

A CORRESPONDENCE OF SIMPLE ALCAHESTIC GROUP FUNCTORS

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August 2008

I certify that I have read this thesis and that in my opinion it is fully adequate,
in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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Representation theory of finite groups associates two classical constructions to a group G , namely the representation ring of G and the Burnside ring of G . These rings share a special structure that comes from three classical maps, namely restriction, conjugation, and transfer maps. These are not the only objects having this structure and the theory of Mackey functors, introduced by Green, unifies the treatment of such objects.

The above constructions share a further structure that comes from two other maps, the inflation map and the deflation map. Unified treatment of the objects having this further structure was introduced by Bouc [4]. These objects are called biset functors.

Between Mackey functors and biset functors there lies more natural constructions, for example the functor of group (co)homology. In order to handle these intermediate structures, Bouc introduced another concept, now known as globally-defined Mackey functors, a name given by Webb.

In this thesis, we unify the above theories by considering the algebra whose module category is equivalent to the category of biset functors and by introducing alcahestic group functors. Our main results classify and describe simple alcahestic group functors and give a criterion of semisimplicity for the categories of these functors.

Keywords: biset functor, Mackey functor, GDMF, (alcahestic) group functor, alchemic algebra, simple functor, induction, coinduction, semisimplicity, mark morphism.

ÖZET

BASİT ÖBEK İZLEÇLERİNİN BİR EŞLEMESİ

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Sonlu öbeklerin adlanım kuramı sonlu bir öbek olan G 'ye iki geleneksel kurulum ilişkilendirmektedir. Bunlar G 'nin adlanım dolamı ve G 'nin Burnside dolamıdır. Bu dolamlar üç geleneksel işlevden gelen özel bir yapıyı paylaşmaktadırlar. Bu işlevler kısıtlav, eşlev ve aktarım işlevleridir. Bu yapıya sahip başka doğal kurulumlarda bulunmaktadır ve Green tarafından ortaya konan Mackey izleçleri kuramı bu tür nesneleri incelemektedir.

Yukarıdaki kurulumlar şişirme ve söndürme işlevlerinden gelen bir yapıyı da paylaşmaktadırlar. Bu ileri yapıya sahip nesnelerin incelenmesini sağlayan iki etki izleçleri Bouc tarafından keşfedilmiştir.

Mackey izleçleri ile iki etki izleçleri arasında bazı doğal kurulumlar bulunmaktadır. Örneğin öbek (eş)benzeti izleci. Bu tür ara yapıları inceleyebilmek için Bouc şimdi küresel tanımlı Mackey izleçleri olarak bilinen kavramı ortaya koymuştur. Bu izleçlerin ismi Webb tarafından verilmiştir.

Bu savda yukarıdaki kuramları birarada inceleyeceğiz. Bunu yapmak için parça ulamı iki etki izleçlerinin ulamına denk olan cebiri düşüneceğiz. Ana sonuçlarımız basit öbek izleçlerini sınıflandıracak ve tanımlarını yapacak. Ayrıca bu tür ulamlar için bir yarı-basitlilik ölçütü ispatlayacağız.

Anahtar sözcükler: İki etki izleci, Mackey izleci, öbek izleci, simya cebiri, basit izleç, yarı basitlilik, işaret yapı dönüşümü.

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Zeynep Naz'a.

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Chapter 1

Introduction

Representation theory of finite groups associates two classical constructions to a group G , namely the representation ring of G and the Burnside ring of G . These rings share a special structure that comes from three classical maps, namely restriction map, conjugation map and transfer map. These are not the only objects having this structure and the theory of Mackey functors, introduced by Green [14], unifies the treatment of such objects.

The theory of Mackey functors is now well understood by the works of Boltje, Bouc, Dress, Lindner, Thévenaz, Webb and others, see [21] for a list of references. Notably, Thévenaz and Webb realized Mackey functors as modules of a finite dimensional algebra, called the Mackey algebra. This enables them to consider simple Mackey functors, projective Mackey functors, indecomposable Mackey functors, etc.

Over the years, the theory has found many applications as well as adaptations to infinite groups. A remarkable application is the theory of canonical induction formulae introduced by Boltje [2]. This theory constructs explicit versions of known induction theorems, including Brauer's induction theorem, which serves as a prototype for potential induction theorems. There is also a version of Alperin's weight conjecture in terms of Mackey functors [18].

Aiming to study the structure coming from the above three maps, the theory of Mackey functors could not capture a further structure given by a pair of maps, the inflation map and the deflation map. Unified treatment of the objects having this further structure was introduced by Bouc [4]. His main observation is that all of these five maps are induced by certain bisets, sets with double action, and conversely any biset is composed of these five basic bisets. Hence these objects are called biset functors.

The theory of biset functors has found several immediate applications where the main one is to the Dade group of a p -group. Along with the results of Alperin, Carlson, Dade, Mazza, Puig and Thévenaz, the final determination of the structure of the Dade group is done by Bouc using the techniques of biset functors.

Between Mackey functors and biset functors there lies more natural constructions, for example the functor of group (co)homology. In order to handle these intermediate structures, Bouc introduced another concept, now known as globally-defined Mackey functors (GDMFs), a name given by Webb. This intermediate theory is successfully applied by Webb to determine stable decompositions of classifying spaces.

For almost all of the above applications, one of the main tools is the knowledge of the simple functors. Being the building blocks of all functors, simple functors carry crucial information regarding the structure of the functor in question. Simple Mackey functors are described by Thévenaz and Webb [17], whereas simple GDMFs are described by Bouc [4]. Note that the methods used in each case differ from one another. Also there are descriptions of certain types of GDMFs by Webb [20], namely of global Mackey functors and inflation functors, see Section 3.2.2 for definitions.

In this thesis, we unify the above theories by considering the algebra whose module category is equivalent to the category of biset functors. One of our main results unifies the classification and description of above simple functors.

Now we need to introduce some notation in order to give a summary of the

thesis. Let G and H be finite groups. A (finite) (G, H) -biset is a (finite) set with a left G -action and a commuting right H -action. Consider the category $\mathcal{C}$ whose objects are finite groups with morphisms given by bisets. The composition in the category $\mathcal{C}$ is given by the usual amalgamated product of bisets, see Section 2.1. A **biset functor** over a commutative ring R is a functor $\mathcal{C} \rightarrow R\text{-mod}$.

The functor category of biset functors is abelian. By a result of Mitchell [16], this category is equivalent to the module category of the so-called category algebra of $\mathcal{C}$. Here the category algebra of a category $\mathcal{D}$ is the algebra generated by the morphisms in the category. In our case the category algebra of $\mathcal{C}$ is generated by all bisets. By the above mentioned result of Bouc, this algebra is generated by the five basic type of bisets. Following Barker, we call this algebra the **alchemic algebra**¹.

Note that since the category $\mathcal{C}$ contains infinitely many objects, the alchemic algebra is not finitely generated. Although it is possible to work with this algebra, we restrict ourselves to a fixed finite group. That is, we consider the full subcategory $\mathcal{C}_G$ of $\mathcal{C}$ consisting only of the subquotients of G . Here a subquotient of G is a pair (H^*, H_*) of subgroups of G where $H_* \trianglelefteq H^* \leq G$. We denote the pair (H^*, H_*) by H and write $H \preceq G$. See Section 2.3 for details. With this notation we consider biset functors for G over R . Note further that our results can be generalized to the infinite scenario very easily.

Now the alchemic algebra Γ_G for G is finitely generated, generated by the five type of basic bisets together with the relations induced by the composition of these bisets. Although we have obtained a finitely generated algebra, there are so many relations between the generators that cannot be specified in a tractable way. In order to get rid of this problem, we introduce two new amalgamated variables, namely **tinflation** which is the composition of the transfer biset and the inflation biset, and **destriction**, composition of the deflation biset and the restriction biset. The fifth element remains the same but we call it **isogation**. In

¹The name refers to the five elements of the nature in alchemy; air, fire, water, earth and the fifth element known as *quintessence* or *aether*. Two elements, transfer and inflation, go upwards as the two elements fire and air. Similarly deflation and restriction go downwards as water and earth, see [1].

this way, the relations become not only tractable but also similar to the relations between the generators of the Mackey algebra.

Note that in [21], Webb specified a list of relations that works for globally-defined Mackey functors. Our list of relations, although obtained using the Mackey product, can also be obtained from these relations with some straightforward calculations. Also in [1], Barker constructed the alchemic algebra, without specifying the relations between the five generators explicitly, for a family of finite groups closed under taking subquotients and isomorphisms. This covers our case when the family is the family of subquotients of G .

The first main result of the thesis is a correspondence theorem for simple modules of certain subalgebras of the alchemic algebra. Given a subalgebra Π of the alchemic algebra, a Π -module will be called a **Π -group functor**, or a **group functor over Π** . We call a subalgebra **alcahestic** if it satisfies conditions A1-A4 of Section 3.2.2.2. These are the conditions that are satisfied by the algebras of immediate interest, like the alchemic algebra or the Mackey algebra. In this case, a Π -group functor is called an **alcahestic group functor over Π** .

The General Correspondence Theorem of Section 4.1 states that the parameterizing set for simple alcahestic group functors over Π are determined only by the set of isogations contained in Π . More precisely, let Ω_Π denote the (alcahestic) subalgebra of Π generated by all isogations in Π . Then

Theorem 1 (See Theorem 4.1.2) *Let Π be an alcahestic subalgebra of the alchemic algebra. Then there is a bijective correspondence between*

- *the set of isomorphism classes of simple Π -modules and*
- *the set of isomorphism classes of simple Ω_Π -modules.*

In Section 4.1, we also prove that the algebra Ω_Π is Morita equivalent to a product of certain group algebras. This way, we recover the classification theorems of Bouc, Thévenaz - Webb and Webb.

It is easy to describe the map from the set of isomorphism classes of simple alcahestic group functors over Π to that of the algebra Ω_Π . On the other hand the inverse correspondence necessarily constructs simple Π -modules. Our second main result is the description of the inverse correspondence, and hence the description of simple modules. We need more notations.

Any alcahestic subalgebra Π comes with a triple of subalgebras $(\Delta_\Pi, \Omega_\Pi, \nabla_\Pi)$. The middle algebra is the **isogation algebra** associated to Π and is the one described above. The algebra Δ_Π is the subalgebra of Π generated by all tinflation and isogation maps in Π . It is called the **tinflation algebra** associated to Π . Similarly we have the **destriction algebra** ∇_Π associated to Π .

By definition, each of these algebras are alcahestic and hence by the General Correspondence Theorem, simple modules of each of these algebras are parameterized by the same set. Observing that the isogation algebra Ω_Π is also a quotient of each of Δ_Π and ∇_Π , we see that the set of simple modules of Δ_Π and that of ∇_Π are just the inflations from Ω_Π . Now with the Special Correspondence Theorem below, we obtain two descriptions of simple Π -modules.

Theorem 2 (See Theorem 4.2.3) *Let $\Pi \subset \Theta$ be two alcahestic subalgebras and S be a simple Π -module corresponding to the simple Ω_Π -module $\mathfrak{S}$.*

- *The Θ -module $\text{Ind}_\Pi^\Theta S$ has a unique maximal submodule provided that $\nabla_\Pi = \nabla_\Theta$. In this case, the simple quotient is also parameterized by $\mathfrak{S}$.*
- *The Θ -module $\text{Coind}_\Pi^\Theta S$ has a unique minimal submodule provided that $\Delta_\Pi = \Delta_\Theta$. In this case, the simple submodule is also parameterized by $\mathfrak{S}$.*

It is clear from the statement of this theorem that to get a description of simple Π -modules, we need to know the description of either induction from the destriction algebra together with a description of the maximal submodule of the above induced module or coinduction from the tinflation algebra together with a description of the minimal submodule of the above coinduced module. As an example of the Special Correspondence Theorem, in Chapter 5, we construct simple

biset functors and simple (ordinary) Mackey functors. These two constructions serve as a prototype for any other choices of alcahestic subalgebras.

The final chapter of the thesis concerns the semisimplicity problem of alcahestic subalgebras. It is well-known, by a result of Thévenaz and Webb, that the Mackey algebra is semisimple over a field of characteristic zero. It is also known by a result of Bouc that the alchemic algebra for G is semisimple over a field of characteristic zero if and only if G is cyclic. These results are proved using different methods. By associating a mark morphism to each simple module, we obtain the following criterion of semisimplicity.

Theorem 3 (See Theorem 6.2.4) *Let Π be an alcahestic subalgebra of the alchemic algebra for G over a field of characteristic zero. The following are equivalent.*

- *The algebra Π is semisimple.*
- *The mark morphism associated to any simple Π -module is an isomorphism.*

We end the thesis by an adaptation of this criterion to arbitrary finite dimensional algebras. Note that this completes a well-known result in module theory.

Theorem 4 (See Theorem 6.3.1) *Let A be a finite dimensional algebra over a field and e be an idempotent of A . Let $f = 1 - e$. Then the following are equivalent*

- *The algebra A is semisimple.*
- *The following two conditions hold.*
 1. *The algebras eAe and fAf are semisimple.*
 2. *For any simple gAg -module V , for $g \in \{e, f\}$, there is an isomorphism of A -modules $Ag \otimes_{gAg} V \cong \text{Hom}_{gAg}(gA, V)$.*

Finally I would like to remark that the content of the thesis is a reviewed version of the paper [13].

Chapter 2

Preliminaries and Conventions

2.1 Bisets

In this section, we introduce the concept of bisets. This is the standard theory of bisets and can be found in [5], or in [1].

