SIMPLE SINGULAR IRREDUCIBLE PLANE SEXTICS

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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We consider irreducible complex plane projective curves of degree six with simple singular points only and classify such curves up to equisingular deformation. (We concentrate on the so-called non-special curves, as the special ones are already known). We list all sets of singularities realized by such curves, discuss their relation to the maximizing sets (i.e., those of total Milnor number 19), and, for each set of singularities found, describe the connected components of the moduli space. We also discuss the question of the realizability of a given set of singularities by a real curve.

Keywords: plane sextic, simple singularity.

ÖZET

BASİT TEKİLLİ İNDİRGENEMEZ DÜZLEMSEL ALTINCI DERECEDEN EĞRİLER

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Sadece basit tekilliği olan altıncı dereceden Compleks projektif düzlemsel eğrileri göz önünde bulundurduk ve bu tip eğrileri tekil noktaları koruyan deformasyona göre sınıflandırdık. (Özel olanlar zaten bilindiği için özel olmayan olarak adlandırılan eğrilere odaklandık). Bu tür eğriler tarafından gerçekleştirilebilen tekil kümelerini listeledik, maksimize (toplam Milnor sayısı 19 olanlar) kümelerle olan bağlantılarını araştırdık, ve bulduğumuz her bir kümenin modüli uzayının bağlantılı parçalarını tanımladık. Ayrıca elde edilen her bir tekil kümenin reel bir eğride hayat bulup bulamayacağını irdeledik.

Anahtar sözcükler: düzlemsel altıncı dereceden eğriler, basit tekillik.

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Chapter 1

Introduction

1.1 Area of sextics from the point of equisingular deformation

Equisingular deformation classification of sextics has been a subject of interest for several years. A sextic is a plane curve $D \in \mathbb{P}^2$ of degree six. Equisingular deformation of a curve D is homotopic deformation of the curve, preserving its singularities in their neighbourhoods. The degree six is interesting among simple singular curves, since proof of classification of simple singular sextics is based on K3-surfaces. A sextic is *simple* if all its singular points are simple, *i.e.*, those of type **A**–**D**–**E**, see [1]. Classification of sextics which have non-simple singularities requires completely different techniques and these sextics are well-known.

Simple sextics are questioned from several aspects such as equisingular deformation classification, fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus D)$ of the complement or finding equations for given set of singularities. Fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus D)$ of sextics are widely handled and computed by A.Degtyarev in [2, 3]. Finding an equation for a given set of singularities is an open area such that equations have been found for only a few set of singularities in [4, 5, 6, 7, 8, 9]. Moreover, visualization of sextics is an open problem except for the sextics with a triple point. Degtyarev gave the idea about how they sit in $\mathbb{P}^2$ by means of skeletons in [2]. The other aspect, equisingular deformation classification of sextics is the area of this thesis. This classification is already done for the so called *maximizing* sextics which are the sextics with the maximal total Milnor number $\mu = 19$. Inquiry of existence and uniqueness of simple sextics with a given set of singularities is reduced to an arithmetical problem in [10, 11] and set of singularities which are realized by reducible/irreducible simple sextics are computed and listed by Jin-Gen Yang in [11]. Afterwards, Ichiro Shimada classified maximizing simple sextics in [12] up to equisingular deformation.

In the classification of sextics two new notions come into being; special and non-special simple sextics. An irreducible sextic $D \subset \mathbb{P}^2$ is called *special* (more precisely, $\mathbb{D}_{2n}$-*special*) if its fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ factors to a dihedral group $\mathbb{D}_{2n}$, $n \geqslant 3$. Most of the sets of singularities which have special and non-special realizations constitute *Zariski pairs* (for more detail, see [10, 13]).

In this thesis we completed the classification of non-maximal non-special irreducible simple plane sextics and derived the connection to the maximal ones. Resolution of double covering of $\mathbb{P}^2$ branched at simple sextic gives us a K3-surface and classification of sextics relies on the application of global Torelli theorem for this K3-surface (see §3.1). By this way, topological matter turns out to be an arithmetical problem. Hence integral lattices and quadratic forms become the main tool in the classification.

1.2 Principal results

Primary way that we follow throughout this thesis depends on the steps of enumeration of homological types in [10]. These steps are stated in §3.2.2.

In the application of the steps, main tools are lattices and quadratic forms. For the first part, we used Nikulin's existence theorem in [14] which is the primary way to find existing sets of singularities. However, for uniqueness we used Miranda – Morrison results (see §2.3.1) instead of Nikulin's uniqueness theorem, since

Miranda – Morrison's theorems provide the desired results about uniqueness for almost all set of singularities.

By computer aided calculations we listed all sets of singularities which can be realized by irreducible non-maximal non-special simple plane sextics and obtained 2996 sets of singularities. Among them, all but 12 sets can be realized by real sextics.

Among the list of non-maximal sextics there are some extraordinary sets of singularities. The set $2\mathbf{A}_9$ is the only one non-maximal that can be realized by two disjoint deformation families. The set $2\mathbf{A}_5 \oplus \mathbf{E}_6 \oplus \mathbf{A}_1$ is the only non-real stratum (can not be realized by a deformation class of real sextic) with Milnor number $\mu = 17$. Another one is $\mathbf{A}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_7$: the corresponding stratum is real, but it contains no real curves.

The rest of this thesis is organized as follows. In the second chapter we give definitions and preliminary information about lattices and integral quadratic forms. The part "Lattice extensions" which contain Nikulin's theorem (about existence of a lattice) is also included in this chapter. Moreover the most useful tool for our research, Miranda – Morrison's results, takes a great place here.

In Chapter 3 we give general information about all (special and non-special, maximal and non-maximal) sextics and introduce the notion of strata to specify different classes of sextics. We also applied global Torelli theorem on K3-surface which is the way how we convert a topological matter to an arithmetic algorithm. In the last part of this chapter we gathered the previously known results about maximal sextics done by Yang [11], Shimada [12] and Degtyarev [2, 3].

Chapter 4 and Chapter 5 contain our results on non-maximal sextics and their proofs. Additionally, in Chapter 5 we gave some theorems about real structures.

1.3 Notations

$\underline{\mathbb{G}_n}$: $(= \mathbb{Z}/n\mathbb{Z})$ (reserving $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for p-adic numbers) the cyclic group of order n.
$\underline{\mathbb{D}_{2n}}$: the dihedral group of order $2n$.
$\underline{SL(n, \Bbbk)}$: the group of $(n \times n)$-matrices M over a field $\Bbbk$ such that $\det M = 1$.
$\underline{\mathbb{B}_n}$: the braid group on n stings. The *reduced braid group* (or the *modular group*) is the quotient $\Gamma = \mathbb{B}_3/(\sigma_1\sigma_2)^3$ of $\mathbb{B}_3$ by its center; one has $\Gamma = PSL(2, \mathbb{Z}) = \mathbb{G}_2 * \mathbb{G}_3$. The braid group is generated by the *Artin generators* σ_i, $i = 1, \ldots, n-1$, subject to the relations

$$[\sigma_i, \sigma_j] = 1 \quad \text{if } |i-j| > 1, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

Throughout the thesis, all group actions are right, and we use the notation $(x, g) \mapsto x \uparrow g$. The standard action of $\mathbb{B}_n$ on the free group $\langle \alpha_1, \ldots, \alpha_n \rangle$ is as follows:

$$\sigma_i : \begin{cases} \alpha_i \mapsto \alpha_i \alpha_{i+1} \alpha_i^{-1}, & \\ \alpha_{i+1} \mapsto \alpha_i, & \\ \alpha_j \mapsto \alpha_j, & \text{if } j \neq i, i+1 \end{cases}$$

The element $\rho_n := \alpha_1 \ldots \alpha_n \in \langle \alpha_1, \ldots, \alpha_n \rangle$ is preserved by $\mathbb{B}_n$. Given a pair α_1, α_2, we use the notation $\{\alpha_1, \alpha_2\}_n := \alpha_2^{-1}(\alpha_2 \uparrow \sigma_1^n) \in \langle \alpha_1, \alpha_2 \rangle$ for $n \in \mathbb{Z}$.
$\underline{\mathbb{P}}$: $(= \{2, 3, \ldots\})$ the set of all primes.
$\underline{R^\times}$: the group of units of a commutative ring R. Recall that $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 = \{\pm 1\}$ for $p \in \mathbb{P}$ odd, and $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2 = (\mathbb{Z}/8)^\times \cong \{\pm 1\} \times \{\pm 1\}$ is generated by 7 mod 8 and 5 mod 8. If $m \in \mathbb{Z}$ is prime to p, its class in $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ is the Legendre symbol $(\frac{m}{p}) \in \{\pm 1\}$ if p is odd or $m \bmod 8 \in (\mathbb{Z}/8)^\times$ if $p = 2$.

Chapter 2

Integral Lattices and Quadratic Forms

2.1 Finite quadratic forms (see [14])

A *finite quadratic form* is a finite abelian group $\mathcal{N}$ equipped with a symmetric bilinear form $b\colon \mathcal{N} \otimes \mathcal{N} \to \mathbb{Q}/\mathbb{Z}$ and a *quadratic extension* of b, *i.e.*, a map $q\colon \mathcal{N} \to \mathbb{Q}/2\mathbb{Z}$ such that $q(x+y) - q(x) - q(y) = 2b(x,y)$ for all $x, y \in \mathcal{N}$ (where 2 is the isomorphism $\times 2\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$); clearly, b is determined by q. If q is understood, we abbreviate $b(x,y) = x \cdot y$ and $q(x) = x^2$. In what follows, we consider *nondegenerate* forms only, *i.e.*, such that the associated homomorphism $\mathcal{N} \to \operatorname{Hom}(\mathcal{N}, \mathbb{Q}/\mathbb{Z})$, $x \mapsto (y \mapsto x \cdot y)$ is an isomorphism.

Each finite quadratic form $\mathcal{N}$ splits into orthogonal sum $\mathcal{N} = \bigoplus_{p \in \mathbb{P}} \mathcal{N}_p$ of its p-primary components $\mathcal{N}_p := \mathcal{N} \otimes \mathbb{Z}_p$. The *length* $\ell(\mathcal{N})$ of $\mathcal{N}$ is the minimum number of generators of $\mathcal{N}$. Obviously, $\ell(\mathcal{N}) = \max_{p \in \mathbb{P}} \ell_p(\mathcal{N})$, where $\ell_p(\mathcal{N}) := \ell(\mathcal{N}_p)$. The notation $-\mathcal{N}$ stands for the group $\mathcal{N}$ with the form $x \mapsto -x^2$.

We describe nondegenerate finite quadratic forms by expressions of the form $\langle q_1 \rangle \oplus \ldots \oplus \langle q_r \rangle$, where $q_i := \frac{m_i}{n_i} \in \mathbb{Q}$, g.c.d.$(m_i, n_i) = 1$, $m_i n_i = 0 \bmod 2$; the group is generated by pairwise orthogonal elements $\alpha_1, \ldots, \alpha_r$ (numbered in the order of appearance), so that $\alpha_i^2 = q_i \bmod 2\mathbb{Z}$ and the order of α_i is n_i. (In the 2-torsion, there also may be indecomposable summands of length 2, but we do not need them.) Describing an automorphism σ of such a group, we only list the images of the generators α_i that are moved by σ.

A finite quadratic form is called *even* if $x^2 = 0 \bmod \mathbb{Z}$ for each element $x \in \mathcal{N}$ of order two; otherwise, the form is called *odd*. In other words, $\mathcal{N}$ is odd if and only it contains $\langle \pm\frac{1}{2} \rangle$ as an orthogonal summand.

Given a prime $p \in \mathbb{P}$, the *determinant* $\det_p \mathcal{N}$ is defined as the determinant of the 'matrix' of the quadratic form on $\mathcal{N}_p$ in an appropriate basis (see [14] and [15] for details and alternative definitions). The determinant is well defined modulo squares; if $\mathcal{N}_p$ is nondegenerate, one has $\det_p \mathcal{N} = u/|\mathcal{N}_p|$ for some $u \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$. If $p = 2$, the determinant $\det_2 \mathcal{N}$ is well defined only if $\mathcal{N}_2$ is even. By definition, one always has $|\mathcal{N}| \det_p \mathcal{N} \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$.

The group of q-autoisometries of $\mathcal{N}$ is denoted by $\operatorname{Aut}\mathcal{N}$; obviously, one has $\operatorname{Aut}\mathcal{N} = \prod_{p\in\mathbb{P}} \operatorname{Aut}\mathcal{N}_p$. An element $\xi \in \mathcal{N}_p$ is called a *mirror* if, for some integer k, one has $p^k\xi = 0$ and $\xi^2 = 2u/p^k \bmod 2\mathbb{Z}$, g.c.d.$(u,p) = 1$. If this is the case, the map $x \mapsto 2(x\cdot\xi)/\xi^2 \bmod p^k$ is a well defined functional $\mathcal{N}_p \to \mathbb{Z}/p^k$; hence, one has a *reflection* $t_\xi \in \operatorname{Aut}\mathcal{N}_p$,

$$t_\xi \colon x \mapsto x - \frac{2(x\cdot\xi)}{\xi^2}\xi. \tag{2.1.1}$$

Note that $t_\xi = \mathrm{id}$ whenever $2\xi = 0$ and $\xi^2 = \frac{1}{2} \bmod \mathbb{Z}$.

2.2 Lattices and discriminant forms

2.2.1 Lattices and discriminant forms (see [14])

An (*integral*) *lattice* N is a finitely generated free abelian group equipped with a symmetric bilinear form $b\colon N \otimes N \to \mathbb{Z}$. If b is understood, we abbreviate $b(x,y) = x \cdot y$ and $b(x,x) = x^2$. A lattice N is called *even* if $x^2 = 0 \bmod 2$

for all $x \in N$; it is called *odd* otherwise. The determinant $\det N$ of a lattice N is the determinant of the Gram matrix of b. As the transition matrix from one integral basis to another has determinant ± 1, the determinant $\det N \in \mathbb{Z}$ is well-defined. The lattice N is called *non-degenerate* if $\det N \neq 0$ and *unimodular* if $\det N = \pm 1$. The signature $(\sigma_+ N, \sigma_- N)$ of a non-degenerate lattice N is the pair of the inertia indices of the bilinear form b.

