LIMITING GIBBS MEASURES IN SOME ONE AND TWO DIMENSIONAL MODELS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

LIMITING GIBBS MEASURES IN SOME ONE AND TWO DIMENSIONAL MODELS

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We give the definitions of finite volume Gibbs measure and limit Gibbs states. In one dimensional Ising model with arbitrary boundary conditions we calculate correlation functions in explicit way. In one dimension, conditions for uniqueness of Gibbs state are considered. We also discuss two dimensional Ising model.

Keywords: Hamiltonian, Thermodynamic Limit, Gibbs Measures, Ising Model, Phase Transition.

ÖZET

BAZI BİR VE İKİ BOYUTLU MODELLERDE LİMİT GİBBS ÖLÇÜLERİ

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Sonlu hacimli Gibbs ölçüsünün ve limit Gibbs durumlarının tanımlarını veriyoruz. Keyfi sınır şartlı bir boyutlu Ising modelinde korelasyon fonksiyonlarını açık bir şekilde hesaplıyoruz. Bir boyutta Gibbs durumunun teklik şartları göz önünde bulundurulur. Aynı zamanda iki boyutlu Ising modeli de tartışıyoruz.

Anahtar sözcükler: Hamiltonyan, Termodinamik Limit, Gibbs Ölçümleri, Ising Modeli, Faz Dönüşümleri.

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Contents

Chapter 1

Introduction

The theory of Gibbs measures is a part of Probability Theory as in Classical Statistical Physics. Notion of a Gibbs measure dates back to R.L. Dobrushin (1968-1970) and O.E. Lanford and D. Ruelle (1969). During the three decades since 1968, this notion has received considerable interest both mathematical physicist and probabilists. In probabilistic terms, a Gibbs measure is the distribution of a countably infinite family of random variables which admits on prescribed conditional probabilities. Now let us give an outline of some physical grounds which motivate the definition of, and justify the interest in, Gibbs measures.

Consider a piece of ferromagnetic metal (like iron or nickel) in thermal equilibrium that consists of a very large number of atoms. This atoms are located at sites of crystal lattice and each atoms shows a magnetic moment which can be visualized as vector in $\mathbb{R}^3$. This magnetic moment results from the angular moments of the electrons, that is called spins or spin of the atom. Because of the interaction properties of the electrons in the crystal, the spins of two adjacent atoms tends to be parallel. This tendency is compensated by thermal notion at high temperatures. Also the coupling of moments dominates and gives rise to the phenomenon of spontaneous magnetization at low temperatures which is below a certain value called Curie temperature. Even in the absence of any external field, the atomic spins align and thus induce a macroscopic magnetic field. In a variable external field h, the magnetization of the ferromagnet exhibit a jump discontinuity at $h = 0$.

Second argument in statistical physics for the existence of a phase transition of a real gas applies to a particular mathematical model of a fluid, the "lattice gas". At low pressures a typical pure fluid at a fixed value of temperature, exists as a gas of fairly low density. If the pressure is increased, the density of the gas increases, but it remains homogeneous. However, when the pressure reaches the vapor pressure (a function of temperature) some of the gas begins to condense in to denser, liquid phase. It is only at the vapor pressure that the two phases, gas and liquid, can exist together in the same container, in thermal. At any higher pressure the only equilibrium situation is one in which the container is entirely filled with liquid. It should be noted that the vapor pressure the relative amount of the two phases are arbitrary: almost all the fluid may occupy comparable volumes. The phenomenon of a discontinuous change in physical properties as the pressure passes through a particular value and the possibility of having two phases existing together in single container is known as "phase transition". The analogy between real gases and ferromagnets also extends microscopic level. The gas consists of a huge number of particles. This particles interact via Van der Waals forces and they have a spatial distribution. Imagine that the container of the gas is divided into a large number of cells which are of the same order of magnitude as the particles. We assign to each cell, the number of particles in the cell that is called occupation number. It is also possible to distinguish between particles of different types or orientations.

The general view of this picture is called a lattice gas. If we consider the cells in the container correspond to the ferromagnetic atoms at the occupation numbers correspond to magnetic moment, the spontaneous magnetization of a ferromagnet is similar to the liquid vapor phase transition of lattice gas. The problem is construct a mathematical model which will display a similar phenomenon having same connection with the microscopic physics -the forces between the atoms etc.- that are important for understanding the properties of real gases.

Mathematical Model:

How can a ferromagnet or a lattice gas in thermal equilibrium be described in mathematical terms? This question leads to the concept of a Gibbs measure. We shall proceed in four steps.

Step 1. Configuration space :
Using the common features of a ferromagnet and lattice gas we have two sets. First, there is a large (but finite) set S which labels the components of the system. Secondly, there is a set E which describes the possible states of each component. In the case of a ferromagnet, S consists of the sites of the crystal lattice which is formed by the positions of the atoms. E is the set of all possible orientations of the magnetic moments. We might assume that each moment is only capable of two orientations. Then $E = \{-1, 1\}$, where 1 stands for "spin up" and -1 for "spin down". In the case of a lattice gas, S is the set of all cells which subdivide the volume which is filled with the gas. We can take $E = \{0, 1, ..., N\}$, where N is the maximal number of particles in a cell. In the simplest case we have $E = \{0, 1\}$, where 1 stands for "cell is occupied" and 0 for "cell is empty". Now we can describe a particular state of the total system by a suitable element $\omega = (\omega_i)_{i\in S}$ of the product space $\Omega = E^S$. Ω is called the configuration space.

Step 2. The probabilistic point of view:
The physical systems considered above are characterized by a sharp contrast. The microscopic structure is enormously complex and microscopic measurements are subject to statistical fluctuations. The macroscopic behavior can be described by means of a few parameters such as temperature and pressure, and macroscopic measurements lead to apparently deterministic results. This contrast between the microscopic and macroscopic level may be summarized as follows: The microscopic complexity can be overcome by a statistical approach; the macroscopic determinism then may be regarded as a consequence of a suitable law of large numbers. So it is not suitable to describe the state of the system by a particular element ω of the configuration space Ω. The system's state should be described by a family $(\sigma_i)_{i\in S}$ of E-valued random variables, or by a probability measure μ on Ω. The probability measure μ should be consistent such that μ should take account of the priori assumption that the system is in thermal equilibrium.

Step 3. The Gibbs distribution:
We must choose a suitable probability measure to describe a physical system in equilibrium. But this equilibrium refers to the forces that act on the system. Thus before specifying a probabilistic model of an equilibrium state we need to specify a Hamiltonian U which assigns to each configuration ω a potential energy $U(\omega)$. In the case of a ferromagnet with state space $E = \{-1, 1\}$ it is suitable to consider a Hamiltonian of the form

$$U(\omega) = -\sum_{\{i,j\}\subset S} J(i,j)\omega_i\omega_j - h\sum_{i\in S}\omega_i, \tag{1.1}$$

where $J(i,j) = J(j,i) > 0$ and h is a real number. The term $-J(i,j)\omega_i\omega_j$ represents the interaction energy of the spins ω_i and ω_j. This energy is minimal if $\omega_i = \omega_j$, it means that if ω_i and ω_j are aligned, the interaction is ferromagnetic. And the number h represents the action of an external magnetic field. If $h > 0$, this field is oriented in positive direction of the spins. In the case of a lattice gas we can use the Hamiltonian of the form (1.1). The term $-J(i,j)\omega_i\omega_j$ is only non-zero when the cells i and j are occupied, hence $-J(i,j)$ is the interaction energy of the two particles in these cells, and the condition $J(i,j) > 0$ means that the particles attract each other. And the number h represents the work which is necessary in order to place a particle in the system. After specifying a Hamiltonian U, the equilibrium state of a physical system with Hamiltonian U is described by the probability measure

$$\mu(d\omega) = Z^{-1}\exp[-\beta U(\omega)]d\omega \tag{1.2}$$

on Ω. The notation $d\omega$ refers to a suitable measure on Ω, for example, if Ω is finite, μ is the counting measure, β is a positive number which is proportional to the inverse of the absolute temperature, and $Z > 0$ is a normalizing constant. The above μ is called the Gibbs distribution relative to U.

Step 4. The infinite volume limit:
The set S which we defined above for ferromagnet or lattice gas should be very large in mathematical terms since the number of atoms in ferromagnet and the number of microscopic in a lattice gas are extremely large. It is therefore a common practice in statistical physics to pass to the infinite volume limit $|S| \to \infty$. This limit also referred to as the thermodynamic limit. Instead of performing the same kind of limit over and over it is often preferable to study directly the class of all possible limiting objects. This means that the finite lattice S should be replaced by a countably infinite lattice such as, for example, the d-dimensional integer lattice $\mathbb{Z}^d$. So we are led to a study of systems with infinitely many interacting components. But there is a problem such that we may not describe an equilibrium state of such a system with a suitable probability measure on infinite product space $\Omega = E^{\mathbb{Z}^d}$. Also if S is an infinite lattice and the interaction is spatially homogeneous then a Hamiltonian in the form (1.1) is not well-defined and formula (1.2) makes no sense. Hence we might consider limits of suitable Gibbs distributions as S increases to an infinite lattice. In general doing this procedure is difficult. So we might try to characterize the Gibbs distribution (1.2).

To be specific we let $E = \{-1, 1\}$ or $\{0, 1\}$, S be finite, and U be given by (1.1). We also let Λ be any non-empty subset of S and $\zeta \in E^\Lambda$ and $\eta \in E^{S\setminus\Lambda}$ any two configurations on Λ and the complement $S \setminus \Lambda$ respectively; the combined configuration on S will be denoted $\zeta\eta$. We consider the probability of the event "ζ occurs in Λ" under the hypothesis "η occurs in $S \setminus \Lambda$" relative to the probability measure μ in (1.2). Cancelling all terms which only depend on η, we find that

$$\begin{aligned} \mu(\zeta \text{ in } \Lambda | \eta \text{ in } S \setminus \Lambda) &= \mu(\zeta\eta \text{ in } S) \big/ \mu(\eta \text{ in } S \setminus \Lambda) \\ &= \exp[-\beta U(\zeta\eta)] \Big/ \sum_{\zeta \in e^\Lambda} \exp[-\beta U(\zeta\eta)] \\ &= Z_\Lambda(\eta)^{-1} \exp[-\beta U_\Lambda(\zeta\eta)]. \end{aligned} \tag{1.3}$$

Here

$$U_\Lambda(\zeta\eta) = -\sum_{\{i,j\}\subset\Lambda} J(i,j)\zeta_i\zeta_j - \sum_{i\in\Lambda}\zeta_i\left[h + \sum_{j\in S\setminus\Lambda}\right]$$

considered as a function of ζ, is the Hamiltonian of the subsystem in Λ with "boundary condition" η, and

$$Z_\Lambda(\eta) = \sum_{\zeta\in E^\Lambda} \exp[-\beta U_\Lambda(\zeta\eta)]$$

is a normalizing constant. Conversely, there is only one μ which satisfies (1.3) for all ζ, η, and Λ, namely the Gibbs distribution (1.2) (To see this it is sufficient to put $\Lambda = S$). Since each $\Lambda \subset S$ is automatically finite, we can conclude that the probability measure μ in (1.2) is uniquely determined by the property that each finite subsystem, conditioned on its surroundings, has a Gibbsian distribution relative to the Hamiltonian that belongs to this subsystem. Now the point is that the last property still makes sense when the lattice S is infinite. Hence we will give the following definition.

