x^2+\alpha y-z$	$y=\frac{1}{1+\alpha}(\ddot{x}+\dot{x}+x-x^2)$ $z=\frac{1}{1+\alpha}(-\ddot{x}+\alpha\dot{x}-x+x^2)$
		$\dddot{y}=(\alpha-1)\ddot{y}+(\alpha-1)\dot{y}$ $-\alpha^2y^2+2\alpha y\dot{y}-y^2$	$\dot{y}=\alpha y-x$ $\ddot{y}=(\alpha^2-1)y-\alpha x-z$	$x=-\dot{y}+\alpha y$ $z=-y+\alpha\dot{y}-y$
G	$\dot{x}=\alpha x+z$ $\dot{y}=xz-y$ $\dot{z}=-x+y$	$\dddot{x}=(\alpha-1)\ddot{x}+(\alpha-1)\dot{x}$ $-x-\alpha x^2+x\dot{x}$	$\dot{x}=\alpha x+z$ $\ddot{x}=(\alpha^2-1)x+y+\alpha z$	$y=\ddot{x}-\alpha\dot{x}+x$ $z=\dot{x}-\alpha x$
H	$\dot{x}=-y+z^2$ $\dot{y}=x+\alpha y$ $\dot{z}=x-z$	$\dddot{z}=(\alpha-1)\ddot{z}+(\alpha-1)\dot{z}-z$ $-\alpha z^2+2z\dot{z}$	$\dot{z}=-z+x$ $\ddot{z}=z+z^2-x-y$	$x=\dot{z}+z$ $y=-\ddot{z}-\dot{z}+z^2$

(Continued)

Model	*Equations*	*Jerky Dynamics*	*Transformations*	*Inverse Transformations*
I	$\dot{x}=-\alpha y$ $\dot{y}=x+z$ $\dot{z}=x+y^2-z$	$\dddot{x}=\ddot{x}-\alpha\dot{x}-2\alpha x$ $-\frac{1}{\alpha}\dot{x}^2$	$\dot{x}=-\alpha y$ $\ddot{x}=-\alpha x-\alpha z$	$y=-\frac{1}{\alpha}\dot{x}$ $z=-\frac{1}{\alpha}\ddot{x}-x$
		$\dddot{y}=-\ddot{y}-\alpha\dot{y}-2\alpha y$ $+2y\dot{y}$	$\dot{y}=x+z$ $\ddot{y}=-\alpha y+y^2+x-z$	$x=\frac{1}{2}(\ddot{y}+\dot{y}+\alpha y-y^2)$ $z=\frac{1}{2}(-\ddot{y}+\dot{y}-\alpha y+y^2)$
J	$\dot{x}=\alpha z$ $\dot{y}=-\beta y+z$ $\dot{z}=-x+y+y^2$	$\dddot{y}=-\beta\ddot{y}+(1-\alpha)\dot{y}$ $-\alpha\beta y+2y\dot{y}$	$\dot{y}=-\beta y+z$ $\ddot{y}=(1+\beta^2)y+y^2$ $-x-\beta z$	$x=-\ddot{y}-\beta\dot{y}+y+y^2$ $z=-\ddot{y}-\dot{y}+y\dot{y}+y^2$
K	$\dot{x}=xy-z$ $\dot{y}=x-y$ $\dot{z}=x+\alpha z$	$\dddot{y}=(\alpha-1)\ddot{y}+(\alpha-1)\dot{y}$ $-y+y\ddot{y}+(2-\alpha)y\dot{y}$ $-\alpha y^2+\dot{y}^2$	$\dot{y}=-y+x$ $\ddot{y}=y-x-z+xy$	$x=\dot{y}+y$ $z=-\ddot{y}-\dot{y}+y\dot{y}+y^2$
L	$\dot{x}=y+\alpha z$ $\dot{y}=\beta x^2-y$ $\dot{z}=\gamma-x$	$\dddot{x}=-\ddot{x}-\alpha\dot{x}-\alpha x$ $+2\beta x\dot{x}+\alpha\gamma$	$\dot{x}=y+\alpha z$ $\ddot{x}=-\alpha x+\beta x^2-y$ $+\alpha\gamma$	$y=-\ddot{x}-\alpha x+\beta x^2+\alpha\gamma$ $z=\frac{1}{\alpha}(\ddot{x}+\dot{x}+\alpha$ $x-\beta x^2-\alpha\gamma)$
		$\dddot{z}=-\ddot{z}+(2\beta\gamma-\alpha)\dot{z}$ $-\beta\dot{z}^2-\beta\gamma^2$	$\dot{z}=\gamma-x$ $\ddot{z}=-\alpha z-y$	$x=\gamma-\dot{z}$ $y=-\ddot{z}-\alpha z$
M	$\dot{x}=-z$ $\dot{y}=-x^2-y$ $\dot{z}=\alpha+\beta x+y$	$\dddot{x}=-\ddot{x}-\beta\dot{x}-\beta x$ $+x^2-\alpha$	$\dot{x}=-z$ $\ddot{x}=-\beta x-y-\alpha$	$y=-\ddot{x}-\beta x-\alpha$ $z=-\dot{x}$
N	$\dot{x}=-\alpha y$ $\dot{y}=x+z^2$ $\dot{z}=\beta+y-\gamma z$	$\dddot{z}=-\gamma\ddot{z}-\alpha\dot{z}-\alpha\gamma z$ $+2z\dot{z}+\alpha\beta$	$\dot{z}=-\gamma z+y+\beta$ $\ddot{z}=\gamma^2 z+z^2+x-\gamma y$ $-\beta\gamma$	$x=\ddot{z}+\gamma\dot{z}-z^2$ $y=\dot{z}+\gamma z-\beta$
O	$\dot{x}=y$ $\dot{y}=x-z$ $\dot{z}=x+xz+\alpha y$	$\dddot{x}=x\ddot{x}+(1-\alpha)\dot{x}-x$ $-x^2$	$\dot{x}=y$ $\ddot{x}=x-z$	$y=\dot{x}$ $z=-\ddot{x}+x$

