STUDIES ON SOME COMPLEXES

by

EMİNE YILDIRIM

THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE ABANT İZZET BAYSAL UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

JUNE 2011

Approval of the Graduate School of Natural and Applied Sciences

Prof. Dr. Nihat Çelebi
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Harun Karslı
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Tahire Özen Öztürk
Supervisor

Examining Committee Members

Assist. Prof. Dr. Tahire Özen Öztürk	(AİBÜ)	
Prelector Cesim Çelik	(AİBÜ)	
Assist. Prof. Dr. M. Tamer Koşan	(GYTE)	

ABSTRACT

STUDIES ON SOME COMPLEXES

Yıldırım, Emine
M.Sc., Department of Mathematics
Supervisor: Assist. Prof. Dr. Tahire Özen Öztürk

June 2011, 66 pages

Let R be any associative ring with unity and $\mathcal{X}$ be a class of R-modules. This thesis consists of three sections. In section 1, it is given some necessary definitions and theorems in homological algebra which is used in other sections. In section 2, we define $\mathcal{X}$-injective (projective) and DG($\mathcal{X}$-injective (projective)) complexes which are generalizations of injective (projective) complexes and DG-injective (projective) complexes and obtain some properties. Moreover, we investigate C($\mathcal{X}$-injective (projective)) preenvelopes (precovers). Finally, in section 3, we define an $\mathcal{X}^*$-complex and a class $C(\mathcal{X}^*)$ and investigate $C(\mathcal{X}^*)$-envelope (cover) of a complex.

Keywords: Complex, Injective (projective) complex, DG-injective (projective) complexes, envelope (cover).

ÖZET

BAZI KOMPLEKSLER ÜZERİNE ÇALIŞMALAR

Yıldırım, Emine
Yüksek lisans, Matematik Bölümü
Tez Yöneticisi: Assist. Prof. Dr. Tahire Özen Öztürk

Haziran 2011, 66 sayfa

R birimli ve birleşmeli bir halka ve $\mathcal{X}$, R-modüllerin bir sınıfı olsun. Bu tez üç bölümden oluşur. Bölüm 1'de diğer bölümlerde de kullandığımız homolojik cebirde gerekli olan bazı tanım ve teoremler verildi. Bölüm 2'de injective (projective) kompleksler ve DG-injective (projective) komplekslerin genellemeleri olan $\mathcal{X}$-injective (projective) kompleksleri ve DG($\mathcal{X}$-injective (projective)) kompleksleri tanımladık ve bazı özelliklerini elde ettik. Bununla birlikte C($\mathcal{X}$-injective (projective))-preenvelopes (precovers) konusunu inceledik. Son olarak $\mathcal{X}^*$-kompleks ve $C(\mathcal{X}^*)$ sınıfını tanımladık ve bir kompleksin $C(\mathcal{X}^*)$-envelope (cover)'ını araştırdık.

Anahtar Kelimeler: Kompleks, Injective (projective) kompleks, DG-injective (projective) kompleks, envelope (cover).

To my family and my supervisor

ACKNOWLEDGEMENTS

I would like to express sincere appreciation to my supervisor, Assist. Prof. Dr. Tahire Özen Öztürk for his motivation, helpful discussions, encouragement, patience and constant guidance during this work. I would also like to thank other examining committee members for their suggestions.

I am grateful to express my appreciation to my family, relatives and friends for their continuous helps.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

Definition 1.1. (Category, Additive and Abelian Category) A category $\mathcal{C}$ consists of three ingredients: a class obj $\mathcal{C}$ of objects, a set of morphisms $\mathrm{Hom}(A, B)$ for every ordered pair (A, B) of objects and composition $\mathrm{Hom}(A, B)\times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$, denoted by $(f, g) \mapsto gf$, for every ordered triple A, B, C of objects. (We often write $f : A \to B$ or $A \xrightarrow{f} B$ instead of $f \in \mathrm{Hom}(A, B)$). These ingredients are subject to the following axioms:

(i) The Hom sets are pairwise disjoint; that is, each $f \in \mathrm{Hom}(A, B)$ has a unique domain A and unique target B;

(ii) For each object A, there is an identity morphism $f1_A = f$ and $1_B f = f$ for all $f : A \to B$;

(iii) Composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h(gf) = (hg)f$.

A category $\mathcal{C}$ is said to be *additive* if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group such that if $f, f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g, g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, then $g(f_1 + f_2) = gf_1 + gf_2$ and $(g_1 + g_2)f = g_1 f + g_2 f$. We note that if $\mathcal{C}$ is an additive category, then $\mathrm{Hom}_{\mathcal{C}}(A, B) \neq \varnothing$ since the zero morphism is always in $\mathrm{Hom}_{\mathcal{C}}(A, B)$. This is denoted 0_{AB}. Easily, R-Mod (or Mod-R) is an additive category.

An additive category $\mathcal{C}$ is said to be an *abelian category* if it satisfies the following conditions:

(i) C has products (and coproducts),

(ii) every morphism in C has a kernel and a cokernel where if $f : A \to B$ is a morphism, then a kernel of f, denoted by $ker(f)$ is a morphism $k : K \to A$ such that $fk = 0$ and for each morphism $g : C \to A$ with $fg = 0$, there exists a unique morphism $h : C \to K$ such that $g = kh$. K is denoted by $Ker(f)$. Dually, a cokernel of f, denoted by $coker(f)$ is a morphism $p : B \to C$ such that $pf = 0$ and for each morphism $g : B \to D$ with $gf = 0$, there exists a unique morphism $h : C \to D$ such that $hp = g$. C is denoted by $Coker(f)$,

(iii) for every morphism in C $f : A \to B$, the map $Coker(ker(f)) \to Ker(coker(f))$ is an isomorphism.

For example, R- Mod and Mod-R are abelian categories.

Definition 1.2. (Covariant Functor) If C and $\mathcal{D}$ are categories, then a functor $T : C \to \mathcal{D}$ is a function such that

(i) If $A \in \text{obj } C$, then $T(A) \in \text{obj } \mathcal{D}$,

(ii) If $f : A \to A'$ in C, then $T(f) : T(A) \to T(A')$ in $\mathcal{D}$,

(iii) If $A \xrightarrow{f} A' \xrightarrow{g} A''$ in C, then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathcal{D}$, and $T(gf) = T(g)T(f)$,

(iv) $T(1_A) = 1_{T(A)}$ for every $A \in \text{obj } C$.

Definition 1.3. (Contravariant Functor) A contravariant functor $T : C \to \mathcal{D}$, where C and $\mathcal{D}$ are categories, is a function such that

(i) If $C \in \text{obj}\ \mathcal{C}$, then $T(C) \in \text{obj}\ \mathcal{D}$,

(ii) If $f : C \to C'$ in $\mathcal{C}$, then $T(f) : T(C') \to T(C)$ in $\mathcal{D}$ (note the reversal of arrows),

(iii) If $C \xrightarrow{f} C' \xrightarrow{g} C''$ in $\mathcal{C}$, then $T(C'') \xrightarrow{T(g)} T(C') \xrightarrow{T(f)} T(C)$ in $\mathcal{D}$ and $T(gf) = T(f)T(g)$,

(iv) $T(1_A) = 1_{T(A)}$ for every $A \in \text{obj}\ \mathcal{C}$.

Definition 1.4. (Exact Sequence) A sequence of R-modules and R-homomorphisms $... \to M_2 \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \to ...$ is said to be exact if $Imd_{i+1} = Kerd_i$ for all $i \in \mathbb{Z}$. An exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be a short exact sequence. We also call this short exact sequence an *extension* of A by C.

Definition 1.5. A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is *split* if there exists a map $j : C \to B$ with $pj = 1_C$ or equivalently there exists a map $k : B \to A$ with $ki = 1_A$.

Definition 1.6. (Complex) Let $\mathcal{C}$ be an abelian category whose objects are modules. By a (chain) complex C of R-modules we mean a sequence $C : \ldots \to C_2 \xrightarrow{\ell_2} C_1 \xrightarrow{\ell_1} C_0 \xrightarrow{\ell_0} C_{-1} \xrightarrow{\ell_{-1}} C_{-2} \to \ldots$ of R-modules and R-homomorphisms such that $\ell_{n-1} \circ \ell_n = 0$, for all $n \in \mathbb{Z}$. Notice that $Im\ell_n \subseteq Ker\ell_{n-1}$. Also C is denoted by $((C_n), (\ell_n))$ or (C, ℓ) where the ℓ_n are called the boundary operators of the complex C and ℓ is the differential of the complex C .

Definition 1.7. If $C : \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \longrightarrow \cdots$ is a complex, then $C' : \cdots \longrightarrow C'_2 \xrightarrow{d'_2} C'_1 \xrightarrow{d'_1} C'_0 \xrightarrow{d'_0} C'_{-1} \longrightarrow \cdots$ is said to be a subcomplex of C if $C'_m \subseteq C_m$ for every $m \in \mathbb{Z}$ and if d_m agrees with d'_m on C'_m. If $S \subseteq C$ is a subcomplex, then for any $m \in \mathbb{Z}$, $d_m : C_m \longrightarrow C_{m-1}$ induces a map $\frac{C_m}{S_m} \longrightarrow \frac{C_{m-1}}{S_{m-1}}$. By these maps, we can obtain a complex $\frac{C}{S}$. Let $f : C \longrightarrow D$ be a morphism of complexes, then $d_m(Ker(f_m)) \subseteq Ker(f_{m-1})$. Then we can obtain a complex $Kerf$ with m-th term $Ker(f_m)$. Similarly, we can find a complex $Im(f) \subseteq D$. And also we have a complex $Coker(f) = \frac{D}{Im(f)}$ and $Im(f) \cong C/Ker(f)$. Thus we can see that the family of all complexes in R-Mod is an abelian category, denoted by $C(R\text{-mod})$.

Remark 1.1. *Let C be a complex. Then we have a subcomplex $Z(C) : \cdots \longrightarrow Ker(d_m) \longrightarrow Ker(d_{m-1}) \longrightarrow \cdots$ where $Z(C)_m = Ker(d_m)$ for all $m \in \mathbb{Z}$ where d is the differential of the complex C. Notice that the differential of $Z(C)$ is 0. And a subcomplex $B(C) \subseteq C$ is defined as $B(C)_m = Im(d_{m+1})$. Since $d_m \circ d_{m+1} = 0$, $B(C) \subseteq Z(C)$. The quotient complex $Z(C)/B(C)$ is denoted by $H(C)$ where $H(C)_m = H_m(C) = \frac{Z(C)_m}{B(C)_m}$. Then the complex C becomes exact if $H(C) = 0$, or equivalently $Ker(d_m) = Im(d_{m+1})$ for all $m \in \mathbb{Z}$.*

Definition 1.8. If M is a module, then $\overline{M} : \cdots 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$ where $\overline{M}_1 = M$ and $\overline{M}_0 = M$ and $\underline{M} : \cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$ where $\underline{M}_0 = M$. Then $\underline{M} \subseteq \overline{M}$.

Remark 1.2. *If F is a covariant additive functor in the some category of modules, then $F(C) : \ldots \rightarrow F(C_2) \xrightarrow{F(\ell_2)} F(C_1) \xrightarrow{F(\ell_1)} F(C_0) \rightarrow \ldots$ is also complex where the complex $C : \ldots \rightarrow C_2 \xrightarrow{\ell_2} C_1 \xrightarrow{\ell_1} C_0 \rightarrow \ldots$. Similarly, if F is a contravariant additive functor, then the sequence $F(C) : \ldots \rightarrow F(C_{-1}) \xrightarrow{F(\ell_0)} F(C_0) \xrightarrow{F(\ell_1)} F(C_1) \rightarrow \ldots$ is a complex. For example in the category of R-Modules if A is an R-Module, then $Hom(A, -)$ is a covariant functor and $Hom(-, A)$ is a contravariant functor. Then $Hom(A, C) : \ldots \rightarrow Hom(A, C_2) \xrightarrow{\ell_2^*} Hom(A, C_1) \xrightarrow{\ell_1^*} Hom(A, C_0) \rightarrow \ldots$ where $\ell_n^* f = \ell_n f$ for all $f \in Hom(A, C_n)$ and $Hom(C, A) : \ldots \rightarrow Hom(C_0, A) \xrightarrow{\ell_1^*} Hom(C_1, A) \xrightarrow{\ell_2^*} Hom(C_2, A) \rightarrow \ldots$ where $\ell_n^* f = f \ell_n$ for all $f \in Hom(C_{n-1}, A)$.*

Definition 1.9. (Chain Map) If C and C' are complexes, then a chain map $f : C \longrightarrow C'$ is a sequence of morphisms $f_n : C_n \longrightarrow C'_n$ for all $n \in \mathbb{Z}$ making the following diagram commutative:

$$\begin{tikzcd}
C : \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
C' : \ldots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

It is easy to check that the composite gf of two chain maps $f : C \to C'$ and $g : C' \to C''$ is itself a chain map, where $(gf)_n = g_n f_n$. The identity chain map 1_C on $((C_n), (d_n))$ is the sequence of identity morphisms $1_{C_n} : C_n \to C_n$.

A chain map $f : A \longrightarrow A'$ is an isomorphism if and only if each f_n is an isomorphism.

Definition 1.10. (Homotopy) Let (C, d) and (C', d') be chain complexes. Then the chain maps $f, g : (C, d) \to (C', d')$ are homotopic, denoted by $f \simeq g$, if for all n, there is a map $s = (s_n)$ where $s_n : C_n \to C'_{n+1}$ with

$$f_n - g_n = d'_{n+1} s_n + s_{n-1} d_n.$$

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "{f_{n+1},g_{n+1}}"'] & C_n \arrow[r, "d_n"] \arrow[d, "{f_n,g_n}"'] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r] \arrow[d, "{f_{n-1},g_{n-1}}"] \arrow[dl, "s_{n-1}"'] & {} \\
\arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

A chain map $f : (C, d) \to (C', d')$ is null-homotopic if $f \simeq 0$, where 0 is the zero chain map.

Definition 1.11. (*nth* Homology) If (A, d) is a complex, then its n-th homology is $H_n(A) = \frac{Kerd_n}{Imd_{n+1}}$. The elements of $Kerd_n$ are called n - cycles. The elements of Imd_{n+1} are called n - boundaries. And also we can show that $Kerd_n = Z_n(A) = Z_n$, $Imd_{n+1} = B_n(A) = B_n$ implies that $H_n(A) = \frac{Z_n(A)}{B_n(A)}$.

Theorem 1.1. *(Theorem 6.14 in [9]) Homotopic chain maps induce the same morphism in homology: if $f, g : (C, d) \to (C', d')$ are chain maps and $f \simeq g$, then for all n,*

$$H_n(f) = H_n(g) : H_n(C) \to H_n(C').$$

Definition 1.12. (Cochain Complex) A chain complex of the form $C : \dots C^{-1} \xrightarrow{\partial^{-1}} C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \longrightarrow \dots$ is called a cochain complex where $\partial^n \circ \partial^{n-1} = 0$ for all $n \in \mathbb{Z}$. Then nth cohomology $H^n(C) = \frac{Ker\partial^n}{Im\partial^{n-1}}$.

Definition 1.13. ($\mathcal{F} - precover, \mathcal{F} - cover$) Let C be an abelian category and let $\mathcal{F}$ be a class of objects of C. Then for an object $M \in C$, a morphism $\varphi : F \longrightarrow M$ where $F \in \mathcal{F}$ is called an $\mathcal{F}$-precover of M, if $f : F' \longrightarrow M$ where $F' \in \mathcal{F}$ and the following diagram

$$\begin{tikzcd} F' \arrow[r, "f"] \arrow[d, dotted, "g"'] & M \\ F \arrow[ur, "\varphi"'] & \end{tikzcd}$$

can be completed to a commutative diagram. If when $F = F'$ and the only such g is automorphism of F, then $\varphi : F \longrightarrow M$ is called an $\mathcal{F}$-cover of M.

Dually, we can give the following definition.

Definition 1.14. ($\mathcal{F}$-preenvelope , $\mathcal{F}$-envelope) Let C be an abelian category and let $\mathcal{F}$ be a class of objects of C. Then for an object $M \in C$ a morphism $\varphi : M \longrightarrow F$

where $F \in \mathcal{F}$ is called an $\mathcal{F}$-preenvelope of M, if $f : M \longrightarrow F'$ where $F' \in \mathcal{F}$ and the following diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"'] & F' \\ F \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

can be completed to a commutative diagram. If when $F = F'$ and the only such g is automorphism of F, then $\varphi : M \longrightarrow F$ is called an $\mathcal{F}$-envelope of M.

Definition 1.15. (Resolution) Let C be an abelian category and ε be a class of objects of C. By a left ε-resolution of an object M of C, we will mean a complex $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ (not necessarily exact) with each $F_i \in \varepsilon$ such that $\mathrm{Hom}(\varepsilon, -)$ makes the complex exact. A right ε-resolution of an object M of C is $\mathrm{Hom}(-, \varepsilon)$ exact complex $0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ (not necessarily exact) with each $E^i \in \varepsilon$.

