ON CALCULATION OF REDUCED GRÖBNER-SHIRSHOV BASIS OF SOME CLASSICAL AFFINE WEYL GROUPS

by

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ABSTRACT

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In this thesis we will consider the calculation of reduced Gröbner-Shirshov bases of Weyl groups. This will be the main focus of the thesis. In [1], Bokut & Shiao gave the Gröbner-Shirshov bases of positive definite classical Weyl groups A_l, B_l, D_l using the techniques of *Composition Diamond Lemma* and *Elimination of Leading Word.*

We will give a counter example to a hypothesis introduced by Bokut & Shiao in [1] and later we will calculate the reduced Gröbner-Shirshov basis of the positive degenerate infinite affine Weyl group $\widetilde{A}_n$ which is isomorphic to semi-direct product group $\Sigma_n \ltimes \mathbb{Z}^{n-1}$, where the symmetric group Σ_n acts on $\mathbb{Z}^n$ by the permutation action, and $\mathbb{Z}^{n-1}$ is the fixed subgroup which corresponds to the eigen-value action. Also we classify all reduced elements of the Weyl group $\widetilde{A}_n$.

ÖZET

BAZI AFFİNE WEYL GRUPLARI İÇİN

İNDİRGENMİŞ GRÖBNER-SHIRSHOV TABANLARININ

HESAPLANMASI

Abdulkadir ARIK

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Bu tezde Weyl gruplarının İndirgenmiş Gröbner-Shirshov tabanlarını inceleyeceğiz. Tezimizin asıl amacı budur.

[1] de, Bokut & Shiao "Öncü Kelimelerin Elenmesi" ve "Birleşik Elmas Teoremi" tekniklerini kullanarak, pozitif tanımlı klasik Weyl grupları olan A_l, B_l ve D_l'in Gröbner-Shirshov tabanlarını hesapladılar.

[1] de, Bokut & Shiao tarafından tanıtılan hipoteze karşıt örnek vereceğiz ve daha sonra $\Sigma_n \ltimes \mathbb{Z}^{n-1}$ yarıdirekt çarpımına izomorf olan pozitif dejenere sonsuz afin Weyl grubu $\widetilde{A}_n$'in Gröbner-Shirshov tabanını hesaplayacağız.

Ayrıca Weyl grup $\widetilde{A}_n$'in bütün indirgenmiş elemanlarını sınıflandıracağız.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

A Coxeter group W is an abstract group with a certain type of presentation. In practice, we can think of W as a motion group generated by mirror reflections through a hyperplane with respect to a symmetric bilinear form. By Tits Theorem, we know that a Coxeter group is discrete (possibly infinite). We can associate a Coxeter group by a graph which is called Coxeter graph. If we impose some regularity condition on the bilinear form of the Coxeter graph such as positive definiteness, we get a class of finite type Coxeter groups A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2. If the form on the graph is positive degenerate, in this case we obtain a class of affine type Coxeter groups $\widetilde{A}_l$, $\widetilde{B}_l$, $\widetilde{C}_l$, $\widetilde{D}_l$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, $\widetilde{G}_2$. This point of view is the combinatorial underpinning of the Cartan-Killing classification of complex simple Lie algebras.

In this thesis we will consider the calculation of reduced Gröbner-Shirshov bases of Weyl groups which have Coxeter system structure. This will be the main focus of the thesis. In [1], Bokut & Shiao have given the Gröbner-Shirshov bases of positive definite classical Weyl groups A_l, B_l, D_l using the techniques of *Composition Diamond Lemma* and *Elimination of Leading Word.*

In Chapter 5, we will give a counter-example to a hypothesis introduced by Bokut & Shiao in [1]. In the last chapter of the thesis we will calculate the Gröbner-Shirshov bases of the positive degenerate infinite affine Weyl group $\widetilde{A}_n$ which is isomorphic to semi-direct product group $\Sigma_n \ltimes \mathbb{Z}^{n-1}$.

Additionally in the last section of the thesis, using Composition Diamond Lemma and the Gröbner-Shirshov basis elements we give the reduced elements of affine Weyl group $\widetilde{A}_n$ and a representation of elements with minimal length of in the flag subset $\widetilde{A}_n/A_n$. The reduced elements of these groups are indexing the basis elements of the cohomology algebra of LSU_n/T and ΩSU_n. After this classification, the calculations in the integral cohomology algebra of LSU_n/T and ΩSU_n are possible using cup product algorithms introduced by Özel & Yılmaz in [2].

Some comments about the structure of this thesis are in order. It is written for a reader with a first course in algebraic geometry and some understanding of the structure of Coxeter groups and geometric representation, and applied algebra techniques such as Gröbner bases and Gröbner- Shirshov bases for commutative and free algebras, plus symbolic computation and some mathematical maturity.

Some good general references are Hiller [3] for structure of Coxeter groups and their geometric representation, Cox & Little & O'Shea [11] for basic facts of algebraic geometry and Gröbner bases, Bokut & Shiao [1] for Gröbner-Shirshov bases for free algebras.

Finally we suggest that reduced Gröbner-Shirshov bases for other affine Weyl groups $\widetilde{B}_n, \widetilde{D}_n$ can be calculated.

The organization of this thesis is as follows.

In Chapter 2, we describe Coxeter groups and graphs. We classify all Coxeter groups with respect to bilinear forms on the Coxeter graphs.

In Chapter 3, all details and techniques about finding Gröbner and Gröbner-Shirshov bases for commutative and free algebras are included.

In Chapter 4, we introduce the hypothesis given by Bokut & Shiao and we give a counter example for this hypothesis.

In Chapter 5, we calculate reduced Gröbner-Shirshov basis of affine Weyl group $\widetilde{A}_n$ and classify reduced elements of affine Weyl group $\widetilde{A}_n$.

CHAPTER 2

REFLECTION GROUPS AND COXETER SYSTEMS

For this chapter the fundamental reference is [3].

Definition 2.1 (*Coxeter Systems*) *A group W is called a* Coxeter Group *if there is a subset S of W such that W has the following presentation*

$$< s : s \in S \mid (ss')^{m_{ss'}} = 1 >$$

where $m_{ss'} \in \{2, 3, 4, 5, ...\}$ is the order of ss', $s \neq s'$ and $m_{ss} = 1$. The pair (W, S) is called a Coxeter System*.*

If S is a finite subset, then we say that (W, S) has rank l, we write $S = \{s_1, s_2, ..., s_l\}$, and m_{ij} for $m_{s_i s_j}$.

We say that a Coxeter group W is irreducible if it cannot be written as a product of nontrivial Coxeter subgroups.

We can associate a Coxeter system by a matrix or a graph.

Definition 2.2 (*Coxeter Matrix*) *A* Coxeter matrix *$M = m_{ij}$ is a symmetric matrix with $1'$s along the diagonal and $m_{ij} \in \{2, 3, 4, 5, ...\}$, $i \neq j$.*

Definition 2.3 (*Coxeter Graph*) *The* Coxeter graph *of a Coxeter system (W, S) is a graph Γ_W , with one vertex for each $s \in S$ and an edge from s_i to s_j if $m_{ij} > 2$, which is labelled by m_{ij}.*

From [4] we have the following result.

Proposition 2.4 *If*

$$W = \prod_{\alpha} W_{\alpha},$$

then its Coxeter graph Γ_W *is the disjoint union*

$$\coprod_{\alpha} \Gamma_{W_\alpha}.$$

In particular, a Coxeter group W *is irreducible if and only if* Γ_W *is connected.*

Coxeter groups have a rich internal structure. Now we will discuss the *length* function on Coxeter groups.

Definition 2.5 (*Length Function and Reduced Decomposition*) *If* (W, S) *is a Coxeter system,* length *of* $w \in W$ *is the smallest integer* $n \geq 0$ *such that* w *can be written as a product of* n *elements from* S*. It is denoted by* $\ell(w)$*. If* $w = s_1 s_2 \ldots s_n$ *and* $\ell(w) = n$ *we call this expression a* reduced decomposition *of* w*.*

We have some facts about the length function as follows:

Lemma 2.6 *If* $w, w' \in W$*, then*

$$\ell(w^{-1}) = \ell(w) \tag{2.1}$$

$$\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w') \tag{2.2}$$

As an example, we can consider the symmetric group $\sum_n$ of permutations of n letters. It is a Coxeter system when

$$S = \{(i\; i+1) : 1 \leq i < n\}$$

is chosen as the set of elementary transpositions in the symmetric group.

If $\tau \in \sum_n$, we can define the number of inversion in τ as

$$e_j(\tau) = |\{i > j : \tau(i) < \tau(j)\}|,$$

where $| \quad |$ denotes cardinality. Then the length of τ is equal to

$$\ell(\tau) = \sum_{j=1}^{n-1} e_j(\tau).$$

The Coxeter graph of (Σ_n, S) is

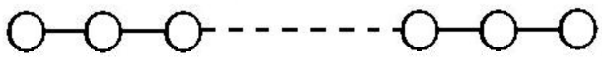


Figure 2.1:

where the ith vortex corresponds to the transposition $(i, i+1)$, $1 \leq i < n$.

As another example we can consider the group $\Theta(n)$ of $n \times n$ signed permutation matrices. It is isomorphic to a semi-direct product $\Sigma_n \ltimes {\mathbb{Z}_2}^n$ where Σ_n acts on $\mathbb{Z}_2^n$ in the obvious way. We define $s_i = (i\, i+1)$, $1 \leq i < n$ and s_n changes the sign of the last coordinate. Let $S =< s_1, ..., s_n >$. Then we get a Coxeter system $(\Theta(n), S)$. We call this group the *hyperoctahedral* group since it is the symmetry group of an n-dimensional octahedron (or cube). The Coxeter graph has n vortexes.

The symmetric group $\sum_n$ has a signature defined by $\sigma(\tau) = (-1)^{\ell(\tau)}$. It is the determinant of τ thought of as a permutation matrix. This generalizes to any Coxeter system.

Lemma 2.7 *If (W, S) is a Coxeter system then there is a signature*

$$\sigma : W \to \mathbb{Z}_2$$

induced from the map $s \to (-1)$, $s \in S$.

Corollary 2.8 *If $w, w' \in W$, then $\ell(ww') \equiv \ell(w) + \ell(w')$ $\mathrm{Mod}\, 2$.*

Corollary 2.9 *If $w \in W$, $s \in S$, then $\ell(ws) = \ell(w) \pm 1$ and $\ell(sw) = \ell(w) \pm 1$.*

The another important property of the length function on Coxeter systems is the *exchange theorem* which will be introduced in the section of root systems.

Now we will give an important classification result.

Theorem 2.10 *If* (W,S) *is a Coxeter system of* rank 2 *then either* W *is a finite dihedral group* D_m *of order* $2m$ *or the infinite dihedral group* D_∞.

Proof. Suppose $S = \{s, s'\}$, $(ss')^m = 1$. If m is infinite then $W = \mathbb{Z}_2 * \mathbb{Z}_2$ where $*$ denotes free product of groups. If m is finite, let $A = ss'$,$B = s$. Then $< A > \cong \mathbb{Z}/m\mathbb{Z}$ is a normal subgroup of W since

$$s(ss')^j s^{-1} = (ss')^{-j}$$

and A and B generate W. Hence $< B > \cdot < A >= W$ and the result readily follows. □

The group D_m is the symmetry group of a regular m-gon. The element A is the rotation through $\frac{2\Pi}{m}$ and B is an appropriate reflection.

We formalize this notion.

If V is a vector space over $\mathbb{R}$ equipped with a symmetric bilinear form $(,)$, and $\alpha \in V$, $(\alpha, \alpha) \neq 0$ then the reflection s_α with respect to α is the unique linear transformation fixing the hyperplane $(x, \alpha) = 0$ and sending α to $-\alpha$.

It is not difficult to check that s_α is given by the formula

$$s_\alpha(x) = x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha.$$

We can easily check that if φ is an automorphism of V preserving the bilinear form on V, then

$$\varphi s_\alpha \varphi^{-1} = s_{\varphi(\alpha)}.$$

We would like to geometrically realize other Coxeter groups as groups generated by reflections. We begin by writing $V = \mathbb{R}^l$, a real vector space of dimension $l = |S|$. Let $e_1, ..., e_l$ be the standard basis of $\mathbb{R}^l$ and we want to define a symmetric bilinear form $(,)$ on $\mathbb{R}^l$ which will ensure that the reflections through the hyperplane perpendicular to e_i satisfy the Coxeter relations. This cannot always be done in a non-degenerate way. In order to motivate the choice of $(,)$ we recall some elementary plane geometry.

Lemma 2.11 *Every orthogonal transformation φ of the Euclidean plane is a reflection or a rotation.*

Corollary 2.12 *If α, β are unit vectors in a Euclidean space V then the composition $s_\beta s_\alpha$ is a rotation through twice the angle between them.*

If two unit vectors α, β make an obtuse angle $\pi - \frac{\pi}{m}$, by Corollary 2.12 $s_\beta s_\alpha$ has finite order m. In addition $(\alpha, \beta) = \cos(\pi - \frac{\pi}{m}) = -\cos(\frac{\pi}{m})$.

Definition 2.13 *If (W, S) is a Coxeter system of* rank l*, then the canonical bilinear form $B(,)$ on $V = \mathbb{R}^l$ is given by*

$$B(e_{s_i}, e_{s_j}) = \begin{cases} -\cos(\frac{\pi}{m_{ij}}) & m_{ij} < \infty \\ -1 & \textit{otherwise.} \end{cases}$$

We can use this form to define the canonical representation ρ of W over $V = \mathbb{R}^l$. Then we have

$$\rho(s)(x) = x - 2B(x, e_s)e_s$$

for $s \in S$, $x \in V = \mathbb{R}^l$.

