DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

FOURIER SERIES EXPANSION METHOD FOR SOLVING SOME PROBLEMS OF ELECTROMAGNETICS

by
Hülya SERT

March, 2010
İZMİR

FOURIER SERIES EXPANSION METHOD FOR SOLVING SOME PROBLEMS OF ELECTROMAGNETICS

A Thesis Thesis Submitted to the
Graduate School of Natural and Applied Sciences of
Dokuz Eylül University
In Partial Fulfillment of the Requirements for
the Degree of Master of Science in Mathematics

by
HÜLYA SERT

March, 2010
İZMİR

M. Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis, entitled **"FOURIER SERIES EXPANSION METHOD FOR SOLVING SOME PROBLEMS OF ELECTROMAGNETICS"** completed by **HÜLYA SERT** under supervision of **PROF. DR. VALERY YAKHNO** and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

....

Supervisor

....

Jury Member

....

Jury Member

Prof. Dr. Mustafa Sabuncu
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my supervisor Prof. Dr. Valery G. YAKHNO. This research project would not have been successful without his continual assistance, invaluable guidance and endless patience throughout the course of this research project. My deepest gratitude are also due to the members of the supervisory committee, Specialist Barış ÇİÇEK and Assistant Şengül KECELLİ without whose knowledge and assistance this project would not have been successful.

I also would like to say my thanks to my husband and to my parents for their understanding and endless love, through the duration of my study.

Hülya SERT

FOURIER SERIES EXPANSION METHOD
FOR SOLVING SOME PROBLEMS OF ELECTROMAGNETICS

ABSTRACT

The explicit formulae for solutions of boundary value problems for the inhomogeneous Poisson, Helmholtz and Maxwell's equations are obtained in this thesis.

The formulae for solutions of the considered problem have been obtained by the Fourier series expansion method. These formulae have been obtained in the form of the Fourier series.

Theory of partial differential equations and ordinary differential equations, theory of generalized functions are actively used in this thesis.

Keywords: Fourier series expansion method, Poisson equation, Helmholtz equation, Maxwell's equation system, eigenvalues and eigenfunctions of the ordinary differential operator.

BAZI ELEKTROMAGNETİK PROBLEMLERİ İÇİN FOURİER SERİSİNE AÇILIM YÖNTEMİ

ÖZ

Bu tezde homojen olmayan Poisson, Helmholtz ve Maxwell denklemlerinin sınır değer problemlerinin çözümleri elde edildi.

Fourier serilerine açılım yöntemi kullanılarak bu problemlerin çözümleri oluşturuldu. Bu çözümler Fourier serileri formundadır.

Kısmi diferansiyel denklemler ve normal diferansiyel denklemler teorileri, genelleştirilmiş fonksiyon teorisi tezde aktif olarak kullanıldı.

Anahtar sözcükler:Fourier serilerine açılım yöntemi, Helmholtz denklemi, Maxwell denklem sistemi,simülasyon, Cramer kuralı, normal diferansiyel denklemlerin özdeğerleri ve özfonksiyonları.

CONTENTS

CHAPTER ONE
INTRODUCTION

Many problems of physics and engineering are stated in terms of the boundary value problems for partial differential equations (Courant and Hilbert (1962), Eom (2003), Kong (1986), Monk (2003), Ramo S., Whinnery J. R., Duzer T. (1994), Tai (1994), Vagner D., Lemrikov B.I. and Wyder P. (2004), Vladimirov (1971)).

One of the methods for solving these problem is the Fourier series expansion method (Courant and Hilbert (1962), Pinsky (1991), Tikhonow and Samarskii (1963), Vladimirov (1971)). This method has the following steps. The first step is solving eigenvalue and eigenfunction problem for an differential operator. The set of eigenfunctions is orthogonal and complete. A solution of the original boundary value problem find in the form of the Fourier series expansion according to eigenfunctions found on the first step of solving. The coefficients of this Fourier series are unknown. These coefficients are determined using the given boundary conditions.

One of the goals of the thesis is to solve the inhomogeneous Poisson equation, Helmholtz equations with Dirichlet, Neumann and mixed boundary conditions by means of the Fourier series expansion method.

The second goal of the thesis is to solve Maxwell's equations with perfect conductor conditions on the boundary of a given rectangle by the reduction of the considered problem to the mixed boundary value problems for the inhomogeneous Poisson and Helmholtz equations. The solutions of the reduced problems are derived in the form of the Fourier series expansion.

The thesis is organized as follows:
The Chapter 2 is devoted to eigenvalue and eigenfunction problems for the ordinary differential equation with the Dirichlet, Neumann and mixed boundary conditions.

In the Chapter 3 the Poisson equation with the Dirichlet and Neumann conditions is considered. These boundary value problems for the Poisson equation are solved by the Fourier series expansion method. The explicit formulae for the solutions are main results of this chapter.

The inhomogeneous Helmholtz equation with Dirichlet, Neumann and mixed boundary conditions is considered in the Chapter 4. The main object of this chapter is boundary value problems for the inhomogeneous Helmholtz equation. The explicit formulae for solutions of these problems are obtained in this chapter. The Fourier series expansion method is used for solving these boundary value problems.

The frequency-dependent Maxwell's equations with perfect conductor boundary conditions in a rectangular domain are considered in the Chapter 5. This boundary value problem for Maxwell's equations is reduced to several boundary value problems for the inhomogeneous Helmholtz equation. The obtained boundary value problems are solved by the Fourier series expansion method. The solutions of the problems are presented in the form of the Fourier series.

CHAPTER TWO
STURM-LIOUVILLE PROBLEMS

2.1 Sturm-Liouville Problem with Dirichlet Boundary Condition

Consider the following Sturm-Liouville Promlems (SLP)

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < b, \tag{2.1.1}$$

$$y(0) = 0, \quad y(b) = 0, \tag{2.1.2}$$

where b is a given positive number, λ is a parameter.
Definition: A number λ for which there exists a nonzero function $y(x)$ satisfying (2.1.1) - (2.1.2) is called an eigenvalue and this nonzero function is called an eigenfunction corresponding to λ.

The main problem of this section is to find all eigenvalues and corresponding to them eigenfunctions.

We have three cases for the parameter λ :

$$Case1 : \lambda < 0; Case2 : \lambda = 0; Case3 : \lambda > 0.$$

1) Case 1: For $\lambda < 0$, we have a general solution for (2.1.1) as in the form:

$$y(x) = c_1 e^{\sqrt{|\lambda|}x} + c_2 e^{-\sqrt{|\lambda|}x}. \tag{2.1.3}$$

Substituting (2.1.3) into (2.1.2), we get

$$y(0) = c_1 + c_2 = 0,$$

$$y(b) = c_1 e^{\sqrt{|\lambda|}b} + c_2 e^{-\sqrt{|\lambda|}b} = 0.$$

This system is a linear algebraic system according to c_1, c_2.
The determinant of this matrix is different from zero:

$$detM = e^{-\sqrt{|\lambda|}b} - e^{\sqrt{|\lambda|}b},$$

$$= 2Sinh\sqrt{|\lambda|}b \neq 0.$$

Using Cramer's rule we find $c_1 = c_2 = 0$. This system has trivial solution, $c_1 = c_2 = 0$, so there is no eigenvalues and eigenfunctions in the region $\lambda < 0$.

2) Case 2: For $\lambda = 0$, we have a general solution for (2.1.1) as in the form:

$$y(x) = c_1 x + c_2. \tag{2.1.4}$$

Substituting (2.1.4) into (2.1.2), we get

$$y(0) = c_1 0 + c_2 = 0, \Rightarrow\ c_2 = 0,$$

using the last equation, we have

$$y(b) = c_1 b = 0\ \Rightarrow\ c_1 = 0,$$

since $b \neq 0$. This implies that for $\lambda = 0$ we have zero solution, so $\lambda = 0$ is not an eigenvalue and eigenfunctions.

3) Case 3: For $\lambda > 0$, we have a general solution for (2.1.1) in the form:

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x). \tag{2.1.5}$$

Substituting (2.1.5) into (2.1.2), we get

$$y(0) = c_1 \cos(\sqrt{\lambda}0) + c_2 \sin(\sqrt{\lambda}0)\ \Rightarrow\ c_1 = 0,$$

using the last equation, we have

$$y(b) = c_2 \sin(\sqrt{\lambda}b).$$

For finding a solution we may choose $c_2 \neq 0$ and implies that $\sin(\sqrt{\lambda}b) = 0$. We find that $\sqrt{\lambda}b = k\pi$ for k=1,2,3,... that is

$$\lambda_k = (\frac{k\pi}{b})^2, \quad for \quad k = 1, 2, 3, ...$$

So we have eigenvalues λ_k and corresponding eigenfunctions $y_k(x) = c_2 \sin(\sqrt{\lambda_k}x)$ for k=1,2,3,... We can normalize this orthogonal functions by choosing $c_2 = \sqrt{\frac{2}{b}}$.

We note that

$$\begin{aligned}|y_k|^2 = \int_0^b \sin^2(\sqrt{\lambda_k}x)dx &= \int_0^b \frac{1-\cos(2\sqrt{\lambda_k}x)}{2}dx \\ &= \frac{1}{2}\int_0^b dx - \frac{1}{2}\int_0^b \cos(2\sqrt{\lambda_k}x)dx \\ &= \frac{1}{2}\int_0^b dx - \frac{1}{2}\int_0^b \cos(2\sqrt{\lambda_k}x)dx \\ &= \frac{b}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^b {c_2}^2 \sin^2(\sqrt{\lambda_k}x)dx &= c_2^2 \int_0^b \sin^2(\sqrt{\lambda_k}x)dx \\ &\Rightarrow c_2^2\frac{b}{2} \\ &= 1\end{aligned}$$

$\Rightarrow c_2 = \sqrt{\frac{2}{b}}$.
if $c_2 = \sqrt{\frac{2}{b}}$, then

$$y_k(x_2) = \sqrt{\frac{2}{b}}\sin(\sqrt{\lambda_k}x_2). \tag{2.1.6}$$

4) Conclusion:The eigenvalue-eigenfunction problem (2.1.1), (2.1.2) has the following solution:

$$\lambda_k = (\frac{k\pi}{b})^2, \quad y_k(x) = \sqrt{\frac{2}{b}}\sin(\sqrt{\lambda_k}x) \quad k = 1, 2, 3, ... \tag{2.1.7}$$

where λ_k are eigenvalues and $y_k(x)$ are eigenfunctions corresponding to these eigenvalues.

2.2 Sturm-Liouville Problem with Neumann Boundary Condition

Consider the following SLP

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < b, \tag{2.2.1}$$

$$y'(0) = 0, \quad y'(b) = 0, \tag{2.2.2}$$

where b is a given positive number, λ is a parameter.
We have three cases for the parameter λ.

$$Case1 : \lambda < 0; Case2 : \lambda = 0; Case3 : \lambda > 0.$$

1) Case 1: For $\lambda < 0$, we have a general solution for (2.2.1) as in the form:

$$y(x) = c_1 e^{\sqrt{|\lambda|}x} + c_2 e^{-\sqrt{|\lambda|}x}, \tag{2.2.3}$$

$$y'(x) = c_1\sqrt{|\lambda|}e^{\sqrt{|\lambda|}x} + c_2(-\sqrt{|\lambda|})e^{-\sqrt{|\lambda|}x}. \tag{2.2.4}$$

Substituting (2.2.4) into (2.2.2), we get:

$$y'(0) = c_1 - c_2 = 0,$$

$$y'(b) = c_1 e^{\sqrt{|\lambda|}b} - c_2 e^{-\sqrt{|\lambda|}b} = 0.$$

$c_1(e^{\sqrt{|\lambda|}b} - e^{-\sqrt{|\lambda|}b}) = 0, \quad e^{\sqrt{|\lambda|}b} - e^{-\sqrt{|\lambda|}b} \neq 0$, then $c_1 = c_2 = 0$. This system has trivial solution, $c_1 = c_2 = 0$, so there is no eigenvalues and eigenfunctions in the region $\lambda < 0$.

2) Case 2: For $\lambda = 0$, we have a general solution for (2.2.1) as in the form:

$$y(x) = c_1 x + c_2. \tag{2.2.5}$$

Substituting (2.2.5) into (2.2.2) we get:

$$y'(0) = c_1 = 0,$$

using the last equation we have

$$y'(b) = c_1 = 0,$$

since $b \neq 0$. This implies that for $\lambda = 0$ we have a solution if we choose $c_2 \neq 0$. So we have $y(x) = c_2$. So for $\lambda = 0$ is an eigenvalue and $y(x) = c_2$ is corresponding eigenfunctions to $\lambda = 0$. We can normalize this eigenfunction by choosing $c_2 = \sqrt{\frac{1}{b}}$.

3) Case 3: For $\lambda > 0$, we have a general solution for (2.2.1) as in the form:

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \tag{2.2.6}$$

$$y'(x) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x). \tag{2.2.7}$$

Substituting (2.2.7) into (2.2.2), we get:

$$y'(0) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}0) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}0) \Rightarrow c_2 = 0,$$

using the last equation we have

$$y'(b) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}b) = 0,$$

for finding a solution we may choose $c_1 \neq 0$ and this implies that $\sin(\sqrt{\lambda}b) = 0$ and $\sqrt{\lambda}b = k\pi$ for k=1,2,3,... that is

$$\lambda_k = (\frac{k\pi}{b})^2, \quad for \quad k = 1, 2, 3, ...$$

So we have eigenvalues λ_k and corresponding eigenvectors $y_k(x) = c_1 \cos(\sqrt{\lambda_k}x)$ for k=1,2,3,... We can normalize this orthogonal functions by choosing $c_1 = \sqrt{\frac{2}{b}}$.

4) Conclusion:The eigenvalue-eigenfunction problem (2.2.1), (2.2.2) has the following solution:

$$\lambda_0 = 0, \quad y_0(x) = \sqrt{\frac{1}{b}}, \tag{2.2.8}$$

$$\lambda_k = (\frac{k\pi}{b})^2, \quad y_k(x) = \sqrt{\frac{2}{b}} \cos(\sqrt{\lambda_k}x), \quad k = 1, 2, 3, ... \tag{2.2.9}$$

where λ_k are eigenvalues and $y_k(x)$ are eigenvectors corresponding to these eigenvalues.

2.3 Sturm-Liouville Problem with Mixed Boundary Condition

Consider the following SLP

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < b, \tag{2.3.1}$$

$$y(0) = 0, \quad y'(b) = 0, \tag{2.3.2}$$

where b is a given positive number, λ is a parameter.
We have three cases for the parameter λ.

$$Case1 : \lambda < 0; Case2 : \lambda = 0; Case3 : \lambda > 0.$$

1) Case 1: For $\lambda < 0$, we have a general solution for (2.3.1) as in the form:

$$y(x) = c_1 e^{\sqrt{|\lambda|}x} + c_2 e^{-\sqrt{|\lambda|}x}, \tag{2.3.3}$$

$$y'(x) = c_1\sqrt{|\lambda|}e^{\sqrt{|\lambda|}x} + c_2(-\sqrt{|\lambda|})e^{-\sqrt{|\lambda|}x}. \tag{2.3.4}$$

Substituting (2.3.3) and (2.3.4) into (2.3.2), we get:

$$y(0) = c_1 + c_2 = 0,$$

$$y'(b) = c_1\sqrt{|\lambda|}e^{\sqrt{|\lambda|}b} - c_2(\sqrt{|\lambda|})e^{-\sqrt{|\lambda|}b} = 0.$$

This system is a linear algebraic system according to c_1, c_2. The determinant of this system, is different from zero:

$$detM = e^{-\sqrt{|\lambda|}b} - e^{\sqrt{|\lambda|}b},$$

$$= 2Sinh\sqrt{|\lambda|}b \neq 0.$$

Using Cramer's rule, we find $c_1 = c_2 = 0$.
This system has trivial solution: $c_1 = c_2 = 0$, so there is no eigenvalues and eigenfunctions in the region $\lambda < 0$.

2) Case 2: For $\lambda = 0$, we have a general solution for (2.3.1) as in the form:

$$y(x) = c_1 x + c_2, \tag{2.3.5}$$

$$y'(x) = c_1. \tag{2.3.6}$$

Substituting (2.3.5) and (2.3.6) into (2.3.2), we get:

$$y(0) = c_1 0 + c_2 = 0 \Rightarrow c_2 = 0,$$

$$y'(b) = c_1 = 0 \Rightarrow c_1 = 0.$$

In this case the problem has trivial solution. $c_1 = c_2 = 0$, so for $\lambda = 0$ isn't an eigenvalue.

3) Case 3: For $\lambda > 0$, we have a general solution for (2.3.1) as in the form:

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \tag{2.3.7}$$

$$y'(x) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x). \tag{2.3.8}$$

Substituting (2.3.7) and (2.3.8) into (2.3.2), we get:

$$y(0) = c_1 \cos(\sqrt{\lambda}0) + c_2 \sin(\sqrt{\lambda}0) = 0,$$

$$y'(b) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}b) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}b) = 0.$$

Hence we have $c_1 = 0$ and $c_2 \cos(\sqrt{\lambda}b) = 0$. But $c_2 \cos(\sqrt{\lambda}b) = 0$ if and only if, either $c_2 = 0$ or $\cos(\sqrt{\lambda}b) = 0$. The condition $c_1 = 0$ and $c_2 = 0$ means $y(x) = 0$. This doesn't yield any eigenvalue. If $y(x) \neq 0$, then $c_2 \neq 0$. Thus $\cos(\sqrt{\lambda}b) = 0$ should hold. This last equation has solutions given by:

$$\lambda_k = (\frac{(2k-1)\pi}{2b})^2 \quad for \quad k = 1, 2, 3, ...$$

and the corresponding eigenfunctions are given by:

$$y_k(x) = c_2 \sin(\sqrt{\lambda_k}x),$$

for k=1,2,3,... We can normalize this orthogonal functions by choosing $c_2 = \sqrt{\frac{2}{b}}$.