Let H and K be two finite groups. An (H, K) -**biset** is a set with a left H -action and a right K -action such that

$$h \cdot (x \cdot k) = (h \cdot x) \cdot k$$

for all elements $h \in H$ and $k \in K$ and $x \in X$.

An (H, K) -biset X is called **transitive** if for any elements $x, y \in X$ there exists an element $h \in H$ and an element $k \in K$ such that $h \cdot x \cdot k$ is equal to y .

We can regard any (H, K) -biset X as a right $H \times K$ -set equipped with the action given by

$$x \cdot (h, k) = h^{-1} \cdot x \cdot k$$

for all $x \in X$ and $h \in H$ and $k \in K$. Clearly, X is a transitive (H, K) -biset if and only if X is a transitive $H \times K$ -set. Hence there is a bijective correspondence between

- (i) isomorphism classes $[X]$ of transitive (H, K) -bisets,
- (ii) conjugacy classes $[L]$ of subgroups of $H \times K$

where the correspondence is given by $[X] \leftrightarrow [L]$ if and only if the stabilizer of a point $x \in X$ is in $[L]$.

Hereafter we denote a transitive biset with point stabilizer equal to L by $(\frac{H \times K}{L})$. Recall that the Grothendieck group of the category of $H \times K$ -sets is called the **Burnside group** of $H \times K$ and is denoted by $B(H \times K)$. It is the free abelian group on the set of isomorphism classes of transitive $H \times K$ -sets. Under the above identifications, we also denote by $B(H \times K)$ the Grothendieck group of the category of (H, K) -bisets.

We define a composition product of bisets as follows. Given finite groups H, K, M and an $H \times K$ -biset X and a $K \times M$ -biset Y . We define the **Mackey product** $X \times_K Y$ of X and Y as the set

$$X \times_K Y := X \times Y / K$$

of K -orbits of the Cartesian product $X \times Y$. Here K acts via $k \cdot (x, y) := (x \cdot k^{-1}, k \cdot y)$. The set $X \times_K Y$ is an $H \times M$ -biset via

$$h \cdot (x, {}_K y) \cdot m := (h \cdot x, {}_K y \cdot m)$$

where $h \in H, m \in M$ and $(x, {}_K y)$ denotes the image of (x, y) . By linear extension, the Mackey product induces a bilinear map

$$B(H \times K) \times B(K \times M) \longrightarrow B(H \times M).$$

For the transitive bisets $(\frac{H \times K}{L})$ and $(\frac{K \times M}{N})$, the Mackey product is explicitly given by

$$\left(\frac{H \times K}{L}\right) \times_K \left(\frac{K \times M}{N}\right) = \sum_{x \in p_2(L) \backslash K / p_1(N)} \left(\frac{H \times M}{L * {}^{(x,1)}N}\right)$$

where the subgroup $L * N$ of $H \times M$ is defined by

$$L * N = \{(h, m) \in H \times M : (h, k) \in L \text{ and } (k, m) \in N \text{ for some } k \in K\}$$

and the subgroup $p_1(N)$ (resp. $p_2(L)$) of K is the projection of N (resp. of L) with K . In other words

$$p_1(L) = \{l \in H : (l, k) \in L \text{ for some } k \in K\}$$

and

$$p_2(L) = \{k \in K : (l, k) \in L \text{ for some } l \in L\}.$$

In [5], Bouc proved that any transitive biset is a Mackey product of five special types of bisets. Next we recall this result. First we introduce some notation. Let H be a finite group and $N \trianglelefteq J$ be subgroups of H and let L, M be two isomorphic finite groups with a fixed isomorphism $\phi : L \rightarrow M$, then the five basic bisets are given as follows.

1. **Induction (or Transfer) (H, J) -biset:** $\text{Ind}_J^H := (\frac{H \times J}{T})$ where $T = \{(j, j) : j \in J\}$.
2. **Inflation $(J, J/N)$ -biset:** $\text{Inf}_{J/N}^J := (\frac{J \times J/N}{I})$ where $I = \{(j, jN) : j \in J\}$.
3. **Isomorphism (M, L) -biset:** $c_{M,L}^\phi = (\frac{M \times L}{C^\phi})$ where $C^\phi = \{(\phi(l), l) : l \in L\}$.
4. **Deflation $(J/N, J)$ -biset:** $\text{Def}_{J/N}^J = (\frac{J/N \times J}{D})$ where $D = \{(jN, j) : j \in J\}$.
5. **Restriction (J, H) -biset:** $\text{Res}_J^H = (\frac{J \times H}{R})$ where $R = \{(j, j) : j \in J\}$.

Remark 2.1.1 *Throughout the thesis, we use notations $\text{Ind}_J^H \text{Res}_J^G, \text{Inf}_J^K \text{Res}_J^G$ etc. instead of $\text{Ind}_J^H \times_J \text{Res}_J^G, \text{Inf}_J^K \times_J \text{Res}_J^G$ etc.*

The following theorem explicitly shows the decomposition of any transitive biset in terms of the above basic bisets. See [5] for a proof of this result.

Theorem 2.1.2 (Bouc) *Let L be any subgroup of $H \times K$. Then*

$$\left(\frac{H \times K}{L} \right) = \text{Ind}_{p_1(L)}^H \text{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} c_{p_1(L)/k_1(L), p_2(L)/k_2(L)}^\phi \text{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \text{Res}_{p_2(L)}^K$$

where the subgroup $k_1(L)$ of H and the subgroup $k_2(L)$ of K are given by

$$k_1(L) = \{h \in H : (h, 1) \in L\} \quad \text{and} \quad k_2(L) = \{k \in K : (1, k) \in L\}.$$

The isomorphism

$$\phi : p_2(L)/k_2(L) \rightarrow p_1(L)/k_1(L)$$

is the one given by associating $lk_2(L)$ to $mk_1(L)$ where for a given element $l \in p_2(L)$ we let m be the unique element in $p_1(L)$ such that $(m, l) \in L$.

2.2 Biset Functors

Let R be a commutative ring. Fix a set $\mathcal{G}$ of representatives of isomorphism classes of finite groups. Let $\mathcal{C} := \mathcal{C}_R^{\mathcal{G}}$ be the following category.

- The objects of $\mathcal{C}$ are the groups in $\mathcal{G}$.
- Given G, H in $\mathcal{G}$, we set

$$\mathrm{Hom}_{\mathcal{C}}(G, H) := RB(H \times G) := R \otimes_{\mathbb{Z}} B(H \times G).$$

- Composition in $\mathcal{C}$ is given by the R -linear extension of the bilinear map induced by the Mackey product.

Now a **biset functor** over R is an R -linear functor $\mathcal{C} \longrightarrow R\text{-mod}$. Biset functors over R form a category together with natural transformations of functors. This category is abelian with point-wise constructions of kernels and cokernels. Therefore it is possible to study subfunctors, quotient functors, simple functors, indecomposable functors, etc.

Classification and description of the simple biset functors is done by Bouc. We shall review his construction briefly.

Let H be a finite group. We denote by E_H the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(H)$ of H in the category $\mathcal{C}$. It is easy to show that E_H decomposes as an R -module as

$$E_H = I_H \oplus R\mathrm{Out}(H)$$

where $\text{Out}(H) = \text{Aut}(H)/\text{Inn}(H)$ is the group of outer automorphisms of H and $R\text{Out}(H)$ is the group algebra of $\text{Out}(H)$ and I_H is a two-sided ideal of E_H (see [5] for explicit description of I_H). Therefore, we obtain an epimorphism

$$E_H \twoheadrightarrow R\text{Out}(H)$$

of algebras. In particular, we can lift any simple $R\text{Out}(H)$ -module V to a simple E_H -module, still denoted by V .

Denote by e_H the evaluation at H functor, that is, let $e_H : \text{Biset}_R \rightarrow E_H\text{-mod}$ be the functor sending a biset functor to its value at the group H . Now let $L_{H,V}$ denote the left adjoint of the functor e_H . Explicitly, for a finite group K , we get

$$L_{H,V}(K) = \text{Hom}_{\mathcal{C}}(H, K) \otimes_{E_H} V.$$

The action of a biset on $L_{H,V}$ is given by composition of morphisms. The functor $L_{H,V}$ has a unique maximal subfunctor

$$J_{H,V}(K) = \left\{ \sum_i \phi_i \otimes v_i \mid \forall \psi \in \text{Hom}_{\mathcal{C}_G}(K, H), \sum_i (\psi \phi_i) v_i = 0 \right\}.$$

Hence taking the quotient of $L_{H,V}$ with this maximal ideal, we obtain a simple biset functor

$$S_{H,V} := L_{H,V} / J_{H,V}.$$

Moreover, we have

Theorem 2.2.1 (Bouc [5]) *Any simple biset functor is of the form $S_{H,V}$ for some finite group H and a simple $R\text{Out}(H)$ -module V .*

As mentioned in the introduction, the main examples of the biset functors are the functor of the Burnside ring and the functor of the representation ring. Further, over a field of characteristic zero, the rational representation ring is an example of a simple biset functor. More precisely, Bouc proved in [5] that the biset functor of rational representation ring $\mathbb{Q}\mathcal{R}_{\mathbb{Q}}$ over $\mathbb{Q}$ is isomorphic to the simple biset functor $S_{1,\mathbb{Q}}$. Another interesting example of a simple biset functor is the functor $S_{C_p \times C_p, \mathbb{Q}}$ defined only over p -groups where p is a prime number. In

[11], it is shown that this functor is isomorphic to the functor of the Dade group $\mathbb{Q}D$ with coefficients extended to the rational numbers $\mathbb{Q}$ and there is an exact sequence of biset functors

$$0 \longrightarrow \mathbb{Q}D \longrightarrow \mathbb{Q}B \longrightarrow \mathbb{Q}\mathcal{R}_{\mathbb{Q}} \longrightarrow 0.$$

Here the map $\mathbb{Q}B \rightarrow \mathbb{Q}\mathcal{R}_{\mathbb{Q}}$ can be chosen as the natural map sending a P -set X to the permutation module $\mathbb{Q}X$. For an improvement of this result to $\mathbb{Z}$, see [7] and for more exact sequences relating these functors, see [9]. Some other well-known examples of biset functors are the functor of units of the Burnside ring [10] and the functor of the group of relative syzygies [6]. For further details also see [5], [8], [11] and [1].

2.3 Subquotients and Amalgamations

For the rest of the thesis, we concentrate on certain algebras defined only for subquotients of a fixed finite group G . We introduce our notation that will be used throughout the thesis.

Recall that a **subquotient** of G is a pair (H^*, H_*) where $H_* \trianglelefteq H^* \leq G$. We write the pair (H^*, H_*) as H and denote the subquotient relation by $H \preceq G$. Here, and afterwards, we regard the group G as the subquotient $(G, 1)$.

The group G acts on the set of its subquotients by conjugation. We write $H \preceq_G G$ to mean that H is taken up to G -conjugacy. Note that we always consider H as the quotient group H^*/H_* . Therefore, for example, what we mean by up to G -conjugation is that the subgroup H^* is taken up to G -conjugacy and the normal subgroup H_* of H^* is taken up to $N_G(H)$ -conjugacy.

The relation $\preceq$ extends to a partial order on the set of all subquotients of G in the following way. Let J and H be two subquotients of G . Then we write $J \preceq H$ if and only if $H_* \leq J_*$ and $H^* \geq J^*$. In this case the pair $(J^*/H_*, J_*/H_*)$ is a subquotient of H . The poset structure is compatible with the G -action, that is, the set of subquotients of G is a G -poset.

Also we say that two subquotients H and K of G are isomorphic if and only if they are isomorphic as groups, that is, if $H^*/H_* \cong K^*/K_*$. In this case, write $H \preceq_* G$ to mean that H runs over a set of representatives of isomorphism classes of subquotients of G , or we simply say that H is taken up to isomorphism.

With this notation, we simplify the decomposition theorem of Bouc by introducing certain amalgamations of the basic bisets. Given a finite group H and a subquotient J of H . We define the **tinflation** (H, J) -biset, denoted Tin_J^H , as the Mackey product of the **transfer** (H, J^*) -biset and the **inflation** (J^*, J) -biset. Similarly, we define the **destriction** (J, H) -biset, denoted Des_J^H , as the Mackey product of the **Deflation** (J, J^*) -biset and the **restriction** (J^*, H) -biset. We also change the name of the remaining biset as the **isogation biset**, (**isomorphisms are sometimes just conjugations**). Explicitly we have

$$\text{Tin}_J^H := \left(\frac{H \times J}{U} \right)$$

where $U = \{(jJ_*, j) : j \in J^*\}$ and

$$\text{Des}_J^H := \left(\frac{J \times H}{V} \right)$$

where $V = \{(jJ_*, j) : j \in J^*\}$. Now Bouc's theorem becomes

$$\left(\frac{H \times K}{L} \right) = \text{Tin}_{p_1(L)/k_1(L)}^H \mathbf{c}_{p_1(L)/k_1(L), p_2(L)/k_2(L)}^\phi \text{Des}_{p_2(L)/k_2(L)}^K.$$

Note that we are also omitting the product sign $\times_H$ while using the abbreviated notations like Tin and Des .

Remark 2.3.1 In [8], the amalgamated variables *tinflation* and *destriction* are abbreviated as *indinf* and *defres*, respectively.

Finally we introduce the intersection of subquotients. Let $H, K \preceq G$. We define the **intersection** $H \sqcap K$ of the subquotients H and K as

$$H \sqcap K = \frac{(H^* \cap K^*)H_*}{(H^* \cap K_*)H_*}.$$

Note that, in general, this intersection is neither commutative nor associative. But, there is an isomorphism of groups

$$\lambda : H \sqcap K \rightarrow K \sqcap H$$

which we call the **canonical isomorphism** between the groups $H \sqcap K$ and $K \sqcap H$. The isomorphism is the one that comes from the Zassenhaus-Butterfly Lemma.