Definition 1.0.1 *Consider a probability measure μ on a product space $\Omega = E^S$, where S is countably infinite and E is any measurable space. Then μ is called a Gibbs measure if, for each finite subset Λ of S μ-almost every configuration η outside Λ, the conditional distribution of the configuration in Λ given η is Gibbsian relative to the Hamiltonian in Λ with boundary condition η. The family $\gamma = (\gamma_\Lambda(\cdot|\eta))_{\eta,\Lambda}$ of all these Gibbsian conditional distributions is called the specification of μ. γ describes the interdependencies between the configurations on different parts of S; these interdependencies are dictated by the interaction between the components of the system.*

Finally after these steps we see that a Gibbs measure is a mathematical idealization of an equilibrium state of a physical system which consists of a very large number of interacting components. On the other hand in the language of probability theory, a Gibbs measure is simply the distribution of a stochastic process which is parametrized by the sites of a spatial lattice, and has the special feature of admitting prescribed versions of the conditional distributions with respect to the configurations outside finite regions.

Phase Transition As A Non-uniqueness Phenomenon

We consider again the spontaneous magnetization of a ferromagnet at low temperature. As a first experiment, we place the ferromagnet in an external magnetic field (which is oriented along one of the axes of the ferromagnetic crystal). Turning the field and waiting until equilibrium, we find that the ferromagnet exhibits a macroscopic magnetic moment in the same direction as the stimulating external field. A second experiment with an external field in the opposite direction produces an equilibrium state with the opposite magnetization as before. Thus the ferromagnet admits two distinct equilibrium states which are compatible with the external conditions and internal structure. Now the properties of material and the experimental conditions are modelled by the system of conditional Gibss distributions (i.e., the specification) and the distinct equilibrium states are modelled by suitable Gibbs measures relative to this specification. If we let $\vartheta(\gamma)$ be the set of all Gibbs measures for a given specification γ then phase transitions depend on the size of the set $\vartheta(\gamma)$. Thus the physical phenomenon of phase transition should be reflected in our mathematical model by the non-uniqueness of the Gibbs measures for a prescribed specification.

Next we introduce some mathematical definitions.

Definition 1.0.2 *Let S be countably infinite set and (E, Ξ) any measurable space. A family $(\sigma_i)_{i \in S}$ of random variables which are defined on some probability space and take values in (E, Ξ) is called a random field. Here S is called the parameter set and (E, Ξ) the state space of the random field. Each element i of S is called a (lattice) site and σ_i the spin at site i.*

Definition 1.0.3 *Let $\Omega = E^S = \{(\sigma_i)_{i\in S} : \sigma_i \in E\}$, then each element in Ω is called a configuration and Ω is the set of all configurations.*

Let $\Sigma = \Xi^S$ be the product σ-algebra on Ω, i.e., the smallest σ-algebra containing the cylinder events (a cylinder event is an event which depends on finitely many coordinates only).

Next, let $\vartheta = \{\Lambda \subset S : \Lambda \neq \emptyset,\ |\Lambda| < \infty\}$ be the countably infinite set of all non-empty finite subsets of S.

Definition 1.0.4 *An interaction potential (or simply a potential) is a family $\Phi = (\Phi_A)_{A\in\vartheta}$ of functions $\Phi_A : \Omega \to \mathbb{R}$ with the following properties:*

1) For each $A \in \vartheta,\ \Phi_A$ is Σ_A-measurable, where $\Sigma_A = \Xi^A$.

2) For all $\Lambda \in \vartheta$ and $\sigma \in \Omega$, the series

$$U_\Lambda^\Phi(\sigma) = \sum_{A\in\vartheta, A\cap\Lambda\neq\emptyset} \Phi_A(\sigma) \tag{1.4}$$

exists. $U_\Lambda^\Phi(\sigma)$ is called the total energy of σ in Λ for Φ. It is also called Hamiltonian in Λ for Φ. Also we can use the notation $h_\Lambda^\Phi(\sigma)$ to denote:

$$h_\Lambda^\Phi(\sigma) = \exp[-U_\Lambda^\Phi(\sigma)]. \tag{1.5}$$

$h_\Lambda^\Phi(\sigma)$ is called the Boltzman factor.

Next, we introduce a probability distribution on the space $\Omega_\Lambda = E^\Lambda = \{(\sigma_i)_{i\in\Lambda} : \sigma_i \in E\}$ defining the probability of a configuration $\sigma^\Lambda \in \Omega_\Lambda$ by:

$$P_\Lambda(\sigma^\Lambda) = Z_\Lambda^{-1} \exp[-\beta U_\Lambda^E(\sigma^\Lambda)]$$

where Z_Λ is a normalizing factor defined by the condition: $\sum\limits_{\sigma^\Lambda \in \Omega_\Lambda} P_\Lambda(\sigma^\Lambda) = 1$. Thus $Z_\Lambda = \sum\limits_{\sigma^\Lambda \in \Omega_\Lambda} \exp[-\beta U_\Lambda^E(\sigma^\Lambda)]$ where $\beta = (kT)^{-1}$, with k is a constant and T is the temperature. Z_Λ is called a partition function.

Definition 1.0.5 *The probability distribution defined above is called a Gibbs probability distribution in Λ corresponding to the given Hamiltonian.*

Let us generalize the above definition with the boundary conditions.

Definition 1.0.6 *Let λ be σ-finite measure (a priori measure) in the set (E, Ξ). We call a potential Φ λ-admissible if*

$$Z_\Lambda^\Phi(\sigma) = \int \lambda^\Lambda(d\zeta) \exp[-U_\Lambda^\Phi(\zeta\sigma_{S\setminus\Lambda})]$$

is finite for all $\Lambda \in \vartheta$ and $\sigma \in \Omega$. $Z_\Lambda^\Phi(\sigma)$ is the partition function in Λ for Φ, σ (and λ).

Definition 1.0.7 *Suppose Φ is a λ-admissible potential and $\sigma \in \Omega$, $\Lambda \in \vartheta$. Then the probability measure for each $A \in \vartheta$*

$$\gamma_\Lambda^\Phi(A|\sigma) = Z_\Lambda^\Phi(\sigma)^{-1} \int \lambda^\Lambda(d\zeta) \exp[-U_\Lambda^\Phi(\zeta\sigma_{S\setminus\Lambda})] 1_A(\zeta\sigma_{S\setminus\Lambda})$$

on (Ω, Σ) is called the Gibbs distribution in Λ with boundary condition $\sigma_{S\setminus\Lambda}$, interaction potential Φ, and single spin measure λ, where

$$1_A(x) = \begin{cases} 1 \quad , & \text{if } x \in A \\ 0 \quad , & \text{othrwise} \end{cases}$$

is the indicator function of A. The λ-specification $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda\in\vartheta}$ is called the Gibbsian specification for Φ and λ. Each random field $\mu \in \vartheta(\Phi)$ is called a Gibbs measure or a Gibbs random field for Φ and λ.

Now let us introduce some basic notations of Gibbs fields used in chapter 2 and chapter 3.

An ordered triple (Ω, Σ, μ) where Ω is a set (we shall usually denote by Ω a set of configurations of a random field), Σ is a σ-algebra of subsets of Ω and μ is a probability measure on σ, is a probability space. If Ω is a topological space, then Σ denotes its Borel σ-algebra $\mathcal{B}(\Omega)$.

μ_0 denotes a free (nonperturbed) measure on Ω (usually independent or Gaussian).

For any random variable such that a measurable function ξ on a probability space (Ω, Σ, μ), its mean (mathematical expectation) is denoted by

$$\langle \xi \rangle = \langle \xi \rangle_\mu = \int_\Omega \xi d\mu. \tag{1.6}$$

$(A_1, ..., A_n)$ is an ordered and $\{A_1, ..., A_n\}$ is an unordered collection of sets $A_i,\ i = 1, ..., n$ (similarly for collections of points).

A partition $\alpha = \{T_1, ..., T_k\}$ of a set A is an unordered collection of non empty mutually disjoint subsets $T_i \subset A,\ i = 1, ..., k$, whose union is $A, \cup_{i=1}^k = A$.

T is a countable (or finite) set supporting a random field. Q is a space supporting a point field. S is a space of values of a field (a space of spins or charges).

Λ, A, R are finite (bounded) subsets of T (or Q).

Ω_Λ denotes the space of configurations of a field in Λ ($\Lambda \subset T$ or $\Lambda \subset Q$) and Σ_Λ denotes a σ- algebra in Ω_Λ.

U_Λ denotes a Hamiltonian (energy) in Λ and $U_{\Lambda,y} = U(\cdot/y)$ is a Hamiltonian (energy) in Λ under the boundary configuration y.

μ_Λ denotes a finite Gibbs modification (in Λ) and $\mu_{\Lambda,y}$ is a Gibbs modification under the boundary configuration y.

Z_Λ denotes a partition function (of Gibbs modification) in Λ and $Z_{\Lambda,y}$ is a partition function in Λ under the boundary configuration.

Chapter 2

Gibbs Modifications

We are occupied with explicit constructions of random fields. They are based on a single method: the so-called Gibbs modification. The probability distribution of a finite system of random variables $\{\xi_1, ..., \xi_n\}$ (of finite field) is most often given in terms of its density $p(x_1, ..., x_n)$ with respect to some (usually Lebesgue) measure on R^n. Gibbs modification is a natural generalization of this method to the case of infinite fields. The simplest infinite random fields are independent and Gaussian and their functionals (until recently, they remained the only well-studied class of fields).

As to the Gibbs method of construction, one first introduces fields with a distribution prescribed by a finite (local) density with respect to independent or Gaussian fields (a finite or local Gibbs modification) and then passes to the weak limit of such distributions (a limit Gibbs modification); this limit measure is already singular with respect to the original distribution, independent or Gaussian one. Random fields originating in this limit (the thermodynamic limit) actually form the main subject of study in the theory of Gibbs fields.

2.1 Random Fields

Some classes of Random fields are the following classes:

(1) *Random fields in a countable set T with values in a metric (complete and separable) space S.*

We briefly begin with the definition of random field:

Definition 2.1.1 *A family $(x_t)_{t \in T}$ of random variables which are defined on some probability space (Ω, Σ, μ) and take values in Ω is called a random field or a spin system.*

The probability space (Ω, Σ, μ) is represented in this case by the set $S^T = \Omega$ of functions (also called configurations) $x = \{x_t,\ t \in T\}$ defined on T, with values in $S \subset \Omega$ (S is often called the set of spins).

We will investigate probability distributions μ defined on the Borel σ-algebra $\mathcal{B}(\mathcal{S}) = \Sigma$ of the space S^T. The collection of random variables x_t, $t \in T$, arising in this way, i.e., the values of the random configuration x at points $t \in T$, forms a random field.

The field of independent and identically distributed variables with the measure μ on $\mathcal{B}(S^T)$ which is defined to be the product of countably many identical copies of some probability measure λ_0 on the space S is an example of such a field.

(2) *Random point fields in a separable metric space Q with values in a space S.*

Definition 2.1.2 *The subset x (at most countable) is called locally finite if any bounded set $\Lambda \in Q$ contains only a finite number of points.*

The role of the probability space is played by the set Ω of all locally finite subsets $x \subset Q$. A metrizable topology can be introduced in the space Ω.