For a lattice N, the bilinear form extends to a $\mathbb{Q}$-valued bilinear form on $N \otimes \mathbb{Q}$. If N is non-degenerate, the dual group $N^\sharp := \operatorname{Hom}(N, \mathbb{Q})$ can be identified with the subgroup $\{x \in N \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in N\}$. The lattice N is a finite index subgroup of $N^\sharp$. The quotient $\operatorname{discr} N := N^\sharp / N$ is called *discriminant group* of N; it is often denoted by $\mathcal{N}$, and we use the shortcut $\operatorname{discr}_p N = \mathcal{N}_p$ for the p-primary components. One has $\det N = (-1)^{\sigma_- N} |\mathcal{N}|$. The group $\mathcal{N}$ inherits from $N \otimes \mathbb{Q}$ a symmetric bilinear form $b \colon \mathcal{N} \otimes \mathcal{N} \to \mathbb{Q}/\mathbb{Z}$, called the *discriminant form*, and, if N is even, a quadratic extension of b. Thus, the discriminant group of an even nondegenerate lattice is always regarded as a finite quadratic form.

The *genus* $g(N)$ of a nondegenerate even lattice N can be defined as the set of isomorphism classes of all even lattices L such that $\operatorname{discr} L \cong \mathcal{N}$ and $\sigma_\pm L = \sigma_\pm N$. If N is indefinite and $\operatorname{rk} N \geqslant 3$, then $g(N)$ is a finite abelian group.

An *isometry* is a homomorphism of lattices preserving the forms. (We do not assume the surjectivity.) The group of auto-isometries of a lattice N is denoted by $O(N)$. There is a natural homomorphism $\mathrm{d} \colon O(N) \to \operatorname{Aut} \mathcal{N}$, and we denote by $\mathrm{d}_p \colon O(N) \to \operatorname{Aut} \mathcal{N}_p$ its restrictions to the p-primary components. For an element $u \in N$ such that $2u/u^2 \in N^\sharp$, the *reflection* $t_u \colon x \mapsto 2u(x \cdot u)/u^2$ is an involutive isometry of N. Each image $\mathrm{d}_p(t_u)$, $p \in \mathbb{P}$, is also a reflection. If $u^2 = \pm 1$ or ± 2, then $\mathrm{d}(t_u) = \mathrm{id}$.

2.2.2 Root lattices (see [16])

In this thesis, a *root lattice* is a negative definite lattice generated by vectors of square (-2) (*roots*). Any root lattice has a unique decomposition into orthogonal sum of indecomposable ones, which are of types $\mathbf{A}_p$, $p \geqslant 1$, $\mathbf{D}_q$, $q \geqslant 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$.

Given a root lattice S, the vertices of the Dynkin graph $\mathfrak{G} := \mathfrak{G}_S$ can be identified with the elements of a basis for S constituting a single Weyl chamber. This identification defines a homomorphism $\operatorname{Sym}\mathfrak{G} \to O(S)$, $s \mapsto s_*$, where $\operatorname{Sym}\mathfrak{G}$ is the group of symmetries of $\mathfrak{G}$. The image consists of the isometries preserving the distinguished Weyl chamber. For indecomposable root lattices, the groups $\operatorname{Sym}\mathfrak{G}$ are as follows:

- $\operatorname{Sym}\mathfrak{G} = 1$ if S is $\mathbf{A}_1$, $\mathbf{E}_7$, or $\mathbf{E}_8$,
- $\operatorname{Sym}\mathfrak{G} \cong \mathbb{S}_6$ if S is $\mathbf{D}_4$, and
- $\operatorname{Sym}\mathfrak{G} = \mathbb{G}_2$ in all other cases.

In the latter case, unless $S = \mathbf{D}_{\text{even}}$, the generator of $\operatorname{Sym}\mathfrak{G}$ induces $-\mathrm{id}$ on $\mathcal{S}$. If $S = \mathbf{E}_8$, then $\mathcal{S} = 0$. For $S = \mathbf{A}_1$, $\mathbf{E}_7$, or $\mathbf{D}_{\text{even}}$, the groups $\mathcal{S}$ are $\mathbb{F}_2$-modules and $-\mathrm{id} = \mathrm{id}$ in $\operatorname{Aut}\mathcal{S}$.

A choice of a Weyl chamber gives rise to a decomposition $O(S) = R(S) \rtimes \operatorname{Sym}\mathfrak{G}$, where $R(S) \subset O(S)$ is the subgroup generated by reflections t_u, $u \in S$, $u^2 = -2$. Furthermore,

$$\operatorname{Ker}[\mathrm{d}\colon O(S) \to \operatorname{Aut}\mathcal{S}] = R(S) \rtimes \operatorname{Sym}_0\mathfrak{G}, \tag{2.2.1}$$

where $\operatorname{Sym}_0\mathfrak{G}$ is the group of permutations of the $\mathbf{E}_8$-type components of $\mathfrak{G}$. Thus, denoting by $\operatorname{Sym}'\mathfrak{G} \subset \operatorname{Sym}\mathfrak{G}$ the group of symmetries acting identically on the union of the $\mathbf{E}_8$-type components, we obtain an isomorphism $\operatorname{Sym}'\mathfrak{G} = \operatorname{Im}\mathrm{d}$. For future references, we combine these statements in a separate lemma.

Lemma 2.2.2. *Let S be a root lattice. Then the epimorphism $\mathrm{d}\colon O(S) \twoheadrightarrow \operatorname{Im}\mathrm{d}$ has a splitting $\operatorname{Im}\mathrm{d} = \operatorname{Sym}'\mathfrak{G}_S \hookrightarrow O(S)$ and one always has $-\mathrm{id} \in \operatorname{Im}\mathrm{d}$.* $\triangleleft$

2.2.3 Lattice extensions (see [14])

An *extension* of a lattice S is an isometry $S \to L$. Two extensions $S \to L_1, L_2$ are (*strictly*) *isomorphic* if there is a bijective isometry $L_1 \to L_2$ identical on S. More generally, given a subgroup $O' \subset O(S)$, two extensions are O'*-isomorphic* if they are related by a bijective isometry that restricts to an element of O' on S.

We use the notation $S \hookrightarrow L$ for finite index extension ($[L : S] < \infty$). There is a unique embedding $L \subset S \otimes \mathbb{Q}$ and, hence, inclusions $S \subset L \subset L^\sharp \subset S^\sharp$. The *kernel* of a finite index extension $S \hookrightarrow L$ is the subgroup $\mathcal{K} = L/S \subset S^\sharp/S = \mathcal{S}$. Since L is an integral lattice, the kernel $\mathcal{K}$ is isotropic, *i.e.*, the restriction to $\mathcal{K}$ of the quadratic form $q : \mathcal{S} \to \mathbb{Q}/2\mathbb{Z}$ is identically zero. Conversely, given an isotropic subgroup $\mathcal{K} \subset \mathcal{S}$, the subgroup $L = \{x \in S^\sharp \,|\, (x \bmod S) \in \mathcal{K}\} \subset S^\sharp$ is an extension of S. Thus, we have the following theorem.

Theorem 2.2.3 (Nikulin [14]). *The map $L \mapsto \mathcal{K} = L/S \subset \mathcal{S}$ establishes a one-to-one correspondence between the set of isomorphism classes of finite index extension $S \hookrightarrow L$ and that of isotropic subgroup $\mathcal{K} \subset \mathcal{S}$. One has $\mathcal{L} = \mathcal{K}^\perp/\mathcal{K}$.* ▷

An isometry $a \in O(S)$ extends to a finite index extension L if and only if $\mathrm{d}(a)$ preserves the kernel $\mathcal{K}$ (as a set). Hence, O'-isomorphism classes of finite index extensions of S correspond to the $\mathrm{d}(O')$-orbits of isotropic subgroups $\mathcal{K} \subset \mathcal{S}$.

Another extreme case is that of a *primitive* extension $S \to L$, *i.e.*, such that the group L/S is torsion free; we use the notation $S \rightarrowtail L$. If L is unimodular, one has $\operatorname{discr} S^\perp \cong -\mathcal{S}$; hence, the genus $g(S^\perp)$ is determined by those of S and L. If L is also indefinite, it is unique in its genus. Then, for each representative $N \in g(S^\perp)$, an extension $S \rightarrowtail L$ with $S^\perp \cong N$ is determined by a bijective anti-isometry $\varphi\colon \mathcal{S} \to \mathcal{N}$ (the graph of φ is the kernel of the finite index extension $S \oplus N \hookrightarrow L$), and the latter induces a homomorphism $\mathrm{d}^\varphi\colon O(S) \to \operatorname{Aut} \mathcal{N}$. If φ is not fixed, this map is well defined up to an inner automorphism of $\operatorname{Aut} \mathcal{N}$.

Theorem 2.2.4 (Nikulin [14]). *Let L be an indefinite unimodular even lattice, $S \subset L$ a nondegenerate primitive sublattice, and $O' \subset O(S)$ a subgroup. Then, the O'-isomorphism classes of primitive extensions $S \rightarrowtail L$ are enumerated by the* *pairs* (N, c_N)*, where* $N \in g(S^\perp)$ *and* $c_N \in \mathrm{d}^\varphi(O')\backslash \operatorname{Aut}\mathcal{N}/\operatorname{Im}\mathrm{d}$ *is a double coset (for given* N *and some anti-isometry* $\varphi\colon \mathcal{S} \to \mathcal{N}$*).* ▷

Theorem 2.2.5 (Nikulin [14])**.** *Let* $S \rightarrowtail L$ *be a lattice extension as in Theorem 2.2.4,* $N = S^\perp$*, and* $\varphi\colon \mathcal{S} \to \mathcal{N}$ *the corresponding anti-isometry. Then, a pair of isometries* $a_S \in O(S)$*,* $a_N \in O(N)$ *extends to* L *if and only if* $\mathrm{d}^\varphi(a_S) = \mathrm{d}(a_N)$*.* ▷

Fix the notation $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$, where $\mathbf{U}$ is the *hyperbolic plane*, $\mathbf{U} = \mathbb{Z}u + \mathbb{Z}v$, $u^2 = v^2 = 0$, $u \cdot v = 1$. For the ease of references, we recast Nikulin's existence theorem from [14] to the particular case of primitive extensions $S \rightarrowtail \mathbf{L}$. Note that we do not need the restriction on the Brown invariant: by the additivity, it would hold automatically.

Theorem 2.2.6 (Nikulin [14])**.** *Given a nondegenerate even lattice* S*, a primitive extension* $S \rightarrowtail \mathbf{L}$ *exists if and only if the following conditions hold:* $\sigma_+S \leqslant 3$*,* $\sigma_-S \leqslant 19$*,* $\ell(\mathcal{S}) \leqslant \delta := 22 - \operatorname{rk} S$*, and*

- *for all odd* $p \in \mathbb{P}$*, either* $\ell_p(\mathcal{S}) < \delta$ *or* $|\mathcal{S}| \det_p \mathcal{S} = (-1)^{\sigma_+S-1} \bmod (\mathbb{Z}_p^\times)^2$*;*
- *either* $\ell_2(\mathcal{S}) < \delta$*, or* $\mathcal{S}_2$ *is odd, or* $|\mathcal{S}| \det_2 \mathcal{S} = \pm 1 \bmod (\mathbb{Z}_2^\times)^2$*.* ▷

2.3 Miranda-Morrison results

2.3.1 Miranda–Morrison results (see [17, 18, 15])

Nikulin have some results on the uniqueness of a lattice N with genus $g(N)$ in [14]. On the other hand, uniqueness of primitive embedding of a lattice S into L(indefinite and unimodular) is not only a matter of having $S^\perp = N$ unique in its genus, but also having some extra properties as stated in Theorem 2.2.4. Nikulin gave some sufficient conditions for uniqueness of N and surjectivity of $O(N) \xrightarrow{\mathrm{d}} \operatorname{Aut}\mathcal{N}$. But these theorems do not cover all cases. Miranda–Morrison stated a stronger theorem and introduced some new techniques to obtain the group $\operatorname{Coker}\mathrm{d}$ for all cases with the following requirement:

($*$) N is a nondegenerate indefinite even lattice, $\operatorname{rk} N \geqslant 3$.

Warning 2.3.1. *The convention used in this thesis (following Nikulin [14] and, eventually, Gauss) differs slightly from that of Miranda–Morrison, where quadratic and bilinear forms are related* via $q(x+y) - q(x) - q(y) = b(x, y)$*. Roughly, the values of all quadratic (but not bilinear) forms in [17, 18, 15], both on lattices and finite groups, should be multiplied by* 2*. In particular, all lattices in [17, 18, 15] are even by definition. Note though that this multiplication by* 2 *is partially incorporated in [17, 18, 15]: for example, the isomorphism class of a finite quadratic form generated by an element* α *with* $q(\alpha) = (u/p^k) \bmod \mathbb{Z}$*, which is* $(2u/p^k) \bmod 2\mathbb{Z}$ *in our notation, is designated by the class of* $2u$ *in* $(\mathbb{Z}_p^\times)/(\mathbb{Z}_p^\times)^2$*.*

Given a lattice N and a prime $p \in \mathbb{P}$, we define the number $e_p := e_p(N) \in \mathbb{N}$ and the subgroup $\tilde{\Sigma}_p := \tilde{\Sigma}_p(N) \subset \Gamma_0 := \{\pm 1\} \times \{\pm 1\}$ as in Equation 2.3.5. Algorithms to calculate $e_p(N)$ and $\tilde{\Sigma}_p(N)$ are given explicitly in [18]. Computations are in terms of $\operatorname{rk} N$, $\det N$, and $\mathcal{N}$ only which means that genus $g(N)$ determines $e_p(N)$, $\tilde{\Sigma}_p(N)$ and moreover $\operatorname{Coker} \mathrm{d}$. By these calculations one has $e_p = 1$ and $\tilde{\Sigma}_p = \Gamma_0$ for almost all primes p.

Theorem 2.3.2 (Miranda–Morrison [17, 18])**.** *For* N *as in* ($*$)*, there is an* $\mathbb{F}_2$*-module* $\mathrm{E}(N)$ *and an exact sequence*

$$O(N) \xrightarrow{\mathrm{d}} \operatorname{Aut} \mathcal{N} \xrightarrow{\mathrm{e}} \mathrm{E}(N) \to g(N) \to 1,$$

where $g(N)$ *is the genus group of* N*. One has* $|\mathrm{E}(N)| = e(N)/[\Gamma_0 : \tilde{\Sigma}(N)]$*, where* $e(N) := \prod_{p \in \mathbb{P}} e_p(N)$ *and* $\tilde{\Sigma}(N) := \bigcap_{p \in \mathbb{P}} \tilde{\Sigma}_p(N)$*.* ▷

The group $\mathrm{E}(N)$ and homomorphism $\mathrm{e}\colon \operatorname{Aut} \mathcal{N} \to \mathrm{E}(N)$ given by Theorem 2.3.2 will be called, respectively, the *Miranda–Morrison group* and *Miranda–Morisson homomorphism* of N. The next statement follows from Theorem 2.2.4, Theorem 2.3.2, and the fact that a unimodular even indefinite lattice is unique in its genus.