Lemma 2.14 *For any $s \in S$, $\rho(s)$ is a reflection that preserves the bilinear form $B(,)$.*

Proof. We compute

$$\begin{aligned} B\left(\rho(s)(x), \rho(s)(y)\right) &= B\left(x - 2B(x, e_s)e_s, y - 2B(y, e_s)e_s\right) \\ &= B(x, y) - 4B(x, e_s)B(y, e_s) + 4B(x, e_s)B(y, e_s) \\ &= B(x, y). \end{aligned}$$

□

In order to check we really get a representation of W, we need the following facts.

Lemma 2.15 *The restriction of $B(,)$ to a 2-dimensional subspace $\mathbb{R}e_s + \mathbb{R}e_{s'}$ is positive. In addition, it is positive-definite if and only if $m_{ss'} < \infty$.*

Lemma 2.16 *For $s, s' \in S$, $(\rho(s)\rho(s'))^{m_{ss'}} = I_V$.*

Let $O(B)$ be the subgroup of the general linear group $GL(V)$ of non-singular linear operators of $V = \mathbb{R}^l$ preserving the symmetric bilinear form B. i.e.

$$O(B) = \{\psi \in GL(V) \,:\, B(\psi(x), \psi(y)) = B(x, y),\, \forall x, y \in V\}.$$

Proposition 2.17 *If (W, S) is a Coxeter system and $B(,)$ is the canonical symmetric bilinear form, then there exists a unique representation $\rho : W \to O(B)$ satisfying the reflection equality $\rho(s)(x) = x - 2B(x, e_s)e_s$, $\forall s \in S, x \in V$.*

In the next section we will show that ρ is faithful(one to one) representation and $\rho(W)$ is discrete in $O(B)$.

Corollary 2.18 *Any Coxeter matrix is the Coxeter matrix of a Coxeter system.*

Lemma 2.19 *Suppose (W, S) is irreducible. If E is an invariant i.e. $\rho(W)(E) = E$, proper subspace of V then $E \subset V^{\perp} = \{x \in V : B(x, y) = 0,\ \forall y \in V\}$.*

Proof. First we prove $e_s \notin E$, for all $s \in S$. Let $S' = \{s \in S : e_s \in E\}$ and $S'' = S - S'$. Since W is irreducible there exist e_s, e_t with $s \in S'$, $t \in S''$ and $B(e_s, e_t) \neq 0$. Then $\rho(t)(e_s) = e_s - 2B(e_s, e_t)e_t$, so that

$$2B(e_s, e_t)e_t = e_s - \rho(t)(e_s)$$

But E is invariant under W, so this element is in E. Hence $e_t \in E$ a contradiction.

Now suppose $x \in E$ is arbitrary. If $s \in S$ then

$$\rho(s)(x) = x - 2B(x, e_s)e_s.$$

It can be written as

$$2B(x, e_s)e_s = x - \rho(s)x \in E$$

provided that $B(x, e_s) = 0$, for all $s \in S$; so $x \in V^{\perp}$ and the proof is complete. $\square$

Corollary 2.20 *Let (W, S) be an irreducible Coxeter system. Then*

(i) If $B(,)$ is degenerate, then ρ is not completely reducible.

(ii) If $B(,)$ is non-degenerate, then ρ is irreducible.

From Maschke Theorem we know that the real representations of finite groups are completely reducible. Using this fact we get the following result.

Corollary 2.21 *If (W, S) is irreducible and $B(,)$ is degenerate, then W is infinite.*

2.1 TITS' Theorem

In this section we prove that the canonical representation ρ of an arbitrary Coxeter system (W, S) is faithful and discrete.

Let V^* be the dual space of V and let ρ^* be the contragradient representation on V^* given by

$$\rho^*(w)(f)(x) = f(w^{-1}x), (w \in W), \quad \forall f \in V^*, x \in V.$$

The first step is to construct an open simplicial cone in V^* whose closure is a *fundamental domain* for the action of W. Let

$$H_s = \{x \in V^* : x(e_s) = 0\}, \quad \text{(hyperplane of s)}$$

$$A_s = \{x \in V^* : x(e_s) > 0\}, \quad \text{(positive half-space of } s\text{)}$$

so that $V^* = A_s \bigcup H_s \bigcup sA_s$ and the closure $\bar{A}_s = A_s \bigcup H_s$. We call

$$C = \bigcap_{s \in S} A_s,$$

the *fundamental chamber* of W. The translates wC, $w \in W$ are the chambers of W. Now we will give two lemmata which are used in the proof of the Tits Theorem.

Lemma 2.22 *Suppose $w \in W_{s,s'}$, where $W_{s,s'} \subset W$ is the dihedral group generated by distinct elements $s, s' \in S$. Then we have either*

(i) $w(A_s \bigcap A_{s'}) \subset A_s$ and $\ell(sw) = \ell(w) + 1$, or

(ii) $w(A_s \bigcap A_{s'}) \subset sA_s$ and $\ell(sw) = \ell(w) - 1$

Lemma 2.23 *If $w \in W$ and $s \in S$, then we have either*

(i) $wC \subseteq A_s$ and $\ell(sw) = \ell(w) + 1$, or

(ii) $wC \subseteq sA_s$ and $\ell(sw) = \ell(w) - 1$.

The main result is in the following.

Theorem 2.24 (*Tits Theorem*) *If (W, S) is a Coxeter system and $w \neq 1$, then $wC \bigcap C = \emptyset$.*

Proof. If $w \neq 1$, then there exists $w' \in W$ such that $w = sw'$ and $\ell(w) = \ell(w') + 1$. By Lemma 2.23, $w'C \subseteq A_s$. Hence $wC = sw'C \subseteq sA_s$ and since $C \subset A_s$, we have the desired result. Then $wC \cap C = \emptyset$. $\square$

So the Tits Theorem has some consequences.

Corollary 2.25 *The Coxeter group W acts simply transitively on the set of chambers $\{wC\}_{w \in W}$.*

Corollary 2.26 *The representations ρ^* and ρ are faithful.*

Corollary 2.27 *The subgroup $\rho^*(W)$ is a discrete subgroup of $GL(V^*)$.*

Proposition 2.28 *The Coxeter group W is finite iff $\bigcup_{w \in W} w\bar{C} = V^*$.*

2.2 Root Systems

Definition 2.29 *Let V be a vector space. A finite set of vectors $\triangle \subset V$ forms a* root system *in V if the following conditions hold.*

(i) If $\alpha, \beta \in \triangle$, then $s_\alpha(\beta) \in \triangle$.

(ii) If $k\alpha \in \triangle$ for $\alpha \in \triangle$ then $k = \pm 1$.

(iii) $\triangle$ generates V.

A base Σ for a root system $\triangle$ is a linearly independent subset of Σ satisfying the following condition: if $\gamma \in \triangle$ then γ is a linear combination of elements of Σ with all positive or all negative coefficients. From [5] we have the following fact.

Proposition 2.30 *Every root system $\triangle$ contains a base Σ.*

Given a pair $(\triangle, \Sigma)$ we get a partition of $\triangle = \triangle^+ \coprod \triangle^-$ into positive and negative roots, so that $\triangle^- = -\triangle^+$. We define *Weyl group* $W(\triangle)$ of $\triangle$ to be the group generated by the reflections s_β, $\beta \in \triangle$ and the *Weyl system* of $(\triangle, \Sigma)$ to be $(W(\triangle), S(\Sigma))$ where $S(\Sigma) = \{s_\alpha : \alpha \in \Sigma\}$. Since W acts faithfully on $\triangle$, W is necessarily finite. If (W, S) is a Coxeter system and $(\triangle, \Sigma)$ is a pair whose Weyl system is (W, S) we sometimes call it a geometric realization of (W, S). The first goal of this section is to understand the length function of W in terms of such a geometric realization.

Let Γ_w denote the set $\triangle^+ \bigcap w^{-1}\triangle^-$, i.e. the set of positive roots sent into negative roots by w and set $\gamma(w) = |\Gamma_w|$. We will show that $\gamma(w)$ provides a geometric description of the length function of W.

Lemma 2.31 *If $\alpha \in \Sigma$, then $\Gamma_{s_\alpha} = \{\alpha\}$, hence $\gamma(s_\alpha) = 1$.*

Lemma 2.32 *If $w \in W$, $\alpha \in \Sigma$ then*

(i) $\gamma(ws_\alpha) = \gamma(w) + 1$ iff $w(\alpha) \in \triangle^+$,

(ii) $\gamma(s_\alpha w) = \gamma(w) + 1$ iff $w^{-1}(\alpha) \in \triangle^+$.

Theorem 2.33 *For all $w \in W$, $\ell(w) = \gamma(w)$.*

Theorem 2.34 *The Weyl system of* $(\triangle, \Sigma)$ *is a Coxeter system.*

Proof. Suppose

$$\begin{aligned} & s_1 ... s_k = 1, \\ & s_i = s_{\alpha_i}, \\ & \alpha_i \in \Sigma, \end{aligned} \tag{2.3}$$

is a relation of minimal length in W that does not follow from $(s_i s_j)^{m_{ij}} = 1$ where m_{ij} is the order of $(s_i s_j)$. Clearly, $s_1 ... s_k = (-1)^k$, so k is even. Let $k = 2m$ and let us consider

$$s_1 ... s_{m+1} = s_{2m} ... s_{m+2}$$

so that $\ell(s_1 ... s_{m+1}) < m + 1$. Hence, there exist $1 \leq i \leq j \leq m$ such that

$$s_{i+1} ... s_{j+1} = s_i ... s_j$$

By considering the substitution of this into (2.3), one deduces that $2m = 2(j - i + l)$ otherwise one could contradict the minimality of (2.3). Since $1 \leq i \leq j \leq m$, we have $j = m$, $i = 1$, $s_2 ... s_m(\alpha_{m+1})$.

The relation (2.3) is equivalent to $s_2 ... s_k s_1 = 1$. Hence, repeating the argument above gives

$$s_3 ... s_{m+1}(\alpha_{m+2}) = \alpha_2$$

$$s_3 ... s_{m+1} s_{m+2} s_{m+1} ... s_3 = s_2$$

or

$$s_3 s_2 s_3 ... s_{m+1} s_{m+2} s_{m+1} ... s_4 = 1.$$

This is also a relation of length $2m$ that cannot be deduced from the Coxeter relations. So mimicking the argument above gives $s_2 ... s_m(\alpha(m+1)) = \alpha_3$. Hence

$\alpha_1 = \alpha_3$ and continuing in this fashion we get

$$s_1 = s_3 = ... = s_{2m-1} \quad \text{and} \quad s_2 = s_4 = ... = s_{2m}.$$

Hence (2.3) is a consequence of $(s_1 s_2)^{m_{12}} = 1$ and the proof is complete. $\square$

It's easy to see that a base Σ for $\triangle$ determines a fundamental chamber C_Σ. Actually there are bijections between

$$\{\text{bases for} \triangle\} \leftrightarrow \{\text{chambers}\} \leftrightarrow W$$

where the last one is 2.25. In particular, there exists a unique element $w_0 \in W$ such that $w_0(\Sigma) = -\Sigma$, so that $w_0(\triangle^+) = \triangle^-$. We call this element the *longest word* of W since $\ell(w_0) = \gamma(w_0) = |\triangle^+|$.

Proposition 2.35 *If $w \in W$, $\ell(ww_0) = \ell(w_0) - \ell(w) = \ell(w_0 w)$.*

Now we will give a result about Γ_w.

Proposition 2.36 *Let $\theta_i = s_1 ... s_{i-1}(\alpha_i),\ 1 \leq i \leq k$, and $w = s_1 ... s_k,\ s_i = s_{\alpha_i}$, be a reduced decomposition of w, then the following sets are equal.*

(i) $\Gamma_{w^{-1}} = \triangle^+ \bigcap w\triangle^-$.

(ii) $\{\theta_i : 1 \leq i \leq k\}$.

(iii) $\{\gamma \in \triangle^+ : s_\gamma w_i = w \text{ for some } i\}$ *where* $w_i = s_1 ... \widehat{s_i} ... s_k$ *and* $\wedge$ *denotes deletion.*

Theorem 2.37 (*Matsumoto's Exchange Condition*) *If (W, S) is a finite Coxeter group, $w = s_1 .... s_q$, $(s_i = s_{\alpha_i})$ and $\ell(sw) < \ell(w)$, $s = s_\alpha \in S$. Then there exists a j, $1 \leq j \leq q$, so that*

$$ss_1 ... s_j = s_1 ... s_{j-1} \quad \text{or} \quad sw = s_1 ... \widehat{s_j} ... s_q$$

Furthermore, if the decomposition is reduced then j is unique.

Definition 2.38 *A root $\alpha \in \triangle$ is called a* simple root *if $\alpha \in \Sigma$.*

Definition 2.39 *A Coxeter group W is called a Weyl group if $\mathbb{Z}$-lattice generated by the simple roots is invariant under the action of W.*

We know that a reflection is given by

$$s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha.$$

So if $\alpha, \beta \in \Sigma$, we must have the condition

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}. \tag{2.4}$$

Reversing the roles of α and β and multiplying we get the condition

$$4\cos^2\frac{\pi}{m_{\alpha\beta}} = \frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} \in \mathbb{Z}.$$

It implies that $m_{\alpha\beta} \in \{2,3,4,6\}$. There is also a more direct geometric argument due to Barlow [6].