We can give an example for a right resolution:

Let an abelian category C have an ε-preenvelope. Let E^0 be an ε-preenvelope of $X \in C$, then there exists ϕ such that $0 \longrightarrow X \xrightarrow{\phi} E^0$ (not necessarily 1 - 1). Then $\frac{E^0}{\phi(X)} \in C$ implies that $\frac{E^0}{\phi(X)}$ has an ε-preenvelope E^1 such that,

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\phi"] & E^0 \arrow[r, "d^0"] \arrow[d, "\pi"'] & E^1 \\ & & \dfrac{E^0}{\phi(X)} \arrow[ur, dotted, "\phi_1"'] & \end{tikzcd}$$

where $\phi_1 \pi = d^0$. Then $d^0 \phi = 0$. If we continue in this way, then we can find that a complex $0 \longrightarrow X \xrightarrow{\varepsilon} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow ...$ (not necessarily exact) such that all $E^i \in \varepsilon$. $\mathrm{Hom}(-, F)$ makes the complex exact when $F \in \varepsilon$. That is, the sequence $... \rightarrow \mathrm{Hom}(E^1, F) \rightarrow \mathrm{Hom}(E^0, F) \rightarrow \mathrm{Hom}(X, F) \rightarrow 0$ is exact. Such complex is a right ε - resolution of X.

Similarly we can give an example for a left resolution:

Let an abelian category C have an ε-precover. Let F_0 be an ε-precover of $X \in C$, then there exists ϕ such that $F_0 \xrightarrow{\phi} X \longrightarrow 0$ (not necessarily onto). Then $Ker(\phi) \in C$ and so it has an ε-precover F_1 such that,

$$\begin{tikzcd}
F_1 \arrow[r, "\lambda_1"] \arrow[dr, dotted, "\phi_1"'] & F_0 \arrow[r, "\phi"] & X \arrow[r] & 0 \\
& Ker\phi \arrow[u, "i"'] & &
\end{tikzcd}$$

where $i\phi_1 = \lambda_1$. Then $\phi\lambda_1 = 0$. If we continue in this way, then we can find that a complex $\cdots \longrightarrow F_2 \xrightarrow{\lambda_2} F_1 \xrightarrow{\lambda_1} F_0 \xrightarrow{\phi} X \longrightarrow 0$ (not necessarily exact) such that all $F_i \in \varepsilon$. $\mathrm{Hom}(F, -)$ makes the complex exact when $F \in \varepsilon$. Such complex is a left ε - resolution of X.

Definition 1.16. (Pushout) Given two morphisms $f : A \rightarrow B$ and $g : A \rightarrow C$ in a category C, a pushout (or fibered sum) is a triple (D, α, β) with $\beta g = \alpha f$ that is a solution to the universal mapping problem: for every triple (Y, α', β') with $\beta' g = \alpha' f$, there exists a unique morphism $\theta : D \rightarrow Y$ making the diagram following commute.

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\beta"] \arrow[ddr, "\beta'"] & \\
B \arrow[r, "\alpha"] \arrow[drr, "\alpha'"'] & D \arrow[dr, dotted, "\theta"] & \\
 & & Y
\end{tikzcd}
$$

Pushouts are unique up to isomorphism when they exist. In the above if we set $D = (B \oplus C)/K$ where $K = \{(f(a), -g(a)) : a \in A\}$ and let $\alpha(b) = (b, 0) + K, \beta(c) = (0, c) + K$, then D with morphisms α, β is a pushout. If g is one to one (onto), then so is α. And also if g is one to one, then $C/A \cong D/B$.

Definition 1.17. (Pullback) Given two morphisms $f : B \to A$ and $g : C \to A$ in a category C, a pullback (or fibered product) is a triple (D, α, β) with $g\alpha = f\beta$ that is a solution to the universal mapping problem: for every triple (X, α', β') with $g\alpha' = f\beta'$, there exists a unique morphism $\theta : X \to D$ making the following diagram commute.

$$
\begin{tikzcd}
X \arrow[dr, dotted, "\theta"] \arrow[drr, "\alpha'"] \arrow[ddr, "\beta'"'] & & \\
 & D \arrow[r, "\alpha"] \arrow[d, "\beta"] & C \arrow[d, "g"] \\
 & B \arrow[r, "f"'] & A
\end{tikzcd}
$$

Pullbacks, when they exist, are unique up to isomorphism. If we set $D = \{(b, c) \in B \oplus C : f(b) = g(c)\}$ and let α and β be projection morphisms, then D with morphisms α, β is a pullback. If g is one to one (onto), then so is β.

Lemma 1.1. *(Wakamatsu's Lemma) If the class $\mathcal{F}$ is closed under extensions and*

$\phi : F \to M$ is an $\mathcal{F}$ - cover of an object M of C, then Ker(ϕ) $\in \mathcal{F}^{\perp}$ and if $\psi : M \to F$ is an $\mathcal{F}$ - envelope, then Coker(ψ) $\in {}^{\perp}\mathcal{F}$.

Proof. Let $A \in \mathcal{F}$ and $0 \to S \to P \to A \to 0$ be an exact sequence, with P projective. We know that; $Ext^1(A, Ker\phi) = 0$ if and only if Hom(P , Kerϕ) $\to$ Hom(S , Kerϕ) $\to O$ epic. Let $f : S \to$ Kerϕ. Then we have the following diagram:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "f"'] & P \arrow[d, "0"] \\ Ker\phi \subseteq F \arrow[r, "\phi"'] & M \end{tikzcd}$$

where $\frac{P}{S} \in \mathcal{F}$. We can form the Pushout diagram as follow;

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "f"'] & P \arrow[d, dotted, "\beta"] \\ F \arrow[r, "\alpha"'] & G \end{tikzcd}$$

where $G = \frac{F \oplus P}{\{(f(s), -i(s)) : s \in S\}}$. Then $\alpha : F \to G$ is an injection and Coker($F \to G$) $\cong \frac{P}{S} \in \mathcal{F}$. Since $\mathcal{F}$ is closed under extensions, $G \in F$. We can suppose $F \subset G$. By the properties of the pushout diagram, we have a linear $h : G \to M$ which agrees with ϕ on F. Since $\phi : F \to M$ is a cover, we have a linear $g : G \to F$ such that $\phi g = h$. Then, since $\phi \circ (g|_F) = \phi$ where $g|_F$ is the restriction of g on F, we have that $g|_F$ is an automorphism of F. If we replace g by $(g|_F)^{-1} \circ g$, we see that we can assume $g|_F = id_F$. So g makes the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \arrow[d] \arrow[dl, dotted] \\ F \arrow[r] & G \end{tikzcd}$$

commutative and hence makes commutative the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \arrow[d] \arrow[dl, dotted] \\ F \arrow[r] & M \end{tikzcd}$$

Thus $\phi g = 0$ implies $\phi g(p) = 0$ for all $p \in P$. Then $g(P) \subseteq Ker(\phi)$. Dually, we can prove that $coker(\psi) \in^{\perp} \mathcal{F}$, if $\psi : M \longrightarrow F$ is an $\mathcal{F}$-preenvelope. $\square$

Definition 1.18. (Projective Module) Let R be a ring. A left R-module P is projective if, whenever p is surjective and h is any map, there exists a lifting g as the following; that is, there exists a map g making the following diagram commute:

$$\begin{tikzcd} & P \arrow[d, "h"] \arrow[dl, dotted, "g"'] & \\ A \arrow[r, "p"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

Definition 1.19. (Injective Module) Let R be a ring. A left R-module E is injective if, whenever i is an injection, a dashed arrow exists making the following diagram commute:

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[ul, dotted, "g"'] \end{tikzcd}$$

Let M be a left R-module and N be a submodule such that the intersection of N with any nonzero submodule of M is nonzero. Then N is called the essential submodule of M, denoted by $N \subseteq^{ess} M$.

If a left R-module M is essential in an injective module E, then E is called the injective hull (or injective envelope) of M, denoted by $E(M)$. Then $E(M)$ is the smallest injective module containing M and the greatest module containing M as essential. And also every R-module has a unique injective envelope up to isomorphism.

Definition 1.20. (Ext) Let an injective resolution of an R-module B is $E : 0 \longrightarrow B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$, then $E^B : 0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$ and $0 \longrightarrow \mathrm{Hom}(A, E^0) \xrightarrow{d_*^0} \mathrm{Hom}(A, E^1) \xrightarrow{d_*^1} \mathrm{Hom}(A, E^2) \longrightarrow \cdots$. Then $Ext_R^n(A, B) = H^n(\mathrm{Hom}_R(A, E^B)) = \frac{kerd_*^n}{imd_*^{n-1}}$, where d_*^n :$\mathrm{Hom}_R(A, E^n) \longrightarrow \mathrm{Hom}_R(A, E^{n+1})$ is defined, as usual, by $d_*^n : f \mapsto d^n f$.

Let C be any abelian category and ε be a class of objects of C. If $Ext^1(X, Y) = 0$ for all $X \in \varepsilon$, then $Y \in \varepsilon^{\perp}$. If $Ext^1(X, Y) = 0$ for all $Y \in \varepsilon$, then $X \in {}^{\perp}\varepsilon$.

Lemma 1.2. *(Proposition 6.40 in [9]) The module $Ext_R^n(A, B)$ is independent of the choice of injective resolution of B.*

Remark 1.3. *By definition we understand $Hom_R(A, B) \cong Ext_R^0(A, B)$.*

Lemma 1.3. *(Corollary 6.46 in [9]) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is a short exact sequence of left R-modules, then for every left R-module A, there is a long exact* *sequence of abelian groups,*

$0 \longrightarrow Hom_R(A,B') \longrightarrow Hom_R(A,B) \longrightarrow Hom_R(A,B'') \longrightarrow Ext^1_R(A,B') \longrightarrow Ext^1_R(A,B)$
$\longrightarrow Ext^1_R(A,B'') \longrightarrow Ext^2_R(A,B') \longrightarrow Ext^2_R(A,B) \longrightarrow Ext^2_R(A,B'') \longrightarrow ....$

Thus, Ext repairs the loss of exactness after applying $Hom_R(A,-)$ to a short exact sequence.

Definition 1.21. (ext) If the chosen projective resolution of an R-module A is P : $... \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ and $P_A : ... \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$, then

$$ext^n_R(A,B) = H_n(Hom_R(P_A,B)) = \frac{ker\ d^*_n}{im\ d^*_{n-1}} ,$$

where $d^*_n : Hom_R(P_{n-1},B) \longrightarrow \mathrm{Hom}_R(P_n,B)$ is defined, as usual, by

$$d^*_n : f \mapsto fd_n .$$

Lemma 1.4. *(Corollary 6.57 in [9]) The module $ext^n_R(A,B)$ is independent of the choice of projective resolution of A.*

Theorem 1.2. *(Theorem 6.67 in [9]) Let A and B be left R-modules, let $E : 0 \longrightarrow B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow ...$ be an injective resolution of B and let $P : ... \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ be a projective resolution of A. Then $H_n(Hom_R(P_A,B)) \cong H^n(Hom_R(A,E^B))$ for all $n \geq 0$ and so*

$$Ext^n_R(A,B) \cong ext^n_R(A,B) .$$

Proposition 1.1. *(Proposition 7.24 in [9]) If $Ext^1_R(C,A) = \{0\}$, then every extension of A by C splits.*

Corollary 1.1. *(Corollary 7.25 in [9])*

(i) *A left R-module P is projective if and only if* $Ext^1_R(P, B) = \{0\}$ *for all every R-module B.*

(ii) *A left R-module E is injective if and only if* $Ext^1_R(A, E) = \{0\}$ *for all every R-module A.*

Definition 1.22. (Suspension) The suspension of a complex C is the complex denoted by $S(C)$ or $C[1]$ where $S(C)_m = C[1]_m = C_{m-1}$ (or $S(C)^m = C[1]^m = C^{m+1}$) and whose differential is $-d$ where d is the differential of C. Then we can define $S^k(C)$ for any $k \in \mathbb{Z}$ as $S^k(C)_m = C[k]_m = C_{m-k}$ (or $S^k(C)^m = C[m]^k = C^{m+k}$) whose differential is $(-1)^k d$ where d is the differential of C.

Definition 1.23. (Mapping Cone) Let $f : (X, \lambda) \longrightarrow (Y, d)$ be a chain map. For each n, define

$$\begin{tikzcd}
X: & \cdots \arrow[r] & X^{-1} \arrow[r, "\lambda^{-1}"] \arrow[d, "f^{-1}"'] & X^0 \arrow[r, "\lambda^0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "\lambda^1"] \arrow[d, "f^1"'] & X^2 \arrow[r] \arrow[d, "f^2"'] & \cdots \\
Y: & \cdots \arrow[r] & Y^{-1} \arrow[r, "d^{-1}"'] & Y^0 \arrow[r, "d^0"'] & Y^1 \arrow[r, "d^1"'] & Y^2 \arrow[r] & \cdots
\end{tikzcd}$$

and $M(f)^n = X^{n+1} \oplus Y^n$. Then the complex $M(f) : \cdots M(f)^{n-1} \stackrel{\partial^{n-1}}{\rightarrow} M(f)^n \stackrel{\partial^n}{\rightarrow}\rightarrow \cdots$ whose boundary operators are $\partial^n(x, y) = (-\lambda^{n+1}x, f^{n+1}(x) + d^n y)$ for $(x, y) \in X^{n+1} \oplus Y^n$ is called a mapping cone. So there exists an exact sequence, $0 \longrightarrow Y \stackrel{i}{\longrightarrow} M(f) \stackrel{p}{\longrightarrow} X[1] \longrightarrow 0$.

Definition 1.24. (Projective Complex) A complex P is said to be a projective complex if for any morphism $P \longrightarrow D$ and any epimorphism $C \longrightarrow D$, the diagram

$$\begin{tikzcd} & P \arrow[dl, dotted] \arrow[d] & \\ C \arrow[r] & D \arrow[r] & 0 \end{tikzcd}$$

can be completed to a commutative diagram by a morphism $P \longrightarrow C$.

Definition 1.25. (Injective Complex) I is said to be an injective complex if for any morphism $C \longrightarrow I$ and any morphism $C \longrightarrow D$, the diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d] & D \arrow[dl, dotted] \\ & I & \end{tikzcd}$$

can be completed to a commutative diagram by a morphism $D \longrightarrow I$.

Example 1.1. Let M be an R-module. Let $\overline{M} : ... \to 0 \to M \to M \to 0 \to ...$ such that $\overline{M}_0 = M$ and $\overline{M}_1 = M$. If M is a projective (injective) module, then $\overline{M}$ is also a projective (injective) complex.

Definition 1.26. (Enough Projective Objects) Let $\mathcal{C}$ be an abelian category. $\mathcal{C}$ is said to have enough projectives, if for every object A of $\mathcal{C}$, there is a projective object $P \in \mathcal{C}$ and an exact sequence $P \to A \to 0$ (epic).

Definition 1.27. (Enough Injective Objects) Let $\mathcal{C}$ be an abelian category. $\mathcal{C}$ said to have enough injectives, if for every object $A \in \mathcal{C}$, there is an injective object $E \in \mathcal{C}$ and exact sequence $0 \longrightarrow A \longrightarrow E$ (monic).

Remark 1.4. *Notice that for all $m \in \mathbb{Z}$, $S^m(\overline{M})$ is an injective complex whenever M is an injective module. And also any direct product of injective complexes is an injective* *complex. Let C be any complex. Then we have an injective linear monomorphism $C^m \longrightarrow I^m$ and then form the corresponding morphism $C \longrightarrow \overline{I^m}$. Then we have $C \longrightarrow \prod\limits_{m\in\mathbb{Z}} \overline{I^m}$ is a monomorphism as a product of injective complexes where $\prod\limits_{m\in\mathbb{Z}} \overline{I^m}$ is an injective complex. Then $\prod\limits_{m\in\mathbb{Z}} \overline{I^m}$ is an injective preenvelope of C. Thus there are enough injective complexes. If C itself is injective, then C is isomorphic to a direct summand of $\prod\limits_{m\in\mathbb{Z}} \overline{I^m}$ and also it is exact. If A is a non zero subcomplexes of a complex C such that $A \cap B \neq 0$ for all nonzero subcomplexes B, then A is called an essential subcomplex of C. There exists an injective complex containing essentially the complex C. It is called the injective envelope of the complex C.*

Definition 1.28. (Ext in Complexes) Let A and B complexes in the category of complexes of left R-modules $\mathcal{C}$. The definition of Ext in complexes is the same as the definition of Ext in R-Mod. We can obtain an injective resolution of the complex B since every complex has an enough injective complex. If the chosen injective resolution of B is $E = 0 \longrightarrow B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$, then $Ext^n(A, B) = H^n(\mathrm{Hom}(A, E^B)) = \frac{kerd_*^n}{imd_*^{n-1}}$, where d_*^n :$\mathrm{Hom}(A, E^n) \longrightarrow \mathrm{Hom}(A, E^{n+1})$ is defined, as usual, by $d_*^n : f \mapsto d^n f$.

Definition 1.29. ($\mathcal{H}om(A, C)$) If $(A, \Delta^{'})$ and $(C, \Delta^{''})$ are complexes, then $\mathcal{H}om(A, C)$ is the complex defined by $\mathcal{H}om(A, C)_n = \prod_{p+q=n} \mathrm{Hom}(A_{-p}, C_q)$ and $D_n : \mathcal{H}om(A, C)_n \longrightarrow \mathcal{H}om(A, C)_{n-1}$ such that $D_n : f \longrightarrow D_n f = \{g_{p,q}\}$ where $g_{p,q} : A_{-p} \longrightarrow C_q$ with $q + p = n - 1$ is defined by $g_{p,q} = \Delta^{''}_{q+1} f_{p,q+1} + (-1)^{p+q} f_{p+1,q} \Delta^{'}_{-p}$.

Definition 1.30. (DG-projective and DG-injective complex) Let ε be the class of exact complexes. We call a complex P DG-projective if each P^n is projective and if $\mathcal{H}om(P, E)$ is exact for all $E \in \varepsilon$.