Definition 2.1.3 *Every probability measure defined on the Borel (with respect to that topology) σ-algebra $\mathcal{B}(\Omega)$ is called a random point field in Q.*

Now suppose that a metrizable space S, also called the space of charges (or labels), is given. We use Ω_S to denote the space of pairs $\{x, s_x\}$, with $x \in \Omega$ and s_x being a function on x taking values from S. Such pairs will be called configurations. As well as in Ω, a metrizable topology can be introduced in the space Ω^S.

Every probability measure on $\mathcal{B}(\Omega)$ determines a labelled random field in Q with values in the space Q of charges.

(3) *Ordinary or generalized fields in R^ν.*

The probability is a topological vector (locally convex) space Ω of functions or distributions (in the sense of Schwartz) defined on R^ν in this case. A random field is given again by a definition of a probability measure on the Borel σ-algebra $\mathcal{B}(\Omega)$.

2.2 Method of Gibbs Modifications

We can construct new measures from an originally given measure μ_0 (or from a family of measures) with Gibbs modification. We first describe a general scheme of dealing with Gibbs modifications.

Finite Gibbs Modifications:

Definition 2.2.1 *Let (Ω, Σ, μ_0) be a measurable space with a finite or σ-finite measure μ_0 (called usually a free measure), and let $U(x)$, $x \in \Omega$, be a real function on Ω (taking possibly the value $+\infty$), often called the interaction energy (or Hamiltonian). The measure μ with density*

$$\frac{d\mu}{d\mu_0}(x) = Z^{-1} \exp\{-U(x)\} \tag{2.1}$$

with respect to the measure μ_0 is called Gibbs modification of the measure μ_0 by means of the interaction U.

Moreover it is assumed that the normalization factor Z (called the partition function) satisfies the stability condition:

$$Z = \int_\Omega \exp\{-U(x)\}\, d\mu_0(x) \neq 0, \infty. \tag{2.2}$$

Using finite Gibbs modification, the measures absolutely continuous with respect to μ_0 arise. A more interesting class of measures, already singular with respect to the original measure μ_0, arises when passing to the weak limit of finite Gibbs modifications.

2.3 Weak Convergence of Measures

Let Ω be a topological space, $\mathcal{B}(\Omega)$ its Borel σ-algebra, and $\Sigma \subset \mathcal{B}$ some of its sub-σ-algebras.

Definition 2.3.1 *Let a directed family $\mathcal{F} = \{\Lambda\}$ of indices be given. Then the measure μ, defined on the σ-algebra $\Sigma \subset \mathcal{B}$ is called the weak limit of the sequence of measures μ_0, $\Lambda \in \mathcal{F}$, defined on Σ if*

$$\int_\Omega f(x)\, d\mu_\Lambda \to \int_\Omega f(x)\, d\mu \tag{2.3}$$

for any bounded continuous Σ-measurable function f given on Ω.

For more general situation we have:

Definition 2.3.2 *Let a complete family $\{\Sigma_\Lambda,\ \Lambda \in \mathcal{F}\}$, where $\Sigma_{\Lambda_1} \subset \Sigma_{\Lambda_2}$, $\Lambda_1 < \Lambda_2$, of sub-σ-algebras of the σ-algebra $\mathcal{B}$ (such that $\mathcal{B}$ coincides with the smallest Σ-algebra containing the algebra $\mathfrak{A} = \cup_{\Lambda\in\mathcal{F}}\Sigma_\Lambda$) be given. The σ-algebras Σ_Λ are called local σ-algebras and any function f, defined on Ω and measurable with respect to some of the local algebras, is called a local function (function f, measurable with respect to a σ-algebra Σ_A, $A \in \mathcal{F}$, is denoted by f_A).*

With the following two definitions we generalize the Definition (2.3.1).

Definition 2.3.3 *A finitely additive measure μ defined on the algebra $\mathfrak{A}$ so that its restriction $\mu|_{\Sigma_\Lambda}$ to any σ-algebra Σ_Λ is a σ-additive measure on Σ_Λ is called a cylinder measure.*

Definition 2.3.4 *Let a finite or σ-finite measure be given on each σ-algebra Σ_Λ. A cylinder measure μ on $\mathfrak{A}$ is called the weak local limit of the measures μ_Λ if*

$$\lim_{\Lambda} \int_\Omega f(x)\, d\mu_\Lambda \to \int_\Omega f(x)\, d\mu \tag{2.4}$$

for any bounded continuous local function f defined on Ω.

It means that, a cylinder measure (or its extension to a measure on the σ-algebra $\mathcal{B}$) is the weak local limit of measures $\{\mu_\Lambda,\ \Lambda \in \mathcal{F}\}$ if, for each $\Lambda_0 \in \mathcal{F}$, the restrictions $\mu_\Lambda|_{\Sigma_{\Lambda_0}} = \mu_\Lambda^{\Lambda_0}$, $\Lambda \in \mathcal{F}$, of the measures μ_Λ to the σ-algebra Σ_{Λ_0} weakly converge to $\mu|_{\Sigma_{\Lambda_0}} = \mu_{\Lambda_0}$.

If we consider the case $\Omega = S^T$ (with T is countable set and S is a metric space), the index Λ runs over finite subsets of T, and $\Sigma_\Lambda = \varphi_\Lambda^{-1}(\mathcal{B}(S^\Lambda))$ where $\varphi_\Lambda : S^T \to S^\Lambda$ is the restriction mapping, then the convergence (2.4) is called the weak convergence of finite-dimensional distributions if, μ_Λ are probability measures.

The following proposition gives us the relation between the definitions (2.3.1) and (2.3.4).

Proposition 2.3.5 *Let a family $\{\Sigma_\Lambda,\ \Lambda \in \mathcal{F}\}$ of σ-algebras be such that the set $C_0(\Omega)$ of bounded continuous local functions is dense everywhere in the space $C(\Omega)$ of all bounded continuous functions defined on Ω (in the uniform metric in $C(\Omega)$). Then the necessary and sufficient condition for a measure μ on $\mathcal{B}(\Omega)$ to be the local limit of probability measures $\{\mu_\Lambda\}$ (defined each on the σ-algebra Σ_Λ) is that their arbitrary extensions $\tilde{\mu}$ to probability measures on the σ-algebra $\mathcal{B}(S)$ weakly converge to μ.*

2.4 Limit Gibbs Modifications

Definition 2.4.1 *Let a complete directed family $\{\Sigma_\Lambda,\ \Lambda \in \mathcal{F}\}$ of sub-σ-algebra $\mathcal{B}(\Omega)$ be given and let a free measure μ_Λ^0 and a Hamiltonian U_Λ be defined for each Λ so that the stability condition (2.2) is satisfied. A cylinder measure μ on the algebra $\mathfrak{A} = \cup\Sigma_\Lambda$ (or its σ-additive extension to the σ-algebra $\mathcal{B}(S)$) is called a limit Gibbs measure (or a limit Gibbs modification) of the measure μ_Λ^0 (by means of the energies U_Λ).*

The theory of Gibbs measures depends on the choice of σ-algebras Σ_Λ, measures μ_Λ^0, and Hamiltonians U_Λ. Let us describe the ways of such a choice of Σ_Λ, μ_Λ^0, and U_Λ for random fields in a countable set T.

Gibbs modifications of fields in a countable set T: Let a set of configurations $S^\Lambda = \{x^\Lambda = (x_t,\ t \in \Lambda)\}$ be defined in finite $\Lambda \subset T$. And endow it with a Tikhonov topology and Borel σ-algebra $\mathcal{B}(S^\Lambda)$. The restriction mapping

$$\varphi : x \to x^\Lambda = x|_\Lambda \tag{2.5}$$

defines a σ-algebra $\Sigma_\Lambda = \varphi_\Lambda^{-1}(\mathcal{B}(S^\Lambda)) \subset \mathcal{B}(S^T)$ that will be identified with $\mathcal{B}(S^\Lambda)$. The set $\{\Sigma_\Lambda,\ \Lambda \in T\}$ is obviously complete in $\mathcal{B}(S^T)$.

Remark 2.4.2 *The set $C_0(S^T) \in C(S^T)$ of bounded continuous local functions on S^T is dense everywhere in $C(S^T)$ (according to Stone's theorem).*

Hamiltonians U_Λ are usually defined with help of functions Φ_A on Ω such that Φ_A are measurable with respect to σ-algebra Σ which is called potential $\{\Phi_A; A \subset T,\ |A| < \infty\}$. Φ_A can be viewed as a function defined on the space S^A. For any finite A, we put

$$U_\Lambda = \sum_{A \in \Lambda} \Phi_A. \tag{2.6}$$

Instead of the explicit formula for the potential, formal Hamiltonian (formal sum)

$$U = \sum_{A} \Phi_A \tag{2.7}$$

is used.

Remark 2.4.3 *In many cases, the free measures μ_Λ^0 are restrictions of some probability measure μ_0 defined on S^T to the respective σ-algebras $\Sigma_A \in \mathcal{B}$.*

In such cases the measure $\tilde{\mu}_\Lambda$ given on the σ-algebra $\mathcal{B}(S^T)$ by

$$\frac{d\tilde{\mu}_\Lambda}{d\mu_0}(x) = Z_\Lambda^{-1} \exp\{-U_\Lambda(x)\}$$

is investigated instead of a Gibbs modification μ_Λ defined by the formula (2.1).

The measure $\tilde{\mu}$ is a natural extension of the measure μ_Λ to the whole σ-algebra $\mathcal{B}(S^T)$. This measure also called a finite Gibbs modification of the measure μ_0. A limit Gibbs measure μ on the space S^T (i.e., the weak limit of the measures μ_Λ) is the weak limit of the measures $\tilde{\mu}$, $\Lambda \nearrow T$.

2.5 Weak Compactness of Measures

Let $\mathcal{A}$ be some collection of measures defined on the whole Borel σ-algebra $\mathcal{B}(\Omega)$ of a topological space Ω or on its sub-σ-algebra $\Sigma \subset \mathcal{B}(\Omega)$. By the weak compactness of the set $\mathcal{A}$, it is sequentially compact such that there is a weakly converging sequence $\mu_n \to \mu,\ n \to \infty,\ \mu_n \in B$, in any infinite subset $B \in \mathcal{A}$.

Lemma 2.5.1 *For weak compactness of the set $\mathcal{A}$ in a complete separable metric space Ω and $\Sigma = \mathcal{B}(\Omega)$, the following conditions are sufficient:*

1) *Each measure $\mu \in \mathcal{A}$ is a probability measure, and there is a compact function $h > 0$ defined on Ω such that*

$$\int_\Omega h(x) d\mu < C$$

for any measure $\mu \in \mathcal{A}$ where C does not depend on μ. A function h on Ω is called compact if the set $\{x \in \Omega,\ h(x) < a\}$ is compact for any $a > 0$.