If we define the co-root of β by

$$\beta^V = \frac{2\beta}{(\beta,\beta)}$$

then we can rewrite the crystallographic condition 2.4 as

$$(\alpha, \beta^V) \in \mathbb{Z}$$

for $\alpha, \beta \in \triangle$. We can define the *dual basis* $\{\omega_\alpha\}_{\alpha\in\Sigma}$ given by the following condition

$$(\omega_\alpha, \beta^V) = \delta_{\alpha\beta} \qquad \alpha, \beta \in \Sigma.$$

The ω_α 's are called *fundamental weights.* We see that the fundamental chamber determined by simple roots in Σ is precisely the positive cone determined by the fundamental weights ω_α.

We also observe that if $m_{\alpha\beta}$ is odd then s_α and s_β are conjugate,so α and β must have the same length. In particular the two simple roots for D_m, m odd, must have the same length.

Let us recall the geometric realization of Σ_n. Let V be a real vector space with standard basis $e_1, ..., e_n$. We let $\triangle = \{e_i - e_j : 1 \leq i \neq j \leq n\}$ and $\Sigma = \{\alpha_i = e_i - e_{i+1} : 1 \leq i < n\}$. Clearly, Σ is a basis for $\overline{V} = \{\Sigma x_i e_i : \Sigma x_i = 0\}$ of dimension $n-1$. Since $s_{e_i - e_j}$ acts by permuting coordinates, $W(\triangle)$ preserves $\overline{V}$. Here is a picture of $\triangle$ for Σ_3.

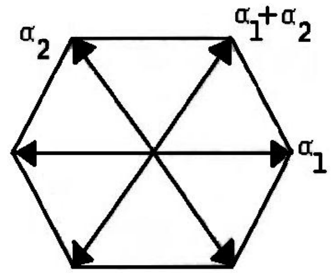


Figure 2.2:

Finally, we briefly indicate how root systems arise in the structure theory for finite simple Lie algebras $\mathfrak{g}$ over $\mathbb{C}$. Let $\mathfrak{h}$ be a maximal abelian *Cartan* subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ acts on $\mathfrak{g}$ by the adjoint representation ad : $\mathrm{ad}(h)(x) = [h, x]$ where $[,]$ denotes commutator operation in Lie algebra. The eigenspace of the adjoint action associated with the zero is $\mathfrak{h}$.

Definition 2.40 *Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. If $\alpha \in \mathfrak{h}^*$ gives a non-zero eigenspace $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \mathrm{ad}(h)(x) = \alpha_(h)x \ \forall h \in \mathfrak{h}\}$ we call α a* root.

So there is a Cartan decomposition of the Lie algebra as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle} \mathfrak{g}_\alpha.$$

The set of roots is clearly finite and actually forms a root system in $\mathfrak{h}^*$.

Indeed, according to a theorem of Serre a simple Lie algebra can be constructed from a given root system. This is how the Cartan-Killing classification of complex simple Lie algebra works. We discuss such classification results in the next section. For more details on the structure of Lie algebras, the best reference is [7].

2.3 Forms

Definition 2.41 *If* $(\ ,\ ) : V \times V \to R$ *is a bilinear form on* V*, then* $(\ ,\ )$ *is* symmetric *if* $(x, y) = (y, x)$ *for all* $x, y \in V$*.*

The form $(\ ,\ )$ *is* positive (semi-definite) *if* $(x, x) \geq 0$*, for all* $x \in V$ *and* positive definite *if it is positive and* $(x, x) = 0$ *implies* $x = 0$*.*

The form $(\ ,\ )$ *is* non-degenerate *or* (non-singular) *if* $(x, y) = 0$ *for all* $y \in V$*, implies* $x = 0$*.*

Since $Hom(V \otimes V, \mathbb{R}) \cong Hom(V, V^*)$, by Riesz representation theorem, a form is equivalent to a map $V \to V^*$ given by $x \to (x, \cdot)$.

Clearly, a form $(\ ,\ )$ is non-degenerate if and only if the associated map is an isomorphism.

The map $q(x) = (x, x)$ is called the *quadratic form associated* to $(\ ,\ )$. If $(\ ,\ )$ is symmetric, then the quadratic form determines it via the *polarization identity*

$$(\alpha, \beta) = \frac{1}{4}\, q(\alpha + \beta) - \frac{1}{4}\, q(\alpha - \beta).$$

Suppose $\{e_1, e_2, ..., e_n\}$ is a basis for V. Then the matrix representation of $(\ ,\ )$ is a real $n \times n$ matrix with entries $q_{i,j} = (e_i, e_j)$. If $x, y \in V$ have their coordinates arranged in column vectors X, Y respectively, then

$$(x, y) = X^t \cdot B \cdot Y,$$

where $\cdot$ denotes the matrix multiplication. Similarly,

$$q(x) = X^t \cdot B \cdot X = \sum_{i,j} q_{ij} X_i X_j.$$

Lemma 2.42 *If $(\, , \,)$ is positive and $(x,x) = 0$, then $(x,y) = 0$ for all $y \in V$.*

Corollary 2.43 *A form $(\, , \,)$ is positive definite iff it is positive and non-degenerate.*

In our work on the affine Weyl Groups we need a result on the kernel of the quadratic form $q(x)$. First we have

Lemma 2.44 *If $\{a_1, a_2, ..., a_k\} \subseteq V$, $(\, , \,)$ is a positive form and $(a_i, a_j) \leq 0$ for all $i \neq j$, then $q(\sum_{i=1}^{k} c_i a_i) = 0$ implies $q(\sum_{i=1}^{k} |c_i| a_i) = 0$.*

Definition 2.45 *We will say that a form is* irreducible *if V has no a non-trivial decomposition $V' \oplus V''$ with $(v', v'') = 0$, for all $v' \in V'$, $v'' \in V''$.*

For example, the canonical bilinear form on representation of an irreducible Coxeter group is irreducible.

Corollary 2.46 *Suppose that $(\, , \,)$ is a positive, irreducible symmetric bilinear form on V, and $\{e_1, e_2, ..., e_k\}$ is a basis for V such that $q_{ij} = (e_i, e_j) \leq 0$, $i \neq j$. Then $Ker(q)$ is of dimension 0 or 1. If $dim(Ker(q)) = 1$, then $Ker(q)$ is generated by a vector with positive coordinates.*

Proof. Suppose q is not positive definite and $0 \neq \sum_{i=1}^{n} c_i e_i \in Ker(q)$. Then by Lemma 2.44, $\sum_{i=1}^{n} |c_i| e_i \in Ker(q)$ so that $\sum_{i=1}^{n} |c_i| q_{ij} = 0$, for all j.

Let $I = \{i : c_i \neq 0\}$. Since

$$0 = \sum_{i=1}^{n} |c_i| q_{ij} = \sum_{i \in I} |c_i| q_{ij} \quad \text{if} \quad j \in I,$$

so that $q_{ij} \leq 0$, for all $i \in I$, we must have $q_{ij} = 0$. By irreducibility of $(\, , \,)$, we can conclude $I = \{1, 2, ..., n\}$. We claim now $Ker(q)$ is one-dimensional. Otherwise, $dim(Ker(q) \bigcap \{x_i = 0\}) \geq 1$, contradicting the above argument that all coordinates must be non-zero. Now the last assertion is also clear. $\square$

Corollary 2.47 *If $(\, , \,)$ is a positive, irreducible symmetric bilinear form on V, and $\{e_1, e_2, ..., e_k\}$ is a basis for V such that $q_{ij} = (e_i, e_j) \leq 0$, then for any k, the form restricted to $\{e_i\}_{i \neq k}$ is positive definite.*

In the classification result we exploited a criterion for the positive semi-definiteness of a form.

Proposition 2.48 *If $(\ ,\ )$ is a positive, irreducible symmetric bilinear form on V, and $\{e_1, e_2, ..., e_k\}$ is a basis for V such that $q_{ij} = (e_i, e_j) \leq 0$, then $q(x) = (x, x) = \sum_{i,j} q_{ij} x_i x_j$ is positive degenerate iff there exist positive numbers $\{z_1, z_2, ..., z_n\}$ such that*

$$\sum_i z_i q_{ij} = 0 \quad 1 \leq j \leq n.$$

Proof. The necessity of the condition follows from Lemma 2.44. From [6] we have an identity

$$\sum_{i,j} q_{ij} X_i X_j = -\frac{1}{2} \sum_{i,j} z_i z_j q_{ij} \left(\frac{X_i}{z_i} - \frac{X_j}{z_j}\right)^2.$$

The right-hand side is ≥ 0, since $q_{ij} \leq 0$ but vanishes at $X_i = z_i$. Hence q is positive degenerate. □

Finally, we have a result which will be used in the proof of Witt's Theorem.

Proposition 2.49 *If W is a group acting irreducibly on a real vector space V and containing a reflection s_α then any two positive definite W-invariant forms are equivalent.*

2.4 Classification of Finite Weyl Groups

In this section we will classify root systems and Coxeter groups using some graph theory and quadratic forms discussed in the last section. We begin with a result of Witt from [8].

Theorem 2.50 *Let W be a Coxeter group. Then W is finite if and only if the canonical bilinear form $B(,)$ is positive definite.*

The Witt's Theorem tells us that the problem of finding all irreducible Coxeter groups is equivalent to the combinatorial problem of enumerating connected Coxeter graphs with positive definite canonical forms B_Γ.

Definition 2.51 *A Coxeter graph is called* simply-laced *if all the edges are labelled by* 3. *In other words* $s_i s_j s_i = s_j s_i s_j$

We note that all edges in a simply-laced Coxeter graph will be unlabelled.

Here first our aim is to classify simply-laced Coxeter graphs Γ with quadratic form q_Γ positive and degenerate. Actually this kind of Coxeter graphs classifies Affine Weyl groups which will be discussed in the next section.

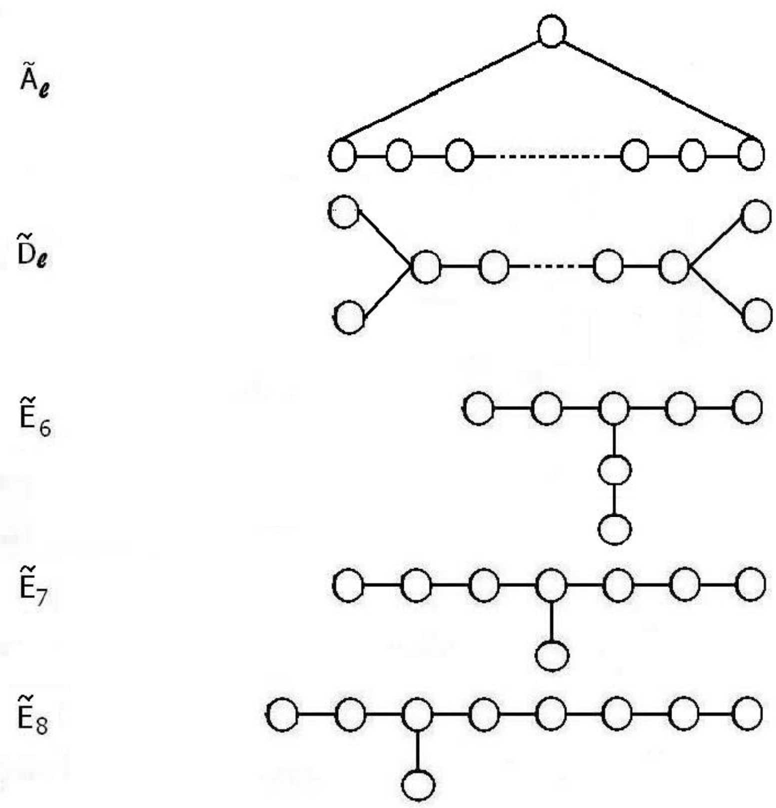
Now we need some basic definitions on the graph theory.

Definition 2.52 *We write* $\Gamma < \Gamma'$ *if* Γ *is isomorphic to a proper subgraph of* Γ'.

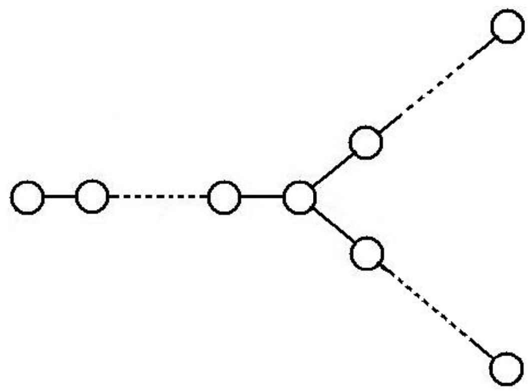
Definition 2.53 *Two graphs* Γ, Γ' *are said to be* comparable *if* $\Gamma < \Gamma'$ *or* $\Gamma = \Gamma'$ *or* $\Gamma > \Gamma'$.