We call a complex I DG-injective if each I^n is injective and $\mathcal{H}om(E, I)$ is exact for all $E \in \varepsilon$.

Every projective(injective) complex is a DG-projective(injective) complex. Moreover, every right(left) bounded complex is a DG-injective(projective) complex.

Definition 1.31. (Inverse system) Given a partially ordered set I and a category C, *an inverse system* in C is an ordered pair $((M_i)_{i\in I}, (\psi_i{}^j)_{j\geq i})$, abbreviated $\{M_i, \psi_i{}^j\}$, where $(M_i)_{i\in I}$ is an indexed family of objects in C and $(\psi_i{}^j : M_j \to M_i)_{j\geq i}$ is an indexed family of morphisms for which $\psi_i{}^i = 1_{M_i}$ for all i, and such that the following diagram commutes whenever $k \geq j \geq i$.

$$\begin{tikzcd} M_k \arrow[r, "\psi_i{}^k"] \arrow[d, "\psi_j{}^k"] & M_i \\ M_j \arrow[ur, "\psi_i{}^j"'] & \end{tikzcd}$$

Definition 1.32. (Inverse limit) Let I be a partially ordered set, let C be a category, and let $\{M_i, \psi_i{}^j\}$ be an inverse system in C over I. The *inverse limit* is an object $\varprojlim M_i$ and a family of projections $(\alpha_i : \varprojlim M_i \to M_i)_{i\in I}$ such that

(i) $\psi_i{}^j \alpha_j = \alpha_i$ whenever $i \leq j$,

(ii) for every $X \in obj(C)$ and all morphisms $f_i : X \to M_i$ satisfying $\psi_i{}^j f_j = f_i$ for all $i \leq j$, there exists a unique morphism $\theta : X \to \varprojlim M_i$ making the diagram commute.

$$
\begin{tikzcd}
\varprojlim M_i \arrow[dr, "\alpha_i"] \arrow[ddr, "\alpha_j"'] & & X \arrow[ll, dotted, "\theta"'] \arrow[dl, "f_i"'] \arrow[ddl, "f_j"] \\
& M_i & \\
& M_j \arrow[u, "\psi_i{}^j"'] &
\end{tikzcd}
$$

Definition 1.33. (Direct system) Given a partially ordered set I and a category C, *an direct system* in C is an ordered pair $((M_i)_{i\in I}, (\varphi_j{}^i)_{i\leq j})$, abbreviated $\{M_i, \varphi_j{}^i\}$, where $(M_i)_{i\in I}$ is an indexed family of objects in C and $(\varphi_j{}^i : M_i \to M_j)_{i\leq j}$ is an indexed family of morphisms for which $\varphi_i{}^i = 1_{M_i}$ for all i, and such that the following diagram commutes whenever $i \leq j \leq k$.

$$
\begin{tikzcd}
M_i \arrow[r, "\psi_k{}^i"] \arrow[d, "\psi_j{}^i"] & M_k \\
M_j \arrow[ur, "\psi_k{}^j"'] &
\end{tikzcd}
$$

Definition 1.34. (Direct limit) Let I be a partially ordered set, let C be a category, and let $\{M_i, \varphi_j{}^i\}$ be an direct system in C over I. The *direct limit* is an object $\varinjlim M_i$ and insertion morphisms $(\alpha_i : M_i \to \varinjlim M_i)_{i\in I}$ such that

(i) $\alpha_j\varphi_j{}^i = \alpha_i$ whenever $i \leq j$,

(ii) for every $X \in obj(C)$ and all morphisms $f_i : M_i \to X$ satisfying $f_j\varphi_j{}^i = f_i$ for all $i \leq j$, there exists a unique morphism $\theta : \varinjlim M_i \to X$ making the diagram commute.

$$
\begin{tikzcd}
\varinjlim M_i \arrow[rr, dotted, "\theta"] & & X \\
& M_i \arrow[ul, "\alpha_i"'] \arrow[ur, "f_i"] \arrow[d, "\varphi_j^i"] & \\
& M_j \arrow[uul, "\alpha_j"] \arrow[uur, "f_j"'] &
\end{tikzcd}
$$

Definition 1.35. Let $S, T : \mathcal{A} \to \mathcal{B}$ be covariant functors. A natural transformation $\tau : S \to T$ is a one parameter family of morphisms in $\mathcal{B}$

$$\tau = (SA \to TA)_{A \in obj(\mathcal{A})}$$

making the following diagram commute for all $f : A \to A'$ in $\mathcal{A}$:

$$
\begin{tikzcd}
SA \arrow[r, "\tau_A"] \arrow[d, "Sf"] & TA \arrow[d, "Tf"] \\
SA' \arrow[r, "\tau_{A'}"] & TA'
\end{tikzcd}
$$

A *natural isomorphism* is a natural transformation τ for which each τ_A is an isomorphism.

Proposition 1.2. *(Proposition 5.21 in [9]) If* $\{M_i, \psi^j{}_i\}$ *is an inverse system of left R-*

modules, then there is a natural isomorphism

$$Hom_R(A, \varprojlim M_i) \cong \varprojlim Hom_R(A, M_i)$$

for every left R-module A.

Proposition 1.3. *(Proposition 5.26 in [9]) If $\{M_i, \varphi^j{}_i\}$ is a direct system of left R-modules, then there is an isomorphism*

$$\theta : Hom_R(\varinjlim M_i, B) \to \varprojlim Hom_R(M_i, B)$$

for every left R-module B.

Remark 1.5. *By propositions above we can see that*

$$Ext^j(N, \varprojlim M_i) \cong \varprojlim Ext^j(N, M_i)$$

and

$$Ext^j(\varinjlim M_i, N) \cong \varprojlim Ext^j(M_i, N)$$

for all $j \geq 0$.

Theorem 1.3. *(Theorem 1.5.6. in [3]) Let $\mathcal{F}' = ((M_i'), (f_{ji}'))$, $\mathcal{F} = ((M_i), (f_{ji}))$, $\mathcal{F}'' = ((M_i''), (f_{ji}''))$ be inductive systems and suppose there are maps $\mathcal{F}' \xrightarrow{\{\sigma_i\}} \mathcal{F} \xrightarrow{\{\tau_i\}} \mathcal{F}''$ such that $M_i' \xrightarrow{\sigma_i} M_i \xrightarrow{\tau_i} M_i''$ is exact for each $i \in I$, that is, $\mathcal{F}' \xrightarrow{\sigma_i} \mathcal{F} \xrightarrow{\tau_i} \mathcal{F}''$ is exact, then $\varinjlim M_i' \xrightarrow{\varinjlim \sigma_i} \varinjlim M_i \xrightarrow{\varinjlim \tau_i} \varinjlim M_i''$ is exact.*

Theorem 1.4. *(Theorem 1.5.13. in [3]) Let $\mathcal{F}' = ((M_i'), (f_{ij}'))$, $\mathcal{F} = ((M_i), (f_{ij}))$, $\mathcal{F}'' = ((M_i''), (f_{ij}''))$ be inverse systems and suppose there are maps $\mathcal{F}' \xrightarrow{\{\sigma_i\}} \mathcal{F} \xrightarrow{\{\tau_i\}} \mathcal{F}''$ such that $0 \to M_i' \xrightarrow{\sigma_i} M_i \xrightarrow{\tau_i} M_i''$ is exact for each i, then the induced sequence $0 \to \varprojlim M_i' \xrightarrow{\varprojlim \sigma_i}$*

$\varprojlim M_i \xrightarrow{\varprojlim \tau_i} \varprojlim M_i''$ *is exact.*

If furthermore the set of indices is $\mathbb{N}$ *and if the maps* f'_{ij} *are surjective, then when* $0 \to M'_i \to M_i \to M''_i \to 0$ *are exact for each i, the induced sequence*

$$0 \to \varprojlim M'_i \to \varprojlim M_i \to \varprojlim M''_i \to 0$$

is exact.

Lemma 1.5. *(Corollary 5.2.7 in [3]) Let* $\mathcal{F}$ *be a class of R-modules that is closed under direct limits, and M be an R-module. If M has an* $\mathcal{F}$*-precover, then it has an* $\mathcal{F}$*-cover.*

Lemma 1.6. *(Corollary 6.3.5 in [3]) Let* $\mathcal{F}$ *be a class of R-modules that is closed under summands and inverse limits, and M be an R-module. If M has an* $\mathcal{F}$*-preenvelope, then it has an* $\mathcal{F}$*-envelope.*

CHAPTER 2

A GENERALIZATION OF INJECTIVE AND PROJECTIVE COMPLEXES

Let $\mathcal{X}$ be a class of R-modules. In this section $\mathcal{X}$-injective (projective) and DG($\mathcal{X}$-injective (projective)) complexes as a generalizations of injective (projective) complexes are defined and given some properties. Moreover, it is investigated $C(\mathcal{X} - injective\ (projective))$-preenvelopes (precovers). However, $\mathcal{X}$-injective and $\mathcal{X}$-projec tive is studied in [8]. We extend the definition to injective (projective) complexes.

Definition 2.1. ($\mathcal{X}^* - complex$) Let $\mathcal{X}$ be a class of R-modules. A complex $C : \ldots \longrightarrow C^{n-1} \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow \ldots$ is called an $\mathcal{X}^* -$ (*cochain*) complex if $C^i \in \mathcal{X}$ for all $i \in \mathbb{Z}$. A complex $C : \ldots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \ldots$ is called an $\mathcal{X}^* -$ (*chain*) complex if $C_i \in \mathcal{X}$ for all $i \in \mathbb{Z}$. The class of all $\mathcal{X}^* - complexes$ is denoted by $C(\mathcal{X}^*)$.

Definition 2.2. ($\mathcal{X}$-injective and $\mathcal{X}$-projective module) A left R-module E is $\mathcal{X}$-injective, if $Ext^1(B/A, E) = 0$ where for every module $B/A \in \mathcal{X}$, whenever i is an injection and f is any map, there exists a map g making the following diagram commutes;

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[ul, dotted, "g"']
\end{tikzcd}$$

A left R-module P is $\mathcal{X}$-projective, if $Ext^1(P, X) = 0$ for every module $X \in \mathcal{X}$ where $X = Kerp$, whenever p is surjective and h is any map, there exists a map g making the following diagram commutes;

$$\begin{tikzcd} & P \arrow[d, "h"] \arrow[dl, dotted, "g"'] & \\ A \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

Definition 2.3. ($\mathcal{X} - injective$ and $\mathcal{X}$-projective complexes) A complex C is called an $\mathcal{X}$-injective complex, if $Ext^1(Y/X, C) = 0$ where for every complex $Y/X \in C(\mathcal{X}^*)$. Thus the following diagram commutes as follows;

$$\begin{tikzcd} & C & \\ 0 \arrow[r] & X \arrow[u, "f"] \arrow[r, "\phi"'] & Y \arrow[ul, dotted, "\widetilde{f}"'] \end{tikzcd}$$

such that $\widetilde{f}\phi = f$ where ϕ is a monomorphism.

Dually we can define an $\mathcal{X} - projective$ complex. A complex C is called an $\mathcal{X} - projective$ complex, if $Ext^1(C, X) = 0$ for every complex $X \in C(\mathcal{X}^*)$ where $Ker\phi = X \in C(\mathcal{X})$. By diagram we can show as follows;

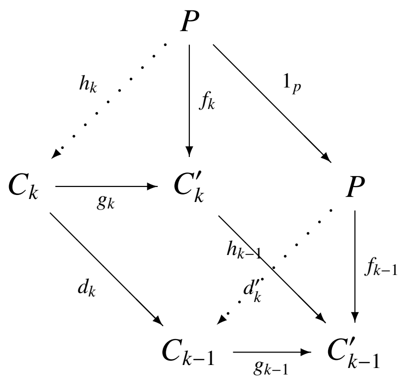
$$\begin{tikzcd} & C \arrow[d, "f"] \arrow[dl, dotted, "\widetilde{f}"'] & \\ A \arrow[r, "\phi"'] & B \arrow[r] & 0 \end{tikzcd}$$

such that $\phi\widetilde{f} = f$ where ϕ is onto.

Example 2.1. If P is an $\mathcal{X}-projective$ ($\mathcal{X}-injective$) module, then $\overline{P}: ... \longrightarrow 0 \longrightarrow P \longrightarrow P \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ is an $\mathcal{X}-projective$ ($\mathcal{X}-injective$) complex. Also, the finite direct sum of $\mathcal{X}-projective$ ($\mathcal{X}-injective$) complexes is again $\mathcal{X}-projective$ ($\mathcal{X}-injective$).

Note that if P is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) module and P is not in the class $\mathcal{X}$, then $\overline{P}$ is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) complex, but not an $\mathcal{X}^*$-complex. So $\mathcal{X}$-injective ($\mathcal{X}$-projective) complex may not be an $\mathcal{X}^*$-complex.

Proof. Consider the following diagram;



where $g : C \longrightarrow C' \longrightarrow 0$ is an epic morphism such that $Ker g \in \mathcal{X}$ and $f : \overline{P} \longrightarrow C$ is a morphism and d, d' are differentials of C, C', respectively. Since P is an $\mathcal{X}$-projective module, there exists a morphism $h_k : P \longrightarrow C_k$ with $g_k h_k = f_k$.

Define $h_{k-1} : P \longrightarrow C_{k-1}$ with $h_{k-1} = d_k h_k$. Then $g_{k-1} h_{k-1} = f_{k-1}$. Thus we have a chain map $h : \overline{P} \longrightarrow C$, as required. Dually, we can prove that if P is an $\mathcal{X}$-injective module, then $\overline{P}$ is an $\mathcal{X}$-injective complex. □

Definition 2.4. Let ε be the class of exact complexes. Then we can define ε_1 such that ε_1 is the class of exact complexes whose kernels are in $\mathcal{X}$.

Definition 2.5. A complex I is called $DG(\mathcal{X}-injective)$, if each I^n is $\mathcal{X}$-injective and $\mathcal{H}om(E, I)$ is exact for all $E \in \varepsilon_1$.

A complex I is called $DG(\mathcal{X}-projective)$, if each I^n is $\mathcal{X}-projective$ and $\mathcal{H}om(I,E)$ is exact for all $E \in \varepsilon_1$.

Lemma 2.1. *Let $A \xrightarrow{\beta} B \xrightarrow{\theta} C$ be an exact sequence of modules (complexes) where $Ker\beta \in \mathcal{X}$ $(C(\mathcal{X}^*))$. Then for all $\mathcal{X}$-projective modules (complexes) I,*

$$Hom(I,A) \longrightarrow Hom(I,B) \longrightarrow Hom(I,C)$$

is exact.

Proof. $0 \longrightarrow Ker\theta \xrightarrow{i} B \xrightarrow{\theta} C$ is exact, so

$$0 \longrightarrow Hom(I,Ker\theta) \longrightarrow Hom(I,B) \longrightarrow Hom(I,C)$$

is exact.

We have the following commutative diagram;

$$\begin{tikzcd} A \arrow[r, "\beta"] & Im\beta \arrow[r] & 0 \\ & I \arrow[u, "g"'] \arrow[ul, dotted, "f"] & \end{tikzcd}$$

such that $\beta f = g$.

Since I is $\mathcal{X}$-projective module (complex) and $Ker\beta \in \mathcal{X}$ $(C(\mathcal{X}^*))$.

Therefore, $\mathrm{Hom}(I,A) \longrightarrow \mathrm{Hom}(I,B) \longrightarrow \mathrm{Hom}(I,C)$ is exact. □

Dually, we can give the following lemma.

Lemma 2.2. *Let $A \xrightarrow{\beta} B \xrightarrow{\theta} C$ be an exact sequence of modules (complexes) where*

$\frac{C}{Im\theta} \in \mathcal{X}$ $(C(\mathcal{X}^*))$. *Then for all $\mathcal{X}$-injective modules (complexes) I,*

$$Hom(C, I) \longrightarrow Hom(B, I) \longrightarrow Hom(A, I)$$

is exact.

Proof. $A \xrightarrow{\beta} B \xrightarrow{\theta} Im\theta \longrightarrow 0$ is exact, so

$$0 \longrightarrow Hom(Im\theta, I) \longrightarrow Hom(B, I) \longrightarrow Hom(A, I)$$

is exact.

We have the following commutative diagram;

$$\begin{tikzcd}
0 \arrow[r] & Im\theta \arrow[r, "i"] \arrow[d, "g"] & C \arrow[dl, dotted, "f"] \\
 & I &
\end{tikzcd}$$

such that $fi = g$ since I is $\mathcal{X}$-injective module (complex) and $\frac{C}{Im\theta} \in \mathcal{X}$ $(C(\mathcal{X}^*))$.

Thus, Hom(C, I) $\longrightarrow$Hom(B, I) $\longrightarrow$Hom(A, I) is exact. □

Example 2.2. Let $I = ... \longrightarrow 0 \longrightarrow I^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ where I^0 is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) module. Then I is DG($\mathcal{X}$-injective (projective)) complex.

Proof. Let

$$E : ... \longrightarrow E^{-1} \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow ...$$

is exact and $Kerd^n \in \mathcal{X}$, then

$$\mathcal{H}om(E, I) \cong ... \longrightarrow Hom(E^2, I^0) \longrightarrow Hom(E^1, I^0) \longrightarrow Hom(E^0, I^0) \longrightarrow ...$$

By Lemma 2.2, $\mathcal{H}om(E, I)$ is exact, the required is obtained. $\square$

Lemma 2.3. *If a complex $X : \ldots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots$ is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) complex, then for all $n \in \mathbb{Z}$, X_n is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) module.*

Proof. Let $0 \longrightarrow N \xrightarrow{i} M$ and $\frac{M}{N} \in \mathcal{X}$ and $\alpha : N \to X_n$ be linear form the pushout;

$$\begin{tikzcd}
N \arrow[r, "i"] \arrow[d, "\alpha"'] & M \arrow[d, dotted, "\gamma_n"'] \\
X_n \arrow[r, dotted, "\theta_n"] & \frac{X_n \oplus M}{A}
\end{tikzcd}$$

where $A = \{(\alpha(n), -i(n)) : n \in N\}$.

