

CONTINUOUS REVIEW LIMITED BACKORDER (r, Q) INVENTORY SYSTEM
WITH STRATEGIC CUSTOMERS AND INFORMATION AVAILABILITY

by

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to whom I owe everything-my love and father, Deniz



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ABSTRACT

CONTINUOUS REVIEW LIMITED BACKORDER (r, Q) INVENTORY SYSTEM WITH STRATEGIC CUSTOMERS AND INFORMATION AVAILABILITY

The motivation of this study is to characterize the equilibrium behavior of a continuous review, limited-backorder (r, Q) inventory model that accommodates heterogeneous, waiting-time-sensitive customers. We assume that arriving customers may be presented with one of three information levels upon arrival: no information, stock-out information, or complete stock-availability information. The cumulative potential arrival process is modeled as a homogeneous Poisson process; however, the interaction of heterogeneous waiting sensitivities and varying information availability results in a state-dependent Poisson process in equilibrium. Both deterministic and exponential procurement lead-time settings are investigated.

First, we derive the conditional waiting-time distribution for a continuous review, unlimited-backorder (r, Q) policy with homogeneous Poisson arrivals and deterministic procurement lead times, an analysis that, to the best of our knowledge, does not currently exist in the literature.

Second, for any continuous review (r, Q) model, we introduce an analogous bulk queueing representation. Using the supplementary variable technique, we formulate the Forward Kolmogorov equations corresponding to this bulk queueing system, which features a state-dependent Poisson arrival process, an infinite number of servers, fixed batch service of size Q , and deterministic service times. We obtain closed-form solutions to these equations in the case where at most one server is operational. For cases

involving multiple operational servers, we develop a finite-difference-based numerical technique to solve the equations and approximate steady-state behavior. We note that the proposed analogy, the derived Forward Kolmogorov equations, the analytical solutions for the single-server case, and the numerical technique are all novel contributions to the literature.

Next, we focus on the continuous review, limited-backorder (r, Q) model with deterministic procurement lead times. Arriving customers are assumed to be presented with one of the three information levels: no information, stock-out information, or complete stock availability. Customers are heterogeneous with respect to their waiting-time sensitivity and upon arrival, decide strategically whether to purchase or balk based on the available information and the state of the system. For the case with at most one outstanding order, we provide exact equilibrium behavior at arrival, departure, and arbitrary epochs, as well as exact characterizations of inventory metrics and conditional or unconditional waiting-time distributions. For cases with multiple outstanding orders, we employ our finite-difference-based technique to obtain satisfactory numerical approximations for the equilibrium behavior, inventory metrics, and waiting-time characterizations.

Finally, for the exponential lead-time case with at most one outstanding order, we derive the exact equilibrium behavior, inventory metrics, and conditional or unconditional waiting-time characterizations. For cases with multiple outstanding orders, we propose a combinatorial method to determine the exact equilibrium behavior and waiting-time distributions. We note that the equilibrium behaviors and waiting-time characterizations developed in this study do not appear to exist in the existing literature.

ÖZET

BEKLEME SÜRESİNE DUYARLI HETEROJEN MÜŞTERİLERİ İÇEREN SÜREKLİ İZLENEN SINIRLI STOK GERİ SİPARİŞLİ (r, Q) ENVANTER MODELİ

Bu çalışmanın motivasyonu, bekleme süresine duyarlı heterojen müşterileri içeren sürekli gözden geçirmeli, sınırlı stok geri siparişli (r, Q) envanter modelinin denge davranışını karakterize etmektir. Çalışmada, müşterilerin sisteme geldiklerinde üç bilgi düzeyinden biriyle karşılaşabilecekleri varsayılmaktadır: bilgi yokluğu, stok tükenme bilgisi veya tam stok durumu bilgisi. Toplam potansiyel müşteri geliş süreci homojen bir Poisson süreci olarak modellenmiştir; ancak, heterojen bekleme duyarlılıkları ile bilgiye erişim farklılıklarının etkileşimi sonucu, dengede duruma bağlı bir Poisson süreci ortaya çıkmaktadır. Çalışmada hem deterministik hem de üstel tedarik süresi durumları incelenmiştir.

Öncelikle, homojen Poisson müşteri gelişleri ve deterministik tedarik süreleri altında, sürekli gözden geçirmeli ve sınırsız geri siparişli (r, Q) politikasında bekleme süresinin koşullu dağılımı türetilmiştir. Bildiğimiz kadarıyla, bu analiz literatürde daha önce yer almamaktadır.

İkinci olarak, herhangi bir sürekli gözden geçirmeli (r, Q) modeli için buna karşılık gelen analogik bir toplu (bulk) kuyruk sistemi sunulmuştur. Ek değişken tekniği kullanılarak, bu toplu kuyruk sistemi için ileri Kolmogorov denklemleri formüle edilmiştir. Bu sistem, duruma bağlı Poisson geliş süreci, sonsuz sayıda sunucu, her birinin sabit büyüklükte Q müşteriden oluşan toplu hizmet verdiği deterministik hizmet süreleri ile tanımlanmaktadır. En fazla bir sunucunun çalıştığı durumda bu denklemlere kapalı

formda çözümler elde edilmiştir. Birden fazla sunucunun faal olduğu durumlarda ise, denklemleri çözmek ve durağan davranışı yaklaşık olarak elde etmek için sonlu farklar tabanlı sayısal bir yöntem geliştirilmiştir. Sunulan analogi, türetilen ileri Kolmogorov denklemleri, tek sunuculu durumda elde edilen analitik çözümler ve geliştirilen sayısal yöntem literatüre yeni katkılar olarak değerlendirilmektedir.

Sonrasında, deterministik tedarik sürelerine sahip sürekli gözden geçirmeli, sınırlı geri siparişli (r, Q) modeli incelenmiştir. Burada da, müşterilerin sisteme geldiklerinde üç bilgi düzeyinden biriyle karşılaştıkları varsayılmaktadır: bilgi yokluğu, stok tükenme bilgisi veya tam stok bilgisi. Müşteriler, bekleme sürelerine olan duyarlılıkları ile heterojendir ve mevcut bilgi düzeyi ile sistem durumuna bağlı olarak satın alma veya vazgeçme kararı vermektedirler. En fazla bir açık siparişin bulunduğu durumda, sistemin geliş, ayrılış ve keyfi zaman noktalarındaki tam denge davranışı, stokla ilgili ölçütler (envanter metrikleri) ve koşullu/koşulsuz bekleme süresi karakterizasyonları tam olarak verilmiştir. Birden fazla açık siparişin bulunduğu durumlarda ise, denge davranışını, stok metriklerini ve bekleme süresi karakterizasyonlarını yaklaşık olarak elde etmek için sonlu farklara dayalı sayısal yöntemimiz kullanılmıştır.

Son olarak, üstel tedarik süresine sahip ve en fazla bir açık siparişin bulunduğu durumda, tam denge davranışı, stok metrikleri ve koşullu/koşulsuz bekleme süresi karakterizasyonları türetilmiştir. Birden fazla açık siparişin bulunduğu durumlar için ise, tam denge davranışı ile bekleme süresi dağılımlarını elde etmek üzere kombinatoryal bir yöntem önerilmiştir. Bu çalışmada geliştirilen denge davranışları ve bekleme süresi karakterizasyonlarının mevcut literatürde yer almadığı görülmektedir.

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LIST OF SYMBOLS

$\mathbf{A}_{ij}$	$(i, j)^{th}$ entry of the generator matrix $\mathbf{A}$
$\mathcal{B}$	Number of backorders allowed
$\mathcal{B}$	Random vector of path IN takes to hit r after a replenishment order arrives
C	Random variable denoting unit waiting cost
$\mathcal{C}$	Expected cycle time in the inventory system
$\mathcal{C}$	Capacity of a queueing system
$\mathbf{d}$	Vector of unknown discretized steady-state density values
$F(x)$	The distribution function of C
$f_{n,m}(\cdot; t)$	Transient density function corresponding to the system state with n customers in queue and m servers active at time t
$G(x)$	The distribution function of lead-time random variable
h	Holding cost per unit per unit of time
$h(x)$	The hazard rate function of lead-time random variable
IN	Net inventory at the steady-state
$IN(t)$	Net inventory at time t
IO	Number of outstanding items at the steady-state
$IO(t)$	Number of outstanding items at time t
IP	Inventory position at the steady-state
$IP(t)$	Inventory position at time t
K	Fixed ordering cost
$\mathbf{L}$	Lead-time random variable
L	Expectation of lead-time random variable
$\mathcal{L}$	Laplace Transformation operator
$\mathbf{M}$	Linearization matrix
M_n	State-dependent Poisson process
$\mathcal{N}$	Expected number of orders given per unit of time
Q	Fixed order quantity

r	Reorder level
R	Valuation of the item by customers
$(r, Q, \mathcal{B})$	(r, Q) inventory policy with maximum number of backorders
$\mathcal{B}$	
p	The price of the item
$p_{n,m}(\cdot)$	Steady-state density function corresponding to the system state with n customers in queue and m servers active
$\mathbf{v}$	Normalizing column vector
v	Retailer's purchase price
W_n^a	Random variable denoting waiting time of a customer who arrived when $IN = n$
β	Supremum of the support of C
Δ	Random variable denoting the time it takes for IN to hit r once a replenishment arrives
Λ	Potential arrival rate per unit of time
λ_n	Arrival rate per unit of time when $IN = n$
Ξ_i	Domain of the steady-state density function which expresses density of i customers in the queueing system
Π	Steady-state vector of probabilities at arbitrary epochs
Π^a	Steady-state vector of probabilities at arrival epochs
Π^d	Steady-state vector of probabilities at departure epochs
π_n	Steady-state probability of observing n customers in the system at an arbitrary instance
π_n^a	Steady-state probability of observing n customers in the system at an arrival instance
π_n^d	Steady-state probability of observing n customers in the system at a departure instance
ω_j^*	Random vector of paths that IN takes until a customer who observes $IN = j$ just consequent to their arrival obtains their product

LIST OF ACRONYMS/ABBREVIATIONS

cdf	Cumulative distribution function
FIFO	First in first out
FKE	Forward Kolmogorov equations
PASTA	Poisson Arrivals See Time Averages
pdf	Probability density function



1. INTRODUCTION

Most inventory systems experience intense arrival of customers that have unit-sized demands, but replenishment orders are placed in bulk. The reason behind this phenomenon is that generally the fixed-cost of ordering is higher compared to unit item price. In addition, it may not be possible to place frequent orders of small size due to various exogenous factors. Hence, inventory policies with bulk replenishments are practically useful for systems involving a high demand rate and a considerably high fixed-cost of ordering. One type of such ordering policy is called the reorder point and fixed-order quantity model, also known as the (r, Q) policy where r denotes the reorder point and Q denotes the fixed-order quantity. In the continuous review setting, an order of fixed size Q is placed, which arrives according to the procurement lead time distribution, whenever the sum of products in stock, customers waiting and products that will arrive equals r .

With the increasing rate of availability of information in service systems, the quantification of the effect of providing customers with different levels of information is sought. Depending on the service provider and the system, an arriving customer may be provided with different levels of information. As an example, in an inventory system, upon arrival, a customer may not be provided with any information, may only be provided with the information that there is available stock or not, or complete stock availability information. Furthermore, heterogeneous customer impatience is a valuable phenomenon to incorporate into mathematical models to depict reality. Heterogeneous customer impatience may be modeled as the heterogeneity of customer valuations per unit of waiting time they incur. Combining the information availability concept with the customer perspective on waiting sensitivity, an arriving customer decides to obtain service if their marginal gain outweighs their perceived loss estimate based on their expected waiting time estimate, which is formed according to the available information level. Consequently, the motivation of this study is to analyze the equilibrium behavior of (r, Q) models experiencing impatient and heterogeneous customers who may be

presented with various levels of information.

An appropriate way to involve customer heterogeneity in the mathematical models is to assume a potential arrival process that models the arrival of an arbitrary customer to the system. Then, in light of the information presented to the customers, the arriving customer chooses to balk or purchase the product. Hence, depending on information availability, customer heterogeneity, and the state of the system, the arrival process becomes a state-dependent process. To model the equilibrium behavior of a system with customer heterogeneity with respect to waiting times and various information scenarios, we are required to model the system with state-dependent arrival rates and incorporate comprehensive waiting-time characterizations. The Poisson point process possesses elegant properties that we can exploit in this setting to model the cumulative arrival process. Therefore, in the study, cumulative arrival processes are assumed to be state-dependent Poisson processes.

In some inventory systems, there may be a limit on the number of customers who are allowed to wait. From the retailer's perspective, the reason could be to keep the system stable or to sustain customer satisfaction from the service by bounding the waiting times. From another point of view, if waiting times are getting longer, customers may also choose not to wait for the service. Leaving practical reasons aside, the analysis of the equilibrium behavior of the (r, Q) system becomes increasingly difficult if we do not limit the maximum number of customers that may be present in the system at any given time.

We would like to note that currently in the literature there is no work that presents exact equilibrium behavior on (r, Q) models with state-dependent arrival rates. We will analyze deterministic and exponential procurement lead-time settings. We would like to note that for the deterministic lead-time setting, steady-state behavior is only present for the unlimited backorder and non-state-dependent arrival case. Also for the unlimited backorder and non-state-dependent arrival case, waiting-time distribution function of an arbitrary customer was derived around 1980s through *Renewal Theoretic*

arguments. For deterministic procurement lead-times, non-state-dependent arrivals, lost sales and at most one order outstanding case, in [1] some inventory state metrics are available, yet they do not derive the stationary behavior of inventory state variables. For the exponential lead-time setting, no metrics or steady-state behavior are available. Hence, there is no literature on continuous review (r, Q) models with a limited number of backorders. Our contributions, which currently do not exist in the literature to the best of our knowledge, may be summarized as follows:

- The conditional distribution function of waiting times in a continuous-review, unlimited-backorder (r, Q) model with deterministic procurement lead times and a homogeneous Poisson arrival process,
- An analogous bulk queueing system corresponding to any continuous-review (r, Q) policy with an arbitrary arrival process, a general procurement lead-time distribution, and limited backorders,
- The global balance equations of a queueing system with state-dependent Poisson arrivals, servers that serve customers in fixed batches of Q , deterministic service times, an infinite number of servers, and arbitrary system capacity,
- The solutions to the global balance equations and the exact, complete steady-state characterization of the aforementioned bulk queueing system for a single-server setting with a capacity of up to $2Q$ customers,
- A finite-difference-based numerical technique for solving the global balance equations when multiple servers are operational, along with approximations of the corresponding steady-state probability density functions,
- The exact steady-state distribution of any inventory state variable at arrival, departure, or arbitrary epochs; the exact conditional or unconditional distributions of customer waiting times at arrival or arbitrary epochs; and any performance measure for an (r, Q) inventory system with state-dependent Poisson arrivals, deterministic procurement lead times, and at most one outstanding order,
- An approximation of the steady-state distributions of inventory state variables at arbitrary, arrival, or departure epochs, as well as the conditional or unconditional distributions of customer waiting times at arrival or arbitrary epochs, for cases

where more than one order may be outstanding,

- The exact equilibrium behavior and complete steady-state characterization of a continuous-review (r, Q) model with deterministic or exponential procurement lead times, at most one outstanding order, and a homogeneous Poisson arrival process, where arriving customers receive either no information, stock-out information, or complete stock-availability information, and behave heterogeneously with respect to their waiting-time sensitivity,
- The approximate equilibrium behavior and steady-state characterization of a continuous-review (r, Q) model with deterministic procurement lead times, multiple outstanding orders, and a homogeneous Poisson arrival process, where arriving customers receive either no information, stock-out information, or complete stock-availability information, and behave heterogeneously with respect to their waiting-time sensitivity,
- A combinatorial approach for evaluating the exact equilibrium behavior and complete steady-state characterization of a continuous-review (r, Q) model with exponential procurement lead times, multiple outstanding orders, and a homogeneous Poisson arrival process, where arriving customers receive either no information, stock-out information, or complete stock-availability information, and behave heterogeneously with respect to their waiting-time sensitivity.

In general, (r, Q) inventory models and bulk queueing systems with stochastic subprocesses are known to be difficult to analyze. We note that the analysis of continuous review (r, Q) models with a limited number of backorders is not present in the literature. We provide the methodology for steady-state characterization of a continuous review (r, Q) model with a limited number of backorders, experiencing a state-dependent Poisson process as the cumulative arrival process and deterministic or exponential procurement lead-times. The steady-state characterization of the aforementioned (r, Q) model allows us to model the equilibrium behavior when it experiences heterogeneous customers with sensitivity to waiting. Next, we summarize the contents of each chapter.

In Chapter 2, we begin with a review of the literature on inventory systems with time-sensitive customers, queueing models with bulk and state-dependent arrivals, and reorder point inventory models involving customer heterogeneity. In Chapter 3, we introduce the notation and our mathematical model to incorporate customer heterogeneity with respect to waiting sensitivity.

For a continuous review (r, Q) policy with unlimited number of backorders, homogeneous Poisson process as the cumulative arrival process and deterministic procurement lead-times, in [2] the waiting time distribution for an arbitrary customer exists. The derivation uses findings from earlier papers on stationary distributions of inventory state variables and is based on *Renewal Theoretic* arguments. In Chapter 4, we provide a simpler derivation of the waiting-time distribution function of an arbitrary customer. Furthermore, we provide the conditional waiting time distribution function of an arbitrary customer, which does not exist in the literature to the best of our knowledge.

In Chapter 5, in our first proposition, we form the analogy of any continuous review, limited backorder (r, Q) inventory model with a corresponding bulk queueing model. We present analogous stochastic processes. In relation to the continuous review (r, Q) model with state-dependent arrivals and deterministic procurement lead-times, we are required to derive the global balance equations for a bulk queueing model that experiences a state-dependent Poisson process as the cumulative arrival process with an infinite number of servers that provide bulk service in fixed size of Q customers. Using the supplementary variable technique, we extend the Forward Kolmogorov equations to an uncountable state space. After taking limits with respect to time, the set of Forward Kolmogorov equations form a set of closed integral-partial differential difference equations. We note that the set of Forward Kolmogorov equations does not exist in the literature. For one server setting, we solve the set of equations by the method of multi-dimensional Laplace Transformations or the Method of Characteristics technique. Let $Q \geq 2$ be a fixed integer. The results imply that, for any pair of reorder point and backorder amount that restricts the system to have at most one outstanding

order, we are able to derive the exact values of: the steady-state distribution of the inventory state variables at arrival, departure, or arbitrary epochs. Furthermore, we are able to compute the expected cycle length, conditional or unconditional customer waiting time distributions in arrival or arbitrary epochs. In fact, a complete steady-state characterization of the queueing or inventory model is available for the one server or one outstanding order cases, respectively. We conclude the chapter with a finite-difference-based numerical technique that allows one to produce an approximation for the cases where more than one server may be operating. The technique consists of discretization of the set of partial-differential integral difference equations. We note that the algorithm and technique are not present in the literature and appear to be new contributions to the literature, since we have modified the method to obtain the steady-state behavior of a semi-markovian process.

In Chapter 6, we derive the equilibrium behavior of continuous review, limited backorder (r, Q) models with a homogeneous Poisson process as the cumulative arrival process, deterministic procurement lead-time setting; however, customers are presented with three different levels of information and are heterogeneous with respect to waiting. For at most one outstanding-order case, we present complete steady-state behavior and metrics. For more than one outstanding-order case, we provide an approximation. In Chapter 7, we provide the corresponding equilibrium behaviors and metrics for exponential service times. In Chapter 7, we present a combinatorial method to find the exact equilibrium behavior and complete waiting time characterizations for the cases where more than one order may be outstanding. The findings of Chapters 6 and 7 appear to not exist in the literature.

We conclude our study with Chapter 8, where we present our experiments and demonstrate our findings with some examples. We note that each chapter where a mathematical model is present, i.e. Chapters 4, 5, 6 and 7, findings that do not exist in the literature are present. In the next chapter, we begin with a review of the literature.

2. LITERATURE REVIEW

We organize the literature review into three parts: Literature on inventory systems with heterogeneous or time-sensitive customers, literature on certain related queueing systems, especially bulk and state-dependent arrival queueing systems, and literature on reorder point inventory models that elaborate on waiting times and state dependent demand rates. A work that we discuss may be on the intersection of two or all three parts, since all three are intimately related areas, as we will demonstrate. Further, some works discussed may involve queueing inventory models, which are essentially a queueing system appended to an inventory system, but modeling or solution approaches may be related to ours.

2.1. Inventory Systems with Heterogeneous or Impatient Customers

There are myriad ways for customers to behave and perform valuations, hence the customer heterogeneity and policies that address this issue. Arriving customers may even have different classes that reflect how they act. Nowadays, customer heterogeneity and impatience have been hot topics; thus, modeling efforts for this phenomenon have become increasingly popular. Stochastic inventory models may be utilized to model the equilibrium behavior of an inventory system experiencing heterogeneous or impatient customers, if present, or to determine the optimal parameters of a desired inventory policy. Extensions are limitless considering either modeling or optimization aspects, since models may include various modifications on behavior, policies, and constraints.

2.1.1. Inventory Systems with Heterogeneous Customers

As in our work, customers may decide whether to join or not through a utility function that involves several criteria depending on the information availability to the customer. In [3] a continuous review stochastic control model is formed that allows one to control the service rate that is assumed to be a time-dependent Poisson process.

However, the only random variable in the utility function of a typical customer is the waiting time that is scaled by a constant, which makes the utility different from our approach. The arrivals to the system are assumed to form a homogeneous Poisson process. However, under these assumptions, arriving customers may decide not to join the system based on their utilities which are affected by the waiting time to be observed by customers. In this setting, the optimal policy turns out to be a barrier policy, i.e. the firm produces at maximal capacity whenever net inventory falls below a threshold or does not produce if net inventory is above that threshold. This type of behavior and policies are mentioned to be existent and practically useful in the microchip industries. The same model is also formulated for an n-firm game setting to model the market competition.

In a production inventory queueing system that operates as a make-to-stock [4] formulates a joint pricing and inventory control model. As in our study, the customer utility function includes a random variable with a generalized distribution. Further, production times are generalized distributions, and as in our setting, the arrival process is a homogeneous Poisson process. Customers are assumed to receive the product immediately if available. The system behavior is analyzed at arrival epochs, rendering a Markov process. Through use of Markov regenerative processes and fixed-point equations an equilibrium behavior is obtained. The model is used to obtain profit or welfare maximizing equilibrium strategies.

The main paper that inspired our work was [5] where a continuous review $(S - 1, S)$ policy is being investigated considering arbitrarily distributed replenishment lead times, a homogeneous Poisson process as the demand process, and complete backordering. Customer utility functions include an i.i.d. general distributed random variable that allows an arriving customer to establish their unit waiting cost. All arriving customers are assumed to have the same information level, which may be of type: No information, lead time information, stock-out information, and complete information. Preceding information levels refer to arriving customers making inferences on their waiting times based on: long-run averages, expected lead time, long-run averages de-

pending on observing stock-out upon arrival, and current system state upon arrival, respectively. For each information level, equilibrium behavior and expected waiting times are derived. For the constant lead time case, the distribution function of customer waiting times is derived.

2.1.2. Inventory Systems with Impatient Customers

Another production inventory system with a make-to-stock operation has been analyzed with a continuous review, prioritized base-stock policy on [6]. Two types of customers arrive at the system according to two separate independent homogeneous Poisson streams, demanding different kinds of parts produced by a single machine. However, only one customer type is time sensitive and, consequently, experiences partial backorders. Time sensitive customers do not enter the system if the backlog level for the desired part type is at the level l . Ignoring the setup times and negative exponential production times, a matrix-analytic method has been employed to derive the steady-state distribution of the system. A more refined barrier policy is derived, and the cost functions' structures are examined in detail. The policy iteration algorithm has been employed to find base stock levels that minimize the long-run average cost subject to a desired performance measure.

A queueing inventory system that operates under a $M/M/1/N$ and (s, S) policy with all negative exponential lifetimes is modeled on [7]. Customer impatience is incorporated into the model as another exponential random variable that leads to a waiting customer's departure from the queue before obtaining service if his/her patience runs out. The stationary behavior is obtained by utilizing the rate matrix and balance equations, since the stochastic process of interest evolves in a Markovian manner. The same model is formulated and solved for customers with random order size. Performance metrics such as the average inventory on hand or expected waiting times are calculated. The same framework is applied for a homogeneous discrete compound Poisson process as for the arrival process.

2.2. Queueing Models with Bulk and State Dependent Arrivals

A stochastic process may become increasingly complex through what may seem like small modifications. Multiple processes that evolve through quantifiable randomness may interact and form a unique stochastic process. In terms of continuous random variables with real values, *exponential* or sometimes referred to as the *negative exponential* random variable has a special place due to the memorylessness property. The memorylessness(forgetfulness) property allows one to perform the analysis in a Markovian manner, which was first established by *Andrey Andreyevich Markov* at the beginning of 1900s. There is a simple yet elegant technique to perform inferences on stationary behaviors of stochastic processes, either discrete time or continuous time, that evolve through interaction of sub processes that evolve in a Markovian way. However, when a subprocess evolves in a non-Markovian manner, then the analysis becomes extremely difficult compared to the prior case mentioned. For non-Markovian evolving processes, three techniques have been widely utilized. Namely, *Embedded Chain* methods, *Erlang's Method* which was due to *Agner Krarup Erlang* who is known to be the father of the *Queueing Theory*, and at last but not least the *Supplementary Variable* technique. The Supplementary Variable technique was introduced by *David George Kendall* and *D.R. Cox* around the mid 20th century. In [8] a detailed explanation of the technique can be found. Essentially, it boils down to rendering the process evolve in a Markovian manner through defining and utilizing supplementary variables that remove the memory from the system. *Andrey Nikolaevich Kolmogorov*, the father of modern probability theory, derived forward or backward equations that are utilized to formulate the *global balance equations* that govern the relationship between system states. In our work, we will provide a rigorous analysis using the supplementary variable technique for our system of interest.

The system of interest in this study is referred to as a bulk queueing model in the literature. In [9] a detailed analysis of several bulk queueing models is presented. The methods mentioned in [8] are demonstrated in several systems, and analogies between various queueing models are presented.

In a non-bulk queueing system where the arrival process is state-dependent Markovian, a generalized service time distribution with infinite server setting [10] utilizes supplementary variable technique and derive the Forward Kolmogorov Equation set corresponding to that system. Then, they solve the equations and provide one with the stationary distribution.

As mentioned, analysis of bulk queueing systems is extremely difficult in contrast to an equivalent non-bulk queueing system. With a generalized bulk interarrival time distribution, Markovian bulk service provided in fixed batches of K customers and an infinite server setting, [11] derives the transient and stationary *partial binomial moments* for the number of busy servers and waiting customers.

In addition, assuming a homogeneous Poisson process as the arrival process, exponential batch service with fixed size R and infinite server setting [12] explores the transient analysis of system behavior. They assumed a specific customer impatience where a customer may choose to obtain individual service or batch service according to a policy. They derive stationary waiting times in the queue; however, explicit evaluation of the results becomes increasingly difficult as R becomes larger than or equal to 2, again indicating a batch service policy.

Finally, [13] examines several non-batch queueing models with customer impatience. Customer impatience is incorporated into the models as a random variable that signifies how long the customer is willing to wait for the server to return from vacation. The systems analyzed include: $M/M/1$, $M/G/1$, $M/M/c$ with server vacation times that have exponential or generalized distributions, as well as customer impatience that has distributions of generalized type or negative exponential. Generating functions are utilized to derive steady-state distributions and some related metrics such as the mean busy duration and the proportion of customers served.

2.3. Reorder Point Inventory Models with Customer Impatience and Heterogeneity

In this section, we present two reorder point models, namely (r, Q) and (s, S) , which are identical in the continuous review and unit-sized demand case. However, when demands are allowed to be non-unit sized or periodic review is present, order sizes cannot be fixed on each cycle since (s, S) policy requires one to order up to level S whenever inventory position falls to or below the level s . Keeping this in mind, we present several reorder point models, not only restricting ourselves to models with fixed order size, since modeling approaches or solution procedures may be similar and useful.

In [1], a rigorous treatment is present for a number of variants of the reorder point inventory models. The book contains a vast chapter on stochastic demands lot size reorder point models. They provide readers with approximations and exact results, depending on the metric one is interested in, on complete backorder or lost sales cases. For lost sales case with Poisson demands, constant procurement lead times, and under the condition that no more than a single order is outstanding, they derive the expected length of time out of stock per cycle, expected number of lost sales, expected number of unit years of stock held per cycle, and average annual cost. They do not utilize the stationary distribution of the underlying processes and note that the theory for more than a single order outstanding is not present.

A special case of the (s, S) inventory model is the $(S - 1, S)$ inventory model, where the reorder point is $s = S - 1$. The $(S - 1, S)$ policy is also known as one-for-one replenishment, as a replenishment order must be placed upon the arrival of each customer. The variants of the inventory policy may involve periodic review, or non-unit sized demands. In [14] a continuous review $(S - 1, S)$ inventory policy is being investigated. The cumulative demands form a homogeneous compound Poisson process, and the replenishment lead times are arbitrarily distributed. They derive the stationary distribution of the net inventory and several performance metrics. A decade

later, in [15] under continuous review, a compound Poisson cumulative arrival process, and a deterministic procurement lead time setting with inventory policy $(S - 1, S)$, the exact customer waiting time distribution is derived.

For a continuous review, an instantaneous replenishment and unlimited backorders (s, S) inventory system with an arbitrary inter-arrival distribution between unit demands [16] provides us with the transient and steady-state distribution of the inventory position. It is shown that under arbitrary inter-arrival time distributions between unit-sized demands, the stationary distribution of the inventory position is discrete uniform in the set $\{s + 1, s + 2, \dots, S - 1, S\}$.

In an (s, S) policy with continuous review, unlimited backorders, constant procurement lead times, general inter-arrival and demand distributions [17] provides the transient and stationary distributions of inventory position and on hand inventory through *Renewal Theoretic* arguments. In addition, under various assumptions regarding lead times or demand distributions, the functional forms of several measures related to inventory policy are derived. Then, these measures are utilized to form algorithms that provide one with approximations or exact results regarding optimal order quantities depending on different settings.

In an (s, S) policy with continuous review, unlimited backorders, and constant procurement lead times, if cumulative demands form a compound renewal process, the steady-state distribution function of waiting time is derived in [2]. The arguments provided in the paper are formed on the Renewal Theoretic arguments supplied in [17]. Also, [2] provides waiting time metrics for the case of a partial fulfillment, since demand sizes are no longer restricted to be unity and some part of a demand may be backordered.

In an (R, s, Q) system where R denotes the review periods, i.e. the system is observed every R time units, s denotes the reorder point, and Q denotes the fixed reorder quantity [18] provides an approximation for the distribution function of the customer

waiting time under compound renewal demand and stochastic lead times. They derive an approximation to determine the minimal reorder level subject to a predetermined maximum average waiting time. They tested their findings with a discrete time model.

Under a periodic review order-up-to level S policy with homogeneous Poisson process as the arrival process, Gamma distributed demand sizes and general distributed lead times [19] develop an approximation to evaluate the distribution function of customer waiting times. Due to the assumptions, the cumulative demand process turns out to be a Compound Poisson Process. The generalized distribution of lead times causes increased complexity; they impose the assumption that outstanding orders arrive in chronological order that they are given. This assumption inevitably enforces a dependence between lead time random variables.

In a continuous review, lost sales (s, S) inventory system with emergency ordering and negative exponentially distributed demand sizes, [20] forms an exact model for negative exponential replenishment times with different rates for expedited and regular ordering. Arrivals are assumed to form a homogeneous compound Poisson process, and the system point method with a level crossing approach is utilized to derive the stationary distribution of the net inventory. They mention that convexity of the cost function is difficult and impractical to prove for the aforementioned policy, i.e. (s_1, Q_1, s_2, Q_2) , and utilize numerical methods to minimize the expected cost with and without a service level constraint.

Again, for a continuous review lost sales (s, S) inventory model with renewal demands and negative exponential lead times, [21] derives the steady state balance equations for the net inventory through utilizing semi-regenerative processes and embedded Markov renewal processes. They propose a numerical procedure to compute the steady state distribution.

In a periodic review, stochastic demand and zero lead time setting an (R, s, S) inventory policy has been modeled utilizing the discrete Markov chain method on [22].

For a given S , they derive an efficient recursive algorithm that allows one to derive the steady state distribution of the net inventory for all possible s values without solving the balance equations repeatedly. The algorithm allows one to iteratively use the steady state distribution of net inventory corresponding to a previously calculated s value. They used the analytical model to compute optimal parameters for a real life point-of-use hospital inventory system subject to a service level constraint.