Definition 2.54 *Let* $\mathcal{S}$ *and* $\mathcal{L}$ *be two classes of graphs. We say that* $\mathcal{S}$ *separates* $\mathcal{L}$ *if every graph* $\Gamma \in \mathcal{L}$ *is comparable to some graph in* $\mathcal{S}$*. Then we have*

Theorem 2.55 *The following graphs* $\widetilde{A} - \widetilde{D} - \widetilde{E}$ *separate all connected graphs.*



Proof. Let Γ be a connected graph. If Γ contains a cycle then $\Gamma \geq \widetilde{A}_l$, for some $l \geq 2$. So we can assume Γ is a tree. If there is a vertex of degree ≥ 4 where degree of a vertex is equal to the number of edges incident at the vertex, then $\Gamma \geq \widetilde{D}_4$. Now we can assume each vertex has degree ≤ 3. If there is more than one vertex of degree 3, then $\Gamma \geq \widetilde{D}_n$ for some $n \geq 5$. Finally, we can assume Γ has exactly one vertex of degree 3. Suppose Γ has rays of length $r \geq q \geq p \geq 2$ such that counting from the vertex with degree 3.



If $p = q = 2$, then $\Gamma < \widetilde{D}_l$. If $p = 2$, $q = 3$, then Γ is comparable to $\widetilde{E}_8$. If $p = 2$, $q = 4$, then Γ is comparable to $\widetilde{E}_7$. If $p = 2$, $r \geq q \geq 5$, then $\Gamma > \widetilde{E}_7$. Finally, if $p \geq 3$, $\Gamma \geq \widetilde{E}_6$ and we have shown our original Γ is comparable to an $\widetilde{A}$-$\widetilde{D}$- $\widetilde{E}$ graph. $\square$

The following result tells us why the graphs $\widetilde{A} - \widetilde{D} - \widetilde{E}$ are needed.

Lemma 2.56

(a) Any connected simply-laced Coxeter graph properly containing one of the graphs $\widetilde{A} - \widetilde{D} - \widetilde{E}$ is not positive.

(b) Any connected simply-laced Coxeter graph properly contained in one of the graphs $\widetilde{A} - \widetilde{D} - \widetilde{E}$ is non-degenerate.

Proof. Let Γ be a connected simply-laced Coxeter graph properly containing one of the graphs $\widetilde{A} - \widetilde{D} - \widetilde{E}$ and let Γ_0 be a proper subgraph of type $\widetilde{A} - \widetilde{D} - \widetilde{E}$.

Then we can choose an edge as in Figure 1 from p to q such that $p \in \Gamma_0, q \notin \Gamma_0$. Let $X = (X_i)_{i \in \Gamma_0}$ be a vector of real numbers such that $q_{\Gamma_0}(X) = 0$. Then let us define $Y = (Y_i)_{i \in \Gamma_0}$ by

$$Y_i = \begin{cases} X_i & \text{if } i \in \Gamma_0, \\ \pm 2X_p & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

Then

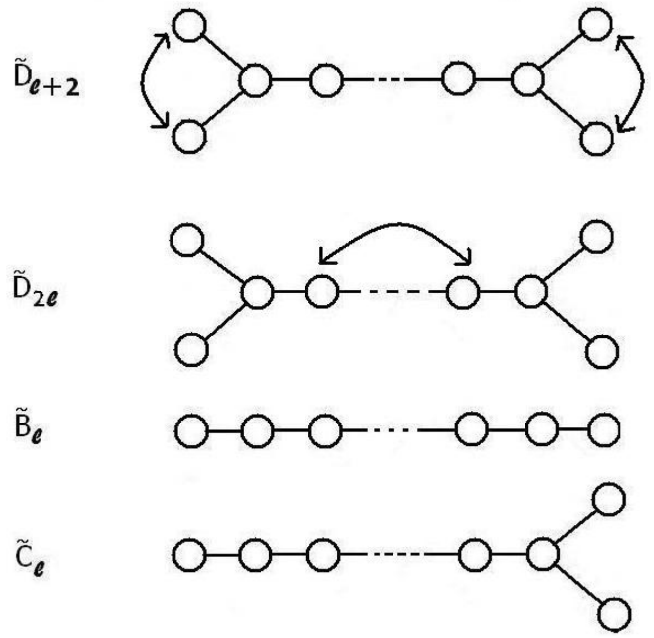
$$\begin{aligned} q_\Gamma(Y) &\leq q_{\Gamma_0}(X) - Y_p X_q + {Y_q}^2 \\ &= -2{X_p}^2 + {X_p}^2 \\ &= -{X_p}^2 < 0. \end{aligned}$$

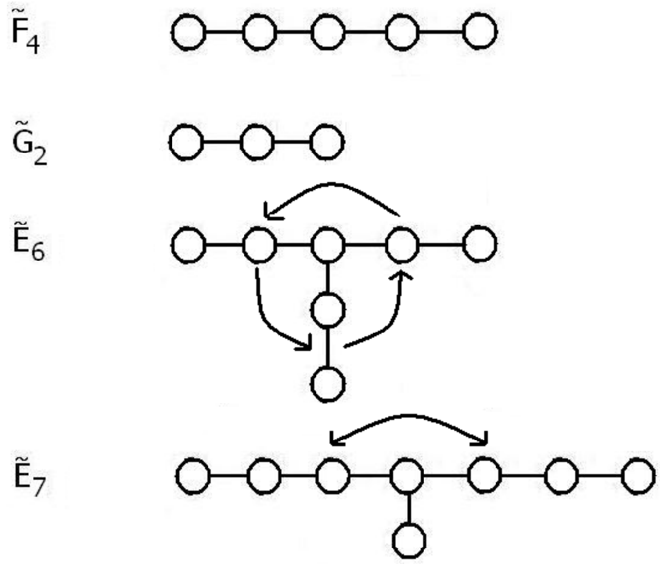
So q_Γ is not positive. The second part of the claim comes from Corollary 2.47. □

By Theorem 2.55 and Lemma 2.56 we have the following result.

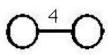
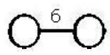
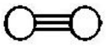
Theorem 2.57 *The only connected, simply-laced, positive degenerate Coxeter graphs are* $\widetilde{A} - \widetilde{D} - \widetilde{E}$.

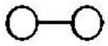
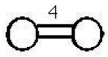
The graphs in Theorem 2.55 admit automorphisms. We are interested in symmetries of the graphs with at least one fixed point. We can use these symmetries to fold the graph introducing multiple edges. Here are some examples:





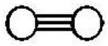
We have also an analogue of Theorem 2.55.

Let W be the set of connected graphs with ≤ 3 edges between any two vortexes. We call these *Weyl graphs.* Let us replace the edges ,  by ,  respectively.

We see that  is a subgraph of .

Now we give an analogue of Theorem 2.55.

Theorem 2.58 *The class of graphs $\widetilde{A}_l$, $\widetilde{B}_l$, $\widetilde{C}_l$, $\widetilde{D}_l$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, $\widetilde{G}_2$ separate W.*

Proof. Let $\Gamma \in W$. If Γ contains a  it is comparable to $\widetilde{G}_2$. By Theorem 2.55, we can assume that Γ has an edge .

Suppose it is a terminal edge. If there is a vortex with degree 3, then $\Gamma \geq \widetilde{C}_l$. Otherwise, Γ is comparable with $\widetilde{B}_l$. If  is not terminal, then Γ is comparable with $\widetilde{F}_4$. $\Box$

The analogue of Lemma 2.56 can be given as follows.

Lemma 2.59

(a) Any graph in W properly containing one of the graphs $\widetilde{A}_l$, $\widetilde{B}_l$, $\widetilde{C}_l$, $\widetilde{D}_l$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, $\widetilde{G}_2$ is not positive.

(b) Any graph in W properly contained in one of the graphs $\widetilde{A}_l$, $\widetilde{B}_l$, $\widetilde{C}_l$, $\widetilde{D}_l$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, $\widetilde{G}_2$ is non-degenerate.

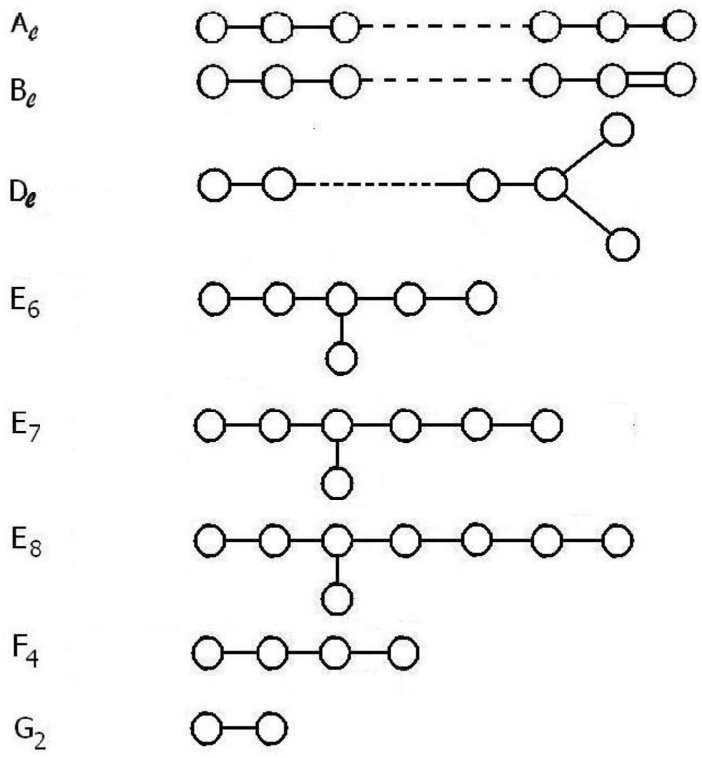
By Proposition 2.48, Theorem 2.58 and Lemma 2.59, we have the following result.

Theorem 2.60 *The only connected positive degenerate Weyl graphs are* $\widetilde{A} - \widetilde{G}$.

The last theorem gives the standard classification of affine Weyl groups which will be discussed in the next section.

The following theorem gives us the standard classification of finite Weyl groups.

Theorem 2.61 *The only connected positive definite Weyl graphs (or Weyl groups) are as follows.*



Proof. These are precisely the connected graphs obtained from $\widetilde{A} - \widetilde{G}$ by removing one vertex. Hence they are positive definite, by Lemma 2.59. Using Theorem 2.58, we have the desired result. □

2.5 Affine Weyl Groups

We have exploited the class of positive degenerate Coxeter groups to get a classification of finite Coxeter groups. Actually these groups have an important geometric interpretation in their own right.

Using Corollary 2.46, we have the following result.

Proposition 2.62 *If W is an irreducible Coxeter group and $B(,)$ on V is a positive degenerate form then the subspace $V^{\perp}$ is a line and is generated by a vector $c = \sum c_s e_s$ with $c_s > 0$ for all $s \in S$.*

Let

$$V^{\perp} = < c > \quad \text{with} \quad c = \sum c_s e_s \quad \text{such that} \quad \sum_{s \in S} c_s = 1,\, c_s > 0.$$

Also let

$$A = \{x \in V^* : x(c) = 1\} \quad \text{and} \quad T = \{x \in V^* : x(c) = 0\}.$$

We see that A is an affine space with space of translations T. The annihilator space $T = (V/V^{\perp})^*$ has a positive definite form from $B(,)$. By Lemma 2.19, A admits an action of W since $wv = v$ for all $v \in V^{\perp}\, w \in W$. Hence a group satisfying the hypotheses of Proposition 2.62 with rank $l+1$ can be viewed as an affine group of motions on a Euclidean affine space of dimension l.

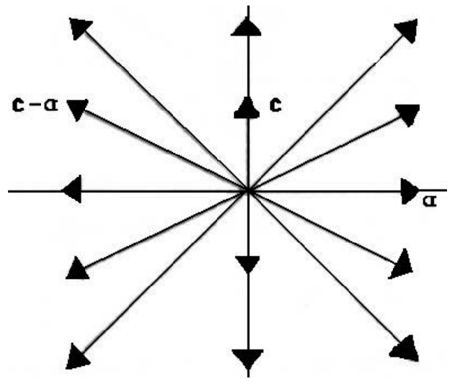
This suggests realization of affine Weyl group $\widetilde{\phi}$ as a group of motions in the representation space of the canonical representation of the Weyl group ϕ which is classified by graphs given in Theorem 2.61.

This process is as follows.

Let α_0 denote the highest root of representation of ϕ. To the new vortex in the graph of $\widetilde{\phi}$ we associate the affine reflection $s_0 = s_{(\alpha_0,1)}$ through the hyperplane $\{x \in V : (x, \alpha_0) = 1\}$. Then $\{s_0, s_1, ..., s_l\}$ generates $\widetilde{\phi}$. If $Q^{\vee} = \sum_{i=1}^{l} \mathbb{Z}{\alpha_i}^{\vee}$ is a $\mathbb{Z}$-lattice of co-roots of representation, then there is semi-direct product decomposition $\widetilde{\phi} = \phi \ltimes Q^{\vee}$.

This also suggests how to construct a root system for $\widetilde{\phi}$.

Let $\Sigma = \{c - \alpha_0, \alpha_1, ..., \alpha_l\}$ in V. Then Σ generates an infinite root system $\triangle$ in V. Not every root is a Weyl conjugation of a simple root in Σ ; e.g. integral multiples of nc are not roots. So they are called *imaginary roots*. Here is a picture of $\triangle$ for $\widetilde{A}_l$.



These root systems actually arise from the Cartan decomposition of *Kac-Moody Lie Algebras*. See [12],[13] and [14] for details.

CHAPTER 3

GRÖBNER-SHIRSHOV BASIS FOR FREE ALGEBRAS AND GROUPS

For this section the fundamental reference is [1].

We begin with definition of linearly ordered set.