Corollary 2.3.3 (Miranda–Morrison [17, 18])**.** *Let* L *be a unimodular even lattice and* $S \subset L$ *a primitive sublattice such that* $N := S^\perp$ *is as in* ($*$)*. Then the strict isomorphism classes of primitive extensions* $S \rightarrowtail L$ *are in a canonical one-to-one correspondence with the Miranda–Morrison group* $\mathrm{E}(N)$*.* ▷

Generalizing, fix an anti-isometry $\varphi\colon \mathcal{S} \to \mathcal{N}$ and consider the induced map $\mathrm{d}^\varphi\colon O(S) \to \operatorname{Aut}\mathcal{N}$, see §2.2.3. Since $\operatorname{Im}\mathrm{d} \subset \operatorname{Aut}\mathcal{N}$ is a normal subgroup with abelian quotient, this map factors to a homomorphism $\mathrm{d}^\perp\colon O(S) \to \operatorname{Aut}\mathcal{N} \to \mathrm{E}(N)$ independent of φ. Then, the following statement is an immediate consequence of Theorems 2.2.4 and 2.3.2.

Corollary 2.3.4. *Let $S \subset L$ be as in Corollary 2.3.3, and let $O' \subset O(S)$ be a subgroup. Then the O'-isomorphism classes of primitive extensions $S \rightarrowtail L$ are in a one-to-one correspondence with the $\mathbb{F}_2$-module $\mathrm{E}(N)/\mathrm{d}^\perp(O')$.* ◁

Theorem 2.3.2 and Corollary 2.3.3 cover most of our needs. However, in a few special cases, we need the more advanced treatment of [15]. Introduce the groups

$$\Gamma_{p,0} := \{\pm 1\} \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \subset \Gamma_p := \{\pm 1\} \times \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2, \quad p \in \mathbb{P},$$

and

$$\Gamma_{\mathbb{A},0} := \prod_p \Gamma_{p,0} \subset \Gamma_{\mathbb{A}} := \Gamma_{\mathbb{A},0} \cdot \sum_p \Gamma_p \subset \Gamma := \prod_p \Gamma_p.$$

(Since the groups involved are multiplicative, although abelian, we follow [15] and use $\cdot$ to denote the sum of subgroups. However, we retain the notation $\sum$ and $\prod$ to distinguished between direct sums and products. Thus, the adelic version $\Gamma_{\mathbb{A}}$ is the set of sequences $\{(s_p, t_p)\} \in \Gamma$ such that $(s_p, t_p) \in \Gamma_{p,0}$ for almost all p.) Let also $\Gamma_{\mathbb{Q}} := \{\pm 1\} \times \mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \subset \Gamma_{\mathbb{A}}$. Then $\Gamma_{\mathbb{A},0} \cdot \Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{A}}$ and the intersection $\Gamma_{\mathbb{A},0} \cap \Gamma_{\mathbb{Q}}$ is the group $\Gamma_0 = \{\pm 1\} \times \{\pm 1\}$ introduced above. We recall that $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is the $\mathbb{F}_2$-module on the basis $\{-1\} \cup \mathbb{P}$, *i.e.*, it is the set of all square free integers.

On various occasions we will also consider the following subgroups:

- $\Gamma_p^{++} := \{1\} \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \subset \Gamma_{p,0}$;
- $\Gamma_{2,2} \subset \Gamma_2^{++}$ is the subgroup generated by $(1, 5)$;
- $\Gamma_{\mathbb{Q}}^{--} \subset \Gamma_{\mathbb{Q}}$ is the subgroup generated by $(-1, -1)$ and $(1, p)$, $p \in \mathbb{P}$;
- $\Gamma_0^{--} := \Gamma_{\mathbb{Q}}^{--} \cap \Gamma_0 \subset \Gamma_0$ is the subgroup generated by $(-1, -1)$.

We denote by $\iota_p\colon \Gamma_p \hookrightarrow \Gamma_{\mathbb{A}}$, $p \in \mathbb{P}$, and $\iota_{\mathbb{Q}}\colon \Gamma_{\mathbb{Q}} \hookrightarrow \Gamma_{\mathbb{A}}$ the inclusions. The images $\iota_{\mathbb{Q}}(1,q)$ and $\iota_q(1,q)$, $q \in \mathbb{P}$, differ by an element of $\prod_p \Gamma_p^{++}$, *viz.*, by the sequence $\{(1,s_p)\}$, where $s_q = 1$ and s_p is the class of q in $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ for $p \neq q$.

Defined and computed in [15] are certain $\mathbb{F}_2$-modules

$$\Sigma_p^\sharp(N) := \Sigma^\sharp(N \otimes \mathbb{Z}_p) \subset \Sigma_p(N) := \Sigma(N \otimes \mathbb{Z}_p),$$

which depend on the genus of N only. One has $\Sigma_p^\sharp \subset \Gamma_{p,0}$, $\Sigma_p \subset \Gamma_p$, and $\Sigma_p \subset \Gamma_{p,0}$ for almost all p. (In fact, for almost all $p \in \mathbb{P}$ one has $\Sigma_p^\sharp = \Sigma_p = \Gamma_{p,0}$.) Hence,

$$\Sigma^\sharp(N) := \prod_p \Sigma_p^\sharp(N) \subset \Gamma_{\mathbb{A},0}, \qquad \Sigma(N) := \prod_p \Sigma_p(N) \subset \Gamma_{\mathbb{A}}.$$

In these notations, the invariants used in Theorem 2.3.2 are

$$e_p(N) = [\Gamma_{p,0} : \Sigma_p^\sharp(N)], \qquad \tilde{\Sigma}_p(N) = \Sigma_0^\sharp(N \otimes \mathbb{Z}_p) := \varphi_p^{-1}(\Sigma_p^\sharp(N)), \tag{2.3.5}$$

where $\varphi_p\colon \Gamma_0 \to \Gamma_{p,0}$ is the projection, and $\mathrm{E}(N)$ is the quotient $\Gamma_{\mathbb{A},0}/\Sigma^\sharp(N) \cdot \Gamma_0$. (Clearly, $\tilde{\Sigma}(N) = \Sigma^\sharp(N) \cap \Gamma_0$.) Unfortunately, the map $\prod_p \operatorname{Aut} \mathcal{N}_p \to \mathrm{E}(N)$ given by Theorem 2.3.2 does not respect the product structures. The following statement refines Theorem 2.3.2, separating the genus group and the p-primary components.

Theorem 2.3.6 (Miranda–Morrison [15]). *Let N be as in $(*)$. Then*:

1. *there is an isomorphism $g(N) = \Gamma_{\mathbb{A}}/\Sigma(N) \cdot \Gamma_{\mathbb{Q}}$ (hence, N is unique in its genus if and only if $\Gamma_{\mathbb{A}} = \Sigma(N) \cdot \Gamma_{\mathbb{Q}}$);*
2. *there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{Aut} \mathcal{N} = \prod_p \operatorname{Aut} \mathcal{N}_p & \xrightarrow{\ \gamma\ } & \prod_p \Sigma_p(N)/\Sigma_p^\sharp(N) \\ \big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \operatorname{Coker} \mathrm{d} & \xrightarrow{\ \cong\ } & \Sigma(N)/\Sigma^\sharp(N) \cdot (\Sigma(N) \cap \Gamma_{\mathbb{Q}}), \end{array}$$

where all maps are epimorphisms, γ is the product of certain epimorphisms $\gamma_p\colon \operatorname{Aut} \mathcal{N}_p \twoheadrightarrow \Sigma_p(N)/\Sigma_p^\sharp(N)$, $p \in \mathbb{P}$, and β is the quotient projection. ▷

2.3.2 A few simple consequences

The homomorphism γ in Theorem 2.3.6(2) is easily computed on reflections: for a mirror $\xi \in \mathcal{N}_r$, $r \in \mathbb{P}$, modulo $\Sigma_r^\sharp(N)$ one has

$$\gamma_r(t_\xi) = (-1, mr^k), \quad \text{where} \quad \xi^2 = \frac{2m}{r^k} \bmod 2\mathbb{Z},\ \text{g.c.d.}(m,r) = 1,\ k \in \mathbb{N}.$$

If $r = 2$ and $\xi^2 = 0 \bmod \mathbb{Z}$, this value is only well defined $\bmod \Gamma_2^{++}$; if $r = 2$ and $\xi^2 = \frac{1}{2} \bmod \mathbb{Z}$, it is well defined $\bmod \Gamma_{2,2}$. In these two cases, the disambiguation of $\gamma_r(t_\xi)$ needs more information about ξ and N: one needs to represent ξ in the form $\frac{1}{2}x$ for some $x \in N \otimes \mathbb{Z}_2$. Given another prime p, consider the homomorphism $\chi_p \colon \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \twoheadrightarrow \{\pm 1\}$,

$$\chi_p(m) := \left(\frac{m}{p}\right) \quad \text{if } p \neq 2, \qquad \chi_2(m) := m \bmod 4,$$

and define the *p-norm* $|\xi|_p \in \{\pm 1\}$ and the 'Kronecker symbol' $\delta_p(\xi) \in \{\pm 1\}$ *via*

$$|\xi|_p := \begin{cases} \chi_p(q^k), & \text{if } r \neq p, \\ \chi_p(m), & \text{if } r = p, \end{cases} \qquad \delta_p(\xi) = (-1)^{\delta_{p,r}},$$

where $\delta_{p,r}$ is the conventional Kronecker symbol. (If $p = 2$ and $\xi^2 = 0 \bmod \mathbb{Z}$, then $|\xi|_2$ is undefined.) Finally, introduce a few *ad hoc* notations for a lattice N:

- the group $\mathrm{E}_p(N) = \{\pm 1\}$ if $p = 1 \bmod 4$ and $e_p(N) \cdot |\tilde{\Sigma}_p(N)| = 8$; in all other cases, $\mathrm{E}_p(N) = 1$;
- the map $\bar{\gamma}_p$ sending a mirror ξ to $|\xi|_p \in \mathrm{E}_p(N)$, with the convention that $\bar{\gamma}_p(\xi) = 1$ whenever $\mathrm{E}_p(N) = 1$;
- the map $\bar{\beta}_p$ sending a mirror ξ to an element of Γ_0: if $p = 1 \bmod 4$, then $\bar{\beta}_p(\xi) = (\delta_p(\xi) \cdot |\xi|_p, 1)$; otherwise, $\bar{\beta}_p(\xi) = \delta_p(\xi) \times |\xi|_p$.

Following [15], we say that a lattice N is *p-regular*, $p \in \mathbb{P}$, if $\Sigma_p^\sharp(N) = \Gamma_{p,0}$, *i.e.*, if $e_p(N) = 1$. We will also say that the prime p is *regular* with respect to N; otherwise, p is *irregular*. In several statements below, we make a technical assumption that $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$; this inclusion does hold for the transcendental lattices of all primitive homological types (see §3.2) except $\mathbf{S} = \mathbf{A}_{15} \oplus \mathbf{A}_3$, see [15].

Lemma 2.3.7. *Let N be a lattice as in* (∗), $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, *and assume that N has one irregular prime p. Then $\mathrm{E}(N) = \mathrm{E}_p(N)$ and $\mathrm{e}(t_\xi) = \bar\gamma_p(\xi)$ for a mirror ξ.*

Lemma 2.3.8. *Let N be a lattice as in* (∗), $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, *and assume that N has two irregular primes p, q. Then*

$$\mathrm{E}(N) = \mathrm{E}_p(N) \times \mathrm{E}_q(N) \times (\Gamma_0/\tilde\Sigma_p(N) \cdot \tilde\Sigma_q(N))$$

and one has $\mathrm{e}(t_\xi) = \bar\gamma_p(\xi) \times \bar\gamma_q(\xi) \times (\bar\beta_p(\xi) \cdot \bar\beta_q(\xi))$ for a mirror $\xi \in \mathcal{N}$, provided that $\xi^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$ or $q = 2$.

Corollary 2.3.9. *Under the hypotheses of Lemma 2.3.8, assume, in addition, that $|\mathrm{E}(N)| = |\mathrm{E}_p(N)| = 2$. Then $\mathrm{E}(N) = \mathrm{E}_p(N)$ and $\mathrm{e}(t_\xi) = |\xi|_p$ for a mirror ξ.* ◁

Proof of Lemmas 2.3.7 and 2.3.8. Let $\Gamma'_{p,0} := \Gamma_{p,0}$ for $p \neq 2$ and $\Gamma'_{2,0} := \Gamma_{2,0}/\Gamma_{2,2}$, so that we can identify $\Gamma'_{p,0} \cong \{\pm 1\} \times \{\pm 1\}$ for all $p \in \mathbb{P}$. If $p \neq 1 \bmod 4$, the map $\varphi_p \colon \Gamma_0 \to \Gamma'_{p,0}$ is an epimorphism; if $p = 1 \bmod 4$, one has $\varphi_p(\Gamma_0) = \{\pm 1\} \times \{1\}$. Modulo $\Gamma_{\mathbb{Q}}^{--}$, the image $\gamma(t_\xi)$ equals $\bar\gamma(t_\xi) := \{(\delta_s(\xi), |\xi|_s)\} \in \prod \Gamma'_{s,0}$.

Now, the first statement of each lemma is a computation of the group $\mathrm{E}(N) = \Gamma_{\mathbb{A},0}/\Sigma^\sharp(N) \cdot \Gamma_0$, which can be done in $\Gamma'_{p,0}$ or $\Gamma'_{p,0} \times \Gamma'_{q,0}$; our group $\mathrm{E}_p(N)$ is the quotient $\Gamma_{p,0}/\Sigma_p^\sharp(N) \cdot \operatorname{Im} \varphi_p$. The second statement is the computation of the image of $\bar\gamma(\xi)$ in $\mathrm{E}(N)$: the maps $\bar\gamma_p$ and $\bar\beta_p$ are the projections $\Gamma_{p,0} \to \mathrm{E}_p(N)$ and $\Gamma_{p,0} \to \operatorname{Im} \varphi_p$, respectively. For the latter, we use the following fact, see [15]: if a prime $p = 1 \bmod 4$ is irregular for N and $\Sigma_p^\sharp(N) \not\subset \operatorname{Im} \varphi_p$, then $\Sigma_p^\sharp(N)$ is generated by $(-1, -1)$. □

2.3.3 The positive sign structure

A positive sign structure on N is a choice of an oriented maximal positive definite subspace of $N \otimes \mathbb{R}$. The map $\det_+ \colon O(N) \to \{\pm 1\}$ sends any automorphism to $+1$ if it preserves the positive sign structure and -1 otherwise. Hence $O_+(N) = \operatorname{Ker} \det_+$ is the group preserving positive sign structure.