Another queueing inventory system with all negative exponential life times, continuous review (s, S) policy, and $M/M/1$ queue with multiple vacations, i.e. the server leaves for a vacation when net inventory level drops to 0 and leaves for another vacation if net inventory is still 0 upon arrival, and impatient customers are investigated by [23]. The customers decide not to join the system when the server is on vacation with an i.i.d. Bernoulli random variable. Impatience times are distributed according to a negative exponential random variable, and a customer may leave the system if they are waiting for service or the server is on vacation. The steady state balance equations of the system is constructed utilizing a matrix geometric approach and solved numerically through truncating the matrix. They use the numerical approximation of the steady state distribution to derive several performance measures.

For a discrete compound Poisson process as the demand process, continuous review, lost sales (s, S) inventory policy with at most one outstanding order, [24] provides time homogeneous transition probabilities of the starting inventory at replenishment epochs. The balance equations are derived utilizing the renewal function and the fact that restricting the inventory policy to have at most one outstanding order and lost sales generates a renewal process in this setting. The process is said to be cumbersome, and as a solution an approximation is derived. However, when customers are allowed to wait, the system carries memory and the backorder case cannot be handled properly. The compounding distribution and (s, S) policy innately require one to have no fixed order size. Our methodology and our setting allow us to handle both the limited backorder case and the lost case. Also, our method involves allowing the outstanding orders to be more than 1 but numerical techniques are required.

Lastly, [25] provides a simulation model for a limited backorder (r, Q) inventory policy, i.e. (Q, s, R) , with deterministic procurement lead times and involving three types of customers. Customer arrivals occur according to three Poisson streams with different intensities, and each type is differentiated based on their tolerance to waiting. Impatience is modeled using a negative exponential random variable with different rates. Partial backordering may or may not be possible, which is incorporated into the model through a Bernoulli random variable. Each customer is capable of demanding more than one unit, and the distribution associated with demand sizes for each customer is a shifted Poisson random variable with customer type specific rates.

In this study, we will investigate a continuous review, limited backorder (r, Q) inventory policy, which we will denote as $(r, Q, \mathcal{B})$. In the notation, $\mathcal{B}$ represents the maximum number of backorders allowed. For $\mathcal{B} = \infty$, i.e. an unlimited number of backorders, homogeneous Poisson process as the cumulative demand process, and deterministic procurement lead times, we derive the conditional distribution function of waiting time conditioned on the value of net inventory. The result is then used to compute the distribution function for waiting for an arbitrary customer. We note that the conditional waiting time distribution and consequently the derivation of the waiting time distribution for an arbitrary customer are new contributions to the literature. We investigate the equilibrium behavior of the inventory policy $(r, Q, \mathcal{B})$ at three different levels of information provided to arriving customers: No information, partial information on whether an item is available or not, and complete information on net inventory. Arriving customers are assumed to be heterogeneous with respect to their sensitivity to the expected unit time of waiting they incur that is incorporated in the model as a pre-specified arbitrarily distributed random variable. We form an analogy of the inventory policy $(r, Q, \mathcal{B})$ with a corresponding batch queueing model experiencing state-dependent Poisson arrivals. We derive the Forward Kolmogorov Equations using the method of Supplementary Variables to formulate the global balance equations of the system of interest. We solve the equations analytically under a capacity constraint and propose a numerical method to solve the equations whenever the capacity constraint is exceeded. The analogy, the global balance equations or solution procedures

do not exist in the literature. We use the equilibrium behavior of the queueing system to obtain the steady-state behavior of the $(r, Q, \mathcal{B})$ inventory policy under partial and complete information scenarios. We derive analytical steady-state distributions of the state variables of the inventory policy at arrival, departure, or arbitrary epochs. We also derive the conditional waiting time distributions at arrival epochs or arbitrary epochs. The results are used to formulate the expected long-term and conditional waiting times at arrival instants or at arbitrary times.

Finally, we will analyze the equilibrium behavior of a continuous review $(r, Q, \mathcal{B})$ inventory policy under state-dependent Poisson arrivals and exponential procurement lead times. We first derive the stationary distribution of the net inventory at arbitrary epochs or arrival epochs. In addition, we derive the conditional waiting times at arrival epochs and use the result to obtain the stationary expected waiting time at arrival epochs. We utilized our findings to derive the conditional waiting times at arbitrary epochs and consequently stationary expected waiting time at arbitrary epochs. We also derive the conditional waiting time distributions at arrival epochs or arbitrary epochs. Conditional waiting-time distributions are combined with the stationary distribution of the net inventory to obtain the stationary waiting-time distributions. The mentioned results are used to obtain the equilibrium behavior of the inventory system when arriving customers are heterogeneous with respect to their waiting times. We would like to note that the stationary distribution of the net inventory at arrival or arbitrary epochs in these settings is not present in the literature. Furthermore, the waiting time metrics aforementioned above are not present in the literature to the best of our knowledge.

3. A MATHEMATICAL DESCRIPTION OF THE MODEL

In this section, we introduce the notation that includes variables, parameters, and significant performance measures. Although we explain our notation in the text, we summarize it in Table 3.1.

Table 3.1. Notation.

Variable/Parameter	Explanation
r	Reorder point
$\mathcal{B}$	Maximum number of backorders allowed
$IN(t)$	Net inventory at time t
$IO(t)$	Number of outstanding items at time t
$IP(t)$	Inventory position at time t
IN	Net inventory at the steady state
IO	Number of outstanding items at the steady state
IP	Inventory position at the steady state
W_n^a	Waiting time of a customer who observed $IN = n$ upon arrival
$\mathbf{L}$	Lead time random variable
L	Expectation of lead time random variable
$G(x)$	The distribution function of $\mathbf{L}$
$h(x)$	The hazard rate function of $\mathbf{L}$
R	Valuation of the item by customers
p	The price of the item
C	Random variable denoting unit waiting cost
Λ	Potential arrival rate per unit time
λ_n	Arrival rate per unit time when $IN = n$
K	Fixed Ordering Cost
$\mathcal{N}$	Expected Number of orders given per unit of time

In this work, we consider a single item, continuous review inventory model with the (r, Q) policy, also known as the reorder point fixed order quantity policy. That is, a fixed order of size Q is given whenever the inventory position hits r , known as the reorder point. In the study, we will also examine the (r, Q) inventory policy with a limited backorder case, and let us denote $\mathcal{B}$ as the parameter that indicates the maximum number of backorders to be allowed. That is, if a customer arrives and observes the number of units currently ordered as $\mathcal{B}$, the customer cannot enter the system. Let us denote the (r, Q) inventory policy with a limit $\mathcal{B}$ on the number of backorders as $(r, Q, \mathcal{B})$. Typically, in an inventory policy, if backorders are allowed, then $\mathcal{B} = \infty$. On the other hand, if backorders are not allowed, then $\mathcal{B} = 0$.

Let the time-dependent state variables $IN(t)$, $IO(t)$, and $IP(t)$ denote the net inventory, the number of items outstanding, and the inventory position at time t , respectively. The definitions of the state variables innately require that for $t \geq 0$, $IP(t) = IN(t) + IO(t)$. Furthermore, we define IN , IO , IP as random variables that denote the steady state net inventory level, the number of outstanding units, and the inventory position, respectively. Assuming that a steady state distribution exists, then again the relationship $IP = IN + IO$ holds.

Given that a customer observed $IN = n$ upon their arrival, W_n^a denotes the random variable that indicates their waiting time. Also, $w_n^a = \mathbb{E}[W_n]$, its expected value. Successive replenishment lead times of orders of size Q are i.i.d. random variables with the cumulative distribution function $G(x)$. We let $\mathbf{L}$ denote the lead time random variable with $\mathbb{E}[\mathbf{L}] = L$. Further, we let $h(x) = \mathbb{P}\{\mathbf{L} = x | \mathbf{L} \geq x\}$ denote the hazard rate function for the lead time random variable. Throughout the study, we assume that procurement lead-times are degenerate random variables, i.e. they are deterministic, with $\mathbf{L} = L$.

An arriving customer decides on their utility based on their valuation of the product to be purchased (R), the price of the product (p) and their waiting time scaled by the random variable C that denotes the unit waiting cost assigned by that

customer. In the study, customers are homogeneous with respect to their valuation and the price of the product. However, customers are heterogeneous in their sensitivity to the expected time units of waiting that they incur. We use the random variable C to denote the distribution of the cost assigned per unit of time waiting incurred by a potential customer. Let $F(x)$ denote the cumulative distribution function of C , and without loss of generality we assume that C has support over the interval $[0, \beta]$. Consequently, a potential arriving customer evaluates their utility for purchasing as follows:

$$R - p - C\mathbb{E}[\text{Waiting time until obtaining product}]. \quad (3.1)$$

A customer will decide to enter the system and is willing to purchase the product if their utility is non-negative. The potential arrival process is assumed to be a homogeneous Poisson process with rate Λ ; however, an arriving customer may decide not to enter the system depending on their utility, resulting in state-dependent effective arrival rates denoted as λ_n . In Equation (3.1), the estimation of the expected waiting time to be observed by an arriving customer is performed according to the accessible information level of that customer. Namely, for a given system, we assume that all customers have access to one of the following three information levels:

- i. No information: arriving customers have no information regarding the state of the system and make their waiting time inferences based on steady state averages.
- ii. Incomplete information: arriving customers can observe if the system is in stock-out state or not, and make their waiting time inference based on steady-state averages.
- iii. Complete information: arriving customers know the net inventory level and use their individual exact waiting time estimates to form their decisions.

We would like to note that in Equation (3.1) the expected waiting time estimate formed by a customer depends on the information availability of the system. Upon arrival

of a customer, assuming that the customer has access to either partial or complete information, the purchasing decision is made based on the current state of the system. Consequently, we require the steady-state distribution of the net inventory level and expected conditional waiting times at arrival epochs. We would like to note that arrivals do not see time averages in the state-dependent arrival or limited backorder cases, as opposed to the PASTA (Poisson Arrivals see time averages) property proven in [26]. We will derive the steady-state distribution of a desired system for arrival epochs, departure epochs, and arbitrary instants.

The inventory manager decides on the quantity of reorder r , fixed order quantity Q and the allowed number of backorders $\mathcal{B}$ for a given information level by solving the following equation:

$$\max_{r, Q, \mathcal{B}} (p - v)\Lambda_{eff} - h\mathbb{E}[(IN)^+] - K\mathcal{N} \quad (3.2)$$

where p is the price of the product, v is the unit cost of the product, h is the unit holding cost per unit time, and K is the fixed cost of ordering. Λ_{eff} is defined as the effective arrival rate per unit time at arbitrary instants for a specified information level and $(x)^+ = x \cdot u_0(x)$, where $u_c(x)$ is the shifted Heaviside Step Function, i.e. $u_c(x) = \mathbb{1}_{\{x \geq c\}}$ for an arbitrary $c \in \mathbb{R}$. $\mathcal{N}$ is defined as the expected number of orders given per unit of time. We let Π , Π^a , and Π^d denote the steady-state distribution of the net inventory at arbitrary instants, arrival epochs, and departure epochs, respectively. In addition, let us denote the individual elements of the vectors as: $\Pi \equiv \{\pi_n : n \leq r + Q\}$, $\Pi^a \equiv \{\pi_n^a : n \leq r + Q\}$, and $\Pi^d \equiv \{\pi_n^d : n \leq r + Q\}$.

An arriving customer evaluates their utility in Equation (3.1) using Π^a and W_n^a , consequently the values of λ_n are determined. However, in Equation (3.2) the inventory manager evaluates Λ_{eff} and $\mathbb{E}[(IN)^+]$ utilizing Π , and $\mathcal{N}$ utilizing Π^d . The need to make decisions based on different distributions is demonstrated by a clear example in [5]. For the inventory policy $(r, Q, \mathcal{B})$, an arriving customer cannot observe $IN = \mathcal{B}$

at their arrival instant, i.e. $\pi_{-\mathcal{B}}^a = 0$. However, at arbitrary instants, backorders may be equal to $\mathcal{B}$, which implies that $\pi_{-\mathcal{B}} > 0$. Then, we are required to derive and use the steady-state distributions at arrival and arbitrary epochs to find the equilibrium behavior of the system.

In the next chapter, we present our derivations on the conditional and unconditional waiting times of customers in an unlimited backorder (r, Q) model. The findings will then be used to analyze the equilibrium behavior of the case of no information with unlimited backorder setting in Section 6.2.

4. CONDITIONAL WAITING TIME DISTRIBUTION IN AN (r, Q) INVENTORY POLICY

In this chapter, we assume that an inventory policy $(r, Q, \mathcal{B} = \infty)$, i.e. an inventory policy (r, Q) with an unlimited number of backorders, with continuous review, homogeneous Poisson arrivals, and deterministic replenishment lead times is of interest. We derive the distribution function of waiting times for a customer conditioned on the value of IN . We note that the conditional waiting time distribution does not exist in the literature to the best of our knowledge. We utilize the conditional waiting time distribution to derive the unconditional waiting time distribution. The stationary waiting time distribution was derived in [2], yet is derived through Renewal Theoretic arguments. Therefore, our derivations appear to be simpler and new contributions to the literature. We begin by obtaining the equilibrium behavior of the inventory system and then derive the conditional and stationary distribution functions of the waiting time.

4.1. Limiting Distributions of State Variables

For an inventory managed with the (r, Q, ∞) policy, where cumulative arrivals form a homogeneous Poisson process with intensity λ , and procurement lead times are deterministic and an unlimited number of backlogs are available, the distributions of IN and IP can be found in [27]. Nevertheless, we will derive the steady-state distributions of the state variables for the sake of the arguments that follow. Currently, let us disregard the information case and derive the limiting distributions of the state variables.

At an arbitrary time t we know that equation

$$IP(t) = IN(t) + IO(t) \tag{4.1}$$

is valid. $IP(t)$ denotes the net inventory position plus the orders in the pipeline, i.e. the number of products for which their orders were given but did not arrive until the time t . Whenever a customer arrives at the system IN decreases by one. At every Q^{th} customer, an order of size Q is given which increases IO and IP by Q . Without loss of generality, assume that for a non-negative integer r and $Q \geq 2$, the system begins to operate with $IN(0) = r + Q$, $IO(0) = 0$, which implies that $IP(0) = r + Q$ due to Equation (4.1). Furthermore, due to our assumptions at the beginning we may safely state

$$IN(t + L) = IP(t) - (D(t + L) - D(t)) \quad (4.2)$$

where $D(t)$ is the counting process that characterizes the number of arrivals occurring between the time period $[0, t]$. It must be noted here that $D(0) = 0$, and $\mathbb{P}\{D(t + L) - D(t) = k\} = e^{-\lambda L}(\lambda L)^k / k!$. Since the process $D(t)$ constitutes a Poisson point process, it has the stationary and independent increments property. Since our arrival process is independent of other processes, has the stationary and independent increments property, we may define

$$D(t + L) - D(t) = \mathcal{D}$$

where $\mathcal{D}$ denotes the random variable that signifies the number of customers that arrive to the system at an interval of length L . Then, $\mathcal{D} \sim \text{Poisson}(\lambda L)$. Further, let us derive the stationary distribution of the Inventory Position at an arbitrary time. We know that an order of size Q is given whenever IP hits r , which is essentially triggered by the arrival of a customer. Therefore, IP constitutes a stochastic process that takes integer values between $r + 1$ and $r + Q$. One may observe that IP forms a continuous time Markov chain (CTMC) with holding time parameters $\lambda(i) = \lambda$ for $i \in \{r + 1, \dots, r + Q\}$. The process moves circularly, assuming that the process starts in an arbitrary state $r + j$ where $j \in \{1, \dots, Q\}$, then it must first visit every state between $r + 1$ and $r + j$, including $r + 1$, and then it moves back to $r + Q$ where it will visit

$r + j$ again after the arrival of $Q - j$ customers. Therefore, the balance equations for the CTMC may be written as

$$\lambda\pi_i = \lambda\pi_{i+1},$$

where $i \in \{r + 1, \dots, r + Q - 1\}$ and

$$\lambda\pi_{r+1} = \lambda\pi_{r+Q}.$$

Therefore upon employing the normalization equation to obtain the stationary distribution, we obtain $\pi_i = 1/Q$, for $i \in \{r + 1, \dots, r + Q\}$. IP is a discrete uniform random variable. Since in Equation (4.2) $IP(t)$ is completely independent of $D(t + L) - D(t)$, which signifies the number of arrivals in the interval $(t, t + L]$, we may take the limits with respect to t on both sides of Equation (4.2) to eliminate the time dimension. Then, assuming that the process is not explosive, i.e. $\lambda > 0$ and $L > 0$ is finite, Equation (4.2) becomes:

$$IN = IP - \mathcal{D}, \tag{4.3}$$

where $IP \sim U[r + 1, r + Q]$, $\mathcal{D} \sim Poisson(\lambda L)$ and the two are mutually independent. Thus,

$$\mathbb{P}\{IN = k\} = \mathbb{P}\{IP - \mathcal{D} = k\}.$$

Let us utilize the law of total probability to exploit the independence as well as the small cardinality of the support of the random variable IP to obtain

$$\begin{aligned}
\mathbb{P}\{IN = k\} &= \sum_{i=r+1}^{r+Q} \mathbb{P}\{i - \mathcal{D} = k, IP = i\} \\
&= \frac{1}{Q} \sum_{i=r+1}^{r+Q} \mathbb{P}\{\mathcal{D} = k - i\},
\end{aligned}$$

where of course $k \in \{x : x \leq r + Q, x \in \mathbb{Z}\}$. Then, we have the limiting distributions of the underlying stochastic processes available. In the next section, we derive the conditional and unconditional waiting time distribution functions.

4.2. Waiting Time Analysis

We derive the expected waiting time, conditional and unconditional distribution functions of the waiting times of the customers in this section. First, we derive the expected waiting time. Upon employing Little's Law and using Equation (4.3):

$$\begin{aligned}
\lambda \bar{w}_N(\lambda) &= \mathbb{E}[(-IN)^+] \\
&= \mathbb{E}[(\mathcal{D} - IP)^+] \\
&= \mathbb{E}[(IP - \mathcal{D})^+ + \mathcal{D} - IP] \\
&= \mathbb{E}[(IP - \mathcal{D})^+] + \mathbb{E}[\mathcal{D}] - \mathbb{E}[IP] \\
&= \mathbb{E}[(IP - \mathcal{D})^+] + \mathbb{E}[\mathcal{D}] - \mathbb{E}[IP] \\
&= \mathbb{E}[\mathbb{E}[(IP - \mathcal{D})^+ | IP]] + \lambda L - \frac{2r + Q + 1}{2} \\
&= \sum_{i=r+1}^{r+Q} \sum_{j=0}^i \frac{(i - j) \cdot \mathbb{P}\{\mathcal{D} = j\}}{Q} + \lambda L - \frac{2r + Q + 1}{2} \\
&= \frac{1}{Q} \sum_{i=r+1}^{r+Q} \sum_{j=0}^i \frac{(i - j) e^{-\lambda L} (\lambda L)^j}{j!} + \lambda L - \frac{2r + Q + 1}{2} \\
&= \frac{1}{Q} \sum_{i=r+1}^{r+Q} \sum_{j=0}^i \frac{i e^{-\lambda L} (\lambda L)^j}{j!} - \frac{1}{Q} \sum_{i=r+1}^{r+Q} \sum_{j=0}^i \frac{j e^{-\lambda L} (\lambda L)^j}{j!} + \lambda L - \frac{2r + Q + 1}{2} \\
&= \frac{1}{Q} \sum_{i=r+1}^{r+Q} \sum_{j=0}^i \frac{i e^{-\lambda L} (\lambda L)^j}{j!} - \frac{\lambda L}{Q} \sum_{i=r+1}^{r+Q} \sum_{j=0}^{i-1} \frac{e^{-\lambda L} (\lambda L)^j}{j!} + \lambda L - \frac{2r + Q + 1}{2}
\end{aligned}$$

$$= \frac{1}{Q} \sum_{i=r+1}^{r+Q} i \mathbb{P}\{\mathcal{D} \leq i\} - \frac{\lambda L}{Q} \sum_{i=r+1}^{r+Q} \mathbb{P}\{\mathcal{D} \leq i-1\} + \lambda L - \frac{2r+Q+1}{2}, \quad (4.4)$$

where $\bar{w}_N(\lambda) = \mathbb{E}[W]$. Hence, performing algebraic operations on Equation (4.4) we obtain:

$$\bar{w}_N(\lambda) = \frac{1}{\lambda Q} \sum_{i=r+1}^{r+Q} i \mathbb{P}\{\mathcal{D} \leq i\} - \frac{L}{Q} \sum_{i=r+1}^{r+Q} \mathbb{P}\{\mathcal{D} \leq i-1\} + L - \frac{2r+Q+1}{2\lambda}. \quad (4.5)$$

Here we would like to note that, since the cumulative arrival process forms a homogeneous Poisson process, the PASTA property holds as proven in [26].

To derive the conditional distribution of the waiting times, we will utilize the basic properties of the Poisson Point Process and a characteristic of the (r, Q) inventory system. For simplicity, assume that the effective arrival rate of the inventory system is equal to λ . First of all, let us consider Equation (4.2), and re-parametrize the index set as follows for a clear description of the method:

$$IN(t) = IP(t-L) - (D(t) - D(t-L)). \quad (4.6)$$

On Equation (4.6) the only modification required is $t \geq L$ due to re-indexing. However, we will be deriving the stationary waiting time distributions and therefore are not concerned about behavior as time progresses. Denote the distribution function of the waiting times as $F(w)$ for $w \geq 0$. Since our arrival process is a Poisson Point Process, arrivals also see time averages. Now, let us make some observations. First, $F(L) = 1$, as it takes L units of time for a customer to receive their product at maximum. $F(0)$ is equal to a customer not waiting, which means that the customer found a product on the shelf upon arrival. Then $\frac{dF}{dw}$ is a mixture of discrete and continuous distributions. By our previous observation, $F(0) = \sum_{i=1}^{r+Q} \mathbb{P}\{IN = i\}$.

Furthermore, let us assume that a customer did not find any products on arrival and observed that IN was 0 upon arrival. Then, for $t \geq L$, $L \geq w \geq 0$ say that this customer arrived at time t to the system and we would like to evaluate the probability

that she obtains service in less than or equal to w time units. Employing Equation (4.6), we observe that $IP(t-L) - (D(t) - D(t-L)) = 0$. Then, focusing on the process $IP(\cdot)$, there are exactly Q possible values which $IP(t-L)$ may take and the stationary distribution is discrete uniform between $[r+1, r+Q]$. If we assume $IP(t-L) = r+Q$, then it is known that there were $r+Q$ customer arrivals in the time period $(t-L, t]$. For this specific customer to receive service before time $t+w$, implying that the service time is less than or equal to w , at least Q of the $r+Q$ customers must have arrived in the time period $(t-L, t-L+w]$ so that an order will be placed until time $t-L+w$ which would arrive before time $t+w$ since the procurement lead time is deterministic and equal to L . Now for this specific example, i.e. a customer arrives at t , observes $IN(t) = 0$ just before her arrival, and $IP(t-L) = r+Q$, which implies that $D(t) - D(t-L) = r+Q$. Then, if we evaluate $\mathbb{P}\{D(t-L+w) - D(t-L) \geq Q | D(t) - D(t-L) = r+Q\}$, we may evaluate the probability that this specific customer receives service in at most w time units. Since $D(t) - D(t-L)$ is known to be a Poisson point process with rate λL , we know that

$$\begin{aligned} & \mathbb{P}\{D(t-L+w) - D(t-L) \geq Q | D(t) - D(t-L) = r+Q\} \\ &= \sum_{i=Q}^{r+Q} \binom{r+Q}{i} (w/L)^i (1-w/L)^{r+Q-i}. \end{aligned}$$

As a remark, we are interested in stationary waiting time distributions, which means that we will drop the time subscript in what follows. Then, to generalize our observation for a customer arriving at time $t+w$, if the customer will observe $IN = n$ upon their arrival where $n \in \{x : x \in \mathbb{Z}_0, x \leq 0\}$ and $s \in \{x : x \in \mathbb{N}, x \leq Q\}$, $D = r + s - n$, then at least $Q \left\lceil \frac{(1-n)}{Q} \right\rceil - Q + s$ of the $r + s - n$ customers should arrive between $[t-L, t+w-L]$. Therefore, since the demand process is a Poisson point process, we have upon conditioning on IN :

$$\begin{aligned} &= \mathbb{P}\{W \leq w, IN = n\} \\ &= \sum_{i=1}^Q \mathbb{P}\{W \leq w, IN = n | IP = r+i\} \mathbb{P}\{IP = r+i\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{Q} \sum_{i=1}^Q \mathbb{P} \left\{ D(t) - D(t-L) = r+i-n, \right. \\
&\quad \left. D(t+w-L) - D(t-L) \geq Q \left\lceil \frac{(1-n)}{Q} \right\rceil - Q + i \right\} \\
&= \frac{1}{Q} \sum_{i=1}^Q \mathbb{P} \left\{ D(t+w-L) - D(t-L) \geq Q \left\lceil \frac{(1-n)}{Q} \right\rceil - Q + i \right. \\
&\quad \left. D(t) - D(t-L) = r+i-n \right\} \\
&\cdot \mathbb{P} \left\{ D(t) - D(t-L) = r+i-n \right\} \\
&= \frac{1}{Q} \sum_{i=1}^Q \sum_{j=Q \lceil \frac{(1-n)}{Q} \rceil - Q + i}^{r+i-n} \binom{r+i-n}{j} \left(\frac{w}{L} \right)^j \left(1 - \frac{w}{L} \right)^{r+i-n-j} \frac{e^{-\lambda L} (\lambda L)^{r+i-n}}{(r+i-n)!}.
\end{aligned} \tag{4.7}$$

Which presents one with the conditional waiting time distribution of the process IN . Utilizing the conditional waiting time distribution, we may obtain the stationary distribution function of the waiting times:

$$\begin{aligned}
\mathbb{P}\{W \leq w\} &= \sum_{n=-\infty}^{r+Q} \mathbb{P}\{W \leq w, IN = n\} \\
&= \sum_{n=1}^{r+Q} \mathbb{P}\{IN = n\} + \sum_{n=-\infty}^0 \mathbb{P}\{W \leq w, IN = n\} \\
&= \sum_{n=1}^{r+Q} \frac{1}{Q} \sum_{i=r+1}^{r+Q} \mathbb{P}\{\mathcal{D} = i - n\} \\
&+ \frac{1}{Q} \sum_{n=0}^{\infty} \sum_{i=1}^Q \sum_{j=Q \lceil \frac{(1+n)}{Q} \rceil - Q + i}^{r+i+n} \binom{r+i+n}{j} \left(\frac{w}{L} \right)^j \left(1 - \frac{w}{L} \right)^{r+i+n-j} \\
&\cdot \frac{e^{-\lambda L} (\lambda L)^{r+i+n}}{(r+i+n)!}.
\end{aligned} \tag{4.8}$$

In [2], one may find a substitute for Equation (4.8) derived through *Renewal Theoretic* arguments. Their formula is more compact, and allows one to compute the distribution function of the stationary waiting times when non-unit sized demands occur or the successive interarrival times form a renewal process. However, the formula requires

multiple fold convolution and does not provide one with the conditional distribution function of the waiting times. To the best of our knowledge, Equations (4.7) and (4.8) appear to be new contributions to the literature.

We note that Equation (4.4) will be used in Chapter 6 to determine the equilibrium behavior for the no information case where an unlimited number of backorders is allowed. However, for the limited backorder case, the results of this section do not apply.

In the next chapter, we begin by forming the analogy between the $(r, Q, \mathcal{B})$ model and the bulk queueing model $M_n/L^Q/\dots$. Then, we formulate the global balance equations of the queueing system with the supplementary variable technique. We solve the equations under a capacity constraint. For the cases where an analytical approach fails, we propose a numerical technique to find a satisfactory approximation of the steady-state behavior.

5. $(r, Q, \mathcal{B})$ MODEL WITH STATE DEPENDENT ARRIVALS AND CONSTANT PROCUREMENT LEAD-TIMES

In this chapter, we will explore the steady-state behavior of a continuous review $(r, Q, \mathcal{B})$ model, and throughout this chapter, we assume that cumulative arrivals form a state-dependent Poisson process, and procurement lead times are deterministic. We begin by introducing an analogy with a bulk queueing system, namely $M_n/L^Q/\infty$. We derive the global balance equations corresponding to that queueing system by the use of the supplementary variable technique. We generalize the global balance equations to characterize any continuous review $(r, Q, \mathcal{B})$ inventory policy, i.e. limited backorder (r, Q) inventory policy. We solve the equations analytically for at most one outstanding-order case and provide the necessary numerical procedures to solve the equations for more than one outstanding-order case and obtain an approximate solution. We note that neither the analogy nor the global balance equations or solutions exist in the literature to the best of our knowledge.

5.1. Bulk Queueing System Analogy

In this section, we begin by demonstrating the analogous queueing system corresponding to a $(r, Q, \mathcal{B})$ policy.

Proposition 5.1. *An $(r, Q, \mathcal{B})$ inventory policy with continuous review, an arbitrary arrival process, say $\mathbb{A}$, and arbitrary procurement lead time distribution, say $\mathbb{B}$, and limited backorders, denoted as $\mathcal{B}$, is analogous to the queueing system $\mathbb{A}/\mathbb{B}^Q/\eta(r, Q, \mathcal{B})/r + Q + \mathcal{B}$ with a first-in first-out (FIFO) service policy, where in Kendall's notation $\mathbb{A}$ denotes the arrival process of the queueing system, $\mathbb{B}$ denotes the service time distribution where service is provided in fixed batches of Q customers, i.e. an available server begins serving customers when the number of customers waiting for service becomes Q and finishes their service altogether. The number of identical servers present is denoted*

as a known function of $r, Q, \mathcal{B}$, i.e. $\eta(r, Q, \mathcal{B}) = \lfloor (r + Q + \mathcal{B})/Q \rfloor$, and the maximum number of customers that may be present in the system is $r + Q + \mathcal{B}$.

Proof. It has been noted that the number of identical servers present is denoted as a function of $r, Q, \mathcal{B}$, since at maximum there can be $\eta(r, Q, \mathcal{B}) = \lfloor \frac{r+Q+\mathcal{B}}{Q} \rfloor$ simultaneous operating servers in the system. Inside the floor function, the nominator, i.e. $r + Q + \mathcal{B}$, denotes the maximum number of customers allowed in the system. In other words, an arriving customer who finds $r + Q + \mathcal{B}$ customers in the system leaves immediately.

Some observations are required to form the queueing system analogy with the proposed (r, Q) inventory policy. Without loss of generality, assume that $IP(0) = r + Q$ and $IN(0) = r + Q$. To demonstrate our proposition, let the sequence $\mathbb{A}_1 \equiv \{T_i : i \geq 1\}$ denote the cumulative realized arrival times of demands to the inventory system, i.e. T_1 denotes the arrival time for the first demand, T_2 denotes the arrival time for the second demand, and T_i denotes the arrival time for the i^{th} demand. We assume that $T_0 = 0$. For $i \geq 2$, we would like to note that $T_i - T_{i-1}$ represents the inter-arrival time of the i^{th} demand, but they are not necessarily identically distributed since we will incorporate a state-dependent arrival process. Further, let $\mathbb{B}_1 \equiv \{S_i : i \geq 1\}$ denote a countable collection of the successive realized replenishment lead times. For our setting, S_i are known to be i.i.d. Now, assume that streams $\mathbb{A}_1$ and $\mathbb{B}_1$ govern the cumulative arrival process of the queueing system, and service times of the servers for a fixed batch of Q customers, respectively. Finally, let the stochastic process $X(t)$ denote the number of customers in the queueing system at time $t \geq 0$, and assume $X(0) = 0$.

We propose that the sample paths of the processes $X(t)$ and $r + Q - IN(t)$ are equivalent, whenever the two streams described above coincide. To illustrate, $X(T_1) = 1$ and $IN(T_1) = r + Q - 1$, which implies that $r + Q - IN(T_1) = 1$. Also, $X(T_Q) = Q$ and $IN(T_Q) = r$. Assuming that $S_1 > T_{Q+1} - T_Q$, $X(T_Q + S_1) = 0$ and $IN(T_Q + S_1) = r + Q$, again indicating that $r + Q - IN(T_Q + S_1) = 0 = X(T_Q + S_1)$. However, since we assumed λ_n to be the state-dependent rate of demands per unit time when $IN = n$, it is required that $\tilde{\lambda}_{r+Q-n} = \lambda_n$ for $n \leq r + Q$, where $\tilde{\lambda}_i$ denotes the rate of arrival of

customers per unit time in the queueing system when there are i customers present in the system, $i \geq 0$.