(Continued).

Model	*Equations*	*Jerky Dynamics*	*Transformations*	*Inverse Transformations*
P	$\dot{x}=\alpha y+z$ $\dot{y}=-x+y^2$ $\dot{z}=x+y$	$\dddot{y}=2y\ddot{y}+(1-\alpha)\dot{y}$ $-y+2\dot{y}^2-y^2$	$\dot{y}=y^2-x$ $\ddot{y}=2y^3-\alpha y$ $-2yx-z$	$x=-\dot{y}+y^2$ $z=-\ddot{y}+2y\dot{y}-\alpha y$
Q	$\dot{x}=-z$ $\dot{y}=x-y$ $\dot{z}=\alpha x+y^2+\beta z$	$\dddot{y}=(\beta-1)\ddot{y}+(\beta-\alpha)\dot{y}$ $-\alpha y-y^2$	$\dot{y}=-y+x$ $\ddot{y}=y-x-z$	$x=\dot{y}+y$ $z=-\ddot{y}-\dot{y}$
R	$\dot{x}=\alpha-y$ $\dot{y}=\beta+z$ $\dot{z}=xy-z$	$\dddot{x}=-\ddot{x}-\alpha x+x\dot{x}-\beta$	$\dot{x}=-y+\alpha$ $\ddot{x}=-z-\beta$	$y=-\dot{x}+\alpha$ $z=-\ddot{x}-\beta$
S	$\dot{x}=-x-\alpha y$ $\dot{y}=x+z^2$ $\dot{z}=\beta+x$	$\dddot{z}=-\ddot{z}-\alpha\dot{z}-\alpha z^2+\alpha\beta$	$\dot{z}=x+\beta$ $\ddot{z}=-x-\alpha y$	$x=\dot{z}-\beta$ $y=\frac{1}{\alpha}(-\ddot{z}-\dot{z}+\beta)$
TR	$\dot{x}=-y-z$ $\dot{y}=x$ $\dot{z}=\alpha(y-y^2)-\beta z$	$\dddot{y}=-\beta\ddot{y}-\dot{y}-(\alpha+\beta)y$ $+\alpha y^2$	$\dot{y}=x$ $\ddot{y}=-y-z$	$x=\dot{y}$ $z=-\ddot{y}-y$
GS	$\dot{x}=y$ $\dot{y}=z$ $\dot{z}=-ax-by-cz+x^2$	$\dddot{x}=-c\ddot{x}-b\dot{x}-ax+x^2$	$\dot{x}=y$ $\ddot{x}=z$	$y=\dot{x}$ $z=\ddot{x}$

Table 5.2: Basic Classes of Dissipative Jerky Dynamics.