Proposition 2.3.10 (Miranda–Morrison [15])**.** *Let N be a lattice as in* $(*)$*. Then one has $\tilde{\Sigma}(N) \subset \Gamma_0^{--}$ if and only if $\det_+ a = 1$ for all $a \in \operatorname{Ker}[\mathrm{d}\colon O(N) \to \operatorname{Aut}\mathcal{N}]$.* ▷

Thus, if $\tilde{\Sigma}(N) \subset \Gamma_0^{--}$, there is a well defined descent $\det_+\colon \operatorname{Im}\mathrm{d} \to \{\pm 1\}$. The next lemma computes the values of $\det_+$ on reflections.

Lemma 2.3.11. *Let N be a lattice as in $(*)$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that there is a prime p such that $\tilde{\Sigma}_p(N) \subset \Gamma_0^{--}$. Then, for a mirror $\xi \in \mathcal{N}$ such that $t_\xi \in \operatorname{Im}\mathrm{d}$ and $\xi^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$, one has $\det_+ t_\xi = \delta_p(\xi) \cdot |\xi|_p$.*

Proof. The proof is similar to that of Lemmas 2.3.7 and 2.3.8: we *assume* that the element $\bar{\gamma}(t_\xi) \cdot \iota_{\mathbb{Q}}(\delta_p(\xi), \delta_p(\xi))$ representing t_ξ lies in $\Sigma^\sharp(N) \cdot \Gamma_0$ and compute its image in $\Sigma^\sharp(N) \cdot \Gamma_0 / \Sigma^\sharp(N) \cdot \Gamma_0^{--} = \{\pm 1\}$. This can be done in $\Gamma_{p,0}$. □

Proposition 2.3.10 can be restated in a form closer to Theorem 2.3.2: introducing the group $\mathrm{E}_+(N) := \Gamma_{\mathbb{A},0} / \Sigma^\sharp(N) \cdot \Gamma_0^{--}$, one has an exact sequence

$$O_+(N) \xrightarrow{\mathrm{d}} \operatorname{Aut}\mathcal{N} \xrightarrow{\mathrm{e}_+} \mathrm{E}_+(N) \to g(N) \to 1. \qquad (2.3.12)$$

The groups $\mathrm{E}_+(N)$, as well as a few other counterparts, are also computed in [18]: for the order $|\mathrm{E}_+(N)|$, one merely replaces $\tilde{\Sigma}(N)$ with $\tilde{\Sigma}(N) \cap \Gamma_0^{--}$ in Theorem 2.3.2. In the special case of at most two irregular primes, the computation is very similar to §2.3.2. For an irregular prime p, denote $\tilde{\Sigma}_p^+(N) := \tilde{\Sigma}_p(N) \cap \Gamma_0^{--} \subset \Gamma_0^{--}$ and introduce the groups $\mathrm{E}_p^+(N)$ and maps $\bar{\gamma}_p^+$, $\bar{\beta}_p^+$ defined on the set of mirrors and taking values in $\mathrm{E}_p^+(N)$ and $\Gamma_0^{--} = \{\pm 1\}$, respectively, as follows:

- if $p = 1 \bmod 4$, then $\mathrm{E}_p^+(N) = \mathrm{E}_p(N)$, $\bar{\gamma}_p^+ = \bar{\gamma}_p$, and $\bar{\beta}_p^+(\xi) = \delta_p(\xi) \cdot |\xi|_p$;
- if $p \neq 1 \bmod 4$, then $\mathrm{E}_p^+(N) = \Gamma_0 / \tilde{\Sigma}_p(N) \cdot \Gamma_0^{--}$ (if $p \neq 2$ or $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, one has $\mathrm{E}_p^+(N) = \{\pm 1\}$ if $e_p(N) \cdot |\tilde{\Sigma}_p^+(N)| = 4$ and $\mathrm{E}_p^+(N) = 1$ otherwise);
- if $p \neq 1 \bmod 4$ and $\mathrm{E}_p^+(N) \neq 1$, then $\bar{\gamma}_p^+(\xi) = \delta_p(\xi) \cdot |\xi|_p$ and $\bar{\beta}_p^+(\xi) = |\xi|_p$;
- if $p \neq 1 \bmod 4$ and $\mathrm{E}_p^+(N) = 1$, then $\bar{\gamma}_p^+(\xi) = 1$ and $\bar{\beta}_p^+(\xi)$ is the image of $\bar{\beta}(\xi) = \delta_p(\xi) \times |\xi|_p$, see §2.3.2, under the projection $\Gamma_0 \to \Gamma_0 / \tilde{\Sigma}_p(N) = \Gamma_0^{--}$.

(In the last case, one has $\bar{\beta}_p^+(\xi) = |\xi|_p$ unless $p = 2$.) The proof of the next two statements repeats literally that of Lemmas 2.3.7 and 2.3.8.

Lemma 2.3.13. *Let N be a lattice as in ($*$), $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has a single irregular prime p. Then one has $\mathrm{E}_+(N) = \mathrm{E}_p^+(N)$ and $\mathrm{e}_+(t_\xi) = \bar{\gamma}_p^+(\xi)$ for a mirror $\xi \in \mathcal{N}$ such that $\xi^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$.* $\triangleleft$

Lemma 2.3.14. *Let N be a lattice as in ($*$), $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has two irregular primes p, q. Then*

$$\mathrm{E}_+(N) = \mathrm{E}_p^+(N) \times \mathrm{E}_q^+(N) \times (\Gamma_0^{--}/\tilde{\Sigma}_p^+(N) \cdot \tilde{\Sigma}_q^+(N))$$

and one has $\mathrm{e}_+(t_\xi) = \bar{\gamma}_p^+(\xi) \times \bar{\gamma}_q^+(\xi) \times (\bar{\beta}_p^+(\xi) \cdot \bar{\beta}_q^+(\xi))$ for a mirror $\xi \in \mathcal{N}$ such that $\xi^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$ or $q = 2$. $\triangleleft$

Corollary 2.3.15. *Under the hypotheses of Lemma 2.3.14, assume, in addition, that $|\mathrm{E}_+(N)| = |\mathrm{E}_p^+(N)| = 2$. Then $\mathrm{E}_+(N) = \mathrm{E}_p^+(N)$ and $\mathrm{e}(t_\xi) = \bar{\gamma}_p^+(\xi)$ for a mirror $\xi \in \mathcal{N}$ such that $\xi^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$.* $\triangleleft$

Chapter 3

Sextics in the Aritmetical Way

3.1 Simple sextics

Let $D \subset \mathbb{P}^2$ be a reduced simple plane sextic. The minimal resolution of singularities X of the double covering of $\mathbb{P}^2$ ramified at D is a $K3$-surface. It is well known that the intersection index form $H_2(X) \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$ is (the only) even unimodular lattice of signature $(\sigma_+, \sigma_-) = (3, 19)$. We fix the notation $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$.

For each simple singular point P of D, the components of the exceptional divisor $E \subset X$ over P span a root lattice in $\mathbf{L}$ (see §2.2.2). The (obviously orthogonal) sum of these sublattices is denoted by $\mathbf{S}(D)$ and is referred to as the *set of singularities* of D. (Recall that the types of the individual singular points are uniquely recovered from $\mathbf{S}(D)$, see §2.2.2.) The rank $\operatorname{rk} \mathbf{S}(D)$ equals the total Milnor number $\mu(D)$. Since $\mathbf{S}(D) \subset \mathbf{L}$ is negative definite, one has $\mu(D) \leqslant 19$. If $\mu(D) = 19$, the sextic D is called *maximizing*; both the inequality and the term apply to simple sextics only.

A sextic D is said to be of *torus type* if its defining polynomial f can be written in the form $f = f_2^3 + f_3^2$, where f_2 and f_3 are homogenous polynomials of degree 2 and 3, respectively. A representation $f = f_2^3 + f_3^2$ as above, up to the obvious equivalence, is called a *torus structure* on D. According to [19], an irreducible sextic D may have one, four, or twelve distinct torus structures, and we call D a 1-, 4-, or 12-torus sextic, respectively. An irreducible sextic is of torus type if and only if it is $\mathbb{D}_6$-special, see [19]. In this case, the group $\pi_1(\mathbb{P}^2 \smallsetminus D)$ factors to Γ. Note that torus type sextics form the majority of special sextics.

Denote by $\mathcal{M} \cong \mathbb{P}^{27}$ the space of all plane sextics. This space is subdivided into equisingular strata $\mathcal{M}(\mathbf{S})$; we consider only those with $\mathbf{S}$ simple. The space of all simple sextics and each of its strata $\mathcal{M}(\mathbf{S})$ are further subdivided into families $\mathcal{M}_*(\mathbf{S})$, where the subscript $*$ refers to the sequence of invariant factors of a certain finite group, see §3.2 for the precise definition. Our primary concern are the spaces

- $\mathcal{M}_1(\mathbf{S})$: non-special irreducible sextics, see Theorem 3.2.7, and
- $\mathcal{M}_3(\mathbf{S})$: irreducible 1-torus sextics, see Theorem 3.2.8.

In this notation, irreducible 4- and 12-torus sextics constitute $\mathcal{M}_{3,3}$ and $\mathcal{M}_{3,3,3}$, respectively, whereas irreducible $\mathbb{D}_{2n}$-special sextics, $n = 5, 7$, constitute $\mathcal{M}_n$. For each subscript $*$, we denote by $\bar{\mathcal{M}}_*(\mathbf{S})$ and $\partial\mathcal{M}_*(\mathbf{S}) := \bar{\mathcal{M}}_*(\mathbf{S}) \smallsetminus \mathcal{M}_*(\mathbf{S})$ the closure and boundary of $\mathcal{M}_*(\mathbf{S})$ *in* $\mathcal{M}_*$.

If $\mathbf{S}$ is a simple set of singularities, the dimension of the *equisingular moduli space* $\mathcal{M}(\mathbf{S})/PGL(3,\mathbb{C})$ equals $19 - \mu(\mathbf{S})$, as follows from the theory of $K3$-surfaces.

The coordinatewise conjugation $(z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$ in $\mathbb{P}^2$ induces a real structure (*i.e.*, anti-holomorphic involution) $\operatorname{conj}\colon \mathcal{M} \to \mathcal{M}$, which takes a sextic to its conjugate. A sextic $D \in \mathcal{M}$ is *real* if $\operatorname{conj}(D) = D$. A connected component $\mathcal{C} \subset \mathcal{M}_*(\mathbf{S})$ is *real* if it is preserved by conj as a set; this property of $\mathcal{C}$ is independent of the choice of coordinates in $\mathbb{P}^2$. Clearly, any connected component containing a real curve is real. The converse is not true; in the realm of irreducible sextics, the only exception is $\mathcal{M}_1(\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5)$, see Proposition 4.1.3.

Most results of the thesis are stated in terms of degenerations/perturbations of sets of singularities and/or sextics (or, equivalently, in terms of adjacencies of the equisingular strata of $\mathcal{M}$). As shown in [20], the deformation classes of perturbations of a simple singular point P of type $\mathbf{S}$ are in a one-to-one correspondence with the isomorphism classes of primitive extensions $\mathbf{S}' \rightarrowtail \mathbf{S}$ of root lattices, see §2.2.2 and §2.2.3. Thus, by a *degeneration* of sets of singularities we merely mean a class of primitive extensions $\mathbf{S}' \rightarrowtail \mathbf{S}$ of root lattices. Recall (see [21]) that $\mathbf{S}'$ admits a degeneration to $\mathbf{S}$ if and only if the Dynkin graph of $\mathbf{S}'$ is an induced subgraph of that of $\mathbf{S}$. A degeneration $D' \rightarrowtail D$ of simple sextics gives rise to a degeneration $\mathbf{S}(D') \rightarrowtail \mathbf{S}(D)$. According to [22], the converse also holds: given a simple sextic D and a root lattice $\mathbf{S}'$, any degeneration $\mathbf{S}' \rightarrowtail \mathbf{S}(D)$ is realized by a degeneration $D' \rightarrowtail D$ of simple sextics, so that $\mathbf{S}(D') = \mathbf{S}'$.

3.2 Moduli space of sextics and homological types

3.2.1 Homological type of a sextic

Consider a simple sextic $D \subset \mathbb{P}^2$. Recall (see §3.1) that we denote by $X \to \mathbb{P}^2$ the minimal resolution of singularities of the double covering of $\mathbb{P}^2$ ramified at D, and that the set of singularities of D can be identified with the sublattice $\mathbf{S} \subset \mathbf{L}$ spanned by the classes of the exceptional divisors. Let $\tau\colon X \to X$ be the deck translation of the covering.

Lemma 3.2.1. *The induced action of τ on the Dynkin graph $\mathfrak{G} := \mathfrak{G}_{\mathbf{S}}$ preserves the components of $\mathfrak{G}$; it acts by the only nontrivial symmetry on the components of type $\mathbf{A}_{p\geqslant 2}$, $\mathbf{D}_{\mathrm{odd}}$, or $\mathbf{E}_6$, and by the identity otherwise.* ◁

Remark 3.2.2. *In other words, $\tau\colon \mathfrak{G} \to \mathfrak{G}$ can be characterized as the 'simplest' symmetry of $\mathfrak{G}$ inducing $-\mathrm{id}$ on $\operatorname{discr} \mathbf{S}$.*

In addition to $\mathbf{S}$, we have the class $h \in \mathbf{L}$ of the pull-back of a generic line in $\mathbb{P}^2$. Obviously, h is orthogonal to $\mathbf{S}$ and $h^2 = 2$. Let $\mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h$. The triple $\mathcal{H} := (\mathbf{S}, h, \mathbf{L})$, *i.e.*, the lattice extension $\mathbf{S}_h \hookrightarrow \mathbf{L}$ regarded up to isometries of $\mathbf{L}$

preserving $\mathbf{S}$ (as a set) and h, is called the *homological type* of D. This extension is subject to certain restrictions, which are summarized in the following definitions.

Definition 3.2.3. *Let* $\mathbf{S}$ *be a root lattice. A* homological type *(extending* $\mathbf{S}$*) is an extension* $\mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h \hookrightarrow \mathbf{L}$ *satisfying the following conditions:*

1. *any vector* $v \in (\mathbf{S} \otimes \mathbb{Q}) \cap \mathbf{L}$ *with* $v^2 = -2$ *is in* $\mathbf{S}$*;*
2. *there is no vector* $v \in \tilde{\mathbf{S}}_h$ *with* $v^2 = 0$ *and* $v \cdot h = 1$*.*

Given a homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$, we let

- $\tilde{\mathbf{S}}_h := (\mathbf{S}_h \otimes \mathbb{Q}) \cap \mathbf{L}$ be the primitive hull of $\mathbf{S}_h$, and
- $\mathbf{T} := \mathbf{S}_h^\perp$ with $\mathcal{T} = \operatorname{discr} \mathbf{T}$ be the *transcendental lattice*.