We would like to note that the stochastic process $IO(t)$ is equivalent to the process $Q \lfloor X(t)/Q \rfloor$, assuming that there cannot be more than $Q - 1$ customers waiting in the queue, i.e. $\eta(r, Q, \mathcal{B})$ is equal to the value described above. This is because $IO(t)$ denotes the number of units that are ordered and awaited in the inventory system, which translates into the number of customers residing on servers in the queueing system. Lastly, the stochastic process $IP(t) \equiv r + Q - (X(t) \bmod Q)$, because $(X(t) \bmod Q)$ denotes the number of customers in the queue, which is the same as the number of demands that have arrived until the last replenishment order. $\square$

In the next section, we will extend the Forward *Kolmogorov* equations into uncountable state space with the help of the supplementary variable technique and formulate the global balance equations. Before we begin the section, let us present the outline of Section 5.2. We begin the section with an explanation of the notation, required probability functions, and introduction of the required supplementary variables. The section consists of 4 subsections. In Subsection 5.2.1, we derive the time-dependent Forward Kolmogorov equations by using the supplementary variable technique. In Subsection 5.2.2, we destroy the time dimension and obtain the steady-state global balance equations which are integral partial-differential difference equations. We continue with Subsection 5.2.3, where we analytically solve the Forward Kolmogorov equations on systems that have one server. Finally, in Subsection 5.2.4 we propose a finite difference based numerical technique to discretize the Forward Kolmogorov equations and obtain a suitable approximation of the steady-state behavior for systems where more than one server exists.

5.2. Global Balance Equations of $M_n/L^Q/\infty$

In this section, we provide the global balance equations of the $M_n/L^Q/\infty$ queueing system. That is, there are infinitely many identical servers with a degenerate (deter-

ministic) service time distribution and providing service in a fixed size of Q customers. M_n denotes the state dependent Markovian arrival process, i.e. the next inter-arrival time is distributed as a negative exponential random variable with rate λ_n customers per unit time when there are n customers in the system. We provide notation, methodology, and global balance equations, which turn out to be a set of partial differential integral difference equations. Then we utilize the global balance equations to determine the steady-state behavior and several interesting metrics.

Let the stochastic process $X(t)$ denote the number of customers in the system at time $t \geq 0$. $X(t)$ is a semi-Markov process, since the degenerate service time distribution brings in memory to the system, i.e. the process $X(t)$ does not evolve in a Markovian way. That is why traditional Markovian analysis methods fail, and we employ the supplementary variable technique to derive the global balance equations.

The supplementary variable technique may be employed through the forward or backward *Kolmogorov* equations, which are essentially derived by conditioning on the last event that has taken place or the next event that will take place, respectively. However, one must define different supplementary variables and might be required to utilize hazard rate or probability density function of the inter-arrival time random variable depending on the approach. We develop the forward *Kolmogorov* equation set for the system by conditioning on the last event that could have taken place a sufficiently short time ago and the system state that leads us to the state we are interested in before the transition. Therefore, we utilize the elapsed service or waiting times as our supplementary variables.

Let y_i denote the elapsed service time of the server i , i.e. for how long Q customers were in service with the server i , and $G(y)$ the cumulative distribution function (CDF) of the service time random variable Y . Also, denote the probability density function (pdf) and the hazard rate function of the random variable Y by $g(y)$ and $h(y)$, respectively. We express the elapsed waiting time of the i^{th} customer in the queue as x_i , i.e. the elapsed time since the arrival of that customer. Let

$$f_{n,m}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t)$$

represent the probability density that there are n customers waiting in the queue with elapsed waiting times $x_1 \leq x_2 \leq \dots \leq x_n$, m servers operating with elapsed service times $y_1 \leq y_2 \leq \dots \leq y_m \leq L$, and at time $t \geq 0$. We would like to indicate that $0 \leq n \leq Q - 1$ and $m \geq 0$. Additionally, we would like to indicate that if $m \geq 1$, that is, a server is active, then it is always the case that $x_n \leq y_1$. This is because the service routine was assumed to be FIFO, or in other words, the elapsed service time of the most recent batch must be larger than or equal to the elapsed waiting time of the longest waiting customer in the queue.

Let $d\psi_1 d\psi_2 \dots d\psi_n$ and $d\xi_1 d\xi_2 \dots d\xi_m$ denote differential time elements such that

$$f_{n,m}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t) d\psi_1 d\psi_2 \dots d\psi_n d\xi_1 d\xi_2 \dots d\xi_m$$

approaches the probability that at time t there are n customers waiting in the queue where j^{th} most recently arrived customer's elapsed waiting time is between $[x_j, x_j + d\psi_j)$ for $j \in \{1, 2, \dots, n\}$, and m servers occupied with server i 's elapsed service time is between $[y_i, y_i + d\xi_i)$ for $i \in \{1, 2, \dots, m\}$.

Assume that the service times are deterministic and are equal to $L > 0$. Then, pdf, cdf, and hazard rate function may be defined as follows:

$$g(y) = \mathbb{P}\{Y = y\} = \begin{cases} 0 & \text{if } y \neq L \\ 1 & \text{if } y = L \end{cases},$$

$$G(y) = \mathbb{P}\{Y \leq y\} = \begin{cases} 0 & \text{if } y < L \\ 1 & \text{if } y \geq L \end{cases},$$

$$h(y) = \mathbb{P}\{Y = y | Y \geq y\} = \begin{cases} 0 & \text{if } y < L \\ 1 & \text{if } y = L \end{cases}. \quad (5.1)$$

We will examine each possible case and derive the forward equations corresponding to our system. From now on, we consider a sufficiently small, yet positive, constant δ to model the change. Conventionally, as in the *Bachmann-Landau* notation, let the notation $f(x) = o(\delta)$ stand for

$$\lim_{\delta \rightarrow 0^+} \frac{|f(\delta)|}{\delta} = 0, \quad (5.2)$$

which may be found in [28], but we only consider unilateral limiting behavior due to reasons that will be explained later.

5.2.1. Forward Kolmogorov Equations

- **Case 1:** $(n = 0), (m = 0)$ (Boundary Condition)

$$f_{0,0}(t) = f_{0,0}(t - \delta)[1 - \lambda_0\delta] + f_{0,1}(L; t - \delta)\delta h(L) + o(\delta). \quad (5.3)$$

- **Case 2:** $(Q - 1 \geq n \geq 1), (m \geq 0), (x_1 = 0), (y_1 > 0)$ (Boundary Condition)

$$\begin{aligned} & f_{n,m}(0, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t) \delta d\psi_2 \cdots d\psi_n d\xi_1 d\xi_2 \cdots d\xi_m \\ &= f_{n-1,m}(x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) [\lambda_{(n-1+mQ)}\delta] d\psi_2 \\ & \cdots d\psi_n d\xi_1 \cdots d\xi_m + o(\delta). \end{aligned} \quad (5.4)$$

- **Case 3:** $(n = 0), (m \geq 1), (y_1 = 0)$ (Boundary Condition)

$$f_{0,m}(0, y_2, y_3, \dots, y_m; t) \delta d\xi_2 \cdots d\xi_m$$

$$\begin{aligned}
&= \int \cdots \int_{0 \leq x_1 \leq \cdots \leq y_2 - \delta} f_{Q-1, m-1}(x_1, \dots, x_{Q-1}, y_2 - \delta, y_3 - \delta, \dots, y_m - \delta; t - \delta) \\
&\cdot dx_1 dx_2 \cdots dx_{Q-1} \lambda_{mQ-1} \delta d\xi_2 \cdots d\xi_m + o(\delta).
\end{aligned} \tag{5.5}$$

- **Case 4:** $(Q - 1 \geq n \geq 1), (m = 0), (x_1 > 0)$

$$\begin{aligned}
&f_{n,0}(x_1, x_2, x_3, \dots, x_n; t) d\psi_1 \cdots d\psi_n \\
&= f_{n,0}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta; t - \delta) d\psi_1 \cdots d\psi_n [1 - \lambda_n \delta] \\
&+ f_{n,1}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta, L; t - \delta) h(L) \delta d\psi_1 \cdots d\psi_n \\
&+ o(\delta).
\end{aligned} \tag{5.6}$$

- **Case 5:** $(n = 0), (m \geq 1), (y_1 > 0)$

$$\begin{aligned}
&f_{0,m}(y_1, y_2, y_3, \dots, y_m; t) d\xi_1 \cdots d\xi_m = \\
&= f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) d\xi_1 \cdots d\xi_m \left[1 - \lambda_{mQ} \delta - \sum_{i=1}^m h(y_i - \delta) \delta \right] \\
&+ f_{0,m+1}(y_1 - \delta, \dots, y_m - \delta, L; t - \delta) h(L) \delta d\xi_1 \cdots d\xi_m \\
&+ o(\delta).
\end{aligned} \tag{5.7}$$

- **Case 6:** $(Q - 1 \geq n \geq 1), (m \geq 1), (y_1 > 0), (x_1 > 0)$

$$\begin{aligned}
&f_{n,m}(x_1, x_2, \dots, x_n, y_1, y_2, y_3, \dots, y_m; t) d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m = \\
&= f_{n,m}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) d\psi_1 \cdots d\psi_n \\
&\cdot d\xi_1 \cdots d\xi_m \left[1 - \lambda_{(n+m)Q} \delta - \sum_{i=1}^m h(y_i - \delta) \delta \right] \\
&+ f_{n,m+1}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta, L; t - \delta) h(L) \delta \\
&\cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m + o(\delta).
\end{aligned} \tag{5.8}$$

In the next section we derive the integral-partial differential difference equations which will constitute the global balance equations for stationary behavior.

5.2.2. Integral-Partial Differential Difference Equations

For each of the cases mentioned in Subsection 5.2.1, we perform the necessary manipulations to derive the steady-state global balance equations. For $0 \leq n \leq Q - 1$, $m \geq 0$, the steady-state probability density functions of the system are represented as:

$$\lim_{t \rightarrow \infty} f_{n,m}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t) = p_{n,m}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m).$$

For validity of the operations that will follow, we assume that the time-dependent probability density functions are uniformly continuous with respect to t . We note that uniform continuity with respect to t is not an unrealistic assumption and, as an intuitive checking mechanism, does not raise a contradiction, since the results turn out to be valid if the system is stable. Also, for the stability of the system, we assume that $\lim_{n \rightarrow \infty} \lambda_n$ is finite.

- **Case 1:** ($n = 0$), ($m = 0$) (Boundary Condition)

$$f_{0,0}(t) = f_{0,0}(t - \delta)[1 - \lambda_0\delta] + f_{0,1}(L; t - \delta)\delta h(L) + o(\delta).$$

Simple algebraic manipulations yields one with:

$$\frac{f_{0,0}(t) - f_{0,0}(t - \delta)}{\delta} = -\lambda_0 f_{0,0}(t - \delta) + f_{0,1}(L; t - \delta)h(L) + \frac{o(\delta)}{\delta}.$$

Let us apply $\lim_{\delta \rightarrow 0^+}$ on both sides, and noting that on Equation (5.1), $h(L) = 1$:

$$\frac{\partial f_{0,0}(t)}{\partial t} = -\lambda_0 f_{0,0}(t) + f_{0,1}(L; t).$$

Now let us take the limit as $t \rightarrow \infty$. We have assumed that pdf is uniformly continuous with respect to t , which allows us to interchange the limit and derivative operators on the left-hand side:

$$\frac{\partial}{\partial t} \lim_{t \rightarrow \infty} f_{0,0}(t) = -\lambda_0 \lim_{t \rightarrow \infty} f_{0,0}(t) + \lim_{t \rightarrow \infty} f_{0,1}(L; t).$$

We would like to note that the partial derivative operator is equivalent to total derivative operator in this case, since there is only one independent variable. However, for the consistency of the notation, we employ the partial derivative operator.

$$\frac{\partial}{\partial t} p_{0,0} = -\lambda_0 p_{0,0} + p_{0,1}(L),$$

noting that $p_{0,0}$ is a constant, this yields us with the final equation for the Case 1:

$$\lambda_0 p_{0,0} = p_{0,1}(L). \quad (5.9)$$

- **Case 2:** $(Q - 1 \geq n \geq 1)$, $(m \geq 0)$, $(x_1 = 0)$, $(y_1 > 0)$ (Boundary Condition)

$$\begin{aligned} & f_{n,m}(0, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t) \delta d\psi_2 \cdots d\psi_n d\xi_1 d\xi_2 \cdots d\xi_m \\ &= f_{n-1,m}(x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) [\lambda_{(n-1+mQ)} \delta] d\psi_2 \\ & \cdots d\psi_n d\xi_1 \cdots d\xi_m + o(\delta). \end{aligned}$$

Let us divide both sides by δ , and take the limit of both sides as $\delta \rightarrow 0^+$.

$$f_{n,m}(0, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t) = f_{n-1,m}(x_2, \dots, x_n, y_1, y_2, \dots, y_m; t) \lambda_{(n-1+mQ)},$$

noting that differential time elements cancel each other. Taking the limit as $t \rightarrow \infty$:

$$p_{n,m}(0, x_2, \dots, x_n, y_1, y_2, \dots, y_m) = \lambda_{(n-1+mQ)} p_{n-1,m}(x_2, \dots, x_n, y_1, y_2, \dots, y_m). \quad (5.10)$$

- **Case 3:** ($n = 0$), ($m \geq 1$), ($y_1 = 0$) (Boundary Condition)

$$\begin{aligned} & f_{0,m}(0, y_2, y_3, \dots, y_m; t) \delta d\xi_2 \cdots d\xi_m. \\ = & \int \cdots \int_{0 \leq x_1 \leq \cdots \leq y_2 - \delta} f_{Q-1,m-1}(x_1, \dots, x_{Q-1}, y_2 - \delta, y_3 - \delta, \dots, y_m - \delta; t - \delta) \\ & \cdot dx_1 dx_2 \cdots dx_{Q-1} \lambda_{mQ-1} \delta d\xi_2 \cdots d\xi_m \end{aligned}$$

We divide both sides by δ , and take the limit as $\delta \rightarrow 0^+$. At this point, we require the transient density to be uniformly continuous with respect to the supplementary variables y_i to exchange the integral operator with the limit operator. This requirement is satisfied since the domain of the function is compact, i.e. $[0, L]^{Q-1+mQ}$, and it is assumed that the function is continuous over the domain. Then,

$$\begin{aligned} & f_{0,m}(0, y_2, y_3, \dots, y_m; t) \\ = & \lambda_{mQ-1} \int \cdots \int_{0 \leq x_1 \leq \cdots \leq y_2} f_{Q-1,m-1}(x_1, \dots, x_{Q-1}, y_2, y_3, \dots, y_m; t) \\ & \cdot dx_1 dx_2 \cdots dx_{Q-1}. \end{aligned}$$

Taking the limit as $t \rightarrow \infty$ yields us with:

$$\begin{aligned} & p_{0,m}(0, y_2, y_3, \dots, y_m) \\ = & \lambda_{mQ-1} \int \cdots \int_{0 \leq x_1 \leq \cdots \leq y_2} p_{Q-1,m-1}(x_1, \dots, x_{Q-1}, y_2, y_3, \dots, y_m) \\ & \cdot dx_1 dx_2 \cdots dx_{Q-1}. \end{aligned} \quad (5.11)$$

- **Case 4:** $(Q - 1 \geq n \geq 1)$, $(m = 0)$, $(x_1 > 0)$

$$\begin{aligned}
& f_{n,0}(x_1, x_2, x_3, \dots, x_n; t) d\psi_1 \cdots d\psi_n \\
&= f_{n,0}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta; t - \delta) d\psi_1 \cdots d\psi_n [1 - \lambda_n \delta] \\
&+ f_{n,1}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta, L; t - \delta) h(L) \delta d\psi_1 \cdots d\psi_n \\
&+ o(\delta).
\end{aligned}$$

Simple algebraic manipulations yields us with,

$$\begin{aligned}
& \frac{f_{n,0}(x_1, x_2, x_3, \dots, x_n; t) - f_{n,0}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta; t - \delta)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n \\
&= -\lambda_n f_{n,0}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta; t - \delta) d\psi_1 \cdots d\psi_n \\
&+ f_{n,1}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta, L; t - \delta) h(L) d\psi_1 \cdots d\psi_n \\
&+ \frac{o(\delta)}{\delta}.
\end{aligned}$$

We add and subtract the transient pdfs on the left-hand side numerator, fixing one variable at a time, before taking the limits. This trick allows us to obtain the partial differentials of the transient pdf. Namely,

$$\begin{aligned}
& \frac{f_{n,0}(x_1, x_2, x_3, \dots, x_n; t) - f_{n,0}(x_1 - \delta, x_2, x_3, \dots, x_n; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n \\
&+ \frac{f_{n,0}(x_1 - \delta, x_2, x_3, \dots, x_n; t) - f_{n,0}(x_1 - \delta, x_2 - \delta, x_3, \dots, x_n; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n \\
&+ \frac{f_{n,0}(x_1 - \delta, x_2 - \delta, x_3, \dots, x_n; t) - f_{n,0}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n + \cdots \\
&+ \frac{f_{n,0}(x_1 - \delta, x_2 - \delta, \dots, x_n; t) - f_{n,0}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n \\
&+ \frac{f_{n,0}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta; t) - f_{n,0}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta; t - \delta)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n
\end{aligned}$$

$$\begin{aligned}
&= -\lambda_n f_{n,0}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta; t - \delta) d\psi_1 \cdots d\psi_n \\
&+ f_{n,1}(x_1 - \delta, x_2 - \delta, x_3 - \delta, \dots, x_n - \delta, L; t - \delta) h(L) d\psi_1 \cdots d\psi_n \\
&+ \frac{o(\delta)}{\delta}.
\end{aligned}$$

Taking the limits of both sides as $\delta \rightarrow 0^+$:

$$\begin{aligned}
&\sum_{i=1}^n \frac{\partial f_{n,0}(x_1, x_2, \dots, x_n; t)}{\partial x_i} + \frac{\partial f_{n,0}(x_1, x_2, \dots, x_n; t)}{\partial t} \\
&= -\lambda_n f_{n,0}(x_1, x_2, x_3, \dots, x_n; t) \\
&+ f_{n,1}(x_1, x_2, x_3, \dots, x_n, L; t)
\end{aligned}$$

Again, taking limits of both sides with respect to t , as t tends to ∞ , and for the last term on the left-hand side involving partial derivative with respect to t switching the partial derivative operator with the limit operator due to reasons mentioned above,

$$\begin{aligned}
\sum_{i=1}^n \frac{\partial p_{n,0}(x_1, x_2, \dots, x_n)}{\partial x_i} &= -\lambda_n p_{n,0}(x_1, x_2, x_3, \dots, x_n) \\
&+ p_{n,1}(x_1, x_2, x_3, \dots, x_n, L)
\end{aligned} \tag{5.12}$$

- **Case 5:** $(n = 0), (m \geq 1), (y_1 > 0)$

$$\begin{aligned}
&f_{0,m}(y_1, y_2, y_3, \dots, y_m; t) d\xi_1 \cdots d\xi_m = \\
&= f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) d\xi_1 \cdots d\xi_m \left[1 - \lambda_{mQ} \delta - \sum_{i=1}^m h(y_i - \delta) \delta \right] \\
&+ f_{0,m+1}(y_1 - \delta, \dots, y_m - \delta, L; t - \delta) h(L) \delta d\xi_1 \cdots d\xi_m \\
&+ o(\delta).
\end{aligned}$$

Performing simple algebraic operations and the trick mentioned in Case 4,

$$\begin{aligned}
&\frac{f_{0,m}(y_1, y_2, \dots, y_m; t) - f_{0,m}(y_1 - \delta, y_2, \dots, y_m; t)}{\delta} d\xi_1 \cdots d\xi_m \\
&+ \frac{f_{0,m}(y_1 - \delta, y_2, \dots, y_m; t) - f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m; t)}{\delta} d\xi_1 \cdots d\xi_m
\end{aligned}$$

$$\begin{aligned}
& + \dots \\
& + \frac{f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m; t) - f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t)}{\delta} d\xi_1 \dots d\xi_m \\
& + \frac{f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t) - f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta)}{\delta} \\
& \cdot d\xi_1 \dots d\xi_m \\
& = f_{0,m}(y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) d\xi_1 \dots d\xi_m \left[-\lambda_{mQ} - \sum_{i=1}^m h(y_i - \delta) \right] \\
& + f_{0,m+1}(y_1 - \delta, \dots, y_m - \delta, L; t - \delta) h(L) d\xi_1 \dots d\xi_m \\
& + \frac{o(\delta)}{\delta}.
\end{aligned}$$

Taking the limits of the both sides as $\delta \rightarrow 0^+$,

$$\begin{aligned}
\sum_{i=1}^m \frac{\partial f_{0,m}(y_1, y_2, \dots, y_m; t)}{\partial y_i} + \frac{\partial f_{0,m}(y_1, y_2, \dots, y_m; t)}{\partial t} &= -\lambda_{mQ} f_{0,m}(y_1, y_2, \dots, y_m; t) \\
&+ f_{0,m+1}(y_1, \dots, y_m, L; t).
\end{aligned} \tag{5.13}$$

Here we note that $\lim_{\delta \rightarrow 0^+} h(L - \delta) = 0$, since the function has a jump discontinuity from the left hand-side at the point L . Taking the limit as $t \rightarrow \infty$, and interchanging the limit and partial derivative operator of the last term on the left-hand side,

$$\sum_{i=1}^m \frac{\partial p_{0,m}(y_1, y_2, \dots, y_m)}{\partial y_i} = -\lambda_{mQ} p_{0,m}(y_1, y_2, \dots, y_m) + p_{0,m+1}(y_1, \dots, y_m, L). \tag{5.14}$$

- **Case 6:** $(Q - 1 \geq n \geq 1), (m \geq 1), (y_1 > 0), (x_1 > 0)$

$$\begin{aligned}
& f_{n,m}(x_1, x_2, \dots, x_n, y_1, y_2, y_3, \dots, y_m; t) d\psi_1 \dots d\psi_n d\xi_1 \dots d\xi_m = \\
& = f_{n,m}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) d\psi_1 \dots d\psi_n \\
& \cdot d\xi_1 \dots d\xi_m \left[1 - \lambda_{(n+mQ)} \delta - \sum_{i=1}^m h(y_i - \delta) \delta \right] \\
& + f_{n,m+1}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta, L; t - \delta) h(L) \delta \\
& \cdot d\psi_1 \dots d\psi_n d\xi_1 \dots d\xi_m + o(\delta).
\end{aligned}$$

Simple algebraic operations provide us with,

$$\begin{aligned}
& \frac{f_{n,m}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t) - f_{n,m}(x_1 - \delta, x_2, \dots, x_n, y_1, y_2, \dots, y_m; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m \\
& + \frac{f_{n,m}(x_1 - \delta, x_2, \dots, x_n, y_1, \dots, y_m; t) - f_{n,m}(x_1 - \delta, x_2 - \delta, \dots, x_n, y_1, \dots, y_m; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m + \cdots \\
& + \frac{f_{n,m}(x_1 - \delta, \dots, x_n, y_1, \dots, y_m; t) - f_{n,m}(x_1 - \delta, \dots, x_n - \delta, y_1, \dots, y_m; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m \\
& + \frac{f_{n,m}(x_1 - \delta, \dots, x_n - \delta, y_1, \dots, y_m; t) - f_{n,m}(x_1 - \delta, \dots, x_n - \delta, y_1 - \delta, \dots, y_m; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m + \cdots \\
& + \frac{f_{n,m}(x_1 - \delta, \dots, y_1 - \delta, \dots, y_m; t) - f_{n,m}(x_1 - \delta, \dots, y_1 - \delta, \dots, y_m - \delta; t)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m \\
& + \frac{f_{n,m}(x_1 - \delta, \dots, y_1 - \delta, \dots, y_m; t) - f_{n,m}(x_1 - \delta, \dots, y_1 - \delta, \dots, y_m - \delta; t - \delta)}{\delta} \\
& \cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m \\
& = f_{n,m}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta; t - \delta) d\psi_1 \cdots d\psi_n \\
& \cdot d\xi_1 \cdots d\xi_m \left[-\lambda_{(n+mQ)} - \sum_{i=1}^m h(y_i - \delta) \right] \\
& + f_{n,m+1}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta, y_1 - \delta, y_2 - \delta, \dots, y_m - \delta, L; t - \delta) h(L) \\
& \cdot d\psi_1 \cdots d\psi_n d\xi_1 \cdots d\xi_m + \frac{o(\delta)}{\delta}.
\end{aligned}$$

Taking the limit of both sides as $\delta \rightarrow 0^+$:

$$\begin{aligned}
& \sum_{i=1}^n \frac{\partial f_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)}{\partial x_i} + \sum_{j=1}^m \frac{\partial f_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)}{\partial y_j} \\
& + \frac{\partial f_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)}{\partial t} \\
& = -\lambda_{(n+mQ)} f_{n,m}(x_1, \dots, x_n, y_1, y_2, \dots, y_m; t) + f_{n,m+1}(x_1, \dots, x_n, y_1, \dots, y_m, L; t).
\end{aligned}$$

Taking the limit of both sides as $t \rightarrow \infty$:

$$\begin{aligned} & \sum_{i=1}^n \frac{\partial p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)}{\partial x_i} + \sum_{j=1}^m \frac{\partial p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)}{\partial y_j} \\ &= -\lambda_{(n+mQ)} p_{n,m}(x_1, \dots, x_n, y_1, y_2, \dots, y_m) + p_{n,m+1}(x_1, \dots, x_n, y_1, \dots, y_m, L). \end{aligned} \quad (5.15)$$

Finally, we have the normalization equation:

$$\sum_{n=0}^{Q-1} \sum_{m=0}^{\infty} \int \cdots \int p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) dx_1 \cdots dx_n dy_1 \cdots dy_m = 1. \quad (5.16)$$

Equations (5.9)-(5.15) with the normalization condition, i.e. Equation (5.16), form the global balance equations for the queueing system $M_n/L^Q/\infty$. We have formulated the equations for the uncapacitated case; however, if it is desired to impose a capacity constraint on the system, the equation system does not change yet is truncated. Denote the maximum number of customers, in the queue and servers combined, allowed in the system to be $\mathcal{C} \in \mathbb{N}$. Then, the global balance equations are obtained by setting $\lambda_n = 0$ for $n \geq \mathcal{C}$ and fixing

$$p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) = 0, \quad (5.17)$$

for all pairs $(n, m) \in \{(n, m) : (n, m) \in \mathbb{N}^2, n + mQ > \mathcal{C}, n < Q\}$. We are able to obtain functional forms of densities analytically for the cases where $\mathcal{C} \leq 2Q - 1$. We note that the capacity of the queueing system, i.e. $\mathcal{C}$, is directly related to the amount of backorders allowed in the inventory policy $(r, Q, \mathcal{B})$. In fact, $r + Q + \mathcal{B} = \mathcal{C}$. Therefore, it is crucial for us to be able to translate the set of Forward Kolmogorov Equations into a capacitated case.

As a remark, we have assumed that the cumulative arrivals form a Poisson point process. Therefore, the occurrence of two arrivals in a short time interval of length, say $\delta > 0$, is $o(\delta)$. Then we may safely state that for any steady-state probability

density function $p_{n,m}(\cdot)$, the probability density of observing any two equal-valued supplementary variables is equal to 0. However, the limits as two subsequent independent variables approach each other exist and are not necessarily equal to 0. In the next proposition, we state a result regarding continuity of the steady-state probability density functions.

Proposition 5.2. *For the steady-state probability density $p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)$, if $n \geq 1$, then for $x_1 \in \mathbb{R}_0^+$ and the set of variables $x_2, \dots, x_n, y_1, \dots, y_m$ where the function is defined, the limit*

$$\lim_{x_2 \rightarrow x_1^+} p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m), \quad (5.18)$$

exists and is not equal to 0. However, the function has value 0 if $x_1 = x_2$, which implies that the function is right discontinuous and has a jump discontinuity in that hyperplane. In fact, one may state that:

$$\lim_{x_n \rightarrow x_{n-1}^+} \cdots \lim_{x_2 \rightarrow x_1^+} \lim_{x_1 \rightarrow 0^+} p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) = p_{0,m}(y_1, \dots, y_m) \prod_{i=0}^{n-1} \lambda_{i+mQ}. \quad (5.19)$$

Additionally, for a steady-state probability density of type $p_{0,m}(y_1, \dots, y_m)$ and for the set of variables where the function is defined, the limit:

$$\lim_{y_2 \rightarrow y_1^+} p_{0,m}(y_1, \dots, y_m), \quad (5.20)$$

exists and is equal to 0. Therefore, the steady-state probability density functions of type $p_{0,m}(y_1, \dots, y_m)$ are right continuous.

Proof. As noted above, it is widely known that for a Poisson point process, the probability of the occurrence of more than one event on a short interval of length $\delta > 0$ is $o(\delta)$. In Equations (5.18) and (5.19), the situation of interest is the occurrence of two arrivals at an instant, which is impossible due to the definition of the Poisson point process. We note that the mentioned stochastic process is not a compound Poisson process, where the situation would be different in that case. Therefore, for any steady-

state probability density we may safely state that if two or more independent variables are equal:

$$p_{n,m}(x_1, x_1, \dots, x_n, y_1, \dots, y_m) = 0.$$

Now we are required to show the value of the limits presented in Equations (5.18) and (5.19). To show the statement in (5.18), we may appeal to Equation (5.10), and let x_2 approach x_1 from the right. This operation does not violate any assumptions required to derive the original equation. Therefore, the steady-state probability density is defined and is not equal to 0, because the ordering relationships among the independent variables are preserved. It is trivial to observe why the steady-state probability density is not equal to 0 for $x_1 = 0$, since iteratively tracing the boundary equations backward results in a multiple of the normalization constant π_0 , which is strictly positive, unless the system is not degenerate. For the other cases, i.e. where $x_1 \neq 0$, observe that the system has a strictly positive probability density to be in that state. In other words, when the supplementary variables are not equal to each other, there is a strictly positive probability density to observe that state of the system.

Now, to show the statement in (5.19), we again invoke Equation (5.10) and may take the limits in a consecutive manner to agree with our result in Equation (5.19). No assumptions are violated, since the limits are one-sided. Therefore, the steady-state probability density has a jump-discontinuity at any point where two or more independent variables are equal to each other. Also, this statement provides another supporting argument for the first statement, since assuming that x_2 approaching x_1 from the right and x_1 approaching 0 from the right would result in the state of the system x_i approaching x_{i-1} from the right after the arrival of $i - 2$ customers, for an arbitrary positive integer value of $i < Q - 1$.

Now, let us show the limit presented in Equation (5.20). We know that for a small yet positive δ , if $y_2 - y_1 = \delta$, then it implies that $Q + 1$ arrivals have taken place

in an interval of length δ . The probability of occurrence of such an event is $o(\delta)$, and therefore the value of the function is also equal to 0. To observe why the limit is also equal to 0, we employ Equation (5.11). As we let y_2 approach y_1 from the right and y_1 approach 0 from the right, the integral term tends to 0. If it is not possible for y_2 to approach y_1 from the right and y_1 to approach 0 from the right simultaneously, then it is not possible to observe any y_i to approach y_{i-1} from the right, since the system must first visit the former state to arrive at the latter. $\square$

In addition, we make the equations more compact through hierarchical functional relationships that we notice to exist in the provided set of equations, and we provide a novel numerical method to solve the equations for the cases where $\mathcal{C} \geq 2Q$.

5.2.3. Analytical Solutions for a Subset of the Integral-Partial Differential Difference Equations

In this section, we will provide some observations and solve the set of equations derived in the previous subsection. We will begin by defining the unilateral multi-dimensional Laplace transformations. We will provide solutions to the set of equations analytically by multi-dimensional unilateral Laplace transformation under a capacity constraint. Also, we will attack the set of equations using the Method of Characteristics technique to show that the solutions coincide.