Model	Basic classes coefficients = values for which there is irregular behavior				*Transformation*
$\boxed{JD_1}$	$\dddot{\xi} = k_1\ddot{\xi} + k_2\xi + \xi\dot{\xi} + k_3$				
I	$k_1 = -1$	$k_2 = -2\alpha = -0.4$	$k_3 = -2\alpha^2 = -0.08$		$\xi = 2y - \alpha$
J	$k_1 = -\beta = -2$	$k_2 = -\alpha\beta = -4$	$k_3 = \alpha\beta(1-\alpha) = -4$		$\xi = 2y + (1-\alpha)$
L	$k_1 = -1$	$k_2 = -\alpha = -3.9$	$k_3 = \alpha(2\beta\gamma - \alpha) = -8.19$		$\xi = 2\beta x - \alpha$
N	$k_1 = -\gamma = -2$	$k_2 = -\alpha\gamma = -4$	$k_3 = \alpha(2\beta - \alpha\gamma) = -4$		$\xi = 2z - \alpha$
R	$k_1 = -1$	$k_2 = -\alpha = -0.9$	$k_3 = -\beta = -0.4$		$\xi = x$
SJ	$k_1 = -A = -2.017$	$k_2 = -1$	$k_3 = 0$		$\xi = v$
$\boxed{JD_2}$	$\dddot{\xi} = k_1\ddot{\xi} + k_2\dot{\xi} + \xi^2 + k_3$				
M	$k_1 = -1$	$k_2 = -\beta = -1.7$	$k_3 = -\alpha - \frac{\beta^2}{4} = -2.4225$		$\xi = x - \frac{\beta}{2}$
Q	$k_1 = \beta - 1 = -0.5$	$k_2 = \beta - \alpha = -2.6$	$k_3 = -\frac{\alpha^2}{4} = -2.4025$		$\xi = -y - \frac{\alpha}{2}$
S	$k_1 = -1$	$k_2 = -\alpha = -4$	$k_3 = -\alpha^2\beta = -16$		$\xi = -\alpha z$
TR	$k_1 = -\beta = -0.2$	$k_2 = -1$	$k_3 = -\frac{1}{4}(\alpha+\beta)^2 = -0.0858$		$\xi = \alpha y - \frac{1}{2}(\alpha+\beta)$
GS	$k_1 = -c$	$k_2 = -b$	$k_3 = -\frac{a^2}{4}$		$\xi = x - \frac{a}{2}$
$\boxed{JD_3}$	$\dddot{\xi} = k_1\ddot{\xi} + k_2\dot{\xi} + k_3\xi^2 + \xi\dot{\xi} + k_4$				
F	$k_1 = \alpha - 1 = -0.5$	$k_2 = \alpha - \frac{1}{\alpha} - 1 = -2.5$	$k_3 = -\frac{\alpha}{2} = -0.25$	$k_4 = \frac{1}{2\alpha} = 1$	$\xi = 2x + \frac{1}{\alpha}$
G	$k_1 = \alpha - 1 = -0.6$	$k_2 = \alpha - \frac{1}{2\alpha} - 1 = -1.85$	$k_3 = -\alpha = -0.4$	$k_4 = \frac{1}{4\alpha} = 0.625$	$\xi = x + \frac{1}{2\alpha}$
H	$k_1 = \alpha - 1 = -0.5$	$k_2 = \alpha - \frac{1}{\alpha} - 1 = -2.5$	$k_3 = -\frac{\alpha}{2} = -0.25$	$k_4 = \frac{1}{2\alpha} = 1$	$\xi = 2z + \frac{1}{\alpha}$
$\boxed{JD_4}$	$\dddot{\xi} = k_1\ddot{\xi} + k_2\dot{\xi} + k_3\xi^2 + \xi\ddot{\xi} + k_4$				
O	$k_1 = -\frac{1}{2}$	$k_2 = 1 - \alpha = -1.7$	$k_3 = -1$	$k_4 = \frac{1}{4}$	$\xi = x + \frac{1}{2}$
$\boxed{JD_5}$	$\dddot{\xi} = k_1\dot{\xi} + k_2\xi^2 + k_3\dot{\xi}^2 + \xi\ddot{\xi}$				
D	$k_1 = -1$	$k_2 = -1$	$k_3 = -\alpha = -3$		$\xi = x$
$\boxed{JD_6}$	$\dddot{\xi} = k_1\ddot{\xi} + k_2\dot{\xi} + k_3\xi^2 + k_4\dot{\xi}^2 + \xi\ddot{\xi} + k_5$				
P	$k_1 = -1$	$k_2 = 1 - \alpha = -1.7$	$k_3 = -\frac{1}{2}$	$k_4 = 1$	$\xi = 2y + 1$
	$k_5 = \frac{1}{2}$				
$\boxed{JD_7}$	$\dddot{\xi} = k_1\ddot{\xi} + k_2\dot{\xi} + k_3\xi^2 + k_4\dot{\xi}^2 + k_5\xi\dot{\xi} + \xi\ddot{\xi} + k_6$				
K	$k_1 = \alpha - \frac{1}{2\alpha} - 1 = -2.37$	$k_2 = \alpha - \frac{1}{\alpha} - \frac{1}{2} = -3.53$	$k_3 = -\alpha = -0.3$	$k_4 = 1$	$\xi = y + \frac{1}{2\alpha}$
	$k_5 = 2 - \alpha = 1.7$			$k_6 = \frac{1}{4\alpha} = 0.83$	