One has $\sigma_+\mathbf{T} = 2$ and there is a positive sign structure on $\mathbf{T}$ which is called an *orientation* of $\mathcal{H}$ and denoted by $\mathfrak{o}$. The homological type of a plane sextic D has a canonical orientation, *viz.* the one given by the real and imaginary parts of the class of a holomorphic form ω on X.

An *automorphism* of a homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is an autoisometry of $\mathbf{L}$ preserving $\mathbf{S}$ (as a set) and h. The group of automorphisms of $\mathcal{H}$ is denoted by $\operatorname{Aut} \mathcal{H}$. Let $\operatorname{Aut}_+ \mathcal{H} \subset \operatorname{Aut} \mathcal{H}$ be the subgroup of the automorphisms inducing id on $\mathbf{T}$. On the other hand, we have the group $\operatorname{Aut}_h \tilde{\mathbf{S}}_h \subset O(\tilde{\mathbf{S}}_h)$ of the isometries of $\tilde{\mathbf{S}}_h$ preserving h. There are obvious homomorphisms

$$\operatorname{Aut}_+ \mathcal{H} \hookrightarrow \operatorname{Aut} \mathcal{H} \to \operatorname{Aut}_h \tilde{\mathbf{S}}_h \hookrightarrow O(\mathbf{S}), \tag{3.2.4}$$

where the latter inclusion is due to item 1 in Definition 3.2.3, as $\mathbf{S} \subset \tilde{\mathbf{S}}_h$ is recovered as the sublattice generated by the roots orthogonal to h. If the primitive extension $\tilde{\mathbf{S}}_h \rightarrowtail \mathbf{L}$ is defined by an anti-isometry $\varphi\colon \operatorname{discr} \tilde{\mathbf{S}}_h \to \mathcal{T}$ (see §2.2.3), so that we have a homomorphism $\mathrm{d}^\varphi\colon \operatorname{Aut}_h \tilde{\mathbf{S}}_h \to \operatorname{Aut} \mathcal{T}$, then, for $\epsilon = +$ or empty,

$$\operatorname{Im}[\operatorname{Aut}_\epsilon \mathcal{H} \to \operatorname{Aut}_h \tilde{\mathbf{S}}_h] = (\mathrm{d}^\varphi)^{-1}\mathrm{d}(O_\epsilon(\mathbf{T})). \tag{3.2.5}$$

Theorem 3.2.6 (see [10]). *The map sending a plane sextic $D \subset \mathbb{P}^2$ to the pair of homological type (S, h, L) and orientation $\mathfrak{o}$ of the space ω establishes a one-to-one correspondence between the set $\pi_0(\mathcal{M}_*(\mathbf{S}))$ and the set of pairs $(\mathcal{H}, \mathfrak{o})$. Changing orientation of ω corresponds to conjugation map on the strata $\mathcal{M}_*(\mathbf{S})$ and hence $\pi_0(\mathcal{M}_*(\mathbf{S})/\operatorname{conj})$ corresponds to $\mathcal{H}$.*

In §3.1 the space $\mathcal{M}(\mathbf{S})$ is defined as the set of all plane sextics with set of singularities $\mathbf{S}$ and subdivided into families $\mathcal{M}_*(\mathbf{S})$. In moduli space, the notation * stands for the information of whether they are non-special or special sets of singularities and if they are special, what kind of special sextics they are. Correspondingly, in homological types, the subscript is the sequence of invariant factors of the kernel $\mathcal{K}$ of the finite index extension $\mathbf{S}_h \hookrightarrow \tilde{\mathbf{S}}_h$. Our primarily used spaces $\mathcal{M}_1(\mathbf{S})$ and $\mathcal{M}_3(\mathbf{S})$ correspond to homological types with $\mathcal{K} = 0$ and $\mathcal{K} = \mathbb{Z}_3$, respectively.

A homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is called *primitive* if $\mathbf{S}_h \subset \mathbf{L}$ is a primitive sublattice, *i.e.*, if $\mathcal{K} = 0$. In this case, one has $\operatorname{discr} \tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{2} \rangle$ and the above inclusion $\operatorname{Aut}_h \tilde{\mathbf{S}}_h \hookrightarrow O(\mathbf{S})$ is an isomorphism.

Theorem 3.2.7 (see [19]). *A simple plane sextic D is irreducible and non-special if and only if its homological type is primitive.* ▷

Theorem 3.2.8 (see [19]). *A simple plane sextic D is irreducible and p-torus, $p =$ 1, 4, or 12, if and only if the kernel $\mathcal{K}$ of the extension $\mathbf{S}_h \hookrightarrow \tilde{\mathbf{S}}_h$ is, respectively, $\mathbb{G}_3$, $\mathbb{G}_3 \oplus \mathbb{G}_3$, or $\mathbb{G}_3 \oplus \mathbb{G}_3 \oplus \mathbb{G}_3$.* ▷

There is a similar characterization of other special sextics: a sextic is irreducible and $\mathbb{D}_{2n}$-special, $n > 3$, if and only if the kernel $\mathcal{K}$ is $\mathbb{G}_n$; one necessarily has $n = 5$ or 7. Note that these statements cover all possibilities for the kernel $\mathcal{K}$ free of 2-torsion, and $\mathcal{K}$ has 2-torsion if and only if the sextic is reducible, see, *e.g.*, [2].

3.2.2 Extending a fixed set of singularities $\mathbf{S}$ to a sextic

By Theorem 3.2.6, realizing a set of singularities $\mathbf{S}$ by a sextic becomes a matter of extending $\mathbf{S}$ to a homological type. Hence, the question is about the embedding $\mathbf{S} \hookrightarrow \mathbf{L}$.

For a fixed $\mathbf{S}$ there are three items on realization by sextics (see [10]):

1. "Existence of a sextic having set of singularities $\mathbf{S}$" can be determined by:
   (a) enumerating lattices $\tilde{\mathbf{S}}_h$ extending $\mathbf{S}$. It is determined by a choice of $\mathcal{K} \subset \mathcal{S}$,
   (b) determining the existence of primitive extension $\tilde{\mathbf{S}}_h \rightarrowtail \mathbf{L}$. It can be calculated by Theorem 2.2.6.
2. "Number of classes of sextics by which $\mathbf{S}$ is realized, up to complex conjugation" can be determined by:
   (a) enumerating number of isomorphism classes of $\mathbf{T}$,
   (b) enumerating bi-cosets of $\mathrm{Aut}_h \mathcal{S} \times \mathrm{Aut}_{\mathbf{T}} \mathcal{T}$.
   For unique realization of $\mathbf{S}$ (covering steps (a),(b)), by Corollary 2.3.3 it is sufficient to have $|\mathrm{E}(\mathbf{T})| = 1$ by calculations in [18]. In the case $|\mathrm{E}(\mathbf{T})| > 1$, one needs to look for the match of images of d and $\mathrm{d}^{\perp}$. If all automorphisms in $\mathrm{Aut}(\mathcal{S})$ has preimage in $O(\mathbf{S})$ or in $O(\mathcal{T})$ then $\mathbf{S}$ has unique realization with fixed complement $\mathbf{T}$.

$$\begin{array}{ccc} O(\mathbf{S}) \xrightarrow{\mathrm{d}^{\perp}} \mathrm{Aut}(\mathcal{S}) \cong & \mathrm{Aut}(\mathcal{T}) & \xleftarrow{\mathrm{d}} O(\mathbf{T}) \\ & \Big\downarrow \mathrm{e} & \\ & \mathrm{E}(\mathbf{T}), & \end{array}$$

3. Real or complex realization of each class of sextics.
   Details about this item are given in §5.1. The necessary and sufficient condition for realization of $\mathbf{S}$ by a real sextic is having an involutive orientation reversing automorphism on the homological type of $\mathbf{S}$ (see Theorem 5.1.1).

3.3 Maximizing sextics

This section is based on the studies on the sets of singularities realized by simple plane sextics (see [11]), the deformation classification of this list (see [12]) and fundamental groups of irreducible sextics (see [2, 3]). For the classification of sextics there is an alternative way for triple points in [2].

The relevant part of these results is collected in Tables 3.1, 3.2 (irreducible maximizing non-special sextics) and Table 3.3 (irreducible maximizing 1-torus sextics). In the tables, the column (r, c) refers to the numbers of real (r) and pairs of complex conjugate (c) curves realizing the given set of singularities; thus, the total number of connected components of the stratum $\mathcal{M}_1(\mathbf{S})$ (or $\mathcal{M}_3(\mathbf{S})$ for Table 3.3) is $n := r + 2c$. Some sets of singularities are prefixed with a link of the form $^{[n]}$: this link refers to the listings of the fundamental groups found below. Some pairs of singular points are marked with a *. This marking is related to the real structures; it is explained in §5.1.2.

The fundamental groups of most irreducible maximizing sextics are computed in [2, 3].

The known fundamental groups $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ of the maximizing non-special irreducible sextics D are as follows (depending on the set of singularities):

1. for $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$, the group is the central product
$$\pi_1 = SL(2, \mathbb{F}_5) \odot \mathbb{G}_{12} := \big(SL(2, \mathbb{F}_5) \times \mathbb{G}_{12}\big)/(-\mathrm{id} = 6),$$
where $-\mathrm{id}$ is the generator of the center $\mathbb{G}_2 \subset SL(2, \mathbb{F}_5)$;
2. for $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$, the group is $\pi_1 = SL(2, \mathbb{F}_{19}) \times \mathbb{G}_6$;
3. for $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$, the group is $\pi_1 = SL(2, \mathbb{F}_5) \rtimes \mathbb{G}_6$, the generator of $\mathbb{G}_6$ acting on $SL(2, \mathbb{F}_5)$ by (any) order 2 outer automorphism;
4. for $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$, $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$, $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$, $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, only for the *real* curve the group $\pi_1 = \mathbb{G}_6$ is known;
5. for $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, only for one of the two curves the group $\pi_1 = \mathbb{G}_6$ is known;

Table 3.1: The spaces $\mathcal{M}_1(\mathbf{S})$, $\mu(\mathbf{S}) = 19$, with a triple point in $\mathbf{S}$

	Singularities	(r,c)
	$2\mathbf{E}_8 \oplus \mathbf{A}_3$	$(1,0)$
	$2\mathbf{E}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{A}_4$	$(0,1)$
	$\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus 2\mathbf{A}_2$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5$	$(0,1)$
	$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{D}_{11}$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{D}_9 \oplus \mathbf{A}_2$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{D}_7 \oplus \mathbf{A}_4$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6$	$(0,1)$
	$\mathbf{E}_8 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{A}_{11}$	$(0,1)$
	$\mathbf{E}_8 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_1$	$(1,1)$
	$\mathbf{E}_8 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{A}_8 \oplus \mathbf{A}_3$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
	$\mathbf{E}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0,1)$
	$\mathbf{E}_8 \oplus \mathbf{A}_7 \oplus 2\mathbf{A}_2$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(0,1)$
	$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,1)$
	$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
	$\mathbf{E}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
[1]	$\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2^*$	$(1,0)$
	$\mathbf{E}_7 \oplus 2\mathbf{E}_6^*$	$(1,0)$
	$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_6$	$(0,1)$
	$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
	$\mathbf{E}_7 \oplus \mathbf{A}_{12}$	$(0,1)$
	$\mathbf{E}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(2,0)$
	$\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$	$(0,1)$
	$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
	$\mathbf{E}_7 \oplus 2\mathbf{A}_6$	$(0,1)$
[2]	$\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1,0)$
	$2\mathbf{E}_6^* \oplus \mathbf{A}_7$	$(1,0)$
	$2\mathbf{E}_6^* \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1,0)$
[3]	$2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1,0)$
	$\mathbf{E}_6 \oplus \mathbf{D}_{13}$	$(1,0)$
	$\mathbf{E}_6 \oplus \mathbf{D}_{11} \oplus \mathbf{A}_2$	$(1,0)$

Singularities	(r,c)
$\mathbf{E}_6 \oplus \mathbf{D}_9 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{A}_6$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_8$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus 2\mathbf{A}_4$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{13}$	$(0,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_1$	$(0,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$	$(2,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$	$(2,0)$
$\mathbf{D}_{19}$	$(1,0)$
$\mathbf{D}_{17} \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_{15} \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{D}_{13} \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{D}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_{11} \oplus \mathbf{A}_8$	$(1,0)$
$\mathbf{D}_{11} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_{11} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1,0)$
$\mathbf{D}_9 \oplus \mathbf{A}_{10}$	$(1,0)$
$\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{D}_9 \oplus 2\mathbf{A}_4^* \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_7 \oplus \mathbf{A}_{12}$	$(1,1)$
$\mathbf{D}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{D}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$	$(2,0)$
$\mathbf{D}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_7 \oplus 2\mathbf{A}_6$	$(0,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_{14}$	$(0,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$	$(1,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus 2\mathbf{A}_2^*$	$(1,0)$
$\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_4$	$(2,0)$
$\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1,0)$

Table 3.2: The spaces $\mathcal{M}_1(\mathbf{S})$, $\mu(\mathbf{S}) = 19$, with double points only

Singularities	(r,c)	Singularities	(r,c)
$\mathbf{A}_{19}$	$(2,0)$	$\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{A}_{18} \oplus \mathbf{A}_1$	$(1,1)$	[6] $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$	$(0,1)$
$\mathbf{A}_{16} \oplus \mathbf{A}_3$	$(2,0)$	[6] $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$	$(2,0)$
$\mathbf{A}_{15} \oplus \mathbf{A}_4$	$(0,1)$	[6] $\mathbf{A}_{10} \oplus 2\mathbf{A}_4^* \oplus \mathbf{A}_1$	$(1,1)$
[6] $\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(0,3)$	$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1,0)$
[6] $\mathbf{A}_{13} \oplus \mathbf{A}_6$	$(0,2)$	[6] $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(2,0)$
$\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$	[4] $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$	$(1,1)$
[6] $\mathbf{A}_{12} \oplus \mathbf{A}_7$	$(0,1)$	[6] $\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0,1)$
[4] $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1,1)$	[4] $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1,0)$	[6] $\mathbf{A}_7 \oplus 2\mathbf{A}_6$	$(0,1)$
[6] $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
[5] $\mathbf{A}_{11} \oplus 2\mathbf{A}_4^*$	$(2,0)$	[6] $\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1,0)$
$\mathbf{A}_{10} \oplus \mathbf{A}_9$	$(2,0)$	[6] $2\mathbf{A}_6^* \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(2,0)$
[4] $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$(1,1)$	[6] $\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4^*$	$(2,0)$

6. for the thirteen sets of singularities marked with [6] in Table 3.2, the fundamental group is still unknown.

In all other cases, the fundamental group is abelian: $\pi_1 = \mathbb{G}_6$.