Let us begin by defining the one-dimensional unilateral Laplace transformation operator denoted as $\mathcal{L}\{\}$. Namely, for an arbitrary real-valued function f with domain $\mathbb{R}_0^+$:

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{f(x)\}(s) = \int_0^\infty f(x)e^{-sx}dx. \quad (5.21)$$

Additionally, the transformation depicted in Equation (5.21) extends to multiple dimensions for an arbitrary real-valued function g involving two independent variables x

and y , and domain $(\mathbb{R}_0^+)^2$:

$$\mathcal{L}_x\{g\}(s) = \mathcal{L}_x\{g(x, y)\}(s) = \int_0^\infty g(x, y)e^{-sx}dx, \quad (5.22)$$

where the subscript x of $\mathcal{L}$ operator denotes the variable of integration, or the independent variable that transformation is applied on. The multi-dimensional Laplace transformation is also commutative:

$$\begin{aligned} & \mathcal{L}_{xy}\{g(x, y)\}(s, q) \\ &= \int_0^\infty \int_0^\infty g(x, y)e^{-sx}e^{-qy}dxdy \\ &= \int_0^\infty \int_0^\infty g(x, y)e^{-sx}e^{-qy}dydx. \\ &= \mathcal{L}_{yx}\{g(x, y)\}(s, q) \end{aligned}$$

Also, on unilateral Laplace transformations of a function with one independent variable the identity:

$$\mathcal{L}\left\{\frac{df(x)}{dx}\right\}(s) = \int_0^\infty \frac{df(x)}{dx}e^{-sx}dx = s\mathcal{L}\{f(x)\}(s) - f(0), \quad (5.23)$$

holds under some regularity conditions. One may easily prove the result through integration by parts. For multiple-dimensions the Equation (5.23) translates into:

$$\begin{aligned} & \mathcal{L}_{x_1}\left\{\frac{\partial f(x_1, \dots, x_n)}{\partial x_1}\right\}(s) \\ &= \int_0^\infty \frac{\partial f(x_1, \dots, x_n)}{\partial x_1}e^{-sx_1}dx_1 \\ &= s\mathcal{L}_{x_1}\{f(x_1, \dots, x_n)\}(s) - f(0, x_2, \dots, x_n). \end{aligned} \quad (5.24)$$

The result may be proven again using integration by parts. We will extend the results for functions that involve more than two independent variables. For the uniqueness and existence properties of multi-dimensional Laplace transforms, we refer to [29].

Let us make some observations and deductions based on the functional equations provided in the previous subsection.

Proposition 5.3. *Assume that for a real-valued function $f(x_1, \dots, x_n)$ with support $\chi \equiv \{(x_1, \dots, x_n) : x_i \in \mathbb{R}_0^+ \ \forall i \in \{1, \dots, n\}, x_i \leq x_j \ \forall (i, j) \in \{(q, s) : q < s, q \in \{1, \dots, n\}, s \in \{1, \dots, n\}\}\}$, all partial derivatives exist and for a real-valued constant c , the functional equation:*

$$\sum_{i=1}^n \frac{\partial f(x_1, \dots, x_n)}{\partial x_i} = -cf(x_1, \dots, x_n), \quad (5.25)$$

holds. Then, on the original support χ , the function $f(x_1, \dots, x_n)$ must satisfy the relationship:

$$f(x_1, \dots, x_n) = e^{-cx_1} f(0, x_2 - x_1, \dots, x_n - x_1).$$

Proof. Let us apply the n -dimensional unilateral Laplace transform operator on both sides of Equation (5.25). By additivity of the Laplace transformations, on the left-hand side we have the following terms:

$$\sum_{i=1}^n \mathcal{L}_{x_1 \dots x_n} \left\{ \frac{\partial f(x_1, \dots, x_n)}{\partial x_i} \right\} (s_1, \dots, s_n).$$

By invoking Equation (5.24), this is precisely equal to:

$$\begin{aligned} &= \mathcal{L}_{x_1 \dots x_n} \{f(x_1, \dots, x_n)\} (s_1, \dots, s_n) \sum_{i=1}^n s_i - \mathcal{L}_{x_2 \dots x_n} \{f(0, x_2, \dots, x_n)\} (s_2, \dots, s_n) \\ &- \mathcal{L}_{x_1 x_3 \dots x_n} \{f(x_1, 0, x_3, \dots, x_n)\} (s_1, s_3, \dots, s_n) - \dots \\ &- \mathcal{L}_{x_1 \dots x_{n-1}} \{f(x_1, \dots, x_{n-1}, 0)\} (s_1, \dots, s_{n-1}). \end{aligned} \quad (5.26)$$

In Equation (5.26), all of the $(n-1)$ dimensional Laplace transformations vanish except the transformation that does not involve the independent variable x_1 . Let us demon-

strate this fact on an arbitrary $(n - 1)$ -dimensional Laplace transform in Equation (5.26) that does not involve the independent variable x_i where $i \neq 1$. Namely,

$$\begin{aligned}
&= \mathcal{L}_{x_1 \dots x_{i-1}, x_{i+1}, \dots, x_n} \{f(x_1, \dots, x_i = 0, \dots, x_n)\}(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \\
&= \int \dots \int_{0 \leq x_1 \leq \dots \leq x_{i-1} = 0 \leq x_{i+1} \leq \dots \leq x_n < \infty} f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \prod_{j \neq i} e^{-s_j x_j} \\
&\cdot dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n
\end{aligned}$$

By noting that the first $(i - 1)$ integrals are taken over a point, i.e. a measure zero, they vanish. Hence, returning to our main argument, in Equation (5.26) the left-hand side consists of two distinct Laplace transformed terms. By taking the right-hand side into account, we have the following:

$$\begin{aligned}
&\mathcal{L}_{x_1 \dots x_n} \{f(x_1, \dots, x_n)\}(s_1, \dots, s_n) \sum_{i=1}^n s_i - \mathcal{L}_{x_2 \dots x_n} \{f(0, x_2, \dots, x_n)\}(s_2, \dots, s_n) \\
&= -c \mathcal{L}_{x_1 \dots x_n} \{f(x_1, \dots, x_n)\}(s_1, \dots, s_n).
\end{aligned}$$

Then by collecting distinct transformed terms on each side, we have:

$$\begin{aligned}
&\mathcal{L}_{x_1 \dots x_n} \{f(x_1, \dots, x_n)\}(s_1, \dots, s_n) (c + \sum_{i=1}^n s_i) \\
&= \mathcal{L}_{x_2 \dots x_n} \{f(0, x_2, \dots, x_n)\}(s_2, \dots, s_n)
\end{aligned}$$

Dividing each side by $(c + \sum_{i=1}^n s_i)$, and applying inverse Laplace transformation operator with respect to independent variable x_1 , i.e. $\mathcal{L}^{-1}(x_1)$:

$$\begin{aligned}
&\mathcal{L}_{x_2 \dots x_n} \{f(x_1, \dots, x_n)\}(s_2, \dots, s_n) \\
&= e^{-x_1(c + \sum_{i=2}^n s_i)} \mathcal{L}_{x_2 \dots x_n} \{f(0, x_2, \dots, x_n)\}(s_2, \dots, s_n).
\end{aligned}$$

Applying $(n - 1)$ fold inverse Laplace transformation operator on both sides, we finally have:

$$f(x_1, \dots, x_n) = e^{-cx_1} f(0, x_2 - x_1, x_3 - x_1, \dots, x_n - x_1) \prod_{i=2}^n u_{x_1}(x_i),$$

where $u_{x_1}(x_i)$ is conventionally defined as the Heaviside Step function, which is due to *Oliver Heaviside*. That is, for an arbitrary real-valued constant $c \in \mathbb{R}$,

$$u_c(x) = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x \geq c \end{cases}.$$

We observe that over the set of admissible values, i.e. the set χ , the Heaviside Step functions are all equal to 1, and that is what was desired to be shown. □

By Proposition 5.3, for the capacitated case, let us make the set of integral partial differential equations more compact. Before we state the next lemma, let us present an example that shows the hierarchical interconnected relationship between the last $Q + 1$ steady-state probability density functions.

Let $\mathcal{C} = 4, Q = 2$ and $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ be finite real-valued scalars. Let us explicitly state the boundary and non-boundary partial differential equations corresponding to the steady-state probability densities: $p_{0,1}(y_1), p_{1,1}(x_1, y_1)$ and $p_{0,2}(y_1, y_2)$. Namely:

$$\frac{\partial p_{0,1}(y_1)}{\partial y_1} = -\lambda_2 p_{0,1}(y_1) + p_{0,2}(y_1, L).$$

$$\frac{\partial p_{1,1}(x_1, y_1)}{\partial x_1} + \frac{\partial p_{1,1}(x_1, y_1)}{\partial y_1} = -\lambda_3 p_{1,1}(x_1, y_1).$$

$$p_{1,1}(0, y_1) = \lambda_2 p_{0,1}(y_1).$$

$$\frac{\partial p_{0,2}(y_1, y_2)}{\partial y_1} + \frac{\partial p_{0,2}(y_1, y_2)}{\partial y_2} = 0.$$

$$p_{0,2}(0, y_2) = \lambda_3 \int_0^{y_2} p_{1,1}(x_1, y_2) dx_1.$$

By Proposition 5.3, we know that the probability density functions $p_{1,1}(x_1, y_1)$ and $p_{0,2}(y_1, y_2)$ satisfy:

$$p_{1,1}(x_1, y_1) = e^{-\lambda_3 x_1} p_{1,1}(0, y_1 - x_1),$$

$$p_{0,2}(y_1, y_2) = p_{0,2}(0, y_2 - y_1).$$

Taking the boundary conditions of both probability density functions into account:

$$p_{1,1}(x_1, y_1) = e^{-\lambda_3 x_1} \lambda_2 p_{0,1}(y_1 - x_1),$$

$$\begin{aligned} p_{0,2}(y_1, y_2) &= \lambda_3 \int_0^{y_2 - y_1} p_{1,1}(x_1, y_2 - y_1) dx_1 \\ &= \lambda_2 \lambda_3 \int_0^{y_2 - y_1} e^{-\lambda_3 x_1} p_{0,1}(y_2 - y_1 - x_1) dx_1, \end{aligned}$$

where we have substituted the integrand with respect to the first deduction. In fact, this idea can be generalized for any value of Q . In the next lemma, we generalize this phenomenon.

Lemma 5.4. *Assume that the maximum number of customers that may simultaneously reside in the system is finite, and equal to $C \geq Q$. Then, the mathematical forms of steady-state density functions corresponding to the system states with more than $C - Q + 1$ customers may be defined completely by the density function corresponding to the $C - Q$ customers. In other words, the last Q density functions can explicitly be*

derived by the use of the last $(Q + 1)^{st}$ density function.

Proof. Let $\omega = (\mathcal{C} \bmod Q)$. Without loss of generality, assume that $\omega > 0$, if $\omega = 0$ nothing changes except that for the steady-state pdf of the last density function there are no supplementary variables that indicate the elapsed waiting times in the queue, i.e. x_i s. Let $\xi = \lfloor \mathcal{C}/Q \rfloor$. Note that $\omega + Q\xi = \mathcal{C}$. Then, for the non-boundary conditions of the steady-state density function $p_{\omega,\xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi)$ by Equation (5.15) we have:

$$\sum_{i=1}^{\omega} \frac{\partial p_{\omega,\xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi)}{\partial x_i} + \sum_{j=1}^{\xi} \frac{\partial p_{\omega,\xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi)}{\partial y_j} = 0.$$

By Proposition 5.3 we know that the pdf $p_{\omega,\xi}(\cdot)$ must satisfy:

$$p_{\omega,\xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi) = p_{\omega,\xi}(0, x_2 - x_1, \dots, x_\omega - x_1, y_1 - x_1, \dots, y_\xi - x_1). \quad (5.27)$$

By combining our findings in Equations (5.27) and (5.10), it is known that:

$$p_{\omega,\xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi) = \lambda_{(\mathcal{C}-1)} p_{\omega-1,\xi}(x_2 - x_1, \dots, x_\omega - x_1, y_1 - x_1, \dots, y_\xi - x_1).$$

Repeating this process $\omega - 1$ more times results in:

$$\begin{aligned} & p_{\omega,\xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi) \\ &= \left(\prod_{i=0}^{\omega-1} \lambda_{(i+Q\xi)} \right) \left(\prod_{j=1}^{\omega-1} e^{-\lambda_{(\omega-j+Q\xi)}[x_{j+1}-x_j]} \right) p_{0,\xi}(y_1 - x_\omega, \dots, y_\xi - x_\omega). \end{aligned} \quad (5.28)$$

By using Equations (5.11) and (5.28) we obtain:

$$\begin{aligned} & p_{\omega,\xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi) \\ &= \left(\prod_{i=0}^{\omega-1} \lambda_{(i+Q\xi)} \right) \left(\prod_{j=1}^{\omega-1} e^{-\lambda_{(\omega-j+Q\xi)}[x_{j+1}-x_j]} \right) \lambda_{(Q\xi-1)} e^{-\lambda_{Q\xi}(y_1-x_\omega)} \\ & \cdot \int \cdots \int_{0 \leq x'_1 \leq \dots \leq x'_{Q-1} \leq y_2 - y_1} p_{Q-1,\xi-1}(x'_1, \dots, x'_{Q-1}, y_2 - y_1, \dots, y_\xi - y_1) \end{aligned}$$

$$\cdot dx'_1 \dots dx'_{Q-1}. \quad (5.29)$$

Let us use Proposition 5.3 and Equation (5.10) on the integrand of Equation (5.29). After $(Q - \omega)$ steps, we obtain the following:

$$\begin{aligned} & p_{\omega, \xi}(x_1, \dots, x_\omega, y_1, \dots, y_\xi) \\ = & \left(\prod_{i=0}^{Q-1} \lambda_{(i+\omega+(Q-1)\xi)} \right) \left(\prod_{j=1}^{\omega-1} e^{-\lambda_{(\omega-j+Q\xi)}[x_{j+1}-x_j]} \right) e^{-\lambda_{Q\xi}(y_1-x_\omega)} \\ & \cdot \int \dots \int_{0 \leq x'_1 \leq \dots \leq x'_{Q-1} \leq y_2 - y_1} \left(\prod_{j=1}^{Q-\omega} e^{-\lambda_{(Q\xi-i+1)}[x'_{i+1}-x'_i]} \right) e^{-\lambda_{(Q\xi-1)}x'_1} \\ & \cdot p_{\omega, \xi-1}(x'_{Q-\omega} - x'_{Q-\omega-1}, \dots, y_2 - y_1 - x'_{Q-\omega-1}, \dots, y_\xi - y_1 - x'_{Q-\omega-1}) \\ & \cdot dx'_1 \dots dx'_{Q-1}. \end{aligned} \quad (5.30)$$

To further simplify it, in Equation (5.30), the inner $(Q-\omega-2)$ integrals can be explicitly evaluated. The demonstrated steps lead to the expression of the last Q steady-state probability density functions as the last $(Q+1)^{st}$ density function, i.e. $p_{\omega, \xi-1}$, which is what was desired to be shown. □

The above result implies that there is no need to derive the mathematical forms of the last Q density functions. In fact, upon substituting the last $(Q+1)^{st}$ probability density function instead of the last probability density function on the non-boundary Forward Kolmogorov Equation, i.e. either Equation (5.15) or Equation (5.14), yields an autonomous partial differential equation. However, the boundary values are dependent on the other probability density functions or on multiple-fold integrals of other probability density functions, making the equation difficult to solve. To overcome this problem, we will employ some mathematical tricks to remove the partial differential operators and then attack the problem numerically. First, let us demonstrate under what conditions it is possible to recover mathematical forms of probability density functions.

Proposition 5.5. *For a system with $\mathcal{C} \leq 2Q - 1$, i.e. at most one server operating, the mathematical forms of probability density functions can be recovered from the set of Forward Kolmogorov Equations.*

Proof. Without loss of generality, assume that $\mathcal{C} = 2Q - 1$. For the cases where $\mathcal{C} < 2Q - 1$, the same solution procedures are able to recover the probability density functions, and the solution becomes simpler. Also, assume that the vector of λ_i s are consisting of positive, finite real values, except $\lambda_{\mathcal{C}} = 0$. Consider the steady-state probability density function $p_{0,1}(y_1)$, and the non-boundary functional Equation (5.14), which has the following form for $y_1 > 0$:

$$\frac{\partial p_{0,1}(y_1)}{\partial y_i} = -\lambda_Q p_{0,1}(y_1).$$

It is known that for a real-valued constant γ , $p_{0,1}(y_1) = \gamma e^{-\lambda_Q y_1}$. To determine the value of γ , Equation (5.9) or Equation (5.11) can be utilized. Essentially, it does not matter which equation is chosen, since the chosen equation will provide us with the normalizing constant. Conventionally, let us use Equation (5.9) and choose π_0 as our normalization constant. Then,

$$p_{0,1}(L) = \gamma e^{-\lambda_Q L} = \lambda_0 \pi_0.$$

Implying that:

$$\gamma = \lambda_0 \pi_0 e^{\lambda_Q L}.$$

Hence, we are able to recover the first density function that will allow us to solve the rest of the system:

$$p_{0,1}(y_1) = \lambda_0 \pi_0 e^{-\lambda_Q (y_1 - L)}. \quad (5.31)$$

For steady-state probability density functions of type $p_{i,1}(\cdot)$ where $i \geq 1$, the non-boundary Forward Kolmogorov Equation takes the form:

$$\sum_{j=1}^i \frac{\partial p_{i,1}(x_1, \dots, x_i, y_1)}{\partial x_j} + \frac{\partial p_{i,1}(x_1, \dots, x_i, y_1)}{\partial y_1} = -\lambda_{i+Q} p_{i,1}(x_1, \dots, x_i, y_1). \quad (5.32)$$

Hence, using Lemma 5.4 it is known that for $i \geq 1$:

$$p_{i,1}(x_1, \dots, x_i, y_1) = \lambda_0 \pi_0 \left(\prod_{j=0}^{i-1} \lambda_{Q+j} \right) \left(\prod_{j=0}^{i-1} e^{-\lambda_{(Q+i-j)}[x_{j+1}-x_j]} \right) e^{-\lambda_Q(y_1-L-x_i)}, \quad (5.33)$$

where $x_0 = 0$ and $\lambda_{2Q-1} = 0$ due to our assumptions. Then, for the remaining probability density functions, for the non-boundary Forward Kolmogorov Equations, i.e. Equation (5.12), we need to solve a Volterra Integral Equation of the 2nd kind. The solution may be obtained through multi-dimensional unilateral Laplace transformations or the Method of Characteristics. Let us provide the solution using the Laplace transformation approach. For $p_{1,0}(x_1)$, Equation (5.12) takes the form:

$$\frac{\partial p_{1,0}(x_1)}{\partial x_1} = -\lambda_1 p_{1,0}(x_1) + p_{1,1}(x_1, L). \quad (5.34)$$

Let us Laplace transform both sides with respect to x_1 and substitute the boundary value through Equation (5.10):

$$s\mathcal{L}\{p_{1,0}(x_1)\}(s) - \lambda_0 \pi_0 = -\lambda_1 \mathcal{L}\{p_{1,0}(x_1)\}(s) + \mathcal{L}\{p_{1,1}(x_1, L)\}(s).$$

It is known from Equation (5.33) that for $x_1 \leq L$, $p_{1,1}(x_1, L) = \lambda_0 \pi_0 \lambda_Q e^{-(\lambda_{Q+1}-\lambda_Q)x_1}$. Hence,

$$\mathcal{L}\{p_{1,1}(x_1, L)\}(s) = \frac{\lambda_0 \pi_0 \lambda_Q}{s + \lambda_{Q+1} - \lambda_Q},$$

which allows us to have the algebraic equation:

$$(\lambda_1 + s)\mathcal{L}\{p_{1,0}(x_1)\}(s) = \lambda_0\pi_0 + \frac{\lambda_0\pi_0\lambda_Q}{s + \lambda_{Q+1} - \lambda_Q}.$$

Through re-arranging the equation:

$$\mathcal{L}\{p_{1,0}(x_1)\}(s) = \frac{\lambda_0\pi_0}{(\lambda_1 + s)} + \frac{\lambda_0\pi_0\lambda_Q}{(s + \lambda_{Q+1} - \lambda_Q)(s + \lambda_1)},$$

which can be back-transformed easily. Back transformation from the frequency domain yields:

$$p_{1,0}(x_1) = \lambda_0\pi_0 e^{-\lambda_1 x_1} + \frac{\lambda_0\pi_0\lambda_Q}{-\lambda_{Q+1} + \lambda_Q + \lambda_1} (e^{(-\lambda_{Q+1} + \lambda_Q)x_1} - e^{-\lambda_1 x_1}), \quad (5.35)$$

provided that $\lambda_{Q+1} \neq \lambda_Q$. However, this solution is only valid for $x_1 \leq L$. It is possible for a single customer in the system to wait infinitely long, which leads us to examine the case where $x_1 \geq L$. In a mathematical approach, in the steady-state the only possible path for the process to arrive at $p_{1,0}(x_1 + t)$ where $x_1 \geq L$ is from $p_{1,0}(x_1)$, for a positive t . This is only possible through no arrivals in t time units, which has probability $e^{-\lambda_1 t}$. Independence of the inter-arrival time distribution requires one to multiply the two probabilities. Then, for $x_1 \geq L$, Equation (5.34) takes the form:

$$\frac{\partial p_{1,0}(x_1)}{\partial x_1} = -\lambda_1 p_{1,0}(x_1), \quad (5.36)$$

since the last term on the right hand-side vanishes. The physical reasoning behind this phenomenon is the fact that the customer who have waited for x_1 time units must have arrived after the beginning of service for the current batch. Therefore, the ordinal relationship between variables must be preserved. Then, for $x_1 \geq L$, Equation (5.36) may be solved via the Integrating Factor Technique:

$$(p_{1,0}(t)e^{\lambda_1 t})' = 0,$$

for $x_1 \geq L$, integrating both sides from L to x_1 and applying the Fundamental Theorem of Calculus, we obtain:

$$p_{1,0}(x_1) = p_{1,0}(L)e^{-\lambda_1(x_1-L)}. \quad (5.37)$$

The final ingredient remaining is to find $p_{1,0}(L)$. We note that on Equation (5.35), the value of $p_{1,0}(L)$ may be determined. The question is whether the function is right-continuous at the point $x_1 = L$. The function is left differentiable at $x_1 = L$ due to Equation (5.34) and it is right differentiable at $x_1 = L$ due to Equation (5.36). Hence, the function is semi-differentiable at the point $x_1 = L$, implying that the function must be continuous at the point $x_1 = L$. However, we note that the function $p_{1,0}(x_1)$ is not differentiable at the point $x_1 = L$. Therefore, using Equation (5.35) the value of $p_{1,0}(L)$ may be determined easily. The function is continuous, yet piecewise defined.

The remaining part is to recover other steady-state probability density functions. It turns out that other pdfs are also semi-differentiable functions and are piecewise defined. Let $\zeta \in \{2, \dots, Q-1\}$ be fixed for what follows. Then, $p_{\zeta,0}(x_1, \dots, x_\zeta)$ behaves in accordance with Equation (5.12), for values of $x_\zeta \leq L$. The multi-dimensional Laplace transformation operator is able to recover the form of the function for values of $x_\zeta \leq L$, repeating the procedures above. For the values of $x_\zeta > L$, we note that Equation (5.12) becomes:

$$\sum_{i=1}^{\zeta} \frac{\partial p_{\zeta,0}(x_1, \dots, x_\zeta)}{\partial x_i} = -\lambda_\zeta p_{\zeta,0}(x_1, \dots, x_\zeta).$$

By appealing to Proposition 5.3, it is known that for $x_\zeta > L$, the steady-state probability density function $p_{\zeta,0}(x_1, \dots, x_\zeta)$ admits a solution of type:

$$p_{\zeta,0}(x_1, \dots, x_\zeta) = e^{-\lambda_\zeta x_1} p_{\zeta,0}(0, x_2 - x_1, \dots, x_\zeta - x_1),$$

utilizing Equation (5.10), we know that:

$$p_{\zeta,0}(x_1, \dots, x_\zeta) = e^{-\lambda_\zeta x_1} \lambda_{\zeta-1} p_{\zeta-1,0}(x_2 - x_1, \dots, x_\zeta - x_1), \quad (5.38)$$

and the steady-state probability density function on the right hand-side of Equation

(5.38) can be recovered at an early step, since we begin iterating from $p_{1,0}(\cdot)$. However, we note that this approach does not satisfy the boundary value for $p_{\zeta,0}(x_1, \dots, x_{\zeta-1}, L)$ when $x_1 > 0$. Hence, the continuity assumption is violated. One is required to modify the function for the continuity over the multi-dimensional plane. We note that the explicit derivation of the steady-state probability density functions for values involving $x_\zeta > L$ is not required, since the non-boundary partial differential-integral Forward Kolmogorov Equation could be used to integrate the function over the domain where $x_\zeta > L$.

From another perspective, we will also utilize the Method of Characteristics to obtain the solutions of the Forward Kolmogorov Equation set, and verify that the results match.

Consider for $\phi \in \{1, \dots, Q-1\}$, the steady-state probability density function $p_{\phi,1}(x_1, \dots, x_\phi, y_1)$ and the corresponding non-boundary Forward Kolmogorov Equation (5.15). The equation takes the form:

$$\sum_{i=1}^{\phi} \frac{\partial p_{\phi,1}(x_1, \dots, x_\phi, y_1)}{\partial x_i} + \frac{\partial p_{\phi,1}(x_1, \dots, x_\phi, y_1)}{\partial y_1} = -\lambda_{(\phi+Q)} p_{\phi,1}(x_1, \dots, x_\phi, y_1). \quad (5.39)$$

Let us parameterize the independent variables with respect to t . For $L \geq t \geq 0$, let $x_1(t) = t$, and $x_i(t) = t + s_i$ for $i \in \{2, \dots, \phi\}$ and $y_1(t) = t + s_{\phi+1}$, where $s_2 \leq s_3 \leq \dots \leq s_{\phi+1} \leq L - t$ be real valued positive constants. Then, we transform the original function the following way:

$$\Upsilon(t) = p_{\phi,1}(t, t + s_2, \dots, t + s_\phi, t + s_{\phi+1}) = p_{\phi,1}(x_1(t), \dots, x_\phi(t), y_1(t)). \quad (5.40)$$

For the function $\Upsilon(t)$, let us employ the Total Differential Operator:

$$\begin{aligned} d\Upsilon(t) &= \sum_{i=1}^{\phi} \frac{\partial p_{\phi,1}(x_1(t), \dots, x_\phi(t), y_1(t))}{\partial x_i(t)} \frac{\partial x_i(t)}{\partial t} dt + \frac{\partial p_{\phi,1}(x_1(t), \dots, x_\phi(t), y_1(t))}{\partial y_1(t)} \frac{\partial y_1(t)}{\partial t} dt. \end{aligned}$$

We note that the partial derivatives of parametrized independent variables with respect to t are all equal to 1 since s'_i 's are constant. Dividing both sides by dt , we obtain:

$$\frac{d}{dt}\Upsilon(t) = \sum_{i=1}^{\phi} \frac{\partial p_{\phi,1}(x_1(t), \dots, x_{\phi}(t), y_1(t))}{\partial x_i(t)} + \frac{\partial p_{\phi,1}(x_1(t), \dots, x_{\phi}(t), y_1(t))}{\partial y_1(t)}. \quad (5.41)$$

We replace the left-hand side of Equation (5.39) with the left-hand side of Equation (5.41) and parametrize the original independent variables as prescribed above to obtain the ordinary differential equation:

$$\frac{d}{dt}\Upsilon(t) = -\lambda_{(\phi+Q)}\Upsilon(t).$$

By multiplying both sides with the integrating factor $e^{\lambda_{(\phi+Q)}t}$, taking the product formula for derivatives into account, integrating the function from 0 to t and employing the Fundamental Theorem of Calculus, we obtain:

$$\Upsilon(t) = e^{-\lambda_{(\phi+Q)}t}\Upsilon(0). \quad (5.42)$$

Reversing the parameterization:

$$p_{\phi,1}(x_1, \dots, x_{\phi}, y_1) = e^{-\lambda_{(\phi+Q)}x_1}p_{\phi,1}(0, x_2 - x_1, x_{\phi} - x_1, y_1 - x_1), \quad (5.43)$$

which after employing Equation (5.10) and substituting the function on the right-hand side coincides with the result obtained in Proposition 5.3. Repeated application of this result and using the explicit mathematical form of $p_{0,1}(y_1)$ as demonstrated above produce all probability density functions where the server is active.

For $p_{1,0}(x_1)$, non-boundary Forward Kolmogorov Equation was depicted as Equation (5.34). Multiplying both sides by the integrating factor $e^{\lambda_1 x_1}$ and rearranging the equation, we obtain:

$$(p_{1,0}(x_1)e^{\lambda_1 x_1})' = e^{\lambda_1 x_1}p_{1,1}(x_1, L).$$

Integrating both sides from 0 to x_1 , applying the Fundamental Theorem of Calculus and simple rearrangements, we have the following:

$$p_{1,0}(x_1) = e^{-\lambda_1 x_1} \left(p_{1,0}(0) + \int_0^{x_1} e^{\lambda_1 \psi} p_{1,1}(\psi, L) d\psi \right).$$

The result obtained in Equation (5.37) agrees with this result, since the value of the terms inside the parentheses become constant after $x_1 \geq L$ due to reasons mentioned above.

Let $\kappa \in \{2, \dots, Q-1\}$ be fixed. Consider the steady-state probability density function $p_{\kappa,0}(x_1, \dots, x_\kappa)$ and the Forward Kolmogorov Equation that characterizes its non-boundary behavior, i.e. Equation (5.12). Namely,

$$\sum_{i=1}^{\kappa} \frac{\partial p_{\kappa,0}(x_1, \dots, x_\kappa)}{\partial x_i} = -\lambda_\kappa p_{\kappa,0}(x_1, \dots, x_\kappa) + p_{\kappa,1}(x_1, \dots, x_\kappa, L)$$

Let us perform the parameterization in Equation (5.40) on the aforementioned equation which yields:

$$\frac{d}{dt} \Upsilon(t) = -\lambda_\kappa \Upsilon(t) + p_{\kappa,1}(t, \dots, t + s_\kappa, L),$$

where $\Upsilon(t) = p_{\kappa,0}(x_1(t), \dots, x_\kappa(t))$. Multiplying both sides with the integrating factor $e^{\lambda_\kappa t}$, rearranging the equation and integrating from 0 to t provides us with:

$$\int_0^t (\Upsilon(\psi) e^{\lambda_\kappa \psi})' d\psi = \int_0^t e^{\lambda_\kappa \psi} p_{\kappa,1}(\psi, \dots, \psi + s_\kappa, L) d\psi.$$

Employing the Fundamental Theorem of Calculus, and simple rearrangement of the equation yields:

$$\Upsilon(t) = e^{-\lambda_\kappa t} \left(\Upsilon(0) + \int_0^t e^{\lambda_\kappa \psi} p_{\kappa,1}(\psi, \dots, \psi + s_\kappa, L) d\psi \right).$$

Reverse parameterization of the equation allows us to obtain:

$$\begin{aligned}
& p_{\kappa,0}(x_1, \dots, x_\kappa) \\
&= e^{-\lambda_\kappa x_1} p_{\kappa,0}(0, x_2 - x_1, \dots, x_\kappa - x_1) \\
&+ e^{-\lambda_\kappa x_1} \int_0^{x_1} e^{\lambda_\kappa \psi} p_{\kappa,1}(\psi, \dots, \psi + x_\kappa - x_1, L) d\psi,
\end{aligned}$$

where we note that this solution is only valid for values of $x_\kappa \leq L$, since the integrand vanishes when $x_\kappa \geq L$ due to the change in the original Forward Kolmogorov Equation, i.e. the probability density function in the integrand vanishes on the original Forward Kolmogorov Equation. We cannot directly utilize the solution procedure depicted in Equation (5.43), since the continuity of the function is violated. The explicit determination of the steady-state probability density for values of $x_\kappa \geq L$ requires one to define a function of $s_2, \dots, s_\kappa$ which is constant with respect to the parameter t . It is possible to use the non-boundary partial-differential equation corresponding to the steady-state probability density $p_{\kappa,0}(x_1, \dots, x_\kappa)$ to determine integrals over regions involving $x_\kappa \geq L$. Noting the hierarchical relationship on Equation (5.10), and substituting the first term in the parentheses:

$$\begin{aligned}
& p_{\kappa,0}(x_1, \dots, x_\kappa) \\
&= e^{-\lambda_\kappa x_1} \lambda_{\kappa-1} p_{\kappa-1,0}(x_2 - x_1, \dots, x_\kappa - x_1) \\
&+ e^{-\lambda_\kappa x_1} \int_0^{x_1} e^{\lambda_\kappa \psi} p_{\kappa,1}(\psi, \dots, \psi + x_\kappa - x_1, L) d\psi.
\end{aligned}$$

We note that the function $p_{\kappa,1}(\cdot)$ in the integrand is known due to the procedure prescribed in Equation (5.43). The solutions coincide, and it is possible to evaluate the mathematical forms of all steady-state probability density functions. $\square$

Lemma 5.6. *Assume that the maximum number of customers that may simultaneously reside in the system is finite, and is less than or equal to $2Q - 1$. Then, the steady-state probability distribution of the number of customers at arbitrary instants, at arrival instants, at batch departure instants can be derived. In fact, any desired steady-state metric is available directly through the derived steady-state probability density functions.*

Proof. Let π_i , π_i^a , π_i^d denote the probability that there are i customers in the system in steady-state at arbitrary instants, just before an arrival and just consequent to a departure, respectively. Furthermore, let the vectors Π , Π^a and Π^d denote the collection of the former individual probability mass values, respectively. Let the maximum number of customers that may simultaneously reside in the system be equal to $\mathcal{C} \leq 2Q - 1$. The system trivially admits a unique steady-state behavior, since the stochastic process that governs the number of customers in the system at the steady-state is an ergodic process. For a given vector of λ_i s, by Proposition 5.5, the steady-state probability density functions are available. For each element of π_i where $i \in \{1, \dots, \mathcal{C}\}$ we first evaluate the invariant measure that is not normalized yet consists of the currently undetermined normalization constant π_0 :

$$\hat{\pi}_i = \int \cdots \int_{\Xi_i} p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) dx_1 \dots dx_n \dots dy_1 \dots dy_m,$$

where $n = (i \bmod Q)$, $m = \lfloor \frac{i}{Q} \rfloor$ and $\Xi_i \equiv \{(x_1, \dots, x_n, y_1, \dots, y_m) : x_i \in \mathbb{R}_0^+ \ \forall i \in \{1, \dots, n\}, x_i \leq x_j \ \forall (i, j) \in \{(q, s) : q < s, q \in \{1, \dots, n\}, s \in \{1, \dots, n\}\}, y_j \in [0, L] \ \forall j \in \{1, \dots, m\}, y_i \leq y_j \ \forall (i, j) \in \{(q, s) : q < s, q \in \{1, \dots, m\}, s \in \{1, \dots, m\}\}, x_n \leq y_1\}$. In the case where $m = 0$, the upper limit of the integral is equal to ∞ , otherwise it is equal to L . After evaluating each $\hat{\pi}_i$, it is possible to determine the normalization constant:

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{\mathcal{C}} \hat{\pi}_i}.$$

In order to scale the invariant measure, and obtain the unique probability mass distribution of the number of customers in the system at arbitrary instants, for $i \in \{1, \dots, \mathcal{C}\}$, we simply have to do the following:

$$\pi_i = \pi_0 \hat{\pi}_i.$$

It is entirely possible and simple to check whether the proposed distribution is a valid probability distribution, and it can be observed by summing up all π_i s. Next, we derive the steady-state probability distribution just before an arrival instant. Let π_i^a denote the probability that an arriving customer that will enter the system will observe i customers residing in the system, except themselves. Again, we first have to evaluate the invariant measures that we denote by $\hat{\pi}_i^a$ which are calculated by evaluating the integral:

$$\hat{\pi}_i^a = \int \cdots \int_{\Xi_i^a} p_{n,m}(0, \dots, x_n, y_1, \dots, y_m) dx_2 \dots dx_n \dots dy_1 \dots dy_m, \quad (5.44)$$

where $n = 1 + (i \bmod Q)$, $m = \lfloor \frac{i}{Q} \rfloor$ and $\Xi_i^a \equiv \{(x_2, \dots, x_n, y_1, \dots, y_m) : x_i \in \mathbb{R}_0^+ \ \forall i \in \{2, \dots, n\}, \ x_i \leq x_j \ \forall (i, j) \in \{(q, s) : q < s, q \in \{2, \dots, n\}, s \in \{2, \dots, n\}\}, \ y_j \in [0, L] \ \forall j \in \{1, \dots, m\}, \ y_i \leq y_j \ \forall (i, j) \in \{(q, s) : q < s, q \in \{1, \dots, m\}, s \in \{1, \dots, m\}\}, x_n \leq y_1\}$. In the case where $m = 0$, the upper limit of the integral is equal to ∞ , otherwise it is equal to L . Here we would like to note that the integral on the right-hand side of Equation (5.44) is precisely equal to:

$$\begin{aligned} \hat{\pi}_i^a &= \int \cdots \int_{\Xi_i^a} p_{n,m}(0, \dots, x_n, y_1, \dots, y_m) dx_2 \dots dx_n \dots dy_1 \dots dy_m \\ &= \lambda_i \pi_i, \end{aligned}$$

from the hierarchical relationship imposed by the Forward Kolmogorov Equations (5.10) and (5.11). Hence, for $i \in \{0, \dots, \mathcal{C} - 1\}$, we obtain the probability mass function at arrival epochs by:

$$\pi_i^a = \frac{\lambda_i \pi_i}{\sum_{j=0}^{C-1} \lambda_j \pi_j}.$$

Then, the probability mass vector at arrival epochs are formed by appending individual elements to each other, i.e. $\Pi^a \equiv \{\pi_i^a : i \in \{1, \dots, C-1\}\}$. We note that π_C^a does not exist, since an arriving customer that is admitted to the system cannot observe C customers residing in the system, excluding the customer who has just arrived.