CHAPTER THREE
FOURIER SERIES EXPANSION METHOD FOR SOLVING THE DIRICHLET AND MIXED PROBLEMS OF THE POISSON EQUATION IN A RECTANGLE

In this section the Dirichlet and Mixed problems for the Poisson equation with the boundary condition in a rectangle will be studied. We will find an explicit formulae for a solution.

3.1 Statement of the Dirichlet Problem for the Poisson Equation

Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1, 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following Dirichlet problem for the Poisson equation in the rectangle $\mathbb{D}$:

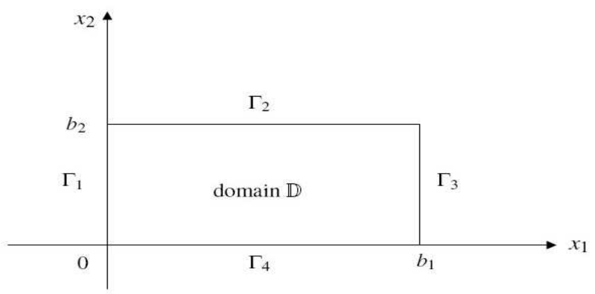
$$\frac{\partial^2 u(x_1, x_2)}{\partial x_1^2} + \frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} = f(x_1, x_2), \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2, \tag{3.1.1}$$

$$u|_\Gamma = 0 \Leftrightarrow \quad u(0, x_2) = 0, \qquad u(b_1, x_2) = 0, \qquad 0 < x_2 < b_2, \tag{3.1.2}$$

$$u(x_1, 0) = 0, \qquad u(x_1, b_2) = 0, \qquad 0 < x_1 < b_1, \tag{3.1.3}$$

where Γ is the boundary of $\mathbb{D}$. The Dirichlet problem consists of finding $u(x_1, x_2)$, function satisfying (3.1.1) - (3.1.3). The boundary Γ may be presented in the form:

Remark: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_1 = \{(x_1, x_2) : x_1 = 0, x_2 \in (0, b_2)\}$, $\Gamma_2 = \{(x_1, x_2) : x_1 \in (0, b_1), x_2 = b_2\}$, $\Gamma_3 = \{(x_1, x_2) : x_1 = b_1, x_2 \in (0, b_2)\}$, $\Gamma_4 = \{(x_1, x_2) : x_1 \in (0, b_1), x_2 = 0\}$.



3.2 Fourier Series Expansion Method for Solving the Dirichlet Problem

For solving the Dirichlet problem, we apply the Fourier series expansion method. This method has several steps. On the first step, we need to solve the eigenvalue-eigenfunction problem of finding all eigenvalues and corresponding to them eigenfunctions of the following equation:

$$X''(x_1) + \alpha X(x_1) = 0, \tag{3.2.1}$$

with boundary conditions

$$X(0) = 0, \quad X(b_1) = 0, \tag{3.2.2}$$

where $x_1 \in (0, b_1)$. $b_1 > 0$ is the given number.
As we showed in Chapter 2 of the thesis, this problem has the following solution:

$$\alpha = \alpha_k = (\frac{k\pi}{b_1})^2, \ for \quad k = 1, 2, 3, \dots \tag{3.2.3}$$

are eigenvalues and corresponding to them eigenfunctions have the form:

$$X_k(x_1) = \sqrt{\frac{2}{b_1}} \sin(\sqrt{\alpha_k} x_1). \tag{3.2.4}$$

We note that the norm of these eigenfunctions are equal to 1. Eigenfunctions $X_k(x_1)$ for k=1,2,3,..., satisfy (3.2.1) and (3.2.2). The solution of (3.1.1) - (3.1.3). We will find the solution in the form:

$$u(x_1, x_2) = \sum_{k=1}^{\infty} X_k(x_1) Y_k(x_2). \tag{3.2.5}$$

The function $f(x_1, x_2)$ can be written in the form of Fourier series expansion:

$$f(x_1, x_2) = \sum_{k=1}^{\infty} X_k(x_1) f_k(x_2), \tag{3.2.6}$$

where $f_k(x_2) = \int_0^{b_1} f(x_1,x_2)X_k(x_1)dx_1, for\, k = 1,2,3,\ldots$. Substituting (3.2.5) and (3.2.6) into (3.1.1), we have

$$\begin{aligned}
\sum_{k=1}^{\infty}[Y_k(x_2)X_k''(x_1) + Y_k''(x_2)X_k(x_1)] &= \sum_{k=1}^{\infty} X_k(x_1)f_k(x_2),\\
\Rightarrow \sum_{k=1}^{\infty}[-Y_k(x_2)\alpha_k X_k(x_1) + Y_k''(x_2)X_k(x_1)] &= \sum_{k=1}^{\infty} X_k(x_1)f_k(x_2),\\
\Rightarrow \sum_{k=1}^{\infty}[X_k(x_1)Y_k''(x_2) - \alpha_k X_k(x_1)Y_k(x_2)] &= \sum_{k=1}^{\infty} X_k(x_1)f_k(x_2),\\
\Rightarrow \sum_{k=1}^{\infty} X_k(x_1)f_k(x_2) &= f(x_1,x_2).
\end{aligned}$$

Also the equation satisfies (3.1.1) - (3.1.3).

$$u(x_1,x_2) = \sum_{k=1}^{\infty} X_k(x_1)Y_k(x_2), \tag{3.2.7}$$

where $X_k(x_1)$ k=1,2,3,... are set of eigenfunctions of (3.2.1), (3.2.2); $Y_k(x_2)$ for k=1,2,3,... are unknown functions. For finding $Y_k(x_2)$, we have the following boundary value problem:

$$Y_k''(x_2) - \alpha_k Y_k(x_2) = f_k(x_2), \tag{3.2.8}$$

$$Y_k(0) = 0, \quad Y_k(b_2) = 0, \tag{3.2.9}$$

where $x_2 \in (0,b_2)$; $\alpha_k = (\frac{k\pi}{b_2})^2 > 0$ are eigenvalues of (3.2.8), (3.2.9); $b_2 > 0$ is a given number. A general solution of the (3.2.8), when $f_k(x_2) \equiv 0$, is given by

$$Y_k(x_2) = c_{1k}e^{\sqrt{|\alpha_k|}x_2} + c_{2k}e^{-\sqrt{|\alpha_k|}x_2}. \tag{3.2.10}$$

To solve of the non-homogenous problem, we will use variation of the parameters method. Using the variation of the parameters method, we find the following equalities:

$$\begin{aligned}
Y_k'(x_2) &= \frac{\partial(c_{1k})}{\partial(x_2)}e^{\sqrt{\alpha_k}x_2} + \sqrt{\alpha_k}e^{\sqrt{\alpha_k}x_2}c_{1k}(x_2)\\
&+ \frac{\partial(c_{2k})}{\partial(x_2)}e^{-\sqrt{\alpha_k}x_2} - \sqrt{\alpha_k}e^{-\sqrt{\alpha_k}x_2}c_{2k}(x_2).
\end{aligned}$$

$$\begin{aligned}
Y_k''(x_2) &= \frac{\partial^2(c_{1k})}{\partial {x_2}^2} e^{\sqrt{\alpha_k}x_2} + 2\frac{\partial(c_{1k})}{\partial(x_2)}\sqrt{\alpha_k}e^{\sqrt{\alpha_k}x_2} \\
&+ \alpha_k e^{\sqrt{\alpha_k}x_2}(c_{1k})(x_2) + \frac{\partial^2(c_{2k})}{\partial {x_2}^2} e^{-\sqrt{\alpha_k}x_2} \\
&- 2\frac{\partial(c_{2k})}{\partial(x_2)}\sqrt{\alpha_k}e^{-\sqrt{\alpha_k}x_2} + \alpha_k e^{-\sqrt{\alpha_k}x_2}(c_{2k})(x_2).
\end{aligned}$$

We can write the following linear algebraic system:

$$c_{1k}'(x_2)e^{\sqrt{\alpha_k}x_2} + c_{2k}'(x_2)e^{-\sqrt{\alpha_k}x_2} = 0,$$

$$\sqrt{\alpha_k}c_{1k}'(x_2)e^{\sqrt{\alpha_k}x_2} - \sqrt{\alpha_k}c_{2k}'(x_2)e^{-\sqrt{\alpha_k}x_2} = f_k(x_2). \tag{3.2.11}$$

Using Cramer's rule we find $c_{1k}(x_2)$ and $c_{2k}(x_2)$ for k=1,2,3,....

$$\mathbf{A} = \begin{pmatrix} e^{\sqrt{|\alpha_k|}x_2} & e^{-\sqrt{|\alpha_k|}x_2} \\ \sqrt{\alpha_k}e^{\sqrt{\alpha_k}x_2} & -\sqrt{\alpha_k}e^{-\sqrt{\alpha_k}x_2} \end{pmatrix}$$

$$detA = -\sqrt{\alpha_k} - \sqrt{\alpha_k} = -2\sqrt{\alpha_k} \neq 0.$$

$$\mathbf{B} = \begin{pmatrix} 0 & e^{-\sqrt{\alpha_k}x_2} \\ f_k(x_2) & -\sqrt{\alpha_k}e^{-\sqrt{\alpha_k}x_2} \end{pmatrix}$$

$$detB = -e^{-\sqrt{\alpha_k}x_2} f_k(x_2).$$

$$\mathbf{C} = \begin{pmatrix} e^{\sqrt{\alpha_k}x_2} & 0 \\ \sqrt{\alpha_k}e^{\sqrt{\alpha_k}x_2} & f_k(x_2) \end{pmatrix}$$

$$detC = e^{\sqrt{\alpha_k}x_2} f_k(x_2).$$

$$c_{1k}'(x_2) = \frac{-e^{-\sqrt{\alpha_k}x_2} f_k(x_2)}{-2\sqrt{\alpha_k}},$$

$$c_{2k}'(x_2) = \frac{e^{\sqrt{\alpha_k}x_2} f_k(x_2)}{-2\sqrt{\alpha_k}}.$$

Integrating the last two relations with respect to x_2 we find:

$$c_{1k}(x_2) = \frac{1}{2\sqrt{\alpha_k}}\int_0^{x_2} e^{-\sqrt{\alpha_k}\xi} f_k(\xi)d\xi + a_k,$$

$$c_{2k}(x_2) = -\frac{1}{2\sqrt{\alpha_k}}\int_0^{x_2} e^{\sqrt{\alpha_k}\xi} f_k(\xi)d\xi + b_k.$$

Here a_k and b_k are arbitrary constants. We choose these constants to satisfy boundary conditions. Using the last two formulaes for c_{1k}, c_{2k}, we find the following equation for $Y_k(x_2)$:

$$\begin{aligned} Y_k(x_2) &= \frac{1}{2\sqrt{\alpha_k}}\Big[\int_0^{x_2} e^{\sqrt{\alpha_k}(-\xi+x_2)}f_k(\xi)d\xi - \int_0^{x_2} e^{\sqrt{\alpha_k}(\xi-x_2)}f_k(\xi)d\xi\Big] \\ &+ a_k e^{\sqrt{\alpha_k}x_2} + b_k e^{-\sqrt{\alpha_k}x_2} \\ &= \frac{1}{\sqrt{\alpha_k}}\int_0^{x_2} \sinh(\sqrt{\alpha_k}(-\xi+x_2))f_k(\xi)d\xi \\ &+ a_k e^{\sqrt{\alpha_k}x_2} + b_k e^{-\sqrt{\alpha_k}x_2}. \end{aligned}$$

Using the first boundary condition, we have the following equality:

$$Y_k(0) = 0 \Rightarrow a_k + b_k = 0 \Rightarrow b_k = -a_k.$$

Using the second boundary condition, we have the following equality:

$$\begin{aligned} Y_k(b_2) &= 0, \\ 0 &= \frac{1}{\sqrt{\alpha_k}}\Big[\int_0^{b_2} \sinh(\sqrt{\alpha_k}(-\xi+b_2))f_k(\xi)d\xi\Big] + 2a_k \sinh(\sqrt{\alpha_k}b_2), \end{aligned}$$

where $e^{\sqrt{\alpha_k}b_2} - e^{-\sqrt{\alpha_k}b_2} = 2\sinh(\sqrt{\alpha_k}b_2)$. If we divide by $-2\sinh(\sqrt{\alpha_k}b_2)$ both side of the equation above, then we find the following equations:

$$a_k = -\frac{1}{2\sqrt{\alpha_k}\sinh(\sqrt{\alpha_k}b_2)}\Big[\int_0^{b_2} \sinh(\sqrt{\alpha_k}(-\xi+b_2))f_k(\xi)d\xi\Big],$$

$$b_k = \frac{1}{2\sqrt{\alpha_k}\sinh(\sqrt{\alpha_k}b_2)}\Big[\int_0^{b_2} \sinh(\sqrt{\alpha_k}(-\xi+b_2))f_k(\xi)d\xi\Big].$$

Substituting a_k and b_k into the equation of $Y_k(x_2)$, we find the following equation:

$$Y_k(x_2) = \frac{1}{\sqrt{\alpha_k}}\int_0^{x_2} \sinh(\sqrt{\alpha_k}(-\xi+x_2))f_k(\xi)d\xi,$$

$$-\frac{\sinh(\sqrt{\alpha_k}x_2)}{\sqrt{\alpha_k}}\int_0^{b_2} \frac{\sinh(\sqrt{\alpha_k}(-\xi+b_2))}{\sinh(\sqrt{\alpha_k}b_2)}f_k(\xi)d\xi.$$

We find the solution of (3.1.1) - (3.1.3) as follows:

$$u(x_1,x_2) = \sum_{k=1}^{\infty}\sqrt{\frac{2}{b_1}}\sin(\sqrt{\alpha_k}x_1)\Big\{\frac{1}{\sqrt{\alpha_k}}\int_0^{x_2} \sinh(-\xi+x_2)f_k(\xi)d\xi$$

$$-\frac{\sinh(\sqrt{\alpha_k}x_2)}{\sqrt{\alpha_k}}\int_0^{b_2} \frac{\sinh(\sqrt{\alpha_k}(-\xi+b_2))}{\sinh(\sqrt{\alpha_k}b_2)}f_k(\xi)d\xi\Big\}. \qquad (3.2.12)$$

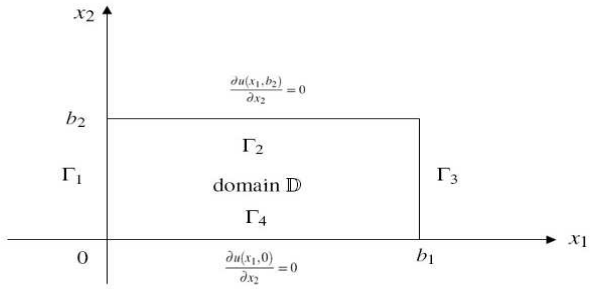
3.3 Statement of the Mixed Problem for the Poisson Equation

Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : \ 0 < x_1 < b_1, \ 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following mixed problem for Poisson equation in the rectangle $\mathbb{D}$:

$$\frac{\partial^2 u(x_1, x_2)}{\partial x_1^2} + \frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} = f(x_1, x_2), \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2, \tag{3.3.1}$$

$$u(0, x_2) = 0, \qquad u(b_1, x_2) = 0, \qquad 0 \leq x_2 \leq b_2, \tag{3.3.2}$$

$$\frac{\partial u(x_1, 0)}{\partial x_2} = 0, \quad \frac{\partial u(x_1, b_2)}{\partial x_2} = 0, \quad 0 \leq x_1 \leq b_1. \tag{3.3.3}$$



3.4 Fourier Series Expansion Method for Solving the Mixed Problem

The solution of (3.3.1) - (3.3.3) find in the form:

$$u(x_1, x_2) = \sum_{k=0}^{\infty} X_k(x_1) Y_k(x_2), \tag{3.4.1}$$

where $X_k(x_1)$, k=0,1,2,... are set of unknown functions; $Y_k(x_2)$, k=0,1,2,... are eigenfunctions of the following eigenvalue-eigenfunction problem:

$$Y''(x_2) - \lambda Y(x_2) = 0, \tag{3.4.2}$$

with boundary conditions

$$Y'(0) = 0, \quad Y'(b_2) = 0. \tag{3.4.3}$$

The eigenvalues of the problem are $\lambda = \lambda_k = -(\frac{k\pi}{b_2})^2$, k=0,1,2,... and corresponding to the eigenvectors

$$Y_0(x_2) = \sqrt{\frac{1}{b_2}},\ k = 0, \tag{3.4.4}$$

$$Y_k(x_2) = \sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2}x_2), \quad k = 1, 2, 3, ... \tag{3.4.5}$$