2) *There are a nonnegative measure μ_0 on $\mathcal{B}(\Omega)$ and a μ_0-integrable function $\varphi(x) \geq 0$ such that any measure $\mu \in \mathcal{A}$ is absolutely continuous with respect to μ_0 and*

$$\left| \frac{d\mu}{d\mu_0}(x) \right| < \varphi(x), \quad x \in \Omega.$$

Definition 2.5.2 *Let $\{\mu_n,\ \Lambda \in \mathcal{F}\}$ be a family of measures defined on the σ-algebra Σ_Λ from a complete family $\{\Sigma_\Lambda,\ \Lambda \in \mathcal{F}\}$ of sub-σ-algebras of the σ-algebra $\mathcal{B}(\Omega)$ where $\mathcal{F}$ is a directed family of indices. A family $\{\mu_n,\ \Lambda \in \mathcal{F}\}$ is called weakly locally compact if the set $\{\mu_\Lambda^{\Lambda_0},\ \Lambda_0 < \Lambda\}$ of restrictions of measures $\{\mu_n\}$ to the σ-algebra Σ_{Λ_0} is weakly compact for any $\Lambda_0 \in \mathcal{F}$.*

Lemma 2.5.3 *Let a family $\{\mu_\Lambda,\ \Lambda \in \mathcal{F}\}$ of measures be locally compact. Then, in any increasing sequence $\Lambda_1 < \Lambda_2 < ... < \Lambda_n < ...$ of indices where the sequence* *of σ-algebras Σ_{Λ_n}, $n = 1, 2, \ldots$, is complete, there is a subsequence having the same property and a cylinder measure μ on $\mathfrak{A} = \cup\Sigma_\Lambda$ such that*

$$\mu = \lim_{h\to\infty} \mu_{i_k} \qquad (\mu_\Lambda = \mu_{\Lambda_n}). \tag{2.8}$$

2.6 Gibbs Modifications Under Boundary Conditions And Definition Of Gibbs Fields By Means Of Conditional Distributions

We now give more general definition of the limit Gibbs field in the case of fields in a countable set T with values in a metric space S. We suppose that a finite or σ-finite measure is defined on S, and we choose, for any finite set $\Lambda \subset T$, the measure $\mu_0^\Lambda = \lambda_0^\Lambda$ where the product of $|\Lambda|$ copies of the measure λ, as the free measure μ_Λ^0 on the space S^Λ.

Further, we suppose that a potential $\{\Phi_A; A \subset T,\ |A| < \infty\}$ of a finite range is given, and that Hamiltonian $U_\Lambda = \sum\limits_{A\subseteq\Lambda} \Phi_A$ determined by it satisfies the stability condition

$$0 < \int_{S^\Lambda} \exp\{-U_\lambda(x)\} d\lambda_0^\Lambda < \infty$$

for any finite $\Lambda \subset T$.

Let μ_Λ be a Gibbs modification of the measure λ_0^Λ, and for any $\Lambda_0 \subset \Lambda$, we use $\mu_\Lambda^{\Lambda_0}(\cdot/\overline{x}^{\Lambda\setminus\Lambda_0})$ to denote the conditional probability distribution on the set of configurations $x^{\Lambda_0} \in S^{\Lambda_0}$ under the condition that a configuration $\overline{x}^{\Lambda\setminus\Lambda_0} \in S^{\Lambda\setminus\Lambda_0}$, in the set $\Lambda \setminus \Lambda_0$, is fixed. The density of the measure $\mu_\Lambda^{\Lambda_0}(\cdot/\overline{x}^{\Lambda\setminus\Lambda_0})$ with respect to the measure $\lambda_0^{\Lambda_0}$ equals

$$\frac{d\mu_{\Lambda}^{\Lambda_0}(x^{\Lambda_0/\overline{x}^{\Lambda\setminus\Lambda_0}})}{d\lambda_0^{\Lambda_0}} = Z_{\Lambda_0}^{-1}(\overline{x}^{\Lambda\setminus\Lambda_0})\exp\{-U_{\Lambda_0}(x^{\Lambda_0}/\overline{x}^{\Lambda\setminus\Lambda_0})\} \tag{2.9}$$

with

$$Z_{\Lambda_0}(\overline{x}^{\Lambda\setminus\Lambda_0}) = \int_{S^{\Lambda_0}} \exp\{-U\Lambda_0(x^{\Lambda_0}/\overline{x}^{\Lambda\setminus\Lambda_0})\}d\lambda_0^{\Lambda_0}$$

$$U_{\Lambda_0}(x^{\Lambda_0}/\overline{x}^{\Lambda\setminus\Lambda_0}) = U_{\Lambda_0}(x^{\Lambda_0}) + \sum_{\substack{A:A\cap\Lambda_0\neq\emptyset\\ A\cap(\Lambda\setminus\Lambda_0)\neq\emptyset}} \Phi_A(x^{\Lambda_0}\cup\overline{x}^{\Lambda\setminus\Lambda_0}) \tag{2.10}$$

where $(x^{\Lambda_0}\cup\overline{x}^{\Lambda\setminus\Lambda_0})$ denotes the configuration in Λ whose restrictions to Λ_0 and $\Lambda\setminus\Lambda_0$ are equal to $\overline{x}^{\Lambda_0}$ and $\overline{x}^{\Lambda\setminus\Lambda_0}$ respectively. The second expression in (2.10) is called the energy of the interaction with a boundary configuration.

For a fixed Λ_0 and a sufficiently large $\Lambda\supset\Lambda_0$ so that $\rho(\Lambda_0,\ T\setminus\Lambda) > d$ with given metric on T, for some constant $d > 0$, the energy $U_{\Lambda_0}(x^{\Lambda_0}/\overline{x}^{\Lambda\setminus\Lambda_0})$ does not depend on the whole configuration Λ_0 but only its restriction $\overline{x}^{\partial_d\Lambda_0}$ to the d-neighbourhood of Λ_0 where $\partial_d\Lambda_0 = \{t\in T\setminus\Lambda_0,\ \rho(t,\Lambda_0)\leq d\}$.

We denote this energy by

$$U_{\Lambda_0}(x^{\Lambda_0}/\overline{x}^{\Lambda\setminus\Lambda_0}) \tag{2.11}$$

and we denote the Gibbs modification of the measure $\lambda_0^{\Lambda_0}$ by means of the Hamiltonian (1.14) by $\mu^{\Lambda_0}_{\overline{x}^{\partial_d\Lambda_0}}$. The measure $\mu^{\Lambda_0}_{\overline{x}^{\partial_d\Lambda_0}}$ is called the Gibbs distribution on Λ_0 with the boundary configuration $\overline{x}^{\partial_d\Lambda_0}$ in the neighbourhood $\partial_d\Lambda_0$.

By the formula (2.9) we get the following:

Definition 2.6.1 *A probability measure μ on the space S^T is called a Gibbs distribution in T (for a given potential $\{\Phi_A\}$) if, for any finite $\Lambda\subset T$ and any configuration $\overline{x}\in S^{T\setminus\Lambda}$, the conditional distribution $\mu(\cdot/x^{T\setminus\Lambda}=\overline{x})$ on the set S^Λ coincides, under the condition that the external configuration $x^{T\setminus\Lambda}$ is fixed and equal to $\overline{x}$, with the measure $\mu_{\overline{x}^{\partial_d\Lambda}}$ given by*

$$\mu(\cdot/x^{T\setminus\Lambda} = \overline{x}) = \mu^{\Lambda}_{\overline{x}^{\partial_d\Lambda}} \tag{2.12}$$

with $\overline{x}^{\partial_d\Lambda}$ *being a restriction of* $\overline{x}$ *to* $\partial_d\Lambda$.

From the definition (2.3.4) d-Markov property holds. We will give the definition of the Markov property for the Ising model.

Let $\Lambda \subset T$ be a finite set and let some probability distribution $q = q^{\partial_d\Lambda}$ on the set $S^{\partial_d\Lambda}$ of boundary configurations $\overline{x} = \overline{x}^{\partial_d\Lambda}$ be given then the measure

$$\mu^{\Lambda}_{q} = \int_{S^{\partial_d\Lambda}} \mu^{\Lambda}_{\overline{x}} dq(\overline{x}) \tag{2.13}$$

on S^{Λ} is called a Gibbs distribution with a q-random boundary configuration in Λ.

Proposition 2.6.2 *For a measure* μ *on the space* S^T *to be Gibbsian, it is necessary that, for any increasing sequence* $\Lambda_n \nearrow T$, $n \to \infty$, *of finite sets* Λ_n, *there is a sequence of distributions* $q_n = q^{\partial_d\Lambda_n}$ *defined each on the set* $S^{\partial_d\Lambda}$ *of boundary configurations, so that the weak local limit of measures* $\mu^{\Lambda_n}_{q_n}$ *coincides with* μ, *it means that*

$$\lim_{n\to\infty} \mu^{\Lambda_n}_{q_n} = \mu, \tag{2.14}$$

and it is sufficient that the condition (2.14) is satisfied for some increasing sequence $\Lambda_n \nearrow T$.

Proof.

Necessity. We choose a distribution $q = \mu|_{S^{\partial_d\Lambda}}$ on the set $S^{\partial_d\Lambda}$ induced by the measure μ for any $\Lambda \subset T$. Then it is obvious that $\mu^{\Lambda}_{q} = \mu|_{S_\Lambda}$, and (2.14) is satisfied.

Sufficiency. For any $\Lambda_0 \subset \Lambda$ such that $\rho(\Lambda_0,\ T-\Lambda) > d$ and any distribution q of boundary configurations $\tilde{x} \in S^{\partial_d \Lambda}$, the conditional distribution $\mu_q^\Lambda(\cdot/\tilde{x}^{\Lambda\setminus\Lambda_0})$ on S^{Λ_0} generated by the Gibbs measure μ_q^Λ on Λ with random boundary configuration coincides with the measure $\mu^{\Lambda_0}_{\tilde{x}^{\partial_d \Lambda}}$, where $\tilde{x}^{\partial_d \Lambda_0}$ is the restriction of the configuration $\tilde{x}^{\Lambda\setminus\Lambda_0}$ to $\partial_d \Lambda_0$, i.e.,

$$\mu_q^\Lambda(\cdot/\tilde{x}^{\Lambda\setminus\Lambda_0}) = \mu^{\Lambda_0}_{\tilde{x}^{\partial_d \Lambda_0}} \tag{2.15}$$

Now let $\Lambda_n \nearrow T$ and q_n be sequences such that (2.14) is satisfied. Since the equality (2.15) is satisfied for any fixed $\Lambda_0 \subset T$ and all sufficiently large Λ_n, it is also valid for the limit measure μ.

Corollary 2.6.3 *Let a family $\{\mu_{\tilde{x}}^\Lambda\}$ of Gibbs modifications be such that there is a unique limit*

$$\mu = \lim_{\Lambda \nearrow T} \mu_{\tilde{x}}^\Lambda$$

for any sequence $\Lambda \nearrow T$ and any choice of boundary configurations $\tilde{x} \in S^{\partial_d \Lambda}$. Then μ is the unique Gibbs measure on S^T.

Chapter 3

Models

Let G be a countably infinite graph of bounded degree with the set V of vertices and the collection of unordered (possibly coinciding) pairs of elements of V (edges). We consider a system on G whose symbol set on V is F, a finite set of at least two elements. A configuration on $A \subseteq V$ is a map

$$\sigma_A : A \to F.$$

A measure μ on F^{ν} is said to be a Markov random field if μ admits conditional probabilities such that for all finite $W \subset V$, all $\xi_w \in F^W$ and all $\sigma_{V-W} \in F^{V-W}$ we have

$$\mu(\xi_W|\sigma_{V-W}) = \mu(\xi_W|\sigma_{V-W}(\partial W)). \tag{3.1}$$

In other words Markov random field property says that the conditional distribution of what we see on W given everything else depends only the values on the boundary ∂W.