Finally, we derive the probability mass function at departure epochs, i.e. $\Pi^d \equiv \{\pi^d : i \in \{0, \dots, Q-1\}\}$. We note that π_i^d denotes the probability mass of a departing batch to leave behind i customers. To do so, for $i \in \{0, \dots, Q-1\}$, we evaluate the integral to determine the invariant measure:

$$\hat{\pi}_i^d = \int \cdots \int_{\Xi_i^d} p_{i,m}(x_1, \dots, x_i, y_1, \dots, y_{m-1}, L) dx_1 \dots dx_i \dots dy_1 \dots dy_m,$$

where $\Xi_i^d \equiv \{(x_1, \dots, x_n, L) : x_i \in \mathbb{R}_0^+ \ \forall i \in \{1, \dots, n\}, x_i \leq x_j \ \forall (i, j) \in \{(q, s) : q < s, q \in \{1, \dots, n\}, s \in \{1, \dots, n\}\}\}$. Then by normalization we obtain the probability measures:

$$\pi_i^d = \frac{\hat{\pi}_i^d}{\sum_{j=0}^{Q-1} \hat{\pi}_j^d}.$$

We have obtained the steady-state probability vectors. We note that it is possible to obtain the expected conditional waiting times of the customers in the queue, or at arbitrary instants, by using the probability density functions. As stated above, any desired metric regarding the system may be evaluated using steady-state probability density functions. One may also derive the random variables that indicate customer waiting times in queue at arbitrary instants, or at arrival instants. However, the mentioned processes require single or multiple-fold convolution with the inter-arrival time distributions, or one-fold convolution with a generalized phase-type distribution, which is also known as the special case of a hypoexponential distribution. $\square$

In fact, one may modify the Forward Kolmogorov equations to accept $2Q$ customers but restrict the system to have only one server. That is, the system mentioned may begin another service at the instant of the departure of a batch if there are Q customers in the queue, which is the maximum number of customers allowed in the queue. The system is solvable using the Forward Kolmogorov Equations; however, only small modifications are required in the equations that govern the behavior of the state of the system with Q customers. Additionally, one more density is required to be evaluated, yet it can be evaluated using the techniques described above. The reason why we cannot solve the single-server case for more than $2Q$ customers is the ordering relationship between the variables cease to exist if more than $2Q$ customers are allowed in the system.

Corollary 5.7. *For a given set of finite, positive real valued $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and L , the complete steady-state characterization of the queueing system $M_n/L^3/1/5$, i.e. $Q = 3$ case, is available.*

Proof. We appeal to Proposition 5.5 for the derivation of the steady-state probability density functions. Without loss of generality, for the solution procedures that will follow, we assume that all λ_i are distinct. If λ_i are not distinct, the same solution procedures are still applicable yet may require small modifications. Step by step, let us derive explicit functional forms of steady-state probability density functions. First, for a clear depiction of the global balance equations, we provide the Integral-Partial Differential Difference Equations corresponding to the system of interest:

$$\begin{aligned}
 \lambda_0 \pi_0 &= p_{0,1}(L), \\
 p_{1,0}(0) &= \lambda_0 \pi_0, \\
 \frac{\partial p_{1,0}(x_1)}{\partial x_1} &= -\lambda_1 p_{1,0}(x_1) + p_{1,1}(x_1, L), \\
 p_{2,0}(0, x_2) &= \lambda_1 p_{1,0}(x_2), \\
 \frac{\partial p_{2,0}(x_1, x_2)}{\partial x_1} + \frac{\partial p_{2,0}(x_1, x_2)}{\partial x_2} &= -\lambda_2 p_{2,0}(x_1, x_2) + p_{2,1}(x_1, x_2, L), \\
 p_{0,1}(0) &= \lambda_2 \int_0^\infty \int_0^{x_2} p_{2,0}(x_1, x_2) dx_1 dx_2,
 \end{aligned}$$

$$\begin{aligned}
\frac{\partial p_{0,1}(y_1)}{\partial y_1} &= -\lambda_3 p_{0,1}(y_1), \\
p_{1,1}(0, y_1) &= \lambda_3 p_{0,1}(y_1), \\
\frac{\partial p_{1,1}(x_1, y_1)}{\partial x_1} + \frac{\partial p_{1,1}(x_1, y_1)}{\partial y_1} &= -\lambda_4 p_{1,1}(x_1, y_1), \\
p_{2,1}(0, x_2, y_1) &= \lambda_4 p_{1,1}(x_2, y_1), \\
\sum_{i=1}^2 \frac{\partial p_{2,1}(x_1, x_2, y_1)}{\partial x_i} + \frac{\partial p_{2,1}(x_1, x_2, y_1)}{\partial y_1} &= 0.
\end{aligned}$$

We know that:

$$p_{0,1}(y_1) = \lambda_0 \pi_0 e^{-\lambda_3(y_1 - L)}.$$

By employing Equation (5.33):

$$\begin{aligned}
p_{1,1}(x_1, y_1) &= \lambda_0 \lambda_3 \pi_0 e^{-(\lambda_4 - \lambda_3)x_1} e^{-\lambda_4(y_1 - L)}, \\
p_{2,1}(x_1, x_2, y_1) &= \lambda_0 \lambda_3 \lambda_4 \pi_0 e^{-\lambda_4 x_1} e^{-(\lambda_4 - \lambda_3)x_2} e^{-\lambda_4(y_1 - L)}.
\end{aligned}$$

The remaining two steady-state probability density functions to be determined are $p_{1,0}(x_1)$ and $p_{2,0}(x_1, x_2)$. For $p_{1,0}(x_1)$, substituting the mathematical form of $p_{1,1}(x_1, L)$ on the non-boundary condition of $p_{1,0}(x_1)$:

$$\frac{\partial p_{1,0}(x_1)}{\partial x_1} = -\lambda_1 p_{1,0}(x_1) + \lambda_0 \lambda_3 \pi_0 e^{-(\lambda_4 - \lambda_3)x_1}.$$

Applying one-dimensional unilateral Laplace transformation on both sides and taking the left boundary condition into account:

$$s\tilde{p}_{1,0}(s) = -\lambda_1 \tilde{p}_{1,0}(s) + \lambda_0 \pi_0 + \frac{\lambda_0 \lambda_3 \pi_0}{s + \lambda_4 - \lambda_3},$$

where $\tilde{p}_{1,0}(s) = \mathcal{L}\{p_{1,0}(x_1)\}(s)$. Then, the corresponding algebraic equation in the

frequency domain is:

$$\tilde{p}_{1,0}(s) = \frac{\lambda_0 \pi_0}{s + \lambda_1} + \frac{\lambda_0 \lambda_3 \pi_0}{(s + \lambda_4 - \lambda_3)(s + \lambda_1)}.$$

Applying the inverse Laplace transform operator on both sides:

$$p_{1,0}(x_1) = \lambda_0 \pi_0 e^{-\lambda_1 x_1} + \frac{\lambda_0 \lambda_3 \pi_0}{\lambda_4 - \lambda_3 + \lambda_1} (e^{-x_1(\lambda_4 - \lambda_3)} - e^{-\lambda_1 x_1}),$$

for $x_1 \leq L$, due to reasons mentioned in Proposition 5.5. For $x_1 \geq L$, we have:

$$p_{1,0}(x_1) = \left[\lambda_0 \pi_0 e^{-\lambda_1 L} + \frac{\lambda_0 \lambda_3 \pi_0}{\lambda_4 - \lambda_3 + \lambda_1} (e^{-L(\lambda_4 - \lambda_3)} - e^{-\lambda_1 L}) \right] e^{-\lambda_1(x_1 - L)}.$$

For $p_{2,0}(x_1, x_2)$ upon substituting $p_{2,1}(x_1, x_2, L)$ on the non-boundary condition, a Volterra Integral Equation of 2nd kind is formed:

$$\frac{\partial p_{2,0}(x_1, x_2)}{\partial x_1} + \frac{\partial p_{2,0}(x_1, x_2)}{\partial x_2} = -\lambda_2 p_{2,0}(x_1, x_2) + \lambda_0 \lambda_3 \lambda_4 \pi_0 e^{-\lambda_4 x_1} e^{-(\lambda_4 - \lambda_3)x_2}$$

Let us apply two dimensional unilateral commutative Laplace transformation operator on both sides and rearrange the resulting algebraic equation:

$$\tilde{\tilde{p}}_{2,0}(s, q) = \frac{\tilde{p}_{2,0}(0, q)}{s + q + \lambda_2} + \frac{\lambda_0 \lambda_3 \lambda_4 \pi_0}{(s + \lambda_4)(q + \lambda_4 - \lambda_3)(s + q + \lambda_2)}.$$

where $\tilde{\tilde{p}}_{2,0}(s, q) = \mathcal{L}_{x_1 x_2} \{p_{2,0}(x_1, x_2)\}(s, q)$ and $\tilde{p}_{2,0}(0, q) = \mathcal{L}_{x_2} \{p_{2,0}(0, x_2)\}(q)$. We apply the one-dimensional inverse Laplace transform operator on both sides with respect to s .

$$\tilde{p}_{2,0}(x_1, q) = \tilde{p}_{2,0}(0, q) e^{-x_1(q + \lambda_2)} + \frac{\lambda_0 \lambda_3 \lambda_4 \pi_0}{(-\lambda_4 + q + \lambda_2)(q + \lambda_4 - \lambda_3)} [e^{-\lambda_4 x_1} - e^{-(q + \lambda_2)x_1}],$$

now we inverse transform both sides with respect to q , and substitute the boundary

condition:

$$\begin{aligned}
p_{2,0}(x_1, x_2) &= e^{-\lambda_2 x_1} p_{1,0}(x_2 - x_1) u_{x_1}(x_2) \\
&+ e^{-\lambda_4 x_1} \frac{\lambda_0 \lambda_3 \lambda_4 \pi_0}{2\lambda_4 - \lambda_2 - \lambda_3} \left[e^{(\lambda_4 - \lambda_2)x_2} - e^{(-\lambda_4 + \lambda_3)x_2} \right] \\
&- e^{-\lambda_2 x_1} \frac{\lambda_0 \lambda_3 \lambda_4 \pi_0}{2\lambda_4 - \lambda_2 - \lambda_3} \left[e^{(\lambda_4 - \lambda_2)(x_2 - x_1)} - e^{(-\lambda_4 + \lambda_3)(x_2 - x_1)} \right] u_{x_1}(x_2),
\end{aligned}$$

for $x_2 \leq L$.

For obtaining the steady-state probability distribution of the number of customers in the system at arbitrary instants,

$$\begin{aligned}
\pi_1 &= \int_0^\infty p_{1,0}(x_1) dx_1 \\
&= \int_0^L p_{1,0}(x_1) dx_1 + p_{1,0}(L)/\lambda_1 \\
&= \lambda_0 \pi_0 \frac{(1 - e^{-\lambda_1 L})}{\lambda_1} + \frac{\lambda_0 \pi_0 \lambda_3 (1 - e^{-L(\lambda_4 - \lambda_3)})}{(\lambda_4 - \lambda_3 + \lambda_1)(\lambda_4 - \lambda_3)} \\
&- \frac{\lambda_0 \pi_0 \lambda_3 (1 - e^{-\lambda_1 L})}{(\lambda_4 - \lambda_3 + \lambda_1) \lambda_1} \\
&+ \frac{\lambda_0 \pi_0}{\lambda_1} \left\{ e^{-\lambda_1 L} + \frac{\lambda_3}{\lambda_4 - \lambda_3 + \lambda_1} (e^{-L(\lambda_4 - \lambda_3)} - e^{-\lambda_1 L}) \right\}.
\end{aligned}$$

We note that π_2 is already available as the left-hand boundary condition of $p_{0,1}(y_1)$:

$$\begin{aligned}
\pi_2 &= \int_0^\infty \int_0^{x_2} p_{2,0}(x_1, x_2) dx_1 dx_2 \\
&= p_{0,1}(0) \\
&= \frac{\lambda_0 \pi_0 e^{\lambda_3 L}}{\lambda_2}.
\end{aligned}$$

For calculating π_3 :

$$\pi_3 = \int_0^L p_{0,1}(y_1) dy_1$$

$$= \frac{\lambda_0 \pi_0 (e^{\lambda_3 L} - 1)}{\lambda_3}.$$

The remaining steady-state probabilities are:

$$\begin{aligned} \pi_4 &= \int_0^L \int_0^{y_1} p_{1,1}(x_1, y_1) dx_1 dy_1 \\ &= \frac{\lambda_0 \lambda_3 \pi_0 e^{\lambda_3 L}}{\lambda_4 - \lambda_3} \left\{ \frac{1 - e^{-\lambda_3 L}}{\lambda_3} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4} \right\}, \end{aligned}$$

and

$$\begin{aligned} \pi_5 &= \int_0^L \int_0^{y_1} \int_0^{x_2} p_{2,1}(x_1, x_2, y_1) dx_1 dx_2 dy_1 \\ &= \lambda_0 \pi_0 e^{\lambda_3 L} \left\{ L - \frac{1 - e^{-\lambda_3 L}}{\lambda_3} \right\} - \frac{\lambda_0 \lambda_3 \pi_0 e^{\lambda_3 L}}{\lambda_4 - \lambda_3} \left\{ \frac{1 - e^{-\lambda_3 L}}{\lambda_3} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4} \right\}. \end{aligned}$$

Hence by the normalization condition:

$$\begin{aligned} \pi_0 &= \frac{1}{1 + \pi_1/\pi_0 + \pi_2/\pi_0 + \pi_3/\pi_0 + \pi_4/\pi_0 + \pi_5/\pi_0} \\ &= \left[1 + \lambda_0 \frac{(1 - e^{-\lambda_1 L})}{\lambda_1} + \frac{\lambda_0 \lambda_3 (1 - e^{-L(\lambda_4 - \lambda_3)})}{(\lambda_4 - \lambda_3 + \lambda_1)(\lambda_4 - \lambda_3)} \right. \\ &\quad - \frac{\lambda_0 \lambda_3 (1 - e^{-\lambda_1 L})}{(\lambda_4 - \lambda_3 + \lambda_1) \lambda_1} \\ &\quad + \frac{\lambda_0}{\lambda_1} \left\{ e^{-\lambda_1 L} + \frac{\lambda_3}{\lambda_4 - \lambda_3 + \lambda_1} (e^{-L(\lambda_4 - \lambda_3)} - e^{-\lambda_1 L}) \right\} \\ &\quad + \frac{\lambda_0 e^{\lambda_3 L}}{\lambda_2} \\ &\quad + \frac{(e^{\lambda_3 L} - 1)}{\lambda_3} \\ &\quad + \frac{\lambda_0 \lambda_3 e^{\lambda_3 L}}{\lambda_4 - \lambda_3} \left\{ \frac{1 - e^{-\lambda_3 L}}{\lambda_3} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4} \right\} \\ &\quad \left. + \lambda_0 e^{\lambda_3 L} \left\{ L - \frac{1 - e^{-\lambda_3 L}}{\lambda_3} \right\} - \frac{\lambda_0 \lambda_3 e^{\lambda_3 L}}{\lambda_4 - \lambda_3} \left\{ \frac{1 - e^{-\lambda_3 L}}{\lambda_3} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4} \right\} \right]^{-1}, \end{aligned}$$

which is a non-trivial function of λ_i s and L . The steady-state probabilities at the arrival or departure epochs can be computed as prescribed in Proposition 5.5. $\square$

The derivations demonstrated in Corollary 5.7 could also be performed by using the Method of Characteristics. We will attack the problem with the Method of Characteristics in the next subsection.

Before we begin the next subsection, we comment on the importance of the exact results of this subsection. The procedures prescribed in this subsection allow one to completely characterize a $(r, Q, \mathcal{B})$ model that experiences state-dependent arrivals and deterministic procurement lead-times for any value of $Q \in \mathbb{N}$, provided that $r + \mathcal{B} \leq Q - 1$. As an example, one may choose $Q = 100$, and say $r = 20$. Then, for any value of $\mathcal{B} \leq 79$, the complete steady-state characterization of the inventory model with state-dependent arrival rates is available, by virtue of the steady-state probability density functions of the analogous queueing model. Hence, for an inventory system that has a relatively insignificant holding cost compared to the fixed ordering cost, it is more sensible to order large quantities.

In the next subsection, we will provide numerical solution techniques for cases where $\mathcal{C} \geq 2Q$, i.e. instances where it is allowed for more than one server to operate simultaneously in the queueing system. We note that condition $\mathcal{C} \geq 2Q$ is equivalent to $r + Q + \mathcal{B} \geq 2Q$, i.e. there can be two or more outstanding orders of size Q in the inventory system $(r, Q, \mathcal{B})$. The analysis of the queueing system becomes difficult when more than one server is allowed to operate. Hence, we need to approximate the steady-state behavior of the queueing system in those cases. We will discretize the Forward Kolmogorov equations utilizing a finite-difference technique, and then propose a numerical method to approximate the steady-state behavior of the queueing system.

5.2.4. Numerical Solution Procedure for the Integral-Partial Differential Difference Equations in the Case of Multiple Servers

For the cases where $\mathcal{C} \geq 2Q$, it is not possible to obtain an analytic solution. Therefore, we employ numerical techniques on the Forward Kolmogorov Equations to obtain the steady-state probability densities. We begin this subsection by explaining

the reasoning behind the requirement to implement numerical techniques to obtain the steady-state behavior of the aforementioned queueing system in the cases where $\mathcal{C} \geq 2Q$. Then we elaborate on the Finite Difference technique. After demonstrating different types of differencing, we will discretize the Forward Kolmogorov Equation set. We will provide some tricks that will render the numerical procedure more stable and efficient.

The Forward Kolmogorov Equations do not consist of an equation that involves separate terms with partial differential operators and integrals, indicating that individual equations are either integral equations or partial differential equations. Whenever it is allowed for more than one server to be active, the equation system becomes immensely difficult to solve in contrast to the former case. The solution procedures demonstrated in the previous subsection are no longer applicable; however, the hierarchical relationships are still preserved. Namely, steady-state probability density functions corresponding to system states that involve more than $\mathcal{C} - Q$ customers may be expressed as the probability density function of the system state with $\mathcal{C} - Q$ customers. This result is presented in Lemma 5.4. However, when the non-boundary Forward Kolmogorov Equation, i.e. Equation (5.14) or Equation (5.15), is examined and through substitution with regard to Lemma 5.4, a complicated functional equation is formed. In other words, the steady-state probability density function corresponding to the system state $\mathcal{C}$ may be expressed as the steady-state probability density function of the system state with $\mathcal{C} - Q$ customers; however, the substituted functional term involves multiple-integrals and exponential terms in the integrand. Although it is a functional equation consisting of a single probability density function, the integral transformation techniques are not directly applicable, since boundaries of integrals contain independent variables and the domain of the function is a compact set, i.e. closed and bounded.

Moreover, we note that if it were possible to obtain an analytical solution to the functional partial differential integral equation, then it would be possible to determine the exact mathematical forms of all steady-state probability density functions. The

reasoning behind this statement is simply an application of the Lemma 5.4, since one may recursively iterate backward and determine the mathematical forms by the techniques in the last subsection, i.e. multi-dimensional unilateral Laplace transforms or the Method of Characteristics.

A rigorous treatment of the Finite Difference Technique may be found in [30]. The idea behind the finite difference formula is discretization of the derivative of a function. Essentially, assume that for a one-dimensional function f the first derivative exists for all points in its domain. The first derivative of the function at an arbitrary real-valued point x_0 may be defined as:

$$\frac{df}{dx}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

where one may let h approach 0 from right or left, in the case of one sided differentiation. Also, let us assume that f is twice differentiable. Then, one may express f by its *Taylor Series Expansion* around the point x_0 to obtain the finite difference approximation. It is possible to express f through forward, backward, or central differences. To obtain the Forward Difference equation, also known as the *Euler's Method*, for a fixed, positive yet small enough h , we expand the first order Taylor Series around the point x_0 to obtain:

$$\frac{df}{dx}(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + \mathcal{O}(h^2/2), \quad (5.45)$$

where $\mathcal{O}$ is defined conventionally as follows: Assume that as x approaches x_0 in the real line, independent of x , $\exists k \in \mathbb{R}$ such that for two arbitrary real valued functions f^* and g :

$$\lim_{x \rightarrow x_0} \frac{|f^*(x)|}{|g(x)|} = k \implies f^*(t) = \mathcal{O}(g(t)),$$

where a more refine definition may be found again in [28]. Additionally, due to the relations between the orders of convergence, one may write $o(h/2)$ instead of $\mathcal{O}(h^2/2)$ on the right-hand side of Equation (5.45). Then, for a fixed, small enough and positive h , by rearranging Equation (5.45) and disregarding the error term the following forms the forward difference approximation for the function f at the point x_0 :

$$\frac{df}{dx}(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}.$$

Correspondingly, for a fixed value of h that is positive yet small enough, the backward finite difference is defined as:

$$\frac{df}{dx}(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h},$$

where the error term is again of the order $\mathcal{O}(h^2/2)$, as h tends to 0 from right.

Forward and backward finite differences are used extensively, since inherently some problems require one to iterate forward or backward depending on the differential equation and boundary values. However, we will employ the central finite difference technique, since it offers greater numerical stability. The central finite difference is defined simply as a linear combination of forward and backward difference:

$$\frac{df}{dx}(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h},$$

where the error term is now of order $\mathcal{O}(h^2/4)$.

For functions involving more than two independent variables, and consequently partial derivatives, the central differencing takes the form:

$$\left. \frac{\partial f(x, y)}{\partial x} \right|_{x=x_0, y=y_0} \approx \frac{f(x_0 + h, y_0) - f(x_0 - h, y_0)}{2h},$$

where the order of approximation is the same as in the one-dimensional case. Now we will explain the discretization procedure for the integrals. We will employ *Trapezoidal*

rule to discretize integral terms and perform small modifications in the method for the cases where a kernel exists in the integrand to render the numerical integration more refined.

Assume that we have a real-valued, one-dimensional function f with bounded support $[0, \beta]$. Then, assume that we want to evaluate:

$$\int_0^\beta f(x)dx.$$

We begin by selecting an equally distanced set of points, i.e. forming a uniform sized mesh. Assume that we choose $n \in \mathbb{N}$ equal-sized intervals, i.e. each interval having length β/n . We have $n + 1$ points, and one may assume that the function behaves linearly between these points, i.e. the function is formed by linearly interpolating each consequent pair $(x_i, f(x_i))$, where $i \in \{1, \dots, n+1\}$ and $x_i = (i-1)\beta/n$. Then, geometrically, we have formed n trapezoids. Assuming that function behaves approximately linear between each pair, then the following integral may be expressed as:

$$\int_{x_i}^{x_{i+1}} f(t)dt \approx \frac{\beta}{2n} [f(x_{i+1}) + f(x_i)],$$

where we note that each interval has length $c = \beta/n$. We know that for a bounded function and our choice of n , the following holds:

$$\begin{aligned} \int_0^\beta f(t)dt &= \sum_{i=1}^n \int_{x_i}^{x_{i+1}} f(t)dt \\ &\approx \frac{\beta}{2n} \sum_{i=1}^n [f(x_{i+1}) + f(x_i)] \\ &= \frac{c}{2} [f(0) + f(\beta)] + c \sum_{i=1}^{n-1} f(ic) \end{aligned} \tag{5.46}$$

In some cases, the integrands involve kernels. Namely, it may be required to discretize:

$$\int_0^\beta g(x)f(x)dx,$$

where g is a known real-valued function, i.e. kernel in this case. Then, it is possible to use Equation (5.46) with a small modification as follows:

$$\int_0^\beta g(x)f(x)dx \approx \frac{c}{2}[f(0)g(0) + f(\beta)g(\beta)] + c \sum_{i=1}^{n-1} f(ic)g(ic). \quad (5.47)$$

However, this modification ignores the information provided by the function g . Since $g(x)$ is known, such as e^x , with a little effort we can incorporate more information, which would render the discretization more refined and stable. We simply further partition each interval into $k+1$ subintervals, each of which has length $c' = \beta/(nk)$. In each subinterval, the known values of f at the end points of the intervals can be linearly interpolated to determine the value of f in the subinterval. Namely, for $i \in \{1, \dots, n\}$ and $j \in \{0, \dots, k\}$:

$$f(jc' + (i-1)c) = \frac{f(ic) - f((i-1)c)}{c}jc' + f((i-1)c).$$

Then, for $i \in \{1, \dots, n\}$, a known kernel $g(\cdot)$, and Trapezoidal rule, we can discretize the integral over an arbitrary interval the following way:

$$\begin{aligned} & \int_{(i-1)c}^{ic} f(s)g(s)ds \\ &= \frac{1}{2} \sum_{j=0}^k \left[\frac{f(ic) - f((i-1)c)}{c}jc' + f((i-1)c) \right] g(jc' + (i-1)c). \end{aligned} \quad (5.48)$$

Hence, over the whole domain Trapezoidal rule takes the form:

$$\begin{aligned} & \int_0^\beta f(s)g(s)ds \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^k \left[\frac{f(ic) - f((i-1)c)}{c}jc' + f((i-1)c) \right] g(jc' + (i-1)c) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^k \left[\frac{f(ic) - f((i-1)c)}{k} j + f((i-1)c) \right] g(jc' + (i-1)c) \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^k \left[f(ic) j/k \right] g(jc' + (i-1)c) \\
&+ \frac{1}{2} \sum_{i=1}^n f((i-1)c) \sum_{j=0}^k \left[1 - j/k \right] g(jc' + (i-1)c) \\
&= \frac{1}{2} f(0) \sum_{j=0}^k \left[1 - j/k \right] g(jc' + (i-1)c) \\
&+ \frac{1}{2} f(\beta) \sum_{j=0}^k \frac{j}{k} g(jc' + (i-1)c) \\
&+ \sum_{i=1}^{n-1} f(ic) \sum_{j=0}^k \left[\frac{j}{k} g(jc' + (i-1)c) + \left\{ 1 - \frac{j}{k} \right\} g(jc' + ic) \right]. \tag{5.49}
\end{aligned}$$

As a matter of fact, through the Finite Difference formula, we linearize a set of partial differential integral difference equations. Normally, the system consists of continuous and almost everywhere differentiable probability density functions. However, to make the system solvable numerically and depict the behavior of the functions, we will choose a uniformly distanced set of values to create a uniform grid. Then, by the finite difference formula, we will relate certain functions of these steady-state probability density points to each other, creating a linear system of equations.