The eigenfunctions $\{Y_k(x_2)\}$, k=0,1,2,... is a basis i.e., the function $f(x_1, x_2)$ can be written in the form of the following Fourier series expansion:

$$f(x_1, x_2) = \sum_{k=0}^{\infty} f_k(x_1) Y_k(x_2), \tag{3.4.6}$$

where $f_k(x_1) = \int_0^{b_2} f(x_1, x_2) Y_k(x_2) dx_2$, k=0,1,2,... Substituting (3.4.6) and (3.4.1) into (3.3.1) we have

$$\sum_{k=0}^{\infty} [X_k''(x_1) Y_k(x_2) - X_k(x_1) Y_k''(x_2) - f_k(x_1) Y_k(x_2)]$$

$$= \sum_{k=0}^{\infty} f_k(x_1) Y_k(x_2)$$

or

$$\sum_{k=0}^{\infty} [X_k''(x_1) + \lambda_k X_k(x_1) - f_k(x_1)] Y_k(x_2)$$

$$= \sum_{k=0}^{\infty} f_k(x_1) Y_k(x_2).$$

As a result, we have the following boundary value problem for unknown function $X_k(x_1)$:

$$X_k''(x_1) + \lambda_k X_k(x_1) = f_k(x_1), \quad k = 0, 1, 2, 3, ... \tag{3.4.7}$$

$$X_k(0) = 0, \quad X_k(b_1) = 0. \tag{3.4.8}$$

We can solve the problem using variation of parameters method. A general solution of $X_k(x_1)$, we find in the form:

$$X_k(x_1) = a_k(x_1) e^{\sqrt{|\lambda_k|} x_1} + b_k(x_1) e^{-\sqrt{|\lambda_k|} x_1} \quad k = 1, 2, 3, ...,$$

and

$$X_0(x_1) = a_0(x_1) + b_0(x_1).x_1,$$

where $a_k(x_1), b_k(x_1), a_0(x_1), b_0(x_1)$ are unknown functions. For finding these function we have these systems:

$$a_0'(x_1) + x_1 b_0'(x_1) = 0,$$

$$0.a_0'(x_1) + 1.b_0'(x_1) = f_0(x_1).$$

And

$$e^{\sqrt{|\lambda_k|x_1}}.a_k'(x_1) + e^{-\sqrt{|\lambda_k|x_1}}.b_k'(x_1) = 0,$$

$$\sqrt{|\lambda_k|}.e^{\sqrt{|\lambda_k|x_1}}.a_k'(x_1) + (-\sqrt{|\lambda_k|}).e^{-\sqrt{|\lambda_k|x_1}}.b_k'(x_1) = f_k(x_1).$$

Using Cramer's rule for the first system, we find:

$$a_0'(x_1) = -f_0(x_1).x_1,$$

$$b_0'(x_1) = f_0(x_1).$$

Integrating the last two relations with respect to x_1, we find:

$$a_0(x_1) = -\int_0^{x_1} f_0(\xi)\xi d\xi + a_0,$$

$$b_0(x_1) = \int_0^{x_1} f_0(\xi)\xi d\xi + b_0.$$

Here a_0 and b_0 are arbitrary constants. We choose these constants to satisfy boundary conditions. Using the last formulaes for $a_0(x_1), b_0(x_1)$ we find the following equation for $X_0(x_1)$:

$$X_0(x_1) = \int_0^{x_1} f_0(\xi)(x_1 - \xi)d\xi + a_0 + b_0 x_1.$$

Using the first boundary condition, we write the following equality:

$$X_0(0) = 0 \Rightarrow a_0 = 0.$$

Using the second boundary condition, we write the following equality:

$$X_0(b_1) = \int_0^{b_1} f_0(\xi)(b_1 - \xi)d\xi + b_1.b_0 = 0 \Rightarrow b_0 = -\frac{1}{b_1}\int_0^{b_1} f_0(\xi)(b_1 - \xi)d\xi.$$

We find the following result of $X_0(x_1)$:

$$X_0(x_1) = \int_0^{x_1}(x_1 - \xi)f_0(\xi)d\xi - \frac{x_1}{b_1}\int_0^{b_1}(b_1 - \xi)f_0(\xi)d\xi. \tag{3.4.9}$$

Let now k=1,2,3,... Using Cramer's rule for the second system, we find:

$$a'_k(x_1) = \frac{1}{2\sqrt{|\lambda_k|}} f_k(x_1) e^{-\sqrt{|\lambda_k|}x_1},$$

$$b'_k(x_1) = -\frac{1}{2\sqrt{|\lambda_k|}} f_k(x_1) e^{\sqrt{|\lambda_k|}x_1}.$$

Integrating the last two relations with respect to x_1, we find:

$$a_k(x_1) = \frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_1} f_k(\xi) e^{-\sqrt{|\lambda_k|}\xi} d\xi + c_k,$$

$$b_k(x_1) = -\frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_1} f_k(\xi) e^{\sqrt{|\lambda_k|}\xi} d\xi + d_k.$$

Here c_k and d_k are arbitrary constants. We choose these constants to satisfy boundary conditions. Using the last formula for $a_k(x_1)$, $b_k(x_1)$ we find the following equation for $X_k(x_1)$:

$$\begin{aligned} X_k(x_1) &= \frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_1} f_k(\xi) e^{\sqrt{|\lambda_k|}(x_1-\xi)} d\xi \\ &- \frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_1} f_k(\xi) e^{-\sqrt{|\lambda_k|}(x_1-\xi)} d\xi \\ &+ c_k e^{\sqrt{|\lambda_k|}x_1} + d_k e^{-\sqrt{|\lambda_k|}x_1}, \\ &= \frac{1}{\sqrt{|\lambda_k|}} \int_0^{x_1} \sinh(\sqrt{|\lambda_k|}(x_1-\xi)) f_k(\xi) d\xi + c_k e^{\sqrt{|\lambda_k|}x_1} \\ &+ d_k e^{-\sqrt{|\lambda_k|}x_1}. \end{aligned}$$

Using the first boundary condition, we write the following equality:

$$X_k(0) = c_k + d_k = 0 \Rightarrow -c_k = d_k.$$

Using the second boundary condition, we write the following equality:

$$\begin{aligned} X_k(b_1) &= \frac{1}{\sqrt{|\lambda_k|}} \int_0^{b_1} \sinh(\sqrt{|\lambda_k|}(b_1-\xi)) f_k(\xi) d\xi \\ &+ 2c_k \sinh(\sqrt{|\lambda_k|}b_1). \end{aligned}$$

$$c_k = -\frac{1}{2\sqrt{|\lambda_k|}} \int_0^{b_1} \frac{\sinh(\sqrt{|\lambda_k|}(b_1-\xi))}{\sinh(\sqrt{|\lambda_k|}b_1)} f_k(\xi) d\xi.$$

We find the following result of $X_k(x_1)$:

$$X_k(x_1) = \frac{1}{\sqrt{|\lambda_k|}} \int_0^{x_1} \sinh(\sqrt{|\lambda_k|}(x_1 - \xi)) f_k(\xi) d\xi - \frac{\sinh(\sqrt{|\lambda_k|} x_1)}{\sqrt{|\lambda_k|}}$$

$$\int_0^{b_1} \frac{\sinh(\sqrt{|\lambda_k|}(b_1 - \xi))}{\sinh(\sqrt{|\lambda_k|} b_1)} f_k(\xi) d\xi, \tag{3.4.10}$$

where k=1,2,3,... We find the following solution of (3.3.1) - (3.3.3) as follows:

$$\begin{aligned} u(x_1, x_2) &= \sqrt{\frac{1}{b_2}} \left[\int_0^{x_1} (x_1 - \xi) f_0(\xi) d\xi - \frac{x_1}{b_1} \int_0^{b_1} (b_1 - \xi) f_0(\xi) d\xi \right] \\ &+ \sum_{k=1}^{\infty} \left[\frac{1}{\sqrt{|\lambda_k|}} \int_0^{x_1} \sinh(\sqrt{|\lambda_k|}(x_1 - \xi)) f_k(\xi) d\xi \right. \\ &- \left. \frac{\sinh(\sqrt{|\lambda_k|} x_1)}{\sqrt{|\lambda_k|}} \int_0^{b_1} \frac{\sinh(\sqrt{|\lambda_k|}(b_1 - \xi))}{\sinh(\sqrt{|\lambda_k|} b_1)} f_k(\xi) d\xi \right] \end{aligned}$$

$$\sqrt{\frac{2}{b_2}} \cos(\sqrt{|\lambda_k|} x_2). \tag{3.4.11}$$

CHAPTER FOUR
FOURIER SERIES EXPANSION METHOD FOR SOLVING THE DIRICHLET, NEUMANN AND MIXED PROBLEMS OF THE HELMHOLTZ EQUATION IN A RECTANGLE

In this section the Dirichlet, Neumann and Mixed problems for the Helmholtz equation with boundary condition in a rectangle will be studied. We find a solution in the form of Fourier series expansion.

4.1 Statement of the Dirichlet Problem for the Helmholtz Equation

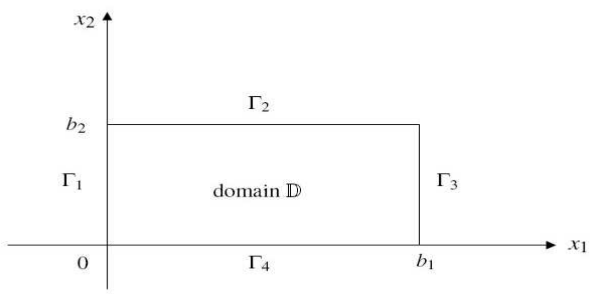
Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following Dirichlet problem for the Helmholtz equation in the rectangle $\mathbb{D}$:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + a^2 u = f(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{D}, \tag{4.1.1}$$

$$u|_\Gamma = 0 \Leftrightarrow \quad u(0, x_2) = 0, \qquad u(b_1, x_2) = 0, \tag{4.1.2}$$

$$u(x_1, 0) = 0, \qquad u(x_1, b_2) = 0, \tag{4.1.3}$$

where Γ is the boundary of $\mathbb{D}$ and $a > 0$ is a given real number, $f(x_1, x_2)$ is a given function.
Find a function $u(x_1, x_2)$ satisfying (4.1.1) - (4.1.3).



4.2 Fourier Series Expansion Method for Solving the Dirichlet Problem

The solution of this problem has several steps. On the first step, we need to solve the eigenvalue-eigenfunction problem of finding all eigenvalues and corresponding to them eigenfunctions of the following equation:

$$X''(x_1) + \alpha X(x_1) = 0, \tag{4.2.1}$$

with boundary conditions

$$X(0) = 0, \quad X(b_1) = 0, \tag{4.2.2}$$

where $x_1 \in (0, b_1)$. $b_1 > 0$ is the given number.
As we showed is the first part of the thesis this problem has the following solution:

$$\alpha = \alpha_k = (\frac{k\pi}{b_1})^2, \ for \quad k = 1, 2, 3, \ldots \tag{4.2.3}$$

are eigenvalues and corresponding to them eigenfunctions have the form:

$$X_k(x_1) = \sqrt{\frac{2}{b_1}} \sin(\sqrt{\alpha_k} x_1). \tag{4.2.4}$$

We note that the norm of these eigenfunctions is equal to 1. Eigenfunctions $X_k(x_1)$ for k=1,2,3,..., satisfy (4.2.1) and (4.2.2). The solution of the problem in the form:

$$u(x_1, x_2) = \sum_{k=1}^{\infty} X_k(x_1) Y_k(x_2), \tag{4.2.5}$$

where $X_k(x_1)$, k=1,2,3,...are set of eigenfunctions of (4.2.1). $Y_k(x_2)$, k=1,2,3,...are unknown functions. The Fourier series expansion of $f(x_1, x_2)$ is given in the form:

$$f(x_1, x_2) = \sum_{k=1}^{\infty} X_k(x_1) f_k(x_2),$$

where $f_k(x_2) = \int_0^{b_1} f(x_1, x_2) X_k(x_1) dx_1$. Substituting $f_k(x_2)$ and $u(x_1, x_2)$ into (4.1.1), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} [Y_k(x_2) X_k''(x_1) + Y_k''(x_2) X_k(x_1) + a^2 X_k(x_1) Y_k(x_2)] &= \sum_{k=1}^{\infty} X_k(x_1) f_k(x_2), \\
\Rightarrow \sum_{k=1}^{\infty} [-Y_k(x_2) \alpha_k X_k(x_1) + Y_k''(x_2) X_k(x_1) + a^2 X_k(x_1) Y_k(x_2)] &= \sum_{k=1}^{\infty} X_k(x_1) f_k(x_2), \\
\Rightarrow \sum_{k=1}^{\infty} [X_k(x_1) Y_k''(x_2) + (a^2 - \alpha_k) X_k(x_1) Y_k(x_2)] &= \sum_{k=1}^{\infty} X_k(x_1) f_k(x_2), \\
\Rightarrow \sum_{k=1}^{\infty} X_k(x_1) f_k(x_2) &= f(x_1, x_2).
\end{aligned}$$

For finding $Y_k(x_2)$ we have the following problem:

$$Y_k''(x_2) + (a^2 - \alpha_k)Y_k(x_2) = f_k(x_2), \tag{4.2.6}$$

$$Y_k(0) = 0, \quad Y_k(b_2) = 0, \tag{4.2.7}$$

where $x_2 \in (0, b_2)$; $\alpha_k = (\frac{k\pi}{b_2})^2 > 0, \alpha_k \neq 0$; $b_2 > 0$ is given number. As we showed in Chapter 2 of the thesis this problem has the following solution. Firstly, we have three cases for the parameter $\lambda_k = (a^2 - \alpha_k)$:
Case 1: $(a^2 - \alpha_k) < 0$; Case 2: $(a^2 - \alpha_k) = 0$; Case 3: $(a^2 - \alpha_k) > 0$.

a)If $\lambda_k = a^2 - \alpha_k < 0$, a general solution of the problem (4.2.6), we find in the form:

$$Y_k(x_2) = c_{1k}(x_2)e^{\sqrt{|\lambda_k|}x_2} + c_{2k}(x_2)e^{-\sqrt{|\lambda_k|}x_2}, \tag{4.2.8}$$

where $c_{1k}(x_2)$ and $c_{2k}(x_2)$ are unknown functions. Using variation of parameters method, we get

$$c_{1k}(x_2) = \frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_2} e^{-\sqrt{|\lambda_k|}\xi} f_k(\xi)d\xi + A_k,$$

$$c_{2k}(x_2) = -\frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_2} e^{\sqrt{|\lambda_k|}\xi} f_k(\xi)d\xi + B_k,$$

where A_k, B_k are arbitrary constants, we will choose A_k, B_k to satisfy (4.2.8). For this we substitute $c_{1k}(x_2)$ and $c_{2k}(x_2)$ into (4.2.8), then we can find the following equation:

$$\begin{aligned} Y_k(x_2) &= (\frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_2} e^{-\sqrt{|\lambda_k|}\xi} f_k(\xi)d\xi + A_k)e^{\sqrt{|\lambda_k|}x_2} \\ &+ (-\frac{1}{2\sqrt{|\lambda_k|}} \int_0^{x_2} e^{\sqrt{|\lambda_k|}\xi} f_k(\xi)d\xi + B_k)e^{-\sqrt{|\lambda_k|}x_2}, \\ &= \frac{1}{\sqrt{|\lambda_k|}} \int_0^{x_2} \sinh(\sqrt{|\lambda_k|}(-\xi + x_2)) f_k(\xi)d\xi + A_k e^{\sqrt{|\lambda_k|}x_2} \\ &+ B_k e^{-\sqrt{|\lambda_k|}x_2}. \end{aligned}$$

Using the first boundary condition of (4.2.7), we have:

$$Y_k(0) = 0 \Rightarrow A_k + B_k = 0 \Rightarrow B_k = -A_k.$$

Using the second boundary condition of (4.2.7), we find:

$$\begin{aligned} Y_k(b_2) &= 0, \\ 0 &= \frac{1}{\sqrt{|\lambda_k|}}\left[\int_0^{b_2} \sinh(\sqrt{|\lambda_k|}(-\xi + b_2)) f_k(\xi) d\xi\right] + 2A_k \sinh(\sqrt{|\lambda_k|} b_2) \end{aligned}$$

where $2\sinh(\sqrt{|\lambda_k|}b_2) = e^{\sqrt{|\lambda_k|}b_2} - e^{-\sqrt{|\lambda_k|}b_2}$.

$$\begin{aligned} -2A_k \sinh(\sqrt{|\lambda_k|}b_2) &= \frac{1}{\sqrt{|\lambda_k|}}\left[\int_0^{b_2} \sinh(\sqrt{|\lambda_k|}(-\xi + b_2)) f_k(\xi) d\xi\right], \\ -\frac{A_k \sinh(\sqrt{|\lambda_k|}b_2)}{\sinh(\sqrt{|\lambda_k|}b_2)} &= \frac{1}{2\sqrt{|\lambda_k|}}\left[\int_0^{b_2} \frac{\sinh(\sqrt{|\lambda_k|}(-\xi + b_2))}{\sinh(\sqrt{|\lambda_k|}b_2)} f_k(\xi) d\xi\right]. \end{aligned}$$

Finally, we find the following relation for A_k and B_k:

$$A_k = -\frac{1}{2\sqrt{|\lambda_k|}}\left[\int_0^{b_2} \frac{\sinh(\sqrt{|\lambda_k|}(-\xi + b_2))}{\sinh(\sqrt{|\lambda_k|}b_2)} f_k(\xi) d\xi\right],$$

$$B_k = \frac{1}{2\sqrt{|\lambda_k|}}\left[\int_0^{b_2} \frac{\sinh(\sqrt{|\lambda_k|}(-\xi + b_2))}{\sinh(\sqrt{|\lambda_k|}b_2)} f_k(\xi) d\xi\right].$$