The fundamental groups of sextics of torus type are large and more difficult to describe. To simplify the description, we introduce a few *ad hoc* groups:

$$G(\bar{s}) := \bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| \rho_3^4 = (\alpha_1\alpha_2)^3,\ \{\alpha_2 \uparrow \sigma_1^i, \alpha_3\}_{s_i} = 1,\ i = 0, \ldots, 5 \bigr\rangle, \tag{3.3.1}$$

where $\bar{s} = (s_0, \ldots, s_5) \in \mathbb{Z}^6$ is an integral vector,

$$\begin{aligned} L_{p,q,r} := \bigl\langle \alpha_1, \alpha_2 \bigm| (\alpha_1\alpha_2\alpha_1)^3 = \alpha_2\alpha_1\alpha_2,\ \{\alpha_2, (\alpha_1\alpha_2)\alpha_1(\alpha_1\alpha_2)^{-1}\}_p \\ = \{\alpha_1, \alpha_2\alpha_1\alpha_2^{-1}\}_q = \{\alpha_2, (\alpha_1\alpha_2^2)\alpha_1(\alpha_1\alpha_2^2)^{-1}\}_r = 1 \bigr\rangle, \end{aligned} \tag{3.3.2}$$

where $p, q, r \in \mathbb{Z}$, and

$$\begin{aligned} E_{p,q} := \bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| \rho_3\alpha_2\rho_3^{-1} = \alpha_2^{-1}\alpha_1\alpha_2 = \rho_3^{-1}\alpha_3\rho_3, \\ \rho_3^4 = (\alpha_1\alpha_2)^3,\ \{\alpha_2, \alpha_3\}_p = \{\alpha_1, \alpha_3\}_q = 1 \bigr\rangle, \end{aligned} \tag{3.3.3}$$

where $p, q \in \mathbb{Z}$. Then, the fundamental groups of the maximizing irreducible 1-torus sextics are as follows:

Table 3.3: The spaces $\mathcal{M}_3(\mathbf{S})$, $\mu(\mathbf{S})=19$

Singularities	(r,c)	Singularities	(r,c)
[1] $(3\mathbf{E}_6)\oplus\mathbf{A}_1$	$(1,0)$	$(\mathbf{A}_{17}\oplus\mathbf{A}_2)$	$(1,0)$
[2] $(2\mathbf{E}_6\oplus\mathbf{A}_5)\oplus\mathbf{A}_2$	$(2,0)$	$(\mathbf{A}_{14}\oplus\mathbf{A}_2)\oplus\mathbf{A}_3$	$(1,0)$
[3] $(2\mathbf{E}_6\oplus 2\mathbf{A}_2^*)\oplus\mathbf{A}_3$	$(1,0)$	$(\mathbf{A}_{14}\oplus\mathbf{A}_2)\oplus\mathbf{A}_2\oplus\mathbf{A}_1$	$(1,0)$
$(\mathbf{E}_6\oplus\mathbf{A}_{11})\oplus\mathbf{A}_2$	$(1,0)$	$(\mathbf{A}_{11}\oplus 2\mathbf{A}_2^*)\oplus\mathbf{A}_4$	$(1,0)$
$(\mathbf{E}_6\oplus\mathbf{A}_8\oplus\mathbf{A}_2)\oplus\mathbf{A}_3$	$(1,0)$	$(2\mathbf{A}_8)\oplus\mathbf{A}_3$	$(1,0)$
$(\mathbf{E}_6\oplus\mathbf{A}_8\oplus\mathbf{A}_2)\oplus\mathbf{A}_2\oplus\mathbf{A}_1$	$(1,1)$	[6] $(\mathbf{A}_8\oplus\mathbf{A}_5\oplus\mathbf{A}_2)\oplus\mathbf{A}_4$	$(0,1)$
[4] $(\mathbf{E}_6\oplus\mathbf{A}_5\oplus 2\mathbf{A}_2^*)\oplus\mathbf{A}_4$	$(2,0)$	[5] $(\mathbf{A}_8\oplus 3\mathbf{A}_2^*)\oplus\mathbf{A}_4\oplus\mathbf{A}_1$	$(1,0)$
$\mathbf{D}_5\oplus(\mathbf{A}_8\oplus 3\mathbf{A}_2^*)$	$(1,0)$		

1. for $(3\mathbf{E}_6)\oplus\mathbf{A}_1$, the group is $\pi_1=\mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^3$;
2. for $(2\mathbf{E}_6\oplus\mathbf{A}_5)\oplus\mathbf{A}_2$, the groups are $E_{3,6}$, see (3.3.3), and $L_{3,6,0}$, see (3.3.2);
3. for $(2\mathbf{E}_6\oplus 2\mathbf{A}_2)\oplus\mathbf{A}_3$, the group is $E_{4,3}$, see (3.3.3);
4. for $(\mathbf{E}_6\oplus\mathbf{A}_5\oplus 2\mathbf{A}_2)\oplus\mathbf{A}_4$, the groups are $L_{5,6,3}$ and $G(6,5,3,3,5,6)$, see (3.3.2) and (3.3.1), respectively;
5. for $(\mathbf{A}_8\oplus 3\mathbf{A}_2)\oplus\mathbf{A}_4\oplus\mathbf{A}_1$, the group is

$$\pi_1=\bigl\langle\alpha_1,\alpha_2,\alpha_3 \bigm| [\alpha_2,\alpha_3]=\{\alpha_1,\alpha_2\}_3=\{\alpha_1,\alpha_3\}_9=1,\\ \alpha_3\alpha_1\alpha_2^{-1}\alpha_3\alpha_1\alpha_3(\alpha_3\alpha_1)^{-2}\alpha_2=(\alpha_1\alpha_3)^2\alpha_2^{-1}\alpha_1\alpha_3\alpha_2\alpha_1\bigr\rangle;$$

6. for the set of singularities $(\mathbf{A}_8\oplus\mathbf{A}_5\oplus\mathbf{A}_2)\oplus\mathbf{A}_4$, the group is unknown.

In all other cases, the fundamental group is $\pi_1=\Gamma$. In each of items 2 and 4, it is not known whether the two groups are isomorphic. The groups corresponding to distinct sets of singularities (listed above) are distinct, except that it is not known whether the group in item 5 is isomorphic to Γ.

Fundamental groups of non-maximizing sextics (see §4.1) are as follows.

Corollary 3.3.4. *Let $D\subset\mathbb{P}^2$ be a non-special irreducible simple plane sextic. If $\mu(D)=19$, the fundamental group $\pi_1:=\pi_1(\mathbb{P}^2\smallsetminus D)$ is as shown in Tables 3.1 and 3.2. Otherwise, one has*

- $\pi_1=SL(2,\mathbb{F}_3)\times\mathbb{G}_2$ *for* $2\mathbf{D}_7\oplus 2\mathbf{A}_2$, $\mathbf{D}_7\oplus\mathbf{D}_4\oplus 3\mathbf{A}_2$, *and* $2\mathbf{D}_4\oplus 4\mathbf{A}_2$,

- $\pi_1 = SL(2, \mathbb{F}_5) \odot \mathbb{G}_{12}$, *for* $2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$,

and $\pi_1 = \mathbb{G}_6$ *in all other cases.*

Chapter 4

Principal Results

4.1 Statements

There are 110 maximizing sets of simple singularities realized by non-special irreducible sextics as shown in Tables 3.1, 3.2. We have studied non-special non-maximizing sets of singularities in the way guided in [10]. We found that 2996 sets of simple singularities are realized by non-maximizing non-special irreducible sextics. (This statement is almost contained in [11], although no distinction between special and non-special curves is made there.) The corresponding counts for irreducible 1-torus sextics are 15 and 105, respectively, see [9].

Instead of listing such a huge list, by the following theorem one can find all existing non-maximal sets of singularities which are realized by irreducible non-special sextics. Furthermore, it reveals the relation between maximal and non-maximal sextics.

Theorem 4.1.1 (see §4.2.1 and §5.1)**.** *The space $\mathcal{M}_1(\mathbf{S})$ is nonempty if and only if either $\mathbf{S}$ is in one of the following two exceptional degeneration chains*

$$2\mathbf{D}_8 \rightarrowtail \mathbf{D}_9 \oplus \mathbf{D}_8 \rightarrowtail 2\mathbf{D}_9, \qquad 2\mathbf{D}_4 \oplus 4\mathbf{A}_2 \rightarrowtail \mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2 \rightarrowtail 2\mathbf{D}_7 \oplus 2\mathbf{A}_2$$

or $\mathbf{S}$ degenerates to one of the maximizing sets of singularities listed in Tables 3.1, 3.2.

Table 4.1: Disconnected spaces $\mathcal{M}_1(\mathbf{S})$, $\mu(\mathbf{S}) < 19$

Singularities	(r, c)	Singularities	(r, c)
$\mathbf{E}_8 \oplus 2\mathbf{A}_5$	$(0, 1)$	$\mathbf{D}_6 \oplus 2\mathbf{A}_6$	$(0, 1)$
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5$	$(0, 1)$	$\mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1$	$(0, 1)$
$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0, 1)$	$2\mathbf{A}_9$	$(2, 0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$	$(0, 1)$	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(1, 0)$
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5$	$(0, 1)$	$3\mathbf{A}_6$	$(0, 1)$
$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	$(0, 1)$	$2\mathbf{A}_6 \oplus 2\mathbf{A}_3$	$(0, 1)$
$\mathbf{E}_6 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_1$	$(0, 1)$	$2\mathbf{A}_7 \oplus \mathbf{A}_4$	$(0, 1)$

Table 4.1 contains 14 sets of singularities among all non-maximal sextics with the speciality of having disconnected $\mathcal{M}_1(\mathbf{S})$ spaces.

Theorem 4.1.2. *The numbers (r, c) of connected components of $\mathcal{M}_1(\mathbf{S})$ are as shown in Tables 3.1, 3.2, and 4.1; in all other cases, $\mathcal{M}_1(\mathbf{S})$ is connected and contains real curves.*

Two lines in Table 4.1 deserve separate statements: to our knowledge, phenomena of this kind have not been observed before.

Proposition 4.1.3 (see §5.1.3)**.** *The space $\mathcal{M}_1(\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5) = \mathcal{M}(\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5)$ is connected (hence, its only component is real), but it contains no real curves.*

Proposition 4.1.4 (see §4.2.3)**.** *Let $\mathbf{S}_0 := 2\mathbf{A}_9$, $\mathbf{S}_1 := \mathbf{A}_{19}$, and $\mathbf{S}_2 := \mathbf{A}_{10} \oplus \mathbf{A}_9$. The space $\mathcal{M}_1(\mathbf{S}_i)$, $i = 0, 1, 2$, consists of two connected components $\mathcal{M}_1^{\pm}(\mathbf{S}_i)$, each containing real curves, so that $\partial\mathcal{M}_1^{\epsilon}(\mathbf{S}_0) = \mathcal{M}_1^{\epsilon}(\mathbf{S}_1) \cup \mathcal{M}_1^{\epsilon}(\mathbf{S}_2)$ for each $\epsilon = \pm$.*

Corollary 4.1.5 (see §4.2.4)**.** *With the same six exceptions as in Theorem 4.1.1, any non-special irreducible simple sextic degenerates to a maximizing sextic with these properties, see Tables 3.1 and 3.2.*

Table 4.2: Exceptional sets of singularities (see §4.2.1)

[1] $\mathbf{E}_6 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	[3] $\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus 2\mathbf{A}_2$	[3] $\mathbf{A}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$
[1] $\mathbf{A}_5 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	[3] $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5$	[4] $2\mathbf{A}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$
[2] $3\mathbf{A}_4 \oplus 3\mathbf{A}_2$	[3] $2\mathbf{A}_7 \oplus 2\mathbf{A}_2$	[5] $2\mathbf{A}_9$

4.2 Proofs of principal results

4.2.1 Proof of Theorem 4.1.1

We apply Theorem 3.2.6 to any possible set of singularities

$$\mathbf{S} = \bigoplus_{i=1}^{18} a_i \mathbf{A}_i \oplus \bigoplus_{j=4}^{18} d_j \mathbf{D}_j \oplus \bigoplus_{k=6}^{8} e_k \mathbf{E}_k$$

with

$$\operatorname{rank} \mathbf{S} = \sum_{i=1}^{18} i a_i + \sum_{j=4}^{18} j d_j + \sum_{i=6}^{8} k e_k \leq 18.$$

Following the steps in the first item of 3.2.2, it is sufficient to find if there exists an embedding $\tilde{\mathbf{S}}_h \hookrightarrow \mathbf{L}$. Space $\mathcal{M}_1(\mathbf{S})$ is the set of non-special sextics and by Theorem 3.2.7, their homological types are primitive ($\mathcal{K} = 0$). Hence we have $\operatorname{discr} \tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{2} \rangle$. Using Theorem 2.2.6 we list all sets of singularities extending to a primitive homological type. The resulting list is compared against the list of all perturbations of the maximizing sets obtained and the relation in the statement is observed.

4.2.2 Proof of Theorem 4.1.2

Let $\mathbf{S}$ be one of the sets of singularities found in Theorem 4.1.1 with $\mu(\mathbf{S}) \leqslant 18$, and let $\mathbf{T}$ be a representative of the genus $g(\mathbf{S}_h^\perp)$. In most cases, Theorem 2.3.2 gives us $\mathrm{E}(\mathbf{T}) = 0$ and, due to Corollary 2.3.3, a primitive homological type extending $\mathbf{S}$ is unique up to strict isomorphism. In the remaining cases, it suffices to show that the map $\mathrm{d}^\perp \colon O(\mathbf{S}) \to \mathrm{E}(\mathbf{T})$, see §2.3.1 and second item in §3.2.2, is onto; this will also imply the uniqueness of $\mathbf{T}$ in its genus.

There are 32 sets of singularities containing a point of type $\mathbf{A}_4$ and satisfying the hypotheses of Lemma 2.3.7 or Corollary 2.3.9 (with $p = 5$); in these cases, a nontrivial symmetry of any of the type $\mathbf{A}_4$ points maps to the generator $-1 \in \mathrm{E}(\mathbf{T}) = \{\pm\}$. The remaining nine sets of singularities are collected in Table 4.2, with references to the list below, where we indicate the Miranda–Morrison homomorphism $\mathrm{e}\colon \operatorname{Aut}\mathcal{T} \to \mathrm{E}(\mathbf{T})$ (given by Lemma 2.3.8) and automorphism(s) of $\mathbf{S}$ generating $\mathrm{E}(\mathbf{T})$.