By the following diagram;

$$\begin{tikzcd}
0 \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_n \arrow[r] \arrow[d] & \frac{M \oplus X_n}{A} \arrow[r] \arrow[d] & \frac{M}{N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_{n-1} \arrow[r] & X_{n-1} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We have the exact sequence $0 \longrightarrow X \longrightarrow T \longrightarrow S \longrightarrow 0$ where

$T : ... \longrightarrow X_{n+1} \longrightarrow X_{n+2} \longrightarrow \frac{M \oplus X_n}{A} \longrightarrow X_{n-1}...$ and

$S : ... \longrightarrow 0 \longrightarrow 0 \longrightarrow \frac{M}{N} \longrightarrow 0....$

Since X is a $\mathcal{X} - injective$ complex, $Ext^1(S, X) = 0$, and so

$0 \to \mathrm{Hom}(S, X) \to \mathrm{Hom}(T, X) \to \mathrm{Hom}(X, X) \to Ext^1(S, X) = 0$. Therefore there exists $\beta_n : T_n = \frac{M \oplus X_n}{A} \longrightarrow X_n$ such that $\beta_n \theta_n = 1$. So,

$$\beta_n \theta_n(\alpha(n)) = \alpha(n)$$

$$\beta_n((\alpha(n), 0) + A) = \alpha(n)$$

$$\beta_n((0, i) + A) = \alpha(n)$$

$$\beta_n \gamma_n i(n) = \alpha(n)$$

And hence $\beta_n \gamma_n i = \alpha$. So, X_n is a $\mathcal{X} - injective$ module. □

Remark 2.1. *Let $X : ... \to X_{n+1} \to X_n \to X_{n-1} \to ...$ be a complex such that X_n are $\mathcal{X}$-injective ($\mathcal{X}$-projective) modules for all $n \in \mathbb{Z}$. Then it need not to be that X is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) complex.*

Let $R \in \mathcal{X}$ and $\mathcal{X}$-injective module and $f : R \to R \oplus R$ be a morphism such that $f(a) = (0, a)$ and $g : R \oplus R \to R$ be a morphism such that $g(a, b) = a$. then $gf = 0$ where $g \neq 0$. So, we have the following diagram;

$$\begin{array}{ccccccc}
...0 & \longrightarrow & R & \xrightarrow{f} & R\oplus R & \longrightarrow & 0... \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & \downarrow \\
...0 & \longrightarrow & R\oplus R & \xrightarrow{1} & R\oplus R & \longrightarrow & 0... \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\
...0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0...
\end{array}$$

So, $\underline{R}$ cannot be an $\mathcal{X}$-injective complex. Dually, we can give an example for $\mathcal{X}$-projectivity.

Lemma 2.4. *Let I be an $\mathcal{X}$-injective (projective) module. Then $\underline{I}$ is in $\varepsilon_1^{\perp}({}^{\perp}\varepsilon_1)$.*

Proof. Let

$$0 \longrightarrow \underline{I} \longrightarrow I^1 \xrightarrow{\lambda_1} I^2 \xrightarrow{\lambda_2} I^3 \longrightarrow \cdots$$

be a injective resolution of I.

Then $Hom(E, I^1) \overset{\lambda_1{}^*}{\to} Hom(E, I^2) \overset{\lambda_2{}^*}{\to} Hom(E, I^3)$ where $E : ... \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to ...$ and $E \in \varepsilon_1$.

Now we will show that $Ext^1(E, \underline{I}) = 0$. Let $u \in Ker(\lambda_2{}^*)$. Then $\lambda_2{}^*(u) = 0$ and hence $\lambda_2(u) = 0$. Since $\lambda_2{}^{-2} = 1$, $u^{-2} = 0$. We have the diagram as follow;

$$
\begin{tikzcd}
E : \ldots \arrow[r] & E^{-3} \arrow[r, "d^{-3}"] \arrow[d] & E^{-2} \arrow[r, "d^{-2}"] \arrow[d, "u^{-2}"] & E^{-1} \arrow[r, "d^{-1}"] \arrow[d, "u^{-1}"] & E^{0}\ldots \arrow[d] \\
I^2 : \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & I^0 \arrow[r, "1"] \arrow[d, "1"] & I^0 \arrow[r] \arrow[d] & 0\ldots \arrow[d] \\
I^3 : \ldots \arrow[r] & I^0 \arrow[r] & I^0 \arrow[r] & 0 \arrow[r] & 0\ldots
\end{tikzcd}
$$

where $u^{-1}d^{-2} = 0$. Since $E^{-2} \overset{d^{-2}}{\to} E^{-1} \overset{d^{-1}}{\to} E^0$ is exact and I^0 is $\mathcal{X}$-injective, $\mathrm{Hom}(E^0, I^0) \overset{d^{-1^*}}{\to} \mathrm{Hom}(E^{-1}, I^0) \overset{d^{-2^*}}{\to} \mathrm{Hom}(E^{-2}, I^0)$ is exact. Since $u^{-1} \in Ker(d^{-2^*}) = Im(d^{-1^*})$, there exists $f^0 \in Hom(E^0, I^0)$ such that $u^{-1} = f^0 d^{-1}$. So, we have the following diagram;

$$
\begin{tikzcd}
E : \ldots \arrow[r] & E^{-3} \arrow[r, "d^{-3}"] \arrow[d] & E^{-2} \arrow[r, "d^{-2}"] \arrow[d] & E^{-1} \arrow[r, "d^{-1}"] \arrow[d, "f^{-1}"] & E^{0} \arrow[r] \arrow[d, "f^{0}"] & E^{1}\ldots \arrow[d] \\
I^1 : \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & I^0 \arrow[r, "1"] \arrow[d, "1"] & I^0 \arrow[r] \arrow[d] & 0\ldots \arrow[d] \\
I^2 : \ldots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^0 \arrow[r] & 0 \arrow[r] & 0\ldots
\end{tikzcd}
$$

where $f^0 d^{-1} = f^{-1} = u^{-1}$ and $f^{-1} d^{-2} = 0$.

So, we have a morphism f such that $\lambda_1{}^*(f) = u$. $\square$

Lemma 2.5. *${\varepsilon_1}^{\perp}({}^{\perp}\varepsilon_1)$ is extension closed.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence where A, C $\in \varepsilon_1^{\perp}$ and $X \in \varepsilon_1$. Then we have the exact sequence $0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \to Ext^1(X, A) \to Ext^1(X, B) \to Ext^1(X, C) \to ...$

Since $Ext^1(X, A) = Ext^1(X, C) = 0$, then $Ext^1(X, B) = 0$. So, $B \in \varepsilon_1^{\perp}$.

Dually, we can prove ${}^{\perp}\varepsilon_1$ is extension closed. □

Corollary 2.1. *Let $I = ... \to 0 \to I^0 \to I^1 \to ... \to I^n \to 0 \to ...$ where I^i $(0 \leq i \leq n)$ is an $\mathcal{X}$-injective (projective) module, then I is an ${\varepsilon_1}^{\perp}({}^{\perp}\varepsilon_1)$.*

Proof. We will use an induction on n. Let $n = 0$. Then by Lemma 2.4, $\underline{I^0} : ... \to 0 \to I^0 \to 0 \to ...$ is in ${\varepsilon_1}^{\perp}({}^{\perp}\varepsilon_1)$.

Assume that for $n - 1$, the induction is satisfied. Then we have the following exact sequence;

$$0 \to A \to I \to I/A \to 0$$

where $A = ... \to 0 \to I^0 \to 0 \to ....$ By Lemma 2.5, $I \in {\varepsilon_1}^{\perp}({}^{\perp}\varepsilon_1)$ □

Since every right(left) bounded complex is a direct(inverse) limit of a bounded complex, we can obtain the following result.

Corollary 2.2. *Every left(right) bounded complex I where I_i is an $\mathcal{X} - injective$ $(projective)$ module is in ${\varepsilon_1}^{\perp}$ $({}^{\perp}\varepsilon_1)$.*

Proof. Let $I = ... \to 0 \to I^0 \to I^1 \to I^2 \to ...$ be a left bounded complex. $I_i : ... \to 0 \to I^0 \to ... \to I^i \to 0 \to ...$ where $i \geq 0$. Then $\varprojlim I_i = I$. By Remark 1.5, $Ext^1(A, \varprojlim I_i) \cong \varprojlim Ext^1(A, I_i) = 0$ where $A \in \varepsilon_1$. So, $I \in \varepsilon_1^{\perp}$.

Dually, we can prove for ${}^{\perp}\varepsilon_1$. □

Lemma 2.6. *If $I \in \varepsilon_1^{\perp}$, then each I^n is $\mathcal{X}$-injective for each $n \in \mathbb{Z}$.*

Proof. Let $S \subseteq M$ be a submodule of a module M with $\frac{M}{S} \in \mathcal{X}$ and $\alpha : S \longrightarrow I_n$ be linear form the pushout;

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "\alpha"'] & M \arrow[d, dotted, "i_1"] \\ I^n \arrow[r, dotted, "f^n"] & \frac{I^n \oplus M}{A} = I^n \oplus_S M \end{tikzcd}$$

where $A = \{(\alpha(s), -s) : s \in S\}$. Thus i_2 is one-to-one the same as i. Then $\overline{I} : ... \longrightarrow I^{n-1} \longrightarrow I^n \oplus_S M \longrightarrow I^{n+1} \longrightarrow I^{n+2} \longrightarrow ...$ is a complex.

$$\begin{tikzcd} 0 \arrow[r] & I^{n-1} \arrow[r] \arrow[d] & I^{n-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^n \arrow[r] \arrow[d] & I^n \oplus_S M \arrow[r] \arrow[d] & \frac{M}{S} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^{n+1} \arrow[r] & I^{n+1} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Therefore, we have an exact sequence $0 \longrightarrow I \longrightarrow \overline{I} \longrightarrow E \longrightarrow 0$ where $E : ... \longrightarrow 0 \longrightarrow \frac{M}{S} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ and so, we have an exact sequence $0 \longrightarrow \mathrm{Hom}(E, I) \longrightarrow \mathrm{Hom}(\overline{I}, I) \longrightarrow \mathrm{Hom}(I, I) \longrightarrow Ext'(E, I) = 0$ since $I \in \varepsilon_1^{\perp}$.

This implies that we can find $\overline{f} : \overline{I} \longrightarrow I$ with $\overline{f}f = 1$. Therefore, there exists a function $\overline{f}^n : I^n \oplus_S M \longrightarrow I^n$ with $\overline{f}^n f^n = 1$. So,

$$\overline{f}^n f^n(\alpha(s)) = \alpha(s)$$

$$\overline{f}^n((\alpha(s), 0) + A) = \alpha(s)$$

$$\overline{f}^n((0, s) + A) = \alpha(s)$$

$$\overline{f}^n i_1 i(s) = \alpha(s)$$

and hence $\overline{f}_n i_1 i = \alpha$ and each $I^n \in \mathcal{X} - injective$. $\square$

Lemma 2.7. *(Lemma 3.2 in [4]) Let $f : X \longrightarrow Y$ be a morphism of complexes. Then the exact sequence* $0 \longrightarrow Y \longrightarrow M(f) \longrightarrow X[1] \longrightarrow 0$ *associated with the mapping cone $M(f)$ splits if and only if f is homotopic to* 0.

Proof.

($\Longrightarrow$)

Since the sequence splits, there exists a morphism $g : X[1] \longrightarrow M(f)$ with $g(x) = (x, -s(x))$ where s is a morphism from $X[1]$ to Y. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y^{n-1} \arrow[r, "i^{n-1}"] \arrow[d, "\partial_1^{n-1}"'] & X^n \oplus Y^{n-1} \arrow[r, shift left, dotted, "p^{n-1}"] \arrow[d, "\theta^{n-1}"'] & X^n \arrow[l, shift left, dotted, "g^n"] \arrow[r] \arrow[d, "-\partial^n"] & 0 \\
0 \arrow[r] & Y^n \arrow[r, "i^n"'] & X^{n+1} \oplus Y^n \arrow[r, shift left, dotted, "p^n"] & X^{n+1} \arrow[l, shift left, dotted, "g^{n+1}"] \arrow[r] & 0
\end{tikzcd}$$

where $\theta^{n-1}(x, y) = (-\partial^n x, f^n(x) + \partial_1^{n-1} y)$ for all $(x, y) \in X^n \oplus Y^{n-1}$. By the commutative diagram, we can find that for all $x \in X^n$

$$\theta^{n-1}(g^n(x)) = -g^{n+1}\partial^n(x)$$

$$\theta^{n-1}(x, -s^n(x)) = (-\partial^n(x), s^{n+1}\partial^n(x))$$

$$(-\partial^n x, f^n(x) + \partial_1^{n-1}(-s^n(x))) = (-\partial^n(x), s^{n+1}\partial^n(x)).$$

This implies that $f^n(x) = \partial_1^{n-1}s^n(x) + s^{n+1}\partial^n(x)$, for all $x \in X^n$. Therefore, $f^n = \partial^{n-1}s^n + s^{n+1}\partial^n$ which implies that f is homotopic to 0.

($\Longleftarrow$)

Suppose that f is homotopic to 0. If s is such a homotopy, then we can construct a function,

$$g : X[1] \longrightarrow M(f)$$

$$x \longmapsto (x, -s(x))$$

where $s^n : X^n \longrightarrow Y^{n-1}$ such that $f^n = \partial^{n-1}s^n + s^{n+1}\partial^n$. Then $p^{n-1}g^n(x) = p^{n-1}(x, -s^n(x)) = I(x)$ for all $x \in X^n$ and so $p^{n-1}g^n = I$. So the exact sequence $0 \longrightarrow Y \longrightarrow M(f) \longrightarrow X[1] \longrightarrow 0$ splits. □

Lemma 2.8. *Let X and I are complexes. If $Ext^1(X, I[n]) = 0$ for all $n \in \mathbb{Z}$, then $\mathcal{H}om(X, I)$ is exact.*

Proof. Since $Ext^1(X, I[n]) = 0$, if $f : X[-1] \to I[n]$ is a morphism, then $0 \to I[n] \to M(f) \to X \to 0$ splits.

By Lemma 2.7, $f : X[-1] \to I[n]$ is homotopic to zero for all n.

So $f^1 : X \to I[n+1]$ is homotopic to zero for all $n \in \mathbb{Z}$. Thus $\mathcal{H}om(X, I)$ is exact.

□

Corollary 2.3. $\varepsilon_1{}^{\perp}({}^{\perp}\varepsilon_1) \subseteq DG(\mathcal{X} - injective\ (projective))$.

Proof. It follows from Lemma 2.6 and Lemma 2.8. □

Corollary 2.4. *Every left (right) bounded complex I where I_i is an $\mathcal{X}$-projective (injective) module is a DG($\mathcal{X}$-projective (injective)) complex.*

Proof. It follows from Corollary 2.2 and Corollary 2.3. □

Lemma 2.9. *Let $f : X \longrightarrow Y$ a chain morphism, Y is an $\mathcal{X}^*$-complex and X is an $\mathcal{X}$-projective complex. Then f is a homotopic to zero. Moreover if a complex has a $C(\mathcal{X} - projective)$-precover in $C(\mathcal{X}^*)$, then such precovers are homotopic.*

Proof. Let $id : Y \longrightarrow Y$ and the exact sequence $0 \longrightarrow Y[-1] \longrightarrow M(id)[-1] \longrightarrow Y \longrightarrow 0$.

Since X is an $\mathcal{X} - projective$ complex, we have the following commutative diagram;

$$\begin{tikzcd}
M(id)[-1] \arrow[r, "\pi"'] & Y \arrow[r] & 0 \\
X \arrow[u, dotted, "g"'] \arrow[ur, "f"'] & &
\end{tikzcd}$$

where $\pi g = f$. Let $\pi' : M(id)[-1] \longrightarrow Y[-1]$ be a projection. If we take an $s = \pi' g$, then for all $n \in \mathbb{Z}$, $s^{n+1}\lambda^n + \gamma^{n-1}s^n = f^n$ where λ and γ are boundary maps of the complexes of X and Y, respectively. So, f is homotopic to zero. □

Lemma 2.10. *Let $f : X \longrightarrow Y$ be a chain morphism, X is an $\mathcal{X}^*$-complex and Y is an $\mathcal{X} - injective$ complex. Then f is homotopic to zero. Moreover if a complex has a $C(\mathcal{X} - injective)$-preenvelope in $C(\mathcal{X}^*)$, then such preenvelopes are homotopic.*

Proof. Let $id : X \longrightarrow X$, then we have the following exact sequence;

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"] & M(id) \arrow[r] \arrow[dl, dotted, "g"] & X[1] \arrow[r] & 0 \\
& Y & & &
\end{tikzcd}$$

where $gi = f$.