Substituting A_k and B_k into the equation for $Y_k(x_2)$ we can find:

$$\begin{aligned} Y_k(x_2) &= \frac{1}{\sqrt{|\lambda_k|}} \int_0^{x_2} \sinh(-\xi + x_2) f_k(\xi) d\xi, \\ &- \frac{\sinh(\sqrt{|\lambda_k|}x_2)}{\sqrt{|\lambda_k|}} \int_0^{b_2} \frac{\sinh(\sqrt{|\lambda_k|}(-\xi + b_2))}{\sinh(\sqrt{|\lambda_k|}b_2)} f_k(\xi) d\xi. \end{aligned}$$

b)If $\lambda_k = a^2 - \alpha_k > 0$, a general solution of (4.2.6), we find in the form:

$$Y_k(x_2) = c_{1k}(x_2)\sin(\sqrt{\lambda_k}x_2) + c_{2k}(x_2)\cos(\sqrt{\lambda_k}x_2), \tag{4.2.9}$$

where $c_{1k}(x_2)$ and $c_{2k}(x_2)$ are unknown functions. Using variation of parameters method, we get:

$$c_{1k}(x_2) = \frac{1}{\sqrt{\lambda_k}} \int_0^{x_2} \cos(\sqrt{\lambda_k}\xi) f_k(\xi) d\xi + A_k,$$

$$c_{2k}(x_2) = -\frac{1}{\sqrt{\lambda_k}} \int_0^{x_2} \sin(\sqrt{\lambda_k}\xi) f_k(\xi) d\xi + B_k,$$

where A_k, B_k are arbitrary constants. Substituting $c_{1k}(x_2)$ and $c_{2k}(x_2)$ into (4.2.9), we find the following equation:

$$\begin{aligned} Y_k(x_2) &= \frac{1}{\sqrt{\lambda_k}} \int_0^{x_2} (\cos(\sqrt{\lambda_k}\xi)\sin(\sqrt{\lambda_k}x_2) \\ &- \sin(\sqrt{\lambda_k}\xi)\cos(\sqrt{\lambda_k}x_2)) f_k(\xi) d\xi + A_k \sin(\sqrt{\lambda_k}x_2) \\ &+ B_k \cos(\sqrt{\lambda_k}x_2), \\ &= \frac{1}{\sqrt{\lambda_k}} \int_0^{x_2} \sin(\sqrt{\lambda_k}(x_2 - \xi)) f_k(\xi) d\xi + A_k \sin(\sqrt{\lambda_k}x_2) \\ &+ B_k \cos(\sqrt{\lambda_k}x_2). \end{aligned}$$

We choose constants A_k and B_k to satisfy boundary condition (4.2.7).
Using the first boundary condition of (4.2.7), we have:

$$Y_k(0) = 0 \Rightarrow 0 + B_k = 0 \Rightarrow B_k = 0.$$

Using the second boundary condition of (4.2.7), we have:

$$Y_k(b_2) = \frac{1}{\sqrt{\lambda_k}} \int_0^{b_2} \sin(\sqrt{\lambda_k}(b_2 - \xi)) f_k(\xi) d\xi + A_k \sin(\sqrt{\lambda_k}b_2) + B_k \cos(\sqrt{\lambda_k}b_2)$$

$$\begin{aligned} A_k \sin(\sqrt{\lambda_k}b_2) &= -\frac{1}{\sqrt{\lambda_k}} \int_0^{b_2} \sin(\sqrt{\lambda_k}(b_2 - \xi)) f_k(\xi) d\xi, \\ A_k &= -\frac{1}{\sqrt{\lambda_k}\sin(\sqrt{\lambda_k}b_2)} \int_0^{b_2} \sin(\sqrt{\lambda_k}(b_2 - \xi)) f_k(\xi) d\xi. \end{aligned}$$

Substituting A_k into the equation of $Y_k(x_2)$, then we obtain the following equation:

$$\begin{aligned} Y_k(x_2) &= \frac{1}{\sqrt{\lambda_k}} \int_0^{x_2} \sin(\sqrt{\lambda_k}(x_2 - \xi)) f_k(\xi) d\xi \\ &- \frac{\sin(\sqrt{\lambda_k}x_2)}{\sqrt{\lambda_k}\sin(\sqrt{\lambda_k}b_2)} \int_0^{b_2} \sin(\sqrt{\lambda_k}(b_2 - \xi)) f_k(\xi) d\xi. \end{aligned}$$

c)If $\lambda_k = a^2 - \alpha_k = 0$, we have the following boundary value problem:

$$Y_k''(x_2) = f_k(x_2), \tag{4.2.10}$$

$$Y_k(0) = 0,\ Y_k(b_2) = 0. \tag{4.2.11}$$

Integrating twice (4.2.10) and using the first boundary condition (4.2.11), we find:

$$Y_k'(x_2) = Y_k'(0) + \int_0^{x_2} f_k(\xi) d\xi,$$

$$Y_k(x_2) = Y_k(0) + A_k x_2 + \int_0^{x_2} \left(\int_0^{z} f_k(\xi) d\xi \right) dz,$$

$$Y_k(x_2) = A_k x_2 + \int_0^{x_2} (x_2 - \xi) f_k(\xi) d\xi, \tag{4.2.12}$$

where A_k is an arbitrary constant. We choose A_k to satisfy the second boundary condition (4.2.11),

$$Y_k(b_2) = 0 \Leftrightarrow 0 = A_k b_2 + \int_0^{b_2} (b_2 - \xi) f_k(\xi) d\xi.$$

We find

$$A_k = -\frac{1}{b_2} \int_0^{b_2} (b_2 - \xi) f_k(\xi) d\xi.$$

Substituting A_k into (4.2.12), we find:

$$Y_k(x_2) = -\frac{x_2}{b_2} \int_0^{b_2} (b_2 - \xi) f_k(\xi) d\xi + \int_0^{x_2} (x_2 - \xi) f_k(\xi) d\xi.$$

Finally, the solution of (4.1.1) - (4.1.3) is given by:

$$\begin{aligned}
u(x_1, x_2) &= \sum_{k=1}^{\infty} \sqrt{\frac{2}{b_1}} \sin(\sqrt{\alpha_k} x_1) \Bigg[\frac{1}{\sqrt{|\lambda_k|}} \int_0^{x_2} \sinh(\sqrt{|\lambda_k|}(-\xi + x_2)) f_k(\xi) d\xi \\
&- \frac{\sinh(\sqrt{|\lambda_k|} x_2)}{\sqrt{|\lambda_k|}} \int_0^{b_2} \frac{\sinh(\sqrt{|\lambda_k|}(-\xi + b_2))}{\sinh(\sqrt{|\lambda_k|} b_2)} f_k(\xi) d\xi \\
&+ \frac{1}{\sqrt{\lambda_k}} \int_0^{x_2} \sin(\sqrt{\lambda_k}(x_2 - \xi)) f_k(\xi) d\xi \\
&- \frac{\sin(\sqrt{\lambda_k} x_2)}{\sqrt{\lambda_k} \sin(\sqrt{\lambda_k} b_2)} \int_0^{b_2} \sin(\sqrt{\lambda_k}(b_2 - \xi)) f_k(\xi) d\xi \\
& -\frac{x_2}{b_2} \int_0^{b_2} (b_2 - \xi) f_k(\xi) d\xi + \int_0^{x_2} (x_2 - \xi) f_k(\xi) d\xi \Bigg].
\end{aligned} \tag{4.2.13}$$

4.3 Statement of the Problem for the Helmholtz Equation with Mixed boundary conditions

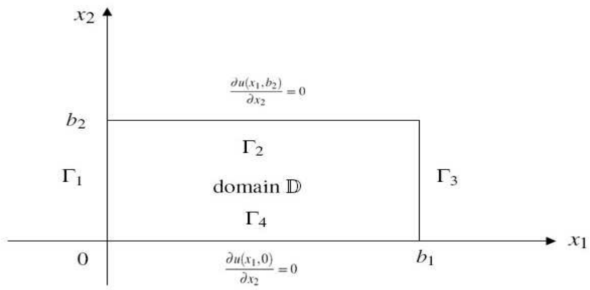
Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following mixed problem for the Helmholtz equation in the rectangle $\mathbb{D}$:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + a^2 u = f(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{D}, \tag{4.3.1}$$

$$u(0, x_2) = 0, \qquad u(b_1, x_2) = 0, \qquad 0 \leq x_2 \leq b_2, \tag{4.3.2}$$

$$\frac{\partial u(x_1, 0)}{\partial x_2} = 0, \quad \frac{\partial u(x_1, b_2)}{\partial x_2} = 0, \quad 0 \leq x_1 \leq b_1, \tag{4.3.3}$$

$a > 0$ is a given real number, $f(x_1, x_2)$ is a given function. Find $u(x_1, x_2)$ satisfying (4.3.1) - (4.3.3).



4.4 Fourier Series Expansion Method for Solving the Mixed Problem

The solution of this problem has several steps. We will find the solution in the form:

$$u(x_1, x_2) = \sum_{k=0}^{\infty} X_k(x_1) Y_k(x_2), \tag{4.4.1}$$

where $X_k(x_1)$, k=0,1,2,... are set of unknown functions; $Y_k(x_2)$, k=0,1,2,... are eigenfunctions of the following eigenvalue-eigenfunction problem:

$$Y''(x_2) + \lambda Y(x_2) = 0, \tag{4.4.2}$$

$$Y'(0) = 0, \quad Y'(b_2) = 0, \tag{4.4.3}$$

where $x_2 \in (0, b_2)$; $b_2 > 0$ is given number; $\lambda = \lambda_k = (\frac{k\pi}{b_2})^2$, for k=0,1,2,... are eigenvalues corresponding to eigenvectors

$$Y_0(x_2) = \sqrt{\frac{1}{b_2}}, \; k = 0, \tag{4.4.4}$$

$$Y_k(x_2) = \sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2}x_2), \quad k = 1, 2, 3, ... \tag{4.4.5}$$

The Fourier series expansion of $f(x_1, x_2)$ is given in the form:

$$f(x_1, x_2) = \sum_{k=0}^{\infty} f_k(x_1)Y_k(x_2), \tag{4.4.6}$$

where $f_k(x_1) = \int_0^{b_2} f(x_1, x_2)Y_k(x_2)dx_2$. Substituting (4.4.6) and (4.4.1) into (4.3.1), we have

$$\sum_{k=0}^{\infty}[X_k''(x_1)Y_k(x_2) + X_k(x_1)Y_k''(x_2) + a^2X_k(x_1)Y_k(x_2) - f_k(x_1)Y_k(x_2)]$$

$$= \sum_{k=0}^{\infty} f_k(x_1)Y_k(x_2)$$

or

$$\sum_{k=0}^{\infty}[X_k''(x_1) - \lambda_k X_k(x_1) + a^2X_k(x_1) - f_k(x_1)]Y_k(x_2)$$

$$= \sum_{k=0}^{\infty} f_k(x_1)Y_k(x_2).$$

Since $\{Y_k(x_2)\}$ is orthonormal set of functions we find the following boundary value problem:

$$X_k''(x_1) + \alpha_k X_k(x_1) = f_k(x_1),\ k = 0, 1, 2, 3, ... \tag{4.4.7}$$

$$X_k(0) = 0,\ X_k(b_1) = 0, \tag{4.4.8}$$

where $x_1 \in (0, b_1)$. $\alpha_k = a^2 - \lambda_k = a^2 - (\frac{k\pi}{b_2})^2 > 0$; $b_2 > 0$ is given number for k=0,1,2,.... As we showed in Chapter 2 of the thesis this problem has the following solution. Firstly, we have three cases for the parameter α_k.
Case 1: $\alpha_k < 0$; Case 2: $\alpha_k = 0$; Case 3: $\alpha_k > 0$.

a)Let us consider the Case: $\alpha_k < 0, \quad k = n+1, n+2, \ldots$. A general solution of (4.4.7) as follows:

$$X_k(x_1) = a_k(x_1)e^{\sqrt{|\alpha_k|}x_1} + b_k(x_1)e^{-\sqrt{|\alpha_k|}x_1},$$

where $a_k(x_1)$ and $b_k(x_1)$ are unknown functions. If we use the variation of the parameters method, then we can write following equation system:

$$a_k'(x_1)e^{\sqrt{|\alpha_k|}x_1} + b_k'(x_1)e^{-\sqrt{|\alpha_k|}x_1} = 0,$$

$$\sqrt{|\alpha_k|}a'_k(x_1)e^{\sqrt{|\alpha_k|}x_1} - \sqrt{|\alpha_k|}a'_k(x_1)e^{-\sqrt{|\alpha_k|}x_1} = f_k(x_1). \quad (4.4.9)$$

This is a linear algebraic system. We apply Cramer's rule to find $a_k(x_1)$, $b_k(x_1)$, are unknown functions.
Using Cramer's rule we find:

$$a'_k(x_1) = \frac{1}{2\sqrt{|\alpha_k|}}e^{-\sqrt{|\alpha_k|}x_1}f_k(x_1),$$

$$b'_k(x_1) = -\frac{1}{2\sqrt{|\alpha_k|}}e^{\sqrt{|\alpha_k|}x_1}f_k(x_1).$$

Integrating $a'_k(x_1)$ and $b'_k(x_1)$

$$a_k(x_1) = \frac{1}{2\sqrt{|\alpha_k|}}\int_0^{x_1} f_k(\xi)e^{-\sqrt{|\alpha_k|}\xi}d\xi + c_k,$$

$$b_k(x_1) = -\frac{1}{2\sqrt{|\alpha_k|}}\int_0^{x_1} f_k(\xi)e^{\sqrt{|\alpha_k|}\xi}d\xi + d_k.$$

Here c_k and d_k are arbitrary constants. We choose these constants to satisfy boundary conditions. We can write the following equation for $X_k(x_1)$:

$$\begin{aligned} X_k(x_1) &= \frac{1}{2\sqrt{|\alpha_k|}}\int_0^{x_1} f_k(\xi)e^{\sqrt{|\alpha_k|}(x_1-\xi)}d\xi - \frac{1}{2\sqrt{|\alpha_k|}} \\ &\cdot \int_0^{x_1} f_k(\xi)e^{-\sqrt{|\alpha_k|}(x_1-\xi)}d\xi \\ &+ c_k e^{\sqrt{|\alpha_k|}x_1} + d_k e^{-\sqrt{|\alpha_k|}x_1} \end{aligned}$$

or

$$X_k(x_1) = \frac{1}{\sqrt{|\alpha_k|}}\int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_k(\xi)d\xi + c_k e^{\sqrt{|\alpha_k|}x_1} + d_k e^{-\sqrt{|\alpha_k|}x_1}.$$

Using the first boundary condition, we have:

$$X_k(0) = c_k + d_k = 0 \Rightarrow -c_k = d_k.$$

Using the second boundary condition, we have:

$$X_k(b_1) = \frac{1}{\sqrt{|\alpha_k|}}\int_0^{b_1} \sinh(\sqrt{|\alpha_k|}(b_1-\xi))f_k(\xi)d\xi + 2c_k\sinh(\sqrt{|\alpha_k|}b_1),$$

$$c_k = -\frac{1}{2\sqrt{|\alpha_k|}}\int_0^{b_1} \frac{\sinh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)}f_k(\xi)d\xi.$$

We find the following result of $X_k(x_1)$:

$$X_k(x_1) = \frac{1}{\sqrt{|\alpha_k|}} \int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1 - \xi)) f_k(\xi) d\xi$$

$$- \frac{\sinh(\sqrt{|\alpha_k|} x_1)}{\sqrt{|\alpha_k|}} \int_0^{b_1} \frac{\sinh(\sqrt{|\alpha_k|}(b_1 - \xi))}{\sinh(\sqrt{|\alpha_k|} b_1)} f_k(\xi) d\xi, \tag{4.4.10}$$

where $k = n+1, n+2, \ldots$.

b)Let us consider the $Case$: $\alpha_k > 0, \ k = 0, 1, 2, \ldots, n-1.$
A general solution of $X_k(x_1)$, we find in the form:

$$X_k(x_1) = a_k(x_1) \sin(\sqrt{\alpha_k} x_1) + b_k(x_1) \cos(\sqrt{\alpha_k} x_1).$$

If we use the variation of the parameters method, we find following equations:

$$a'_k(x_1) \sin \sqrt{\alpha_k} x_1 + b'_k(x_1) \cos \sqrt{\alpha_k} x_1 = 0,$$

$$\sqrt{\alpha_k} a'_k(x_1) \cos \sqrt{\alpha_k} x_1 - \sqrt{\alpha_k} b'_k(x_1) \sin - \sqrt{\alpha_k} x_1 = f_k(x_1). \tag{4.4.11}$$

This is a linear algebraic system. We apply Cramer's rule to find $a_k(x_1)$, $b_k(x_1)$, for k=1,2,3,....

$$a'_k(x_1) = \frac{1}{\sqrt{\alpha_k}} \cos(\sqrt{\alpha_k} x_1) f_k(x_1),$$

$$b'_k(x_1) = \frac{-1}{\sqrt{\alpha_k}} \sin(\sqrt{\alpha_k} x_1) f_k(x_1).$$

Integrating $a'_k(x_1)$ and $b'_k(x_1)$, we find:

$$a_k(x_1) = \frac{1}{\sqrt{\alpha_k}} \int_0^{x_1} \cos(\sqrt{\alpha_k} \xi) f_k(\xi) d\xi + A_k,$$

$$b_k(x_1) = -\frac{1}{\sqrt{\alpha_k}} \int_0^{x_1} \sin(\sqrt{\alpha_k} \xi) f_k(\xi) d\xi + B_k.$$

We can find the following equation:

$$X_k(x_1) = \frac{1}{\sqrt{\alpha_k}} \int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1 - \xi)) f_k(\xi) d\xi + A_k \sin(\sqrt{\alpha_k} x_1)$$

$$+ B_k \cos(\sqrt{\alpha_k} x_2)$$

is an eigenfunction corresponding to α_k, where A_k and B_k are unknown parameters.