Definition 3.1 (*A Linearly Ordered Set*) A linearly ordered set (or totally ordered set) *is a set plus a relation on the set (called a total order) that satisfies the conditions for a partial order plus an additional condition known as the comparability condition. A relation $\leq$ is a total order on a set S if the following properties hold.*

1. *Reflexivity: $a \leq a$ for all $a \in S$.*
2. *Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$.*
3. *Transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$.*
4. *Comparability (trichotomy law):For any $a, b \in S$ either $a \leq b$ and $b \leq a$.*

The first three are the axioms of a partial order, while addition of the trichotomy law defines a total order.

Every finite totally ordered set is well ordered. Any two totally ordered sets are order isomorphic, and therefore have the same order type.

Let S be a linearly ordered set, k is a field, $k\langle S\rangle$ is the formal free associative algebra over S and k where generators are the letters in the totally ordered set S.

Let S^* be the set of words which are obtained from letters in S performing free product.

Suppose that the words in S^* are degree-lexicographic ordered.i.e. comparing two words first by lengths and then lexicographically.

Definition 3.2 *Any polynomial $f \in k\langle S\rangle$ is a linear sum of words in S^*.*

In this section the words plays the crucial roles as like as the role of monomials in the last section.

Since any polynomial in $f \in k\langle S\rangle$ consists of finitely many words and the degree-lexicographic order is total, any polynomial $f \in k\langle S\rangle$ has a leading word.

Let $\bar{f}$ be the leading word of f.

Definition 3.3 *If $\bar{f}$ in f has a coefficient* 1 *then we say that f is called* monic.

Definition 3.4 (*Composition of Intersection*) *A composition of intersection of two monic polynomials relative to some word w is defined as $(f, g)_w = fb - ag$ whenever $w = \bar{f}b = a\bar{g}$, $deg(\bar{f}) + deg(\bar{g}) > deg(w)$.*

Definition 3.5 (*Composition of Including*) *A composition of including $(f, g)_w$ of two monic polynomials relative to some word w is defined as $(f, g)_w = f - agb$ whenever $w = \bar{f} = a\bar{g}b$.*

Definition 3.6 (*Elimination of the Leading Word*) *In the last case $(f, g)_w = f - agb$ is called the* elimination of the leading word (ELW) of g in f.

Definition 3.7 (*Trivial Relative*) *A composition $(f, g)_w$ is called* trivial relative *to some $R \subset k\langle X\rangle$ if $(f, g)_w = \sum \alpha_i a_i t_i b_i$, where $t_i \in R$, $a_i, b_i \in S^*$ and $a_i\bar{t}_i b_i < w$.*

In particular, if $(f, g)_w$ goes to zero by the ELW 's of R then $(f, g)_w$ is trivial relative to R.

Definition 3.8 (*Gröbner-Shirshov Bases*) *A* Gröbner-Shirshov basis *is a subset R of $k\langle S\rangle$ if any composition of polynomials from R is trivial relative to R.*

By the algebra $\langle S|R\rangle$ with generators S and defining relations R, we mean the *factor algebra* of $k\langle S\rangle$ by the ideal generated by R.

The following lemma goes back to the *Diamond Lemma* of [15] and the *Composition Lemma* of [16].

Lemma 3.9 (***Composition-Diamond Lemma***) *R is a Gröbner-Shirshov basis if and only if the set*

$$\{u \in S^* | u \neq a\bar{f}b, \ \forall f \in R\}$$

of R reduced words, consists of a linear basis of the algebra $\langle S|R\rangle$.

If a subset R of $k\langle S\rangle$ is not a Gröbner-Shirshov basis then we can add all non-trivial compositions of polynomials of R to R and by continuing this process (possibly infinitely) many times in order to get a Gröbner-Shirshov basis R^{comp} that contains R. This procedure is called *Buchberger-Shirshov Algorithm.* For the more details of this useful procedure, see [17].

Definition 3.10 (***Reduced Gröbner-Shirshov Basis***) *A Gröbner-Shirshov basis is called* reduced *if any $s \in S$ is a linear combination of $S \backslash \{\{s\}-$ reduced words $\}$.*

We note that any ideal of $k\langle S\rangle$ has a unique reduced Gröbner-Shirshov basis.

If R is a set of *semigroup relations* i.e.

$$\{u - v : \ u, v \in S^*\},$$

then any nontrivial composition will have the same form. As a result the set R^{comp} consists of semigroup relations too.

Let $A = \text{smg}\langle S|R\rangle$ be a semigroup presentation. Then R is a subset of $k\langle S\rangle$ and we can find a Gröbner-Shirshov basis R^{comp}. The last set doesn't depend on k, and it consist of semigroup relations. we will call $\mathbb{R}^{comp}$ a Gröbner-Shirshov basis of A. It is the same as Gröbner-Shirshov basis of the semigroup algebra $kA = \langle S|R\rangle$.

The same terminology is valid for any group presentation meaning that we include in this presentation all trivial group relations of the form $ss^{-1} = 1$, $s^{-1}s = 1$, $s \in S$.

In the next two chapters we will consider the calculation of Gröbner-Shirshov basis of Weyl groups. This will be the main focus of the thesis. In [1], Bokut &

Shiao gave the Groebner-Shirshov basis of positive definite classical Weyl groups A_l, B_l, D_l using the techniques of *Composition Diamond Lemma* and *Elimination of Leading Word*. In the first next chapter we will verify that the Gröbner-Shirshov bases of Weyl groups A_l, B_l, D_l are right using the *Composition of Intersection*, *Composition of Including* and definition of Gröbner-Shirshov basis. In the last two chapters of the thesis we will give a counter-example to a hypothesis introduced by Bokut & Shiao in [1] and later we will calculate the Gröbner-Shirshov basis of the positive degenerate infinite affine Weyl group $\widetilde{A}_l$ which is isomorphic to semi-direct product group $\Sigma_l \ltimes \mathbb{Z}^{l-1}$.

CHAPTER 4

A COUNTER-EXAMPLE TO THE HYPOTHESIS OF BOKUT&SHIAO THEOREM

In [1] they formulate a general hypothesis on Gröbner-Shirshov bases for any Coxeter group.

Let W be a Coxeter group. Let S be a linear ordered set with l-generators, $M = [m_{ss'}]$ be a $l \times l$ Coxeter matrix associated with W. Then we have

$$W = \text{smg}\langle s^2 = 1,\ (ss')^{m_{ss'}} = 1,\ s \neq s',\ \forall s, s' \in S \text{ and finite } m_{ss'} \rangle.$$

Let us define for finite $m_{ss'}$,

$$\begin{aligned} m(s,s') &= ss'... \quad \text{(there are } m_{ss'} \text{ alternative letters } s, s'), \\ (m-1)(s,s') &= ss'... \quad \text{(there are } m_{ss'} - 1 \text{ alternative letters } s, s'), \end{aligned}$$

and so on. For example, if $m_{ss'} = 2$ then $m(s,s') = ss', (m-1)(s,s') = s$. If $m_{ss'} = 3$, then $m(s,s') = ss's, (m-1)(s,s') = ss'$.

The relation $m_{ii} = 1$ means that $(s_i)^2 = 1$ for all i; the generators are involutions.

If $m_{ij} = 2$, then the generators s_i and s_j commute.

In this notation defining relations of W can be presented in the form

$$s^2 = 1,\ m(s,s') = m(s',s),\ s > s' \tag{4.1}$$

for all $s, s' \in S$ and finite $m_{ss'}$.

Definition 4.1 *Two words in S are* equivalent *if they are equal module commutativity relations from* (5.1).

To be more precise, it means that they are equal in the so called free partially commutative semigroup (algebra)generated by S and with commutativity relations from (5.1). Later on the word problem can be solved in this semigroup(or algebra).

Definition 4.2 *Two relations $a = b$ and $c = d$ of W are called* equivalent *if a, b are equivalent to c, d respectively.*

4.1 Hypothesis

The Gröbner-Shirshov bases of W consist of initial relations (5.1) and relations that are equivalent to the following ones:

$$(m-1)(s,s')(m-1)(s_{i_1},s_{i_2})...(m-1)(s_{i_{2k-1}},s_{i_{2k}})m(s_{i_{2k+1}}s_{i_{2k+2}})$$

$$= m(s',s)(m-1)(s_{i_1},s_{i_2})...(m-1)(s_{i_{2k-1}},s_{i_{2k}})(m-1)(s_{i_{2k+1}}s_{i_{2k+2}}), \quad (5.2)$$

where $s > s'$, $s_{i_1} < s_{i_2}, ..., s_{i_{2k-1}} < s_{i2k}, s_{i_{2k+1}} < s_{i_{2k+2}}$, and any neighbor pairs $(s', s)(s_{i_1}, s_{i_2}), ..., (s_{i_{2k+1}}s_{i_{2k+2}})$ are different, and

$$\begin{cases} s_{i_2} = s', & \text{if } m_{ss'} \text{ is even} \\ s_{i_2} = s, & \text{if } m_{ss'} \text{ is odd} \\ \dots & \dots \quad \dots \\ s_{i_{2k+2}} = s_{i_{2k}}, & \text{if } m_{s_{2k-1}s_{2k}} \text{ is even} \\ s_{i_{2k+2}} = s_{i_{2k-1}}, & \text{if } m_{s_{2k-1}s_{2k}} \text{ is odd.} \end{cases}$$

In [1] they are checking that the hypothesis given by them holds for Gröbner-Shirshov bases for finite Coxeter groups. In the following we will a point out a misunderstanding in the notation of the hypothesis.

For example, let consider the Coxeter matrix $A_4 = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{bmatrix}$.

Using above notation $s_i s_i = 1$ for $i = 1, 2, 3, 4$.

$$m(s_2, s_1) = s_2 s_1 s_2 = m(s_1, s_2) = s_1 s_2 s_1 \quad \text{because} \quad m_{21} = m_{12} = 3$$

$$m(s_3, s_2) = s_3 s_2 s_3 = m(s_2, s_3) = s_2 s_3 s_2 \quad \text{because} \quad m_{32} = m_{23} = 3$$

$$m(s_3, s_1) = s_3 s_1 = m(s_1, s_3) = s_1 s_3 \quad \text{because} \quad m_{31} = m_{13} = 2$$

$$m(s_4, s_3) = s_4 s_3 s_4 = m(s_3, s_4) = s_3 s_4 s_3$$

$$m(s_4, s_2) = s_4 s_2 = m(s_2, s_4) = s_2 s_4$$

$$m(s_4, s_1,) = s_4 s_1 = m(s_1, s_4) = s_1 s_4.$$

These are all initial relations. According to these relations the new notation is above.

The extra elements for A_4 are:

$$(m-1)(s_3, s_2) m(s_1, s_3) = m(s_2, s_3)(m-1)(s_1, s_3)$$

which gives $s_3 s_2 s_1 s_3 = s_2 s_3 s_2 s_1$, and

$(m-1)(s_4, s_3)(m-1)(s_2, s_4) m(s_1, s_4) = m(s_3, s_4)(m-1)(s_2, s_4)(m-1)(s_1, s_4)$ which gives $s_4 s_3 s_2 s_1 s_4 = s_3 s_4 s_3 s_2 s_1$.

Pay attention to the second element (s_2, s_4) that shows the complexity of this notation in the article [1] since we have

$(m-1)(s_4, s_3) m(s_2, s_4) = m(s_3, s_4)(m-1)(s_2, s_4)$ giving $s_4 s_3 s_2 s_4 = s_4 s_4 s_3 s_2$.

So the second component of a pair is determined by being even or odd of the associated entry of the Coxeter matrix if we take the previous pair as an index. But the first component of the chosen can be any element less than the element in the second component. If we write the pairs the procedure in the article, the second element of the pair (s_1, s_2) must be s_2 but it says that the element must be s or s'.

4.2 A Contrary Example for this Hypothesis

The Coxeter matrix of affine Weyl group $\widetilde{A}_3$ is $M = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{bmatrix}$.

Initial relations are

$(s_0)^2 = (s_1)^2 = (s_2)^2 = (s_3)^2 = 1,$

$s_0s_1s_0 = s_1s_0s_1,$

$s_0s_2 = s_2s_0,$

$s_0s_3s_0 = s_3s_0s_3,$

$s_1s_2 = s_1s_2s_1s_2,$

$s_1s_3 = s_3s_1,$

$s_2s_3s_2 = s_3s_2s_3.$

The Gröbner-Shirshov Basis for W consists of the elements

$s_0s_0 - 1,$

$s_1s_1 - 1,$

$s_2s_2 - 1,$

$4s_3s_3 - 1,$

$s_1s_0s_1 - s_0s_1s_0,$

$s_2s_1s_2 - s_1s_2s_1,$

$s_3s_2s_3 - s_2s_3s_2,$

$s_3s_0s_3 - s_0s_3s_0,$

$s_3s_1 - s_1s_3,$

$s_2s_0 - s_0s_2$

$s_3s_0s_1s_0 - s_1s_3s_0s_1,$

$s_3s_0s_2s_3s_2 - s_0s_3s_0s_2s_3,$

$s_3s_0s_2s_3s_0 - s_2s_3 * s_0s_2s_3,$

$s_3s_2s_1s_3 - s_2s_3s_2s_1,$

$s_3s_0s_1s_3 - s_0s_3s_0s_1,$

$s_2s_1s_0s_2 - s_1s_2s_1s_0,$

$s_3s_0s_2s_1s_3s_2s_1 - s_0s_3s_0s_2s_1s_3s_2,$

$s_3s_0s_1s_2s_3s_2 - s_0s_3s_0s_1s_2s_3,$

$s_3s_2s_1s_0s_3s_0 - s_2s_3s_2s_1s_0s_3,$

$s_3s_0s_2s_1s_3s_0s_1 - s_2s_3s_0s_2s_1s_3s_0,$

$s_3s_0s_1s_2s_3s_0s_2 - s_0s_3s_0s_1s_2s_3s_0,$

$s_3s_0s_2s_1s_3s_0s_2s_1s_0 - s_0s_3s_0s_2s_1s_3s_0s_2s_1,$

$s_3s_0s_1s_2s_1s_3s_2s_1 - s_0s_3s_0s_1s_2s_1s_3s_2,$

$s_3s_0s_2s_1s_0s_3s_0s_1 - s_2s_3s_0s_2s_1s_0s_3s_0,$

$s_3s_0s_1s_2s_1s_3s_0s_2s_1s_0 - s_0s_3s_0s_1s_2s_1s_3s_0s_2s_1,$

$s_3s_0s_1s_2s_3s_0s_1s_2s_1 - s_0s_3s_0s_1s_2s_3s_0s_1s_2,$

$s_3s_0s_1s_2s_1s_3s_0s_1s_2s_1s_0 - s_0s_3s_0s_1s_2s_1s_3s_0s_1s_2s_1.$

Let us check if these words are appropriate or not, for the hypothesis.