APPENDIX

Mathematica Source Codes

This section provides listings of relevant Mathematica source codes of some figures in Chapter 4.
Following third code to plot the first return map (Fig. 4.5) requires installing any version of Dynamical Systems Package (DynPac) before all else.

1. Bifurcation diagram

$\alpha = 0.75;$

$\mathbf{Pas}\beta = \mathbf{0.01};$

$\mathbf{TablePlot} = \{\};$

$\mathbf{For}[\beta = 0.58, \beta \leq 0.81, \beta = \beta + \mathbf{Pas}\beta,$

$\mathbf{solution} = \mathbf{NDSolve}[\{x'[t] == y[t], x[0] == -1.005, y'[t] == z[t], y[0] == 0.05,$

$z'[t] == -\alpha y[t] - \beta(x[t]\text{^}2 - 1) - z[t], z[0] == 0.05\}, \{x, y, z\}, \{t, 0, 500\},$

$\mathbf{Method} \rightarrow \mathbf{ExplicitRungeKutta}];$

$\mathbf{Pas}\beta = \mathbf{Which}[\beta < 0.65, 0.0001, (\beta \geq 0.65)\&\&(\beta < 0.75), 0.0001, \beta \geq 0.75, 0.0001];$

$\mathbf{Print}[\text{"}\beta\text{= "}, \beta];$

$\mathbf{Pas} = \mathbf{0.05};$

$\mathbf{Tendance} = \mathbf{0};$

$\mathbf{BorneBasse} = \mathbf{400};$

$\mathbf{BorneHaute} = \mathbf{500};$

$\mathbf{Precedent} = \mathbf{Evaluate}[\{x[\mathbf{BorneBasse}]\}/.\mathbf{solution}][[1]][[1]];$

$\mathbf{Tendance} = \mathbf{Which}[\mathbf{Precedent} \geq \mathbf{Evaluate}[\{x[\mathbf{BorneBasse} + \mathbf{Pas}]\}/.\mathbf{solution}][[1]][[1]],$

```
−1, Precedent < Evaluate[{x[BorneBasse + Pas]}/.solution][[1]][[1]], 1];
For[Valeur = BorneBasse + Pas, Valeur ≤ BorneHaute, Valeur = Valeur + Pas,
Actuel = Evaluate[{x[Valeur]}/.solution][[1]][[1]];
If[(Actuel ≥ Precedent), Goto[NonMaximum]];
If[Actuel < Precedent&&Tendance == −1, Goto[ActualisePrecedent]];
TablePlot =
Append[TablePlot, {N[FromDigits[RealDigits[β]], 6],
N[FromDigits[RealDigits[Evaluate[{x[Valeur − Pas]}/.solution][[1]][[1]]]],
6]}];
Tendance = −1;
Goto[ActualisePrecedent];
Label[NonMaximum];
Tendance = 1;
Goto[ActualisePrecedent];
Label[RechercheFine];
Print["t= "; Valeur];
Label[PremieresValeurs];
If[Valeur ≠ BorneBasse,
Tendance = Which[Actuel < Precedent, −1, Actuel ≥ Precedent, 1]];
Label[ActualisePrecedent];
Precedent = Actuel;
Label[FinTraitement]; ]]


ListPlot[TablePlot,
PlotStyle → {PointSize[0.005], Black},
ImageSize → {400, 400}, PlotRange → {{0.655, 0.775}, {1.7, 1.91}}, Frame → True,
```