Using the first boundary condition, we have

$$X_k(0) = 0 \Rightarrow 0 + B_k = 0 \Rightarrow B_k = 0.$$

Using the second boundary condition, we have

$$X_k(b_1) = \frac{1}{\sqrt{\alpha_k}} \int_0^{b_1} \sin(\sqrt{\alpha_k}(b_1 - \xi) f_k(\xi) d\xi + A_k \sin(\sqrt{\alpha_k} b_1) = 0$$

$$-A_k sin(\sqrt{\alpha_k} b_1) = \frac{1}{\sqrt{\alpha_k}} \int_0^{b_1} \sin(\sqrt{\alpha_k}(b_1 - \xi) f_k(\xi) d\xi$$

$$A_k = -\frac{1}{\sqrt{\alpha_k}\sin(\sqrt{\alpha_k} b_1)} \int_0^{b_1} \sin(\sqrt{\alpha_k}(b_1 - \xi) f_k(\xi) d\xi.$$

Substituting A_k into the equation of $X_k(x_1)$, then we find the following result of $X_k(x_1)$:

$$\begin{aligned} X_k(x_1) &= \frac{1}{\sqrt{\alpha_k}} \int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1 - \xi)) f_k(\xi) d\xi \\ &- \frac{\sin(\sqrt{\alpha_k} x_1)}{\sqrt{\alpha_k}\sin(\sqrt{\alpha_k} b_1)} \int_0^{b_1} \sin(\sqrt{\alpha_k}(b_1 - \xi) f_k(\xi) d\xi, \end{aligned}$$

where $k = 0, 1, 2, \ldots, n-1$.

c)If $a^2 - \lambda_k = 0$, then a general solution of $X_n(x_1)$ we find in the form:

$$X_n(x_1) = a_n(x_1) + b_n(x_1).x_1.$$

If we use the variation of the parameters method, we find following equations:

$$a_n'(x_1) + x_1 b_n'(x_1) = 0,$$

$$0.a_n'(x_1) + 1.b_n'(x_1) = f_n(x_1).$$

This is a linear algebraic system. We apply Cramer's rule to find $a_n(x_1), b_n(x_1), fork = n$.

Using Cramer's rule we find:

$$a_n'(x_1) = -f_n(x_1).x_1,$$

$$b_n'(x_1) = f_n(x_1).$$

Integrating $a'_n(x_1)$ and $b'_n(x_1)$ then

$$a_n(x_1) = -\int_0^{x_1} \xi f_n(\xi) d\xi + A_n,$$

$$b_n(x_1) = \int_0^{x_1} f_n(\xi) d\xi + B_n.$$

We find the following result of $X_n(x_1)$:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi) f_n(\xi) d\xi + A_n + B_n x_1.$$

Using the first boundary condition, we have the following equality:

$$X_n(0) = 0 \Rightarrow A_n = 0.$$

Using the second boundary condition, we have the following equality:

$$X_n(b_1) = \int_0^{b_1} (b_1 - \xi) f_n(\xi) d\xi + b_1.B_n = 0 \Rightarrow B_n = -\frac{1}{b_1} \int_0^{b_1} (b_1 - \xi) f_n(\xi) d\xi.$$

We find the following result of $X_n(x_1)$:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi) f_n(\xi) d\xi - \frac{x_1}{b_1} \int_0^{b_1} (b_1 - \xi) f_n(\xi) d\xi, \tag{4.4.12}$$

where $k = n$. Finally, the solution of (4.3.1) - (4.3.3) is given by:

$$\begin{aligned}
u(x_1, x_2) &= \Bigg[\sum_{k=0}^{n-1} \frac{1}{\sqrt{\alpha_k}} \int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1 - \xi)) f_k(\xi) d\xi \\
&- \frac{\sin(\sqrt{\alpha_k} x_1)}{\sqrt{\alpha_k} \sin(\sqrt{\alpha_k} b_1)} \int_0^{b_1} \sin(\sqrt{\alpha_k}(b_1 - \xi)) f_k(\xi) d\xi \Bigg] \\
&. \sqrt{\frac{2}{b_2}}. \cos(\frac{k\pi}{b_2} x_2) + \left[\int_0^{x_1} (x_1 - \xi) f_n(\xi) d\xi - \frac{x_1}{b_1} \int_0^{b_1} (b_1 - \xi) f_n(\xi) d\xi \right] \\
&. \sqrt{\frac{1}{b_2}} + \Bigg[\sum_{k=n+1}^{\infty} \frac{1}{\sqrt{|\alpha_k|}} \int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1 - \xi)) f_k(\xi) d\xi \\
&- \frac{\sinh(\sqrt{|\alpha_k|} x_1)}{\sqrt{|\alpha_k|}} \int_0^{b_1} \frac{\sinh(\sqrt{|\alpha_k|}(b_1 - \xi))}{\sinh(\sqrt{|\alpha_k|} b_1)} f_k(\xi) d\xi \Bigg] \\
&\qquad .\sqrt{\frac{2}{b_2}}. \cos(\frac{k\pi}{b_2} x_2).
\end{aligned} \tag{4.4.13}$$

4.5 Statement of the Neumann Problem for the Helmholtz Equation

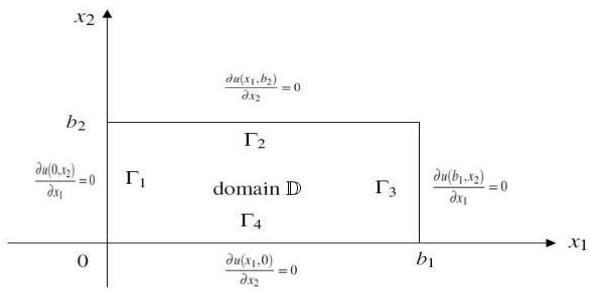
Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\, 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following Neumann problem for the Helmholtz equation in the rectangle $\mathbb{D}$:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + a^2 u = f(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{D}, \tag{4.5.1}$$

$$\frac{\partial u(0, x_2)}{\partial x_1} = 0, \quad \frac{\partial u(b_1, x_2)}{\partial x_1} = 0, \quad 0 \le x_2 \le b_2, \tag{4.5.2}$$

$$\frac{\partial u(x_1, 0)}{\partial x_2} = 0, \quad \frac{\partial u(x_1, b_2)}{\partial x_2} = 0, \quad 0 \le x_1 \le b_1, \tag{4.5.3}$$

$a > 0$ is a given real number, $f(x_1, x_2)$ is a given function. Find $u(x_1, x_2)$ satisfying (4.5.1) - (4.5.3).



4.6 Fourier Series Expansion Method for Solving the Neumann Problem

The solution of this problem has several steps. We will find the solution in the form:

$$u(x_1, x_2) = \sum_{k=0}^{\infty} X_k(x_1) Y_k(x_2), \tag{4.6.1}$$

where $X_k(x_1)$, k=0,1,2,... are set of unknown functions; $Y_k(x_2)$, k=0,1,2,... are eigenfunctions of the following eigenvalue-eigenfunction problem:

$$Y''(x_2) + \lambda Y(x_2) = 0, \tag{4.6.2}$$

$$Y'(0) = 0, \quad Y'(b_2) = 0, \tag{4.6.3}$$

where $x_2 \in (0, b_2)$; $b_2 > 0$ is given number; $\lambda = \lambda_k = (\frac{k\pi}{b_2})^2$, for k=0,1,2,... are eigenvalues corresponding to eigenvectors

$$Y_0(x_2) = \sqrt{\frac{1}{b_2}}, \tag{4.6.4}$$

$$Y_k(x_2) = \sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2}x_2), \quad k = 1, 2, 3, ... \tag{4.6.5}$$

The Fourier series expansion of $f(x_1, x_2)$ is given in the form:

$$f(x_1, x_2) = \sum_{k=0}^{\infty} f_k(x_1) Y_k(x_2), \tag{4.6.6}$$

where $f_k(x_1) = \int_0^{b_2} f(x_1, x_2) Y_k(x_2) dx_2$.

Substituting (4.6.1) and (4.6.6) into (4.5.1), we have

$$\sum_{k=0}^{\infty} [X_k''(x_1)Y_k(x_2) + X_k(x_1)Y_k''(x_2) + a^2 X_k(x_1)Y_k(x_2) - f_k(x_1)Y_k(x_2)]$$

$$= \sum_{k=0}^{\infty} f_k(x_1)Y_k(x_2)$$

or

$$\sum_{k=0}^{\infty} [X_k''(x_1) - \lambda_k X_k(x_1) + a^2 X_k(x_1) - f_k(x_1)]Y_k(x_2)$$

$$= \sum_{k=0}^{\infty} f_k(x_1)Y_k(x_2).$$

Since $\{Y_k(x_2)\}$ is orthonormal set of functions, we find the following boundary value problem:

$$X_k''(x_1) + \alpha_k X_k(x_1) = f_k(x_1),\ k = 0, 1, 2, ... \tag{4.6.7}$$

$$X_k'(0) = 0,\ X_k'(b_1) = 0, \tag{4.6.8}$$

where $x_1 \in (0, b_1)$. $\alpha_k = a^2 - \lambda_k$; $b_1 > 0$ is given number.

There are two possibilities: Either

Case 1: $\alpha_k > 0,\ k = 0, 1, 2, \ldots, n-1;$

Case 2: $\alpha_k = 0,\ k = n;$

Case 3: $\alpha_k < 0,\ k = n+1, n+2, \ldots,$

or

Case 1: $\alpha_k > 0,\ k = 0, 1, 2, \ldots, n-1;$

Case 2: $\alpha_k < 0,\ k = n, n+1, \ldots.$

a)Let us consider the $Case\ 1: \quad \alpha_k > 0,\ k = 0, 1, 2, \ldots, n-1$. A general solution of (4.6.7), we find in the form:

$$X_k(x_1) = A_k(x_1)\sin(\sqrt{\alpha_k}x_1) + B_k(x_1)\cos(\sqrt{\alpha_k}x_1).$$

If we use the variation of the parameters method, we find the following equations:

$$(A_k)'(x_1)\sin\sqrt{\alpha_k}x_1 + (B_k)'(x_1)\cos\sqrt{\alpha_k}x_1 = 0,$$

$$\sqrt{\alpha_k}(A_k)'(x_1)\cos\sqrt{\alpha_k}x_1 - \sqrt{\alpha_k}(B_k)'(x_1)\sin\sqrt{\alpha_k}x_1 = f_k(x_1). \tag{4.6.9}$$

This is a linear algebraic system. Using Cramer's rule, we find

$$(A_k)'(x_1) = \frac{1}{\sqrt{\alpha_k}}\cos(\sqrt{\alpha_k}x_1)f_k(x_1),$$

$$B_k'(x_1) = \frac{-1}{\sqrt{\alpha_k}}\sin(\sqrt{\alpha_k}x_1)f_k(x_1).$$

Integrating the last two relations with respect to x_1, we find

$$A_k(x_1) = \frac{1}{\sqrt{\alpha_k}}\int_0^{x_1}\cos(\sqrt{\alpha_k}\xi)f_k(\xi)d\xi + A_k,$$

$$B_k(x_1) = -\frac{1}{\sqrt{\alpha_k}}\int_0^{x_1}\sin(\sqrt{\alpha_k}\xi)f_k(\xi)d\xi + B_k.$$

Here A_k and B_k are arbitrary constants. We choose these constants to satisfy boundary conditions. Using the last formulaes for $A_k(x_1)$, $B_k(x_1)$ following equation for $X_k(x_1):$

$$X_k(x_1) = \frac{1}{\sqrt{\alpha_k}}\int_0^{x_1}\sin(\sqrt{\alpha_k}(x_1-\xi))f_k(\xi)d\xi + A_k\sin(\sqrt{\alpha_k}x_1)$$

$$+B_k\cos(\sqrt{\alpha_k}x_1),$$

where A_k and B_k are unknown parameters. Differentiating the last formulae, we have

$$X_k'(x_1) = \int_0^{x_1} \cos(\sqrt{\alpha_k}(x_1-\xi)) f_k(\xi)d\xi + \sqrt{\alpha_k}A_k\cos(\sqrt{\alpha_k}x_1)$$

$$-\sqrt{\alpha_k}B_k\sin(\sqrt{\alpha_k}x_1).$$

Using the first boundary condition, we have

$$X_k'(0) = 0 \Rightarrow A_k = 0.$$

Using the second boundary condition, we have

$$X_k'(b_1) = \int_0^{b_1} \cos(\sqrt{\alpha_k}(b_1-\xi)) f_k(\xi)d\xi - \sqrt{\alpha_k}B_k\sin(\sqrt{\alpha_k}b_1) = 0.$$

From the last equality, we find

$$B_k = \frac{1}{\sqrt{\alpha_k}} \int_0^{b_1} \frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)} f_k(\xi)d\xi.$$

As a result of it, the formulae for $X_k(x_1)$ has the form:

$$X_k(x_1) = \frac{1}{\sqrt{\alpha_k}}\left[\int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1-\xi)) f_k(\xi)d\xi\right.$$

$$\left.+\cos(\sqrt{\alpha_k}x_1)\int_0^{b_1} \frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)} f_k(\xi)d\xi\right],$$

where $k = 0, 1, 2, \ldots, n-1$.

b)If $a^2 - \lambda_n = 0$, then a general solution of $X_n(x_1)$ we find in the form:

$$X_n(x_1) = A_n(x_1) + B_n(x_1).x_1.$$

$$A_n'(x_1) + x_1 B_n'(x_1) = 0,$$
$$0.A_n'(x_1) + 1.B_n'(x_1) = f_n(x_1).$$

This is a linear algebraic system. To find $A_n(x_1)$, $B_n(x_1)$ using Cramer's rule we have:

$$A_n'(x_1) = -f_n(x_1).x_1,$$

$$B_n'(x_1) = f_n(x_1).$$

Integrating $A'_n(x_1)$ and $B'_n(x_1)$, we find

$$A_n(x_1) = -\int_0^{x_1} \xi f_n(\xi)d\xi + d_n,$$

$$B_n(x_1) = \int_0^{x_1} f_n(\xi)d\xi + c_n.$$

As a result of it, we have:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi)f_n(\xi)d\xi + d_n + c_n x_1,$$

$$X'_n(x_1) = \int_0^{x_1} f_n(\xi)d\xi + c_n.$$

Using the first boundary condition, we have the following equality:

$$X'_n(0) = 0 \Rightarrow c_n = 0.$$

Using the second boundary condition, we have the following equality:

$$X'_n(b_1) = \int_0^{b_1} f_n(\xi)d\xi = 0$$

Remark: The necessary condition for existence of the solution:

$$\int_0^{b_1} f_n(r)dr = 0.$$

We find the following result of $X_k(x_1)$:

$$X_k(x_1) = \int_0^{x_1} (x_1 - \xi)f_n(\xi)d\xi + d_n,$$

where $k = n$.

c)Let us consider the Case 2: $\alpha_k < 0, \ \ k = n+1, n+2, \dots.$

A general solution of (4.6.7), we find in the form:

$$X_k(x_1) = A_k(x_1)e^{\sqrt{|\alpha_k|}x_1} + B_k(x_1)e^{-\sqrt{|\alpha_k|}x_1},$$

where $A_k(x_1)$ and $B_k(x_1)$ are unknown functions. If we use the variation of the parameters method, then we can write following equation system:

$$A'_k(x_1)e^{\sqrt{|\alpha_k|}x_1} + B'_k(x_1)e^{-\sqrt{|\alpha_k|}x_1} = 0,$$

$$\sqrt{|\alpha_k|}A'_k(x_1)e^{\sqrt{|\alpha_k|}x_1} - \sqrt{|\alpha_k|}A'_k(x_1)e^{-\sqrt{|\alpha_k|}x_1} = f_k(x_1). \tag{4.6.10}$$

This is a linear algebraic system. Using Cramer's rule, we find

$$A'_k(x_1) = \frac{1}{2\sqrt{|\alpha_k|}}e^{-\sqrt{|\alpha_k|}x_1}f_k(x_1),$$

$$B'_k(x_1) = -\frac{1}{2\sqrt{|\alpha_k|}}e^{\sqrt{|\alpha_k|}x_1}f_k(x_1).$$

Integrating the last two relations with respect to x_1, we find

$$A_k(x_1) = \frac{1}{2\sqrt{|\alpha_k|}}\int_0^{x_1} f_k(\xi)e^{-\sqrt{|\alpha_k|}\xi}d\xi + c_k,$$

$$B_k(x_1) = -\frac{1}{2\sqrt{|\alpha_k|}}\int_0^{x_1} f_k(\xi)e^{\sqrt{|\alpha_k|}\xi}d\xi + d_k.$$

Here c_k and d_k are arbitrary constants. We choose these constants to satisfy boundary conditions. Using the last two formulaes for $A_k(x_1)$, $B_k(x_1)$, then we find following equation for $X_k(x_1)$:

$$\begin{aligned} X_k(x_1) &= \frac{1}{2\sqrt{|\alpha_k|}}\int_0^{x_1} f_k(\xi)e^{\sqrt{|\alpha_k|}(x_1-\xi)}d\xi \\ &- \frac{1}{2\sqrt{|\alpha_k|}}\int_0^{x_1} f_k(\xi)e^{-\sqrt{|\alpha_k|}(x_1-\xi)}d\xi \\ &+ c_k e^{\sqrt{|\alpha_k|}x_1} + d_k e^{-\sqrt{|\alpha_k|}x_1} \end{aligned}$$

or

$$X_k(x_1) = \frac{1}{\sqrt{|\alpha_k|}}\int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_k(\xi)d\xi + c_k e^{\sqrt{|\alpha_k|}x_1} + d_k e^{-\sqrt{|\alpha_k|}x_1}.$$

$$X'_k(x_1) = \int_0^{x_1} \cosh(\sqrt{|\alpha_k|}(x_1-\xi))f_k(\xi)d\xi + \sqrt{|\alpha_k|}c_k e^{\sqrt{|\alpha_k|}x_1} - \sqrt{|\alpha_k|}d_k e^{-\sqrt{|\alpha_k|}x_1}.$$

Using the first boundary condition, we have the following equality:

$$X'_k(0) = 0 \Rightarrow c_k - d_k = 0 \Rightarrow c_k = d_k.$$

Using the second boundary condition, we have the following equality:

$$X_k'(b_1) = \int_0^{b_1} \cosh(\sqrt{|\alpha_k|}(b_1-\xi))f_k(\xi)d\xi + 2\sqrt{|\alpha_k|}c_k \sinh(\sqrt{|\alpha_k|}b_1) = 0.$$

From the last equality, we find

$$c_k = -\frac{1}{2\sqrt{|\alpha_k|}} \int_0^{b_1} \frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_k(\xi)d\xi.$$

As a result of it, we have

$$\begin{aligned} X_k(x_1) &= \frac{1}{\sqrt{|\alpha_k|}} \int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_k(\xi)d\xi \\ &- \frac{\cosh\sqrt{|\alpha_k|}x_1}{\sqrt{|\alpha_k|}} \int_0^{b_1} \frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_k(\xi)d\xi \end{aligned}$$

$$X_k(x_1) = \frac{1}{\sqrt{|\alpha_k|}} \Bigg[\int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_k(\xi)d\xi$$

$$- \cosh\sqrt{|\alpha_k|}x_1 \int_0^{b_1} \frac{\cosh(\sqrt{|\alpha_k|}(b_1-\tau))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_k(\xi)d\xi \Bigg], \tag{4.6.11}$$

where k=n+1,n+2,.... Finally, the solution of (4.5.1) - (4.5.3) is given by:

$$\begin{aligned} u(x_1,x_2) &= \sum_{k=0}^{n-1} \frac{1}{\sqrt{\alpha_k}} \Bigg[\int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1-\xi))f_k(\xi)d\xi \\ &+ \cos(\sqrt{\alpha_k}x_1) \int_0^{b_1} \frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)} f_k(\xi)d\xi \Bigg] \\ &. \sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2}x_2) \\ &+ \Bigg[\int_0^{x_1} (x_1-\xi)f_n(\xi)d\xi + d_n \Bigg] \sqrt{\frac{1}{b_2}} \\ &+ \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{|\alpha_k|}} \Bigg[\int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_k(\xi)d\xi \\ &- \cosh\sqrt{|\alpha_k|}x_1 \int_0^{b_1} \frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_k(\xi)d\xi \Bigg] \\ &. \sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2}x_2). \end{aligned}$$

CHAPTER FIVE
SOLVING MAXWELL'S EQUATION SYSTEM IN A RECTANGLE WITH THE PERFECT CONDUCTOR CONDITIONS

In this chapter the frequency-dependent Maxwell's system is reduced to three Helmholtz equations. We will use perfect conductor conditions to state problem. We show that this problem is reduced to several boundary value problems. The solutions of obtained problems are found in the form of the Fourier series expansion.