$s_3s_0s_1s_0 - s_1s_3s_0s_1 = (m-1)(s_3,s_1)m(s_0,s_1) - m(s_1,s_3),$

$s_3s_0s_2s_3s_2 - s_0s_3s_0s_2s_3 = (m-1)(s_3,s_0)m(s_2,s_3) - m(s_0,s_3)(m-1)(s_2,s_3),$

$s_3s_0s_2s_3s_0 - s_2s_3 * s_0s_2s_3 = (m-1)(s_3,s_2)m(s_0,s_3) - m(s_2,s_3)(m-1)(s_0,s_3),$

$s_3s_2s_1s_3 - s_2s_3s_2s_1 = (m-1)(s_3,s_2)m(s_0,s_3) - m(s_2,s_3)(m-1)(s_0,s_3),$

$s_3s_0s_1s_3 - s_0s_3s_0s_1 = (m-1)(s_3,s_0)m(s_1,s_3) - m(s_0,s_3)(m-1)(s_1,s_3),$

$s_2s_1s_0s_2 - s_1s_2s_1s_0 = (m-1)(s_2,s_1)m(s_0,s_2) - m(s_1,s_2)(m-1)(s_0,s_2),$

$s_3s_0s_2s_1s_3s_2s_1 - s_0s_3s_0s_2s_1s_3s_2 \equiv (m-1)(s_3,s_0)(m-1)(s_2,s_3)m(s_1,s_2) - m(s_0,s_3)(m-1)m(s_2,s_3)(m-1)(s_1,s_2),$

$s_3s_0s_1s_2s_3s_2 - s_0s_3s_0s_1s_2s_3 \equiv (m-1)(s_3,s_2)(m-1)(s_1,s_3)m(s_2,s_3) - m(s_2,s_3)(m-1)(s_1,s_3)(m-1)(s_2,s_3),$

$s_3s_2s_1s_0s_3s_0 - s_2s_3s_2s_1s_0s_3 \equiv (m-1)(s_3,s_2)(m-1)(s_1,s_3)m(s_0,s_3) - m(s_2,s_3)(m-1)(s_1,s_3)(m-1)(s_0,s_3).$

We see that the elements above in the Gröbner-Shirshov basis can be written as the procedure in the hypothesis. But we have a problem for the following element.

Let consider $s_3s_0s_2s_1s_3s_0s_1 - s_2s_3s_0s_2s_1s_3s_0$. Since the second term of the element begins with s_2s_3, we must start by $(m-1)(s_3, s_2) = s_3s_2$.

Since $m_{43} = 3(s_3$ and s_2 are 4th and 3rd elements respectively) is an odd number, we should continue by $m(a, s_0)$ where a is a reflection less than s_0. It is not allowed in the hypothesis. Actually this term can be written as $(m-1)(s_3, s_2)(m-1)(s_0, s_3)m(s_1, s_0) - m(s_2, s_3)(m-1)(s_0, s_3)$

$(m-1)(s_1, s_0)$, but it does not obey the procedure in the hypothesis.

Similarly the following elements can be written as the procedure in the hypothesis.

$s_3s_0s_1s_2s_3s_0s_2 - s_0s_3s_0s_2s_1s_3s_0s_2s_1 \equiv (m-1)(s_3, s_0)(m-1)(s_1, s_3)(m-1)(s_2, s_3)m(s_0, s_2) - m(s_0, s_3)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_0, s_2)$,

$s_3s_0s_2s_1s_3s_0s_2s_1s_0 - s_0s_3s_0s_2s_1s_3s_0s_2s_1s_1 \equiv (m-1)(s_3, s_0)(m-1)(s_2, s_3)(m-1)(s_1, s_2)m(s_0, s_1) - m(s_0, s_3)(m-1)(s_2, s_3)(m-1)(s_1, s_2)(m-1)(s_0, s_1)$,

$s_3s_0s_1s_2s_1s_3s_2s_1 - s_0s_3s_0s_1s_2s_1s_3s_2 \equiv (m-1)(s_3, s_0)(m-1)(s_1, s_3)(m-1)(s_2, s_3)m(s_1, s_2) - m(s_0, s_3)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_1, s_2)$.

The element $s_3s_0s_2s_1s_0s_3s_0s_1 - s_2s_3s_0s_2s_1s_0s_3s_0$ is written as $(m-1)(s_3, s_2)(m-1)(s_1, s_3)(m-1)(s_0, s_3)m(s_1, s_0) - m(s_3, s_2)(m-1)(s_1, s_3)(m-1)(s_0, s_3)(m-1)(s_1, s_0)$, but the last term of the expression does not obey the procedure of the hypothesis.

The remaining elements obey the rule in the hypothesis.

$s_3s_0s_1s_2s_1s_3s_0s_2s_1s_0 - s_0s_3s_0s_1s_2s_1s_3s_0r - 2s_1 \equiv (m-1)(s_3, s_0)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_1, s_2)m(s_0, s_1) - m(s_0, s_3)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_1, s_2)(m-1)(s_0, s_1)$,

$s_3s_0s_1s_2s_3s_0s_1s_2s_1 - s_0s_3s_0s_1s_2s_3s_0s_1s_2 \equiv (m-1)(s_3, s_0)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_0, s_2)m(s_1, s_2) - m(s_0, s_3)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_0, s_2)(m-1)(s_1, s_2)$,

$s_3s_0s_1s_2s_1s_3s_0s_1s_2s_1s_0 - s_0s_3s_0s_1s_2s_1s_3s_0s_1s_2s_1 \equiv (m-1)(s_3, s_0)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_0, s_2)(m-1)(s_1, s_2)m(s_0, s_1) - m(s_0, s_3)(m-1)(s_1, s_3)(m-1)(s_2, s_3)(m-1)(s_0, s_2)(m-1)(s_1, s_2)m(s_0, s_1)$.

So we conclude that two elements can not be expressed as the procedure in the hypothesis.

With this counter-example, we obtain that the hypothesis does not hold for positive degenerate affine Weyl groups.

CHAPTER 5

CALCULATION OF REDUCED GRÖBNER-SHIRSHOV BASIS FOR AFFINE WEYL GROUP A_N

In this chapter our aim is to calculate Grobner-Shirshov basis of positive degenerate infinite affine Weyl group $\widetilde{A}_n$ which is isomorphic to semi-direct product of permutation group S_n with translation group $\mathbb{Z}^{n-1}$, where $\mathbb{Z}^{n-1}$ is the eigenmodule of the action of the permutation group on its representation module $\mathbb{Z}^n$. It has a presentation with generators $\{s_0, s_1, \ldots, s_n\}$ and relations

$$s_i s_i = 1,\ i = 0, \ldots, n;$$

$$s_i s_j = s_j s_i,\ i = 0, \ldots, n-2, j-i > 1;$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ i = 0, \ldots, n-1 \text{ and}$$

$$s_0 s_n s_0 = s_n s_0 s_n.$$

Here we will choose the linear ordering $s_0 > s_1 > s_2 > \cdots > s_n$ on the generators on the contrary of the ordering chosen by Bokut and Shiao in [1].

Theorem 5.1 *The affine Weyl group $\widetilde{A}_n$ is generated by $\{s_0, s_1, s_2, \cdots, s_n\}$ and $s_{i,j} = s_i s_{i+1} \ldots s_j$, where $i < j$; and $s_{i,i} = s_i$, $s_{i,i-1} = 1$ with defining relations;*

$\widetilde{A}1$: $s_i{}^2 = 1 \dashrightarrow s_i s_i = 1$, *where* $0 \leq i \leq n$,

$\widetilde{A}2$: $s_i s_j = s_j s_i$, *where* $i = 0, \ldots, n-2$, $j - i > 1$,

$\widetilde{A}3$: $s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$ *where* $1 \leq i \leq n-1$,

$\widetilde{A}4$: $s_0 s_n s_0 = s_n s_0 s_n$,

$\widetilde{A}5$: $s_{i,j} s_i = s_{i+1} s_{i,j}$, *where* $i = 1, \ldots, n-2, j = i+2, \ldots, n$,

$\widetilde{A}6$: $s_0 s_n s_j = s_j s_0 s_n$, *where* $j = 2, \ldots, n-2$,

$\widetilde{A}7$: $s_0s_ns_{j+1}s_j\cdots s_{k+1}s_ks_{k+1} = s_js_os_ns_{j+1}s_j\cdots s_{k+1}s_k$, *where* $k = j,\ldots,n-1$,

$\widetilde{A}8$: $s_0s_ns_{n-1}\cdots s_js_0 = s_ns_0s_ns_{n-1}\cdots s_j$, *where* $j = n-1,\ldots,2$,

$\widetilde{A}9$: $s_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_j = s_ns_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}$, *where* $j = 1,2,\ldots,n$,

$\widetilde{A}10$: $s_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_ks_j =$
$s_ns_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_k$, *where* $j = 1,2,\ldots,n$; $k = n,n-1,\ldots,j+2$,

$\widetilde{A}11$: $s_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_2s_js_{j-1}\cdots s_ks_{k-1}s_k$
$=s_ns_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_2s_js_{j-1}\cdots s_ks_{k-1}$, *where* $j = 1,2,\ldots,n; k = j,\ldots,n$,

$\widetilde{A}12$: $s_0s_1s_ns_0s_j\cdots s_2s_n = s_1s_0s_1s_ns_0s_j\cdots s_2$, *where* $j = n-2,\ldots,2$,

$\widetilde{A}13$: $s_0s_1s_ns_0s_j\cdots s_2s_3s_4\cdots s_ks_{k+1}s_k = s_1s_0s_1s_ns_0s_j\cdots s_ks_{k+1}\cdots s_2$, *where* $j = n-1,\ldots,2$,

$\widetilde{A}14$: $s_0s_1s_ns_0s_j\cdots s_2s_3s_4\cdots s_ks_{k+1}s_0s_k = s_1s_0s_1s_ns_0s_j\cdots s_ks_{k+1}s_0\cdots s_2$, *where* $j = n-1,\ldots,2$,

$\widetilde{A}15$: $s_0s_ns_{n-1}s_1s_0\cdots s_ks_{k+1}s_ks_{k+1} = s_ns_0s_ns_{n-1}s_1s_0\cdots s_ks_{k+1}s_k$, *where* $k = 1,\ldots,n-1$,

$\widetilde{A}16$: $s_0s_ns_1s_0\cdots s_{k+1}s_k = s_ns_0s_ns_1s_0\cdots s_{k+1}s_k$, *where* $k = 1,\ldots,n-1$,

$\widetilde{A}17$: $s_0s_1s_ns_0s_ns_{n-1}\cdots s_ks_{k+1}s_k = s_1s_0s_1s_ns_0s_ns_{n-1}\cdots s_ks_{k+1}$, *where* $k = n-1,\ldots,2$.

So the Gröbner-Shirshov basis for $\widetilde{A}_n$ consists of the relations given by $\widetilde{A}1-\widetilde{A}17$.