Let $i' : X[1] \longrightarrow M(id)$ be an injection and $s : X[1] \longrightarrow Y$ such that $s = gi'$ with $s^n = g^{n-1}i'$.

$$\begin{tikzcd}
X^{n-2} \oplus X^{n-1} \arrow[r, "u^{n-2}"] \arrow[d, "g^{n-2}"] & X^{n-1} \oplus X^{n} \arrow[r, "u^{n-1}"] \arrow[d, "g^{n-1}"] & X^{n} \oplus X^{n+1} \arrow[d, "g^{n}"] \\
Y^{n-2} \arrow[r, "\gamma^{n-2}"] & Y^{n-1} \arrow[r, "\gamma^{n-1}"] & Y^{n}
\end{tikzcd}$$

$$s^{n+1}\lambda^n+\gamma^{n-1}s^n = g^n i'\lambda^n+\gamma^{n-1}g^{n-1}i' = g^n(i'\lambda^n+g^n u^{n-1}i' = g^n i'\lambda^n + u^{n-1}i') = g^n i = f^n$$

□

Now we will investigate when every left (right) bounded complex has a $C(\mathcal{X}-projective$ $(injective))$-precover (preenvelope).

Lemma 2.11. *Let $\mathcal{X}$ be closed under finite direct sum. Let every R-module has an epic $\mathcal{X}$-projective precover in $\mathcal{X}$ with kernel in $\mathcal{X}$. Then every bounded complex has an exact $C(\mathcal{X}-projective)$ precover. Thus every bounded $\mathcal{X}$-projective complex in $C(\mathcal{X}^*)$ is exact.*

Proof. Let $Y(n) : ... \to 0 \to Y^0 \to Y^1 \to ... \to Y^n \to 0 \to ....$
We will use induction on n. Let $n = 0$, then we have the following commutative diagram;

$$
\begin{tikzcd}
D(0): ... \arrow[r] & 0 \arrow[r] \arrow[d] & P^0 \arrow[r, "id"] \arrow[d, "f^0"] & P^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
Y(0): ... \arrow[r] & 0 \arrow[r] & Y^0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

where $D(0)$ is exact and $Ker(D(0) \to Y(0)) \in \mathcal{X}$.

Let $n = 1$, then we have the following commutative diagram;

$$
\begin{tikzcd}
D(1): ...0 \arrow[r] & P^0 \arrow[r, "\lambda_1^0"] \arrow[d, "f^0"] & P^0 \oplus P^1 \arrow[r, "\lambda^1"] \arrow[d, "(0{,}f^1)"] & P^1 \arrow[r] \arrow[d] & 0... \\
Y(1): ...0 \arrow[r] & Y^0 \arrow[r, "a^0"] & Y^1 \arrow[r] & 0 \arrow[r] & 0...
\end{tikzcd}
$$

where $D(1)$ is exact and $Ker(D(1) \to Y(1)) \in \mathcal{X}$.

We also have the following commutative diagram;

$$
\begin{tikzcd}
P^0 \arrow[r, "s^0"] \arrow[d, "f^0"] & P^1 \arrow[d, "f^1"] \\
Y^0 \arrow[r, "a^0"] & Y^1
\end{tikzcd}
$$

$\lambda_1^0(x) = (x, s^0(x))$ and $\lambda^1(x,y) = s^0(x) - y$

We assume that the following diagram $D(n) \to Y(n)$ which is commutative;

$$\begin{tikzcd}
...0 \arrow[r] & P^0 \arrow[r, "\lambda^0"] \arrow[d, "f^0"] & P^0 \oplus P^1 \arrow[r, "\lambda^1"] \arrow[d, "(0,f^1)"] & \cdots \arrow[r] & P^{n-1} \oplus P^n \arrow[r, "\lambda^n"] \arrow[d, "(0,f^n)"] & P^n \arrow[r] \arrow[d] & 0 \\
...0 \arrow[r] & Y^0 \arrow[r, "a^0"] & Y^1 \arrow[r, "a^1"] & \cdots \arrow[r] & Y^n \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where $D(n)$ is exact and $Ker(D(n) \to Y(n)) \in \mathcal{X}$. And we also have the following commutative diagrams;

$$\begin{tikzcd}
D(n) \arrow[r, "s"] \arrow[d] & \overline{P^{n+1}} \arrow[d] \\
Y(n) \arrow[r] & \underline{Y^{n+1}}
\end{tikzcd}$$

So, $D(n) \to \overline{P^{n+1}}$;

$$\begin{tikzcd}
...0 \arrow[r] & P^0 \arrow[r, "\lambda^0"] \arrow[d] & P^0 \oplus P^1 \arrow[r, "\lambda^1"] \arrow[d] & \cdots \arrow[r, "\lambda^{n-1}"] & P^{n-1} \oplus P^n \arrow[r, "\lambda^n"] \arrow[d, "s^1"] & P^n \arrow[r] \arrow[d, "s^2"] & 0 \\
...0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & P^{n+1} \arrow[r, "1"] & P^{n+1} \arrow[r] & 0
\end{tikzcd}$$

where $s^2\lambda^n = s^1$ and $s^1\lambda^{n-1} = 0$.

$$\begin{tikzcd}
P^{n-1} \oplus P^n \arrow[r, "s^1"] \arrow[d, "(0, f^n)"] & P^{n+1} \arrow[d, "f^{n+1}"] \\
Y^n \arrow[r, "a^n"] & Y^{n+1}
\end{tikzcd}$$

where $f^{n+1}s^1 = a^n(0, f^n)$.

$$\begin{tikzcd}
P^n \arrow[r, "s^2"] \arrow[d] & P^{n+1} \arrow[d, "f^{n+1}"] \\
0 \arrow[r] & Y^{n+1}
\end{tikzcd}$$

where $f^{n+1}s^2 = 0$. Hence, $D(n+1) \to Y(n+1)$;

$$\begin{tikzcd}
...0 \arrow[r] & P^0 \arrow[r, "\lambda^0"] \arrow[d, "f^0"] & P^0 \oplus P^1... \arrow[r] \arrow[d, "(0,f^1)"] & P^{n-1} \oplus P^n \arrow[r, "\lambda_1^n"] \arrow[d, "(0,f^n)"] & P^n \oplus P^{n+1} \arrow[r, "\lambda^{n+1}"] \arrow[d, "(0,f^{n+1})"] & P^{n+1}... \\
...0 \arrow[r] & Y^0 \arrow[r, "a^0"] & Y^1... \arrow[r, "a^1"] & Y^n \arrow[r, "a^n"] & Y^{n+1} \arrow[r] & 0...
\end{tikzcd}$$

where $D(n+1)$ is exact and $Ker(D(n+1) \to Y(n+1)) \in \mathcal{X}$, $\lambda_1^n(x,y) = (\lambda^n(x,y), s^1(x,y))$, $\lambda^{n+1}(x,y) = s^2(x) - y$.

Therefore, $Y(n)$ has an $C(\mathcal{X} - projective)$ precover. $\square$

Lemma 2.12. *Let every R-module has a monic $\mathcal{X}$-injective preenvelope in $\mathcal{X}$ with cokernel in $\mathcal{X}$. Then every bounded complex has an $C(\mathcal{X} - injective)$ preenvelope. Thus every bounded $\mathcal{X}$-injective complex in $C(\mathcal{X}^*)$ is exact.*

Proof. Let $Y(n) : ... \to 0 \to Y_n \to Y_{n-1} \to ... \to Y_0 \to 0 \to ....$
We will use induction on n. Let $n = 0$, then we have the following commutative diagram;

$$\begin{tikzcd}
Y(0) : ... \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
E(0) : ... \arrow[r] & 0 \arrow[r] & E_0 \arrow[r, "id"] & E_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where $E(0)$ is exact and cokernel in $\mathcal{X}$.

Let $n = 1$, then we have the following commutative diagram;

$$\begin{tikzcd}
Y(1) : ...0 \arrow[r] & 0 \arrow[r] \arrow[d] & Y_1 \arrow[r, "a_1"] \arrow[d, "(f_1{,}0)"] & Y_0 \arrow[r] \arrow[d, "f_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
E(1) : ...0 \arrow[r] & E_1 \arrow[r, "\lambda_2"] & E_1 \oplus E_0 \arrow[r, "\lambda_1"] & E_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where $E(1)$ is exact and cokernel in $\mathcal{X}$.

We also have the following commutative diagram;

$$\begin{tikzcd}
\underline{Y_1} \arrow[r] \arrow[d] & \underline{Y_0} \arrow[d] \\
\overline{E_1} \arrow[r] & \overline{E_0}
\end{tikzcd}$$

where $E_1 \xrightarrow{\lambda} E_0$ such that $f_0 a_1 = \lambda f_1$. $\lambda_2(x) = (x, -f(x))$ and $\lambda_1(x, y) = f(x) + y$. We assume that the following diagram which is commutative;

$$\begin{tikzcd}
Y(n) : ...0 \arrow[r] & 0 \arrow[r] \arrow[d] & Y_n \arrow[r, "a_n"] \arrow[d, "{(f_n,0)}"] & \cdots \arrow[r] & Y_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
E(n) : ...0 \arrow[r] & E_n \arrow[r, "\lambda'_n"] & E_n \oplus E_{n-1} \arrow[r, "\lambda_{n-1}"] & \cdots \arrow[r] & E_0 \arrow[r] & 0
\end{tikzcd}$$

where $E(n)$ is exact and cokernel in $\mathcal{X}$.

And we have the following commutative diagram $Y(n+1) \to E(n+1)$;

$$\begin{tikzcd}
...0 \arrow[r] & 0 \arrow[r] \arrow[d] & Y_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, "{(f_{n+1},0)}"] & Y_n... \arrow[r, "a_n"] \arrow[d, "{(f_n,0)}"] & Y_0 \arrow[r] \arrow[d, "f^0"] & 0... \\
...0 \arrow[r] & E_{n+1} \arrow[r, "\lambda_{n+1}"] & E_{n+1} \oplus E_n \arrow[r, "\lambda_n"] & E_n \oplus E_{n-1}... \arrow[r] & E_0 \arrow[r] & 0...
\end{tikzcd}$$

where $E(n+1)$ is exact and cokernel in $\mathcal{X}$. We also have the following commutative diagrams;

$$\begin{tikzcd} \underline{Y_{n+1}} \arrow[r] \arrow[d] & Y(n) \arrow[d] \\ \overline{E_{n+1}} \arrow[r] & E(n) \end{tikzcd}$$

$$\begin{tikzcd} Y_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, "f_{n+1}"] & Y_n \arrow[d, "(f_n{,}0)"] \\ E_{n+1} \arrow[r, "s_{n+1}"] & E_n \oplus E_{n-1} \end{tikzcd}$$

where $(f_n, 0)a_{n+1} = s_{n+1}f_{n+1}$.

$$\begin{tikzcd} E_{n+1} \arrow[r, "t_{n+1}"] \arrow[d, "1"] & E_n \arrow[d, "\lambda_n^1"] \\ E_{n+1} \arrow[r, "s_{n+1}"] & E_n \oplus E_{n-1} \end{tikzcd}$$

where $\lambda_n^1 t_{n+1} = s_{n+1}1$.

Then, $\lambda_{n+1}(x) = (x, -t_{n+1}(x))$, $\lambda_n(x, y) = s_{n+1}(x) + \lambda_n^1(y)$.

Therefore, $Y(n)$ has an $C(\mathcal{X} - injective)$ preenvelope. □

Theorem 2.1. *Let R be a noetherian ring and $C(\mathcal{X}^*)$ be closed under direct limit. Let every R-module have a monic $\mathcal{X}$-injective preenvelope in $\mathcal{X}$. Then every right bounded complex has a C($\mathcal{X}$-injective) preenvelope.*

Proof. Let $Y : ... \to Y_2 \to Y_1 \to Y_0 \to 0 \to ...$ and $E(n)$ be a C($\mathcal{X}$-injective) preenvelope of $Y(n) : ... \to 0 \to Y_n \to ... \to Y_1 \to Y_0 \to 0 \to ....$ Then $\varinjlim Y(n) = Y$. Since $0 \to Y(n) \to E(n)$ is exact, then by Theorem 1.3 $0 \to \varinjlim Y(n) \to \varinjlim E(n)$ is exact with cokernel $\varinjlim \frac{E(n)}{Y(n)} \in C(\mathcal{X}^*)$. Since R is noetherian, then $Ext'(\frac{A}{B}, \varinjlim E(n)) \cong \varinjlim Ext'(\frac{A}{B}, E(n)) = 0$ where $\frac{A}{B} \in C(\mathcal{X}^*)$ by Lemma 3.1.16 in [3]. So $\varinjlim E(n)$ is a C($\mathcal{X}$-injective) preenvelope of Y. □

Theorem 2.2. *Let R be a noetherian ring and $C(\mathcal{X}^*)$ be closed under both inverse and direct limit. Let every R-module have a monic $\mathcal{X}$-injective preenvelope in $\mathcal{X}$. Then every complex has a C($\mathcal{X}$-injective) preenvelope.*

Proof. Let $Y : ... \to Y_2 \to Y_1 \to Y_0 \to Y_{-1} \to ...$ and $Y(n) : ... \to Y_1 \to Y_0 \to ... \to Y_{-n} \to 0 \to ....$ Then $\varprojlim Y(n) = Y$.
By Theorem 2.1, $Y(n)$ has a C($\mathcal{X}$-injective) preenvelope $D(n)$ such that $0 \to Y(n) \to D(n)$ is monic and $\frac{D(n)}{Y(n)} \in C(\mathcal{X})$. So, $0 \to \varprojlim Y(n) \to \varprojlim D(n)$ is monic by Theorem 1.4 and $\varprojlim \frac{D(n)}{Y(n)} \in C(\mathcal{X})$.
By Remark 1.5 $Ext'(\frac{A}{B}, \varprojlim D(n)) \cong \varprojlim Ext'(\frac{A}{B}, D(n)) = 0$ where $\frac{A}{B} \in C(\mathcal{X}^*)$. Therefore, $\varprojlim D(n)$ is a C($\mathcal{X}$-injective) preenvelope of Y. □

Lemma 2.13. *(A generalization of comparison theorem) Let R-Mod have an $\mathcal{X}$-precover (preenvelope with monic) with onto. Given a module M, then*

- **(i)** *We have an exact complex $C : \ldots \longrightarrow C_2 \xrightarrow{\ell_2} C_1 \xrightarrow{\ell_1} C_0 \xrightarrow{\ell_0} M \longrightarrow 0$ such that C_0 is a $\mathcal{X}$-precover (preenvelope) of M, $C_i \in \mathcal{X}$ is a $\mathcal{X}$-precover (preenvelope) of $Ker\ell_{i-1}$ for all $i \in \mathbb{Z}$ and so C is a $\mathcal{X}^* -$ complex.*

- **(ii)** *Let D be an $\mathcal{X}^* -$ complex and C be a complex as above. If $f_{-1} : M' \longrightarrow M$ is a morphism, then we can compose a commutative diagram;*

$$\begin{tikzcd}
\cdots \arrow[r] & D_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & D_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & D_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & M' \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r, "c_2"] & C_1 \arrow[r, "c_1"] & C_0 \arrow[r, "c_0"] & M \arrow[r] & 0
\end{tikzcd}$$

such that f is a chain map where $\overline{f}_{-1} = f_{-1}$. Moreover, any two such chain maps are homotopic and the dual of the comparison theorem is also true.