5.1 Maxwell's equations system

Let $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ be a space variable and $t \in \mathbf{R}$ be a time variable. The time-dependent Maxwell's system is given by (Eom, 2004):

$$curl_x H(x,t) = \epsilon \frac{\partial(E(x,t))}{\partial t} + J(x,t), \tag{5.1.1}$$

$$curl_x E(x,t) = -\mu \frac{\partial(H(x,t))}{\partial t}, \tag{5.1.2}$$

$$div_x(\epsilon E(x,t)) = \rho(x,t), \tag{5.1.3}$$

$$div_x(\mu H(x,t)) = 0, \tag{5.1.4}$$

where $H = (H_1, H_2, H_3)$, $E = (E_1, E_2, E_3)$, $J = (J_1, J_2, J_3)$ are vector functions; $H_k(x,t), E_k(x,t), J_k(x,t), \rho(x,t)$ (where $k = 1, 2, 3$) are scalar functions; μ and ϵ are positive constants. μ denotes the magnetic permeability, ϵ denotes the electric permittivity, H denotes magnetic field, E denotes electric field, J denotes current density, ρ denotes charge density. Applying Fourier transform with respect to t we find the frequency-dependent Maxwell's system:

$$curl_x \tilde{H}(x,\omega) = (-iw)\epsilon \tilde{E}(x,\omega) + \tilde{J}(x,\omega), \tag{5.1.5}$$

$$curl_x \tilde{E}(x,\omega) = (iw)\mu \tilde{H}(x,\omega), \tag{5.1.6}$$

$$div_x(\tilde{E}(x,\omega)) = \frac{\tilde{\rho}(x,\omega)}{\epsilon}, \tag{5.1.7}$$

$$div_x(\tilde{H}(x,w)) = 0, \tag{5.1.8}$$

where $i^2 = -1$, and ω denotes the frequency and J, $\tilde{\rho}$, ϵ and μ are given.

$$curl_x \tilde{H}(x,\omega) = (\frac{\partial \tilde{H}_3}{\partial x_2} - \frac{\partial \tilde{H}_2}{\partial x_3}, \frac{\partial \tilde{H}_1}{\partial x_3} - \frac{\partial \tilde{H}_3}{\partial x_1}, \frac{\partial \tilde{H}_2}{\partial x_1} - \frac{\partial \tilde{H}_1}{\partial x_2}),$$

$$curl_x \tilde{E}(x,\omega) = (\frac{\partial \tilde{E}_3}{\partial x_2} - \frac{\partial \tilde{E}_2}{\partial x_3}, \frac{\partial \tilde{E}_1}{\partial x_3} - \frac{\partial \tilde{E}_3}{\partial x_1}, \frac{\partial \tilde{E}_2}{\partial x_1} - \frac{\partial \tilde{E}_1}{\partial x_2}),$$

$$div_x(\tilde{E}(x,\omega)) = \frac{\partial \tilde{E}_1}{\partial x_1} + \frac{\partial \tilde{E}_2}{\partial x_2} + \frac{\partial \tilde{E}_3}{\partial x_3},$$

$$div_x(\tilde{H}(x,w)) = \frac{\partial \tilde{H}_1}{\partial x_1} + \frac{\partial \tilde{H}_2}{\partial x_2} + \frac{\partial \tilde{H}_3}{\partial x_3}.$$

The Conservation law of charge is in the following form:

$$div_x \tilde{J}(x,w) + (-iw)\,\tilde{\rho}(x,w) = 0. \tag{5.1.9}$$

Remark:

$$\begin{aligned} F_t[H(x,t)](\omega) &= \int_{-\infty}^{\infty} H(x,t)e^{i\omega t}dt \\ &= \tilde{H}(x,\omega) \end{aligned}$$

$$\begin{aligned} F_t[\frac{\partial^m}{\partial t^m} H(x,t)](\omega) &= (-i\omega)^m F_t[H(x,t)](\omega) \\ &= (-i\omega)^m \tilde{H}(x,\omega) \end{aligned}$$

$$F_t[div_x(\epsilon E(x,t)] = div_x(\epsilon \tilde{E}(x,\omega))$$

where $\tilde{E}(x,\omega)$, be the Fourier transform image of the electric field $E(x,t)$, $\tilde{H}(x,\omega)$, be the Fourier transform image of the magnetic field $H(x,t)$.

Applying $curl$ operator to eqn (5.1.5) we find

$$curl curl \tilde{H}(x,\omega) - \omega^2 \epsilon(x)\mu \tilde{H}(x,\omega) = curl \tilde{J}(x,\omega),$$

$$(\nabla_x div_x - \Delta_x)\tilde{H}(x,\omega) - \omega^2 \epsilon(x)\mu \tilde{H}(x,\omega) = curl \tilde{J}(x,\omega).$$

From $\Rightarrow$ $div_x \tilde{H}(x,\omega) = 0$ we have

$$\Delta_x \tilde{H}(x,\omega) + \omega^2 \epsilon(x)\mu \tilde{H}(x,\omega) = -curl \tilde{J}(x,\omega), \tag{5.1.10}$$

$$curl_x \tilde{J}(x,\omega) = (\frac{\partial \tilde{J}_3}{\partial x_2} - \frac{\partial \tilde{J}_2}{\partial x_3}, \frac{\partial \tilde{J}_1}{\partial x_3} - \frac{\partial \tilde{J}_3}{\partial x_1}, \frac{\partial \tilde{J}_2}{\partial x_1} - \frac{\partial \tilde{J}_1}{\partial x_2})$$

from (5.1.5), we have

$$(-i\omega)\epsilon\tilde{E}(x,\omega) = curl_x\tilde{H}(x,\omega) - \tilde{J}(x,\omega)$$

or

$$\tilde{E}(x,\omega) = -\frac{1}{i\omega\epsilon}[curl_x\tilde{H}(x,\omega) - \tilde{J}(x,\omega)]. \tag{5.1.11}$$

We assume that the vector functions $\tilde{H}$, $\tilde{E}$ $\tilde{J}$ and the scalar function $\tilde{\rho}$ are independent of x_3. These functions depend on the variable x_1, x_2 and the frequency ω.

5.2 The Problem for the Maxwell's equations in a rectangle with the Perfect Conductor Conditions

Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle, Γ be the boundary of $\mathbb{D}$. We assume that $\tilde{J} = 0$ on the boundary Γ. Let's consider the system (5.1.5) - (5.1.8) in $\mathbb{D}$ with the following perfect conductor conditions on the boundary Γ:

$$\tilde{E} \times \nu = 0, \tag{5.2.1}$$

$$\tilde{H} \times \nu = J_s, \tag{5.2.2}$$

$$\epsilon\tilde{E} \cdot \nu = \rho_s, \tag{5.2.3}$$

$$\mu\tilde{H} \cdot \nu = 0, \tag{5.2.4}$$

where ν is a unit normal vector at the surface of the conductor (on the boundary Γ); J_s is the surface current density on the boundary Γ; ρ_s is the surface change density on the boundary Γ. The electric field is normal and the magnetic field tangential at the surface of a perfect conductor. From eqn (5.1.5) we have

$$curl\tilde{H} \times \nu = (-i\omega)\epsilon\tilde{E} \times \nu + \tilde{J} \times \nu.$$

Let us consider the last equation on the boundary Γ, i.e.

$$(curl\tilde{H} \times \nu)|_\Gamma = (-i\omega)\epsilon(\tilde{E} \times \nu)|_\Gamma + \tilde{J} \times \nu|_\Gamma.$$

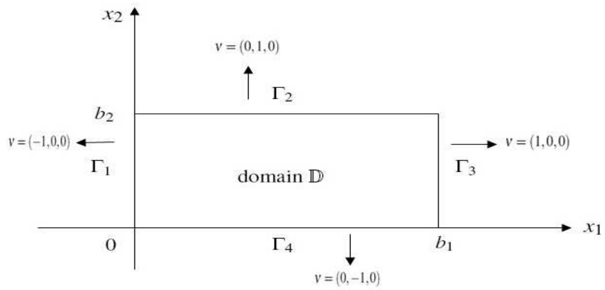
Since $\tilde{E} \times \nu = 0$, $\tilde{J} \times \nu = 0$, we find

$$(curl\tilde{H} \times \nu)|_\Gamma = 0. \tag{5.2.5}$$

We have from eqn (5.2.4) that

$$\tilde{H} \cdot \nu|_{\Gamma} = 0. \tag{5.2.6}$$

The boundary Γ may be presented in the form: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_1 = \{(x_1, x_2) : x_1 = 0,\, x_2 \in (0, b_2)\}$, $\Gamma_2 = \{(x_1, x_2) : x_1 \in (0, b_1),\, x_2 = b_2\}$, $\Gamma_3 = \{(x_1, x_2) : x_1 = b_1,\, x_2 \in (0, b_2)\}$, $\Gamma_4 = \{(x_1, x_2) : x_1 \in (0, b_1),\, x_2 = 0\}$.



Let's consider the normal unit vector ν on the $\Gamma_1 : \nu = (-1, 0, 0)$. The condition $\tilde{H} \cdot \nu|_{\Gamma_1} = 0$ can be written in the form:

$$\tilde{H} \cdot \nu|_{\Gamma_1} = 0 \;\Rightarrow\; (\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) \cdot (-1, 0, 0,)|_{\Gamma_1} = -\tilde{H}_1|_{\Gamma_1} = 0, x_2 \in (0, b_2). \tag{5.2.7}$$

$$curl\tilde{H} = (\frac{\partial \tilde{H}_3}{\partial x_2}, -\frac{\partial \tilde{H}_3}{\partial x_1}, \frac{\partial \tilde{H}_2}{\partial x_1} - \frac{\partial \tilde{H}_1}{\partial x_2}).$$

Using the independence $\tilde{H}_k$, k=1,2,3, from x_3 and the condition (5.1.5) we find

$$curl\tilde{H}|_{\Gamma_1} = (\frac{\partial \tilde{H}_3}{\partial x_2}, -\frac{\partial \tilde{H}_3}{\partial x_1}, \frac{\partial \tilde{H}_2}{\partial x_1})|_{\Gamma_1}.$$

Using(5.2.5), we find

$$curl\tilde{H}|_{\Gamma_1} \times \nu = 0 \;\Leftrightarrow\; (0, -\frac{\partial \tilde{H}_2}{\partial x_1}, -\frac{\partial \tilde{H}_3}{\partial x_1})|_{\Gamma_1} = 0,$$

or

$$\frac{\partial \tilde{H}_2}{\partial x_1}|_{\Gamma_1} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_1}|_{\Gamma_1} = 0.$$

As a result, we find the boundary condition on the Γ_1 as follows:

$$\tilde{H}_1|_{\Gamma_1} = 0, \quad \frac{\partial \tilde{H}_2}{\partial x_1}|_{\Gamma_1} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_1}|_{\Gamma_1} = 0. \tag{5.2.8}$$

Let's consider the normal unit vector $\nu = (1, 0, 0)$ on the Γ_3, we find from (5.2.6)

$$\tilde{H} \cdot \nu|_{\Gamma_3} = 0 \;\Rightarrow\; (\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) \cdot (1, 0, 0,)|_{\Gamma_3} = \tilde{H}_1|_{\Gamma_3} = 0. \tag{5.2.9}$$

Hence

$$curl\tilde{H}|_{\Gamma_3} = (\frac{\partial \tilde{H}_3}{\partial x_2}, -\frac{\partial \tilde{H}_3}{\partial x_1}, \frac{\partial \tilde{H}_2}{\partial x_1}).$$

Using (5.2.5), we find

$$(curl\tilde{H} \times \nu)|_{\Gamma_3} = 0 \Leftrightarrow (0, \frac{\partial \tilde{H}_2}{\partial x_1}, -\frac{\partial \tilde{H}_3}{\partial x_1})|_{\Gamma_3} = 0$$

or

$$\frac{\partial \tilde{H}_2}{\partial x_1}|_{\Gamma_3} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_1}|_{\Gamma_3} = 0.$$

As a result, we find the boundary condition on the Γ_3 as follows:

$$\tilde{H}_1|_{\Gamma_3} = 0, \quad \frac{\partial \tilde{H}_2}{\partial x_1}|_{\Gamma_3} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_1}|_{\Gamma_3} = 0. \tag{5.2.10}$$

Let's consider the normal unit vector $\nu = (0, 1, 0)$ on the Γ_2, we have from (5.2.6)

$$\tilde{H} \cdot \nu|_{\Gamma_2} = 0. \tag{5.2.11}$$

Applying $curl$ to $\tilde{H}$ we have

$$curl\tilde{H}|_{\Gamma_2} = (\frac{\partial \tilde{H}_3}{\partial x_2}, -\frac{\partial \tilde{H}_3}{\partial x_1}, -\frac{\partial \tilde{H}_1}{\partial x_2}).$$

Using(5.2.5), we find

$$(curl\tilde{H} \times \nu)|_{\Gamma_2} = 0.$$

This means that

$$\Leftrightarrow \frac{\partial \tilde{H}_1}{\partial x_2}|_{\Gamma_2} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_2}|_{\Gamma_2} = 0.$$

As a result, we find the boundary condition on the Γ_2 as follows:

$$\tilde{H}_2|_{\Gamma_2} = 0, \quad \frac{\partial \tilde{H}_1}{\partial x_2}|_{\Gamma_2} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_2}|_{\Gamma_2} = 0. \tag{5.2.12}$$

Let's consider the normal unit vector $\nu = (0, -1, 0)$ on the Γ_4.
Using (5.2.6) we have

$$\tilde{H} \cdot \nu|_{\Gamma_4} = 0. \tag{5.2.13}$$

We find $curl\tilde{H}$ on the Γ_4,

$$curl\tilde{H}|_{\Gamma_4} = (\frac{\partial \tilde{H}_3}{\partial x_2}, -\frac{\partial \tilde{H}_3}{\partial x_1}, -\frac{\partial \tilde{H}_1}{\partial x_2}).$$

From (5.2.5), we find

$$(curl\tilde{H} \times \nu)|_{\Gamma_4} = 0.$$

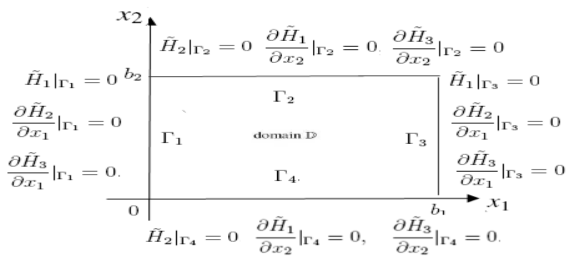
Hence

$$\frac{\partial \tilde{H}_1}{\partial x_2}|_{\Gamma_4} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_2}|_{\Gamma_4} = 0.$$

As a result, we find the boundary condition on the Γ_4 as follows:

$$\tilde{H}_2|_{\Gamma_4} = 0, \quad \frac{\partial \tilde{H}_1}{\partial x_2}|_{\Gamma_4} = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_2}|_{\Gamma_4} = 0. \tag{5.2.14}$$

From eqn (5.2.8), (5.2.10), (5.2.12), (5.2.14) and boundary conditions, we have the following three problems for the Helmholtz equation:



5.2.1 Mixed Problem 1 for the Helmholtz equation

Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle. Find $\tilde{H}_1(x_1, x_2)$ satisfying the Helmholtz equation

$$\frac{\partial^2 \tilde{H}_1}{\partial x_1^2} + \frac{\partial^2 \tilde{H}_1}{\partial x_2^2} + a^2 \tilde{H}_1 = f_1^{\tilde{H}}, \quad (x_1, x_2) \in \mathbb{D}, \tag{5.2.15}$$

and the following boundary conditions

$$\tilde{H}_1(0, x_2) = 0,\ \tilde{H}_1(b_1, x_2) = 0, \tag{5.2.16}$$

$$\frac{\partial \tilde{H}_1}{\partial x_2}(x_1, 0) = 0, \quad \frac{\partial \tilde{H}_1}{\partial x_2}(x_1, b_2) = 0, \tag{5.2.17}$$

where $a^2 = \epsilon\mu\omega^2$, $f_1^{\tilde{H}} = -(curl\tilde{J}_1) = -\frac{\partial \tilde{J}_3}{\partial x_2}$.