Since the subgroup H_* of H^* is contained in $(H^* \cap K_*)H_*$ and the group $(H^* \cap K^*)H_*$ is contained in H^* , the pair $((H^* \cap K^*)H_*/H_*, (H^* \cap K_*)H_*/H_*)$ is a subquotient of H , that is, $H \sqcap K \preceq H$. Similarly we have $K \sqcap H \preceq K$.

Chapter 3

Group Functors

In this chapter, we will introduce the *group functors* and give several examples. Throughout the chapter we fix a finite group G and a field R .

3.1 Alchemic algebra

Our main framework will be the following finite dimensional algebra

$$\Gamma_R(G) = \bigoplus_{H, K \leq G} RB(H \times K)$$

defined for G over R . Here the product is given by the R -linear extension of the Mackey product. When there is no risk of confusion, we will write Γ instead of $\Gamma_R(G)$.

Our first aim is to analyze this algebra by giving an alternative description in terms of generators and relation. We also determine a free basis for the algebra, which will be the main result of this section.

3.1.1 A characterization

It is evident that the alchemic algebra Γ has a basis consisting of the isomorphism classes of transitive (H, K) -bisets as H and K runs over subquotients of G . Hence by Theorem 2.1.2, it is generated by the five basic bisets, namely by transfer, inflation, isomorphism, deflation and restriction bisets. Following Barker, [1], we call this algebra the **alchemic algebra** for G over R .

It is possible to *define* the alchemic algebra by forgetting the bisets altogether. In order to do this, we can consider the algebra generated freely over R by the five types of variables corresponding to the five basic bisets. Then the alchemic algebra is the quotient of this algebra by the ideal generated by relations between the variables induced by the Mackey product of bisets. But the relations obtained in this way are not tractable. To get the relations in a tractable way, we use our new amalgamated variables. Instead of the five variables, we consider the amalgamated variables corresponding to tinflation, destruction and isogation. In this way, we obtain a set of relations that is very similar to the defining relations of the well-known Mackey algebra.

Explicitly, consider the algebra freely generated over R by the following three types of variables.

V1. Tin_J^H for each $J \preceq H \preceq G$,

V2. Des_J^H for each $J \preceq H \preceq G$,

V3. $c_{M,L}^\phi$ for each $M, L \preceq G$ such that $M \cong L$ and for each isomorphism $\phi : M \rightarrow L$.

Then we let $\tilde{\Gamma}_R(G)$, written $\tilde{\Gamma}$, be the quotient of this algebra by the ideal generated by the following relations.

R1. Let $h : H \rightarrow H$ denotes the inner automorphism of H induced by conjugation by $h \in H$. Then $c_{H,H}^h = \text{Tin}_H^H = \text{Des}_H^H$.

R2. Let $L \preceq J$ and $\psi : M \rightarrow S$ be an isomorphism. Then

- (i) $c_{S,M}^\psi c_{M,L}^\phi = c_{S,L}^{\psi \circ \phi}$,
- (ii) $\text{Tin}_J^H \text{Tin}_L^J = \text{Tin}_L^H$,
- (iii) $\text{Des}_L^J \text{Des}_J^H = \text{Des}_L^H$.

R3. Let $K \preceq G$ and let $\alpha : H \rightarrow K$ be an isomorphism and let ${}^\alpha J$ denote $\alpha(J^*)/\alpha(J_*)$, then

- (i) $c_{K,H}^\alpha \text{Tin}_J^H = \text{Tin}_{\alpha J}^K c_{\alpha J,J}^\alpha$,
- (ii) $\text{Des}_I^K c_{K,H}^\alpha = c_{I, \alpha^{-1}I}^\alpha \text{Des}_{\alpha^{-1}I}^H$.

R4. (Mackey relation) Let $I \preceq H$. Then

$$\text{Des}_I^H \text{Tin}_J^H = \sum_{x \in I^* \backslash H/J^*} \text{Tin}_{I \cap^x J}^I c^{x \circ \lambda} \text{Des}_{J \cap I^x}^J$$

Here $c^{x \circ \lambda} := c_{I \cap^x J, J \cap I^x}^{x \circ \lambda}$ and λ is the canonical isomorphism introduced in the previous section.

R5. $1 = \sum_{H \preceq G} c_H$ where $c_H := c_{H,H}^1$.

R6. All other products of the generators are zero.

Even it is clear from the construction, the following proposition formally shows that the algebra $\tilde{\Gamma}$ is isomorphic to the alchemic algebra Γ .

Proposition 3.1.1 *The algebras Γ and $\tilde{\Gamma}$ are isomorphic.*

Proof. The correspondence

$$\text{Tin}_J^H c_{J,I}^\phi \text{Des}_I^K \longrightarrow \left(\frac{H \times K}{A} \right)$$

where $A = \{(h, k) \in J^* \times I^* : hJ_* = \phi(kI_*)\}$ extends linearly to a map $\alpha : \tilde{\Gamma} \rightarrow \Gamma$. We must show that α is an algebra isomorphism. Indeed, α is an isomorphism of R -modules by Theorem 2.1.2. Thus, it suffices to check that it respects the

multiplication. We shall only check the Mackey relation. The others can be checked similarly and are left to the reader. First note that the images of tinflation and destruction are

$$\alpha(\text{Tin}_J^H) = \text{Ind}_{J^*/H_*}^H \inf_{(J^*/H_*)/(J_*/H_*)}^{J^*/H_*} c_{(J^*/H_*)/(J_*/H_*), J}^\lambda =: \left(\frac{H \times J}{T'} \right)$$

and

$$\alpha(\text{Des}_J^H) = c_{J, (J/H_*)/(J_*/H_*)}^{\lambda^{-1}} \text{Def}_{(J/H_*)/(J_*/H_*)}^{J^*/H_*} \text{Res}_{J^*/H_*}^H =: \left(\frac{J \times H}{R'} \right)$$

where λ is the canonical map $J \rightarrow (J^*/H_*)/(J_*/H_*)$ and

$$T' = \{(jH_*, j'J_*) \in J^*/H_* \times J : (jH_*)J_*/H_* = \lambda(j'J_*)\}$$

and

$$R' = \{(jJ_*, j'H_*) \in J \times J^*/H_* : (j'H_*)J_*/H_* = \lambda(jJ_*)\}.$$

Hence, we must show that

$$\alpha(\text{Des}_I^H \text{Tin}_J^H) = \left(\frac{I \times H}{R'} \right) \times_H \left(\frac{H \times J}{T'} \right).$$

By the Mackey product formula, we have

$$\left(\frac{I \times H}{R'} \right) \times_H \left(\frac{H \times J}{T'} \right) = \sum_{x \in p_2(R') \setminus H/p_1(T')} \left(\frac{I \times J}{R' *_{(x,1)} T'} \right)$$

where

$$R * T = \{(iI_*, jJ_*) \in I \times J : (iI_*, hH_*) \in R \text{ and } (hH_*, jJ_*) \in T \text{ for some } hH_* \in H\}.$$

Straightforward calculations show that $p_1(R * T)/k_1(R * T) = I \sqcap J$ and $p_2(R * T)/k_2(R * T) = J \sqcap I$, and hence the Mackey relation. $\square$

Hereafter, we shall identify Γ and $\tilde{\Gamma}$ via the above isomorphism α .

3.1.2 A free basis

Now let us describe the free basis of the alchemic algebra consisting of the isomorphism classes of transitive bisets in terms of the new variables. Clearly any

transitive biset corresponds to a product of tinflation, isogation and destruction, in this order. We are aiming to find an equivalence relation on the set $\mathcal{B} = \{\text{Tin}_J^H \text{c}_{J,I}^\phi \text{Des}_I^K : J \preceq H, I \preceq K, \phi : I \rightarrow J\}$ such that under α , the equivalence classes of the relation correspond to the isomorphism classes of transitive bisets.

Given two subquotients H and K of G . Also given subquotients J, A of H and subquotients I, C of K such that there are isomorphisms $\phi : I \rightarrow J$ and $\psi : C \rightarrow A$. We say that the triples (J, I, ϕ) and (A, C, ψ) are (H, K) –**conjugate** if there exist $k \in K$ and $h \in H$ such that

1. the equalities ${}^h J = A$ and ${}^k I = C$ hold and
2. (compatibility of ϕ and ψ) the following diagram commutes.

$$\begin{array}{ccc} I & \xrightarrow{\phi} & J \\ \downarrow k & & \downarrow h \\ C & \xrightarrow{\psi} & A \end{array}$$

We denote by $[J, I, \phi]$ the (H, K) –conjugacy class of (J, I, ϕ) . Then we obtain

Theorem 3.1.2 *Letting H and K run over the subquotients of G and $[J, I, \phi]$ run over the (H, K) –conjugacy classes of triples $(J \preceq H, I \preceq K, \phi : I \rightarrow J)$, the elements $\text{Tin}_J^H \text{c}_{J,I}^\phi \text{Des}_I^K$ run, without repetitions, over a free R –basis of the alchemic algebra Γ .*

Proof. We are to show that (H, K) –conjugacy classes of the triples (J, I, ϕ) are in one-to-one correspondence with the isomorphism classes of transitive (H, K) –bisets. This follows from the following lemma. $\square$

Lemma 3.1.3 *Let $H, K \preceq G$. Then there is a one-to-one correspondence between*

- (i) The (H, K) -conjugacy classes $[J, I, \phi]$ of triples (J, I, ϕ) ,
- (ii) The isomorphism classes $[X]$ of transitive (H, K) -bisets

where the correspondence is given by associating $[J, I, \phi]$ to the isomorphism class of the biset $\alpha(\text{Tin}_J^H \text{c}_{J,I}^\phi \text{Des}_I^K)$.

Proof. Let (J, I, ϕ) and (A, C, ψ) be two (H, K) -conjugate triples. Then we have to show that the transitive bisets $\alpha(\text{Tin}_J^H \text{c}_{J,I}^\phi \text{Des}_I^K)$ and $\alpha(\text{Tin}_A^H \text{c}_{A,C}^\psi \text{Des}_C^K)$ are isomorphic. Let us write

$$\alpha(\text{Tin}_J^H \text{c}_{J,I}^\phi \text{Des}_I^K) = \left(\frac{H \times K}{\mathfrak{a}} \right)$$

and

$$\alpha(\text{Tin}_A^H \text{c}_{A,C}^\psi \text{Des}_C^K) = \left(\frac{H \times K}{\mathfrak{b}} \right)$$

for some subgroups $\mathfrak{a}, \mathfrak{b} \in H \times K$ given explicitly in the proof of Theorem 3.1.1. Let $h \in H$ and $k \in K$ such that

$${}^h J = A \quad \text{and} \quad {}^k I = C.$$

We shall show that ${}^{(h,k)} \mathfrak{a} = \mathfrak{b}$. Let $(j, i) \in \mathfrak{a}$. Then by the definition of α , we have $jJ_* = \phi(iI_*)$. Clearly, $({}^h j, {}^k i) \in A \times C$. So it suffices to show ${}^h jA_* = \psi({}^k iC_*)$. But,

$${}^h(jJ_*) = h j J_* h^{-1} = h j h^{-1} h J_* h^{-1} = {}^h j A_*,$$

and

$${}^h \phi(iI_*) = h \phi(k^{-1}(k i k^{-1})(k I_* k^{-1})k) h^{-1} = \psi({}^k i C_*)$$

by the compatibility of ϕ and ψ . Hence ${}^h j A_* = \psi({}^k i C_*)$, as required.

Conversely, let $\mathfrak{a}, \mathfrak{b} \in H \times K$ be two conjugate subgroups of $H \times K$. Then we are to show that the triples $(p_1(\mathfrak{a})/k_1(\mathfrak{a}), p_2(\mathfrak{a})/k_2(\mathfrak{a}), \phi)$ and $(p_1(\mathfrak{b})/k_1(\mathfrak{b}), p_2(\mathfrak{b})/k_2(\mathfrak{b}), \psi)$ are (H, K) -conjugate. Here ϕ and ψ are the canonical isomorphisms introduced in Theorem 2.1.2.

Now let ${}^{(h,k)} \mathfrak{a} = \mathfrak{b}$ for some $h \in H$ and $k \in K$. Then, clearly,

$${}^h(p_1(\mathfrak{a}), k_1(\mathfrak{a})) = (p_1(\mathfrak{b}), k_1(\mathfrak{b}))$$

and

$${}^k(p_2(\mathfrak{a}), k_2(\mathfrak{a})) = (p_2(\mathfrak{b}), k_2(\mathfrak{b})).$$

So, it suffices to show that the diagram

$$\begin{array}{ccc} p_2(\mathfrak{a})/k_2(\mathfrak{a}) & \xrightarrow{\phi} & p_1(\mathfrak{a})/k_1(\mathfrak{a}) \\ \downarrow k & & \downarrow h \\ p_2(\mathfrak{b})/k_2(\mathfrak{b}) & \xrightarrow{\psi} & p_1(\mathfrak{b})/k_1(\mathfrak{b}) \end{array}$$

commutes.