FrameLabel → {"β", "x_{max}"}, RotateLabel → False]

2. Time series, Phase space, Poincaré Section

```
Manipulate[Module[{sol1, tbl, sol2}, α = 2.03;
sol1 = NDSolve[{x'[t] == y[t], y'[t] == z[t], z'[t] == −z[t] − αy[t] − β(x[t]^2 − 1),
x[0] == −1.005, y[0] == 0.05, z[0] == 0.05}, {x[t], y[t], z[t]}, {t, 0, 400},
MaxSteps → Infinity];
If[selection == "Poincaré section",
sol2 = NDSolve[{x'[t] == y[t], y'[t] == z[t], z'[t] == −z[t] − αy[t] − β(x[t]^2 − 1),
x[0] == −1.005, y[0] == 0.05, z[0] == 0.05}, {x[t], y[t], z[t]}, {t, 0, 4000},
MaxSteps → Infinity];
tbl = Table[Evaluate[{First[x[t]/.sol2], First[y[t]/.sol2]}],
{t, 100, 4000, ControlActive[0, 10]}]];
Which[selection == "time series",
Plot[Evaluate[x[t]/.sol1], {t, 100, 300}, PlotStyle → RGBColor[1, 0.47, 0],
ImageSize → {550, 375}], selection == "phase space",
ParametricPlot[Evaluate[{First[x[t]/.sol1], First[y[t]/.sol1]}],
{t, 100, 200}, ImageSize → {550, 375}, PlotPoints → 1000],
selection == "Poincaré section",
ListPlot[tbl, ImageSize → {550, 375},
PlotStyle → {PointSize[0.0075], RGBColor[1, 0, 0]},
PlotRange → {{−2, 2}, {−2, 2}}]]],
{{β, −1.7, "bif param"}, −2, 2, 0.1, Appearance → "Labeled"},
{{selection, "time series"}, {"time series", "phase space", "Poincaré section"}},
```

TrackedSymbols :→ {β, selection}]

3. First return map

```
sysid
intreset;
plotreset;
setstate[{x, y, z}]; setparm[{α, β}]; sysname = " ";
slopevec = {y, z, −z − α * y − β(x^2 − 1)};
parmval = {2.03, −1.9};
t0 = 0.0; h = 0.01; initvec = {−1.005, 0.05, 0.05};
integrate[initvec, t0, h, 500];
bigsol = integrate[lastx, t0, h, 30000];
bigsol[[1]]
bigsol[[10000]]
bigsol[[30001]]
modsol = stripsol[stripsol[stripsol[bigsol, 1], 1], 0];
modsol[[1]]
modsol[[10000]]
modsol[[30001]]
maxQ[left_, mid_, right_]:=((mid > left)&&(mid > right))
maxQ[1., 2., 1.5]
maxQ[1., 2., 3.]
findmax[xlist_]:=Module[{len, ans, i, modxlist}, modxlist = Flatten[xlist];
len = Length[modxlist];
```

```
ans = {};Do[(left = modxlist[[i]];mid = modxlist[[i + 1]];right = modxlist[[i + 2]];
If[maxQ[left, mid, right], ans = Append[ans, mid]]), {i, 1, len − 2}];ans]
maxlist = findmax[modsol];
Length[maxlist]
Max[maxlist]
−Max[−maxlist]
plotlist[maxlist_]:=Module[{ans, i, len}, len = Length[maxlist];ans = {};
Do[ans = Append[ans, {maxlist[[i]], maxlist[[i + 1]]}], {i, 1, len − 1}];ans]
plotter = plotlist[maxlist];
lorgraph = ListPlot[plotter, PlotRange → {{0, 2.2}, {0, 2.2}},
AxesLabel → {"x(n)", "x(n+1)"}, PlotLabel → "First return map", ImageSize → 300,
AspectRatio → 1];
```

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