1. $\mathrm{e}\colon t_\xi \mapsto \delta_3(\xi) \cdot \delta_5(\xi) \cdot |\xi|_5 \in \{\pm 1\}$; a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$;
2. $\mathrm{e}\colon t_\xi \mapsto (\delta_3(\xi) \cdot \delta_5(\xi) \cdot |\xi|_5, |\xi|_5) \in \{\pm 1\} \times \{\pm 1\}$; a symmetry of $\mathbf{A}_4$ *and* a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$ (two generators);
3. $\mathrm{e}\colon t_\xi \mapsto \delta_2(\xi) \cdot \delta_3(\xi) \cdot |\xi|_2 \cdot |\xi|_3 \in \{\pm 1\}$; a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$ or a symmetry of $\mathbf{A}_4$, $\mathbf{A}_5$, or $\mathbf{E}_6$;
4. $\mathrm{e}\colon t_\xi \mapsto \delta_3(\xi) \cdot \delta_7(\xi) \cdot |\xi|_3 \cdot |\xi|_7 \in \{\pm 1\}$; a transposition $\mathbf{A}_1 \leftrightarrow \mathbf{A}_1$;
5. $\mathrm{e}\colon t_\xi \mapsto |\xi|_5 \in \{\pm 1\}$; none.

The last case $\mathbf{S} = 2\mathbf{A}_9$ is special: the map $\mathrm{d}^\perp\colon O(\mathbf{S}) \to \mathrm{E}(\mathbf{T})$ is not surjective and there are two deformation families, as stated. Still, we can assert that $\mathbf{T}$ is unique in its genus: $\mathrm{E}(\mathbf{T})$ is generated by $\mathrm{e}(t_\xi)$, where $\xi^2 = \frac{4}{5} \bmod 2\mathbb{Z}$.

This concludes the proof of the fact that any set of singularities $\mathbf{S} \neq 2\mathbf{A}_9$ with $\mu(\mathbf{S}) \leqslant 18$ extends to a unique primitive homological type $\mathcal{H}$; hence, the space $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ is connected. To complete the proof of the theorem, we need to analyze whether $\mathcal{M}_1(\mathbf{S})$ contains a real curve and, if it does not, whether $\mathcal{H}$ is symmetric. This is done in §5.1.2 below. □

4.2.3 Proof of Proposition 4.1.4

One has $\mathbf{T} \cong \mathbb{Z}b \oplus \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$, with $a_1^2 = a_2^2 = 10$, $b^2 = -2$. The group $\mathcal{T}$ is $\langle\frac{2}{5}\rangle \oplus \langle\frac{2}{5}\rangle \oplus \langle\frac{1}{2}\rangle \oplus \langle\frac{1}{2}\rangle \oplus \langle\frac{3}{2}\rangle$, and $\operatorname{Aut}\mathcal{T}$ is generated by

$$\sigma_{1,2}\colon \alpha_{1,2} \mapsto -\alpha_{1,2}, \quad \sigma_3\colon \alpha_1 \leftrightarrow \alpha_2, \quad \sigma_4\colon \alpha_3 \leftrightarrow \alpha_4.$$

Since $|\mathrm{E}(\mathbf{T})| = 2$, the image $\operatorname{Im} \mathrm{d}$ is generated by σ_1, σ_2, and $\sigma_3\sigma_4$, and this subgroup coincides with the image of $O(\mathbf{S})$.

In other words, in the group $\mathcal{T}$, the map $\frac{1}{2}a_1 \mapsto \frac{1}{2}a_2$, $\frac{1}{5}a_1 \mapsto \pm\frac{1}{5}a_2$ establishes a well defined bijection between the two non-characteristic elements of square $\frac{1}{2}$ and the two pairs of opposite elements of square $\frac{2}{5}$. A similar bijection in $-\mathcal{S}$ is given by the orthogonal sum decomposition $\mathbf{S} = \mathbf{A}_9 \oplus \mathbf{A}_9$, and the two homological types differ by whether the isometry $\varphi : -\mathcal{S} \oplus \langle \frac{3}{2} \rangle \to \mathcal{T}$ does or does not respect these bijections. Hence $\mathbf{S}_0 := 2\mathbf{A}_9$ has two connected components represented by $\mathcal{M}_1^{\pm}(\mathbf{S}_0)$ which stands by the item (2)(b) in §3.2.2. In order to understand and fix these spaces, let us consider the lattices $\mathbf{S}_h \oplus (\mathbf{S}_h)^{\perp}_{\mathbf{L}} = \mathbf{S}_h \oplus \mathbf{T} \subset \mathbf{L}$ and the graph $\mathcal{P}$ of φ in their discriminant form $\mathcal{S} \oplus \mathcal{T}$.

Recall that the singularity $\mathbf{A}_n$ is a root lattice with n generators $\{e_1, e_2, ..., e_n\}$ where $e_i^2 = -2$, $e_i \cdot e_j = 1$ for $i = j \pm 1$, $e_i \cdot e_j = 0$ otherwise. Let c' be the sum $c' = \sum_{i=1}^{n} i \cdot e_i$. Then one can take c' as the 'generator' of the discriminant form of $\mathbf{A}_n$. Assume that c_1,c_2 are the 'generators' (in the sense just described) of the discriminant of the two copies of $2\mathbf{A}_9$. One has

$$\mathbf{S}_h \oplus \mathbf{T} \cong 2\mathbf{A}_9 \oplus \mathbb{Z}h \oplus \mathbb{Z}b \oplus \mathbb{Z}a_1 \oplus \mathbb{Z}a_2. \tag{4.2.1}$$

Let us fix these spaces with the following corresponding isotropic subgroups:

$$\mathcal{M}_1^{+}(\mathbf{S}_0) \longleftrightarrow < \frac{3c_1 + a_1}{10}, \frac{3c_2 + a_2}{10} >$$

$$\mathcal{M}_1^{-}(\mathbf{S}_0) \longleftrightarrow < \frac{3c_1 + a_1}{5}, \frac{3c_1 + a_2}{2}, \frac{3c_2 + a_2}{5}, \frac{3c_2 + a_1}{2} >$$

On the other hand $\mathbf{S}_0 = 2\mathbf{A}_9$ degenerates to two different maximizing set of singularities $\mathbf{S}_1 = \mathbf{A}_{19}$ and $\mathbf{S}_2 = \mathbf{A}_9 \oplus \mathbf{A}_{10}$. Degeneration of $\mathcal{M}_1(\mathbf{S}_0)$ becomes in the following way:

$\underline{\mathcal{M}_1^{\pm}(\mathbf{S}_0) \longrightarrow \mathcal{M}_1^{\pm}(\mathbf{S}_1) :}$ Similar to $2\mathbf{A}_9$, homological types arising from the set $\mathbf{A}_{19}$ differs by the item (2)(b) in §3.2.2. According to that, homological types differ by whether the isometry $-\mathcal{S} \oplus \langle \frac{3}{2} \rangle \to \mathcal{T}$ does or does not respect the bijections of 4 and 5-primary parts as shown in Equation 4.2.2 and Equation 4.2.3.

One has $\mathbf{A}_{19} \supset 2\mathbf{A}_9 \oplus \langle -20 \rangle$. Let c be the 'generator' of the discriminant of $\mathbf{A}_{19}$ in the same sense as described above and p be the generator of $\langle -20 \rangle$. Then

$c = 2c_2 + p$ and the lattice $\mathbf{S}_h \oplus \mathbf{T}$ is

$$\mathbf{A}_{19} \oplus \mathbb{Z}h \oplus \mathbb{Z}r \oplus \mathbb{Z}s \supset 2\mathbf{A}_9 \oplus \mathbb{Z}h \oplus \mathbb{Z}p \oplus \mathbb{Z}r \oplus \mathbb{Z}s$$

where $p^2 = -20$, $r^2 = 2$, $s^2 = 20$. Hence one has such relations with generators in 4.2.1:

$$p = 3a_1 - 3a_2 + 10b,$$

$$r = a_1 - a_2 + 3b,$$

$$s = a_1 + a_2.$$

Isotropic subgroup are generated by $\{\frac{3c+s}{5}, \frac{3c+s+2r}{4}\}$ and $\{\frac{3c+s}{5}, \frac{3c-s+2r}{4}\}$ in $(+)$ and $(-)$ signed cases, respectively.

$$\mathcal{M}_1^+(\mathbf{S}_1): \quad \frac{3c+s}{5} \cong \frac{3c_2+a_2}{5} \mod \mathbb{Z} \tag{4.2.2}$$

$$\frac{3c+s+2r}{4} \cong \frac{3c_2+a_2}{2} \mod \mathbb{Z}$$

which corresponds to $\mathcal{M}_1^+(\mathbf{S}_0)$.

$$\mathcal{M}_1^-(\mathbf{S}_1): \quad \frac{3c+s}{5} \cong \frac{3c_2+a_2}{5} \mod \mathbb{Z} \tag{4.2.3}$$

$$\frac{3c-s+2r}{4} \cong \frac{3c_2+a_1}{2} \mod \mathbb{Z}$$

which corresponds to $\mathcal{M}_1^-(\mathbf{S}_0)$.

$\underline{\mathcal{M}_1^{\pm}(\mathbf{S}_0) \longrightarrow \mathcal{M}_1^{\pm}(\mathbf{S}_2):}$ The space $\mathcal{M}_1(\mathbf{S}_2)$ has two connected components by (2)(a) in §3.2.2 which differ by the lattice $\mathbf{T}$, say $\mathcal{M}_1^+(\mathbf{S}_2)$ is matched with $\mathbf{T}^+ \cong \langle 22 \rangle \oplus \langle 10 \rangle$ and $\mathcal{M}_1^-(\mathbf{S}_2)$ is matched with $\mathbf{T}^- \cong \langle 2 \rangle \oplus \langle 110 \rangle$. Since $\mathbf{A}_{10} \supset \mathbf{A}_9 \oplus \langle -110 \rangle$, the lattices $\mathbf{S}_h \oplus \mathbf{T}$ for two different realization are as follows:

$$\mathcal{M}_1^+(\mathbf{S}_2): \mathbf{A}_9 \oplus \mathbf{A}_{10} \oplus \mathbb{Z}h \oplus \mathbf{T}^+ \supset 2\mathbf{A}_9 \oplus \mathbb{Z}h \oplus \mathbb{Z}k \oplus \mathbb{Z}m \oplus \mathbb{Z}a_1$$

$$\mathcal{M}_1^-(\mathbf{S}_2): \mathbf{A}_9 \oplus \mathbf{A}_{10} \oplus \mathbb{Z}h \oplus \mathbf{T}^- \supset 2\mathbf{A}_9 \oplus \mathbb{Z}h \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}x,$$

where $y^2 = k^2 = -110$, $m^2 = 22$, $z^2 = 2$, $x^2 = 110$. Relations of these generators with the generators in 4.2.1 are

$$k = 10b + a_2,$$

$$y = 15a_1 + 22a_2 + 60b.$$

Isotropic subgroup are generated by $\{\frac{c_2+k}{10}\}$ and $\{\frac{c_2+y}{10}\}$ in $(+)$ and $(-)$ signed cases, respectively.

$$\mathcal{M}_1^+(\mathbf{S}_2): \quad \frac{c_2+k}{5} = \frac{c_2+10b+a_2}{5} \cong \frac{3c_2+a_2}{5} \bmod \mathbb{Z}$$

$$\frac{c_2+k}{2} = \frac{c_2+10b+a_2}{2} \cong \frac{3c_2+a_2}{2} \bmod \mathbb{Z}$$

which corresponds to $\mathcal{M}_1^+(\mathbf{S}_0)$.

$$\mathcal{M}_1^-(\mathbf{S}_2): \quad \frac{c_2+y}{5} = \frac{c_2+15a_1+22a_2+60b}{5} \cong \frac{3c_2+a_2}{5} \bmod \mathbb{Z}$$

$$\frac{c_2+y}{2} = \frac{c_2+15a_1+22a_2+60b}{2} \cong \frac{3c_2+a_1}{2} \bmod \mathbb{Z}$$

which corresponds to $\mathcal{M}_1^-(\mathbf{S}_0)$.

□

4.2.4 Proof of Corollary 4.1.5

Unless $\mathbf{S} = 2\mathbf{A}_9$, the statement follows immediately from Theorem 4.1.1. Indeed, there is a degeneration $\mathbf{S} \rightarrowtail \mathbf{S}'$ to a maximizing set of singularities $\mathbf{S}'$. Due to [22, Proposition 5.1.1], there is a degeneration $D \rightarrowtail D'$ of *some* sextics $D \in \mathcal{M}_1(\mathbf{S})$ and $D' \in \mathcal{M}_1(\mathbf{S}')$. Since $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ is connected, a degeneration exists for *any* sextic $D \in \mathcal{M}_1(\mathbf{S})$. The exceptional case $\mathbf{S} = 2\mathbf{A}_9$ with disconnected moduli space is given by Proposition 4.1.4, see §4.2.3 above. □

Chapter 5

Real Structures

5.1 Real sextics

5.1.1 Conventions for real sextics

A *real structure* on a complex analytic variety X is an anti-holomorphic involution $c : X \to X$ and *real variety* is the pair (X, c) for complex variety X and real structure c. The fixed point set $X_{\mathbb{R}} = \operatorname{Fix} c$ of the involution is the real part of the complex variety X.

Let (X, c) be a real surface. Any curve $D \subset X$ is real curve if $c(D) = D$. For the double covering $\bar{X}$ branched over real curve D, c lifts to two different real structures on $\bar{X}$ and they differ by the deck translation.

Theorem 5.1.1. *A homological type $\mathcal{H}$ is realized by a real sextic if and only if $\mathcal{H}$ admits an* involutive *orientation reversing automorphism.*

Proof. The necessity is obvious: the real structure on $\mathbb{P}^2$ lifts to a real structure on the covering $K3$-surface X, which induces an involutive automorphism of the homological type.

For the converse, let $a \in \operatorname{Aut} \mathcal{H}$ be an automorphism as in the statement. Due to Lemma 2.2.2, the restriction $a|_{\mathbf{S}}$ has the form $r \circ (-s_*)$, where $r \in \operatorname{Ker} \mathrm{d}$ and s_* is induced by an *involutive* symmetry $s \in \operatorname{Sym}' \mathfrak{G}_{\mathbf{S}}$. Since $\operatorname{Ker} \mathrm{d} \in \operatorname{Aut} \mathcal{H}$ (in the obvious way: automorphisms extend to $\mathbf{S}^\perp$ by the identity, see Theorem 2.2.5), the involution $r^{-1} \circ a$ is also in $\operatorname{Aut} \mathcal{H}$. Let $c := r^{-1} \circ a \circ t_h \in O(\mathbf{L})$; it is still an involution and $c|_{\mathbf{T}} = a|_{\mathbf{T}}$.