A consistent set of conditional distributions for all finite W and all boundary conditions σ_{V-W} is called a specification, denoted by Q. The specification Q is said to be Markovian if every element in Q is Markov (see (3.1)).

If μ is a measure on F^V satisfying all conditional distributions in a Markovian Q, we say that μ is a Gibbs measure for Q. Such measures are automatically Markov random fields and the existence of Gibbs measure follows from standard compactness arguments. The fundamental question is whether a Markovian specification allows the existence of more than one Gibbs measure. If this is the case, we say that we have phase transition. One of the most studied models which can exhibit a phase transition is the Ising model.

3.1 Ising Model

The Ising model is an important mathematical model of a ferromagnetic metal. It is the one of the simplest mathematical models to exhibit a phase transition: at high temperature, there is a unique equilibrium state for the system, but at temperatures below a certain critical temperature, there are several distinct equilibrium states. Now we introduce the Ising model.

Consider the lattice Z^ν of points $t = (t^{(1)}, ..., t^{(n)}) \in R^\nu$ of ν-dimensional real space with integer coordinates. The points in Z^ν whose coordinates have absolute values not grater than N (with an integer $N > 0$) defines a set $\Lambda_N = \Lambda$ which is called cube in Z^ν centered at the origin.

Each function $\sigma^\Lambda = \{\sigma_t,\ t \in \Lambda\}$, which defined on the set Λ and taking values $\sigma_t = \pm 1$, is called a configuration (in the cube) and the set of all such configurations is denoted by Ω_Λ. The number of configurations in Λ equals $2^{|\Lambda|}$, where $|\Lambda|$ is the number of lattice sites in Λ.

We define the energy of the configuration σ^Λ on Ω_Λ by the function

$$U_\Lambda \equiv U_\Lambda(\sigma^\Lambda) = -\left(h \sum_{t\in\Lambda} \sigma_t + \beta \sum_{\langle t,t'\rangle} \sigma_t \sigma_{t'} \right) \tag{3.2}$$

The summation in the second sum in the equation (3.2) is taken over all unordered pairs $\langle t, t'\rangle, t, t' \in \Lambda$, such that $\rho(t, t') = 1$, with

$$\rho(t,t') = \sum_{i=1}^{\nu} \left| t^{(i)} - t'^{(i)} \right|, \tag{3.3}$$

where

$$t = \left(t^{(1)}, ..., t^{(\nu)}\right), t = \left(t'^{(1)}, ..., t'^{(\nu)}\right).$$

Definition 3.1.1 *A physical system with the configuration space Ω_Λ of configurations in Λ and a configuration energy of the form (3.2) is called the Ising model.*

The real numbers h and β are the parameters of the model and they are fixed. We are interested in the case $\beta > 0$ which is called ferromagnetic Ising model.

Now we define the probability of a configuration σ^Λ on the space Ω_Λ by

$$P_\Lambda(\sigma^\Lambda) = Z_\Lambda^{-1} \exp\{-U_\Lambda(\sigma^\Lambda)\} \tag{3.4}$$

where the quantity Z_Λ is called a partition function and can be found by the condition

$$\sum_{\sigma_\Lambda \in \Omega_\Lambda} P_\Lambda(\sigma^\Lambda) = 1$$

and thus

$$Z_\Lambda = \sum_{\sigma_\Lambda \in \Omega_\Lambda} \exp\{-U_\Lambda(\sigma^\Lambda)\}. \tag{3.5}$$

The probability distribution (3.4) is called a Gibbs probability distribution in Λ corresponding to the Ising model.

We can consider the configurations σ^Λ as random variables and the formula (3.4) as the joint probability distribution of these random variables. We denote the mean (value) of an arbitrary function f on the space Ω_Λ under the distribution (3.4) by $\langle f\rangle_\Lambda$. The means $\langle \sigma_T\rangle_\Lambda$ of random variables

$$\sigma_T = \prod_{t\in T}\sigma_t, \qquad \sigma_\emptyset = 1, \tag{3.6}$$

with $T \subset \Lambda$ being an arbitrary subset of Λ, are called correlation functions (or moments) of the distribution (3.4).

For any $T \subset \Lambda$, we use $P_\Lambda^{(T)}$ to denote the joint distribution of the system of random variables $\{\sigma_t,\ t \in T\}$, i.e., the collection of probabilities

$$P_\Lambda^{(T)}(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}) = Pr(\sigma_{t_1} = \overline{\sigma}_{t_1}, ..., \sigma_{t_n} = \overline{\sigma}_{t_n}), \tag{3.7}$$

with $T = \{t_1, ..., t_n\}$ and $\{\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}\}$ being an arbitrary collection of values $\overline{\sigma}_{t_i} = \pm 1,\ i = 1, 2, ..., n$. The probabilities (3.7) can be expressed by means of correlation functions $\langle \sigma_T\rangle_\Lambda$. Indeed

$$\begin{aligned} P_\Lambda^{(T)}(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}) &= \frac{1}{2^n}(-1)^k \left\langle \prod_{i=1}^{n}(\sigma_{t_i} + \overline{\sigma}_{t_i})\right\rangle_\Lambda \\ &= \frac{(-1)^k}{2^n}\sum_{T'\subseteq T} C_{T'}\langle \sigma_{T'}\rangle_\Lambda \end{aligned} \tag{3.8}$$

where k is the number of values $\overline{\sigma}_t$, that equal -1, and

$$C_{T'} = \prod_{t\in T\setminus T'}\overline{\sigma}_t.$$

3.1.1 Thermodynamic Limit

Let the set T be fixed and Λ expand to Z^ν, $\Lambda \nearrow Z^\nu$, i.e., put $N \to \infty$. Now consider the limit

$$\lim_{\Lambda \nearrow Z^\nu} \langle \sigma_T \rangle_\Lambda. \tag{3.9}$$

If we prove the existence of above limits then we may conclude that correlation functions (and finite-dimensional distributions) almost do not depend on Λ for Λ sufficiently large in comparison with T. Such a passage to the limit is called the thermodynamic limit (the limit of a large number of degrees of freedom σ_t). The limits (3.9) are called limit correlation functions and denoted by $\langle \sigma_t \rangle$.

Finite-dimensional distributions also have limits which form a compatible family of finite-dimensional distributions by (3.8), and by Kolmogorov theorem this family defines a system of random variables $\{\sigma_t,\ t \in Z^\nu\}$, called a (limit) Gibbs random field (for the Ising model), as well as their distribution P (a measure) on the space $\Omega = \{-1, 1\}^{Z^\nu}$ of finite configurations in the lattice Z^ν.

The existence of the limit distribution P follows from the following theorem.

Theorem 3.1.2 *The thermodynamic limit (3.9) of correlation functions* $\langle \sigma_T \rangle_\Lambda$ *exists for* $\beta \geq 0$ *and every finite* T.

Proof.

Remark 3.1.3 *In the case of* $\beta = 0, \langle \sigma_T \rangle_\Lambda$ *can be easily evaluated:*

$$\langle \sigma_T \rangle_\Lambda = \left(\frac{e^h - e^{-h}}{e^h + e^{-h}} \right)^{|T|} \tag{3.10}$$

We see that, $\langle \sigma_T \rangle_\Lambda$ does not depend on Λ (for $T \subset \Lambda$). So the thermodynamic limit $\langle \sigma_T \rangle_\Lambda$ exists in this case and equals (3.10). The random variables σ_t are mutually independent, both with respect to the distributions in finite Λ and with respect to the limit distributions.

Remark 3.1.4 *It is sufficient to consider the case* $h \geq 0$*, because of the following property of the Ising model (in the notations given below,* β *and* h *as subscripts indicate the dependence of Gibbs distributions on these parameters):*

$$
\begin{aligned}
P_{\Lambda,\beta,h}(\sigma_\Lambda) &= \frac{\exp\left\{h\sum_{t\in\Lambda}\sigma_t+\beta\sum_{\langle t,t'\rangle}\sigma_t\sigma_{t'}\right\}}{\sum_{\sigma^\Lambda\in\Omega_\Lambda}\exp\left\{h\sum_{t\in\Lambda}\sigma_t+\beta\sum_{\langle t,t'\rangle}\sigma_t\sigma_{t'}\right\}} \\
&= \frac{\exp\left\{-h\sum_{t\in\Lambda}-\sigma_t+\beta\sum_{\langle t,t'\rangle}\sigma_t\sigma_{t'}\right\}}{\sum_{\sigma^\Lambda\in\Omega_\Lambda}\exp\left\{-h\sum_{t\in\Lambda}-\sigma_t+\beta\sum_{\langle t,t'\rangle}\sigma_t\sigma_{t'}\right\}} \\
&= P_{\Lambda,\beta,-h}(-\sigma_\Lambda) \qquad (3.11)
\end{aligned}
$$

where $-\sigma^\Lambda$ *denotes the configuration whose values have an opposite sign to those of the configuration* σ^Λ.

It follows from (3.11) that

$$
\begin{aligned}
\langle\sigma_T\rangle_{\Lambda,\beta,h} &= \frac{\sum_{T\subset\Lambda}\sigma_T\exp\left\{h\sum_{t\in\Lambda}\sigma_t+\beta\sum_{\langle t,t'\rangle}\sigma_t\sigma_{t'}\right\}}{\sum_{\sigma^\Lambda\in\Omega_\Lambda}\exp\left\{h\sum_{t\in\Lambda}\sigma_t+\beta\sum_{\langle t,t'\rangle}\sigma_t\sigma_{t'}\right\}} \\
&= \sum_{T\subset\Lambda}\sigma_T P_{\Lambda,\beta,h}
\end{aligned}
$$

so that

$$
\langle\sigma_T\rangle_{\Lambda,\beta,h} = \begin{cases} \langle\sigma_T\rangle_{\Lambda,\beta,-h} & ,|T| \quad \text{even} \\ -\langle\sigma_T\rangle_{\Lambda,\beta,-h} & ,|T| \quad \text{odd} \end{cases} \qquad (3.12)
$$

In particular, for odd $|T|$,

$$\langle \sigma_T \rangle_{\Lambda,\beta,0} = 0. \tag{3.13}$$

We need two inequalities to prove the theorem and we can consider more general situation. Let Λ be an arbitrary subset of Z^ν, Ω_Λ a set of all configurations $\sigma^\Lambda = \{\sigma_t,\ t \in \Lambda\}$, $\sigma_t = \pm 1$, in Λ, and the energy $U_\Lambda(\sigma^\Lambda)$ of the configuration σ^Λ be of the form

$$U_\Lambda(\sigma^\Lambda) = -\left(\sum_{t\in\Lambda} h_t \sigma_t + \sum_{t,t'} \beta_{t,t'} \sigma_t \sigma_{t'} \right) \tag{3.14}$$

where $h_t \geq 0$ and $\beta_{t,t'} \geq 0$. The distribution P_Λ on Ω_Λ is given as in the equation (3.4), and $\langle\ \rangle_\Lambda$ denotes the mean under the distribution.