Let $(a, c) \in \mathfrak{b}$. Then by the definition of ψ , we have

$$\psi(a k_2(\mathfrak{b})) = c k_1(\mathfrak{b}).$$

But writing $i = a^k$ and j for the unique element $j = c^k$, the left hand side becomes

$$\psi(a k_2(\mathfrak{b})) = \psi(k i k^{-1} k k_2(\mathfrak{a}) k^{-1}) = \psi(k(i k_2(\mathfrak{a})) k^{-1})$$

and the right hand side becomes

$$c k_1(\mathfrak{b}) = h(j k_1(\mathfrak{a})) h^{-1} = h(\phi(j k_2(\mathfrak{a}))) h^{-1}.$$

Thus combining these two equality we get $\psi(k(i k_2(\mathfrak{a})) k^{-1}) = h(\phi(j k_2(\mathfrak{a}))) h^{-1}$, as required. $\square$

3.2 Π -Group Functors : Definition and Examples

3.2.1 Definition

Let $\Pi := \Pi_R(G)$ be a subalgebra of the alchemic algebra $\Gamma_R(G)$ of G over R . We call a (finitely-generated left) Π -module as a **Π -group functor** for G over R . Before giving examples of Π -group functors, we justify the word “functor” in the definition, by imitating Bouc’s definition of biset functors, cf. 2.2.

Consider the category $\mathcal{C}_\Pi$ defined as follows.

- The objects of $\mathcal{C}_\Pi$ are the subquotients of G .
- Given two subquotients H and K of G , the set of morphisms from K to H is

$$\mathrm{Hom}_{\mathcal{C}_\Pi}(K, H) := c_H \Pi c_K.$$

- The composition is induced by the product in the alchemic algebra.

Now a **Π -group functor** for G over R is an R -linear functor $\mathcal{C}_\Pi \rightarrow R\text{-mod}$. Having two definitions of Π -group functors, we have to prove that the two definitions are equivalent. But this is standard since now Π is the category algebra of the category $\mathcal{C}_\Pi$.

Remark 3.2.1 *Note that in the above definition, instead of considering subquotients of a fixed finite group, one can consider a class of finite groups closed under taking subquotients, isomorphisms and extensions. In the literature, most examples of group functors are studied with this refinement. For our purposes, we can stick with a fixed group and as mentioned before, our results can easily be generalized to this global scenario as far as the subalgebra can.*

3.2.2 Examples

Special cases of group functors appear several times under several different names. For the rest of the section, we review these concepts with our point of view which amounts to introduce a more general type of subalgebras that unifies all these concepts. This unification will be the main objective of the rest of the thesis.

3.2.2.1 Biset functors

Let $\Pi = \Gamma$ be the alchemic algebra itself. Then clearly the modules of the alchemic algebra are the biset functors.

3.2.2.2 Alcahestic group functors

We call Π an **alcahestic subalgebra**¹ of Γ if the following four conditions hold:

- A1.** For each subquotient H of G , the element c_H is contained in Π .
- A2.** The algebra Π is generated by the set of tinflations, isogations and destrinctions in it.
- A3.** The Mackey product formula holds in Π , that is, if $x, y \in \Pi$ and if $x \cdot y = \sum_i z_i$ in Γ , then each z_i is contained in Π .
- A4.** For any isogation c_ϕ in Π , its inverse $c_{\phi^{-1}}$ is in Π .

Now an **alcahestic group functor**, abbreviated as **AGF**, is a module of an alcahestic subalgebra. Clearly the alchemic algebra is an alcahestic subalgebra. Actually the subalgebras of immediate interest, for example the subalgebras realizing cohomology functors or homology functors or Brauer character ring functor, are all alcahestic. Next we recall the well-known alcahestic subalgebras.

Globally-defined Mackey functors (GDMF). These functors are also introduced by Bouc in [4] without a name and the name is given by Webb in [21]. Given two classes $\mathcal{X}$ and $\mathcal{Y}$ of finite groups having the following properties:

1. If a finite group H is contained in $\mathcal{X}$ (resp. in $\mathcal{Y}$) then any subquotient of H is also contained in $\mathcal{X}$ (resp. in $\mathcal{Y}$).
2. If H, K are contained in $\mathcal{X}$ (resp. in $\mathcal{Y}$) then any extension of these groups are also contained in $\mathcal{X}$ (resp. in $\mathcal{Y}$).

Define the category $\mathcal{C}_R^{\mathcal{X}, \mathcal{Y}}$ as the category consists of all finite groups with morphisms given by all finite bisets with left point stabilizers in $\mathcal{X}$ and right point stabilizers in $\mathcal{Y}$. The composition is still the Mackey product of

¹In alchemy, alcahest is the universal solvent. An alcahestic subalgebra contains an alcahest that solves its elements.

bisets. Now a **globally-defined Mackey functor** with respect to $\mathcal{X}$ and $\mathcal{Y}$ is an R -linear functor $\mathcal{C}_R^{\mathcal{X}, \mathcal{Y}} \rightarrow R\text{-mod}$.

When we restrict to the category consisting only of subquotients of G , it is clear that the subalgebra associated to given two classes $\mathcal{X}$ and $\mathcal{Y}$ is generated by all isogations, transfers and restrictions together with all inflations from a group in the class $\mathcal{X}$ and all deflations to a group in $\mathcal{Y}$. In particular, it is alcahestic.

Global Mackey functors. A global Mackey functor is a globally-defined Mackey functor in the special case that $\mathcal{X} = \mathcal{Y} = 1$. The corresponding alcahestic subalgebra is called the **global Mackey algebra** and is generated by all transfer, isogation and restriction maps. These objects are introduced by Webb in [20].

Inflation and Deflation functors. An inflation (resp. deflation) functor is a globally-defined Mackey functor with respect to the families $\mathcal{Y} = 1$ and $\mathcal{X}$ equal to all groups (resp. $\mathcal{X} = 1$ and $\mathcal{Y}$ equal to all groups). The corresponding alcahestic subalgebras are generated over the global Mackey algebra, respectively, by all inflations and all deflations. These are also introduced by Webb in [20].

(Ordinary) Mackey functors. The starting point of such a unified approach to representation theory of finite groups is the theory of Mackey functors introduced by Dress and Green. The theory is later developed by many authors, notably by Thévenaz and Webb who realized Mackey functors as modules of the **Mackey algebra**, [19]. Although the Mackey algebra $\mu_R(G)$ is not a subalgebra of the alchemic algebra, it has the crucial properties of an alcahestic subalgebra as it is generated by all transfers and restrictions between subgroups of G together with conjugations on the subgroups, that are the isogations induced by the conjugation action of the group G on the subgroups. All of our results in the next sections are also valid for the Mackey algebra with appropriate changes. See [12] for details.

Restriction, Transfer, Conjugation, etc. functors. Note that the category of (ordinary) Mackey functors is an example of a category of alcahestic

group functors that is not a category of GDMF. Another type of category of AGFs is given by considering only at most one type of generator together with certain isogations. For example, a **restriction functor** is a module over the subalgebra generated by all isogations and all restrictions. This algebra is clearly alcahestic and is called the **restriction algebra**. Similarly we have **transfer functors**, **conjugation functors**, **tinflation functors**, **destriction functors**, etc. together with the corresponding alcahestic subalgebras. These kind of AGFs will play a crucial role in our main results.

3.2.2.3 Removing the conditions

Two of the four conditions in the definition of alcahestic subalgebra are not crucial for the aims of the present thesis and hence can be removed. Removing condition A1 does not create a difficult problem. One can remove this condition and deal with a family of subquotients of G closed under taking subquotients and isomorphisms, instead of all subquotients. For simplicity, we deal with all subquotients.

Condition A2 can also be removed with a recipe of modification of the main results of the thesis. We call a subalgebra that satisfies conditions A3 and A4 of the alcahestic subalgebra as a **pre-alcahestic subalgebra**. Regarding the modification in the classification of simple functors, see Section 4.3.

It seems that it is not possible, for the purposes of the thesis, to remove the other two conditions.

3.2.2.4 Non-alcahestic cases

Biset Algebras. Let K be a subquotient of G . The Burnside group $B(K \times K)$ of (K, K) -bisets is a subalgebra of Γ , called the **biset algebra of K** . It is clearly pre-alcahestic. Note also that for any biset functor F , the evaluation $F(K)$ is either zero or a module for $B(K \times K)$.

Group Algebras. Let H be a subgroup of $\text{Out}(K)$ for some subquotient K of

G . Then the group algebra RH is a subalgebra of Γ and the RH -group functors are just modules of the group algebra.

Chapter 4

Construction of Simple AGFs

This is the main chapter of the thesis. In this chapter we classify and describe the simple alcahestic group functors. In the case of globally-defined Mackey functors, our description differs from that of Bouc's. Also in the case of (ordinary) Mackey functors, the description differs from that of Thévenaz and Webb. In this sense, our approach provides a unified way of describing simple functors. In the next chapter, we construct simple functors in these two cases to demonstrate our method. Note also that our approach can be adapted to pre-alcahestic functors. We shall not work this adaptation here because the results will not be as explicit as in the case of AGFs, see Section 4.3.

4.1 General correspondence theorem

Let Π be an alcahestic subalgebra of the alchemic algebra Γ . We denote by Ω_Π the subalgebra of Π generated by all isogations in Π . We call Ω_Π the **isogation algebra associated to Π** . We write Ω for the isogation algebra associated to the alchemic algebra Γ . Clearly, Ω_Π is alcahestic by definition. The structure of the isogation algebra Ω_Π is easy to describe. We examine this structure since it is crucial in proving our main results.

Let H and K be two subquotients of G . We say that H is **Π -isomorphic** with K , and write $H \cong_{\Pi} K$, if and only if there exists an isomorphism $\phi : H \rightarrow K$ with $c_{K,H}^{\phi}$ in Π . Clearly being Π -isomorphic is an equivalence relation since Π is alcahestic and is determined only by the isogation algebra Ω_{Π} associated to Π .

It is evident that the isogation algebra Ω_{Π} associated to Π has the following decomposition

$$\Omega_{\Pi} = \bigoplus_{I, J \preceq G} c_I \Omega_{\Pi} c_J.$$

Now for a fixed subquotient $H \preceq G$, the following isomorphism holds:

$$\bigoplus_{I, J \cong_{\Pi} H} c_I \Omega_{\Pi} c_J \cong \text{Mat}_n(c_H \Omega_{\Pi} c_H).$$

Here n is the number of subquotients of G which are Π -isomorphic to H . In particular, we see that the isogation algebra Ω_{Π} is Morita equivalent to the algebra $\bigoplus_{H \preceq_{\Pi} G} c_H \Omega_{\Pi} c_H$. Here the sum is over the representatives of the Π -isomorphism classes of subquotients of G . Note that if Π is the alchemic algebra, there is an isomorphism $c_H \Omega c_H \cong R\text{Out}(H)$ of algebras. Recall that $\text{Out}(H)$ is the group of outer automorphisms of H . In general, by the definition of alcahestic subalgebras, $c_H \Omega_{\Pi} c_H$ is isomorphic with the group algebra $R\text{Out}_{\Pi}(H)$ where $\text{Out}_{\Pi}(H)$ denotes the subgroup of $\text{Out}(H)$ generated by isogations in Π . Hence we have proved the following proposition.

Proposition 4.1.1 *Let Π be an alcahestic subalgebra of the alchemic algebra. Then the isogation algebra Ω_{Π} associated to Π is Morita equivalent to the algebra*

$$\bigoplus_{H \preceq_{\Pi} G} R\text{Out}_{\Pi}(H)$$

where the sum is over a set of representatives of the Π -isomorphism classes of subquotients of G and $\text{Out}_{\Pi}(H)$ is the subgroup of $\text{Out}(H)$ generated by all isogations in Π .

As an easy consequence of the above result, the simple modules of the isogation algebra Ω_{Π} are parameterized by the pairs (H, V) where H is a subquotient of G and V is a simple $R\text{Out}_{\Pi}(H)$ -module. We call (H, V) a **simple pair** for Ω_{Π} and

denote by $S_{H,V}^\Omega$ the corresponding simple Ω_Π -module. It is clear that $S_{H,V}^\Omega$ is the Ω_Π -module defined for any subquotient K of G by

$$S_{H,V}^\Omega(K) = \phi V \text{ if there exists an isomorphism } \phi : K \rightarrow H \text{ in } \Pi$$

and zero otherwise. Note that the definition does not depend on the choice of the isomorphism $\phi : K \rightarrow H$ since any two such isomorphisms differ by an inner automorphism of H and the group $\text{Inn}(H)$ acts trivially on V .

More generally, for any alcaestic subalgebra Π , the following theorem holds.

Theorem 4.1.2 (General Correspondence) *Let Π be an alcaestic subalgebra of the alchemic algebra Γ . There is a bijective correspondence between*

- (i) *the isomorphism classes $[S^\Pi]$ of simple Π -modules S^Π ,*
- (ii) *the isomorphism classes $[S_{H,V}^\Omega]$ of simple Ω_Π -modules $S_{H,V}^\Omega$*

given by $[S^\Pi] \leftrightarrow [S_{H,V}^\Omega]$ if and only if H is minimal such that $S^\Pi(H) \neq 0$, and $S^\Pi(H) = V$.

In other words, the general correspondence theorem claims that the parameterizing set for the simple modules of an alcaestic subalgebra is determined only by the isogations contained in the subalgebra.

4.2 Proof of the theorem

We prove the general correspondence theorem in several steps.

4.2.1 A characterization

The first step is to characterize the simple modules in terms of images of tinflation maps and kernels of destruction maps. This characterization is an adaptation

of a similar result of Thévenaz and Webb [17] for ordinary Mackey functors. In particular, this characterization implies that any simple Π -module has a unique minimal subquotient, up to Π -isomorphism, and the evaluation at this subquotient is a simple module.