We would like to make some remarks before we explain the formal procedure to transform the original set of equations into a linear system of equations. By Lemma 5.4, the dimension of the linear system can be significantly reduced, since we ignore the last Q steady-state probability density functions. In addition, the boundary values are not known, they are either functions of the other steady-state probability functions or a scaled version of the normalizing constant π_0 . Let us denote the square invertible linearization matrix by $\mathbf{M}$, where the dimension of $\mathbf{M}$ increases exponentially. The reason behind the exponential growth of the dimensionality of $\mathbf{M}$ can be explained by the number of points required for each steady-state probability density function. For

a uniform-sized grid formed for $p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)$ on the set:

$$\begin{aligned}
\chi_{n,m} \equiv & \{(x_1, \dots, x_n, y_1, \dots, y_m) : x_i \in \mathbb{R}_0^+ \ \forall i \in \{1, \dots, n\}, x_i \leq x_j \\
& \forall (i, j) \in \{(q, s) : q < s, q \in \{1, \dots, n\}, s \in \{1, \dots, n\}\} \\
& , \ y_j \in [0, L] \ \forall j \in \{1, \dots, m\}, y_i \leq y_j \ \forall (i, j) \in \{(q, s) : q < s, q \in \{1, \dots, m\} \\
& , \ s \in \{1, \dots, m\}\}, x_n \leq y_1\}, \tag{5.50}
\end{aligned}$$

consists of $C \binom{n+m+k}{n+m}$ points, provided that each independent variable can be evaluated at $k + 1$ equally distanced points on the compact set $[0, L]$. We will have a column vector that includes evaluations of all steady-state probability densities on their respective uniform-distance mesh. Let us denote the vector of the right-hand side vector of known values by $\mathbf{v}$. Then, by using the set of modified Forward Kolmogorov Equations and the Finite Difference formula, each equation can be discretized. After collecting terms that involve probability density functions on the left-hand side and normalizing terms on the right-hand side, we have a linear system of equations that depicts the hierarchical relationships of the desired queueing system. In each equation, the relationships between the probability densities on the left-hand side can be included in the linearization matrix $\mathbf{M}$ and the scalar constants of the normalization term can be included in the column vector $\mathbf{v}$. Then, the linear system of equations is expressed compactly as:

$$\mathbf{M}\mathbf{d} = \pi_0\mathbf{v}, \tag{5.51}$$

where the column vector $\mathbf{d}$ denotes the collection of steady-state probability density functions evaluated at each point in the grids, which is what is desired to be found. The Forward Kolmogorov equations are complete and form an invariant measure, which implies that linearizing each equation would result in a full rank invertible Linearization Matrix $\mathbf{M}$. The system is then solved by inverting the Linearization matrix, left-multiplying it on both sides of Equation (5.51), resulting in:

$$\mathbf{d} = \pi_0 \mathbf{M}^{-1} \mathbf{v}. \quad (5.52)$$

To convert the invariant measure into a probability measure, we need to find π_0 . To find π_0 , one simply numerically integrates the unnormalized vector of steady-state probability density functions, $\mathbf{d}/\pi_0$, over the admissible regions of each density. That is, for $p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)$ we have:

$$\begin{aligned} & \frac{1}{\pi_0} \int \cdots \int_{\chi_{n,m}} p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) dx_1 \dots dx_n dy_1 \dots dy_m \\ & \approx c^{n+m} \sum_{i_{n+m}=0}^k \sum_{i_{n+m-1}=0}^{i_{n+m}} \cdots \sum_{i_1=0}^{i_2} \tilde{p}_{n,m}(i_1 L/k, i_2 L/k, \dots, i_{n+m} L/k) \\ & = \tilde{\pi}_{n+mQ}, \end{aligned} \quad (5.53)$$

where $\tilde{p}_{n,m}(i_1 L/k, i_2 L/k, \dots, i_{n+m} L/k)$ denotes the numerically estimated steady-state probability density function corresponding to the system state with $n+mQ$ customers. The numerical integration procedure presented in Equation (5.53) can be refined more, but as the partition number k increases, it converges to the refined one at a rate of convergence $o(c^{n+m})$. We note that for the first $Q-1$ steady-state probability density functions, the domains are unbounded. However, the multiple-integral term can be decomposed into two multiple-integral terms where one has a finite upper boundary value equal to L , and the other has an upper boundary equal to ∞ . For $i \in \{1, \dots, Q-1\}$, the steady-state probability density function $p_{i,0}(x_1, \dots, x_i)$ converges to 0 negative exponentially for $x_i \geq L$ as $x_i \rightarrow \infty$, considering that its integral must be less than or equal to 1 and Equation (5.38). Hence, for a steady-state probability density function of type $p_{i,0}(x_1, \dots, x_i)$, we are required to evaluate:

$$\begin{aligned} & \frac{1}{\pi_0} \int \cdots \int_{\chi_{i,0}} p_{i,0}(x_1, \dots, x_i) dx_1 \dots dx_i \\ & \approx c^i \sum_{j_i=0}^k \sum_{j_{i-1}=0}^{j_i} \cdots \sum_{j_1=0}^{j_2} \tilde{p}_{i,0}(j_1 L/k, j_2 L/k, \dots, j_i L/k) \\ & + \frac{1}{\pi_0} \int \cdots \int_{\chi_{i,0} \cap (x_i \geq L)} p_{i,0}(x_1, \dots, x_i) dx_1 \dots dx_i. \end{aligned} \quad (5.54)$$

The remaining integral term in Equation (5.54) can be evaluated using the relation in Equation (5.38). As an example, it can be analytically evaluated for $i = 1$, and is equal to $p_{1,0}(L)/\lambda_1$, and $p_{1,0}(L)$ is already available as $\tilde{p}_{1,0}(L)$. Then, to find the steady-state distribution of the number of customers in the system at arbitrary instants, for $j \in \{0, \dots, \mathcal{C}\}$, we simply have to evaluate:

$$\pi_j = \frac{\tilde{\pi}_j}{1 + \sum_{i=1}^{\mathcal{C}} \tilde{\pi}_i}, \quad (5.55)$$

where $\tilde{\pi}_0 = 1$, since we assumed that $\tilde{\pi}_j = \pi_j/\pi_0$ in Equation (5.53).

We now present the algorithm to form the linearized system of equations, i.e. $\mathbf{M}$, $\mathbf{v}$ and $\mathbf{d}$. After we present the algorithm, we will elaborate on some technical details regarding implementation. Then, we will apply a previously mentioned method, i.e. the Method of Characteristics, to render the set of equations more compact. In some cases, this trick improves the numerical efficiency of the method by replacing the complex hierarchical relationships between the steady-state probability density functions with more straightforward relationships, yet may require evaluation of multiple-fold integrals in other cases. We will explain why such a phenomenon occurs.

The algorithm presented in Figure 5.1 provides us with a linear system of equations, where the techniques for solving the system are widely available. One must be extra careful not to lose any degrees of freedom, since otherwise the Linearization Matrix $\mathbf{M}$ cannot be inverted since the system admits infinitely many solutions. In the estimated values of the steady-state probability density functions, i.e. $\tilde{p}_{n,m}(\cdot)$, if two or more independent variables are equal to each other, the function value can be taken as 0. The need to appoint 0 to functions on these occasions arises from the fact that the partial derivatives of the steady-state probability density functions cannot be defined properly at these points. It is possible to use backward differencing at these points; however, using different kinds of differencing in the same numerical procedure makes the problem numerically less stable. Hence, the numerical stability of the method increases when one disregards the grid points involving equal independent variables. We

note that sometimes backward differencing is obligatory to preserve the required degrees of freedom. As an example, in one-dimensional steady-state probability density functions, at the right boundary value L it is required to use backward differencing for the reasons mentioned in this paragraph. We will provide a detailed example on this section and numerical results in the next sections.

Before we present an example as a corollary with regard to the results of this subsection, we would like to note that the methodology presented in this subsection will allow us to find a suitable approximation for the steady-state behavior of the aforementioned bulk queueing system when there are multiple servers. The bulk queueing system of interest is analogous to the inventory system $(r, Q, \mathcal{B})$. Therefore, by approximating the steady-state behavior of the bulk queueing system when more than one server is operating, we will be able to approximate the steady-state behavior of the $(r, Q, \mathcal{B})$ model when more than one outstanding order is allowed. As an example, the bulk queueing system $M_n/L^2/2/4$ is analogous to the $(r, Q = 2, \mathcal{B})$ model if $r + \mathcal{B} = 2$. For the sake of a concrete example, assume that $L, \lambda_0, \lambda_1, \lambda_2, \lambda_3$ are positive real-valued scalars. Furthermore, assume that $r = 0$ and $\mathcal{B} = 2$. Assume that we present customers with full information regarding the state of the inventory system upon their arrival and that they do not balk if there are any items available. Then, we may set $\lambda_0 = \lambda_1 = \Lambda$ where Λ represents the potential arrival intensity per unit time to the system. Furthermore, $\lambda_2 = \Lambda \hat{p}_0$ and $\lambda_3 = \Lambda \hat{p}_{-1}$ where $\hat{p}_i$ represents the probability of balking for a customer who observes that IN is i on arrival at the system. In the next section, the set of equilibrium balking probabilities, i.e. the vector of $\{\hat{p}_i\}$, will be determined based on the level of information presented to the customers. For the sake of the arguments that will follow, assume that the values of $\hat{p}_i$ for $i = 0, -1$ are known at the moment. Then, by discretizing the set of Forward Kolmogorov equations and numerically solving them, we can obtain an approximation for the steady-state behavior of the analogous bulk queueing system. The approximation can be used by the virtue of Proposition 5.1 to approximate the steady-state behavior of the $(r = 0, Q = 2, \mathcal{B} = 2)$ model. In the next corollary, we present an example demonstration of the numeric methods of this subsection.

```

 $n \leftarrow (\mathcal{C} \bmod Q)$ 
 $m \leftarrow \lfloor \mathcal{C}/Q \rfloor - 1$ 
Form the set of Forward Kolmogorov Equations(FKE) as prescribed
Using Lemma 5.4 substitute  $p_{n,m}$  wherever possible on the set of FKE
 $\mathbf{d} \leftarrow \emptyset$ 
for  $j = 0$  to  $m$  do
  for  $i = 0$  to  $Q$  do
    if  $i + jQ \geq \mathcal{C} - Q + 1$  then
      Terminate
    end if
    Create a uniform sized mesh
    Append the unknowns  $p_{i,j}(\cdot)$  at each of the points in the grid to  $\mathbf{d}$ 
  end for
end for
Create  $\mathbf{M}$ ,  $|\mathbf{d}|$  by  $|\mathbf{d}|$  Linearization Matrix, label column and row names as  $\mathbf{d}$ 
Create  $\mathbf{v}$ , Normalizing column vector of size  $|\mathbf{d}|$ 
for  $j = 0$  to  $m$  do
  for  $i = 0$  to  $Q$  do
    if  $i + jQ \geq \mathcal{C} - Q + 1$  then
      Terminate
    end if
    Use the discretized boundary equation to relate  $p_{i,j}$  to the previous density
    Use the discretized non-boundary equation to relate  $p_{i,j}$  to other densities
    On each equation, record the constants of density functions on  $\mathbf{M}$ 
    On each equation, record the normalizing term on  $\mathbf{v}$ 
  end for
end for

```

Figure 5.1. Algorithm to Linearize the set of Forward Kolmogorov Equations.

Corollary 5.8. *For a given set of finite, positive and real valued $\lambda_0 = \lambda_1 = \Lambda$, λ_2 , λ_3 , and L , the complete steady-state characterization of the queueing system $M_n/L^2/2/4$ is available.*

Proof. We appeal to the algorithm presented in Figure 5.1 combined with the numerical procedures presented in this subsection. We first present the raw set of Forward Kolmogorov equations:

$$\Lambda\pi_0 = p_{0,1}(L).$$

$$p_{1,0}(0) = \Lambda\pi_0.$$

$$\frac{\partial p_{1,0}(x_1)}{\partial x_1} = -\Lambda p_{1,0}(x_1) + p_{1,1}(x_1, L).$$

$$p_{0,1}(0) = \Lambda \int_0^\infty p_{1,0}(x_1) dx_1.$$

$$\frac{\partial p_{0,1}(y_1)}{\partial y_1} = -\lambda_2 p_{0,1}(y_1) + p_{0,2}(y_1, L).$$

$$p_{1,1}(0, y_1) = \lambda_2 p_{0,1}(y_1).$$

$$\frac{\partial p_{1,1}(x_1, y_1)}{\partial x_1} + \frac{\partial p_{1,1}(x_1, y_1)}{\partial y_1} = -\lambda_3 p_{1,1}(x_1, y_1).$$

$$p_{0,2}(0, y_2) = \lambda_3 \int_0^{y_2} p_{1,1}(x_1, y_2) dx_1.$$

$$\frac{\partial p_{0,2}(y_1, y_2)}{\partial y_1} + \frac{\partial p_{0,2}(y_1, y_2)}{\partial y_2} = 0.$$

Let us partition the domain $[0, L]$ into n equal length sub-intervals. Let $c = L/n$. Then we have $n + 1$, $n + 1$, $\frac{n^3+3n+2}{2}$, and $\frac{n^3+3n+2}{2}$ function evaluations to be performed for $p_{1,0}(x_1)$, $p_{0,1}(y_1)$, $p_{1,1}(x_1, y_1)$, and $p_{0,2}(y_1, y_2)$, respectively. In total, we have $n^3 + 5n + 4$ function evaluations. First, let us form the currently unknown density values at each of the aforementioned points in the vector $\mathbf{d}$. Let $\mathbf{d}_i$ denote the i^{th} element of the column vector $\mathbf{d}$. For $0 \leq i \leq n$:

$$\mathbf{d}_{i+1} = p_{1,0}(ic).$$

For $0 \leq i \leq n$:

$$\mathbf{d}_{i+n+2} = p_{0,1}(ic).$$

For $0 \leq j \leq n$ and $0 \leq i \leq j$:

$$\mathbf{d}_{2n+2+ni+i-(i-1)i/2+1-i+j} = p_{1,1}(ic, jc).$$

For $0 \leq j \leq n$ and $0 \leq i \leq j$:

$$\mathbf{d}_{(n^2+7n+6)/2+ni+i-(i-1)i/2+1+j-i} = p_{0,2}(ic, jc).$$

Let $\mathbf{M}$ be a square $(n^2 + 5n + 4)$ by $(n^2 + 5n + 4)$ matrix full of 0s, and label row and column names as the unknowns in the column matrix $\mathbf{d}$. Let $\mathbf{M}_{i,j}$ denote the element that resides in the row i and the column j of $\mathbf{M}$. Finally, let $\mathbf{v}$ be a column vector of 0s and length $n^2 + 5n + 4$.

Let us step-by-step discretize the Forward Kolmogorov Equation set. For $p_{1,0}(x_1)$, we have:

$$p_{1,0}(0) = \Lambda\pi_0.$$

For $1 \leq i \leq n-1$:

$$\frac{p_{1,0}((i+1)c) - p_{1,0}((i-1)c)}{2c} = -\Lambda p_{1,0}(ic) + p_{1,1}(ic, L).$$

For $i = n$:

$$\frac{p_{1,0}(L) - p_{1,0}((n-1)c)}{c} = -\Lambda p_{1,0}(L) + p_{1,1}(L, L).$$

Then,

- $\mathbf{M}_{1,1} = 1, \mathbf{v}_1 = \Lambda.$
- For $1 \leq i \leq n-1$:

$$\mathbf{M}_{i,i-1} = -1,$$

$$\mathbf{M}_{i,i+1} = 1,$$

$$\mathbf{M}_{i,i} = \Lambda c,$$

$$\mathbf{M}_{i,2n+2+ni+i-(i-1)i/2+1-i+n} = -2c.$$
- For $i = n$:

$$\mathbf{M}_{i,i} = \Lambda c + 1,$$

$$\mathbf{M}_{i,i-1} = -1,$$

$$\mathbf{M}_{i,(n^2+7n+6)/2} = -c.$$

For $p_{1,0}(y_1)$, we have:

$$p_{0,1}(0) = \frac{c\Lambda[p_{1,0}(0) + p_{1,0}(L)]}{2} + \sum_{i=1}^{n-1} c\Lambda p_{1,0}(ic) + p_{1,0}(L).$$

For $1 \leq i \leq n-1$:

$$\frac{p_{0,1}((i+1)c) - p_{0,1}((i-1)c)}{2c} = -\lambda_2 p_{0,1}(ic) + p_{0,2}(ic, L).$$

$$p_{0,1}(L) = \Lambda\pi_0.$$

Hence,

- For $i = 0$:
 Set $\mathbf{M}_{n+2,n+2} = 1$,
 $\mathbf{M}_{n+2,1} = -c\Lambda/2$,
 $\mathbf{M}_{n+2,n+1} = -1 - c\Lambda/2$.
- For $2 \leq j \leq n$:
 Set $\mathbf{M}_{n+2,j} = -c\Lambda$.
- For $1 \leq i \leq n-1$:
 Set $\mathbf{M}_{n+i+2,n+i+2} = -2c\lambda_2$,
 $\mathbf{M}_{n+i+2,i+2} = 1$,
 $\mathbf{M}_{n+i+2,i} = -1$,
 $\mathbf{M}_{n+i+2,(n^2+7n+6)/2+ni+i-(i-1)i/2+1+n-i} = -2c$.
- Set $\mathbf{M}_{2n+2,2n+2} = 1$ and $\mathbf{v}_{2n+2} = \Lambda$.

For $p_{1,1}(x_1, y_1)$, let $0 \leq i \leq n$:

$$p_{1,1}(0, ic) = \lambda_2 p_{0,1}(ic).$$

For $1 \leq j \leq n$, $1 \leq i \leq j$:

$$p_{1,1}(ic, jc) = \lambda_2 e^{-ic\lambda_3} p_{0,1}((j-i)c).$$

Then,

- For $0 \leq i \leq n$:

$$\text{Set } \mathbf{M}_{2n+2+1+i, 2n+2+1+i} = 1,$$

$$\mathbf{M}_{n+2+i} = -\lambda_2.$$

- For $1 \leq j \leq n$ and $1 \leq i \leq j$:

$$\text{Set } \mathbf{M}_{2n+2+ni+i-(i-1)i/2+1-i+j, 2n+2+ni+i-(i-1)i/2+1-i+j} = 1,$$

$$\mathbf{M}_{2n+2+ni+i-(i-1)i/2+1-i+j, n+2+(j-i)} = -\lambda_2 e^{-ic\lambda_3}.$$

Now, for $p_{2,0}(y_1, y_2)$:

$$p_{0,2}(0, 0) = 0.$$

Let $1 \leq i \leq n$:

$$p_{0,2}(0, ic) = \lambda_2 \lambda_3 c \left[\frac{p_{0,1}(ic) + p_{0,1}(0)e^{-\lambda_3 ic}}{2} + \sum_{j=1}^{i-1} p_{0,1}(jc) e^{-\lambda_3(i-j)c} \right].$$

Finally, for $1 \leq j \leq n$ and for $1 \leq i \leq j$:

$$p_{0,2}(ic, jc) = p_{0,2}(0, (j-i)c).$$

- Set $\mathbf{M}_{(n^2+7n+8)/2, (n^2+7n+8)/2} = 1,$
- Set $\mathbf{M}_{(n^2+7n+8)/2+1, (n^2+7n+8)/2+1} = 1,$
- Set $\mathbf{M}_{(n^2+7n+8)/2+1, n+3} = -\lambda_2 \lambda_3 c/2,$
- Set $\mathbf{M}_{(n^2+7n+8)/2+1, n+2} = -\lambda_2 \lambda_3 c/2.$
- For $2 \leq i \leq n$:

$$\text{Set } \mathbf{M}_{(n^2+7n+8)/2+i, (n^2+7n+8)/2+i} = 1,$$

$$\mathbf{M}_{(n^2+7n+8)/2+i, n+2+i} = -\lambda_2 \lambda_3 c/2,$$

$$\mathbf{M}_{(n^2+7n+8)/2+i, n+2} = -\lambda_2 \lambda_3 c/2.$$

- For $2 \leq i \leq n$ and $1 \leq j \leq i - 1$:

$$\text{Set } \mathbf{M}_{(n^2+7n+8)/2+i, n+2+j} = -\lambda_2 \lambda_3 c e^{-\lambda_3(i-j)c}.$$

- For $1 \leq j \leq n$, and for $1 \leq i \leq j$:

$$\text{Set } \mathbf{M}_{(n^2+7n+6)/2+ni+i-(i-1)i/2+1+j-i, (n^2+7n+6)/2+ni+i-(i-1)i/2+1+j-i} = 1 \text{ and}$$

$$\mathbf{M}_{(n^2+7n+6)/2+ni+i-(i-1)i/2+1+j-i, (n^2+7n+6)/2+1+j-i} = -1.$$

To evaluate the values of steady-state probability densities, we must first solve the linear system of equations:

$$\mathbf{M}\mathbf{d} = \mathbf{v}.$$

We can invert the Linearization Matrix $\mathbf{M}$ and simply obtain $\mathbf{d} = \mathbf{M}^{-1}\mathbf{v}$. Now, we have to obtain the normalization constant τ by numerically integrating each of the density in her domain. Assume that the numerical integration procedure yields $\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3$, and $\hat{\pi}_4$. Set $\hat{\pi}_0 = 1$. Clearly, by appealing to the Trapezoidal Rule:

$$\hat{\pi}_1 = \frac{c[p_{1,0}(0) + p_{1,0}(L)]}{2} + c \sum_{i=1}^{n-1} p_{1,0}(ic) + \frac{p_{1,0}(L)}{\Lambda}.$$

$$\hat{\pi}_2 = \frac{c[p_{0,1}(0) + p_{0,1}(L)]}{2} + c \sum_{i=1}^{n-1} p_{0,1}(ic).$$

We note that the integrals of the steady-state probability densities $p_{1,1}(x_1, y_1)$ and $p_{0,2}(y_1, y_2)$ must be taken over a two-dimensional plane. Hence, for each area segment, we evaluate the equal weight average over the neighboring points. Namely,

$$\begin{aligned}
\hat{\pi}_3 &= c^2 \left[\frac{p_{1,1}(0,0) + p_{1,1}(L,L)}{6} \right] + \left[\frac{c^2}{6} + \frac{c^2}{4} \right] \sum_{i=1}^{n-1} p_{1,1}(ic, ic) \\
&+ \left[\frac{c^2}{6} + \frac{c^2}{4} \right] \left[p_{1,1}(0, c) + p_{1,1}(L - c, L - c) \right] + \frac{c^2}{2} \sum_{i=1}^{n-1} p_{1,1}(0, ic) \\
&+ \frac{c^2}{4} p_{1,1}(0, L) + \frac{c^2}{2} \sum_{i=1}^{n-1} p_{1,1}(ic, L) + \left[\frac{c^2}{6} + \frac{3c^2}{4} \right] \sum_{i=1}^{n-1} p_{1,1}(ic, ic + c) \\
&+ c^2 \sum_{i=1}^{n-3} \sum_{j=i}^{n-1} p_{1,1}(ic, (j+2)c).
\end{aligned}$$

The integration of $p_{0,2}(y_1, y_2)$ is precisely the same as presented in $\hat{\pi}_3$, where one replaces $p_{1,1}(x_1, y_1)$ with $p_{0,2}(y_1, y_2)$.

Then $\pi_i = \hat{\pi}_i / (\sum_{j=0}^4 \hat{\pi}_j)$. To estimate the waiting times, the integrals in their respective domains must be weighted accordingly, i.e. with the term $L - y_i$. $\square$

In the next chapter, using the findings of this chapter, we will provide the equilibrium behavior of a continuous review $(r, Q, \mathcal{B})$ model where the cumulative arrival process constitutes a homogeneous Poisson process, arriving customers are presented with a pre-specified information level and heterogeneous with respect to their waiting sensitivity and constant procurement lead times.

6. $(r, Q, \mathcal{B})$ MODEL WITH HETEROGENEOUS CUSTOMER WAITING SENSITIVITY AND DETERMINISTIC PROCUREMENT LEAD-TIMES

In this chapter, we use the findings of Chapter 5 to evaluate the equilibrium behavior of a continuous review $(r, Q, \mathcal{B})$ model that experiences customers that are heterogeneous with respect to their waiting sensitivity. The cumulative arrival process is assumed to form a homogeneous Poisson process and lead-times are assumed to be deterministic. We begin by deriving the equilibrium behavior when customers are presented with partial information. Then, we continue with the cases where customers are presented with no information and complete information.

6.1. $(r, Q, \mathcal{B})$ Inventory Policy with Partial Information

We assume that a single product is managed according to a $(r, Q, \mathcal{B})$ inventory policy. The setting is continuous review, and the customers are heterogeneous with respect to their valuation of their waiting times. The cumulative arrival process forms a state-dependent Poisson process, and procurement lead times are assumed to be deterministic. It is assumed that customers are only able to observe whether the net inventory is positive or not. In this subsection, we will provide the steady-state distribution of IN , which may be utilized to derive the steady-state distributions of other inventory state variables, at arbitrary instants, at arrival instants, and at departure instants. Furthermore, we will derive the steady-state distribution functions of customer waiting times at arrival instants and at arbitrary instants. We will use the results to derive the expected conditional waiting times at arrival instants or at arbitrary instants. In addition, we will derive the steady-state waiting time observed by an arbitrary customer or an arriving customer.

An arriving customer decides to purchase the product if their utility presented in Equation (3.1) is non-negative. Customers make inferences about waiting times

based on long-term averages. However, since customers make their decisions regarding entering the system or not upon their arrival, the steady-state behavior has to be examined at arrival instants to obtain the equilibrium behavior. Therefore, the system experiences arrival rates as follows: $\lambda_n = \Lambda$ for $n > 0$ and $\lambda_n = \lambda_e \leq \Lambda$ for $n \leq 0$ where n denotes the state of the net inventory. It is assumed that $R - p \geq 0$, since otherwise no customer would join, due to the non-positive value of their utility. This type of behavior is caused by the two different types of state of system that can be observed by an arriving customer. On the other hand, the retailer decides the values of r , Q and $\mathcal{B}$ based on Equation (3.2). Therefore, for correctly assessing the aforementioned metrics, steady-state behavior must be obtained according to the customer behavior.

Proposition 6.1. *For a continuous review $(r, Q, \mathcal{B})$ inventory policy experiencing homogeneous Poisson arrivals with rate Λ , deterministic procurement lead time distribution and heterogeneous customers that are only presented with stock-out information; the exact mathematical forms of the steady-state probability density functions of the system states can be derived for $r + Q + \mathcal{B} \leq 2Q - 1$. For the other cases, a numerical solution may be obtained. Furthermore, the steady-state arrival rate at stock-out instants, i.e. λ_e , can be calculated as the unique solution of the fixed-point equation:*

$$\lambda = \Lambda F\left(\frac{R - p}{\bar{w}_{PI}^a(\lambda)}\right), \quad (6.1)$$

where F denotes the distribution function of the random variable denoting an arbitrary customer's valuation per unit of waiting time, i.e. $\mathcal{C}$, R denotes the valuation of the item by customers, p denotes the price of the product, and $\bar{w}_{PI}^a(\lambda)$ denotes the steady-state waiting time observed by an arriving customer, given that the customer arrived the system at a stock-out instant, when $\lambda_n = \Lambda$ for $n > 0$, and $\lambda_n = \lambda$ for $n \leq 0$.

Proof. By appealing to Proposition 5.1 the analogous batch queueing system is $M_n/L^Q/\eta(r, Q, \mathcal{B})/r + Q + \mathcal{B}$. Furthermore, for $r + Q + \mathcal{B} \leq 2Q - 1$, the steady-state probability density functions can be derived analytically by Lemma 5.5. The parametric form of $\bar{w}_{PI}^a(\lambda)$ may be evaluated using the explicit functional forms. We would like to note that $\bar{w}_{PI}^a(\lambda) = \mathbb{E}[W^a]/\tau$, where $\tau = \sum_{i=0}^{1-\mathcal{B}} \pi_i^a$. Here, π_i^a denotes

the stationary probabilities that an arriving customer observes IN to be i upon their arrival, implying that τ denotes the probability that an arriving customer observes that the system is at a stock-out instant upon their arrival. Then, the fixed point Equation (6.1) must be solved to obtain λ_e . To show the existence of a solution, we refer to the Brouwer fixed-point theorem. The result and its applicability to this case are discussed in detail in [31]. We note that as λ increases, $\bar{w}_{PI}^a(\lambda)$ decreases. Since $\bar{w}_{PI}^a(\lambda)$ reside on the denominator of the input of the distribution function F , the result must be unique. For the cases where more than one outstanding order is allowed, the numerical solution procedures mentioned in Subsection 5.2.4 must be applied. The output of the numerical procedure is then used to evaluate Equation (6.1), until the fixed point is obtained. $\square$

The result we presented in Proposition 6.1 implies that, as long as the maximum shortfall of IN from $r + Q$ is restricted to be less than or equal to $2Q - 1$, complete and exact mathematical behavior of the system is available. For the cases where the shortfall amount is allowed to be greater than $2Q - 1$, the numerical solution procedures allow one to obtain the steady-state behavior of the system. We note that the distribution of IN is sufficient to characterize IP and IO . The dependence between the state variables is explicitly demonstrated in Proposition 5.1. Now we present how to evaluate the distributions and expected waiting times.

Lemma 6.2. *For the cases where $r + Q + \mathcal{B} \leq 2Q - 1$, the exact form of any steady-state metric can be derived. Specifically, the waiting time distributions and expectations can be obtained at arrival epochs or arbitrary epochs. Additionally, expected cycle length, $\mathcal{C}$, may be evaluated. For $r + Q + \mathcal{B} > 2Q - 1$, the steady-state metrics can be calculated numerically.*

Proof. We appeal to Proposition 6.1 for the characterization of the steady-state probability density functions. Assume that $r + Q + \mathcal{B} \leq 2Q - 1$. Furthermore, assume that in the analogous queueing system, arrival rates per unit time are denoted as the collection of $\tilde{\lambda}_{r+Q-n}$, when $IN = n$. For the partial information case, for $i \geq 0$, let $\tilde{\lambda}_{r+Q+i} = \lambda_e$ and for $-r - Q \leq i < 0$, let $\tilde{\lambda}_{r+Q+i} = \Lambda$. The corresponding state

descriptions of the queueing system are presented in Proposition 5.1. Let us begin by deriving the steady-state distribution at arrival epochs. First, we derive the steady-state distribution of IN at arbitrary instants. Assuming $-\mathcal{B} \leq i \leq r + Q$:

$$\hat{\pi}_i = \int \cdots \int_{\Xi_{r+Q-i}} p_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) dx_1 \dots dx_n dy_1 \dots dy_m,$$

where $n = (r + Q - i \bmod Q)$, $m = \lfloor (r + Q - i)/Q \rfloor$ and $\hat{\pi}_i$ denotes the steady-state invariant measure of the state $IN = i$ with respect to $IN = r + Q$. That is, again, we assume that $\pi_{r+Q} = \mathbb{P}\{IN = r + Q\}$ is our normalizing constant. Also, notice that the equivalent queueing system state corresponding to $IN = i$ is precisely (n, m) , as presented in Lemma 5.6, and Ξ_{r+Q-i} is the corresponding domain of the steady-state probability density function. Then we obtain π_{r+Q} as follows:

$$\pi_{r+Q} = \frac{1}{1 + \sum_{j=-\mathcal{B}}^{r+Q-1} \hat{\pi}_j}.$$

The stationary distribution of IN is then:

$$\pi_i = \hat{\pi}_i \pi_{r+Q},$$

where $\mathcal{B} \leq i \leq r + Q$ and $\hat{\pi}_{r+Q} = 1$. Then,

$$\pi_i^a = \frac{\lambda_i \pi_i}{\sum_{j=1-\mathcal{B}}^{r+Q} \lambda_j \pi_j},$$

where $i \in \{1 - \mathcal{B}, 2 - \mathcal{B}, \dots, r + Q\}$. To compute the distribution function of the waiting times at arbitrary epochs, for $\Omega \in [0, L]$ we note that:

$$\begin{aligned} \mathbb{P}\{W \leq \Omega\} &= \sum_{j=-\mathcal{B}}^{r+Q} \mathbb{P}\{W_j \leq \Omega\} \pi_j \\ &= \sum_{j=1}^{r+Q} \pi_j + \sum_{j=-\mathcal{B}}^0 \mathbb{P}\{W_j \leq \Omega\} \pi_j \\ &= \sum_{j=1}^{r+Q} \pi_j + \sum_{j=-\mathcal{B}}^0 \mathbb{P}\{W \leq \Omega, IN = j\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{r+Q} \pi_j + \sum_{j=-B}^0 \int \cdots \int_{\Xi_{r+Q-j} \cap (y_1 \geq L-\Omega)} p_{n_j, m_j}(x_1, \dots, x_{n_j}, y_1) \\
&\quad \cdot dx_1 \dots dx_{n_j} dy_1,
\end{aligned} \tag{6.2}$$

where $n_j = (r + Q - j \bmod Q)$. The Equation (6.2) follows from the fact that a customer has the probability of experiencing no wait if the net inventory is positive and, consequently, there is an item available. However, when there are no available items, i.e. $IN < 0$, then the customer must wait for the next Q items to arrive which must have been ordered at least $L - \Omega$ time units ago for the customer to wait at most Ω time units. Observe that $\mathbb{P}\{W \leq L\} = 1$, since the resulting formula is simply the normalization condition. We note that W_j denotes the random variable that indicates the waiting time of a customer given that $IN = j$. Then, to compute the distribution function of W_j , we simply scale the integrand:

$$\mathbb{P}\{W_j \leq \Omega\} = \frac{1}{\pi_j} \int \cdots \int_{\Xi_{r+Q-j} \cap (y_1 \geq L-\Omega)} p_{n_j, m_j}(x_1, \dots, x_n, y_1) dx_1 \dots dx_{n_j} dy_1,$$

where $j \leq 0$. To compute the expected steady-state waiting time observed by an arbitrary customer, we simply weigh the integrand accordingly. Namely,

$$\begin{aligned}
\mathbb{E}[W] &= \mathbb{E}[\mathbb{E}[W|IN]] \\
&= \sum_{j=-B}^0 \int \cdots \int_{\Xi_{r+Q-j} \cap (y_1 \geq L-\Omega)} (L - y_1) p_{n_j, 1}(x_1, \dots, x_{n_j}, y_1) dx_1 \dots dx_{n_j} dy_1, \\
&= \bar{w}_{PI}
\end{aligned} \tag{6.3}$$

where the first $r + Q$ terms vanish because the conditional expected values of the waiting times incurred when $IN \geq 1$ are simply 0. For the waiting time distribution conditioned on the value of IN upon arrival,

$$\begin{aligned}
\mathbb{P}\{W^a \leq \Omega\} &= \sum_{j=1-B}^{r+Q} \mathbb{P}\{W_j^a \leq \Omega\} \pi_j^a \\
&= \sum_{j=1}^{r+Q} \pi_j^a + \sum_{j=1-B}^0 \mathbb{P}\{W_j^a \leq \Omega\} \pi_j^a
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{r+Q} \pi_j^a + \sum_{j=1-\mathcal{B}}^0 \mathbb{P}\{W^a \leq \Omega, IN = j\} \\
&= \sum_{j=1}^{r+Q} \pi_j^a + \sum_{j=1-\mathcal{B}}^0 \int \cdots \int_{\Xi_{r+Q-j}^a \cap (y_1 \geq L-\Omega)} p_{n_j, m_j}(0, \dots, x_{n_j}, y_1) \\
&\quad \cdot dx_2 \dots dx_{n_j} dy_1 \\
&= \sum_{j=1}^{r+Q} \pi_j^a + \sum_{j=1-\mathcal{B}}^0 \int \cdots \int_{\Xi_{r+Q-j}^a \cap (y_1 \geq L-\Omega)} p_{n_j, m_j}(0, \dots, x_{n_j}, y_1) \\
&\quad \cdot dx_2 \dots dx_{n_j} dy_1,
\end{aligned} \tag{6.4}$$

where now $n_j = (r + Q - j + 1 \bmod Q)$ and the integrand may be substituted with the steady-state probability density corresponding to the state of the system with one less customer. We would like to state that W_j^a corresponds to the random variable that denotes the waiting time of a customer until obtaining their product, provided that the customer has just arrived and observed the state of the system to be $IN = j$ just prior to their arrival. For the expected waiting time of a customer that has just arrived, and observed the state of the system to be $IN = j \leq 0$ just before their arrival:

$$\mathbb{E}[W_j^a] = \frac{1}{\pi_j \lambda_j} \int \cdots \int_{\Xi_{r+Q-j}^a} (L - y_1) p_{n_j, 1}(0, \dots, x_{n_j}, y_1) dx_2 \dots dx_{n_j} dy_1. \tag{6.5}$$

At last, for the stationary expected waiting time in arrival epochs:

$$\mathbb{E}[W^a] = \sum_{j=\mathcal{B}-1}^0 \mathbb{E}[W_j^a] \pi_j^a, \tag{6.6}$$

where the formula of the expected value of the corresponding random variable can be found in Equation (6.5). Also, to evaluate $\bar{w}_{PI}^a$, we divide $\mathbb{E}[W^a]$ by the probability of IN being nonpositive at an arrival epoch, i.e. the probability of a customer observing the system at a stock-out instant at an arrival epoch:

$$\bar{w}_{PI}^a = \frac{\sum_{j=\mathcal{B}-1}^0 \mathbb{E}[W_j^a] \pi_j^a}{\sum_{i=\mathcal{B}-1}^0 \pi_i^a}. \tag{6.7}$$

Before we tackle the waiting time of customers in the numerical case, we would like to make some remarks. In this case, an arriving customer may experience a waiting time, yet there can be at most one outstanding order standing. This fact implies that an arriving customer may be the first customer to arrive that has experienced stock-out system state upon their arrival, which means that the order was placed at the instant of the arrival of the previous customer. Due to the constraint on the number of backorders allowed, i.e. $r + Q + \mathcal{B} \leq 2Q - 1$, an arriving customer that observes the system in a stock-out state will always receive an item from the arriving outstanding order of size Q , together with all the other customers that were waiting or customers that arrived after the arrival of the mentioned customer but before the arrival of the current outstanding order. Therefore, there cannot be two customers waiting that will receive their orders from two different outstanding orders. For the numerical case, we will demonstrate that this is not the case, yet certain waiting time metrics may be evaluated provided that we are more specific regarding the metric. Before we explore the characterization of the cases where a numerical approach is required, let us present how to evaluate the expected cycle length, $\mathcal{C}$.