Lemma 3.1.5 *The first Griffith inequality*

$$\langle \sigma_T \rangle_\Lambda \geq 0 \tag{3.15}$$

and the second Griffith inequality

$$\langle \sigma_T \sigma_{T'} \rangle_\Lambda - \langle \sigma_{T'} \rangle_\Lambda \geq 0 \tag{3.16}$$

are valid.

Proof. For the first inequality it is sufficient to verify

$$\sum_{\sigma^\Lambda \in \Omega_\Lambda} \sigma_T \exp\{-U_\Lambda(\sigma_\Lambda)\} \geq 0, \tag{3.17}$$

since $\langle \sigma_T \rangle_\Lambda = \sum_{\sigma^\Lambda \in \Omega_\Lambda} \sigma_T Z^{-1} \exp\{-U_\Lambda(\sigma^\Lambda)\}$ and $Z = \sum_{\sigma^\Lambda \in \Omega} \exp\{-U_\Lambda(\sigma^\Lambda)\} \geq 0.$

Let us first expand the exponential function $\exp\{-U_\Lambda(\sigma\Lambda)\}$ in the series $\sum_{n=0}^{\infty} \frac{(-U_\Lambda)^n}{n!}$, and then by removing the parentheses in each term of this series, and, by taking into account that $\sigma_t^2 = 1$, then the left-hand side of the inequality (3.15) becomes

$$\sum_{B\subseteq\Lambda} C_B \sum_{\sigma^\Lambda\in\Omega_\Lambda} \sigma_B \tag{3.18}$$

with $C_B \geq 0$. Since for any $t \in \Lambda$

$$\sum_{\sigma_t=\pm 1} \sigma_t = 0, \tag{3.19}$$

the sum (3.18) is equal to $C_\emptyset$, which proves (3.15).

For the second inequality we investigate two independent samples of distribution P_Λ, i.e., a distribution on the space $\Omega_\Lambda \times \Omega_\Lambda$ of pairs $\{\sigma^\Lambda, \tilde{\sigma}^\Lambda\}$ of configurations of the form

$$\widehat{P}(\sigma^\Lambda, \tilde{\sigma}^\Lambda) = (Z^{-1})^2 \exp\left\{\sum_{t\in\Lambda} h_t(\sigma_t + \tilde{\sigma}_t) + \sum_{t,t'\in\Lambda} \beta_{t,t'}(\sigma_t\sigma_{t'} + \tilde{\sigma}_t\tilde{\sigma}_{t'})\right\}. \tag{3.20}$$

Let us introduce new variables

$$\xi_t = \sigma_t + \tilde{\sigma}_t, \qquad \eta_t = \sigma_t - \tilde{\sigma}_t, \qquad t \in \Lambda,$$

taking values $(\xi_t, \eta_t) = (2,0), (-2,0), (0,2), (0,-2)$. By taking these variables, the probability (3.20) may be written in the form

$$(Z^{-1})^2 \exp\left\{\sum_{t\in\Lambda} h_t\xi_t + \frac{1}{2}\sum_{t,t'\in\Lambda} \beta_{t,t'}(\xi_t\xi_{t'} + \eta_t\eta_{t'})\right\}.$$

Then taking into account that

$$\xi_t\eta_t = 0 \quad \text{and} \quad \sum_{\xi_t=-2,0,2} \xi_t^k = \sum_{\eta_t=-2,0,2} \eta_t^k \geq 0$$

for each $t \in \Lambda$ and each integer $k \geq 0$, and by repeating the proof of the first inequality, we get

$$\langle \xi_T \eta_T \rangle_{\Lambda,\Lambda} \geq 0 \tag{3.21}$$

for all T and T', where ξ_T and η_T are defined as in (3.6), and the mean $\langle\ \rangle_{\Lambda,\Lambda}$ is evaluated under the distribution (3.20).

Notice that

$$\langle \sigma_T \sigma_{T'} \rangle_\Lambda - \langle \sigma_T \rangle_\Lambda \langle \sigma_{T'} \rangle_\Lambda = \frac{1}{2} \langle (\sigma_T - \tilde{\sigma}_T)(\sigma_{T'} - \tilde{\sigma}_{T'}) \rangle_{\Lambda,\Lambda} \tag{3.22}$$

We shall show that the difference $\sigma_T - \tilde{\sigma}_T$ and the sum $\sigma_T + \tilde{\sigma}_T$ can be represented in the form

$$\sigma_T \pm \tilde{\sigma}_T = \sum_{A,B \subseteq T} C^{\pm}_{A,B} \xi_A \eta_B \tag{3.23}$$

with $C^{\pm}_{A,B} \geq 0$. The relations (3.21), (3.22), and (3.23) imply the inequality (3.17). The equation (3.23) can be proved by induction on $|T|$ if we notice that

$$\sigma_{T\cup\{t\}} + \tilde{\sigma}_{T\cup\{t\}} = \frac{1}{2}[(\sigma_T + \tilde{\sigma}_T)\xi_t + (\sigma_T - \tilde{\sigma}_T)\eta_t],$$

$$\sigma_{T\cup\{t\}} - \tilde{\sigma}_{T\cup\{t\}} = \frac{1}{2}[(\sigma_T + \tilde{\sigma}_T)\eta_t + (\sigma_T - \tilde{\sigma}_T)\xi_t]$$

for $t \notin T \subset \Lambda$. The lemma is proved.

Now the derivatives $\dfrac{\partial}{\partial h_t}\langle\sigma_T\rangle_\Lambda$ and $\dfrac{\partial}{\partial \beta_{t,t'}}\langle\sigma_T\rangle_\Lambda$ are equal to

$$\frac{\partial}{\partial h_t}\langle\sigma_T\rangle_\Lambda = \langle\sigma_T\sigma_t\rangle_\Lambda - \langle\sigma_T\rangle_\Lambda\langle\sigma_t\rangle_\Lambda \geq 0,$$

$$\frac{\partial}{\partial \beta_{t,t'}}\langle\sigma_T\rangle_\Lambda = \langle\sigma_T\sigma_t\sigma_{t'}\rangle_\Lambda - \langle\sigma_T\rangle_\Lambda\langle\sigma_t\sigma_{t'}\rangle_\Lambda \geq 0, \tag{3.24}$$

and thus the correlation functions increase when increasing the parameters h_t and $\beta_{t,t'}$. It follows that in the case of the Ising model,

$$\langle\sigma_T\rangle_{\Lambda_1} \leq \langle\sigma_T\rangle_{\Lambda_2} \tag{3.25}$$

for $T \subset \Lambda_1 \subset \Lambda_2$. Indeed with parameters

$$h_t = \begin{cases} h, & t \in T \\ 0, & t \in \Lambda_1 \setminus \Lambda_2 \end{cases}$$

and

$$\beta_{t,t'} = \begin{cases} \beta, & \text{if } t, t' \text{ are nearest neighbours in } \Lambda_1 \\ 0, & \text{otherwise} \end{cases}$$

the mean $\langle\sigma_T\rangle_{\Lambda_1}$ coincides with the mean under the distribution of the form (3.14) in Λ_2.

We get (3.25), by using the monotonicity of $\langle\sigma_T\rangle$ with respect to the parameters h_t and $\beta_{t,t'}$.Since $|\langle\sigma_T\rangle| \leq 1$, the statement of the theorem follows from (3.25).

3.1.2 Markov Property

Let $A \subset Z^\nu$ be a set and its boundary ∂A is defined by

$$\partial A = \{t \in Z^\nu : \rho(t, A) = 1\}, \tag{3.26}$$

such that ∂A is the set of all lattice sites of distance 1 from A.

Let $\Lambda \subset Z^\nu$ be a cube, and let $A, B \subseteq \Lambda$ be such that $A \cap B = \emptyset$ and $\partial A \subset B$. To denote the conditional probability that σ^Λ equals $\overline{\sigma}^A = \{\overline{\sigma}_t,\ t \in A\}$ on the set A under the condition that its values on the set B equal $\tilde{\sigma}^B = \{\tilde{\sigma}_{t'},\ t' \in B\}$, we use

$$P_\Lambda^{(A)}(\overline{\sigma}^A / \tilde{\sigma}^B) = Pr\{\sigma_t = \overline{\sigma}_t, t \in A / \sigma_{t'} = \tilde{\sigma}_{t'}, t' \in B\} \tag{3.27}$$

Lemma 3.1.6 *The following equalities hold true:*

$$\begin{aligned} P_\Lambda^{(A)}(\overline{\sigma}^A / \tilde{\sigma}^B) &= P_\Lambda^{(A)}(\overline{\sigma}^A / \tilde{\sigma}^{\partial A}) \\ &= Z^{-1}(\tilde{\sigma}^{\partial A}) \exp\{-(U_A(\overline{\sigma}^A) + U_{A,\partial A}(\overline{\sigma}^A, \tilde{\sigma}^{\partial A}))\}. \end{aligned} \tag{3.28}$$

Here $U_A(\overline{\sigma}^A)$ is the energy of the configuration $\overline{\sigma}^A$ defined as in (3.2), $U_{A,\partial A}(\tilde{\sigma}^{\partial A})$ is the energy of the interaction between the configurations $\overline{\sigma}^A$ and $\tilde{\sigma}^{\partial A}$:

$$U_{A,\partial A}(\overline{\sigma}^A, \tilde{\sigma}^{\partial A}) = -\beta \sum_{\substack{t \in A, t' \in \partial A \\ \rho(t,t')=1}} \overline{\sigma}_t \tilde{\sigma}_{t'} \tag{3.29}$$

and $Z_A(\tilde{\sigma}^{\partial A})$ is the conditional partition function

$$Z_A(\tilde{\sigma}^{\partial A}) = \sum_{\overline{\sigma}^A} \exp\{-(U_A(\overline{\sigma}^A) + U_{A,\partial A}(\overline{\sigma}^A, \tilde{\sigma}^{\partial A})\}. \tag{3.30}$$

The first equality in (3.28) is called the Markov property of the distribution P_Λ, and the other equality expresses its Gibbs property: the conditional distribution $P_\Lambda^{(A)}$ is similar in form to the distribution (3.4), except that the energy $U_{A,\partial A}$ of the interaction with the boundary configuration $\tilde{\sigma}^{\partial A}$ was added to the energy U_A. The distribution given by the formula on the right-hand side of (3.28) is called the Gibbs distribution in A with the boundary configuration $\tilde{\sigma}^{\partial A}$.