To prove the characterization, we introduce two submodules of a Π -module F , as follows (cf. [17]). Let H be a minimal subquotient for F . In other words, H is a subquotient of G minimal subject to the condition that $F(H) \neq 0$. Define two R -submodules of F by

$$\mathcal{I}_{F,H}(J) = \sum_{I \preceq J, I \cong_{\Pi} H} \text{Im}(\text{tin}_I^J : F(I) \rightarrow F(J))$$

and

$$\mathcal{K}_{F,H}(J) = \bigcap_{I \preceq J, I \cong_{\Pi} H} \text{Ker}(\text{des}_I^J : F(J) \rightarrow F(I)).$$

Proposition 4.2.1 *The R -modules $\mathcal{I}_{F,H}$ and $\mathcal{K}_{F,H}$ are Π -submodules of F , via the induced actions.*

Proof. Let us prove that $\mathcal{I}_{F,H}$ is a submodule of F . The other claim can be proved similarly. Clearly, $\mathcal{I}_{F,H}$ is closed under isogation. It is also clear that $\mathcal{I}_{F,H}$ is closed under tinflation, because of the transitivity of tinflation. So it suffices to show that $\mathcal{I}_{F,H}$ is closed under dstriction, which is basically an application of the Mackey relation. Let $A \preceq K$ be subquotients of G and let f be an element of $\mathcal{I}_{F,H}(K)$. We are to show that $\text{Des}_A^K f$ is an element of $\mathcal{I}_{F,H}(A)$. Write

$$f = \sum_I \text{Tin}_I^K f_I$$

for some $f_I \in F(I)$. Here the sum is over all subquotients of K isomorphic to H . Applying the Mackey relation, we get

$$\begin{aligned} \text{Des}_A^K f &= \sum_I \text{Des}_A^K \text{Tin}_I^K f_I \\ &= \sum_I \sum_{y \in A^* \setminus K/I^*} \text{Tin}_{A \cap y I}^A c^{y\lambda} \text{Des}_{I \cap A^y}^I f_I. \end{aligned}$$

Since H is minimal for F , the last sum contains only the terms $\text{Tin}_{A \sqcap^y I}^A c^{y\lambda}$ where $A \sqcap^y I$ is isomorphic to H . Therefore $\text{Des}_A^K f \in \mathcal{I}_{F,H}(A)$, as required. $\square$

The characterization of simple Π -modules via these subfunctors is as follows, (cf. [17] and [20]).

Proposition 4.2.2 *Let Π be an alcahestic subalgebra of the alchemic algebra Γ . Let S be a Π -module. Let H be a minimal subquotient for S and let V denote the evaluation of S at H . Then S is simple if and only if*

- (i) $\mathcal{I}_{S,H} = S$,
- (ii) $\mathcal{K}_{S,H} = 0$,
- (iii) V is a simple $R\text{Out}_\Pi(H)$ -module.

Proof. It is clear that if S is simple then the conditions (i) and (ii) hold. Also since H is minimal such that $S(H) \neq 0$, any map that decomposes through a smaller subquotient is a zero map. Thus $S(H)$ is a module of the algebra $R\text{Out}_\Pi(H)$. But it has to be simple since any decomposition of the minimal coordinate module gives a decomposition of S .

It remains to show the reverse implication. Suppose the conditions hold. Let T be a subfunctor of S . Since $S(H) = V$ is simple, $T(H)$ is either 0 or V . If $T(H) = V$ then by condition (i), it is equal to S . So, let $T(H) = 0$. Then for any $K \preceq G$, the module $T(K)$ is a submodule of $\mathcal{K}_{S,H}$, because for any $x \in T(K)$ and $H \cong L \preceq K$, we have $\text{Des}_L^K x \in T(L) = 0$. Thus by condition (ii), we obtain $T(K) = 0$, that is, $T = 0$. Thus, any subfunctor of S is either zero or S itself. In other words, S is simple. $\square$

Now it is clear that a simple Π -module S has a unique, up to Π -isomorphism, minimal subquotient, say H . Moreover the coordinate module at H is a simple $R\text{Out}_\Pi(H)$ -module. That is to saying that there is a map from the set of isomorphism classes of simple Π -modules to the set of isomorphism classes of the simple pairs (H, V) for Ω_Π , justifying the existence of the general correspondence of Theorem 4.1.2.

4.2.2 Special correspondence theorem

The second step forward to Theorem 4.1.2 is to describe the behavior of simple modules under induction and coinduction to certain over alcahestic subalgebras. We need two more definitions.

Let Π still denote an alcahestic subalgebra. Define the **destriction algebra** ∇_Π associated to Π as the subalgebra of Π generated by all destriction maps and isogation maps in the algebra Π . Similarly define Δ_Π , the **tinflation algebra**¹ associated to Π . We write ∇ and Δ for the destriction algebra and the tinflation algebra associated to the alchemic algebra Γ , respectively. Clearly, both ∇_Π and Δ_Π are alcahestic subalgebras.

By the general correspondence theorem, simple modules of alcahestic subalgebras containing the same set of isogations are parameterized by the same set of simple pairs. The following theorem shows that in certain cases this correspondence can be made more explicit.

Theorem 4.2.3 (Special Correspondence) *Let $\Pi \subset \Theta$ be two alcahestic subalgebras and $S := S^\Pi$ be a simple Π -module with minimal subquotient H and denote by V the evaluation of S at H .*

- (i) *The Θ -module $\text{Ind}_\Pi^\Theta S$ has a unique maximal submodule provided that $\nabla_\Pi = \nabla_\Theta$. Moreover the minimal subquotient for the simple quotient is H and the evaluation of the simple quotient at H is isomorphic with V .*
- (ii) *The Θ -module $\text{Coind}_\Pi^\Theta S$ has a unique minimal submodule provided that $\Delta_\Pi = \Delta_\Theta$. Moreover the minimal subquotient for the minimal submodule is H and the evaluation of the simple submodule at H is isomorphic with V .*

¹Our notation is consistent with that of ancient alchemists. In alchemy, the symbols of fire and water are Δ and ∇ , respectively. The symbols of air and earth are the same as the symbols of fire and water, respectively, with an extra horizontal line dividing the symbol into two. Moreover quintessence is also known as spirit which has the symbol Ω .

Proof. We only prove part (i). The second part follows from a dual argument. First, observe that the subquotient H is minimal for the induced module $F := \text{Ind}_{\Pi}^{\Theta} S_{H,V}^{\Pi}$ since $\nabla_{\Pi} = \nabla_{\Theta}$. Moreover observe that there is an isomorphism $F(H) \cong V$. Therefore the submodule $\mathcal{K}_{F,H}$ is defined. We claim that $\mathcal{K}_{F,H}$ is the unique maximal submodule of F .

To prove this, let T be a proper submodule of F . We are to show that $T \leq \mathcal{K}_{F,H}$. Since S is generated by its evaluation at H , the Θ -module F is also generated by its evaluation at H . Moreover this module is simple. Therefore $T(H)$ must be the zero module. Now let $K \preceq G$ be such that $T(K) \neq 0$. Then clearly, $T(K) \leq \mathcal{K}_{F,H}(K)$ as $\text{Des}_L^K f \in T(L) = 0$ for any $f \in T(K)$ and $L \cong_{\Pi} H$. Thus $T \leq \mathcal{K}_{F,H}$, as required. $\square$

To prove Theorem 4.1.2, we examine a more specific case of the Special Correspondence Theorem. This special case also initiates the process of describing simple Θ -modules via induction or coinduction using the Special Correspondence Theorem. First we describe simple modules of the restriction algebra and the inflation algebra. For completeness, we include the description of simple Ω_{Π} -modules.

Proposition 4.2.4 *Let Π be an alcahestic subalgebra and (H, V) be a simple pair for its isogation algebra, Ω_{Π} . Then*

- (i) *The Ω_{Π} -module $\mathcal{S}_{H,V}^{\Omega}$ is simple. Moreover any simple Ω_{Π} -module is of this form for some simple pair (H, V) .*
- (ii) *The ∇_{Π} -module $\mathcal{S}_{H,V}^{\nabla} := \inf_{\Omega_{\Pi}}^{\nabla_{\Pi}} \mathcal{S}_{H,V}^{\Omega}$ is simple. Moreover any simple ∇_{Π} -module is of this form for some simple pair (H, V) for Ω_{Π} .*
- (iii) *The Δ_{Π} -module $\mathcal{S}_{H,V}^{\Delta} := \inf_{\Omega_{\Pi}}^{\Delta_{\Pi}} \mathcal{S}_{H,V}^{\Omega}$ is simple. Moreover any simple Δ_{Π} -module is of this form for some simple pair (H, V) for Ω_{Π} .*

Here the inflation functor $\text{Inf}_{\Omega_{\Pi}}^{\nabla_{\Pi}}$ is the generalized restriction induced by the quotient map $\nabla_{\Pi} \rightarrow \nabla_{\Pi}/\mathcal{J}(\nabla_{\Pi})$. The ideal $\mathcal{J}(\nabla_{\Pi})$ is the ideal generated by all

proper restriction maps. We identify the quotient with the isogation algebra Ω_Π in the obvious way.

Proof. The first part follows from the discussion of simple isogation modules in Section 4.1. In part (2), it is clear that the module $\mathcal{S}_{H,V}^\nabla$ is simple. To see that any simple module is of this form, notice that if a ∇ -module D has non-zero evaluations at two non- Π -isomorphic subquotients, then D has a non-zero submodule generated by its evaluation at the smaller subquotient. So any simple ∇_Π -module has a unique, up to Π -isomorphism, non-zero evaluation. Clearly, this evaluation should be simple. a similar argument applies to the second part. $\square$

Remark 4.2.5 *Alternatively, one can apply the special correspondence theorem to obtain simple ∇_Π -modules and simple Δ_Π -modules. In the first case, to identify the submodule $\mathcal{K}$, one should identify Des_I^J with $c_I \text{Des}_I^J c_J$. Similar modification is also needed to identify the submodule $\mathcal{I}$.*

Evidently, for any alcahestic subalgebra Π , the restriction algebra ∇_Π and the tinflation algebra Δ_Π associated to Π are the minimal examples of the subalgebras satisfying the conditions of the first and the second part of the Special Correspondence Theorem, respectively. The following corollary restates the Correspondence Theorem for these special cases. We shall refer to this corollary when ending the proof of the General Correspondence Theorem and also when describing the simple biset functors in the next section.

Corollary 4.2.6 *Let Π and Θ be alcahestic subalgebras of the alchemic algebra such that the restriction algebra ∇_Π associated to Π and the tinflation algebra Δ_Θ associated to Θ are proper subalgebras in the corresponding subalgebra. Let (H, V) be a simple pair for the isogation algebra Ω_Π and (K, W) be a simple pair for the isogation algebra Ω_Θ .*

- (i) *The Π -module $\text{Ind}_{\nabla}^\Pi \mathcal{S}_{H,V}^\nabla$ has a unique maximal subfunctor. Moreover H is a minimal subquotient for the simple quotient and the evaluation of S at H is isomorphic with V .*

- (ii) The Θ -module $\text{Coind}_{\Delta}^{\Theta} \mathcal{S}_{H,V}^{\Delta}$ has a unique minimal subfunctor. Moreover H is a minimal subquotient for the simple subfunctor and the evaluation of S at H is isomorphic with V .

4.2.3 End of proof

Now, we are ready to end the proof of Theorem 4.1.2. By Corollary 4.2.6, we associated a simple module $S_{H,V}^{\Pi}$ to each simple pair (H, V) for the isogation algebra Ω_{Π} . Clearly this is an inverse to the correspondence described above. So it suffices to show that the correspondence is injective. This is equivalent to show that any simple Π -module with minimal subquotient H and $S(H) = V$ is isomorphic with $S_{H,V}^{\Pi}$. So let S be a simple Π -module with this property. Then we are to exhibit a non-zero morphism $S_{H,V}^{\Pi} \rightarrow S$. By our construction of $S_{H,V}^{\Pi}$, it suffices to exhibit a morphism $\phi : \text{Ind}_{\nabla}^{\Pi} \mathcal{S}_{H,V}^{\nabla} \rightarrow S$ such that ϕ_H is non-zero. The morphism exists since

$$\text{Hom}_{\Pi}(\text{Ind}_{\nabla}^{\Pi} \mathcal{S}_{H,V}^{\nabla}, S) \cong \text{Hom}_{\nabla}(\mathcal{S}_{H,V}^{\nabla}, \text{Res}_{\nabla}^{\Pi} S) \cong \text{Hom}_{R\text{Out}_{\Pi}(H)}(V, V) \neq 0.$$

Here the first isomorphism holds because induction is the left adjoint of restriction. On the other hand, the second isomorphism holds since $\mathcal{S}_{H,V}^{\nabla}$ is non-zero only on the isomorphism class of H . Now the identity morphism $V \rightarrow V$ induces a morphism $\phi : \text{Ind}_{\nabla}^{\Pi} \mathcal{S}_{H,V}^{\nabla} \rightarrow S$. Clearly, ϕ_H is non-zero, as required. Therefore we have established the injectivity, as required.

Having the proof of the General Correspondence Theorem, we can restate the Special Correspondence Theorem in a more precise way.

Theorem 4.2.7 (Special Correspondence Theorem) *Let $\Pi \subset \Theta$ be two alchemical subalgebras of the alchemic algebra Γ and (H, V) be a simple pair for the isogation algebra Ω_{Π} .*

- (i) The Θ -module $\text{Ind}_{\Pi}^{\Theta} S_{H,V}^{\Pi}$ has a unique maximal submodule provided that $\nabla_{\Pi} = \nabla_{\Theta}$. Moreover the simple quotient is isomorphic with $S_{H,V}^{\Theta}$.