5.2.2 Mixed Problem 2 for the Helmholtz equation

Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle. Find $\tilde{H}_2(x_1, x_2)$ satisfying the Helmholtz equation

$$\frac{\partial^2 \tilde{H}_2}{\partial x_1^2} + \frac{\partial^2 \tilde{H}_2}{\partial x_2^2} + a^2 \tilde{H}_2 = f_2^{\tilde{H}}, \tag{5.2.18}$$

and the following boundary conditions

$$\frac{\partial \tilde{H}_2}{\partial x_1}(0, x_2) = 0, \quad \frac{\partial \tilde{H}_2}{\partial x_1}(b_1, x_2) = 0, \tag{5.2.19}$$

$$\tilde{H}_2(x_1, 0) = 0,\ \tilde{H}_2(x_1, b_2) = 0, \tag{5.2.20}$$

where $a^2 = \epsilon\mu\omega^2$, $f_2^{\tilde{H}} = -(curl \tilde{J}_2) = \frac{\partial \tilde{J}_3}{\partial x_1}$.

5.2.3 Neumann Problem 3 for the Helmholtz equation

Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle. Find $\tilde{H}_3(x_1, x_2)$ satisfying the Helmholtz equation

$$\frac{\partial^2 \tilde{H}_3}{\partial x_1^2} + \frac{\partial^2 \tilde{H}_3}{\partial x_2^2} + a^2 \tilde{H}_3 = f_3^{\tilde{H}}, \tag{5.2.21}$$

and the following boundary conditions

$$\frac{\partial \tilde{H}_3}{\partial x_1}(0, x_2) = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_1}(b_1, x_2) = 0, \tag{5.2.22}$$

$$\frac{\partial \tilde{H}_3}{\partial x_2}(x_1, 0) = 0, \quad \frac{\partial \tilde{H}_3}{\partial x_2}(x_1, b_2) = 0, \tag{5.2.23}$$

where $a^2 = \epsilon\mu\omega^2$, $f_3^{\tilde{H}} = -(curl \tilde{J}_3) = -\frac{\partial \tilde{J}_2}{\partial x_1} + \frac{\partial \tilde{J}_1}{\partial x_2}$.

5.2.4 Fourier Series Expansion Method for solving the Problem 1

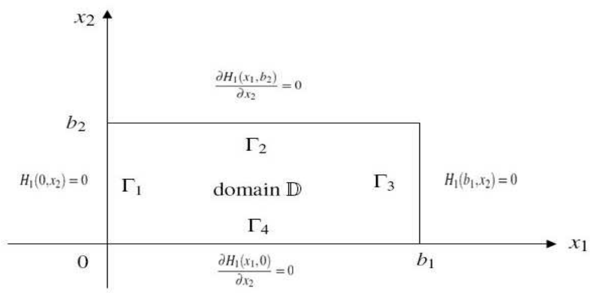
Let $b_1 > 0$, $b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following mixed problem for the Helmholtz equation in the rectangle $\mathbb{D}$:

$$\frac{\partial^2 \tilde{H}_1}{\partial x_1^2} + \frac{\partial^2 \tilde{H}_1}{\partial x_2^2} + a^2 \tilde{H}_1 = f_1^{\tilde{H}}, \qquad (x_1, x_2) \in \mathbb{D}, \tag{5.2.24}$$

$$\tilde{H}_1(0, x_2) = 0, \qquad \tilde{H}_1(b_1, x_2) = 0, \qquad 0 \le x_2 \le b_2, \tag{5.2.25}$$

$$\frac{\partial \tilde{H}_1(x_1, 0)}{\partial x_2} = 0, \quad \frac{\partial \tilde{H}_1(x_1, b_2)}{\partial x_2} = 0, \qquad 0 \le x_1 \le b_1, \tag{5.2.26}$$

$a > 0$ is a given real number, $f_1^{\tilde{H}}(x_1, x_2)$ is a given function. Find $\tilde{H}_1(x_1, x_2)$ satisfying (5.2.24) - (5.2.26).



The solution of this problem has several steps. We will find the solution in the form:

$$\tilde{H}_1(x_1, x_2) = \sum_{k=0}^{\infty} X(x_1) Y(x_2), \tag{5.2.27}$$

where $X_k(x_1)$, k=0,1,2,... are set of unknown functions; $Y_k(x_2)$, k=0,1,2,... are eigenfunctions of the following eigenvalue-eigenfunction problem:

$$Y''(x_2) + \lambda Y(x_2) = 0, \tag{5.2.28}$$

$$Y'(0) = 0, \quad Y'(b_2) = 0, \tag{5.2.29}$$

where $x_2 \in (0, b_2)$; $b_2 > 0$ is given number; $\lambda = \lambda_k = (\frac{k\pi}{b_2})^2$, for k=0,1,2,,... are eigenvalues corresponding to eigenfunctions

$$Y_0(x_2) = \sqrt{\frac{1}{b_2}}, \tag{5.2.30}$$

$$Y_k(x_2) = \sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2} x_2), \quad k = 1, 2, 3, ... \tag{5.2.31}$$

The Fourier series expansion of $f_1^{\tilde{H}}(x_1, x_2)$ is given in the form:

$$f_1^{\tilde{H}}(x_1, x_2) = \sum_{k=0}^{\infty} f_{1k}^{\tilde{H}}(x_1) Y_k(x_2), \tag{5.2.32}$$

where $f_{1k}^{\tilde{H}}(x_1) = \int_0^{b_2} f_1^{\tilde{H}}(x_1, x_2) Y_k(x_2) dx_2$. Substituting (5.2.27) and (5.2.32) into (5.2.24), we have

$$\sum_{k=0}^{\infty} [X_k''(x_1) Y_k(x_2) + X_k(x_1) Y_k''(x_2) + a^2 X_k(x_1) Y_k(x_2) - f_{1k}^{\tilde{H}}(x_1) Y_k(x_2)]$$

$$= \sum_{k=0}^{\infty} f_{1k}^{\tilde{H}}(x_1) Y_k(x_2)$$

or

$$\sum_{k=0}^{\infty} [X_k''(x_1) - \lambda_k X_k(x_1) + a^2 X_k(x_1) - f_{1k}^{\tilde{H}}(x_1)] Y_k(x_2)$$

$$= \sum_{k=0}^{\infty} f_{1k}^{\tilde{H}}(x_1) Y_k(x_2).$$

Since $\{Y_k(x_2)\}$ is orthonormal set of functions we find the following boundary value problem for unknown $X_k(x_1)$:

$$X_k''(x_1) + \alpha_k X_k(x_1) = f_{1k}^{\tilde{H}}(x_1),\ k = 0, 1, 2, ... \tag{5.2.33}$$

$$X_k(0) = 0,\ X_k(b_1) = 0, \tag{5.2.34}$$

where $x_1 \in (0, b_1)$. $\alpha_k = a^2 - \lambda_k$; $b_1 > 0$ is given number.

a)Let us consider the $Case\ 1 : \alpha_k > 0,\ k = 0, 1, 2, \ldots, n-1$. We find the general solution of (5.2.33), (5.2.34) in the form:

$$X_k(x_1) = \frac{1}{\sqrt{\alpha_k}} \left[\int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1 - \xi)) f_{1k}^{\tilde{H}}(\xi) d\xi \right.$$

$$\left. + \sin(\sqrt{\alpha_k} x_1) \int_0^{b_1} \frac{\sin(\sqrt{\alpha_k}(b_1 - \xi))}{\sin(\sqrt{\alpha_k} b_1)} f_{1k}^{\tilde{H}}(\xi) d\xi \right].$$

b)If $a^2 - \lambda_k = 0$, then a general solution of (5.2.33), (5.2.34) we find in the form:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi) f_n(\xi) d\xi - \frac{x_1}{b_1} \int_0^{b_1} (b_1 - \xi) f_{1n}^{\tilde{H}}(\xi) d\xi.$$

c)Let us consider the Case 2: $\alpha_k < 0,\quad k = n+1, n+2, \ldots$. We find the general solution of (5.2.33), (5.2.34) in the form:

$$X_k(x_1) = \frac{1}{\sqrt{|\alpha_k|}} \int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1 - \xi)) f_{1k}^{\tilde{H}}(\xi) d\xi$$

$$-\frac{\sinh(\sqrt{|\alpha_k|}x_1)}{\sqrt{|\alpha_k|}} \int_0^{b_1} \frac{\sinh(\sqrt{|\alpha_k|}(b_1 - \xi))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_{1k}^{\tilde{H}}(\xi) d\xi.$$

Finally, the solution of (5.2.24) - (5.2.26) is given by:

$$\begin{aligned}
\tilde{H}_1(x_1, x_2) &= \sum_{k=0}^{n-1} \frac{1}{\sqrt{\alpha_k}} \Big[\int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1 - \xi)) f_{1k}^{\tilde{H}}(\xi) d\xi \\
&+ \sin(\sqrt{\alpha_k}x_1) \int_0^{b_1} \frac{\sin(\sqrt{\alpha_k}(b_1 - \xi))}{\sin(\sqrt{\alpha_k}b_1)} f_{1k}^{\tilde{H}}(\xi) d\xi \Big] \\
&. \sqrt{\frac{2}{b_2}}. \cos(\frac{k\pi}{b_2} x_2) + \Big[\int_0^{x_1} (x_1 - \xi) f_{1n}^{\tilde{H}}(\xi) d\xi \\
&- \frac{x_1}{b_1} \int_0^{b_1} (b_1 - \xi) f_{1n}^{\tilde{H}}(\xi) d\xi \Big] \sqrt{\frac{1}{b_2}} \\
&+ \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{|\alpha_k|}} \Big[\int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1 - \xi)) f_{1k}^{\tilde{H}}(\xi) d\xi \\
&- \sinh(\sqrt{|\alpha_k|}x_1) \int_0^{b_1} \frac{\sinh(\sqrt{|\alpha_k|}(b_1 - \xi))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_{1k}^{\tilde{H}}(\xi) d\xi \Big]
\end{aligned}$$

$$\sqrt{\frac{2}{b_2}}. \cos(\frac{k\pi}{b_2} x_2). \tag{5.2.35}$$

5.2.5 Fourier Series Expansion Method for solving the Problem 2

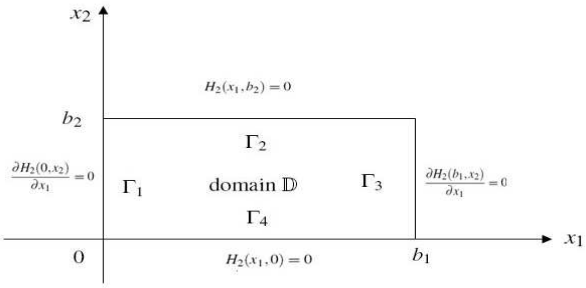
Let $b_1 > 0, b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\ 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following mixed problem for the Helmholtz equation in the rectangle $\mathbb{D}$:

$$\frac{\partial^2 \tilde{H}_2}{\partial x_1^2} + \frac{\partial^2 \tilde{H}_2}{\partial x_2^2} + a^2 \tilde{H}_2 = f_2^{\tilde{H}}, \qquad (x_1, x_2) \in \mathbb{D}, \tag{5.2.36}$$

$$\frac{\partial \tilde{H}_2(0, x_2)}{\partial x_1} = 0, \quad \frac{\partial \tilde{H}_2(b_1, x_2)}{\partial x_1} = 0, \qquad 0 \le x_1 \le b_1, \tag{5.2.37}$$

$$\tilde{H}_2(x_1, 0) = 0, \qquad \tilde{H}_2(x_1, b_2) = 0, \qquad 0 \le x_2 \le b_2, \tag{5.2.38}$$

where $a > 0$ is a given real number, $f_2^{\tilde{H}}(x_1, x_2)$ is a given function. Find $\tilde{H}_2(x_1, x_2)$ satisfying (5.2.36) - (5.2.38).



The solution of this problem has several steps. We will find the solution in the form:

$$\tilde{H}_2(x_1, x_2) = \sum_{k=0}^{\infty} X_k(x_1) Y_k(x_2), \tag{5.2.39}$$

where $X_k(x_1)$ k=0,1,2,... are set of unknown functions; $Y_k(x_2)$ k=0,1,2,... are eigenfunctions of the following eigenvalue-eigenfunction problem:

$$Y''(x_2) + \lambda Y(x_2) = 0, \tag{5.2.40}$$

$$Y(0) = 0, \quad Y(b_2) = 0, \tag{5.2.41}$$

where $x_2 \in (0, b_2)$; $b_2 > 0$ is given number; $\lambda = \lambda_k = (\frac{k\pi}{b_2})^2$, for k=0,1,2,... are eigenvalues corresponding to eigenfunctions

$$Y_k(0) = \sqrt{\frac{1}{b_2}}, \tag{5.2.42}$$

$$Y_k(x_2) = \sqrt{\frac{2}{b_2}} \sin(\sqrt{\lambda_k} x_2). \tag{5.2.43}$$

k=1,2,3,... The Fourier series expansion of $f_2^{\tilde{H}}(x_1, x_2)$ is given in the form:

$$f_2^{\tilde{H}}(x_1, x_2) = \sum_{k=0}^{\infty} f_{2k}^{\tilde{H}}(x_1) Y_k(x_2), \tag{5.2.44}$$

where $f_{2k}^{\tilde{H}}(x_1) = \int_0^{b_2} f_2^{\tilde{H}}(x_1, x_2) Y_k(x_2) dx_2$. Substituting (5.2.39) and (5.2.44) into (5.2.36), we have

$$\sum_{k=0}^{\infty} [X_k''(x_1) Y_k(x_2) + X_k(x_1) Y_k''(x_2) + a^2 X_k(x_1) Y_k(x_2) - f_{2k}^{\tilde{H}}(x_1) Y_k(x_2)]$$

$$= \sum_{k=0}^{\infty} f_{2k}^{\tilde{H}}(x_1) Y_k(x_2)$$

or

$$\sum_{k=0}^{\infty} [X_k''(x_1) - \lambda_k X_k(x_1) + a^2 X_k(x_1) - f_{2k}^{\tilde{H}}(x_1)] Y_k(x_2)$$

$$= \sum_{k=0}^{\infty} f_{2k}^{\tilde{H}}(x_1) Y_k(x_2).$$

Since $\{Y_k(x_2)\}$ is orthonormal set of functions, we find the following boundary value problem for unknown $X_k(x_1)$:

$$X_k''(x_1) + \alpha_k X_k(x_1) = f_{2k}^{\tilde{H}}(x_1),\ k = 0, 1, 2, ... \tag{5.2.45}$$

$$X_k'(0) = 0,\ X_k'(b_1) = 0, \tag{5.2.46}$$

where $x_1 \in (0, b_1)$. $\alpha_k = a^2 - \lambda_k$; $b_1 > 0$ is given number.

a)Let us consider the $Case\ 1:\ \ \alpha_k > 0,\ k = 0, 1, 2, \ldots, n-1$. We find the general solution of (5.2.45), (5.2.46) in the form:

$$X_k(x_1) = \frac{1}{\sqrt{\alpha_k}} \int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1 - \xi)) f_{2k}^{\tilde{H}}(\xi) d\xi$$

$$- \frac{\cos(\sqrt{\alpha_k} x_1)}{\sqrt{\alpha_k}} \int_0^{b_1} \frac{\cos(\sqrt{\alpha_k}(b_1 - \xi))}{\sin(\sqrt{\alpha_k} b_1)} f_{2k}^{\tilde{H}}(\xi) d\xi.$$

b)If $a^2 - \lambda_k = 0$, then we find the general solution of (5.2.45), (5.2.46) in the form:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi) f_{2n}^{\tilde{H}}(\xi) d\xi + d_n.$$

Remark: The necessary condition for existence of the solution:

$$\int_0^{b_1} f_{2n}^{\tilde{H}}(r) dr = 0.$$

We find the following result of $X_n(x_1)$:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi) f_{2n}^{\tilde{H}}(\xi) d\xi + d_n,$$

where d_n is an arbitrary constant.

c)Let us consider the Case 2: $\alpha_k < 0,\ k = n+1, n+2, \ldots$. We find the general solution of (5.2.45), (5.2.46) in the form:

$$X_k(x_1) = \frac{1}{\sqrt{|\alpha_k|}} \int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1-\xi)) f_{2k}^{\tilde{H}}(\xi) d\xi$$

$$+\frac{\sinh(\sqrt{|\alpha_k|}x_1)}{\sqrt{|\alpha_k|}} \int_0^{b_1} \frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_{2k}^{\tilde{H}}(\xi) d\xi.$$

Finally, the solution of (5.2.36) - (5.2.38) is given by:

$$\begin{aligned}
\tilde{H}_2(x_1,x_2) &= \sum_{k=0}^{n-1} \frac{1}{\sqrt{\alpha_k}} \Big[\int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1-\xi)) f_{2k}^{\tilde{H}}(\xi) d\xi \\
&- \cos(\sqrt{\alpha_k}x_1) \int_0^{b_1} \frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)} f_{2k}^{\tilde{H}}(\xi) d\xi \Big] \\
&. \sqrt{\frac{2}{b_2}}. \sin(\frac{k\pi}{b_2}x_2) + \Big[\int_0^{x_1} (x_1-\xi) f_{2n}^{\tilde{H}}(\xi) d\xi + d_n \Big] \sqrt{\frac{1}{b_2}} \\
&+ \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{|\alpha_k|}} \Big[\int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1-\xi)) f_{2k}^{\tilde{H}}(\xi) d\xi \\
&+ \sinh(\sqrt{|\alpha_k|}x_1) \int_0^{b_1} \frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)} f_{2k}^{\tilde{H}}(\xi) d\xi \Big]
\end{aligned}$$

$$\sqrt{\frac{2}{b_2}}. \sin(\frac{k\pi}{b_2}x_2). \tag{5.2.47}$$

5.2.6 Fourier Series Expansion Method for Solving the Problem 3

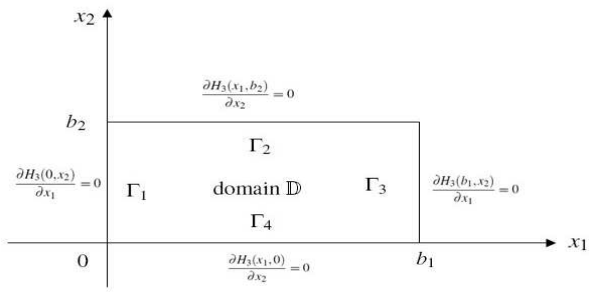
Let $b_1 > 0,\, b_2 > 0$ be given numbers; $\mathbb{D} = \{(x_1,x_2) \in \mathbf{R}^2 : 0 < x_1 < b_1,\, 0 < x_2 < b_2\}$ be a rectangle. Let's consider the following Neumann problem for the Helmholtz equation in the rectangle $\mathbb{D}$:

$$\frac{\partial^2 \tilde{H}_3}{\partial x_1^2} + \frac{\partial^2 \tilde{H}_3}{\partial x_2^2} + a^2 \tilde{H}_3 = f_3^{\tilde{H}}, \qquad (x_1,x_2) \in \mathbb{D}, \tag{5.2.48}$$

$$\frac{\partial \tilde{H}_3(0,x_2)}{\partial x_1} = 0, \quad \frac{\partial \tilde{H}_3(b_1,x_2)}{\partial x_1} = 0, \qquad 0 \le x_2 \le b_2, \tag{5.2.49}$$

$$\frac{\partial H_3(x_1,0)}{\partial x_2} = 0, \quad \frac{\partial \tilde{H}_3(x_1,b_2)}{\partial x_2} = 0, \qquad 0 \le x_1 \le b_1, \tag{5.2.50}$$

where $a > 0$ is a given real number, $f_3^{\tilde{H}}$ is a given function. Find $\tilde{H}_3(x_1,x_2)$ satisfying (5.2.48) - (5.2.50).