Proof. By the formula (3.4) we have

$$\begin{aligned} P_\Lambda^{(A)}(\overline{\sigma}^A/\tilde{\sigma}^B) &= \frac{P_\Lambda^{(A\cup B)}(\overline{\sigma}^A/\tilde{\sigma}^B)}{P_\Lambda^{(B)}(\tilde{\sigma}^B)} \\ &= \frac{\displaystyle\sum_{\sigma^{\Lambda\setminus\{A\cup B\}}} \exp\left\{-U_\Lambda(\overline{\sigma}^A, \tilde{\sigma}^B, \sigma^{\Lambda\setminus\{A\cup B\}})\right\}}{\displaystyle\sum_{\sigma^{\Lambda\setminus\{A\cup B\}},\overline{\sigma}^A} \exp\left\{-U_\Lambda(\overline{\sigma}^A, \tilde{\sigma}^B, \sigma^{\Lambda\setminus\{A\cup B\}})\right\}} \end{aligned} \tag{3.31}$$

where $\sigma^\Lambda = (\overline{\sigma}^A, \tilde{\sigma}^B, \sigma^{\Lambda\setminus\{A\cup B\}})$ and $\sigma^{\Lambda\setminus(A\cup B)}$ is a configuration in the set $\Lambda\setminus(A\cup B)$. Also

$$U_\Lambda(\sigma^\Lambda) = U_A(\overline{\sigma}^A) + U_{A,B}(\overline{\sigma}^A, \tilde{\sigma}^B) + U_B(\tilde{\sigma}^B) + U_{\Lambda\setminus(A\cup B)}(\sigma^{\Lambda\setminus(A\cup B)})$$

$$+ U_{B,\Lambda\setminus(A\cup B)}(\tilde{\sigma}^B, \sigma^{\Lambda\setminus(A\cup B)}) \tag{3.32}$$

with the energies $U_{A,B}$ and $U_{B,\Lambda\setminus(A\cup B)}$ given similarly to (3.29). Accordingly, the denominator of the right-hand side of the (3.31) equals

$$\exp\{-U_B(\tilde{\sigma}^B) Z_{\Lambda\setminus(A\cup B)}(\tilde{\sigma}^B) Z_A(\tilde{\sigma}^B)\},$$

and the nominator is

$$\exp\{-(U_A(\overline{\sigma}) + U_{A,B}(\overline{\sigma}^A, \tilde{\sigma}^B) + U_B(\tilde{\sigma}^B))\} Z_{\Lambda\setminus(A\cup B)}(\tilde{\sigma}^B)$$

with $Z_{\Lambda\setminus(A\cup B)}(\tilde{\sigma}^B)$ and $Z_A(\tilde{\sigma}^B)$ defined similarly to (3.30). Inserting these expressions into (3.31) and noticing that $U_{A,B}(\overline{\sigma}^A, \tilde{\sigma}^B) = U_{A,\partial A}(\overline{\sigma}^A, \tilde{\sigma}^{\partial A})$ and $Z_A(\tilde{\sigma}^B) = Z_A(\tilde{\sigma}^{\partial A})$, then after some cancellations we get (3.28). The lemma is proved.

Definition 3.1.7 *A probability distribution P on the space Ω is said to determine a Gibbs Random field $\{\sigma_t,\ t \in Z^\nu\}$ (for the Ising model) if the conditional distribution $P^{(A)}(\overline{\sigma}^A/\tilde{\sigma}^B)$, generated by the distribution P, coincides with the Gibbs distribution in A, with the boundary configuration $\overline{\sigma}^A$ (see the second equality in (3.28)) for arbitrary finite subsets $A, B \subset Z^\nu$ such that $A \cap B = \emptyset$ and $\partial A \subset B$.*

Thus, according to the first equality in (3.31) and the limit Gibbs distribution constructed above (see(3.9)) defines a Gibbs random field in Z^ν. Are there still other Gibbs fields in Z^ν for the Ising model? It turns out that this depends on the dimension of the lattice Z^ν and on the parameters (h, β). The values of parameters (h, β) for which there exist more than one Gibbs field Z^ν define points of the first order-phase transition in the plane (h, β).

Let us introduce the possible ways of construction of Gibbs fields in Z^ν for the Ising model. Let $\Lambda \subset Z^\nu$ be a cube, $\tilde{\sigma}^{\partial\Lambda}$ be a configuration in the boundary $\partial\Lambda$ of the cube Λ, and let $P_{\Lambda,\tilde{\sigma}^{\partial\Lambda}}(\sigma^\Lambda)$ denote the Gibbs distribution in Λ (on the space Ω) with the boundary configuration $\tilde{\sigma}^{\partial\Lambda}$ (see (3.28)). Let $q^{\partial\Lambda}$ be an arbitrary probability distribution on the set $\Omega_{\partial\Lambda}$ of boundary configurations $\tilde{\sigma}^{\partial\Lambda}$. Let us use the notation $P_{\Lambda,q^{\partial\Lambda}}$ for the distribution

$$P_{\Lambda,q^{\partial\Lambda}}(\sigma^\Lambda) = \langle P_{\Lambda,\tilde{\sigma}^{\partial\Lambda}}(\sigma^\Lambda)\rangle_{q^{\partial\Lambda}} \tag{3.33}$$

on the space Ω_Λ. The distribution (3.33) is called the Gibbs distribution in Λ with a random boundary configuration.

Also the Gibbs distribution P_Λ^{per} with the so-called periodic boundary conditions is often considered as $P_{\Lambda,\tilde{\sigma}^{\partial\Lambda}}$ and $P_{\Lambda,q^{\partial\Lambda}}$. It is defined similarly to the distribution P_Λ (see(3.4)) except for replacing the cube Λ by the torus (by identifying the opposite sides of Λ) and the energy U_Λ in (3.4) by the energy U_Λ^{per} of the interaction of the nearest neighbours on this torus. The Gibbs distribution (3.4) is often called the Gibbs distribution in Λ under the empty boundary conditions.

By the proof of Lemma (3.1.6), we can see that the distributions

$$P_{\Lambda,q^{\partial\Lambda}},\ P_{\Lambda,q^{\partial\Lambda}},\ P_{\Lambda}^{per} \tag{3.34}$$

have the Gibbs property (3.28).

As in the case of Gibbs distributions with the empty boundary conditions, we conclude that the limit $P = \lim\limits_{\Lambda_n \nearrow Z^\nu} P_{\Lambda_n}$ of the sequence P_{Λ_n} of distributions of the form (3.34), with Λ_n being an increasing sequence of cubes, $\Lambda_1 \subset \Lambda_2 \subset ... \subset \cup\Lambda_n = Z^\nu$, defines a Gibbs field in Z^ν.

Lemma 3.1.8 *Every probability distribution P on the space Ω that is a Gibbs random field in Z^ν is the thermodynamic limit of a sequence $P_{\Lambda,q_n^{\partial\Lambda_n}}$ for some choice of $q_n^{\partial\Lambda_n}$.*

Proof. We choose $q^{\partial\Lambda}$ to be the probability distribution on $\Omega_{\partial\Lambda}$ induced by the distribution P for every cube $\Lambda \subset Z^\nu$. $P_{\Lambda,q^{\partial\Lambda}}$ coincides in this case with the distribution induced by P on Ω_Λ. Therefore $P_{\Lambda,q^{\partial\Lambda}} \to P$ (in the sense (3.9)) as $\Lambda \nearrow Z^\nu$.

Theorem 3.1.9 *For ferromagnetic Ising model and for $\nu \geq 2$, the points $(0, \beta)$ with β sufficiently large, $\beta > \beta_1(\nu)$, are points of the first-order phase transition.*

Proof. We denote the Gibbs distribution in Λ with the boundary configuration $\tilde{\sigma}_t \equiv -1$, $t \in \partial\Lambda$ (($-$)-boundary conditions) by $P_{\Lambda,(-)}$.

Lemma 3.1.10 *The inequality*

$$Pr_{\Lambda,(-)}(\sigma_0 = +1) < \frac{1}{3} \tag{3.35}$$

holds uniformly with respect to all cubes $\Lambda \subset Z^\nu,\ 0 \in \Lambda$, *for all sufficiently large* $\beta,\ \beta > \beta_1(\nu)$.

Let us first derive the theorem from this lemma. Consider the Gibbs distribution $P_{\Lambda,(+)}$ with the boundary configuration $\tilde{\sigma}_t \equiv +1,\ t \in \partial\Lambda$ (the (+)-boundary conditions). The symmetry property is valid for $h = 0$, then we have

$$P_{\Lambda,(-)}(\sigma^\Lambda) = P_{\Lambda,(+)}(-\sigma^\Lambda),$$

and hence,

$$Pr_{\Lambda,(+)}(\sigma_0 = -1) < \frac{1}{3}$$

for every Λ, and consequently

$$Pr_{\Lambda,(+)}(\sigma_0 = +1) > \frac{2}{3}. \tag{3.36}$$

The inequalities (3.35) and (3.36) ensure that there are at least two different Gibbs distributions in Z^ν.

Proof of Lemma 3.1.10. To simplify we take the case $\nu = 2$. Let $\tilde{Z}^2$ be the dual lattice obtained from the lattice Z^2 by shifting it by the vector $(\frac{1}{2}, \frac{1}{2})$. For any configuration σ^Λ, we use $\gamma = \gamma(\sigma^\Lambda)$ to denote the collection of those bonds of $\tilde{Z}^2$ that separate two neighbouring sites $t, t' \in \Lambda \cup \partial\Lambda$ with $\sigma_t \neq \sigma_{t'}$, ($\sigma_t = -1$ for $t \in \partial\Lambda$).

We can see that the number of bonds from $\gamma(\sigma^\Lambda)$ attached to a lattice site from $\tilde{Z}^2$ is always even. Thus, the connected components of γ are closed polygons (possibly self-intersecting). We shall call them contours and denote them by $\Gamma_1, ..., \Gamma_n$. We shall show that there is a configuration σ^Λ with $\gamma = \gamma(\sigma^\Lambda)$ for each collection $\gamma = \{\Gamma_1, ..., \Gamma_n\}$ of mutually disjoint contours.

Indeed, it is enough to put $\sigma_t = -1$ for $t \in \Lambda$ that are outside all contours.

Further, we put $\sigma_t = +1$ for the sites that are inside one contour Γ only, $\sigma_t = -1$ for the sites that are encircled by two contours, and so on. Thus, there is a one-to-one correspondence between the configurations σ^Λ and the collections of contours γ.

In addition,

$$U_{\Lambda,(-)}(\sigma^\Lambda) = U_\Lambda(\sigma^\Lambda) + U_{\Lambda,\partial\Lambda}(\sigma^\Lambda, \tilde{\sigma}^{\partial\Lambda} \equiv -1) = 2\beta|\gamma| - \beta|\tilde{\Lambda}|,$$

$$Z_{\Lambda,(-)} = Z_\Lambda(\tilde{\sigma}^{\partial\Lambda} \equiv -1) = \exp\left\{\beta|\tilde{\Lambda}|\right\} \sum_\gamma e^{-2\beta|\gamma|},$$

where $|\gamma|$ is the number of bonds in γ (the length of γ) and $|\tilde{\Lambda}|$ is the number of bonds from $\tilde{Z}^2$ adjacent to at least one site from Λ.

Lemma 3.1.11 *The probability $P_{\Lambda,(-)}(\Gamma)$ of the event that Γ is contained in the collection γ can be estimated by*

$$P_{\Lambda,(-)}(\Gamma) \leq e^{-2\beta|\Gamma|}.$$

Proof. The probability $P_{\Lambda,(-)}(\Gamma)$ equals,

$$\begin{aligned} P_{\Lambda,(-)}(\Gamma) &= \sum_{\gamma:\Gamma\in\gamma} P_{\Lambda,(-)}(\gamma) = \frac{\sum\limits_{\gamma:\Gamma\in\gamma} e^{-2\beta|\gamma|}}{\sum\limits_\gamma e^{-2\beta|\gamma|}} \\ &= \frac{e^{-2\beta|\gamma|}\sum\limits_\gamma{}' e^{-2\beta|\gamma|}}{\sum\limits_\gamma e^{-2\beta|\gamma|}} < e^{-2\beta|\Gamma|}, \end{aligned}$$

where $\sum\limits_\gamma{}'$ denotes the sum over all γ that do not contain Γ or intersect it. The lemma is proved.