Proof. It follows from Theorem 6.16 in [9]. □

Lemma 2.14. *Let X be an $\mathcal{X} - injective$ complex and $\frac{E(X)}{X} \in C(\mathcal{X}^*)$ (or $\frac{Y}{X} \in C(\mathcal{X}^*)$) where $E(X)$ is an injective envelope of X. Then $X = E(X)$ and so, it is an exact complex. (X is a direct summand of Y).*

Proof. We know that every complex has an injective envelope, so, X has an injective envelope $E(X)$. Then $E(X)$ is a injective complex, and so, it is exact. We have the following commutative diagram;

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"'] \arrow[d, "id_x"'] & E(X) \arrow[dl, dotted, "\phi"] \\
& X &
\end{tikzcd}$$

such that $\phi i = id_x$. Therefore X is a direct summand of $E(X)$. So X is an injective complex and hence it is exact. Similarly, if $\frac{Y}{X} \in C(\mathcal{X}^*)$, then we can prove that X is a direct summand of Y. □

Theorem 2.3. *Let $C(\mathcal{X}^*)$ be a class under extensions, quotients and direct limits, then every complex B has an $C(\mathcal{X} - injective)$ preenvelope.*

Proof. We know that B has an injective envelope E. Let $S = \{A : B \subseteq A \subseteq E$ *and* $\frac{A}{B}$ *is an* $C(\mathcal{X}^*)\} \neq \emptyset$. Let S' be an ascending chain. Then $\frac{\cup_{N_i \in S'} N_i^k}{B^k} = \cup_{N_i \in S'} \frac{N_i^k}{B^k} = lim \frac{N_i^k}{B^k}$ is in the class $C(\mathcal{X}^*)$. So S has a maximal element, say T. We shall prove that T is an $\mathcal{X} - injective$ complex. It is enough to show that any exact sequence $0 \longrightarrow T \longrightarrow Y \longrightarrow C \longrightarrow 0$ with $C \in C(\mathcal{X}^*)$ is split. We have the following commutative diagram;

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "\alpha"] & C \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & T \arrow[r, "i"] & E(T) \arrow[r, "\pi"] & \frac{E(T)}{T} \arrow[r] & 0
\end{tikzcd}$$

Let $\beta(C) = \frac{K}{T}$. So, by $\frac{K}{T} \cong \frac{\frac{K}{B}}{\frac{T}{B}}$, we say that $\frac{K}{B}$ is an $\mathcal{X}^* - complex$. Since T is a maximal element of S, $\beta(C) = 0$, and hence $\pi\alpha(Y) = \frac{\alpha(Y)+T}{T} = 0$, and so, $\alpha(Y) \subseteq T$. Therefore, $0 \longrightarrow T \longrightarrow Y \longrightarrow C \longrightarrow 0$ is split exact and T is an $\mathcal{X} - injective$ complex. Moreover T is an $C(\mathcal{X} - injective)$-preenvelope of B by the following diagram;

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d] & T \arrow[dl, dotted] \\
 & Y &
\end{tikzcd}$$

where Y is an $\mathcal{X} - injective$ complex and $\frac{T}{B} \in C(\mathcal{X}^*)$. □

Theorem 2.4. *A complex X is contained in a minimial $\mathcal{X}$ injective complex X'.*

Proof. We know that every complex has an injective envelope. Let $S = \{A : X \subseteq A \subseteq E$ *and* A $\mathcal{X} - injective\ complex\} \neq \emptyset$ and S' be a descending chain of S. We will show that $\cap_{A_\alpha \in S'} \{A_\alpha : \alpha \in I\}$ is an $\mathcal{X} - injective$ complex. Using this pushout diagram we have the following diagram where C with $X^* - complex$,

$$\begin{tikzcd}
0 \arrow[r] & \cap A_\alpha \arrow[r, "\beta"] \arrow[d, "\theta_\alpha"] & Y \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_\alpha \arrow[r, "\gamma_\alpha"] & B_\alpha \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Then the bottom row is split exact. So $0 \longrightarrow \cap A_\alpha \longrightarrow \cap B_\alpha \longrightarrow C \longrightarrow 0$ is split exact. We have the following diagram;

$$\begin{tikzcd}
0 \arrow[r] & \cap A_\alpha \arrow[r, "\beta"] \arrow[d] & Y \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cap A_\alpha \arrow[r, "\gamma"] & \cap B_\alpha \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

By five lemma ϕ is an isomorphism. So, $0 \longrightarrow \cap A_\alpha \longrightarrow Y \longrightarrow C \longrightarrow 0$ is split exact. So, S has a minimal element, say X'. $\square$

CHAPTER 3

C($\mathcal{X}^*$)-COVER AND C($\mathcal{X}^*$)-ENVELOPE

Let R be any associative ring with unity and $\mathcal{X}$ be a class of R-modules of closed under direct sum (and summands) and with extension closed. We prove that every complex has an $C(\mathcal{X}^*)$-cover ($C(\mathcal{X}^*)$-envelope) if every module has an $\mathcal{X}$-cover ($\mathcal{X}$-envelope) where $C(\mathcal{X}^*)$ is the class of complexes of modules in $\mathcal{X}$ such that it is closed under direct and inverse limit. In [6] and [7], respectively, it is proved that if R is a noetherian ring, then every complex of R-modules has a #-injective cover and if R is a Gorenstein ring, then every bounded complex has a Gorenstein projective precover. Using the proof of Lemma 7 in [7], we give a generalization of [6] and [7]. (See [3] and [9] for the other definitions and concepts.) Throughout the section $\mathcal{X}$ denotes a class of R-modules.

Definition 3.1. (#-injective complex) ([1]) A complex of injective modules is called #-injective complex.

Lemma 3.1. *Let for all $n \in \mathbb{Z}$, $0 \longrightarrow P_n \longrightarrow X_n$ is an $\mathcal{X}-$ precover of X_n. Then we have a monic chain map* $0 \longrightarrow P \longrightarrow X$ *where* $P : ... \to P_2 \to P_1 \to P_0 \to ...$ *and* $X : ... \to X_2 \to X_1 \to X_0 \to ...$.

Proof. It is trivial. □

Theorem 3.1. *Let every module have a monic $\mathcal{X}$-precover (or an epic $\mathcal{X}$-preenvelope) then every complex X has a monic $C(\mathcal{X}^*)$-precover (or epic $C(\mathcal{X}^*)$-preenvelope). More-*

over if every module has a monic $\mathcal{X}$-cover (or epic $\mathcal{X}$-envelope), then every complex X has a monic $C(\mathcal{X}^*)$-cover (or epic $C(\mathcal{X}^*)$-envelope).

Proof. Let X be any complex and $0 \to P_n \to X_n$ be an $\mathcal{X}$-precover of X_n. By Lemma 3.1 there exists a complex $P : ... \to P_2 \to P_1 \to P_0 \to ...$ such that $0 \to P \to X$ is a chain map. Let $P' : ... \to P'_2 \to P'_1 \to P'_0 \to ...$ be any $\mathcal{X}^*$-complex and $f' : P' \longrightarrow X$ be a chain map. Then we have the following diagram as follow;

$$\begin{tikzcd}
 & P_{n+1} \arrow[r, "\lambda_{n+1}"] \arrow[dl, "f_{n+1}"'] & P_n \arrow[r, "\lambda_n"] \arrow[dl, "f_n"'] & P_{n-1} \arrow[dl, "f_{n-1}"'] \\
X_{n+1} \arrow[r, "\beta_{n+1}"] & X_n \arrow[r, "\beta_n"] & X_{n-1} & \\
 & P'_{n+1} \arrow[r, "\lambda'_{n+1}"'] \arrow[ul, "f'_{n+1}"] & P'_n \arrow[r, "\lambda'_n"'] \arrow[ul, "f'_n"] & P'_{n-1} \arrow[ul, "f'_{n-1}"]
\end{tikzcd}$$

where since P' is an $\mathcal{X}^*$-complex and $f' : P' \longrightarrow X$ be a chain map, we can find $\theta_n : P'_n \longrightarrow P_n$ for all $n \in \mathbb{Z}$ such that $f_n\theta_n = f'_n$.

Since f' is a chain map, $f'_n\lambda'_{n+1} = \beta_{n+1}f'_{n+1}$, for all $n \in \mathbb{Z}$. Now we will show that θ is a chain map. Since f_n is a chain map,

$$f_n\lambda_{n+1} = \beta_{n+1}f_{n+1},\ f_n\lambda_{n+1}\theta_{n+1} = \beta_{n+1}f_{n+1}\theta_{n+1} = \beta_{n+1}f'_{n+1} = f'_n\lambda'_{n+1} = f_n\theta_n\lambda'_{n+1}$$

Since f_n is $1-1$, $\lambda_{n+1}\theta_{n+1} = \theta_n\lambda'_{n+1}$. So, θ is a chain map. Let $0 \longrightarrow P \longrightarrow X$ be a monic chain map such that $0 \longrightarrow P_n \longrightarrow X_n$ is an $\mathcal{X}$-cover of X_n. Then $0 \longrightarrow P \longrightarrow X$ is an $C(\mathcal{X}^*)$-cover of X. Because we have the following diagram;

$$\begin{tikzcd}
& P_{n+1} \arrow[rr, "\lambda_{n+1}"] \arrow[dl, "f_{n+1}"'] & & P_n \arrow[rr, "\lambda_n"] \arrow[dl, "f_n"'] & & P_{n-1} \arrow[dl, "f_{n-1}"'] \\
X_{n+1} \arrow[rr, "\beta_{n+1}"] & & X_n \arrow[rr, "\beta_n"] & & X_{n-1} & \\
& P_{n+1} \arrow[rr, "\lambda'_{n+1}"'] \arrow[ul, "f'_{n+1}"] & & P_n \arrow[rr, "\lambda'_n"'] \arrow[ul, "f'_n"] & & P_{n-1} \arrow[ul, "f'_{n-1}"]
\end{tikzcd}$$

We can find that for all $n \in \mathbb{Z}$, θ_n is an automorphism. By the proof above $\theta : P \to P$ is a chain map such that $f\theta = f$. Since for all $n \in \mathbb{Z}$, θ_n is a automorphism, θ is an automorphism with $f\theta = f$. So $0 \longrightarrow P \longrightarrow X$ is a cover of X. Dually we can find a $C(\mathcal{X}^*)$-preenvelope (envelope); $X \longrightarrow E \longrightarrow 0$. □

Lemma 3.2. *Let $\mathcal{X}$ be closed under finite sum. If every module has an $\mathcal{X}$-precover, then every bounded complex has an $C(\mathcal{X}^*)$-precover.*

Proof. Let $... \longrightarrow E'_1 \xrightarrow{f_1} E'_0 \xrightarrow{\phi_0} X_0 \longrightarrow 0$ be a minimal $\mathcal{X}$-precover resolvent of X_0. Let A be an $\mathcal{X}^*$-complex and $\beta \in \mathrm{Hom}(A, X(0))$ where $X(0) : ... \longrightarrow 0 \longrightarrow X_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ....$ We have the diagram as follow;

$$\begin{tikzcd}
\cdots \arrow[r] & A_1 \arrow[r, "a_1"] \arrow[d, "\lambda_1"] & A_0 \arrow[r] \arrow[d, "\lambda_0"] & A_{-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & E'_1 \arrow[r, "f_1"] & E'_0 \arrow[r] \arrow[d, "\phi_0"] & 0 & \\
& 0 \arrow[r] & X_0 \arrow[r] & 0 &
\end{tikzcd}$$

where $\phi_0\lambda_0 = \beta_0$ and $\lambda_0 a_1 \in \mathrm{Hom}(A_1, Ker\phi_0)$. Then there exists $\lambda_1 \in \mathrm{Hom}(A_1, E'_1)$, since $E'_1 \longrightarrow Ker\phi_0$ is a precover and similarly, we can define $\lambda_n \in \mathrm{Hom}(A_n, E'_n)$ such that $f_n\lambda_n = \lambda_{n-1}a_n$. Thus, $E' : ... \longrightarrow E'_2 \longrightarrow E'_1 \longrightarrow E'_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ is a $C(\mathcal{X}^*)$-precover of $X(0)$. Let $X(n) : ...0 \longrightarrow X_n \xrightarrow{\ell_n} ... \longrightarrow X_1 \xrightarrow{\ell_1} X_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ....$ We will use an induction on n to prove that $X(n)$ has a $C(\mathcal{X}^*)$-precover. We proved that $X(0)$ has a $C(\mathcal{X}^*)$-precover above. Suppose that $D(n) : ... \longrightarrow D_{n+1} \xrightarrow{d_{n+1}} D_n \longrightarrow .... \longrightarrow D_1 \xrightarrow{d_1} D_0 \longrightarrow 0$ be a $C(\mathcal{X}^*)$-precover of $X(n)$ such that

$$\begin{tikzcd}
D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] \arrow[d, "\phi_n"] & \cdots \arrow[r] & D_1 \arrow[r] \arrow[d, "\phi_1"] & D_0 \arrow[r] \arrow[d, "\phi_0"] & 0 \\
0 \arrow[r] & X_n \arrow[r, "\ell_n"] & \cdots \arrow[r] & X_1 \arrow[r, "\ell_n"] & X_0 \arrow[r] & 0
\end{tikzcd}$$

Let $E_\bullet = ... \to E_2 \overset{t_2}{\to} E_1 \overset{t_1}{\to} E_0 \to 0 \to 0 \to ...$ is a precover of $\underline{X_{n+1}}$ where $t_0 : E_0 \to X_{n+1}$ is a precover of X_{n+1}. Then we have the following commutative diagram;

$$\begin{tikzcd}
E_\bullet \arrow[r, "\nu"] \arrow[d] & D(n) \arrow[d, "\phi"] \\
\underline{X_{n+1}} \arrow[r] & X(n)
\end{tikzcd}$$

where $\ell_{n+1} : X_{n+1} \longrightarrow X_n$. Since $\phi : D(n) \longrightarrow X(n)$ is a $C(\mathcal{X}^*)$-precover of X(n) and $E_\bullet \in C(\mathcal{X}^*)$, we have a morphism $\nu : E_\bullet \longrightarrow D(n)$ where

$$\begin{tikzcd}
E_\bullet = ... \arrow[r] & E_2 \arrow[r, "t_2"] \arrow[d, "\nu_2"] & E_1 \arrow[r, "t_1"] \arrow[d, "\nu_1"] & E_0 \arrow[r] \arrow[d, "\nu_0"] & 0... \\
D(n) = ... \arrow[r] & D_{n+2} \arrow[r, "d_{n+2}"] & D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1}...
\end{tikzcd}$$

...(1)

where $\phi_n\nu_0 = \ell_{n+1}t_0$...(*) Let C be the mapping cone of $\nu : E_\bullet \longrightarrow D(n)$. So we have that $C = ... \longrightarrow D_{n+3} \oplus E_2 \overset{\lambda_{n+3}}{\longrightarrow} D_{n+2} \oplus E_1 \overset{\lambda_{n+2}}{\longrightarrow} D_{n+1} \oplus E_0 \overset{\lambda_{n+1}}{\longrightarrow} D(n) \overset{d_n}{\longrightarrow} D_{n-1} \longrightarrow ... \longrightarrow D_1 \longrightarrow 0$ where

$$\lambda_{n+1}(x, y) = d_{n+1}(x) + \nu_0(y)$$

and for $k \geq 2$

$$\lambda_{n+k}(x, y) = (d_{n+k}(x) + \nu_{k-1}(y), -t_{k-1}(y)).$$

We show that $C \longrightarrow X(n+1)$ is an $C(\mathcal{X}^*)$-precover of $X(n+1)$. Let $A \in C(\mathcal{X}^*)$ where

$$\begin{tikzcd}
A = ... \arrow[r] & A_{n+2} \arrow[r, "a_{n+2}"] \arrow[d] & A_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, "\beta_{n+1}"] & A_n \arrow[r, "a_n"] \arrow[d, "\beta_n"] & \cdots \\
X(n+1) = ... \arrow[r] & 0 \arrow[r] & X_{n+1} \arrow[r, "\ell_{n+1}"] & X_n \arrow[r, "\ell_n"] & \cdots
\end{tikzcd}$$

Then we have the following diagram;

$$\begin{tikzcd}
...A_{n+2} \arrow[r, "a_{n+2}"] \arrow[d, "\theta_{n+2}"] & A_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, "\theta_{n+1}"] & A_n \arrow[r, "a_n"] \arrow[d, "\theta_n"] & \cdots \\
...D_{n+2} \oplus E_1 \arrow[r, "\lambda_{n+2}"] & D_{n+1} \oplus E_0 \arrow[r, "\lambda_{n+1}"] \arrow[d, "(0,t_0)"] & D_n \arrow[r, "d_n"] \arrow[d, "\phi_n"] & \cdots \\
...0 \arrow[r] & X_{n+1} \arrow[r, "\ell_{n+1}"] & X_n \arrow[r, "\ell_n"] & ..
\end{tikzcd}$$

Firstly, we will find θ_{n+1} such that $\lambda_{n+1}\theta_{n+1} = \theta_n a_{n+1}$ and $(0, t_0)\theta_{n+1} = \beta_{n+1}$. We know that $\phi_n\lambda_{n+1} = \ell_{n+1}(0, t_0)$ by (*). Since $\beta_{n+1} : A_{n+1} \longrightarrow X_{n+1}$ and $A_{n+1} \in \mathcal{X}$ and $t_0 : E_0 \longrightarrow X_{n+1}$ is an $\mathcal{X}$-precover of X_{n+1}, there exists $r : A_{n+1} \longrightarrow E_0$ such that $\beta_{n+1} = t_0 r$.
Claim: Let $L^k = Ker(D(k) \to X(k))$ where $k \in \{0, 1, ..., n\}$ and then

$$L^0 : ... \longrightarrow E'_2 \xrightarrow{f_2} E'_1 \xrightarrow{f_1} Ker\phi_0 \longrightarrow 0 \longrightarrow ...,$$

$L^n : ... \longrightarrow D_{n+2} \xrightarrow{d_{n+2}} D_{n+1} \xrightarrow{d_{n+1}} Ker\phi_n \xrightarrow{d_n} Ker\phi_{n-1} \longrightarrow ... \longrightarrow Ker\phi_1 \longrightarrow Ker\phi_0 \longrightarrow 0 \longrightarrow ...$ and also $L^{n+1} : ... \longrightarrow D_{n+2} \oplus E_1 \xrightarrow{\lambda_{n+2}} D_{n+1} \oplus Kert_0 \xrightarrow{\lambda_{n+1}} Ker\phi_n \longrightarrow Ker\phi_{n-1} \longrightarrow ... \longrightarrow Ker\phi_0 \longrightarrow 0 \longrightarrow ...$ where $E_\bullet = ... \longrightarrow E_2 \xrightarrow{t_2} E_1 \xrightarrow{t_1} E_0 \longrightarrow 0$ (where E_0 is in the nth place) is an $C(\mathcal{X}^*)$-precover of $\underline{X_{n+1}}$. We claim that for every $S \in C(\mathcal{X}^*)$, $\mathrm{Hom}(S, L^k)$ is exact where $k \in \{0, 1, ..., n, n+1\}$. Since

$$L^0 : ... \longrightarrow E'_2 \xrightarrow{f_2} E'_1 \xrightarrow{f_1} Ker\phi_0 \longrightarrow 0$$

for every $\mathcal{X}^*$-complex S,

$... \longrightarrow \mathrm{Hom}(S, E'_2) \longrightarrow \mathrm{Hom}(S, E'_1) \longrightarrow \mathrm{Hom}(S, Ker\phi_0) \longrightarrow 0$ is exact. So, $\mathrm{Hom}(S, L^0)$

is exact. It is enough to prove that if Hom(S, L^n) is exact, then Hom(S, L^{n+1}) is exact. We have the following diagram between L^n and L^{n+1}:

$$\begin{tikzcd}
...D_{n+2} \arrow[r, "d_{n+2}"] \arrow[d, "(1,0)"] & D_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "(1,0)"] & Ker\phi_n \arrow[r, "d_n"] \arrow[d] & \cdots \\
...D_{n+2} \oplus E_1 \arrow[r, "\lambda_{n+2}"] & D_{n+1} \oplus Kert_0 \arrow[r, "\lambda_{n+1}"] & Ker\phi_n \arrow[r, "d_n"] & \cdots
\end{tikzcd}$$

Therefore we have a split exact sequence $0 \longrightarrow L^n \longrightarrow L^{n+1} \longrightarrow \frac{L^{n+1}}{L^n} \longrightarrow 0$ where $\frac{L^{n+1}}{L^n} \cong .... \longrightarrow E_2 \longrightarrow E_1 \longrightarrow Kert_0 \longrightarrow 0$ which is Hom$(\mathcal{X}^* - complex, -)$ exact since $... \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow X_{n+1} \longrightarrow 0$ is a minimal $\mathcal{X}$-resolvent. Since Hom(S, L^n) and Hom$(S, \frac{L^{n+1}}{L^n})$ are exact, so Hom(S, L^{n+1}) is exact.