Proof. Let us consider the group algebra with the generators $\{s_0,s_1,s_2,\cdots,s_n\}$ and the following ideals

$\widetilde{A}1$: ${s_i}^2 - 1 \dashrightarrow s_is_i = 1$, where $0 \le i \le n$,

$\widetilde{A}2$: $s_is_j - s_js_i$, where $i = 0,\ldots,n-2$, $j-i > 1$,

$\widetilde{A}3$: $s_{i+1}s_is_{i+1} - s_is_{i+1}s_i$ where $1 \le i \le n-1$,

$\widetilde{A}4$: $s_0s_ns_0 - s_ns_0s_n$,

$\widetilde{A}5$: $s_{i,j}s_i - s_{i+1}s_{i,j}$, where $i = 1,\ldots,n-2, j = i+2,\ldots,n$,

$\widetilde{A}6:\ s_0 s_n s_j - s_j s_0 s_n$, where $j = 2, \ldots, n-2$,

$\widetilde{A}7:\ s_0 s_n s_{j+1} s_j \cdots s_{k+1} s_k s_{k+1} - s_j s_o s_n s_{j+1} s_j \cdots s_{k+1} s_k$, where $k = j, \ldots, n-1$,

$\widetilde{A}8:\ s_0 s_n s_{n-1} \cdots s_j s_0 - s_n s_0 s_n s_{n-1} \cdots s_j$, where $j = n-1, \ldots, 2$,

$\widetilde{A}9:\ s_0 s_n s_{n-1} \cdots s_2 s_1 s_0 s_2 s_1 \cdots s_j s_{j-1} s_j - s_n s_0 s_n s_{n-1} \cdots s_2 s_1 s_0 s_2 s_1 \cdots s_j s_{j-1}$, where $j = 1, 2, \ldots, n$,

$\widetilde{A}10:\ s_0 s_n s_{n-1} \cdots s_2 s_1 s_0 s_2 s_1 \cdots s_j s_{j-1} s_n \cdots s_k s_j -$
$s_n s_0 s_n s_{n-1} \cdots s_2 s_1 s_0 s_2 s_1 \cdots s_j s_{j-1} s_n \cdots s_k$, where $j = 1, 2, \ldots, n$;
$k = n, n-1, \ldots, j+2$,

$\widetilde{A}11:\ s_0 s_n s_{n-1} \cdots s_2 s_1 s_0 s_2 s_1 \cdots s_j s_{j-1} s_n \cdots s_2 s_j s_{j-1} \cdots s_k s_{k-1} s_k$
-$s_n s_0 s_n s_{n-1} \cdots s_2 s_1 s_0 s_2 s_1 \cdots s_j s_{j-1} s_n \cdots s_2 s_j s_{j-1} \cdots s_k s_{k-1}$, where $j = 1, 2, \ldots, n; k = j, \ldots, n$,

$\widetilde{A}12:\ s_0 s_1 s_n s_0 s_j \cdots s_2 s_n - s_1 s_0 s_1 s_n s_0 s_j \cdots s_2$, where $j = n-2, \ldots, 2$,

$\widetilde{A}13:\ s_0 s_1 s_n s_0 s_j \cdots s_2 s_3 s_4 \cdots s_k s_{k+1} s_k - s_1 s_0 s_1 s_n s_0 s_j \cdots s_k s_{k+1} \cdots s_2$, where $j = n-1, \ldots, 2$,

$\widetilde{A}14:\ s_0 s_1 s_n s_0 s_j \cdots s_2 s_3 s_4 \cdots s_k s_{k+1} s_0 s_k - s_1 s_0 s_1 s_n s_0 s_j \cdots s_k s_{k+1} s_0 \cdots s_2$, where $j = n-1, \ldots, 2$,

$\widetilde{A}15:\ s_0 s_n s_{n-1} s_1 s_0 \cdots s_k s_{k+1} s_k s_{k+1} - s_n s_0 s_n s_{n-1} s_1 s_0 \cdots s_k s_{k+1} s_k$, where $k = 1, \ldots, n-1$,

$\widetilde{A}16:\ s_0 s_n s_1 s_0 \cdots s_{k+1} s_k - s_n s_0 s_n s_1 s_0 \cdots s_{k+1} s_k$, where $k = 1, \ldots, n-1$,

$\widetilde{A}17:\ s_0 s_1 s_n s_0 s_n s_{n-1} \cdots s_k s_{k+1} s_k - s_1 s_0 s_1 s_n s_0 s_n s_{n-1} \cdots s_k s_{k+1}$, where $k = n-1, \ldots, 2$.

The ideals from $\widetilde{A}1 - \widetilde{A}4$ comes from the group presentation.

The type $\widetilde{A}5:\ s_{i,j} s_i - s_{i+1} s_{i,j}$, $i = 1, \ldots, n-2, j = i+2, \ldots, n$.

For every i, this element is calculated from

$$< s_i s_{i+1} s_i - s_{i+1} s_i s_{i+1}, s_i s_{i+2} - s_{i+2} s_i >$$

using operation of elimination of leading word.

The following elements of this type are found from

$$< s_i \cdots s_j s_i - s_{i+1} s_i \dots s_j, s_i s_{j+1} - s_{j+1} s_i >$$

performing consecutive operations of elimination of leading word for $j = i+2, i+3, \dots, n$.

In this type elements we have an exceptional situation: the case $i = 0, j = n$. In this case we have $s_0 s_n s_0 = s_n s_0 s_n$ instead of $s_0 s_n - s_n s_0$. It gives us the following type element

$s_0 \cdots s_n s_0 s_n - s_1 s_0 \cdots s_n s_0 =< s_0 \cdots s_{n-1} s_0 - s_1 s_0 \cdots s_{n-1}, s_0 s_n s_0 - s_n s_0 s_n >.$

In the group $\widetilde{A}_4$, this type elements appear as

$s_0 s_1 s_2 s_0 - s_1 s_0 s_1 s_2 =< s_0 s_1 s_0 - s_1 s_0 s_1, s_0 s_2 - s_2 s_0 >$

$s_0 s_1 s_2 s_3 s_0 - s_1 s_0 s_1 s_2 =< s_0 s_1 s_2 s_0 - s_1 s_0 s_1 s_2, s_0 s_3 - s_3 s_0 >$

$s_0 s_1 s_2 s_3 s_4 s_0 s_4 - s_1 s_0 s_1 s_2 s_3 s_4 s_0 =< s_0 s_1 s_2 s_3 s_0 - s_1 s_0 s_1 s_2, s_0 s_4 s_0 - s_4 s_0 s_4 >$

$s_1 s_2 s_3 s_1 - s_2 s_1 s_2 s_3 =< s_1 s_1 s_1 - s_2 s_1 s_2, s_1 s_3 - s_3 s_1 >$

$s_2 s_3 s_4 s_2 - s_3 s_2 s_3 s_4 =< s_2 s_3 s_2 - s_3 s_2 s_3, s_2 s_4 - s_4 s_2 >$

$s_1 s_2 s_3 s_4 s_1 - s_2 s_1 s_2 s_3 s_4 =< s_1 s_2 s_3 s_1 - s_2 s_1 s_2 s_3, s_1 s_4 - s_4 s_1 >.$

In this type we have another relation

$$s_0 s_n s_{n-1} s_n - s_{n-1} s_0 s_n s_{n-1} =< s_0 s_{n-1} - s_{n-1} s_0, s_{n-1} s_n s_{n-1} - s_n s_{n1} s_n > .$$

In the group $\widetilde{A}_4$, this type elements appear as

$$s_0 s_4 s_3 s_4 - s_3 s_0 s_4 s_3 =< s_0 s_3 - s_3 s_0, s_3 s_4 s_3 - s_4 s_3 s_4 > .$$

The type $\widetilde{A}6$: $s_0 s_n s_j - s_j s_0 s_n, j = 2, \dots, n-2.$

This type element is obtained from

$$< s_0 s_j - s_j s_0, s_j s_n - s_n s_j >$$

using operation of elimination of leading word.

In the group $\widetilde{A}_4$, this type elements appear as

$$s_0s_4s_2 - s_2s_0s_4 =< s_0s_2 - s_2s_0, s_2s_4 - s_4s_2 > .$$

The type $\widetilde{A}7$: $s_0s_ns_{j+1}s_j \cdots s_{k+1}s_ks_{k+1} - s_js_os_ns_{j+1}s_j \cdots s_{k+1}s_k, k = j, \ldots, n-1$.

The first element in this type is obtained from

$$< s_0s_ns_j - s_js_0s_n, s_js_{j+1}s_j - s_{j+1}s_js_{j+1} >$$

performing operation of elimination of leading word.

The remaining elements in this type are obtained by performing operation of elimination of leading word to $s_ks_{k+1}s_k - s_{k+1}s_ks_{k+1}$ with the obtained element in the first eliminating operation.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_4s_3s_2s_3 - s_2s_0s_4s_3s_2 =< s_0s_4s_2 - s_2s_0s_4, s_2s_3s_2 - s_3s_2s_3 >$.

The type $\widetilde{A}8$: $s_0s_ns_{n-1} \cdots s_js_0 - s_ns_0s_ns_{n-1} \cdots s_j, j = n-1, \ldots, 2$.

The first element in this type is obtained from

$$< s_0s_ns_0 - s_ns_0s_n, s_0s_{n-1} - s_{n-1}s_0 >$$

performing operation of elimination of leading word.

The remaining elements are obtained from

$$< s_0s_ns_{n-1} \cdots s_js_0 - s_ns_0s_ns_{n-1} \cdots s_j, s_0s_{j-1} - s_{j-1}s_0 >$$

performing the respective operations of elimination of leading word for $j = n-2, \ldots, 2$ respectively.

In the group $\widetilde{A}_4$, this type elements appear as

$$s_0s_4s_3s_0 - s_4s_0s_4s_3 =< s_0s_4s_0 - s_4s_0s_4, s_0s_3 - s_3s_0 >, \quad \text{and}$$

$$s_0s_4s_3s_2s_0 - s_4s_0s_4s_3s_2 =< s_0s_4s_3s_0 - s_4s_0s_4s_3, s_0s_2 - s_2s_0 > .$$

The type $\widetilde{A}9:$ $s_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_j - s_ns_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}, j = 1,2,\ldots,n.$

The first element in this type is obtained from $s_0s_1s_0 - s_1s_0s_1$ by performing the operation of elimination of leading word with the last element in the type $\widetilde{A}8$.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with $s_js_{j+1}s_j - s_{j+1}s_js_{j+1}$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_4s_3s_2s_1s_0s_1 - s_4s_0s_4s_3s_2s_1s_0 =< s_0s_4s_3s_2s_0 - s_4s_0s_4s_3s_2, s_0s_1s_0 - s_1s_0s_1 >,$

$s_0s_4s_3s_2s_1s_0s_2s_1s_2 - s_4s_0s_4s_3s_2s_1s_0s_2s_1 =< s_0s_4s_3s_2s_1s_0s_1 - s_4s_0s_4s_3s_2s_1s_0, s_1s_2s_1 - s_2s_1s_2 >,$

$s_0s_4s_3s_2s_1s_0s_2s_1s_3s_2s_3 - s_4s_0s_4s_3s_2s_1s_0s_3s_2 =< s_0s_4s_3s_2s_1s_0s_2s_1s_2 - s_4s_0s_4s_3s_2s_1s_0s_2s_1, s_2s_3s_2s_3s_2s_3s_0s_4s_3s_2s_1s_0s_2s_1s_3s_2s_4s_3s_4 - s_4s_0s_4s_3s_2s_1s_0s_3s_2s_4s_3 > .$

The type $\widetilde{A}10:$ $s_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_ks_j$ $-s_ns_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_k, j = 1,2,\ldots,n; k = n, n-1,\ldots,j+2.$

The elements in this type are obtained from $s_js_n - s_ns_j$ by performing the operation of elimination of leading word with the each element in the type $\widetilde{A}13$ except last one.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with $s_js_{k-1} - s_{k-1}s_j$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_4s_3s_2s_1s_0s_4s_1 - s_4s_0s_4s_3s_2s_1s_0s_4 =< s_0s_4s_3s_2s_1s_0s_1 - s_4s_0s_4s_3s_2s_1s_0,$

$s_1s_4 - s_4s_1 >$,

$s_0s_4s_3s_2s_1s_0s_4s_3s_1 - s_4s_0s_4s_3s_2s_1s_0s_4s_3 =< s_0s_4s_3s_2s_1s_0s_4s_1 - s_4s_0s_4s_3s_2s_1s_0s_4$, $s_1s_3 - s_3s_1 >$,

$s_0s_4s_3s_2s_1s_0s_2s_1s_4s_2 - s_4s_0s_4s_3s_2s_1s_0s_2s_1s_2 =< s_0s_4s_3s_2s_1s_0s_2s_1s_2 - s_4s_0s_4s_3s_2s_1s_0s_2s_1$, $s_2s_4 - s_4s_2 >$.

The type $\widetilde{A}11$:

$s_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_2s_js_{j-1}\cdots s_ks_{k-1}s_k$

-$s_ns_0s_ns_{n-1}\cdots s_2s_1s_0s_2s_1\cdots s_js_{j-1}s_n\cdots s_2s_js_{j-1}\cdots s_ks_{k-1}, j = 1, 2, \ldots, n; k = j, \ldots, n.$

The first element in this type is obtained from $s_0s_1s_0 - s_1s_0s_1$ by performing the operation of elimination of leading word with the last element in the type $\widetilde{A}10$.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with $s_ks_{k+1}s_k - s_{k+1}s_ks_{k+1}$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_4s_3s_2s_1s_2s_0s_4s_3s_2s_1s_2s_3s_2 - s_4s_0s_4s_3s_2s_1s_2s_0s_4s_3s_2s_1s_2s_3$,

$s_0s_4s_3s_2s_1s_2s_3s_0s_4s_3s_2s_1s_2s_3s_2 - s_4s_0s_4s_3s_2s_1s_2s_3s_0s_4s_3s_2s_1s_2s_3$,

$s_0s_4s_3s_2s_1s_2s_3s_0s_4s_3s_1s_2s_4s_3s_4 - s_4s_0s_4s_3s_2s_1s_2s_3s_0s_4s_3s_1s_2s_4s_3$.