- (ii) The Θ -module $\text{Coind}_{\Pi}^{\Theta} S^{\Pi}$ has a unique minimal submodule provided that $\Delta_{\Pi} = \Delta_{\Theta}$. Moreover the simple submodule is isomorphic with $S_{H,V}^{\Theta}$.

Finally the next result shows that in both cases of the Special Correspondence Theorem, the inverse correspondence is given by restriction.

Theorem 4.2.8 *Let $\Pi \subset \Theta$ be alcahestic subalgebras of the alchemic algebra. Then*

- (i) The Π -module $\text{Res}_{\Pi}^{\Theta} S_{H,V}^{\Theta}$ has a unique maximal submodule, provided that $\Delta_{\Pi} = \Delta_{\Theta}$. Moreover the simple quotient is isomorphic with $S_{H,V}^{\Pi}$.
- (ii) The Π -module $\text{Res}_{\Pi}^{\Theta} S_{H,V}^{\Theta}$ has a unique minimal submodule, provided that $\nabla_{\Pi} = \nabla_{\Theta}$. Moreover this submodule is isomorphic with $S_{H,V}^{\Pi}$.

Proof of the first part of (i) is similar to the proof of Theorem 4.2.3. Indeed the maximal subfunctor of the restricted module $\text{Res}_{\Pi}^{\Theta} S_{H,V}^{\Theta}$ is generated by intersection of kernels of the dstriction maps having range isomorphic to H . Note that this subfunctor is non-zero, in general. The second part of (i) follows from Theorem 4.1.2. Part (ii) can be proved by a dual argument.

In particular, this theorem shows that restriction of a simple Θ -module to a subalgebra Π is indecomposable provided that Π contains either all dstriction maps or tinflation maps in Θ . On the other hand, if none of these conditions holds then the restricted module can be zero, semisimple or indecomposable.

4.3 Pre-alcahestic case

Let Π be a pre-alcahestic subalgebra of the alchemic algebra. For simplicity, suppose Π have (A1). In this case, General Correspondence Theorem does not hold as stated above. In other words, the simple Π -modules are not parameterized by the simple modules of the isogation algebra associated to Π . However we can

still describe the parameterizing set for the simple Π -modules. First let us have an example for which the theorem fails.

Example 1 *Let Π be the biset algebra for H . Then, clearly, the simple Π -modules are the (non-zero) evaluations at H of simple biset functors. On the other hand, the isogation algebra associated to Π is just the group algebra $R\text{Out}(H)$ and it is clear that it has less number of simple modules.*

Let Ξ_Π be the subalgebra of Π that is generated by all elements in Π which does not factor through a smaller group. In other words, $x \in c_H \Pi c_K$ is contained in the generating set of Ξ_Π if and only if the equality $x = y \cdot z$ does not hold for any y and z in Π such that $y \in c_H \Pi c_L$ and $z \in c_L \Pi c_K$ with $|L| < (|H|, |K|)$. Then the simple Π -modules are parameterized by the isomorphism classes of simple Ξ_Π -modules. We shall not go into details of the proof of this claim. For a similar argument, see [3]. Further to this comment, one can also modify the special correspondence theorem in order to obtain descriptions of simple modules.

Chapter 5

Examples of Simple AGFs

In this chapter, we use the method of the previous chapter to construct simple functors for two special choices of alcahestic subalgebras, namely the alchemic algebra and the ordinary Mackey algebra.

The below construction of simple biset functors is a prototype for construction of simple modules of any other alcahestic subalgebra. In this sense we do not need a second example. However there is an important difference between the theory of Mackey functors and the theory of biset functors: Over a field of characteristic zero, the first theory is semisimple whereas the second one is not semisimple, in general. In this chapter, we shall see that the semisimplicity of Mackey functors makes the dual descriptions more explicit whereas, in the next chapter, we will see that the semisimplicity can be read from the two dual constructions of the previous chapter.

5.1 Biset functors

In this section, our aim is to construct simple biset functors using Corollary 4.2.6. In order to use the corollary, we need explicit description of induction from the destruction algebra and coinduction from the tinflation algebra. In the following

theorem, we characterize the evaluations of the induced module $\text{Ind}_{\nabla}^{\Gamma} D$ where D is a ∇ -module.

Theorem 5.1.1 *Let D be a ∇ -module and H be a subquotient of G . Then there is an isomorphism of R -modules*

$$(\text{Ind}_{\nabla}^{\Gamma} D)(H) \cong \left(\bigoplus_{J \preceq H} D(J) \right)_H$$

where the right hand side is the maximal H -fixed quotient of the direct sum.

Proof. Let

$$D_+(H) := \left(\bigoplus_{J \preceq H} D(J) \right)_H.$$

and write $[J, a]_H$ for the image of $a \in D(J)$ in $D_+(H)$. Since H acts trivially, it is clear that $[J, a]_H = {}^h[J, a]_H$ for all $h \in H$. Moreover, D_+H is generated as an R -module by $[J, a]_H$ for $J \preceq_H H$ and $a \in D(J)$. Here $\preceq_H$ means that we take J up to H -conjugacy. In other words,

$$D_+(H) = \bigoplus_{J \preceq_H H} R\{[J, a]_H : a \in D(J)\}.$$

On the other hand,

$$(\text{Ind}_{\nabla}^{\Gamma} D)(H) = \bigoplus_{J \preceq_H H} R\{\text{Tin}_J^H \otimes a : a \in D(J)\}.$$

Now $\text{Tin}_J^H \otimes a = 0$ if and only if $a \in I(\text{Out}(H))D(J)$ where $I(\text{Out}(H))$ is the augmentation ideal of $R\text{Out}(H)$. Therefore, the correspondence $\text{Tin}_J^H \otimes a \longleftrightarrow [J, a]_H$ extends linearly to an isomorphism of R -modules $(\text{Ind}_{\nabla}^{\Gamma} D)(H) \cong D_+(H)$. Evidently, this is an $R\text{Out}(H)$ -module isomorphism. $\square$

Let us describe the action of tinflation, dstriction and isogation on the generating elements $\text{Tin}_J^H \otimes a$ of the biset functor $\text{Ind}_{\nabla}^{\Gamma} D$. Note that we obtain these formulae by multiplying from the left with the corresponding generator and use the defining relations of the alchemic algebra. Let $J \preceq H \preceq G$ and $T \preceq H \preceq K$ and $a \in D(J)$. Finally let $A \preceq G$ such that $\phi : H \cong A$. Then

Tinflation

$$\text{Tin}_H^K(\text{Tin}_J^H \otimes a) = \text{Tin}_J^K \otimes a.$$

Destruction

$$\text{Des}_T^H(\text{Tin}_J^H \otimes a) = \sum_{x \in T^* \setminus H/J^*} \text{Tin}_{T \cap^x J}^T \otimes c^{x\lambda} \text{Des}_{J \cap T^x}^J a.$$

Isogation

$$c_{A,H}^\phi(\text{Tin}_J^H \otimes a) = \text{Tin}_{\phi(J)}^A \otimes c_{\phi(J),J}^\phi a.$$

The other functor, that we will make use of, is coinduction from the tinflation algebra Δ to the alchemic algebra Γ . We can describe its evaluations in terms of fixed-points as follows.

Theorem 5.1.2 *Let E be a Δ -module and $H \preceq G$. Then*

$$(\text{Coind}_\Delta^\Gamma E)(H) \cong \left(\prod_{J \preceq H} E(J) \right)^H$$

where the right hand side is the H -fixed subset of the direct product, and where H acts by permuting the coordinates.

We skip the proof of this theorem since it is similar to the proof of the above theorem. We only describe the actions of tinflation, destruction and isogation on the tuples $(x_J)_{J \preceq H}$. Let $J \preceq H \preceq G$ and $T \preceq H \preceq K$ and $A \preceq G$ such that $\phi : H \cong A$. Then

Tinflation

$$(\text{Tin}_H^K((x_J)_{J \preceq H}))_I = \sum_{y \in I^* \setminus K/H^*} \text{Tin}_{I \cap^y H}^I c^{y\lambda} x_{H \cap I^y}.$$

Here, we write x_L for the L -th coordinate of an element $x \in \text{Coind}_\Delta^\Gamma E(H)$.

Destruction

$$\text{Des}_T^H((x_J)_{J \preceq H}) = (x_J)_{J \preceq T}.$$

Isogation

$$c_{A,H}^\phi((x_J)_{J \preceq H}) = (\phi x_{\phi^{-1}(J)})_{J \preceq A}$$

where $\phi x = c_{J,\phi^{-1}(J)}^\phi x$.

By these theorems together with Corollary 4.2.6, we obtain the following two descriptions of the evaluations of the simple biset functors. For the first description, recall that $\mathcal{S}_{H,V}^\nabla$ denotes the simple ∇ -module corresponding to the simple pair (H, V) where $H \preceq G$ and V is a simple $R\text{Out}(H)$ -module. (Note that $\text{Out}_\Gamma(H) = \text{Out}(H)$.) By Theorem 5.1.1, the evaluation at $K \preceq G$ of the induced module $\text{Ind}_\nabla^\Gamma \mathcal{S}_{H,V}^\nabla$ is given by

$$\text{Ind}_\nabla^\Gamma \mathcal{S}_{H,V}^\nabla(K) = \left(\bigoplus_{L \preceq K, \phi: L \cong H} \phi V \right)_K.$$

To obtain the evaluation at K of the simple biset functor $S_{H,V}$, we take the quotient of the above module by the submodule $\mathcal{K}(K)$, defined in the proof of Theorem 4.2.3. Explicitly, $\mathcal{K}(K)$ is generated by the elements $x \in \text{Ind}_\nabla^\Gamma \mathcal{S}_{H,V}^\nabla(K)$ such that $\text{Des}_L^K x = 0$ for any $L \preceq K$ with $L \cong H$. More explicit conditions on x can be obtained using the generating set $\{\text{Tin}_L^K \otimes v : L \preceq K, L \cong H, v \in S_{H,V}^\nabla(L)\}$.

The second description provides the description of the evaluation of $S_{H,V}$ in terms of the images of tinflation maps. Note that although it is clear that the coordinate modules are just the images of the tinflation maps, the next result gives an explicit generating set.

Corollary 5.1.3 *Let H and K be subquotients of G and suppose that K contains H . Also let V be a simple $R\text{Out}(H)$ -module. Then evaluation at K of $S_{H,V}^\Gamma$ is*

$$S_{H,V}^\Gamma(K) = \sum_{I \preceq K, \phi: H \cong I} \text{Im}(\text{Tin}_I^K)$$

where for $v \in S_{H,V}^\Gamma(I) \cong V$, the J -th coordinate of $\text{Tin}_I^K v$ is given by

$$(\text{Tin}_I^K v)_J = \sum c_{J,I}^{y\lambda} v$$

where the sum is over the representatives of the double cosets $J^* \backslash K / I^*$ such that $I = I \sqcap J^y$ and $J = J \sqcap {}^y I$.

5.2 Ordinary Mackey functors

Our second example is the construction of simple Mackey functors. Since the Mackey algebra does not contain inflation and deflation, our notation simplifies in this case. We first modify our notation according to this simplification.

Following the general convention, we denote by $\mu = \mu_R(G)$ the Mackey algebra for G over R . Then the restriction algebra ∇ associated to μ is just generated by all restriction and conjugation maps. We call it the **restriction algebra** and denote by $\rho := \rho_R(G) = \nabla_\mu$. Similarly, we get the **transfer algebra** $\tau := \tau_R(G) = \Delta_\mu$ and the **conjugation algebra** $\gamma := \gamma_R(G) = \Omega_\mu$. Also in this case, the algebra $c_H \gamma c_H$ becomes the group algebra $R\overline{N}_G(H)$, where $\overline{N}_G(H)$ is the subquotient $N_G(H)/H$ of G . Finally it is clear that two subgroups H and K are μ -isomorphic if and only if they are G -conjugate.

We write $\mathcal{S}_{H,V}^\gamma$ for the conjugation functor defined for $K \leq G$ by

$$\mathcal{S}_{H,V}^\gamma(K) = {}^gV \quad \text{if } K = {}^gH$$

and zero otherwise. We also write $\mathcal{S}_{H,V}^\rho = \inf_\gamma^\rho \mathcal{S}_{H,V}^\gamma$ and $\mathcal{S}_{H,V}^\tau = \inf_\gamma^\tau \mathcal{S}_{H,V}^\gamma$.

As we mentioned earlier, the main reason for including this example is to show the simplification in characteristic zero. In this case Corollary 4.2.6 becomes

Theorem 5.2.1 *Let R be a field of characteristic zero. Then there are the following isomorphisms of Mackey functors hold.*

$$\text{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho \simeq S_{H,V}^\mu \simeq \text{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau$$

with

$$S_{H,V}^\mu(K) \simeq \left(\bigoplus_{L \leq K, L = {}^gH} {}^gV \right)_K \simeq \left(\prod_{L \leq K, L = {}^gH} {}^gV \right)^K$$

for any $K \leq G$.

Chapter 6

Semisimplicity of AGFs

In this chapter, we introduce the mark morphism for certain alcahestic group functors. We also show that it connects our two dual constructions of simple functors in an exact sequence. Furthermore we prove a characterization of semisimplicity of the corresponding alcahestic subalgebra in terms of the mark morphism. As a corollary to this theorem, we are able to obtain alternative proofs of semisimplicity theorems for the alcahestic subalgebras given in Chapter 3. Note that these results are known in each case with different proofs. Our approach gives a unified proof. Finally we end the thesis with an easy adaptation of this characterization for arbitrary finite-dimensional algebras over a field.