Since Hom(A_{n+1}, L^n) is exact where $A_{n+1} \in \mathcal{X}$,

$$Hom(A_{n+1}, D_{n+1}) \xrightarrow{d^*_{n+1}} Hom(A_{n+1}, Ker\phi_n) \xrightarrow{d^*_n} Hom(A_{n+1}, Ker\phi_{n-1})$$

is exact, that is $Kerd^*_n = Imd^*_{n+1}$. Thus $\phi_n(\theta_n a_{n+1} - \nu_0 r) = \phi_n\theta_n a_{n+1} - \phi_n\nu_0 r = \beta_n a_{n+1} - \phi_n\lambda_{n+1}(0, r) = \ell_{n+1}\beta_{n+1} - \ell_{n+1}(0, t_0)(0, r) = \ell_{n+1}\beta_{n+1} - \ell_{n+1}\beta_{n+1} = 0$. So, $\theta_n a_{n+1} - \nu_0 r \in Ker\phi_n$ and also $d^*_n(\theta_n a_{n+1} - \nu_0 r) = d_n(\theta_n a_{n+1} - \nu_0 r) = d_n\theta_n a_{n+1} - d_n\nu_0 r = \theta_{n-1} a_n a_{n+1} - 0 = 0$.

Therefore $\theta_n a_{n+1} - \nu_0 r \in Imd^*_{n+1}$; that is; there exists $s \in$ Hom(A_{n+1}, D_{n+1}) such that

$$\theta_n a_{n+1} - \nu_0 r = d_{n+1} s ...(**)$$

Let

$$\theta_{n+1} : A_{n+1} \longrightarrow D_{n+1} \oplus E_0$$

$$a \longrightarrow (s(a), r(a))$$

Then $(0, t_0)\theta_{n+1}(a) = (0, t_0)(s(a), r(a))$ where $a \in A_{n+1}$

$$= t_0 r(a)$$

$$= \beta_{n+1}(a)$$

Therefore, $(0, t_0)\theta_{n+1} = \beta_{n+1}$. Moreover, $\lambda_{n+1}\theta_{n+1} = \lambda_{n+1}(s, r) = d_{n+1}s + \nu_0 r = \theta_n a_{n+1}$ (by (**) and diagram (1)).

Secondly, let's find θ_{n+2}. We know that $(0, t_0)\lambda_{n+2} = 0$. Since $(0, t_0)(\theta_{n+1}a_{n+2}) = 0$ and $t_1 : E_1 \longrightarrow Kert_0$ is an $\mathcal{X}$-precover, there exists $r_1 : A_{n+2} \longrightarrow E_1$ such that $t_1 r_1 = ra_{n+2}$. Since $\mathrm{Hom}(A_{n+2}, L^n)$ is exact,

$$Hom(A_{n+2}, D_{n+2}) \xrightarrow{d^*_{n+2}} Hom(A_{n+2}, D_{n+1}) \xrightarrow{d^*_{n+1}} Hom(A_{n+2}, Ker\phi_n)$$

is exact. Since $sa_{n+2} + \nu_1 r_1 \in \mathrm{Hom}(A_{n+2}, D_{n+1})$ and $d^*_{n+1}(sa_{n+2} + \nu_1 r_1) = d_{n+1}(sa_{n+2} + d_{n+1}\nu_1 r_1) = (\theta_n a_{n+1} - \lambda_{n+1}(0, r))a_{n+2} + d_{n+1}\nu_1 r_1$ by (**) $= -\lambda_{n+1}(0, ra_{n+2}) + d_{n+1}\nu_1 r_1 - \nu_0 ra_{n+2} + d_{n+1}\nu_1 r_1 - \nu_0 t_1 r_1 + d_{n+1}\nu_1 r_1 = 0$. So $sa_{n+2} + \nu_1 r_1 \in Im(d_{n+2})^*$. Thus there exists $s_1 : A_{n+2} \longrightarrow D_{n+2}$ such that $sa_{n+2} + \nu_1 r_1 = d_{n+2}s_1$. Let $\theta_{n+2} : A_{n+2} \longrightarrow D_{n+2} \oplus E_1; a \longrightarrow (s_1(a), -r_1(a))$. Then $\lambda_{n+2}\theta_{n+2} = \lambda_{n+2}(s_1, r_1) = (d_{n+2}s_1 - \nu_1 r_1, t_1 r_1) = (sa_{n+2}, ra_{n+2}) = \theta_{n+1}a_{n+2}$.

Let's find θ_{n+3}. Since $\mathrm{Hom}(A_{n+3}, L^{n+1})$ is exact,

$$Hom(A_{n+3}, D_{n+3} \oplus E_2) \overset{\lambda^*_{n+3}}{\to} Hom(A_{n+3}, D_{n+2} \oplus E_1) \overset{\lambda^*_{n+2}}{\to} Hom(A_{n+3}, D_{n+1} \oplus Kert_0)$$

is exact. Since $\theta_{n+2}a_{n+3} \in \mathrm{Hom}(A_{n+3}, D_{n+2} \oplus E_1)$ and $\lambda^*_{n+2}(\theta_{n+2}a_{n+3}) = \lambda_{n+2}(\theta_{n+2}a_{n+3}) = \theta_{n+1}(\theta_{n+2}a_{n+3}) = 0$, $\theta_{n+2}a_{n+3} \in Ker\lambda^*_{n+2} = Im\lambda^*_{n+3}$, therefore there exists

$$\theta_{n+3} \in Hom(A_{n+3}, D_{n+3} \oplus E_2)$$

Let's find θ_{n+k} for $k > 3$. Since $\mathrm{Hom}(A_{n+k}, L^{n+1})$ is exact,
$\mathrm{Hom}(A_{n+k}, D_{n+k} \oplus E_{k-1}) \overset{\lambda^*_{n+k}}{\longrightarrow} \mathrm{Hom}(A_{n+k}, D_{n+k-1} \oplus E_{k-2}) \overset{\lambda^*_{n+k-1}}{\longrightarrow} \mathrm{Hom}(A_{n+k},$
$D_{n+k-2} \oplus E_{k-3})$ is exact.
Since $\theta_{n+k-1}a_{n+k} \in \mathrm{Hom}(A_{n+k}, D_{n+k-1} \oplus E_{k-2})$ and $\lambda^*_{n+k-1}(\theta_{n+k-1}a_{n+k})$
$= \lambda_{n+k-1}\theta_{n+k-1}a_{n+k} = \theta_{n+k-2}a_{n+k-1}a_{n+k} = 0$, $\theta_{n+k-1}a_{n+k} \in Ker\lambda^*_{n+k-1}$
$= Im\lambda^*_{n+k}$, therefore there exists $\theta_{n+k} \in \mathrm{Hom}(A_{n+k}, D_{n+k} \oplus E_{k-1})$ such that $\lambda_{n+k}\theta_{n+k} = \theta_{n+k-1}a_{n+k}$. □

Lemma 3.3. *Let $\mathcal{X}$ be closed under finite sum. If every module has an $\mathcal{X}$-preenvelope, then every bounded complex has an $C(\mathcal{X}^*)$-preenvelope.*

Proof. Let $0 \longrightarrow X^0 \overset{\phi^0}{\longrightarrow} E^1_1 \overset{f^1}{\longrightarrow} E^2_1 \longrightarrow ...$ be a minimal $\mathcal{X}$-preenvelope resolvent of X^0. Let A be an $\mathcal{X}^*$-complex and $\beta \in$
$\mathrm{Hom}(X(0), A)$ where $X(0) : ... \longrightarrow 0 \longrightarrow X^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ....$ Then we have the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d, "\phi^0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & E_1^1 \arrow[r, "f^1"] \arrow[d, "\lambda^0"] & E_1^2 \arrow[r, "f^2"] \arrow[d, "\lambda^1"] & E_1^3 \arrow[r, "f^3"] \arrow[d, "\lambda^2"] & \cdots \\
\cdots \arrow[r] & A^{-1} \arrow[r, "a^{-1}"] & A^0 \arrow[r, "a^0"] & A^1 \arrow[r, "a^1"] & A^2 \arrow[r] & \cdots
\end{tikzcd}$$

where $\lambda^0\phi^0 = \beta^0$ and since $\frac{E_1^1}{Im\phi^0} \longrightarrow A^1$ with $x + Imf^0 \mapsto a^0\lambda^0(x)$, there exists λ^1 : $E_1^2 \longrightarrow A^1$ such that $\lambda^1 f^1 = a^0\lambda^0$. Similarly we can define $\lambda^n \in$Hom(E_1^{n+1}, A^n) such that $\lambda^n f^n = a^{n-1}\lambda^{n-1}$.

Let $X(n)$: $...0 \longrightarrow X^0 \xrightarrow{\ell^0} X^1 \xrightarrow{\ell^1} X^2 \xrightarrow{\ell^2} ... \longrightarrow X^n \longrightarrow 0 \longrightarrow ....$ We will use an induction on n to prove that $X(n)$ has an $C(\mathcal{X}^*)$-preenvelope. At the beginning we proved that $X(0)$ has an $C(\mathcal{X}^*)$-preenvelope. Suppose that $D(n)$: $... \longrightarrow 0 \longrightarrow D^0 \xrightarrow{d^0} D^1 \xrightarrow{d^1} .... \longrightarrow D^n \xrightarrow{d^n} D^{n+1} \longrightarrow ...$ be an $C(\mathcal{X}^*)$-preenvelope of $X(n)$ such that

$$\begin{tikzcd}
X(n) : ...0 \arrow[r] & X^0 \arrow[r, "\ell^0"] \arrow[d, "\phi^0"] & X^1... \arrow[r] \arrow[d, "\phi^1"] & X^n \arrow[r, "\ell^n"] \arrow[d, "\phi^n"] & 0... \\
D(n) : ...0 \arrow[r] & D^0 \arrow[r, "d^0"] & D^1... \arrow[r] & D^n \arrow[r, "d^n"] & D^{n+1}...
\end{tikzcd}$$

Let $E^\bullet = ... \longrightarrow 0 \longrightarrow E^1 \xrightarrow{t^1} E^2 \xrightarrow{t^2} E^3 \xrightarrow{t^3} ...$ be an $C(\mathcal{X}^*)$-preenvelope of $\underline{X^{n+1}}$

(where X^{n+1} and E^1 is in nth place) where $t^0 : X^{n+1} \longrightarrow E^1$ is an $\mathcal{X}$-preenvelope of X^{n+1}. Thus we have the morphism ν making the following commutative diagram:

$$\begin{tikzcd} X(n) \arrow[r] \arrow[d] & \underline{X^{n+1}} \arrow[d] \\ D(n) \arrow[r, dotted, "\nu"] & E^\bullet \end{tikzcd}$$

where $t^0\ell^n = \nu^0\phi^n$...(*). Since $X(n) \longrightarrow D(n)$ is an $C(\mathcal{X}^*)$-preenvelope, we have the diagram

$$\begin{tikzcd} D(n) = \ldots \arrow[r] & D^{n-1} \arrow[r, "d^{n-1}"] \arrow[d] & D^n \arrow[r, "d^n"] \arrow[d, "\nu^0"] & D^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "\nu^1"] & \cdots \\ E^\bullet = \ldots \arrow[r] & 0 \arrow[r] & E^1 \arrow[r, "t^1"] & E^2 \arrow[r, "t^2"] & \cdots \end{tikzcd}$$

...(1)

Let C be the mapping cone of $\nu : D(n) \longrightarrow E^\bullet$. So $C = \ldots \longrightarrow 0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \ldots \longrightarrow D^n \xrightarrow{\lambda^n} D^{n+1} \oplus E^1 \xrightarrow{\lambda^{n+1}} D^{n+2} \oplus E^2 \xrightarrow{\lambda_{n+2}} \ldots$ where

$$\lambda^n(x) = (d^n(x), \nu^0(x))$$

and for $k \geq 1$

$$\lambda^{n+k}(x, y) = (-d^{n+k}(x), \nu^k(x) + t^k(y)).$$

We show that $X(n+1) \to C$ is an $C(\mathcal{X}^*)$-preenvelope of $X(n+1)$.

Claim: Let $L^k = coker(X(k) \to D(k))$ where $k \in \{0, 1, ..., n\}$ and then

$$L^0 : ... \to coker(\phi^0) \to E_1^1 \xrightarrow{f^1} E_1^2 \to ...,$$

$$L^{n-1} : ... \longrightarrow coker\phi^{n-1} \xrightarrow{d_1^{n-1}} D^n \xrightarrow{d^n} D^{n+1} \longrightarrow ...$$

$$L^n : ... \longrightarrow coker\phi^n \xrightarrow{d_1^n} D^{n+1} \xrightarrow{d^{n+1}} D^{n+2} \longrightarrow ...$$

$$L^{n+1} : ... \to coker\phi^n \to D^{n+1} \oplus cokert^0 \xrightarrow{\lambda_1^{n+1}} D^{n+2} \oplus E^2 \xrightarrow{\lambda^{n+2}} D^{n+3} \oplus E^3 ...$$

where $\lambda_1^{n+1}(x, y) = \lambda^{n+1}(x, a)$ and $y = a + Imt^0$ for $a \in E^1$. We claim that for every $S \in C(\mathcal{X}^*)$, $\mathrm{Hom}(L^k, S)$ is exact where $k \in \{0, 1, ..., n, n+1\}$. It is understood easily that $\mathrm{Hom}(L^0, S)$ is exact. It is enough to prove that if $\mathrm{Hom}(L^n, S)$ is exact, then $\mathrm{Hom}(L^{n+1}, S)$ is exact. We have the following diagram between L^n and L^{n+1} :

$$\begin{tikzcd}
...coker\phi^n \arrow[r, "d_1^n"] \arrow[d] & D^{n+1} \arrow[r, "d^{n+1}"] \arrow[d] & D^{n+2} \arrow[r] \arrow[d] & D^{n+3}... \arrow[d] \\
...coker\phi^n \arrow[r] & D^{n+1} \oplus cokert^0 \arrow[r, "\lambda_1^{n+1}"] & D^{n+2} \oplus E^2 \arrow[r, "\lambda^{n+2}"] & D^{n+3} \oplus E^3...
\end{tikzcd}$$

And so, $\frac{L^{n+1}}{L^n} : ... \to 0 \to 0 \to ... \to 0 \to cokert^0 \to E^2 \to ...$ is a resolution of $cokert^0$, and hence $\mathrm{Hom}(\frac{L^{n+1}}{L^n}, S)$ is exact where S is an $\mathcal{X}^*$-complex. Since $0 \to L^n \to L^{n+1} \to \frac{L^{n+1}}{L^n} \to 0$ is split exact, we have an exact complex $0 \to \mathrm{Hom}(\frac{L^{n+1}}{L^n}, S) \to \mathrm{Hom}(L^{n+1}, S) \to \mathrm{Hom}(L^n, S) \to 0$. Therefore $\mathrm{Hom}(\frac{L^{n+1}}{L^n}, S)$ and $\mathrm{Hom}(L^n, S)$ are exact, so $\mathrm{Hom}(L^{n+1}, S)$

is exact.

Let $A \in C(\mathcal{X}^*)$. Then we have the diagram as follow:

$$\begin{tikzcd}
...X^0 \arrow[r, "\ell^0"] \arrow[d, "\phi^0"] & ...X^n \arrow[r, "\ell^n"] \arrow[d, "\phi^n"] & X^{n+1} \arrow[r, "\ell^{n+1}"] \arrow[d, "(0,t^0)"] & 0... \arrow[d] \\
...D^0 \arrow[r, "d^0"] \arrow[d, "\theta^0"] & ...D^n \arrow[r, "\lambda^n"] \arrow[d, "\theta^n"] & D^{n+1} \oplus E^1 \arrow[r, "\lambda^{n+1}"] \arrow[d, "\theta^{n+1}"] & D^{n+2} \oplus E^2... \arrow[d, "\theta^{n+2}"] \\
...A^0 \arrow[r, "a^0"] & ...A^n \arrow[r, "a^n"] & A^{n+1} \arrow[r, "a^{n+1}"] & A^{n+2}...
\end{tikzcd}$$

where the $\beta^i : X^i \to A^i$ are maps such that $\theta^0\phi^0 = \beta^0, ..., \theta^n\phi^n = \beta^n$.