The type $\widetilde{A}12$: $s_0s_1s_ns_0s_j\cdots s_2s_n - s_1s_0s_1s_ns_0s_j\cdots s_2, j = n-2, \ldots, 2.$

The first element in this type is obtained from

$$< s_0s_1s_0 - s_1s_0r - 1 - s_0s_ns_0 - s_ns_n >$$

by performing the operation of elimination of leading word.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with $s_0s_ns_0 - s_ns_0s_n$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_1s_4s_0s_4 - s_1s_0s_1s_4s_0 =< s_0s_1s_0 - s_1s_0s_1, s_0s_4s_0 - s_4s_0s_4 >$,

$s_0s_1s_4s_0s_2s_4 - s_1s_0s_1s_4s_0s_2 =< s_0s_1s_2s_0 - s_1s_0s_1s_2, s_0s_4s_0 - s_4s_0s_4 >.$

The type $\widetilde{A}13:$ $s_0s_1s_ns_0s_j\cdots s_2s_3s_4\cdots s_ks_{k+1}s_k - s_1s_0s_1s_ns_0s_j\cdots s_ks_{k+1}\cdots s_2, j = n-1,\ldots,2.$

The first element in this type is obtained from $s_0s_ns_{n-1}s_n - s_{n-1}s_0s_ns_{n-1}$ by performing the operation of elimination of leading word with the last element in the type $\widetilde{A}12$.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with k. element in the type of $\widetilde{A}7$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_1s_4s_0s_2s_3s_4s_3 - s_1s_0s_1s_4s_0s_2s_3s_4 =< s_0s_1s_4s_0s_2s_4 - s_1s_0s_1s_4s_0s_2, s_0s_4s_3s_4 - s_3s_0s_4s_3 >,$

$s_0s_1s_4s_0s_2s_3s_4s_2s_3s_2 - s_1s_0s_1s_4s_0s_2s_3s_4s_2s_3 =< s_0s_4s_1s_2s_3s_0s_4s_3 - s_1s_0s_4s_1s_2s_3s_0s_4, s_0s_4s_3s_2s_3 - s_2s_0s_4s_3s_2 >.$

The type $\widetilde{A}14:$ $s_0s_1s_ns_0s_j\cdots s_2s_3s_4\cdots s_ks_{k+1}s_0s_k - s_1s_0s_1s_ns_0s_j\cdots s_ks_{k+1}s_0\cdots s_2,$ $j = n-1,\ldots,2.$

Each element in this type is obtained from each element in the type of $\widetilde{A}13$ by performing the operation of elimination of leading word with $k-2$. element in the type of $\widetilde{A}8$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_1s_4s_0s_2s_3s_4s_0s_3 - s_1s_0s_1s_4s_0s_2s_3s_4s_0 =< s_0s_4s_1s_2s_3s_0s_4s_3 - s_1s_0s_4s_1s_2s_3s_0s_4, s_0s_4s_3s_0 - s_4s_0s_4s_3 >,$

$s_0s_1s_4s_0s_2s_3s_4s_2s_3s_0s_2 - s_1s_0s_1s_4s_0s_2s_3s_4s_2s_3s_0 =$
$< s_0s_4s_3s_1s_2s_3s_0s_4s_3s_2 - s_1s_0s_4s_3s_1s_2s_3s_0s_4s_3, s_0s_4s_3s_2s_0 - s_4s_0s_4s_3s_2 >.$

The type $\widetilde{A}15:$ $s_0s_ns_{n-1}s_1s_0\cdots s_ks_{k+1}s_ks_{k+1} - s_ns_0s_ns_{n-1}s_1s_0\cdots s_ks_{k+1}s_k, k = 1,\ldots,n-1.$

The first element in this type is obtained from $< s_0s_ns_{n-1}s_n - s_{n-1}s_0s_ns_{n-1}, s_0s_1s_0 - s_1s_0s_1 >$ by performing the operation of elimination of leading word.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with $s_k s_{k+1} s_k - s_{k+1} s_k s_{k+1}$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_4s_3s_1s_0s_1 - s_4s_0s_4s_3s_1s_0 =< s_0s_4s_3s_0 - s_4s_0s_4s_3, s_0s_1s_0 - s_1s_0s_1 >$,

$s_0s_4s_3s_1s_0s_2s_1s_2 - s_4s_0s_4s_3s_1s_0s_2s_1 =< s_0s_4s_3s_1s_0s_1 - s_4s_0s_4s_3s_1s_0, s_1s_2s_1 - s_2s_1s_2 >$,

$s_0s_4s_3s_1s_0s_2s_1s_3s_2s_3 - s_4s_0s_4s_3s_1s_0s_2s_1s_3s_2 =< s_0s_4s_3s_1s_0s_2s_1s_2 - s_4s_0s_4s_3s_1s_0s_2s_1, s_2s_3s_2 - s_3s_2s_3 >$,

$s_0s_4s_3s_1s_0s_2s_1s_3s_2s_4s_3s_4 - s_4s_0s_4s_3s_1s_0s_2s_1s_3s_2s_4s_3 =$
$< s_0s_4s_3s_1s_0s_2s_1s_3s_2s_3 - s_4s_0s_4s_3s_1s_0s_2s_1s_3s_2, s_3s_4s_3 - s_4s_3s_4 >$.

The type $\widetilde{A}16$: $s_0s_ns_1s_0 \cdots s_{k+1}s_k - s_ns_0s_ns_1s_0 \cdots s_{k+1}s_k, k = 1, \ldots, n-1$.

The first element in this type is obtained from

$$< s_0s_ns_0 - s_ns_0s_n, s_ns_0s_1s_0 - s_1s_0s_1 >$$

by performing the operation of elimination of leading word.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with $s_k s_{k+1} s_k - s_{k+1} s_k s_{k+1}$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0s_4s_1s_0s_1 - s_4s_0s_4s_1s_0 =< s_0s_4s_0 - s_4s_0s_4, s_0s_1s_0 - s_1s_0s_1 >$,

$s_0s_4s_1s_0s_2s_1s_2 - s_4s_0s_4s_1s_0s_2s_1$,

$s_0s_4s_1s_0s_2s_1s_3s_2s_3 - s_4s_0s_4s_1s_0s_2s_1s_3s_2$,

$s_0s_4s_1s_0s_2s_1s_3s_2s_4s_3s_4 - s_4s_0s_4s_1s_0s_2s_1s_3s_2s_4s_3$.

The type $\widetilde{A}17$: $s_0s_1s_ns_0s_ns_{n-1} \cdots s_ks_{k+1}s_k - s_1s_0s_1s_ns_0s_ns_{n-1} \cdots s_ks_{k+1}, k = n-1, \ldots, 2$.

The first element in this type is obtained from $s_{n_1}s_ns_{n-1} - s_ns_{n-1}s_n$ by performing the operation of elimination of leading word with the first element in the type $\widetilde{A}15$.

After then each remaining element in this type is obtained from the previous one in this type by performing the operation of elimination of leading word with $s_k s_{k+1} s_k - s_{k+1} s_k s_{k+1}$.

In the group $\widetilde{A}_4$, this type elements appear as

$s_0 s_1 s_4 s_0 s_3 s_4 s_3 - s_1 s_0 s_1 s_3 s_4$,

$s_0 s_1 s_4 s_0 s_3 s_4 s_2 s_3 s_2 - s_1 s_0 s_1 s_3 s_4 s_2 s_3$.

It can be shown that the other obtained elements by using operations of elimination leading word except the elements in types $\widetilde{A}1 - \widetilde{A}17$ can be reduced to 0.

So the Gröbner-Shirshov basis for $\widetilde{A}_n$ consists of the relations given by $\widetilde{A}1 - \widetilde{A}17$.

□

5.1 Representation of Reduced Elements in the Group.

The Composition Diamond Lemma tells that the irreducible elements with respect to the Gröbner-Shirsov basis give us a representation for affine Weyl group $\widetilde{A}_n$. Since we choose the total ordering $s_0 > s_1 > \cdots > s_n$ on the generating set, the elements beginning with s_0 can not have conjugate elements beginning with other generators. Therefore the elements beginning with s_0 give us a representation of elements with minimal length of in the flag subset $\widetilde{A}_n / A_n$.

As a consequence of our calculations it is seen that this representation can be expressed by words which are obtained by juxtaposing of some blocks of the elements.

The first block in the row is a word satisfying the ordering $\{s_0, s_1, \dots s_n\}$ and the following conditions:

(i) $s_i s_j$ not in that block except $s_0 s_n$, for $j - i > 1$, $i = 0, \dots, n-2$,

(ii) $s_0 s_n s_j$ is not contained in the block for $j = 2, \dots, n-2$, and

(iii) if the word is repeated in the sequent block again, the conditions still hold.

By a simple calculation, we have these blocks in the following:

$s_0 s_n s_{n-1} s_1 s_2 \cdots s_{n-2}$,

$s_j s_{j+1} \cdots s_0 s_n s_{n-1} s_1 s_2 \cdots s_{j-1}$,

$s_1 s_2 \cdots s_{n-2} s_0 s_n s_{n-1}$,

$s_{n-1} s_1 s_2 \cdots s_{n-2} s_0 s_n$,

$s_n s_{n-1} s_1 s_2 \cdots s_{n-2} s_0$,

$s_0 s_n s_{n-1} s_{n-2} \cdots s_1$,

$s_j s_{j-1} \cdots s_1 s_0 s_n s_{n-1} s_{n-2} \cdots s_{j+1}$,

$s_{n-1} s_{n-2} \cdots s_1 s_0 s_n$,

$s_n s_{n-1} s_{n-2} \cdots s_1 s_0$,

$s_{n-1} s_0 s_n s_1 s_2 \cdots s_{n-2}$,

$s_j s_{j+1} \cdots s_{n-1} s_0 s_n s_1 s_2 \cdots s_{j-1}$,

$s_1 s_2 \cdots s_{n-2} s_{n-1} s_0 s_n$,

$s_n s_1 s_2 \cdots s_{n-2} s_{n-1} s_0$,

$s_0 s_n s_1 s_2 \cdots s_{n-2} s_{n-1}$,

$s_{n-1} s_n s_0 s_1 s_2 \cdots s_{n-2}$,

$s_j s_{j+1} \cdots s_{n-1} s_n s_0 s_1 s_2 \cdots s_{j-1}$,

$s_1 s_2 \cdots s_{n-2} s_{n-1} s_n s_0$,

$s_0 s_1 s_2 \cdots s_{n-2} s_{n-1} s_n$,

$s_n s_0 s_1 s_2 \cdots s_{n-2} s_{n-1}$.

If we give the examples of this type words in the group $\widetilde{A}_4$ the words beginning with s_0 are the following:

$s_0 s_1 s_2 s_3 s_4$, $s_0 s_4 s_1 s_2 s_3$, $s_0 s_4 s_3 s_1 s_2$, $s_0 s_4 s_3 s_2 s_1$.

The other blocks satisfying the rules above can be easily written as above.

Also there are blocks apart from this type blocks, with the following properties:

(i*) they begin with s_k and end with s_k again,

(ii*) $3 \leq \ell(\text{block}) \leq n$.

These blocks can be written as $s_k x s_k$ where x is the string of generators $\{s_0, \ldots, s_{k-1}, s_{k+1}, \ldots, s_n\}$ with length $3 \leq \ell(x) \leq n$.

If we give the example of this type words in the group $\widetilde{A}_4$ the word beginning with s_0 is $s_0 s_4 s_1 s_0$.

Of course there are a lot of blocks beginning with s_1, s_2, s_3 or s_4. They can be written with respect to the rules above.

The representation of elements is made up of the juxtaposing of obtained blocks. But when the blocks are juxtaposing, it is point out that no leading terms of elements of Gröbner-Shirshov basis must be formed in the strings so that the obtained word can not be reduced.

For example the block $(s_0s_1s_2s_3s_4)$ can take along no block except itself.

Therefore we have the following elements in the flag subset $\widetilde{A}_4/A_4$:

$(s_0s_1s_2s_3s_4)^p$,

$(s_0s_1s_2s_3s_4)^p s_0$,

$(s_0s_1s_2s_3s_4)^p s_0s_1$,

$(s_0s_1s_2s_3s_4)^p s_0s_1s_2$,

$(s_0s_1s_2s_3s_4)^p s_0s_1s_2s_3$.

In the another example the block $(s_0s_4s_1s_2s_3)$ can be followed by itself and the block $s_4s_0s_4$, but we see that the last block can not be followed by a block beginning with s_3 because the block $s_0s_4s_1s_2s_3s_4s_0s_4s_3$ is the leading term of an element in the Gröbner-Shirshov basis. Then the block $s_0s_4s_1s_2s_3s_4s_0s_4$ can be followed by a block beginning with no s_3. For example it can be $s_1s_2s_3s_4s_0$. Since this block can be followed by itself we have the following reduced elements in the flag subset $\widetilde{A}_4/A_4$:

$(s_0s_4s_1s_2s_3)^n$,

$(s_0s_4s_1s_2s_3)^n s_0$,

$(s_0s_4s_1s_2s_3)^n s_0s_4$,

$(s_0s_4s_1s_2s_3)^n s_0s_4s_1$,

$(s_0s_4s_1s_2s_3)^n s_0s_4s_1s_2$,

$(s_0s_4s_1s_2s_3)^n (s_4s_0s_4)(s_1s_2s_3s_4s_0)^p$,

$(s_0s_4s_1s_2s_3)^n (s_4s_0s_4)(s_1s_2s_3s_4s_0)^p s_1$,

$(s_0s_4s_1s_2s_3)^n (s_4s_0s_4)(s_1s_2s_3s_4s_0)^p s_1s_2$,

$(s_0s_4s_1s_2s_3)^n (s_4s_0s_4)(s_1s_2s_3s_4s_0)^p s_1s_2s_3$,

$(s_0s_4s_1s_2s_3)^n (s_4s_0s_4)(s_1s_2s_3s_4s_0)^p s_1s_2s_3s_4$.

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