Let $T_\pm$ be the (± 1)-eigenspaces of the action of c on $\mathbf{T} \otimes \mathbb{R}$. Since c is orientation reversing, one has $\sigma_+ T_\pm = 1$. Hence, one can choose generic vectors $\omega_\pm \in T_\pm$ such that $\omega_+^2 = \omega_-^2 > 0$ and take $\omega := \omega_+ + i\omega_-$ for the class of a holomorphic form. Let, further, S_- be the (-1)-eigenspace of the action of c on $\tilde{\mathbf{S}}_h \otimes \mathbb{R}$. Since $h \in S_-$, one has $\sigma_+ S_- = 1$. By the construction, $-c$ preserves a Weyl chamber of $\mathbf{S}$; hence, condition (1) in Definition 3.2.3 implies that S_- is *not* orthogonal to a vector $v \in \tilde{\mathbf{S}}_h$ of square (-2) and one can find a generic vector $\rho \in S_-$, $\rho^2 > 0$, and take it for the class of a Kähler form. These choices define a 2-polarized $K3$-surface X with $\operatorname{Pic} X = \tilde{\mathbf{S}}_h$ and, by an equivariant version of the global Torelli theorem, c is induced by a real structure on X commuting with the deck translation τ of the ramified covering $X \to \mathbb{P}^2$ defined by h. This real structure descends to $\mathbb{P}^2$ and makes the sextic corresponding to X (*i.e.*, the branch curve) real. $\square$

Let D be a real sextic with the set of singularities $\mathbf{S}$. The real structure c lifts to two real structures on the covering $K3$-surface; they take exceptional divisors to exceptional divisors and, hence, induce two involutive symmetries $c_\pm \colon \mathfrak{G} \to \mathfrak{G}$ of the Dynkin graph $\mathfrak{G} := \mathfrak{G}_{\mathbf{S}}$. Define another symmetry $c_0 \colon \mathfrak{G} \to \mathfrak{G}$ as follows: on each connected component $\mathfrak{G}_i$ of $\mathfrak{G}$ *fixed* by $c_\pm$ and of type other than $\mathbf{D}_{\text{even}}$ let $c_0 = \mathrm{id}$; on all other components, let $c_0 = c_\pm$. In other words, since $c_- = c_+ \circ \tau$, we just let $v \uparrow c_0 = v$ for each vertex v such that $v \uparrow c_+ \neq v \uparrow c_-$, see Lemma 3.2.1.

Corollary 5.1.2. *If a homological type $\mathcal{H}$ is realized by a real sextic (D, c), then any c_0-invariant perturbation $\mathcal{H}'$ of $\mathcal{H}$ is also realized by a real sextic D'.*

Note that we do *not* assert that D' degenerates to D in the class of real sextics. A real perturbation can be found if $\mathcal{H}'$ is invariant under one of $c_\pm$.

Table 5.1: Exceptional sets of singularities

$[3\mathbf{A}_6]_7$	$[\mathbf{E}_6 \oplus \mathbf{A}_5]_3 \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$\star\, 2\mathbf{E}_7 \oplus \mathbf{A}_4$
$[2\mathbf{A}_6]_7 \oplus \mathbf{D}_6$	$[\mathbf{E}_6 \oplus 2\mathbf{A}_5]_3 \oplus \mathbf{A}_1$	$\star\, \mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6$
$[2\mathbf{A}_6]_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_1$	$[\mathbf{E}_7 \oplus \mathbf{A}_7]_2 \oplus \mathbf{A}_4$	$\star\, \mathbf{E}_7 \oplus \mathbf{A}_{11}$
$[2\mathbf{A}_6]_7 \oplus 2\mathbf{A}_3$	$[2\mathbf{A}_7]_2 \oplus \mathbf{A}_4$	$\star\, \mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$
$[2\mathbf{A}_5]_3 \oplus \mathbf{E}_8$	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$\star\, 2\mathbf{D}_9$
$[\mathbf{E}_6 \oplus \mathbf{A}_{11}]_3 \oplus \mathbf{A}_1$	$2\mathbf{D}_7 \oplus 2\mathbf{A}_2$	$\star\, \mathbf{D}_9 \oplus \mathbf{D}_8$
$[\mathbf{E}_6 \oplus \mathbf{A}_5]_3 \oplus \mathbf{E}_7$	$\mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$	$\star\, 2\mathbf{D}_8$
$[\mathbf{E}_6 \oplus \mathbf{A}_5]_3 \oplus \mathbf{A}_7$	$2\mathbf{D}_4 \oplus 4\mathbf{A}_2$	$\star\, \mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$
		$\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3$

Proof of Corollary 5.1.2. Let $c_*\colon \mathbf{L} \to \mathbf{L}$ be the automorphism of $\mathcal{H}$ induced by one of the two lifts of c. Composing c_* with $-\tau_*$ on *some* of the indecomposable summands of $\mathbf{S}$, we can change it to another involutive automorphism c' of $\mathcal{H}$ (see Lemma 2.2.2 and Theorem 2.2.5) inducing c_0 on $\mathfrak{G}$. Then c' preserves $\mathbf{S}'$; hence, $c' \circ t_h$ can be regarded as an involutive orientation reversing automorphism of $\mathcal{H}'$, and Theorem 5.1.1 applies. $\square$

5.1.2 End of the proof of Theorem 4.1.2

It is easily confirmed that most sets of singularities $\mathbf{S}$ with $\mu(\mathbf{S}) \leqslant 18$ are *symmetric* perturbations of maximizing sets of singularities realized by real sextics, see Tables 3.1 and 3.2. (In the tables, marked with a * are pairs of isomorphic singular points permuted by the complex conjugation. These pairs should be taken into account when analyzing symmetric perturbations. Note that singular points of type $\mathbf{D}_{\mathrm{even}}$ do not appear in irreducible maximizing sextics.) Due to Corollary 5.1.2, these sets of singularities are realized by real curves.

The remaining 25 sets of singularities are listed in Table 5.1. Each of these sets $\mathbf{S}$ extends to a unique (up to isomorphism) primitive homological type $\mathcal{H}$, and we denote by $\mathbf{T}$ the corresponding transcendental lattice. In each case, the natural homomorphism $\mathrm{d}\colon O(\mathbf{T}) \to \operatorname{Aut}\mathcal{T}$ is surjective.

By Theorem 2.2.5, the homological type $\mathcal{H}$ is symmetric if and only if there is an isometry $a \in O(\mathbf{T})$ with $\det_+ a = -1$ and such that $\mathrm{d}(a) \in \mathrm{d}^{\varphi}(O(\mathbf{S}))$, where d^φ is induced by any anti-isometry $\varphi\colon \mathcal{S} \oplus \langle \frac{1}{2} \rangle \to \mathcal{T}$. If (and only if) a as above can be chosen *involutive*, then so is $\mathrm{d}(a)$ and, due to Lemma 2.2.2, a extends to $\mathbf{L}$ by an *involutive* isometry of $\mathbf{S}$; hence, $\mathcal{M}_1(\mathbf{S})$ contains real curves, see Theorem 5.1.1.

Lemma 5.1.3. *The first twelve sets of singularities in Table 5.1 (those with a $[\,\cdot\,]_p$ pattern) extend to* asymmetric *primitive homological types.*

Proof. Let $\mathbf{S}$ be one of the sets of singularities in question. Then $\tilde{\Sigma}(\mathbf{T}) \subset \Gamma_0^{--}$, see §2.3.3, and there is a well defined map $\det_+\colon \operatorname{Aut}\mathcal{T} \to \{\pm 1\}$. We use Lemma 2.3.11 (with the 'test prime' p indicated in the table) to show that $\det_+$ takes value $+1$ on the image of $O(\mathbf{S})$. If $p = 7$ (the first four lines), the latter image is generated by reflections t_ξ such that either

- $\xi^2 = \frac{6}{7}$ (a symmetry of the Dynkin graph of $\mathbf{A}_6$), or
- $\xi^2 = \frac{12}{7}$ (interchanging of two copies of $\mathbf{A}_6$), or
- $\xi \in \mathcal{T}_2$ (isometries involving the other singular points);

on the other hand, one has $(\frac{-3}{7}) = (\frac{-6}{7}) = (\frac{2}{7}) = 1$. If $p = 3$ (the next six sets of singularities), the image of $O(\mathbf{S})$ is generated by the following automorphisms a:

- t_ξ with $\xi^2 = \frac{4}{3}$ (a symmetry of the Dynkin graph of $\mathbf{E}_6$ or $\mathbf{A}_5$),
- $t_\xi t_\eta$ with $\xi^2 = \frac{2}{3}$, $\eta^2 = 1$ (interchanging of two copies of $\mathbf{A}_5$),
- $t_\xi t_\eta$ with $\xi^2 = \frac{2}{3}$, $\eta^2 = \frac{1}{4}$ (a symmetry of the Dynkin graph of $\mathbf{A}_{11}$),
- t_ξ with $\xi^2 = \frac{7}{8}$ or $\xi^2 = \frac{6}{7}$ (a symmetry of the Dynkin graph of $\mathbf{A}_7$ or $\mathbf{A}_6$).

In each case, Lemma 2.3.11 (with $p = 3$) implies that $\det_+ a = 1$. Finally, if $p = 2$ (the last two sets of singularities), we have reflections t_ξ such that either

- $\xi^2 = \frac{7}{8}$ (a symmetry of the Dynkin graph of $\mathbf{A}_7$), or
- $\xi^2 = \frac{7}{4}$ (interchanging of two copies of $\mathbf{A}_7$), or
- $\xi^2 = \frac{4}{5}$ (a symmetry of the Dynkin graph of $\mathbf{A}_5$).

Lemma 2.3.11 (with $p = 2$) implies that $\det_+ t_\xi = 1$. $\square$

Listed in the last column in Table 5.1 are the sets of singularities $\mathbf{S}$ extending to symmetric homological types due to Proposition 2.3.10. However, since we want to represent these types by real sextics, we will attempt to find *involutive* orientation reversing automorphisms, see Theorem 5.1.1. A simplest automorphism with this property would be a reflection t_a, $a \in \mathbf{T}$, $a^2 = 2$.

Lemma 5.1.4. *If* $\mathbf{S}$ *is one of the sets of singularities marked with a* $\star$ *in* Table 5.1*, the lattice* $\mathbf{T}$ *contains a vector* a *with* $a^2 = 2$.

Proof. It suffices to find an embedding $\mathbf{S}_h \oplus \mathbb{Z}a \hookrightarrow \mathbf{L}$, $a^2 = 2$, with the image of $\mathbf{S}_h$ primitive. In each case, there is an element $\alpha \in \operatorname{discr} \mathbf{S}_h$ with $\alpha^2 = -\frac{1}{2} \bmod 2\mathbb{Z}$. Let $\beta \in \operatorname{discr}(\mathbb{Z}a) = \langle \frac{1}{2} \rangle$ be the generator, and let $\mathbf{S}_h'$ be the finite index extension of $\mathbf{S}_h$ with the kernel generated by $\alpha + \beta$. On a case-by-case basis one confirms that Theorem 2.2.6 implies the existence of a primitive embedding $\mathbf{S}_h' \hookrightarrow \mathbf{L}$. (In the last case, the set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$, the element α above should be chosen carefully, *viz.* $\alpha = 2\alpha_1 + 4\alpha_2 + \alpha_4$ in $\operatorname{discr} \mathbf{S}_h = \langle \frac{3}{4} \rangle \oplus \langle \frac{9}{8} \rangle \oplus \langle \frac{8}{7} \rangle \oplus \langle \frac{1}{2} \rangle$.) $\square$

The set of singularities $\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$ is considered in Proposition 4.1.3, see §5.1.3 below, and the remaining four deformation families are real and contain real curves; for proof, we construct explicit reflections in $O(\mathbf{T})$.

If $\mathbf{S} = 2\mathbf{D}_7 \oplus 2\mathbf{A}_2$, then $\mathbf{T} = \mathbb{Z}u \oplus \mathbb{Z}v \oplus \mathbb{Z}w$ with $u^2 = 4$, $v^2 = 12$, $w^2 = -6$, and the reflection t_u extends to an involutive automorphism of $\mathcal{H}$ (*via* $-\mathrm{id}$ on one of the $\mathbf{D}_7$ components). Hence, $\mathcal{M}_1(\mathbf{S})$ contains a real curve; by Corollary 5.1.2, so do $\mathcal{M}_1(\mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2)$ and $\mathcal{M}_1(2\mathbf{D}_4 \oplus 4\mathbf{A}_2)$.

Finally, if $\mathbf{S} = \mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3$, then $\mathbf{T} = \mathbb{Z}u \oplus \mathbb{Z}v \oplus \mathbb{Z}w$ with $u^2 = v^2 = 10$, $w^2 = -4$. Since $\mathrm{d}\colon O(2\mathbf{A}_4) \to \operatorname{discr} 2\mathbf{A}_4$ is onto, the reflection t_u extends to an involutive automorphism of $\mathcal{H}$. $\square$

5.1.3 Proof of Proposition 4.1.3

One has $\mathcal{T} = \langle\frac{7}{8}\rangle \oplus \langle\frac{6}{7}\rangle \oplus \langle\frac{4}{3}\rangle \oplus \langle\frac{3}{2}\rangle \oplus \langle\frac{3}{2}\rangle$, and the image of $O(\mathbf{S})$ in $\operatorname{Aut}\mathcal{T}$ is generated by the reflections t_{α_i}, $i = 1, 2, 3$. Furthermore, one has $\tilde{\Sigma}_2(\mathbf{T}) = \Gamma_0^{--}$ and the map $\det_+ \colon \operatorname{Aut}\mathcal{T} \to \{\pm 1\}$ is well defined. Applying Lemma 2.3.11, one finds that $\det_+ t_{\alpha_1} = 1$ and $\det_+ t_{\alpha_2} = \det_+ t_{\alpha_3} = -1$. In particular, it follows that the homological type is symmetric, *i.e.*, $\mathcal{M}_1(\mathbf{S})$ consists of a single real component.

Up to sign, any involutive isometry $a \in O(\mathbf{T})$ with $\det_+ a = -1$ is a reflection, $a = \pm t_x$ for some $x \in \mathbf{T}$, $x^2 > 0$: one can take for x a primitive vector generating the (-1)-eigenlattice of $\pm a$, whichever has rank one. As explained above, t_x must induce $-\mathrm{id}$ in one *and only one* of the components $\mathcal{T}_3$, $\mathcal{T}_7$. Hence, $x^2 = 2^k q$, where $k = 1, 3$ and $q = 3, 7$. (Recall that $x \in (\frac{1}{2}x^2)\mathbf{T}^\sharp$; if $k = 2$, then $\xi := \frac{1}{2}x \in \mathcal{T}_2$ has square $0 \bmod \mathbb{Z}$ and t_ξ is not in the image of $O(\mathbf{S})$.) Obviously, $\eta := \frac{1}{q}x$ is a generator of $\mathcal{T}_q$; on the other hand, one can see that $\eta^2/\alpha^2 \notin (\mathbb{Z}_q^\times)^2$, where $\alpha = \alpha_2$ or α_3 for $q = 7$ or 3, respectively. This is a contradiction. $\square$

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