First, we will find θ^{n+1} such that $\theta^{n+1}\lambda^n = a^n\theta^n$ and $\theta^{n+1}(0, t^0) = \beta^{n+1}$. Moreover $\lambda^n\phi^n = (0, t^0)\ell^n$ by (*). Since $\beta^{n+1} : X^{n+1} \longrightarrow A^{n+1}$ and $A^{n+1} \in \mathcal{X}$ and $t^0 : X^{n+1} \longrightarrow E^1$ is an $\mathcal{X}$-preenvelope, there exists $r : E^1 \longrightarrow A^{n+1}$ such that $\beta^{n+1} = rt^0$.

Since Hom(L^{n-1}, A^{n+1}) is exact where $A^{n+1} \in \mathcal{X}$,

$$... \to Hom(D^{n+1}, A^{n+1}) \overset{(d^{n+1})^*}{\to} Hom(D^n, A^{n+1}) \overset{(d_1{}^{n-1})^*}{\to} Hom(coker\phi^{n-1}, A^{n+1}) \to ...$$

is exact, that is, $Ker(d_1{}^{n-1})^* = Im(d^n)^*$.

Claim: $a^n\theta^n - rv^0 \in Ker(d_1{}^{n-1})^*$.

Since $(d_1{}^{n-1})^*(a^n\theta^n - rv_0) = (a^n\theta^n - rv^0)d_1{}^{n-1} = a^n\theta^n d_1{}^{n-1} - rv^0 d_1{}^{n-1} = a^n\theta^n d_1{}^{n-1}$ by diagram (1). Since $\theta^n d_1^{n-1}(x + Im\phi^{n-1}) = a^{n-1}\theta^{n-1}(x)$ where $x \in D^{n-1}$, $a^n\theta^n d_1{}^{n-1} = 0$. So $a^n\theta^n - rv^0 \in Ker(d_1{}^{n-1})^* = Im(d^n)^*$ and hence there exists s $\in$Hom(D^{n+1}, A^{n+1}) such that $a^n\theta^n - rv^0 = sd^n$ ...(**)

Let $\theta^{n+1} : D^{n+1} \oplus E^1 \to A^{n+1}$ with $(a,b) \mapsto s(a) + r(b)$. Then $\theta^{n+1}(0,t^0)(x) = \theta^{n+1}(0,t^0(x)) = rt^0(x) = \beta^{n+1}(x)$ where $x \in X^{n+1}$. Therefore $\theta^{n+1}(0,t^0) = \beta^{n+1}$.

Moreover, $\theta^{n+1}\lambda^n(x) = \theta^{n+1}(\lambda^n(x)) = \theta^{n+1}(d^n(x), v^0(x)) = a^n\theta^n(x) - rv^0(x) + rv^0(x)$ by (**)

$$= a^n\theta^n(x).$$

Therefore $\theta^{n+1}\lambda^n = a^n\theta^n$.

Secondly, let's find θ^{n+2}. We know that $\lambda^{n+1}(0,t^0) = 0$. Since $\alpha : cokert^0 \longrightarrow A^{n+2}$ with $x + Imt^0 \mapsto -a^{n+1}r(x)$ is a map and $t_1^1 : cokert^0 \longrightarrow E^2$ is an $\mathcal{X}$-preenvelope where $t_1^1\pi = t^1$, there exists $r^1 : E^2 \longrightarrow A^{n+2}$ such that $r^1t^1 = \alpha$. Since $\mathrm{Hom}(L^n, A^{n+2})$ is exact,

$$Hom(D^{n+2}, A^{n+2}) \overset{d^{n+1*}}{\to} Hom(D^{n+1}, A^{n+2}) \overset{d^{n*}}{\to} Hom(D^n, A^{n+2})$$

is exact, and so $Im(d^{n+1})^* = Ker(d^n)^*$.

Claim: $-a^{n+1}s + r^1v^1 \in Ker(d^n)^*$.

$(d^n)^*(-a^{n+1}s + r^1v^1) = (-a^{n+1}s + r^1v^1)d^n = -a^{n+1}sd^n + r^1v^1d^n = -a^{n+1}(a^n\theta^n - rv^0) + r^1v^1d^n$ by(**)

$= a^{n+1}rv^0 + r^1v^1d^n = a^{n+1}rv^0 + r^1t^1v^0$ by diagram (1)

$= a^{n+1}rv^0 + r^1t_1^1\pi v^0 = a^{n+1}rv^0 + \alpha\pi v^0 = a^{n+1}rv^0 - a^{n+1}rv^0 = 0$. Therefore $-a^{n+1}s + r^1v^1 \in Im(d^{n+1})^*$. So there exists $s^1 \in\mathrm{Hom}(D^{n+2}, A^{n+2})$ such that $-a^{n+1}s + r^1v^1 = s^1d^{n+1}$.

Let $\theta^{n+2} : D^{n+2} \oplus E^2 \longrightarrow A^{n+2}$ with $(x,y) \mapsto s^1(x) + r^1(y)$

Then $\theta^{n+2}\lambda^{n+1}(x,y) = \theta^{n+2}(-d^{n+1}(x), v^1(x) + t^1(y)) = s^1(-d^{n+1}(x)) + r^1(v^1(x) + t^1(y)) = -s^1d^{n+1}(x) + r^1v^1(x) + r^1t^1(y) = a^{n+1}s(x) - r^1v^1(x) + r^1v^1(x) + r^1t^1(y) = a^{n+1}s(x) + r^1t^1(y) = a^{n+1}\theta^{n+1}(x,y)$.

Let's find θ^{n+3}. Since $\mathrm{Hom}(L^{n+1}, A^{n+3})$ is exact, $\mathrm{Hom}(D^{n+3} \oplus E^3, A^{n+3})$ $\overset{(\lambda^{n+2})^*}{\to} \mathrm{Hom}(D^{n+2} \oplus E^2, A^{n+3}) \overset{(\lambda_1^{n+1})^*}{\to} \mathrm{Hom}(D^{n+1} \oplus coker(t^0), A^{n+3})$ is exact. Since $a^{n+2}\theta^{n+2} \in$

$\text{Hom}(D^{n+2} \oplus E^2, A^{n+3})$ and

$$(\lambda_1^{n+1})^*(a^{n+2}\theta^{n+2})(x, y) = a^{n+2}\theta^{n+2}\lambda^{n+1}(x, a) = a^{n+2}a^{n+1}\theta^{n+1}(x, a) = 0$$

where $y = a + Im(t^0)$ and $a \in E^1$, we have that

$$a^{n+2}\theta^{n+2} \in Im(\lambda^{n+2})^*$$

Therefore there exists $\theta^{n+3} \in Hom(D^{n+3} \oplus E^3, A^{n+3})$ such that

$$a^{n+2}\theta^{n+2} = \theta^{n+3}\lambda^{n+2}.$$

Let's find θ^{n+k} for $k > 3$. Since $\text{Hom}(L^{n+1}, A^{n+k})$ is exact, $\text{Hom}(D^{n+k} \oplus E^k, A^{n+k}) \overset{(\lambda^{n+k-1})^*}{\rightarrow}$ $\text{Hom}(D^{n+k-1} \oplus E^{k-1}, A^{n+k}) \overset{(\lambda^{n+k-2})^*}{\rightarrow} \text{Hom}(D^{n+k-2} \oplus E^{k-2}, A^{n+k})$ is exact.
Since $a^{n+k-1}\theta^{n+k-1} \in \text{Hom}(D^{n+k-1} \oplus E^{k-1}, A^{n+k})$ and

$$(\lambda^{n+k-2})^*(a^{n+k-1}\theta^{n+k-1}) = a^{n+k-1}\theta^{n+k-1}\lambda^{n+k-2} = a^{n+k-1}a^{n+k-2}\theta^{n+k-2}$$

$= 0$, $a^{n+k-1}\theta^{n+k-1} \in Im(\lambda^{n+k-1})^*$,and so there exists $\theta^{n+k} \in \text{Hom}(D^{n+k} \oplus E^k, A^{n+k})$ such that $a^{n+k-1}\theta^{n+k-1} = \theta^{n+k}\lambda^{n+k-1}$. □

Again using the tecnique of Alina Iacob we can give the following result.

Theorem 3.2. *Let $\mathcal{X}$ be a class of modules with closed under direct sum. If every module has an $\mathcal{X}$-precover, then every right bounded complex has a $C(\mathcal{X}^*)$-precover.*

Proof. Let $X(n) : ... \to 0 \to X_n \to X_{n-1} \to ... \to X_1 \to X_0 \to 0$ and $X = \varinjlim X(n)$. (Then $X = ... \to X_2 \to X_1 \to X_0 \to 0$).
By Lemma 3.2, we know that $X(n)$ has a $C(\mathcal{X}^*)$-precover such that $0 \to L^n \to D(n) \to X(n)$ where L^n is the kernel of $(D(n) \to X(n))$ and $\text{Hom}(S, L^n)$ is exact for any $\mathcal{X}^*$-

complex S.

Let $L = \varinjlim L^n$ and so $L_k = L^k_k$ for any $k \geq 0$.

Using the proof of Lemma 3.2 we can say that $\mathrm{Hom}(S, L_{k+1}) \to \mathrm{Hom}(S, L_k) \to \mathrm{Hom}(S, L_{k-1})$ is exact, that is, $\mathrm{Hom}(S, L)$ is exact. We show that $\varinjlim D(n) \to X$ is a $C(\mathcal{X}^*)$-precover.

Let $D = \varinjlim D(n)$ and $\beta \in \mathrm{Hom}(A, X)$ where A is an $\mathcal{X}^*$-complex. Again using the proof of Lemma 3.2 we can find $\phi \in \mathrm{Hom}(D, X)$. Then $L^0 = Ker\phi_0$, $L^1 = Ker\phi_1$, $L^2 = Ker\phi_2$, ...

We have the diagram;

$$\begin{tikzcd}
\cdots \arrow[r] & A_2 \arrow[r, "a_2"] \arrow[d] & A_1 \arrow[r, "a_1"] \arrow[d] & A_0 \arrow[r, "a_0"] \arrow[d] & A_{-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & D_2 \arrow[r, "d_2"] \arrow[d, "\phi_2"] & D_1 \arrow[r, "d_1"] \arrow[d, "\phi_1"] & D_0 \arrow[r] \arrow[d, "\phi_0"] & 0 & \\
\cdots \arrow[r] & X_2 \arrow[r, "\ell_2"] & X_1 \arrow[r, "\ell_1"] & X_0 \arrow[r] & 0 &
\end{tikzcd}$$

where $\phi_0\alpha_0 = \beta_0$, $\phi_1 r_1 = \beta_1$, ..., $\phi_i r_i = \beta_i$, ...

Then $\phi_0(\alpha_0 a_1 - d_1 r_1) = \phi_0\alpha_0 a_1 - \phi_0 d_1 r_1) = \beta_0 a_1 - \ell_1\phi_1 r_1 = \beta_0 a_1 - \ell_1\beta_1 = 0$.

Thus $\alpha_0 a_1 - d_1 r_1 \in \mathrm{Hom}(A_1, Ker\phi_0 = L_0)$. Since $\mathrm{Hom}(A_1, L)$ is exact, there is $s_1 \in \mathrm{Hom}(A_1, L_1)$ such that $\alpha_0 a_1 - d_1 r_1 = d_1 s_1$. Let $\alpha_1 = s_1 + r_1$. Then $\phi_1\alpha_1 = \phi_1(s_1 + r_1) = \phi_1 s_1 + \phi_1 r_1 = 0$

If we continue in this way, then we can find $\alpha_i \in \mathrm{Hom}(A_i, D_i)$ such that $d_i\alpha_i = \alpha_{i-1}a_i$. Thus $D \to X$ is a $C(\mathcal{X}^*)$-precover. $\square$

Using the modification to complexes of Lemma 1.5 we can give the following result.

Corollary 3.1. *Let $\mathcal{X}$ be a class with closed under direct sum and $C(\mathcal{X}^*)$ be closed under direct limit. If every module has an $\mathcal{X}$-precover, then every bounded above complex has a $C(\mathcal{X}^*)$-cover.*

Theorem 3.3. *Let $\mathcal{X}$ be closed under direct sum and extension closed. Let $C(\mathcal{X}^*)$ be closed under inverse and direct limit. If every module has an $\mathcal{X}$-cover, then every complex has a $C(\mathcal{X}^*)$-cover.*

Proof. First we will show that every complex has $C(\mathcal{X}^*)$-precover if $\mathcal{X}$ is a class of modules with extension closed and closed under direct sum and $C(\mathcal{X}^*)$ is closed under inverse limit. And then modifying Lemma 1.5 we can find a $C(\mathcal{X}^*)$-cover if $C(\mathcal{X}^*)$ is closed under direct limit.

Let $X : ... \to X_2 \to X_1 \to X_0 \to X_{-1} \to ...$ and let $Y^n : ... \to X_1 \to X_0 \to X_{-1} \to ... \to X_{-n} \to 0 \to ....$ So $\varprojlim Y^n = X$.

We see that Y^n has a $C(\mathcal{X}^*)$-precover $D^n : ... \to D_1 \to D_0 \to D_{-1} \to ... \to D_{-n} \to 0 \to ...$ such that

$$D_{-n} = E^0_{-n},$$

$$D_{-n+1} = E^1_{-n} \oplus E^0_{-n+1},$$

$$D_{-n+2} = E^2_{-n} \oplus E^1_{-n+1} \oplus E^0_{-n+2},$$

...

where $... \to E^2_k \to E^1_k \to E^0_k \xrightarrow{\phi_k} X_k \to 0$ is a minimal $\mathcal{X}$-precover resolvent of X_k for $k \geq -n$.

Then $Y^{n+1} = ... \to X_0 \to X_1 \to ... \to X_{-n} \to X_{-n-1} \to 0 \to ...$ and by the proof of Theorem 3.2 the $C(\mathcal{X}^*)$-precover of Y^{n+1}, $D^{n+1} = ... \to E^1_{-n-1} \oplus D_{-n} \to E^0_{-n-1} \to 0 \to ....$

Let $T^n = Ker(\phi : D^n \to Y^n)$ and then again by the proof of Theorem 3.2, $T^{n+1} = ...E^2_{-n-1} \oplus E^1_{-n} \oplus Ker\phi_{-n+1} \to E^1_{-n-1} \oplus Ker\phi_{-n} \to Ker\phi_{-n-1} \to 0,$

$T^n = ... \to E^1_{-n} \oplus Ker\phi_{-n+1} \to Ker\phi_{-n} \to 0.$

Then $Ker(T^{n+1} \to T^n) = ... \to E^2_{-n-1} \to E^1_{-n-1} \to Ker\phi_{-n-1} \to 0$ is the kernel of a $C(\mathcal{X}^*)$-cover of $\underline{Y_{-n-1}}$. By Wakamatsu's Lemma Hom$(S, T^{n+1}) \to$Hom$(S, T^n) \to 0$ is epic for any $\mathcal{X}^*$-complex S. Since we have the exact sequence $0 \to T^n \to D^n \to Y^n$ and D^n is a $C(\mathcal{X}^*)$-cover of Y^n, we have the exact sequence $0 \to$Hom$(S, T^n) \to$Hom(S, D^n) $\to$Hom$(S, Y^n) \to 0$ for any $\mathcal{X}^*$-complex S.

By the modification of Proposition 1.2 and Theorem 1.4 we have the following exact sequence;

$$0 \to Hom(S, \varprojlim T^n) \to Hom(S, \varprojlim D^n) \to Hom(S, \varprojlim Y^n) \to 0$$

So, $\varprojlim D^n$ is a $C(\mathcal{X}^*)$-precover of X. □

Dually, we can say the following theorems without proof;

Theorem 3.4. *Let $\mathcal{X}$ be closed under direct sum. If every module has an $\mathcal{X}$-preenvelope, then every right bounded complex has a $C(\mathcal{X}^*)$-preenvelope.*

Using the modification to complexes of Lemma 1.6 we can give the following result.

Corollary 3.2. *Let $\mathcal{X}$ be a class with closed under direct sum and summands and $C(\mathcal{X}^*)$ be closed under inverse limit. If every module has an $\mathcal{X}$-preenvelope, then every bounded complex has a $C(\mathcal{X}^*)$-envelope.*

Theorem 3.5. *Let $\mathcal{X}$ be closed under direct sum and summands and extension closed and $C(\mathcal{X}^*)$ be closed under inverse and direct limit. If every module has an $\mathcal{X}$-envelope, then every complex has a $C(\mathcal{X}^*)$-envelope.*

REFERENCES

[1] AVRAMOV, L., FOXBY, H.B. *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra 71 :129-155, (1991).

[2] ENOCHS, E.E., BRAVO, D., JENDA, IACOB, A., O.M.G. and RADA, J. *Cotorsion pairs in C(R-Mod)*, submitted.

[3] ENOCHS, E.E., JENDA, O.M.G. *Relative Homological Algebra*, de Gruyter Ex. Math. Volume 30, Walter de Gruyter and Co., Berlin (2000).

[4] ENOCHS, E.E., JENDA, O.M.G. and XU, J. *Orthogonality in the category of complexes* Math. J. Okayama Univ., 38:25-46, (1996).

[5] ENOCHS, E.E., ROZAS, J.R.G. *Exact Envelopes of Complexes*, Communication in Algebra 27 (4), 1615-1627, (1999).

[6] IACOB, A. *DG-injective covers, #-injective covers*, Communications in Algebra, accepted.

[7] IACOB, A. *Gorenstein Flat Dimension of Complexes*, Journal of Mathematics of Kyoto University, 49(4):817-842, (2009).

[8] MAO, L., DING, N. *L-injective Hulls of modules*, Bull. Austral. Math. Soc., 74:37-44, (2006).

[9] ROTMAN, J.J. *An Introduction to Homological Algebra*, Springer, New York (2009).