6.1 Mark morphism

Let Π be an alcahestic subalgebra of Γ . In this section, we write $\Delta := \Delta_\Pi$, $\nabla := \nabla_\Pi$ and $\Omega := \Omega_\Pi$. This will not cause any ambiguity since we will fix the subalgebra Π and only deal with this fixed subalgebra.

To introduce the mark morphism, let D be a ∇ -module. There is a projection map $\pi^D : \text{Res}_\nabla^\Pi \text{Ind}_\nabla^\Pi D \rightarrow D$. We still denote by π^D the induced morphism

$$\text{Res}_\Delta^\Pi \text{Ind}_\nabla^\Pi D \rightarrow \text{Inf}_\Omega^\Delta \text{Res}_\Omega^\nabla D$$

Also there is the following adjunction isomorphism

$$\mathrm{Hom}_{\Pi}(\mathrm{Ind}_{\nabla}^{\Pi} D, \mathrm{Coind}_{\Delta}^{\Pi} \mathrm{Inf}_{\Omega}^{\Delta} \mathrm{Res}_{\Omega}^{\nabla} D) \cong \mathrm{Hom}_{\Delta}(\mathrm{Res}_{\Delta}^{\Pi} \mathrm{Ind}_{\nabla}^{\Pi} D, \mathrm{Inf}_{\Omega}^{\Delta} \mathrm{Res}_{\Omega}^{\nabla} D).$$

Using this adjunction, we obtain a map

$$\beta^D : \mathrm{Ind}_{\nabla}^{\Gamma} D \rightarrow \mathrm{Coind}_{\Delta}^{\Gamma} \mathrm{Inf}_{\Omega}^{\Delta} \mathrm{Res}_{\Omega}^{\nabla} D.$$

Following Boltje [2], we call β^D the **mark morphism** for D .

Remark 6.1.1 *The mark morphism defined as above is a generalization of the well-known mark homomorphism*

$$\beta_G : B(G) \rightarrow \mathcal{B}(G)$$

where $B(G)$ is the Burnside ring of G and $\mathcal{B}(G)$ is the ghost ring of the Burnside ring. The ghost ring $\mathcal{B}(G)$ is defined as the dual of the Burnside ring and is isomorphic to the space of $\mathbb{Z}$ -valued functions constant on conjugacy classes of subgroups of G , i.e. $\mathcal{B}(G) = (\prod_{H \leq G} \mathbb{Z})^G$. The mark morphism is now given by associating a finite G -set X to the function $(|X^H|)_{H \leq G}$. For appropriate choices, this morphism becomes a special case of the morphism between Boltje's plus constructions for ordinary Mackey functors. For a detailed explanation, see [2]. In [12], it is shown that Boltje's plus constructions are nothing but induction and coinduction functors and the mark morphism can be obtained by the above method.

Remark 6.1.2 *The mark morphism can be described explicitly using a basis for the induced and coinduced modules. We shall present this for the alchemic algebra. It can be done for alcahestic algebras in general with more care. Given a subquotient H of K and an element $\mathrm{Tin}_H^K \otimes a$ in $\mathrm{Ind}_{\nabla}^{\Gamma} D(K)$, the morphism β^D is given by*

$$\beta_K^D(\mathrm{Tin}_H^K \otimes a) = (\pi_I(\mathrm{Des}_I^K \mathrm{Tin}_H^K \otimes a))_{I \preceq K}.$$

Now using the Mackey relation, we can calculate β_K^D more explicitly, as follows (cf. [2]).

$$\begin{aligned} \beta_K^D(\mathrm{Tin}_H^K \otimes a) &= \left(\pi_I \left(\sum_{x \in I^* \backslash K/H^*} \mathrm{Tin}_{I \cap^x H}^I c^{x \circ \lambda} \mathrm{Des}_{H \cap I^x}^H a \right) \right)_{I \preceq K} \\ &= \left(\sum_{x \in I^* \backslash K/H^*, I = I \cap^x H} c^{x \circ \lambda} \mathrm{Des}_{H \cap I^x}^H a \right)_{I \preceq K}. \end{aligned}$$

If we put $D = \mathcal{S}_{H,V}^\nabla$, the simple ∇ -module associated to the simple pair (H, V) , then clearly $\text{Inf}_\Omega^\Delta \text{Res}_\Omega^\nabla \mathcal{S}_{H,V}^\nabla = \mathcal{S}_{H,V}^\Delta$. Hence the mark morphism for $\mathcal{S}_{H,V}^\nabla$ is a morphism between the two dual constructions

$$\beta^{H,V} : \text{Ind}_\nabla^\Pi \mathcal{S}_{H,V}^\nabla \rightarrow \text{Coind}_\Delta^\Pi \mathcal{S}_{H,V}^\Delta.$$

Moreover, we have the following exact sequence.

Proposition 6.1.3 *The following spliced sequence is exact.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_{\text{Ind}_\nabla^\Pi \mathcal{S}_{H,V}^\nabla, H} & \longrightarrow & \text{Ind}_\nabla^\Pi \mathcal{S}_{H,V}^\nabla & \xrightarrow{\beta^{H,V}} & \text{Coind}_\Delta^\Pi \mathcal{S}_{H,V}^\Delta \longrightarrow \mathcal{C}_{H,V} \longrightarrow 0 \\ & & & & \searrow & \nearrow & \\ & & & & & S_{H,V}^\Pi & \\ & & & & \nearrow & \searrow & \\ & & 0 & & & & 0 \end{array}$$

where $\mathcal{C}_{H,V} = \text{Coind}_\Delta^\Pi \mathcal{S}_{H,V}^\Delta / \mathcal{I}_{\text{Coind}_\Delta^\Pi \mathcal{S}_{H,V}^\Delta, H}$.

Proof. It suffices to show that the kernel of the mark morphism is the unique maximal subfunctor of $\text{Ind}_\nabla^\Pi \mathcal{S}_{H,V}^\nabla$. The inclusion $\ker \beta^{H,V} \subset \mathcal{K}_{\text{Ind}_\nabla^\Pi \mathcal{S}_{H,V}^\nabla, H}$ is trivial, as the right hand side is maximal and $\beta^{H,V}$ is non-zero. To show the reverse inclusion, let $K \preceq G$ and $a \in \mathcal{K}_{\text{Ind}_\nabla^\Pi \mathcal{S}_{H,V}^\nabla, H}(K)$. Then

$$\beta_K^{H,V}(a) = (\pi_I(\text{Des}_I^K a))_{I \preceq K, I \cong H} = 0$$

as by definition of $\mathcal{K}_{\text{Ind}_\nabla^\Pi \mathcal{S}_{H,V}^\nabla, H}(K)$, we have $\text{Des}_I^K a = 0$ for any $I \cong H$. $\square$

The following is immediate from the exactness of the above sequence.

Corollary 6.1.4 *Let (H, V) be a simple pair for Ω over a field R and $K \preceq G$. Then*

$$\dim_R S_{H,V}^\Pi(K) = \text{rank } \beta_K^{H,V}.$$

6.2 A Semisimplicity criterion

6.2.1 The motivating example

For this section, let $\Pi = \Gamma$. A special case of the mark morphism is the well-known natural morphism, called the **linearization morphism**

$$\mathbb{Q}\text{lin}_G : \mathbb{Q}B(G) \rightarrow \mathbb{Q}R_{\mathbb{Q}}(G)$$

where $B(G)$ is the Burnside group, as above, and $\mathcal{R}_{\mathbb{Q}}(G)$ is the Grothendieck group of rational representation of G . The morphism is given by associating a G -set X to the permutation module $\mathbb{Q}X$. To see that this morphism is a mark morphism, we prove the following identification of the Burnside biset functor B^G .

Proposition 6.2.1 *There is an isomorphism of biset functors $B^G \cong \text{Ind}_{\nabla}^{\Gamma} \mathcal{S}_{1,1}^{\nabla}$ where $\mathcal{S}_{1,1}^{\nabla}$ is the simple ∇ -module having one copy of the trivial module at each trivial coordinate.*

Proof. We need to specify an isomorphism $\Phi : B^G \rightarrow \text{Ind}_{\nabla}^{\Gamma} \mathcal{S}_{1,1}^{\nabla}$ of biset functors. To do this, we specify an isomorphism $\Phi_H : B^G(H) \rightarrow \text{Ind}_{\nabla}^{\Gamma} \mathcal{S}_{1,1}^{\nabla}(H)$ of R -modules for each $H \preceq G$ which is compatible with the actions of tinflation, destriktion and isogation. By Theorem 5.1.1, $\text{Ind}_{\nabla}^{\Gamma} \mathcal{S}_{1,1}^{\nabla}(H)$ is generated by $\{\text{Tin}_{L/L}^H \otimes 1 : L \leq H\}$. Now we define Φ_H by associating $[H/L]$ to $\text{Tin}_{L/L}^H \otimes 1$. Straightforward calculations show that Φ is an isomorphism of biset functors. $\square$

Now the image of the mark morphism $\beta^{1,1}$ is the simple biset functor $S_{1,1}$ by Proposition 6.1.3. It is shown by Bouc [5] that over a field of characteristic zero, the functor of rational representations is simple, isomorphic with $S_{1,1}$. Hence, over a field of characteristic zero, the linearization morphism lin_G (after composing with an inclusion) and the mark morphism $\beta_G^{1,1}$ coincide since they coincide on the trivial subgroup.

Note that the mark morphism is given explicitly by associating $\text{Tin}_{L/L}^H \otimes 1$ to the function $(|K \backslash H/L|)_{K/K \preceq H}$ where $|K \backslash H/L|$ is the number of double coset

representatives of K and L in H . Notice that the mark morphism for the Burnside biset functor is different than the usual mark morphism, which indeed is not a morphism of biset functors. Finally, in [8], Bouc describe the biset functor structure of the kernel of the linearization morphism for p -groups where p is a prime number. Over a field of characteristic zero and for a p -group P , the kernel is known to be isomorphic to the torsion-free Dade group $\mathbb{Q} \otimes D(P)$.

Another corollary of Proposition 6.2.1 is the following well-known result, see [5] and [1].

Corollary 6.2.2 *Let k be a field. Then the Burnside biset functor kB^G is the projective cover of the simple biset functor $S_{1,1}$.*

Indeed it is clear that the simple restriction functor $\mathcal{S}_{1,1}^\nabla$ is projective, which implies that B^G is projective. Also it is indecomposable since the adjunction isomorphism

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_\nabla^\Gamma \mathcal{S}_{1,1}^\nabla, \mathrm{Ind}_\nabla^\Gamma \mathcal{S}_{1,1}^\nabla) \cong \mathrm{Hom}_\nabla(\mathcal{S}_{1,1}^\nabla, \mathrm{Res}_\nabla^\Gamma \mathrm{Ind}_\nabla^\Gamma \mathcal{S}_{1,1}^\nabla)$$

is actually an isomorphism of rings. Further to this, the following dual statement also holds.

Proposition 6.2.3 *There are isomorphisms of biset functors $k\mathcal{B}^G \cong \mathrm{Coind}_\Delta^\Gamma \mathcal{S}_{1,1}^\Delta \cong I_{1,1}$ where $\mathcal{B}^G$ denotes the functor of the ghost ring of Burnside ring and $I_{1,1}$ denotes the injective hull of the simple biset functor $S_{1,1}$. $\square$*

In particular, $S_{1,1}$ is both projective and injective if and only if the mark morphism $\beta^{1,1} : B^G \rightarrow \mathcal{B}^G$ is an isomorphism. This observation inspires the criterion of semisimplicity of the next section.

6.2.2 The criterion

There are semisimplicity results for biset functors and for certain globally-defined Mackey functors. Each result proved independently using different tools. With

the following theorem, we give a criterion that reduces the question of semisimplicity to a question about the mark morphism. After proving the criterion, we shall review the known results according to our approach.

Theorem 6.2.4 *Let G be a finite group and R be a field of characteristic zero and Π be an alcahestic subalgebra of Γ . The following statements are equivalent.*

- (i) *The alcahestic subalgebra $\Pi_R(G)$ for G over R is semisimple.*
- (ii) *The mark morphism $\beta^{H,V} : \text{Ind}_{\nabla}^{\Pi} S_{H,V}^{\nabla} \rightarrow \text{Coind}_{\Delta}^{\Pi} S_{H,V}^{\Delta}$ is an isomorphism for any simple pair (H, V) for Ω_{Π} .*

Proof. It is well-known that a finite dimensional algebra Λ over a field is semisimple if and only if every simple Λ -module is both projective and injective. Hence it is sufficient to prove that this is also equivalent to the second statement of the theorem. First, suppose that β is an isomorphism. We are to show that the simple Π -group functor $S_{H,V}^{\Gamma}$ is both injective and projective. By the exact sequence in Proposition 6.1.3, we get

$$\text{Ind}_{\Delta}^{\Pi} S_{H,V}^{\nabla} \cong S_{H,V}^{\Pi} \cong \text{Coind}_{\nabla}^{\Pi} S_{H,V}^{\Delta}.$$

Now consider any short exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} S_{H,V}^{\Pi} \longrightarrow 0$$

of Π -group functors. By the second isomorphism above we get $\phi \in \text{Hom}_{\Pi}(F, \text{Coind}_{\nabla}^{\Pi} S_{H,V}^{\Delta})$. (Note that since ϕ is a morphism of Π -group functors, preimage of any $v \in S_{H,V}^{\Pi}(X)$ is in the kernel of destruction maps to the isomorphism class of H).