Further, it is easy to convince ourselves that the number of contours Γ of the length n circling a given site $t_0 \in Z^2$ is not greater than $n^2 3^n$. Since the event $\sigma_0 = +1$ under the $(-)$-boundary conditions implies the existence of at least one contour Γ encircling the point 0, we have

$$Pr_{\Lambda,(-)}(\sigma_0 = +1) \leq \sum_{\Gamma:\Gamma \text{ encircles } 0} P_{\Lambda,(-)}(\Gamma) \leq \sum_{n \geq 4} n^2 3^n e^{-2\beta n} < \frac{1}{3}$$

for β large enough. Lemma 3.1.10 and, at the same time, the theorem are proved.

Theorem 3.1.12 *For a ferromagnetic Ising model If $\nu = 1$ then there is a unique Gibbs field.*

Proof. Let $\Lambda = [-N, N] \subset Z^1$ and P_Λ be the Gibbs distribution in Λ (under the empty boundary conditions). The Ising Hamiltonian is

$$U_\Lambda(\sigma^\Lambda) = -\left(-h \sum_{i=-N}^{N} \sigma_i + \beta \sum_{i=-N}^{N} \sigma_i \sigma_{i+1}\right). \tag{3.37}$$

To simplify the formulae, we put $h = 0$. Let us define the matrix $J = \|j_{\sigma\sigma'}\|$ with the matrix elements $j_{\sigma\sigma'} = e^{\beta\sigma\sigma'}$, $\sigma\sigma' = \pm 1$. Then

$$J = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$

is called the transfer matrix of the Ising model.

Now we can find the partition function Z as follows:

$$J^2 = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}^2 = \begin{pmatrix} e^{2\beta} + e^{-2\beta} & 2 \\ 2 & e^{2\beta} + e^{-2\beta} \end{pmatrix}$$

Let $e = (1, 1)$, then

$$J^2 e = \begin{pmatrix} e^{2\beta} + e^{-2\beta} + 2 \\ e^{2\beta} + e^{-2\beta} + 2 \end{pmatrix}$$

and then

$$(J^2 e, e) = 2(e^{2\beta} + e^{-2\beta} + 2) = Z, \quad \text{for} \quad \text{N=1}.$$

We can calculate similarly

$$J^4 = \begin{pmatrix} e^{4\beta} + e^{-4\beta} + 6 & 4e^{2\beta} + 4e^{-2\beta} \\ 4e^{2\beta} + 4e^{-2\beta} & e^{4\beta} + e^{-4\beta} + 6 \end{pmatrix}$$

and,

$$(J^4 e, e) = 2(e^{4\beta} + e^{-4\beta} + 4e^{2\beta} + 4e^{-2\beta} + 6) = Z, \quad \text{for} \quad \text{N=2}.$$

Thus with similar calculations we get

$$Z_\Lambda = (J^{2N} e, e)$$

where $e = (1, 1)$, and we get

$$P_\Lambda^{\{t_1,\ldots,t_n\}}\left(\overline{\sigma}_{t_1}, \ldots, \overline{\sigma}_{t_n}\right) = \frac{\left(e^{\left(\overline{\sigma}_{t_1}\right)}, J^{N_1} e\right)\left(e^{\left(\overline{\sigma}_{t_2}\right)}, J^{t_2 - t_1} e^{\left(\overline{\sigma}_{t_1}\right)}\right) \ldots \left(e, J^{N_2} e^{(\overline{\sigma}_{t_n})}\right)}{(J^{2N} e, e)},$$

where $e^{(1)} = (1, 0)$, $e^{(-1)} = (0, 1)$, $N_1 = t_1 + N$, $N_2 = N - t_n$, $-N \le t_1 < t_2 < \ldots < t_n \le N$.

Now let $g^{(1)}$ and $g^{(2)}$ be two normalized eigenvectors of the transfer matrix J, with eigenvalues λ_1 and λ_2, $\lambda_1 > |\lambda_2| \ge 0$. Using the decompositions

$$e = C_1 g^{(1)} + C_2 g^{(2)} \quad , \quad e^{(\pm 1)} = B_1^{(\pm 1)} g^{(1)} + B_2^{(\pm 1)} g^{(2)}$$

we get

$$\left(J^{2N} e, e\right) \sim C_1^2 \lambda_1^{2N},$$

$$\left(e^{(\overline{\sigma}_{t_1})}, J^{N_1} e\right) \sim B_1^{\left(\overline{\sigma}_{t_1}\right)} C_1 \lambda_1^{N_1} \quad , \quad \left(e, J^{N_2} e^{(\overline{\sigma}_{t_n})}\right) \sim B_1^{(\overline{\sigma}_{t_n})} C_1 \lambda_1^{N_2},$$

for large N and fixed $\{t_1, ..., t_n\}$, and thus

$$\lim_{N\to\infty} P_{\Lambda}^{\{t_1,...,t_n\}}\left(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}\right) = B_1^{\left(\overline{\sigma}_{t_1}\right)} B_1^{(\overline{\sigma}_{t_n})} \prod_{k=2}^{n} \frac{\left(e^{\left(\overline{\sigma}_{t_k}\right)}, J^{t_k - t_{k-1}} e^{\overline{\sigma}_{t_{k-1}}}\right)}{\lambda_1^{t_k - t_{k-1}}}.$$

Similarly we can find the partition function Z, with boundary configurations, and it equals,

$$Z_\Lambda = \left(J^{2(N+1)} e, e\right)$$

and the probabilities $P_{\Lambda, q_n^{\partial\Lambda_n}}^{\{t_1,...,t_n\}}\left(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}\right)$ equal,

$$P_{\Lambda, q_n^{\partial\Lambda_n}}^{\{t_1,...,t_n\}}\left(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}\right) = \frac{\left(e^{\left(\overline{\sigma}_{t_1}\right)}, J^{N_1'} e\right)\left(e^{\left(\overline{\sigma}_{t_2}\right)}, J^{t_2 - t_1} e^{\left(\overline{\sigma}_{t_1}\right)}\right) ... \left(e, J^{N_2'} e^{(\overline{\sigma}_{t_n})}\right)}{\left(J^{2(N+1)} e, e\right)},$$

where $e = (1, 1)$, and $e^{(1)} = (1, 0)$, $e^{(-1)} = (0, 1)$, and $N_1' = t_1 + N + 1$, $N_2' = N + 1 - t_n$, $-N \leq t_1 < t_2 < ... < t_n \leq N$. And the limit for any sequence of Gibbs distributions $P_{\Lambda_n, q_n^{\partial\Lambda_n}}$ is,

$$\lim_{N\to\infty} P_{\Lambda, q_n^{\partial\Lambda_n}}^{\{t_1,...,t_n\}}\left(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}\right) = B_1^{\left(\overline{\sigma}_{t_1}\right)} B_1^{(\overline{\sigma}_{t_n})} \prod_{k=2}^{n} \frac{\left(e^{\left(\overline{\sigma}_{t_k}\right)}, J^{t_k - t_{k-1}} e^{\overline{\sigma}_{t_{k-1}}}\right)}{\lambda_1^{t_k - t_{k-1}}}.$$

3.2 Uniqueness Condition in One Dimension

Now we state the uniqueness theorem for Gibbs specifications in one dimension. Let S be countably finite set (the parameter set), (E, Ξ) be any measurable space (state space) and then $\Omega = E^S$ is the set of all configurations. We let

$$\delta(f) = \sup_{\zeta,\eta\in\Omega} |f(\zeta) - f(\eta)|$$

denote the oscillation of a real function f on Ω.

Proposition 3.2.1 *Let γ be any specification. Suppose there exist a constant $c > 0$ with the following property. For every cylinder event $A \in \Sigma$ there is a set $\Lambda \in \vartheta$ such that*

$$\gamma_\Lambda(A|\zeta) \geq c\ \gamma_\Lambda(A|\eta)$$

for all $\zeta,\ \eta$, where Σ is the product σ-algebra on Ω, and ϑ is the set of all Gibbs measures for the specification (which are defined as in the introduction). Then $|\vartheta(\gamma)| \leq 1$. [2]

Theorem 3.2.2 *Let $S = \mathbb{Z}$ or $\mathbb{N}$, (E, Ξ) be any state space, λ be σ-finite measure (a priori measure) in the set (E, Ξ), and Φ be a λ-admissible potential such that*

$$\sup_{i\in S} \sum_{A\in\vartheta,\ \min A\leq i<\max A} \delta(\Phi_A) < \infty. \tag{3.38}$$

where $\delta(\Phi_A)$ denotes the oscillation of Φ_A. Then $|\vartheta(\gamma)| \leq 1$.

Proof. To prove the first statement we shall check that the Gibbs specification γ^Φ satisfies the hypothesis of proposition (3.2.1). We let s denote the finite supremum in (3.38). We consider an interval Λ in S of the form $\Lambda = (-n, n] \cap S$, $n \geq 1$. For all $\zeta, \eta \in \Omega$ and $\sigma \in E^\Lambda$ we have

$$
\begin{aligned}
\left|U^{\Phi}_{\Lambda}(\sigma\zeta_{S\setminus\Lambda}) - U^{\Phi}_{\Lambda}(\sigma\eta_{S\setminus\Lambda})\right| &\leq \sum_{A\cap\Lambda\neq\emptyset,\ A\setminus\Lambda\neq\emptyset} \delta(\Phi_A) \\
&\leq \sum_{k=\pm n}\ \sum_{\min A\leq k<\max A} \delta(\Phi_A) \\
&\leq 2s.
\end{aligned}
$$

(The term with $k=-n$ does not appear when $S=\mathbb{N}$). Thus

$$
e^{-2s} h^{\Phi}_{\Lambda}(\sigma\eta_{S\setminus\Lambda}) \leq h^{\Phi}_{\Lambda}(\sigma\zeta_{S\setminus\Lambda}) \leq e^{2s} h^{\Phi}_{\Lambda}(\sigma\eta_{S\setminus\Lambda})
$$

where h^{Φ}_{Λ} is the Boltzman factor (1.5). Integrating over σ with respect to λ^{Λ} we obtain

$$
Z^{\Phi}_{\Lambda}(\zeta) \leq e^{2s} Z^{\Phi}_{\Lambda}(\eta)
$$

and

$$
\lambda_{\Lambda}(1_A h^{\Phi}_{\Lambda}|\zeta) \geq e^{-2s} \lambda_{\Lambda}(1_A h^{\Phi}_{\Lambda}|\eta)
$$

for all $A \in \Sigma_A$. Consequently, putting $c = e^{-4s}$ we have

$$
\gamma^{\Phi}_{\Lambda}(A|\zeta) \geq c\ \gamma^{\Phi}_{\Lambda}(A|\eta)
$$

whenever $A \in \Sigma_{\Lambda}$ and $\zeta, \eta \in \Omega$. But for each cylinder event a there is an interval Λ with $A \in \Sigma_{\Lambda}$. The hypothesis of proposition (3.2.1) is verified. Hence $|\vartheta(\Phi)| \leq 1$.

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