Assuming that $r + \mathcal{B} \leq Q - 1$, let the random variable $\mathcal{B}$ denote the state of IN when a replenishment order arrives. Then, the support of the random variable $\mathcal{B}$ is precisely the set $\{r + Q - \mathcal{B}, \dots, r + Q\}$. Furthermore, without loss of generality, we can find the expected time it takes for the system to return to the re-order point, i.e. $IN = r$, for finding the expected cycle length. The time it takes for the inventory system to return to the reorder point can be decomposed into two independent parts:

$$\mathcal{C} = L + \mathbb{E}[\text{Time to place an order once the order arrives}],$$

where L denotes the constant procurement lead-time, and the second component denotes the expected time required for the IN to hit r again, once the replenishment order has arrived. Let the random variable Δ denote the time it takes for IN to hit r , and consequently trigger another replenishment order to be placed, once the current replenishment order arrives. We can find the distribution of $\mathcal{B}$, and condition the

random variable Δ on the value of $\mathcal{B}$ to find the expected value of Δ . Namely,

$$\mathbb{E}[\Delta] = \mathbb{E}\left[\mathbb{E}[\Delta|\mathcal{B}]\right],$$

where the value of $\mathbb{E}[\Delta|\mathcal{B}]$ can be computed trivially. To be rigorous, fix the value of $\mathcal{B}$ to be $k \in \{r + Q - \mathcal{B}, \dots, r + Q\}$. Then:

$$\mathbb{E}[\Delta|\mathcal{B}] = \sum_{i=1}^k \frac{1}{\lambda_i}, \quad (6.8)$$

since the successive inter-arrival times are independent and distributed exponentially. The last ingredient for the expected cycle time is the distribution of $\mathcal{B}$. It can be easily calculated through the steady-state probability density functions of the corresponding queueing system. We only need one observation: The random variable $\mathcal{B}$ in the inventory system $(r, Q, \mathcal{B})$ is precisely equivalent to the random variable $r + Q + \tilde{\mathcal{B}}$, where the random variable $\tilde{\mathcal{B}}$ denotes the number of customers left behind in the queue by a batch that leaves the system. We refer to the Lemma 5.6 for evaluating the vector of steady-state probabilities π^d . Then,

$$\mathbb{P}\{\mathcal{B} = k\} = \pi_{r+Q-k}^d,$$

where π_i^d denotes the steady-state probability that i customers are waiting in the queue at the departure instant of a batch. Combining our findings:

$$\begin{aligned} \mathcal{C} &= L + \mathbb{E}[\Delta] \\ &= L + \mathbb{E}\left[\mathbb{E}[\Delta|\mathcal{B}]\right] \\ &= L + \sum_{k=r+Q-\mathcal{B}}^{r+Q} \mathbb{E}[\Delta|\mathcal{B} = k] \pi_{r+Q-k}^d, \end{aligned}$$

where the formula to compute the conditional expectation is present on the Equation (6.8). We would like to note that in the cases where $r + \mathcal{B} \leq Q - 1$, $\mathcal{N} = 1/\mathcal{C}$ since at most one order can be outstanding, a cycle consists of exactly placement

and replenishment of one order. Now we continue with the cases involving numeric evaluations, i.e. $r + \mathcal{B} \geq Q$.

For the cases where $r + Q + \mathcal{B} \geq 2Q - 1$, it is possible to numerically estimate the probability density functions, as mentioned in Subsection 5.2.4. However, since there may be customers that receive their products from different outstanding orders, it is not straightforward to define the incurred waiting time. If desired, we can evaluate the expected waiting time of customers who will receive their orders from the first batch that will arrive. Informally stating the idea, it is possible to evaluate the waiting time distribution function for a set of customers who receive their orders from the i^{th} batch that will arrive, $i \in \mathbb{N}$. To do so, one must first use the Algorithm in Figure 5.1. Then we use the steady-state probability density estimations as follows:

$$\begin{aligned} \mathbb{P}\{W_i \leq \Omega\} &= \frac{1}{v} \sum_{j=-\mathcal{B}}^{r+Q-Qi} \int \cdots \int_{\Xi_{r+Q-n-m_jQ} \cup (y_{m_j-i+1} \geq L-\Omega)} \\ &\quad \cdot p_{n_j, m_j}(x_1, \dots, x_{n_j}, y_1, \dots, y_{m_j}) \\ &\quad \cdot dx_1 \dots dx_{n_j} dy_1 \dots dy_{m_j}, \end{aligned} \tag{6.9}$$

where $v = \sum_{j=-\mathcal{B}}^{r+Q-Q\tau} \pi_j$, $n_j = (r + Q - j \bmod Q)$, and $m_j = \lfloor (r + Q - j)/Q \rfloor$. It is simple to numerically integrate the estimated steady-state probability density functions, as prescribed in Subsection 5.2.4. To evaluate expected waiting times, we only weigh the integrand with $(L - y_{m_j-i+1})$. $\square$

In the next section, we derive the equilibrium behavior and waiting time characterization for a $(r, Q, \mathcal{B})$ policy with no information.

6.2. $(r, Q, \mathcal{B})$ Inventory Policy with No Information

In this section, we consider a continuous review $(r, Q, \mathcal{B})$ inventory system that experiences a homogeneous Poisson process as the cumulative demand process, deter-

ministic procurement lead times, and arriving customers are heterogeneous and are presented with no information regarding the current state of the system upon their arrival. Throughout the subsection, $\mathcal{B}$ has been assumed to be finite. We derive the equilibrium behavior assuming that the customers behave in accordance with Equation (3.1). Moreover, we derive the steady-state conditional and unconditional expected waiting times at arbitrary or arrival epochs. In addition, we provide the steady-state conditional or unconditional distribution function of the customer waiting times at arbitrary or arrival epochs.

Proposition 6.3. *For a continuous review $(r, Q, \mathcal{B})$ inventory policy experiencing homogeneous Poisson arrivals with rate Λ , deterministic procurement lead time distribution and heterogeneous customers that are presented with no information regarding the state of the system upon their arrival; the exact mathematical forms of the steady-state probability density functions of the system states can be derived for $r + Q + \mathcal{B} \leq 2Q - 1$. For cases where $\mathcal{B}$ is finite and $\mathcal{B} + r > Q - 1$, a numerical approximation may be obtained. Furthermore, the steady-state equilibrium arrival rate λ_{NI} can be calculated as the solution of the fixed-point equation:*

$$\lambda = \Lambda F\left(\frac{R - p}{\bar{w}_{NI}^a(\lambda)}\right), \quad (6.10)$$

where F denotes the distribution function of the random variable denoting the valuation of an arbitrary customer per unit of waiting time, i.e. C , R denotes the valuation of the item by customers, p denotes the price of the product, and $\bar{w}_{NI}^a(\lambda)$ denotes the steady-state expected waiting time experienced by an arbitrary arriving customer. Furthermore, conditional or unconditional steady-state expected waiting times at arbitrary or arrival instants and conditional or unconditional steady-state distribution functions of waiting times are available. Finally, when an unlimited number of backorders is allowed, i.e. $\mathcal{B} = \infty$, the exact equilibrium behavior and waiting time characterizations can be obtained.

Proof. We let $\lambda_n = \lambda_{NI}$ for $1 - \mathcal{B} \leq n \leq r + Q$. We appeal to Proposition 6.1 and Lemma 6.2 for the existence and uniqueness of a solution of the fixed-point Equation

(6.10) and the determination of the equilibrium arrival rate λ_{NI} . Furthermore, the steady-state behavior distribution of IN at arbitrary or arrival epochs is also available in the aforementioned proofs. The complete steady-state waiting time characterization is also available in Lemma 6.2. When the allowed number of backorders is not finite, waiting time metrics are available in Chapter 4. $\square$

In the next section, we present the characterization of an $(r, Q, \mathcal{B})$ inventory policy with full information.

6.3. $(r, Q, \mathcal{B})$ Inventory Policy with Full Information

In this section, it is assumed that a single product is managed according to a continuous review $(r, Q, \mathcal{B})$ inventory policy. The system experiences a homogeneous cumulative arrival process with intensity Λ , and the procurement lead times are deterministic. The customers are heterogeneous with respect to their valuation of their waiting times, are able to observe the exact state of the system upon their arrival and decide to enter the system based on their utilities presented in Equation (3.1). We provide the equilibrium behavior of the inventory system variables, exact waiting time distributions at either arrival or arbitrary epochs, and expected stationary waiting times at arbitrary instants or at arrival instants.

We will provide the equilibrium behavior of the system, which is governed by the customer behavior. The retailer uses Equation (3.2) to decide on the values of r , Q , and $\mathcal{B}$. Next, we state the proposition that allows us to characterize the equilibrium behavior of the system.

Proposition 6.4. *For a continuous review $(r, Q, \mathcal{B})$ inventory policy experiencing homogeneous Poisson arrivals with rate Λ , deterministic procurement lead time distribution and heterogeneous customers that are presented with full information regarding the state of the system; the exact mathematical forms of the steady-state probability density functions of the system states can be derived for $r + Q + \mathcal{B} \leq 2Q - 1$. For the other cases, a numerical solution may be obtained. Furthermore, the steady-state*

arrival rates at the stock-out states, i.e. the vector of λ_n for $n \leq 0$, can be calculated as the solution of the fixed-point equations:

$$\lambda_n = \Lambda F\left(\frac{R - p}{w_n^a}\right), \quad (6.11)$$

where F denotes the distribution function of the random variable denoting the valuation of an arbitrary customer per unit of waiting time, i.e. C , R denotes the valuation of the item by customers, p denotes the price of the product, and w_n^a denotes the steady-state waiting time experienced by an arriving customer who observed $IN = n$ just before their arrival.

Proof. The same reasoning as in Proposition 6.1 can be used to recover the mathematical forms of the steady-state probability density functions; however, now for $i \leq 0$, $\tilde{\lambda}_{r+Q-i} = \lambda_i$, and else $\tilde{\lambda}_{r+Q-i} = \Lambda$. We note that $\tilde{\lambda}_i$ denotes the arrival rate per unit of time when there are i customers in the corresponding queueing system, i.e. $IN = r + Q - i$. The equilibrium arrival rates are required to be found simultaneously, i.e. $\mathcal{B}$ number of Equations of type Equation (6.11) have to be solved at the same time. The reason for obtaining a simultaneous solution is that for a fixed value of $n \leq 0$, the parametric form of w_n depends not only on λ_n , but on all other arrival rates. The existence is easy to show through application of the Brouwer fixed-point theorem extended on multiple dimensions, which may be viewed in detail in [32]. $\square$

Lemma 6.5. *For the cases where $r + Q + \mathcal{B} \leq 2Q - 1$, the exact form of any steady-state metric can be derived. Specifically, the waiting time distributions and expectations can be obtained at arrival epochs or arbitrary epochs. For $r + Q + \mathcal{B} > 2Q - 1$, the steady-state metrics can be calculated numerically.*

Proof. We refer to Lemma 6.2 with a small modification in arrival rates. The solution procedures are exactly the same, but now $\tilde{\lambda}_{r+Q-i} = \Lambda$ for $i \in \mathbb{N}$, and $\tilde{\lambda}_{r+Q-i} = \lambda_i$ for $i \leq 0$. With this modification, an application of Lemma 6.2 provides us with the desired results. $\square$

The results are astounding and, based on our previous findings, there are several additional points that we would like to emphasize. Assume that the value of Q is fixed. Then, as long as the value of $r + \mathcal{B}$ is fixed, it does not affect the parametric forms of the steady-state probability density functions. Let $\Pi(\underline{\lambda}) \equiv \{\pi_i(\underline{\lambda}) : -\mathcal{B} \leq i \leq r + Q\}$ indicate the parametric forms of steady-state probability measures, where π_i is the steady-state probability of observing $IN = i$ at an arbitrary epoch, and $\underline{\lambda}$ is the ordered collection of λ_i s. The result is simply an application of Proposition 5.1 and Lemma 5.5. Although the parametric forms of the steady-state probability density functions remain the same, for the full information case, the number of fixed-point equations to be solved depends on the value of $\mathcal{B}$. Hence, the arrival rate values change. The equilibrium behavior is not the same for a fixed value of Q and $r + \mathcal{B}$.

Corollary 6.6. *Assume that an inventory is managed according to $(r = 0, Q = 3, \mathcal{B} = 2)$ policy with the parameter set Λ, R, p, L . In addition, assume that customers are provided with full information regarding the state variable IN upon their arrival, and evaluate their utilities based on Equation (3.1), where $\mathcal{C}$ is a uniform random variable with support $[0, \beta]$. Then, complete analytical characterization of steady-state behavior of the system variables and customer waiting times are possible.*

Proof. We refer to Corollary 5.7 for the derivation of steady-state probability densities and the limiting distribution at arbitrary epochs corresponding to the analogous queueing system $M_n/L^3/1/5$. Assume that λ_n denotes the arrival rate in the inventory system when $IN = n$, and the corresponding arrival rate in the queueing system is denoted as $\tilde{\lambda}_{r+Q-n}$. We are required to solve Equation (6.11) for $n = 0$ and $n = -1$ simultaneously to obtain λ_0 and λ_{-1} . Due to our assumptions, we know that $\lambda_i = \Lambda$ for $i \geq 1$. Let us We know that:

$$\begin{aligned} \mathbb{E}[W_{-1}^a] &= w_{-1}^a \\ &= \int_0^L \int_0^{y_1} \frac{(L - y_1)p_{2,1}(0, x_2, y_1)}{\pi_{-1}\lambda_{-1}} dx_2 dy_1 \\ &= \lambda_{-1} \int_0^L \int_0^{y_1} \frac{(L - y_1)p_{1,1}(x_2, y_1)}{\pi_{-1}\lambda_{-1}} dx_2 dy_1, \end{aligned}$$

where π_{-1}^a denotes the steady-state probability of an arriving customer observing $IN = -1$ just before their arrival. We have shown before that $\pi_{-1}^a = \lambda_{-1}\pi_{-1}/\lambda_{eff}$, where $\lambda_{eff} = \sum_{i=-1}^3 \lambda_i \pi_i$. Substituting the explicit functional form of the steady-state probability density $p_{1,1}$, which is the steady-state probability density corresponding to the system state $IN = -1$:

$$\begin{aligned}
&= \int_0^L \int_0^{y_1} \frac{(L - y_1)}{\pi_{-1}^a} \pi_0 \lambda_{-1} \lambda_0 \lambda_3 e^{\lambda_0 L} e^{-x_2(\lambda_{-1} - \lambda_0)} e^{-\lambda_0 y_1} dx_2 dy_1, \\
&= \frac{\pi_3}{\pi_{-1} \lambda_{-1}} \left\{ \frac{L \lambda_{-1} \lambda_3 e^{\lambda_0 L} (1 - e^{-\lambda_0 L})}{\lambda_0} - \frac{\lambda_{-1} \lambda_0 \lambda_3 e^{\lambda_0 L}}{\lambda_{-1} - \lambda_0} \left(-e^{-\lambda_0 L} L / \lambda_0 + \frac{1 - e^{-\lambda_0 L}}{\lambda_0^2} \right) \right. \\
&\quad \left. + \frac{\lambda_{-1} \lambda_0 \lambda_3 e^{\lambda_0 L}}{\lambda_{-1} - \lambda_0} \left(-e^{-\lambda_{-1} L} L / \lambda_{-1} + \frac{1 - e^{-\lambda_{-1} L}}{\lambda_{-1}^2} \right) \right\}.
\end{aligned}$$

Now for the effective arrival rate λ_{eff} , we are required to choose a normalization constant; let us proceed with the convention and assign π_3 as our normalization constant. Then, for λ_{eff}/π_3 :

$$\begin{aligned}
\lambda_{eff}/\pi_3 &= \sum_{i=-1}^3 \lambda_i \frac{\pi_i}{\pi_3} \\
&= \lambda_3 + \lambda_2 \lambda_3 \frac{(1 - e^{-\lambda_2 L})}{\lambda_2} + \frac{\lambda_2 \lambda_3 \lambda_0 (1 - e^{-L(\lambda_{-1} - \lambda_0)})}{(\lambda_{-1} - \lambda_0 + \lambda_2)(\lambda_{-1} - \lambda_0)} \\
&\quad - \frac{\lambda_3 \lambda_0 (1 - e^{-\lambda_2 L})}{(\lambda_{-1} - \lambda_0 + \lambda_2)} \\
&\quad + \lambda_3 \left\{ e^{-\lambda_2 L} + \frac{\lambda_0}{\lambda_{-1} - \lambda_0 + \lambda_2} (e^{-L(\lambda_{-1} - \lambda_0)} - e^{-\lambda_2 L}) \right\} \\
&\quad + \lambda_3 e^{\lambda_0 L} + \lambda_3 (e^{\lambda_0 L} - 1) \\
&\quad + \frac{\lambda_{-1} \lambda_3 \lambda_0 e^{\lambda_0 L}}{\lambda_{-1} - \lambda_0} \left\{ \frac{1 - e^{-\lambda_0 L}}{\lambda_0} - \frac{1 - e^{-\lambda_{-1} L}}{\lambda_{-1}} \right\}, \tag{6.12}
\end{aligned}$$

where π_3 is equal to:

$$\begin{aligned}
\pi_3 &= \frac{1}{1 + \pi_2/\pi_3 + \pi_1/\pi_3 + \pi_0/\pi_3 + \pi_{-1}/\pi_3 + \pi_{-2}/\pi_3} \\
&= \left[1 + \lambda_3 \frac{(1 - e^{-\lambda_2 L})}{\lambda_2} + \frac{\lambda_3 \lambda_0 (1 - e^{-L(\lambda_{-1} - \lambda_0)})}{(\lambda_{-1} - \lambda_0 + \lambda_2)(\lambda_{-1} - \lambda_0)} \right]^{-1}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\lambda_3 \lambda_0 (1 - e^{-\lambda_2 L})}{(\lambda_{-1} - \lambda_0 + \lambda_2) \lambda_2} \\
& + \frac{\lambda_3}{\lambda_2} \left\{ e^{-\lambda_2 L} + \frac{\lambda_0}{\lambda_{-1} - \lambda_0 + \lambda_2} (e^{-L(\lambda_{-1} - \lambda_0)} - e^{-\lambda_2 L}) \right\} \\
& + \frac{\lambda_3 e^{\lambda_0 L}}{\lambda_1} \\
& + \frac{(e^{\lambda_0 L} - 1)}{\lambda_0} \\
& + \frac{\lambda_3 \lambda_0 e^{\lambda_0 L}}{\lambda_{-1} - \lambda_0} \left\{ \frac{1 - e^{-\lambda_0 L}}{\lambda_0} - \frac{1 - e^{-\lambda_{-1} L}}{\lambda_{-1}} \right\} \\
& + \left[\lambda_3 e^{\lambda_0 L} \left\{ L - \frac{1 - e^{-\lambda_0 L}}{\lambda_0} \right\} - \frac{\lambda_3 \lambda_0 e^{\lambda_0 L}}{\lambda_{-1} - \lambda_0} \left\{ \frac{1 - e^{-\lambda_0 L}}{\lambda_0} - \frac{1 - e^{-\lambda_{-1} L}}{\lambda_{-1}} \right\} \right]^{-1} \quad (6.13)
\end{aligned}$$

We note that π_{-2} is equal to:

$$\begin{aligned}
\pi_{-2} &= \int_0^L \int_0^{y_1} \int_0^{x_2} p_{2,1}(x_1, x_2, y_1) dx_1 dx_2 dy_1 \\
&= \lambda_3 \pi_3 e^{\lambda_0 L} \left\{ L - \frac{1 - e^{-\lambda_0 L}}{\lambda_0} \right\} - \frac{\lambda_3 \lambda_0 \pi_3 e^{\lambda_0 L}}{\lambda_{-1} - \lambda_0} \left\{ \frac{1 - e^{-\lambda_0 L}}{\lambda_0} - \frac{1 - e^{-\lambda_{-1} L}}{\lambda_{-1}} \right\}.
\end{aligned}$$

Hence, combining our findings:

$$\begin{aligned}
\pi_{-1}^a &= \frac{\pi_{-1} \lambda_{-1}}{\lambda_{eff}} \\
&= \frac{\pi_3 \lambda_{-1} \lambda_3 \lambda_0 e^{\lambda_0 L}}{\lambda_{-1} - \lambda_0} \left\{ \frac{1 - e^{-\lambda_0 L}}{\lambda_0} - \frac{1 - e^{-\lambda_{-1} L}}{\lambda_{-1}} \right\} \lambda_{eff}^{-1}.
\end{aligned}$$

Upon substituting our findings above, the first fixed-point equation is then precisely:

$$\begin{aligned}
\lambda_{-1} &= \Lambda \mathbb{P}\{R - p - \mathcal{C} w_{-1}^a \geq 0\} \\
&= \Lambda \mathbb{P}\{\mathcal{C} \leq \frac{R - p}{w_{-1}^a}\} \\
&= \Lambda F\left(\frac{R - p}{w_{-1}^a}\right) \\
&= \Lambda \frac{(R - p)}{\beta w_{-1}^a} \quad (6.14)
\end{aligned}$$

To obtain the second fixed-point equation, we need to find w_0^a . We know that,

$$\begin{aligned} w_0^a &= \frac{1}{\pi_0 \lambda_0} \int_0^L p_{1,1}(0, y_1)(L - y_1) dy_1 \\ &= \frac{\lambda_3 \pi_3 e^{\lambda_0 L} L (1 - e^{-\lambda_0 L})}{\pi_0 \lambda_0} + \frac{\lambda_0 \lambda_3 \pi_3 e^{\lambda_0 L}}{\pi_0 \lambda_0} \left[\frac{e^{-\lambda_0 L} L}{\lambda_0} - \frac{1 - e^{-\lambda_0 L}}{\lambda_0^2} \right]. \end{aligned}$$

Also π_0^a is equal to:

$$\begin{aligned} \pi_0^a &= \frac{\lambda_0 \pi_0}{\lambda_{eff}} \\ &= \frac{\pi_3 \lambda_3 e^{\lambda_0 L} (1 - e^{-\lambda_0 L})}{\lambda_{eff}}. \end{aligned}$$

The explicit form of π_3 and λ_{eff} is available in Equations (6.13) and (6.12). Then, the second fixed-point equation is:

$$\lambda_0 = \Lambda \frac{(R - p)}{w_0^a \beta}. \quad (6.15)$$

We would like to note that on Equations (6.14) and (6.15) the values of $\lambda_1 = \lambda_2 = \lambda_3 = \Lambda$, due to our assumptions regarding the customer behavior. Solving these two fixed-point equations simultaneously would result in the desired values of λ_0 and λ_{-1} . After the values of λ_0 and λ_{-1} are determined, they can be used to determine the steady-state behavior of the inventory system at arrival, departure, or arbitrary epochs. Furthermore, exact waiting-time distributions are available by applying the Lemma 6.5. Finally, let us evaluate the expected cycle time, $\mathcal{B}$. First, let π_i^d denote the steady state probability that the arrival of an outstanding batch of orders will cause IN to be i . Then,

$$\begin{aligned} \hat{\pi}_3^d &= p_{0,1}(L) \\ &= \lambda_3 \pi_3. \end{aligned}$$

For the steady-state probability that one customer will be waiting for their order upon the arrival of a batch:

$$\begin{aligned}\hat{\pi}_2^d &= \int_0^L p_{1,1}(x_1, L) dx_1 \\ &= \frac{\lambda_3 \pi_3 \lambda_0 (1 - e^{-(\lambda_{-1} - \lambda_0)L})}{\lambda_{-1} - \lambda_0}.\end{aligned}$$

For the steady-state probability that two customers will be waiting for their orders at the replenishment instant:

$$\begin{aligned}\hat{\pi}_1^d &= \int_0^L \int_0^{x_2} p_{2,1}(x_1, x_2, L) dx_1 dx_2 \\ &= \lambda_3 \pi_3 \lambda_0 \left\{ \frac{e^{\lambda_0 L} - 1}{\lambda_0} - \frac{1 - e^{-(\lambda_{-1} - \lambda_0)L}}{\lambda_{-1} - \lambda_0} \right\}.\end{aligned}$$

We are required to normalize the steady-state vector of probabilities by:

$$\pi_i^d = \frac{\hat{\pi}_i^d}{\hat{\pi}_3^d + \hat{\pi}_2^d + \hat{\pi}_1^d},$$

for $i \in \{1, 2, 3\}$. Combining our findings:

$$\mathcal{C} = L + \pi_3^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda} + \pi_1^d \frac{1}{\Lambda}.$$

□

By Corollary 6.6, we have shown that we can analytically characterize a ($r = 0, Q = 3, \mathcal{B} = 2$) inventory policy and derive the equilibrium behavior of the inventory system when full information is provided to customers who are heterogeneous with respect to their waiting times. We would like to note that we only established existence of an equilibrium point; however, our numerical results never disagreed with the fact that the equilibrium point is unique. The formal proof for the uniqueness of the equilibrium point is not present when more than one fixed-point equation is required to be solved. The result that proves the uniqueness of the set of fixed points may

be established via the fact that the set of steady-state probability density functions is unique for a given set of parameters, that is, if the parameter set does not produce a degenerate system. Then, the integral transforms of the set of steady-state probability density functions must be obtained for the expected conditional waiting times. If we were to determine the set of values for λ_n that provides us with a prespecified known set of values for w_n , then if the set of values for λ_n is unique, it would imply that the equilibrium point is unique. The result follows directly from the assumption that CDF $F(\cdot)$ is continuous and consequently monotone. However, integral transforms are applied to more than one steady-state probability density function, are not necessarily monotone, and involve grift functions of all λ_n . Hence, the uniqueness of the equilibrium point is not straightforward to establish.

In the next section, we explain a continuous review $(r, Q, \mathcal{B})$ inventory policy with exponential procurement lead times.

7. $(r, Q, \mathcal{B})$ INVENTORY POLICY WITH EXPONENTIAL LEAD-TIMES AND HETEROGENEOUS CUSTOMERS

In this chapter, we will review a continuous review reorder point, fixed-order quantity inventory model with a limited number of backorders, i.e. a $(r, Q, \mathcal{B})$ inventory model, that involves exponential procurement lead times and state-dependent arrival rates. We will begin by analyzing the steady-state behavior of the analogous queueing system, under a capacitated and uncapacitated setting. Then we use the proposed analogy, i.e. Proposition 5.1, to obtain the steady-state behavior of the corresponding inventory model. We present an analytical method to find the conditional and stationary expected waiting times at arbitrary or arrival epochs. Furthermore, we also derive the conditional and unconditional stationary waiting time distributions. We will present the exact conditional or unconditional waiting time distributions when at most one order is outstanding, and propose a combinatorial method to find the waiting time distributions in the cases where more than one order may be outstanding. Finally, we apply our findings to derive the equilibrium behavior of the inventory system when no-information, partial information, and full information is presented to the arriving customers. Practically, for any information level, the findings of this section can completely characterize the exact steady-state distribution of state variables and the conditional and unconditional distribution of the customer waiting times distribution in a $(r, Q, \mathcal{B})$ model when Q is arbitrary and $r + \mathcal{B} \leq Q - 1$. For the cases where $r + \mathcal{B} \geq Q$, we present a combinatorial method to find the exact conditional or unconditional distribution of waiting times, but the complexity of the method grows exponentially as $r + \mathcal{B}$ increases. We begin by forming an analogy of the inventory system with a bulk queueing system and deriving the steady-state distribution of the corresponding analogous queueing model.

7.1. Steady-State Distribution of the Analogous Queueing System

We begin this section by forming the analogy with the corresponding bulk queueing model: $M_n/M^Q/\dots$

We invoke Proposition 5.1 to establish the fact that the $(r, Q, \mathcal{B})$ inventory system is analogous to the queueing system $M_n/M^Q/\eta(r, Q, \mathcal{B})/r+Q+\mathcal{B}$ in *Kendall's* notation, where the arrival process of the system is a state-dependent Poisson process, the service is provided in batches of exactly Q customers with a sequence of i.i.d. exponential random variables with rate μ , $\eta(r, Q, \mathcal{B}) = \lfloor (r + Q + \mathcal{B})/Q \rfloor$ identical servers and at maximum there can be $r + Q + \mathcal{B}$ customers in the system, i.e. $\lambda_n = 0$ for $n \geq r + Q + \mathcal{B}$. When the number of customers that can be present simultaneously in the system, i.e. the system capacity (denoted as $\mathcal{C}$), is allowed to be infinite, although Markovian analysis methods are applicable, the explicit mathematical forms of the steady-state distribution of the number of customers in the system are not present in the literature. We may obtain an approximation by truncating the system, that is, we assume that $\mathcal{C}$ is finite yet sufficiently large enough. The applicability of standard Markovian analysis methods is due to the fact that the stochastic process denoting the number of customers present in the queueing system of interest forms a CTMC because of exponential lifetimes. Assuming that a steady-state behavior exists and the system is at steady-state, the system does not admit any memory, except the number of customers currently present in the system. For a finite $\mathcal{C}$, the exact steady-state distribution of the system is available. Hence, for the uncapacitated setting, a sufficiently large value of $\mathcal{C}$ would allow one to obtain an analytical approximation of the steady-state behavior. Therefore, we are required to obtain the steady-state behavior of the queueing system $M_n/M^Q/\lfloor \mathcal{C}/Q \rfloor/\mathcal{C}$.