By adjointness of coinduction and restriction we get a morphism $\bar{\phi} \in \text{Hom}_{\nabla}(\text{Res}_{\nabla}^{\Pi} F, S_{H,V}^{\Delta})$. Since $R\text{Out}_{\Pi}(H)$ is semisimple, V is projective as an $R\text{Out}_{\Pi}(H)$ -module and hence the morphism $\bar{\phi}_H : F(H) \rightarrow V$ of $R\text{Out}_{\Pi}(H)$ -modules splits. That is to saying that V is a direct summand of the $R\text{Out}_{\Pi}(H)$ -module $F(H)$. Therefore, there is a morphism $\bar{\psi} : S_{H,V}^{\Delta} \rightarrow \text{Res}_{\Delta}^{\Pi} F$ of Δ -modules

given by sending V to the direct summand of $F(H)$ which is isomorphic to V . Now by adjointness of induction and restriction the map $\bar{\psi}$ induces a map $\psi \in \text{Hom}_\Pi(\text{Ind}_\Delta^\Pi \mathcal{S}_{H,V}^\nabla, F)$. Clearly the composition $\phi \circ \psi$ is the mark homomorphism, which is an isomorphism by our assumption. Hence the composition $\psi \circ \beta^{-1}$ is the required splitting for the above exact sequence.

Similarly it can be shown that any short exact sequence of Π -group functors

$$0 \longrightarrow S_{H,V}^\Pi \xrightarrow{\alpha} F \longrightarrow C \longrightarrow 0$$

splits, that is, $S_{H,V}^\Pi$ is injective.

Conversely, suppose the simple Π -group functor $S_{H,V}^\Pi$ is both injective and surjective. Then both of the sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow \text{Ind}_\Delta^\Pi \mathcal{S}_{H,V}^\nabla \longrightarrow S_{H,V}^\Pi \longrightarrow 0$$

and

$$0 \longrightarrow S_{H,V}^\Pi \longrightarrow \text{Coind}_\nabla^\Pi \mathcal{S}_{H,V}^\Delta \longrightarrow \mathcal{C} \longrightarrow 0$$

splits. But $\text{Ind}_\Delta^\Pi \mathcal{S}_{H,V}^\nabla$ and $\text{Coind}_\nabla^\Pi \mathcal{S}_{H,V}^\Delta$ are indecomposable. Hence the result follows. $\square$

An easy consequence of the above theorem is the following corollary. Let ${}^\text{op} : \Gamma \rightarrow \Gamma$ denote the **opposite map** which extends the correspondence sending a transitive biset $(\frac{G \times H}{U})$ to its **opposite** $(\frac{H \times G}{U^\text{op}})$ where $U^\text{op} = \{(h, g) \in H \times G : (g, h) \in U\}$. For any subalgebra Λ of Γ , we denote by Λ^op the image of Λ under the opposite map.

Corollary 6.2.5 *The algebra Π is semisimple only if $\text{rank}_R \Delta_\Pi = \text{rank}_R \nabla_\Pi^\text{op}$.*

Proof. Let H be a subquotient of G and 1 denotes the trivial $R\text{Out}(H)$ -module. Then it is easy to see that $\text{Ind}_\nabla^\Pi S_{H,1}(K)$ has an R -basis $\{\text{Tin}_L^K \otimes 1 \mid L =_\Pi H\}$.

In particular, its R -dimension is equal to the number of subquotients of K that are Π -isomorphic to H and such that Tin_L^K is in Π . Similarly the R -dimension of the dual construction is equal to the number of subquotients of K that are Π -isomorphic to H and such that Des_L^K is in Π . Hence the two evaluations are isomorphic only if $\text{rank}_R \Delta_\Pi = \text{rank}_R \nabla_\Pi^{\text{op}}$, as required.

Now we can easily prove several semisimplicity theorems.

Theorem 6.2.6 *Let G be a finite group and R be a field of characteristic zero. The followings hold.*

1. (Barker, Bouc) *The alchemic algebra $\Gamma_R(G)$ is semisimple if and only if G is cyclic.*
2. (Webb) *The global Mackey algebra $\tilde{\mu}_R(G)$ is semisimple.*
3. (Thévanaz - Webb) *The (ordinary) Mackey algebra $\mu_R(G)$ is semisimple.*
4. *The category of inflation functors is semisimple only if $G = 1$.*
5. *The category of deflation functors is semisimple only if $G = 1$.*

Proof. The last two parts of the theorem follows immediately from the above corollary. Part (2) and Part (3) follows from [2, Proposition 2.4] which states that the mark morphism for Mackey functors is an isomorphism over a field of characteristic zero. For part (3), since the Mackey algebra is not an actual subalgebra, our theorem cannot be applied directly. However we can easily modify the proof of the theorem so that it holds for the Mackey algebra, too. Hence it remains to prove Part (1).

First, suppose G is not cyclic. We are to show that the mark morphism is not an isomorphism for some pair (H, V) . We have $\text{Ind}_\nabla^\Gamma \mathcal{S}_{1,1}^\nabla \cong RB^G$ by Lemma 6.2.1 which is a projective indecomposable functor. It is easy to see that this functor is not simple since the functor $R\mathcal{R}_\mathbb{Q}$ is a nontrivial quotient of B^G . So the mark morphism for the simple pair $(1, 1)$ is not an isomorphism.

Conversely, suppose that the group G is cyclic. Then we are to show that the mark morphism is an isomorphism for each simple pair (H, V) . But this is equivalent to showing that $\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta$ and $\text{Ind}_\nabla^\Gamma \mathcal{S}_{H,V}^\nabla$ are simple, that is, to showing that the subfunctor $\mathcal{I}_{\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta, H}$ of $\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta$ is not proper and the subfunctor $\mathcal{K}_{\text{Ind}_\nabla^\Gamma \mathcal{S}_{H,V}^\nabla, H}$ is zero. We shall show the first claim. The second one follows from a similar argument. Without loss of generality, we shall prove the claim for the evaluation at G , that is we shall prove that $\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta(G) = \mathcal{I}_{\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta, H}(G)$. For simplicity, write $\mathcal{I}(G)$ for $\mathcal{I}_{\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta, H}(G)$. Now any element of $\mathcal{I}(G)$ is a sum of tinflated elements $\text{Tin}_I^G v_I$ for some $I \cong H$ and where we write v_I for the generator of the one-dimensional $R\text{Out}(H)$ -module $\mathcal{S}_{H,V}^\Delta(I)$. By Corollary 5.1.3, we have

$$(\text{Tin}_I^G v_I)_J = \sum c_{J,I}^{y\lambda} v_I$$

where the sum is over the representatives of the double cosets $J^* \backslash G / I^*$ such that $I = I \cap J^y$ and $J = J \cap {}^y I$. But G is cyclic, hence $J^y = J$ for any $y \in G$. Therefore the J -th coordinate $(\text{Tin}_I^G v_I)_J$ of $\text{Tin}_I^G v_I$ is non-zero only if the equalities $I = I \cap J$ and $J = J \cap I$ hold. Moreover when the equalities hold we have

$$(\text{Tin}_I^G v_I)_J = |J^* \backslash G / I^*| c_{J,I}^\lambda v_I.$$

Clearly $c_{J,I}^\lambda v_I = v_J$ and hence

$$(\text{Tin}_I^G v_I)_J = |J^* \backslash G / I^*| v_J.$$

Finally we have

$$\text{Tin}_I^G v_I = (|J^* \backslash G / I^*| v_J)_{J \in S_I}$$

where $S_I = \{J \preceq G : J \cong H, I = I \cap J, J = J \cap I\}$. Now our aim is to show that the set $\mathcal{B} = \{\text{Tin}_I^G v_I : I \preceq G, I \cong H\}$ forms a basis for the evaluation $\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta(G)$. In order to show this, we first determine the set S_I for any I and then show that the set $\mathcal{B}$ is linearly independent. Since the order of the set $\mathcal{B}$ is equal to the order of the canonical basis of $\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta(G)$, consisting of characteristic functions of subquotients of G , the proof would be established.

Now write $G = C_1 \times C_2 \times \cdots \times C_n$ as a direct product of (cyclic) Sylow subgroups. For simplicity, we shall prove our claim for the case that $H \preceq C_1$. The general case follows from the same argument.

Let $I \preceq G$ be a subquotient isomorphic to H . Then it is easy to see that $J \preceq G$ is in S_I if and only if the equalities $J^* = I^* \times Z$ and $J_* = I_* \times Z$ hold for some $Z \preceq G$ with $(|J^*|, |Z|) = 1$. Indeed the converse statement is trivial and the first implication follows from the equalities

$$\frac{J^*}{J_*} = \frac{(J^* \cap I^*)J_*}{(J^* \cap I_*)J_*} \quad \text{and} \quad \frac{I^*}{I_*} = \frac{(I^* \cap J^*)I_*}{(I^* \cap J_*)I_*}$$

using the fact that all groups involved are cyclic.

Hence we obtain a description of the set S_I . However we can make it more explicit. Let $J \in S_J$ be the subquotient with J^* has minimal order among all $J' \in S_I$. Then clearly $J \preceq C_1$ and $S_J = S_I$. Now we can identify the set S_J with the set of subgroups of $G_1 = G/C_1$. Note further that given $L, N \in S_J$ we have the equality

$$|L^* \backslash G / N^*| = |C_1 / J^*| |L_1^* \backslash G_1 / N_1^*|$$

where $L^* = J^* \times L_1^*$ and $N^* = J^* \times N_1^*$. Indeed we have

$$|L^* \backslash G / N^*| = \frac{|G|}{|L^* N^*|} = \frac{|G| |L^* \cap N^*|}{|L^*| |N^*|}.$$

Now both the denominator and the numerator of the last quotient is divisible by the square of the order of J^* . Cancelling these, we get the above equality.

Hence it remains to show that the elements of the set $\mathcal{B}$ are linearly independent. It is clear that the sets S_I gives a partition of the set of subquotients of G isomorphic to H . Hence it suffices to show that the elements of the set $\mathcal{B}_J = \{\text{Tin}_I^G v_I : I \in S_J\}$ are linearly independent where J is the representative of the set S_J with minimal order of its numerator. But this is clear since for any $I \preceq J$, we have

$$\text{Tin}_I^G v_I = (|C_1 / J^*| |J_1^* \backslash G_1 / I_1^*| v_I)_{I \in S_J}$$

and any linear relation among these elements gives a linear relation between the columns of the double coset matrix $[|L \backslash G_1 / N|]_{L, N \leq G_1}$ which is well-known to be invertible for cyclic groups. Thus we have proved that $\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta(G) = \mathcal{I}(G)$ proving that $\text{Coind}_\Delta^\Gamma \mathcal{S}_{H,V}^\Delta$ is simple, as required. $\square$

6.3 A generalization

We end with a generalization of Theorem 6.2.4 to finite-dimensional algebras over fields. Let A be a finite-dimensional algebra over a field k . Let e be an idempotent of the algebra A . We denote by A_e the truncated subalgebra eAe .

Given a simple A -module V , it is well-known that the restriction eV of V to the subalgebra A_e is either zero or a simple A_e -module. Conversely, given a simple A_e -module W , there exists unique, up to isomorphism, simple A -module $S_{e,W}$ such that $eS_{e,W} \cong W$. Indeed we can construct the simple module $S_{e,W}$ in two ways, as follows. See [15] for further details.

For the first construction, consider the A -module

$$F = \text{Ind}_{A_e}^A W := Ae \otimes_{A_e} W.$$

It is clear that the A -module F satisfies $eF \cong W$. However the A -module F is not necessarily simple. We let K be the sum of all submodules of F annihilated by the idempotent e . Then it is easy to show that the A -submodule K is the unique maximal submodule of the A -module F and it intersects eF trivially. Therefore the quotient F/K is simple satisfying $e(F/K) \cong W$. We leave it as an easy exercise to show that the simple module F/K is unique, up to isomorphism. Note further that the induced module F is indecomposable. To see this consider the adjunction isomorphism

$$\text{Hom}_A(F, F) \cong \text{Hom}_{A_e}(W, eF) \cong \text{Hom}_{A_e}(W, W)$$

which is indeed an isomorphism of rings.

The second way of constructing the simple modules is to consider the dual construction. Let W be as above. Put

$$E = \text{Coind}_{A_e}^A W := \text{Hom}_{A_e}(eA, W).$$

Again we have

$$eE = e \text{Hom}_{A_e}(A, W) \cong \text{Hom}_{A_e}(eAe, W) \cong W$$

since e is the identity element of A_e and we only consider the A_e -invariant maps. As above, the A -module E is not simple, in general. But it is indecomposable, that is its socle is simple with the property that $e \operatorname{soc}(E) \cong W$. Therefore by the uniqueness of the simple module $S_{e,W}$, we have $\operatorname{soc}(E) \cong S_{e,W}$.

Having this notation, we are now ready to prove a generalization of the above criterion.

Theorem 6.3.1 *Let A be a finite dimensional algebra over a field and e be an idempotent of A . Let $f = 1 - e$. Then the following are equivalent*

- (1) *The algebra A is semisimple.*
- (2) *The following two conditions hold.*
 - (a) *The algebras eAe and fAf are semisimple.*
 - (b) *For any simple gAg -module V , for $g \in \{e, f\}$, there is an isomorphism of A -modules $Ag \otimes_{gAg} V \cong \operatorname{Hom}_{gAg}(gA, V)$.*

Proof. It is clear that (1) implies (2). We shall prove that (2) implies (1). Given a simple A -module V , we have $gV \neq 0$ for some $g \in \{e, f\}$. Then by construction, we have $S_{g,gV} \cong V$. But by the assumption we have

$$S_{g,gV} \cong \operatorname{Ind}_{A_g}^A gV.$$

Moreover the simple module gV is projective since the algebra A_g is semisimple. Hence the A -module V is projective. Similarly V is also injective. But V was arbitrary, thus A is semisimple. $\square$

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