The solution of this problem has several steps. We will find the solution in the form:

$$\tilde{H}_3(x_1,x_2)=\sum_{k=0}^{\infty}X_k(x_1)Y_k(x_2), \tag{5.2.51}$$

where $X_k(x_1)$, k=0,1,2,... are set of unknown functions; $Y_k(x_2)$, k=0,1,2,... are eigenfunctions of the following eigenvalue-eigenfunction problem:

$$Y''(x_2)+\lambda Y(x_2)=0, \tag{5.2.52}$$

$$Y'(0)=0, \quad Y'(b_2)=0, \tag{5.2.53}$$

where $x_2\in(0,b_2)$; $b_2>0$ is given number; $\lambda=\lambda_k=(\frac{k\pi}{b_2})^2$, for k=0,1,2,... are eigenvalues corresponding to eigenfunctions

$$Y_0(x_2)=\sqrt{\frac{1}{b_2}},\ k=0, \tag{5.2.54}$$

$$Y_k(x_2)=\sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2}x_2), \quad k=1,2,3,.... \tag{5.2.55}$$

The Fourier series expansion of $f_3^{\tilde{H}}(x_1,x_2)$ is given in the form:

$$f_3^{\tilde{H}}(x_1,x_2)=\sum_{k=0}^{\infty}f_{3k}^{\tilde{H}}(x_1)Y_k(x_2), \tag{5.2.56}$$

where $f_{3k}^{\tilde{H}}(x_1)=\int_0^{b_2}f_3^{\tilde{H}}(x_1,x_2)Y_k(x_2)dx_2$. Substituting (5.2.51) and (5.2.56) into (5.2.48), we have

$$\sum_{k=0}^{\infty}[X_k''(x_1)-\lambda_kX_k(x_1)+a^2X_k(x_1)-f_{3k}^{\tilde{H}}(x_1)]Y_k(x_2)$$

$$=\sum_{k=0}^{\infty}f_{3k}^{\tilde{H}}(x_1)Y_k(x_2).$$

Since $\{Y_k(x_2)\}$ is orthonormal set of functions, we find the following boundary value problem $X_k(x_1)$:

$$X_k''(x_1) + \alpha_k X_k(x_1) = f_{3k}^{\tilde{H}}(x_1),\ k = 0, 1, 2, ... \tag{5.2.57}$$

$$X_k'(0) = 0,\ X_k'(b_1) = 0, \tag{5.2.58}$$

where $x_1 \in (0, b_1)$. $\alpha_k = a^2 - \lambda_k$; $b_1 > 0$ is given number.

a)Let us consider the $Case\ 1:\ \alpha_k > 0,\ k = 0, 1, 2, \ldots, n-1$. We find the general solution of (5.2.57), (5.2.58) in the form:

$$\begin{aligned} X_k(x_1) &= \frac{1}{\sqrt{\alpha_k}}[\int_0^{x_1} \sin(\sqrt{\alpha_k}(x_1 - \xi)) f_{3k}^{\tilde{H}}(\xi) d\xi \\ &+ \cos(\sqrt{\alpha_k} x_1) \int_0^{b_1} \frac{\cos(\sqrt{\alpha_k}(b_1 - \xi))}{\sin(\sqrt{\alpha_k} b_1)} f_{3k}^{\tilde{H}}(\xi) d\xi]. \end{aligned}$$

b)If $a^2 - \lambda_k = 0$, then we find the general solution of (5.2.57), (5.2.58) in the form:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi) f_{3n}^{\tilde{H}}(\xi) d\xi + d_n.$$

Remark: The necessary condition for existence of the solution:

$$\int_0^{b_1} f_{2n}^{\tilde{H}}(r) dr = 0.$$

We find the following result of $X_n(x_1)$:

$$X_n(x_1) = \int_0^{x_1} (x_1 - \xi) f_{2n}^{\tilde{H}}(\xi) d\xi + d_n,$$

where d_n is an arbitrary constant.

c)Let us consider the Case 2: $\alpha_k < 0,\ k = n+1, n+2, \ldots$. We find the general solution of (5.2.57), (5.2.58) in the form:

$$X_k(x_1) = \frac{1}{\sqrt{|\alpha_k|}}[\int_0^{x_1} \sinh(\sqrt{|\alpha_k|}(x_1 - \xi)) f_{3k}^{\tilde{H}}(\xi) d\xi$$

$$- \cosh(\sqrt{|\alpha_k|} x_1) \int_0^{b_1} \frac{\cosh(\sqrt{|\alpha_k|}(b_1 - \xi))}{\sinh(\sqrt{|\alpha_k|} b_1)} f_{3k}^{\tilde{H}}(\xi) d\xi]. \tag{5.2.59}$$

Finally, the solution of (5.2.48) - (5.2.50) is given by:

$$\begin{aligned}
\tilde{H}_3(x_1,x_2) &= \sum_{k=0}^{n-1}\frac{1}{\sqrt{\alpha_k}}\Big[\int_0^{x_1}\sin(\sqrt{\alpha_k}(x_1-\xi))f_{3k}^{\tilde{H}}(\xi)d\xi \\
&+ \cos(\sqrt{\alpha_k}x_1)\int_0^{b_1}\frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)}f_{3k}^{\tilde{H}}(\xi)d\xi\Big]\sqrt{\frac{2}{b_2}} \\
&. \cos(\frac{k\pi}{b_2}x_2)+\Big[\int_0^{x_1}(x_1-\xi)f_{3n}^{\tilde{H}}(\xi)d\xi+d_n\Big]\sqrt{\frac{1}{b_2}} \\
&+ \sum_{k=n+1}^{\infty}\frac{1}{\sqrt{|\alpha_k|}}\Big[\int_0^{x_1}\sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_{3k}^{\tilde{H}}(\xi)d\xi \\
&- \cosh(\sqrt{|\alpha_k|}x_1)\int_0^{b_1}\frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)}f_{3k}^{\tilde{H}}(\xi)d\xi\Big]
\end{aligned}$$

$$\sqrt{\frac{2}{b_2}}.\cos(\frac{k\pi}{b_2}x_2). \tag{5.2.60}$$

5.3 Finding Formulae for Electric Field

From eqn (5.1.11),

$$\tilde{E}(x,\omega) = -\frac{1}{i\omega\epsilon}[curl_x\tilde{H}(x,\omega)-\tilde{J}(x,\omega)]$$

and $curl\tilde{H}$

$$curl\tilde{H} = (\frac{\partial\tilde{H}_3}{\partial x_2}, -\frac{\partial\tilde{H}_3}{\partial x_1}, \frac{\partial\tilde{H}_2}{\partial x_1}-\frac{\partial\tilde{H}_1}{\partial x_2})$$

we have the following three formulaes for the electric field $\tilde{E}(x,\omega)$:

$$\tilde{E}_1(x,\omega) = -\frac{1}{i\omega\epsilon}[\frac{\partial\tilde{H}_3}{\partial x_2}-\tilde{J}_1(x,\omega)], \tag{5.3.1}$$

$$\tilde{E}_2(x,\omega) = -\frac{1}{i\omega\epsilon}[-\frac{\partial\tilde{H}_3}{\partial x_1}-\tilde{J}_2(x,\omega)], \tag{5.3.2}$$

$$\tilde{E}_3(x,\omega) = -\frac{1}{i\omega\epsilon}[\frac{\partial\tilde{H}_2}{\partial x_1}-\frac{\partial\tilde{H}_1}{\partial x_2}-\tilde{J}_3(x,\omega)]. \tag{5.3.3}$$

As a result,we find the following formulaes:

$$
\begin{aligned}
\tilde{E}_1(x,\omega) &= -\frac{1}{i\omega\epsilon}\Big\{\sum_{k=0}^{n-1} -\frac{k\pi}{b_2}\sqrt{\frac{2}{b_2}}\frac{1}{\sqrt{\alpha_k}}\Big[\int_0^{x_1}\sin(\sqrt{\alpha_k}(x_1-\xi))f_{3k}^{\tilde{H}}(\xi)d\xi \\
&+ \cos(\sqrt{\alpha_k}x_1)\int_0^{b_1}\frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)}f_{3k}^{\tilde{H}}(\xi)d\xi\Big] \\
&. \sin(\frac{k\pi}{b_2}x_2) + \sum_{k=n+1}^{\infty} -\frac{k\pi}{b_2}\sqrt{\frac{2}{b_2}}\frac{1}{\sqrt{|\alpha_k|}} \\
&. \Big[\int_0^{x_1}\sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_{3k}^{\tilde{H}}(\xi)d\xi \\
&- \cosh(\sqrt{|\alpha_k|}x_1)\int_0^{b_1}\frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)}f_{3k}^{\tilde{H}}(\xi)d\xi\Big] \\
&\qquad \sin(\frac{k\pi}{b_2}x_2) - \tilde{J}_1(x,\omega)\Big\}. \qquad (5.3.4)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{E}_2(x,\omega) &= -\frac{1}{i\omega\epsilon}\Big\{-\Big[\sum_{k=0}^{n-1}\sqrt{\frac{2}{b_2}}\int_0^{x_1}\cos(\sqrt{\alpha_k}(x_1-\xi))f_{3k}^{\tilde{H}}(\xi)d\xi \\
&- \sqrt{\alpha_k}\sin(\sqrt{\alpha_k}x_1)\int_0^{b_1}\frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)}f_{3k}^{\tilde{H}}(\xi)d\xi\Big]\cos(\frac{k\pi}{b_2}x_2) \\
&- \sqrt{\frac{1}{b_2}}\int_0^{x_1}f_{3n}^{\tilde{H}}(\xi)d\xi - \sum_{k=n+1}^{\infty}\sqrt{\frac{2}{b_2}}\Big[\int_0^{x_1}\cosh(\sqrt{|\alpha_k|}(x_1-\xi))f_{3k}^{\tilde{H}}(\xi)d\xi \\
&- \sqrt{|\alpha_k|}\sinh(\sqrt{|\alpha_k|}x_1)\int_0^{b_1}\frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)}f_{3k}^{\tilde{H}}(\xi)d\xi\Big] \\
&\qquad .\cos(\frac{k\pi}{b_2}x_2) - \tilde{J}_2(x,\omega)\Big\}. \qquad (5.3.5)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{E}_3(x,\omega) &= -\frac{1}{i\omega\epsilon}\Bigg\{\Bigg[\sum_{k=0}^{n-1}\sqrt{\frac{2}{b_2}}\int_0^{x_1}\cos(\sqrt{\alpha_k}(x_1-\xi))f_{2k}^{\tilde{H}}(\xi)d\xi \\
&+ \sqrt{\alpha_k}\sin(\sqrt{\alpha_k}x_1)\int_0^{b_1}\frac{\cos(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)}f_{2k}^{\tilde{H}}(\xi)d\xi\Bigg] \\
&. \sin(\frac{k\pi}{b_2}x_2)+\sqrt{\frac{1}{b_2}}\int_0^{x_1}f_{2n}^{\tilde{H}}(\xi)d\xi \\
&- \sum_{k=n+1}^{\infty}\sqrt{\frac{2}{b_2}}\Bigg[\int_0^{x_1}\cosh(\sqrt{|\alpha_k|}(x_1-\xi))f_{2k}^{\tilde{H}}(\xi)d\xi \\
&+ \sqrt{|\alpha_k|}\cosh(\sqrt{|\alpha_k|}x_1)\int_0^{b_1}\frac{\cosh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)}f_{2k}^{\tilde{H}}(\xi)d\xi\Bigg] \\
&. \sin(\frac{k\pi}{b_2}x_2)+\sum_{k=0}^{n-1}\frac{k\pi}{b_2}\sqrt{\frac{2}{b_2}}\frac{1}{\sqrt{\alpha_k}}\Bigg[\int_0^{x_1}\sin(\sqrt{\alpha_k}(x_1-\xi))f_{1k}^{\tilde{H}}(\xi)d\xi \\
&+ \sin(\sqrt{\alpha_k}x_1)\int_0^{b_1}\frac{\sin(\sqrt{\alpha_k}(b_1-\xi))}{\sin(\sqrt{\alpha_k}b_1)}f_{1k}^{\tilde{H}}(\xi)d\xi\Bigg] \\
&. \sin(\frac{k\pi}{b_2}x_2)+\sum_{k=n+1}^{\infty}\frac{k\pi}{b_2}\sqrt{\frac{2}{b_2}}\frac{1}{\sqrt{|\alpha_k|}}\Bigg[\int_0^{x_1}\sinh(\sqrt{|\alpha_k|}(x_1-\xi))f_{1k}^{\tilde{H}}(\xi)d\xi \\
&- \sinh(\sqrt{|\alpha_k|}x_1)\int_0^{b_1}\frac{\sinh(\sqrt{|\alpha_k|}(b_1-\xi))}{\sinh(\sqrt{|\alpha_k|}b_1)}f_{1k}^{\tilde{H}}(\xi)d\xi\Bigg] \\
&\qquad . \sin(\frac{k\pi}{b_2}x_2)-\tilde{J}_3(x,\omega)\Bigg\}. \qquad (5.3.6)
\end{aligned}
$$

CHAPTER SIX
CONCLUSION

The main result of this thesis are the following;

- The inhomogeneous Poisson and Helmholtz equations with the Dirichlet, Neumann and mixed boundary conditions in given rectangular domains have been solved by the Fourier series expansion method.
- The Fourier series expansion method is implemented for solving Maxwell's equations with perfect conductor boundary conditions in a rectangular domain.
- Maxwell's equations with given boundary conditions have been reduced to several mixed problem for the inhomogeneous Helmholtz equations.
- The obtained problems have been solved using the Fourier series expansion method.
- Solutions of all problems are presented explicitly in the form of the Fourier series expansion.

REFERENCES

Chew, W.C. (1990).*Waves and Fields in Inhomogeneous Media*. *Van Nostrand Reinhold*, New York.

Courant, R., & Hilbert, D. (1962). *Methods of mathematical physics*, *vol. 2*. *Interscience*, New York.

Eom H.J. (2003). *Electromagnetic Wave Theory for Boundary-Value Problems*. *Springer*, Berlin.

Goldberg, J.L. (1992). *Matrix theory with applications*. *McGraw-Hill International Editions*.

Kong J.A. (1986). *Electromagnetic Wave Theory*. *Wiley*, New York.

Monk P. (2003). *Finite Element Methods for Maxwell's Equations*. *Clarendon Press*, Oxford.

Pinsky, Mark. A. (1991). *Partial Differential Equations and Boundary- Value Problems with Applications*. *McGrawHill Inc.*.

Ramo S., & Whinnery J.R., & Duzer T. (1994). *Fields and Waves in Communication Electronics*. *Wiley*, New York.

Tai C.T. (1994). Dyadic Green's Functions in Electromagnetic Theory. *IEES Press*, New Jersey.

Tikhanow, A. N. & Samarskii, A. A. (1963). *Equations of Mathematical Physics*. *Pergamon Press*, London.

Vagner D., & Lembrikov B.I., & Wyder P. (2004). Electrodynamics of Magnetoactive Media. *Springer*, New York.

Vladimirov, V.S. (1971). *Equations of Mathematical Physics*. *Dekker*, New York.

Yakhno,V.G. (2008). Computing and simulation of time-dependent electromagnetic fields in homogeneous anisotropic materials. *International Journal of Engineering Science*. (46), 411-426.