Proposition 7.1. *The steady-state characterization of the number of customers present in the batch queueing system $M_n/M^Q/\lfloor \mathcal{C}/Q \rfloor/\mathcal{C}$ with a first-in-first-out service discipline can be derived by the standard Markovian analysis methods, that is if $\mathcal{C}$ is finite. In Kendall's notation, M_n denotes that the arrival process is a state-dependent Poisson*

process, M^Q denotes that service time is an exponential random variable with rate μ and services are provided in fixed batches of Q customers, $\lfloor \mathcal{C}/Q \rfloor$ and $\mathcal{C}$ denote number of identical servers and maximum number of customers that can be present simultaneously, respectively.

Proof. For simplicity, assume that $\mathcal{C} = r + Q + \mathcal{B}$ is finite. Let us denote the number of customers present in the queueing system at time t by the Stochastic Process $X(t)$. Rather than the transient behavior, we are interested in characterization of the steady-state behavior of $X(t)$, i.e. $\lim_{t \rightarrow \infty} X(t) \equiv \chi$. Assuming that $\lambda_i > 0$ and all rates, including μ , are finite, the process of interest is *ergodic*, as presented in [33], i.e. the aforementioned process has the properties of aperiodicity, positive recurrence, and irreducibility. Denote $(\mathcal{C} + 1)$ by $(\mathcal{C} + 1)$ square generator matrix of the process χ by $\mathbf{A}$. It is known that:

$$\mathbf{A}_{ij} = \begin{cases} -\lambda(i) & \text{if } i = j \\ \lambda(i)Q_{i,j} & \text{if } i \neq j \end{cases}, \quad (7.1)$$

where $\lambda(i) = \lambda_i + \lfloor i/Q \rfloor \mu$ for $i \in \mathbb{N}_0$, and $Q_{i,j} = \lambda_{i,j}/\lambda_i$, where $\lambda_{i,j}$ denotes the flow rate from state i to state j . We note that for $i \geq Q$, $\lambda_{i,(i-Q)} = \lfloor i/Q \rfloor \mu$. Then, by appealing to the proofs of steady-state behavior characterization of a CTMC in [33], we are required to solve:

$$\boldsymbol{\pi}' \mathbf{A} = \underline{\mathbf{0}}, \quad (7.2)$$

where $\boldsymbol{\pi}'$ is the transpose of the collection of steady-state invariant measures as a column vector, and $\underline{\mathbf{0}}$ is a column vector of length $\mathcal{C} + 1$. The equation has infinitely many solutions since the generator matrix $\mathbf{A}$ has rank $\mathcal{C}$, assuming that the system of interest is not degenerate. By coupling Equation (7.2) with the following normalization

condition:

$$\pi' \underline{1} = 1, \quad (7.3)$$

where $\underline{1}$ is a column vector of length $\mathcal{C} + 1$ with all entries equal to 1. Solving Equations (7.2) and (7.3) together, we obtain the unique probability measure corresponding to the stochastic process χ . The process consists of births and deaths; however, deaths occur in batches of size Q . A proof of the global balance identity presented in Equation (7.2) can be provided through Forward or Backward Kolmogorov equations. Simply, it is a combination of the law of total probability, conditioning, and advantageous properties of the Poisson Point Processes. We will not provide a formal proof of the derivation for the identity present in Equation (7.2). Finally, by taking advantage of the Forward or Backward Kolmogorov Equations, the steady-state probability vector at arrival epochs could be derived. Let us denote π_i^a as the steady-state probability that an arriving customer will observe i customers just prior to their arrival. We compute the steady-state probability measures at arrival epochs by the following:

$$\pi_i^a = \frac{\pi_i \lambda_i}{\lambda_{eff}}, \quad (7.4)$$

where $\lambda_{eff} = \sum_{i=0}^{\mathcal{C}-1} \pi_i \lambda_i$, denoting the effective arrival rate. We have presented the steady-state distribution of the number of customers at arrival or in arbitrary epochs, which is what was desired to be shown. $\square$

We note that for an infinite value of $\mathcal{C}$, one may truncate the system, i.e. assume that the system admits a finite yet large enough capacity $\mathcal{C}'$ and solve the corresponding balance equations coupled with the normalization condition. As $\mathcal{C}'$ tends to infinity, the steady-state vector of probabilities for the number of customers in the system would probably converge to the true values. The problem with incrementally increasing the pseudo capacity $\mathcal{C}'$ is that the length of the steady-state vector of probabilities also increases. Therefore, the formal definition of convergence does not suffice in this setting.

In the next section, we will provide a characterization of the equilibrium behavior of the continuous review $(r, Q, \mathcal{B})$ inventory system with heterogeneous customers who are presented with no information.

7.2. $(r, Q, \mathcal{B})$ Inventory System with No Information

In this section, we explore a single item inventory that is managed according to a limited number of backorder (r, Q) policy, i.e. $(r, Q, \mathcal{B})$ policy. We assume that the cumulative arrival process constitutes a homogeneous Poisson process, the lead times are exponentially distributed and the customers are heterogeneous with respect to the unit waiting times they incur. For the case where customers are presented with no information, we derive the equilibrium behavior of the inventory system. It is assumed that an arbitrary arriving customer evaluates their utility with respect to Equation (3.1); however, the waiting time estimate is replaced with the long-run average, since customers are assumed to have no information regarding the state of the system upon their arrival. We will derive the equilibrium behavior, expected waiting times at arrival epochs, or arbitrary epochs. Additionally, we will derive the conditional expected waiting times at arrival epochs and the distribution functions of the conditional waiting times at arrival and arbitrary epochs or stationary waiting times at arbitrary or arrival epochs. We begin with two propositions that will allow us to characterize the net inventory at arrival or at arbitrary epochs and the waiting times.

Proposition 7.2. *For finite values of $r, Q, \mathcal{B}$, assume that a continuous review $(r, Q, \mathcal{B})$ inventory system experiences homogeneous Poisson arrivals as the cumulative arrival process and lead times are exponentially distributed. Then, the steady-state distribution of the stochastic process IN at arbitrary or arrival epochs can be derived analytically.*

Proof. We appeal to Proposition 5.1 to form the analogy with the bulk queueing system $M_n/M^Q/\eta(r, Q, \mathcal{B})/r + Q + \mathcal{B}$. In Proposition 7.1, we have shown that the steady-state distribution of the number of customers present in the bulk queueing system is available. Hence, for $-\mathcal{B} \leq i \leq r + Q$, the steady-state distribution of IN is

precisely:

$$\pi_i = \pi_{r+Q-i}^*, \quad (7.5)$$

where $\pi_i^* = \mathbb{P}\{\lim_{t \rightarrow \infty} X(t) = i\}$, and $X(t)$ denotes the number of customers present in the corresponding analogous queueing system at time $t \geq 0$. In addition for $1 - \mathcal{B} \leq i \leq r + Q$,

$$\pi_i^a = \frac{\lambda_{r+Q-i}^* \pi_{r+Q-i}^*}{\lambda_{eff}^*}, \quad (7.6)$$

where π_i^a denotes the steady-state probability of an arriving customer observing $IN = i$ just before their arrival and λ_j^* denotes the arrival intensity to the queueing system when $X(t) = j$. In the original equation, i.e. Equation (7.6), the nominator included a state-dependent arrival rate corresponding to that state, and the denominator included the effective arrival rate. The simplification is due to the fact that $\lambda_i^* = \lambda$ in this case, where λ_i^* denotes the state-dependent arrival rate of the corresponding queueing system when $X(t) = i$, and consequently $\lambda_{eff}^* = \lambda(1 - \pi_{r+Q+\mathcal{B}}^*)$ and λ terms cancel out. $\square$

Now we provide the proposition regarding the stationary waiting times at arrival and arbitrary epochs.

Proposition 7.3. *For finite values of $r, Q, \mathcal{B}$, assume that a continuous review $(r, Q, \mathcal{B})$ inventory system experiences state-dependent Poisson arrivals as the cumulative arrival process and lead times are exponentially distributed. Then, the distribution functions and expected values of waiting times at arbitrary times or arrival epochs can be derived. Furthermore, the conditional distribution functions and conditional expected values of waiting times can be obtained at arbitrary or arrival epochs. At last, the expected cycle time, $\mathcal{C}$, can be derived when at most one order can be outstanding.*

Proof. For sake of simplicity, without loss of generality, let us assume that at most one order can be outstanding. The case where more than one order is outstanding can also be derived through the same procedure that we will prescribe, yet, as expected, the process is more cumbersome. The assumption that at most one order can be outstanding implies that $r + \mathcal{B} \leq Q - 1$. Let the random variable W_j^a denote the expected waiting time of a customer who observes $IN = j$ just before their arrival, i.e. IN becomes $j - 1$ after their arrival, where $0 \leq j \leq \mathcal{B} - 1$. Furthermore, let ω_j^* denote the random vector of possible paths that IN may take until the customer of interest obtains their product, i.e. the currently outstanding order arrives. In our case, ω_j^* can be represented as a natural number that denotes the number of transitions that IN faces until the customer obtains their product because the possible paths the stochastic process IN takes can be represented as bijective transformation with respect to the number of transitions or equivalently the cardinality of the random path vector ω_j^* . In cases where more than one order may be outstanding, ω_j^* cannot be represented as a natural number, which will be further clarified after we prove the current case. Then, according to the law of total expectation,

$$\mathbb{E}[W_j^a] = \mathbb{E}[\mathbb{E}[W_j^a | \omega_j^*]].$$

First, let us derive the distribution of ω_j^* . Since the lifetimes governing the subprocesses are all exponential, we may take advantage of an important property of the exponential distribution: The minimum of any number of exponential random variables is distributed as another exponential random variable with rate parameter equal to the sum of rates of exponential variables of interest. Then, in the case where at most one order may be outstanding, we may easily evaluate the probability that the customer obtains their product after k transitions. We note that the occurrence of k transitions implies that $\omega_j^* \equiv \{IN = -j - 1, IN = -j - 2, \dots, IN = -j - k\}$, where the last transition is the arrival of the currently outstanding batch of size Q . $\omega_j^* = 1$ implies that no customers have arrived after the arrival of the customer of interest and the order arrived. Therefore, if $\omega_j^* \equiv \{IN = -j - 1, IN = -j - 2, \dots, IN = -j - k\}$ then $|\omega_j^*| = k$ where $|\cdot|$ denotes the cardinality of the vector ω_j^* , and consequently exactly

$k - 1$ customers have arrived after the arrival of the customer of interest but before the arrival of the current outstanding batch. Due to the memorylessness property of the exponential random variables, after each transition the lifetimes reset, or in other words the hazard rate of the random variables of interest are constant. Therefore, for $1 \leq k \leq \mathcal{B} - j$:

$$\mathbb{P}\{|\omega_j^*| = k\} = \frac{\mu}{\mu + \lambda_{-j-k-1}} \prod_{i=0}^{k-2} \frac{\lambda_{-j-1-i}}{\mu + \lambda_{-j-1-i}},$$

where we note that for $k = 1$ the product operator, denoted as capital π , is equal to 1, and $\lambda_{-\mathcal{B}} = 0$, since the maximum number of backorders allowed are $\mathcal{B}$. Furthermore, since at each step we are evaluating the expected value of the minimum of two exponential random variables, the conditional expected value becomes:

$$\mathbb{E}[W_j^a | |\omega_j^*| = k] = \sum_{i=0}^{k-1} \frac{1}{\mu + \lambda_{-j-1-i}}.$$

Therefore, combining our results for $1 - \mathcal{B} \leq j \leq 0$:

$$\mathbb{E}[W_j^a] = \sum_{k=1}^{\mathcal{B}-j} \left\{ \left(\frac{\mu}{\mu + \lambda_{-j-k-1}} \prod_{i=0}^{k-2} \frac{\lambda_{-j-1-i}}{\mu + \lambda_{-j-1-i}} \right) \sum_{i=0}^{k-1} \frac{1}{\mu + \lambda_{-j-1-i}} \right\}. \quad (7.7)$$

Then, the stationary expected waiting time observed by customers at arbitrary instants may be evaluated via:

$$\mathbb{E}[W^a] = \sum_{j=0}^{\mathcal{B}-1} \mathbb{E}[W_j^a] \pi_{-j}^a, \quad (7.8)$$

where π_j^a is as presented in Proposition 7.2. At last, we may evaluate the expected stationary waiting time at an arbitrary epoch by:

$$\mathbb{E}[W] = \sum_{j=0}^{\mathcal{B}-1} \mathbb{E}[W_j^a] \pi_{-j-1}. \quad (7.9)$$

The evaluation of expected stationary waiting time at an arbitrary epoch involves the term $\mathbb{E}[W_j^a]$, since the exponential lifetimes allow us to disregard the past history and consequently it makes no difference whether a customer has just arrived or has been waiting for an arbitrary time t . However, the weights are adjusted accordingly since we are no longer interested in arrival epochs. Finally, we present the conditional and stationary waiting time distributions at arrival or arbitrary epochs. First we observe that,

$$\mathbb{P}\{W_j^a \leq q\} = \sum_{k=1}^{\mathcal{B}-j} \mathbb{P}\{X_k \leq q\} \mathbb{P}\{|\omega_j^*| = k\}, \quad (7.10)$$

where $X_k = \sum_{i=1}^k Y_i$, and $Y_i \sim \text{Exponential}(\mu + \lambda_{-j-i})$. The Equation above is simply a result of conditioning. The random variables Y_i denote the minimum of two exponential variables at step i , and if $|\omega_j^*| = k$ then k -fold convolution with respect to presented random variables are required to be obtained. As mentioned previously, this is precisely a *Hypoexponential* random variable. Then, the waiting time distribution at arrival epochs or arbitrary epochs are evaluated by:

$$\mathbb{P}\{W^a \leq q\} = \sum_{j=0}^{1-\mathcal{B}} \mathbb{P}\{W_j^a \leq q\} \pi_{-j}^a, \quad (7.11)$$

$$\mathbb{P}\{W \leq q\} = \sum_{j=0}^{1-\mathcal{B}} \mathbb{P}\{W_j^a \leq q\} \pi_{-j-1}. \quad (7.12)$$

For the one order outstanding case, we have presented the waiting time information. We have derived the results to apply for the general case, yet currently we replace

$\lambda_i = \lambda$. Now, let us explain how to handle the waiting times in the cases where more than one order can be outstanding.

Before we begin the analysis on waiting times, we note that for finite values of $r, Q, \mathcal{B}$, the steady-state distribution of IN can be evaluated through Proposition 7.2. In the $(r, Q, \mathcal{B})$ inventory system, we assumed that customers are serviced according to the *FIFO* principle. This assumption is entirely realistic; however, as expected, it introduces some complications in the analysis phase. Let us first stress the main issue, which is essentially caused by the nondegenerate probabilistic distribution of the procurement lead times. Arriving customers may be backordered, yet receive their items from an outstanding order of size Q that is currently not existent. Namely, after the arrival of the customer of interest, say the customer of interest is called the customer ϕ , there may be a number of customer arrivals to the system before the customer ϕ obtains their product, and customers that arrive after ϕ may trigger the placement of another order of size Q . Then, it is entirely possible for the customer ϕ to obtain his/her product from the most recently ordered batch of Q products, if the opportunity presents itself. Hence, outstanding orders of size Q no longer arrive in the chronological order they were placed, making tagging of individual customers to currently outstanding orders impossible. However, our analysis method is still possible by conditioning the vector of the random path variable ω_j^* . We are only required to define all possible paths the process might take, which may include non-trivial paths. Needless to say, ω_j^* can no longer be expressed as a one-dimensional bijective transformation of a finite subset of natural numbers. To evaluate the waiting time metrics mentioned above in more than one outstanding order case, one must first enumerate the possible paths the process may take, i.e. determine the support of ω_j^* . Let us denote the set of all possible paths by Ω_j^* , for an arriving customer that experiences $IN = j$ just before their arrival. Then the summations should be taken over Ω_j^* , which cannot be expressed as a natural number in the previous case. The distribution of ω_j^* is trivial due to exponential lifetimes. We utilize the formulas presented above to evaluate the conditional or unconditional expected waiting times and the distribution functions of the waiting times. Finally, we evaluate the expected cycle length in the at most one outstanding order case. Before

we begin, we would like to note that the approach is similar to the one we have used to derive the waiting time characterizations. Let the random variable α^* denote the state of IN at the arrival instant of the current outstanding order. Then, conditioning $\mathcal{C}$ on the value of α^* :

$$\mathcal{C} = \mathbb{E} \left[\mathbb{E}[\text{Cycle Length} | \alpha^*] \right].$$

We note that the inner conditional expectation is trivial. The support of the random variable α^* is precisely $\{r + Q - \mathcal{B}, \dots, r + Q\}$. Then, given that $\alpha^* = k$:

$$\mathbb{E}[\text{Cycle Length} | \alpha^* = k] = \sum_{j=k-Q-r}^r \frac{1}{\mu + \lambda_j} + \sum_{j=1+r}^k \frac{1}{\lambda_j}.$$

At last, combining our findings:

$$\begin{aligned} \mathcal{C} &= \mathbb{E} \left[\mathbb{E}[\text{Cycle Length} | \alpha^*] \right] \\ &= \sum_{i=-\mathcal{B}}^r \frac{\mu}{\mu + \lambda_i} \left\{ \prod_{j=i+1}^r \frac{\lambda_j}{\lambda_j + \mu} \right\} \left\{ \sum_{j=i}^r \frac{1}{\mu + \lambda_j} + \sum_{j=1+r}^{i+Q} \frac{1}{\lambda_j} \right\}, \end{aligned}$$

where the first term inside the summation and the product represent the distribution of the random variable α^* . $\square$

Proposition 7.4. *For finite values of $r, Q, \mathcal{B}$, assume that a continuous review $(r, Q, \mathcal{B})$ inventory system experiences homogeneous Poisson arrivals as the cumulative arrival process with intensity Λ and lead times are i.i.d. and exponentially distributed. If the customers are presented with no information regarding the current state of the system, and customers evaluate their utility with respect to Equation (3.1), then the unique equilibrium arrival rate may be obtained by solving the fixed-point equation:*

$$\lambda = \Lambda F \left(\frac{R - p}{\bar{w}_{NI}^a(\lambda)} \right), \quad (7.13)$$

where $F(\cdot)$ denotes the cumulative distribution function of the random variable C in Equation (3.1), and $\bar{w}_{NI}^a(\lambda)$ denotes the expected waiting time incurred by an arriving customer when customers arrive with an equilibrium intensity λ , R and p are fixed scalars denoting the customer's valuation of the service and the price of the service. In addition, the mathematical forms of conditional or unconditional expected waiting times at arbitrary or arrival epochs and conditional or unconditional distribution functions of the waiting times at arbitrary or arrival epochs are available.

Proof. The existence and uniqueness of a solution for the fixed-point Equation (7.13) can be shown, as we have done previously, which we will not commit. We are required to obtain the parametric form of the stationary distribution of IN in arrival epochs using Proposition 7.2. The expected waiting time incurred by a waiting customer may be evaluated using the results of Proposition 7.3 and the parametric forms of the steady-state distributions. $\bar{w}_{NI}^a(\lambda)$ is precisely equal to $\mathbb{E}[W^a]$. We would like to note that the expected waiting time cannot be computed by using the Little's formula as we presented in the no-information case of the deterministic procurement lead-time setting. This is because the steady-state distribution of the net inventory at arbitrary epochs is no longer equivalent to the steady-state distribution of the net inventory at arrival epochs. The system admits a capacity constraint, and Poisson arrivals no longer see time averages. For the characterization of waiting times, we appeal to Proposition 7.3. $\square$

In the next section, let us present the equilibrium behavior of the $(r, Q, \mathcal{B})$ policy as well as complete waiting time characterizations in the partial information scenario.

7.3. $(r, Q, \mathcal{B})$ Inventory System with Partial Information

In this section, we explore the continuous review $(r, Q, \mathcal{B})$ inventory policy where we assume that the cumulative arrival process is a homogeneous Poisson point process and procurement lead times are i.i.d. exponential random variables. An arriving customer evaluates their utility with respect to Equation (3.1), and is only presented

with the system's stock availability information, i.e. whether there is stock available or not. We first derive the equilibrium behavior of the inventory system, and then present the waiting-time characterizations.

Proposition 7.5. *For finite values of $r, Q, \mathcal{B}$, assume that a continuous review $(r, Q, \mathcal{B})$ inventory system experiences homogeneous Poisson arrivals as the cumulative arrival process with intensity Λ and lead times are i.i.d. and exponentially distributed. If the customers are presented with partial information, and customers evaluate their utility with respect to Equation (3.1), then the unique equilibrium arrival rate at stockout instants, i.e. λ_e , may be obtained by solving the fixed-point equation:*

$$\lambda = \Lambda F\left(\frac{R - p}{\bar{w}_{PI}^a(\lambda)}\right), \quad (7.14)$$

where $F(\cdot)$ denotes the cumulative distribution function of the random variable C in Equation (3.1), and $\bar{w}_{PI}^a(\lambda)$ denotes the expected waiting time incurred by arriving customers when customers arrive with intensities Λ and λ , at available stock and stockout instants, respectively. R and p are fixed scalars that denote the customer's valuation of the service and the price of the service. In addition, the mathematical forms of conditional or unconditional expected waiting times at arbitrary or arrival epochs and conditional or unconditional distribution functions of the waiting times at arbitrary or arrival epochs are available.

Proof. For the existence and uniqueness of the solution of the fixed-point Equation (7.14), we refer to Proposition 6.1. First, let $\lambda_n = \Lambda$ for $n > 0$ and $\lambda_n = \lambda_e$ for $n \leq 0$. The Proposition 7.2 must be used to compute the parametric forms of the steady-state distribution of IN in arrival epochs. Since parametric forms of the steady-state distribution of IN in arrival epochs are available, the expected waiting time experienced by an arriving customer, i.e. $\bar{w}_{PI}^a(\lambda)$ can be evaluated using Proposition 7.3. After computing the solution of Equation (7.14), using λ_e and Proposition 7.3 we can compute desired waiting time distributions or expectations at arbitrary or arrival epochs. $\square$

In the next section, we derive the equilibrium behavior of the continuous review $(r, Q, \mathcal{B})$ inventory policy when arriving customers are presented with complete information.

7.4. $(r, Q, \mathcal{B})$ Inventory System with Complete Information

In this section, we derive the equilibrium behavior of the $(r, Q, \mathcal{B})$ inventory system assuming that the cumulative arrival process is a homogeneous Poisson process, procurement lead times are exponentially distributed random variables and heterogeneous customers are presented with full information. In addition, we provide exact mathematical forms for unconditional and conditional expected waiting times at arbitrary or arrival epochs and unconditional and conditional waiting time distributions at arbitrary or arrival epochs.

Proposition 7.6. *For finite values of $r, Q, \mathcal{B}$, assume that a continuous review $(r, Q, \mathcal{B})$ inventory system experiences homogeneous Poisson arrivals as the cumulative arrival process with intensity Λ and lead times are i.i.d. and exponentially distributed. If the customers are heterogeneous, presented with complete information, and evaluate their utility with respect to Equation (3.1), then the equilibrium arrival rates λ_n for $n \leq 0$ may be obtained by simultaneously solving the set of fixed-point equations:*

$$\lambda_n = \Lambda F\left(\frac{R - p}{w_n^a}\right), \quad (7.15)$$

where $F(\cdot)$ denotes the cumulative distribution function of the random variable C in Equation (3.1), and w_n^a denotes the expected waiting time incurred by an arriving customer when the customer of interest observes $IN = n$ just before their arrival. R and p are fixed scalars that denote the customer's valuation of the service and the price of the service. In addition, the mathematical forms of conditional or unconditional expected waiting times at arbitrary or arrival epochs and conditional or unconditional distribution functions of the waiting times at arbitrary or arrival epochs are available.

Proof. The existence of a solution for the set of fixed-point equations is again a result of the Brouwer's Fixed-Point Theorem. The uniqueness is more difficult to establish, which we will not commit. We set $\lambda_n = A$ for $n > 0$, and let λ_n be the set of unknown intensities for $n < 1$. Proposition 7.3 provides us with the parametric forms of $w_n^a = \mathbb{E}[W_n^a]$. Then, we have $\mathcal{B}$ fixed-point equations to solve simultaneously, where we substitute the parametric form of w_n^a in each of Equation (7.15). After determining the set of rates λ_n , we appeal to Proposition 7.2 for the characterization of the steady-state probability distribution of IN at arrival or arbitrary instants. An application of Proposition 7.3 would yield the desired conditional or unconditional waiting time distribution or expectation at an arbitrary or arrival epoch. $\square$

We note that a uniform grid of values may be formed for each λ_n in the interval $[0, A]$. The grid must then be thoroughly searched with a numerical procedure to obtain the λ_n vector of length $\mathcal{B}$ that satisfies Equation (7.15) for $n \in \{0, \dots, 1 - \mathcal{B}\}$. In addition, an interesting result presented by Proposition 7.6 is that the vector of unknown arrival rates λ_n for $n \leq 0$ is not dependent on the steady-state distribution of IN . This is because conditional expected waiting times incurred at arrival instants are no longer time averages since the system admits no memory, except the number of customers present in the system.

In the next chapter, we present numerical experiments and analysis of all proposed models and findings.

8. NUMERICAL EXPERIMENTS

In this chapter, we will present some illustrative examples and demonstrate how the system behaves for a given set of parameters. The section will be organized as follows: We begin by describing how steady-state system performance measures and the objective function are evaluated. Then, we perform one-way sensitivity analysis for a given $(r, Q, \mathcal{B})$ inventory system. We proceed with a section on providing the sub-optimal triplet $(r, Q, \mathcal{B})$, the corresponding information level, and several performance measures on a subset of triplets for a wide variety of combinations of parameters. First, we elaborate on the calculation procedures.

8.1. Calculation Procedures

In this section, we will explain how steady-state distributions, performance measures, and objective functions are evaluated for a given set of parameters.

For the computation of conditional or unconditional distributions of: steady-state distribution of IN and expected values or distribution functions of waiting times at arrival, or arbitrary epochs we refer to the Lemmas 6.2 and 5.6. Furthermore, throughout the section, we assume that the random variable incorporating customer heterogeneity with respect to their sensitivity to waiting, i.e. C , is distributed as a uniform random variable with support $[0, \beta]$.

Assuming that $r + \mathcal{B} \leq Q - 1$, for the computation of the parametric forms of the steady-state distribution of IN , the expected cycle length and the expected waiting times, we refer to the Lemmas 6.2 and 5.6. If $r + \mathcal{B} > Q$, then we appeal to the Subsection 5.2.4 discretization and numeric evaluation of the steady-state distributions. After solving the fixed-point equations according to the available information type, as presented in Subsections 6.1, 6.2, and 6.3; we determine the equilibrium arrival rates. Another application of the Lemmas 6.2 and 5.6 provides one with steady-state

distributions and waiting-time metrics. To compute the expected number of inventory on hand:

$$\mathbb{E}[(IN)^+] = \sum_{i=0}^{r+Q} i\pi_i,$$

where $(x)^+ = \max(0, x)$, as conventionally defined. Furthermore, for the expected number of backorders:

$$\mathbb{E}[(-IN)^+] = - \sum_{i=-\mathcal{B}}^0 i\pi_i.$$

At last, the effective arrival rate is evaluated by:

$$\Lambda_{eff} = \sum_{i=-\mathcal{B}}^{r+Q} \lambda_i \pi_i.$$

8.2. One Way Sensitivity Analysis

In this section, we perform a one-way sensitivity analysis for the $(r, Q, \mathcal{B})$ model for each of the parameters. In each step, we will fix the value of all parameters except one and then analyze the effect of that parameter on: Equilibrium arrival rates, expected inventory on hand, expected number of backorders, objective function, and expected waiting times. Throughout the section, we assume that full information is presented to the customers in a $(r = 0, Q = 4, \mathcal{B} = 3)$ model. We note that the parametric forms of the steady-state distributions and the conditional expected waiting times are present in the Appendix A.13. Furthermore, we assume that $\Lambda = 1.5$, $L = 1$, $R = 1$, $p = 0.75$, $h = 1$, $\beta = 1$, $v = 0$ and $K = 2$. We begin by analyzing metrics with respect to the valuation of customers, i.e. R .

8.2.1. Effect of R on Metrics

In this subsection, we fix the values of the following parameters to $\Lambda = 1.5$, $L = 1$, $p = 0.75$, $h = 1$, $\beta = 1$, $v = 0$, and $K = 2$. Let $R \in \{i/30 + p : 1 \leq i \leq 30, i \in \mathbb{N}\}$.

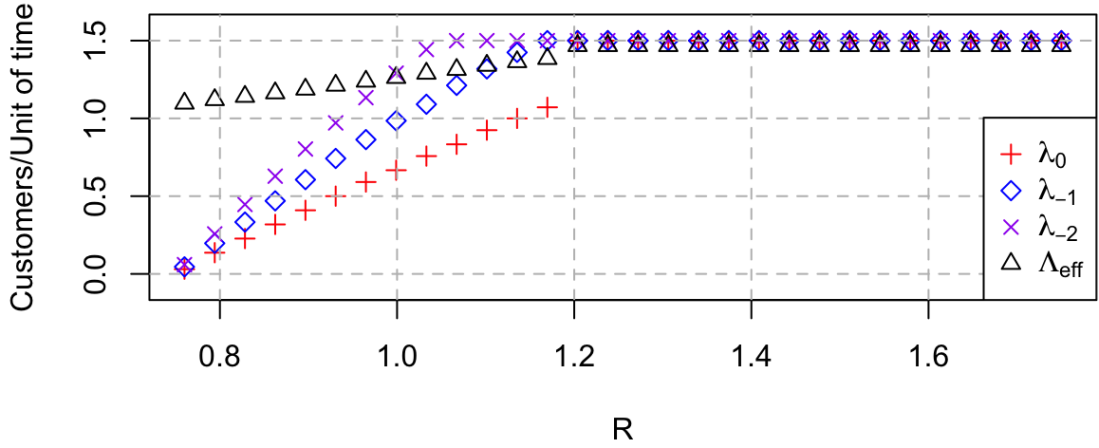


Figure 8.1. R versus Equilibrium Arrival Rates.

In Figure 8.1, it can be seen that as a customer's valuation of the service or product increases, the equilibrium arrival rates also increase. For values of $R \geq 1.2$, all arriving customers choose to wait for the product, i.e. the state-dependent balking probabilities of the arriving customers are equal to 0 when they observe a stock-out state. Moreover, it can be observed that $\lambda_{-2} \geq \lambda_{-1} \geq \lambda_0$. This phenomenon is caused directly by the ordinal relationship between the conditional expected waiting times at arrival epochs: $w_{-2}^a \leq w_{-1}^a \leq w_0^a$. Since at most one order may be outstanding, the arrival of the currently outstanding order will vanish all current backorders. Thus, an arriving customer who observes $IN = -2$ will wait less than customers who are presently waiting for their orders, who observed that IN is -1 or 0 . Intuitively, in equilibrium, upon arrival customers would balk more if their expected waiting times increase.

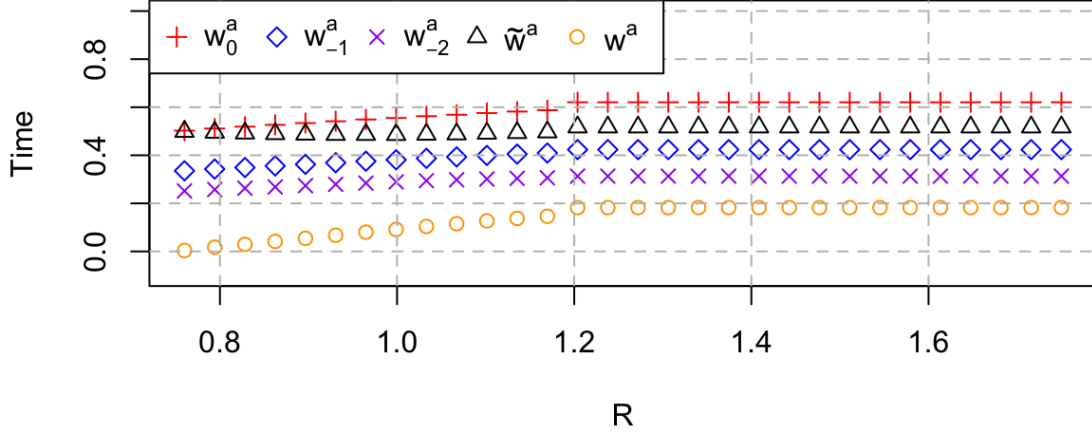
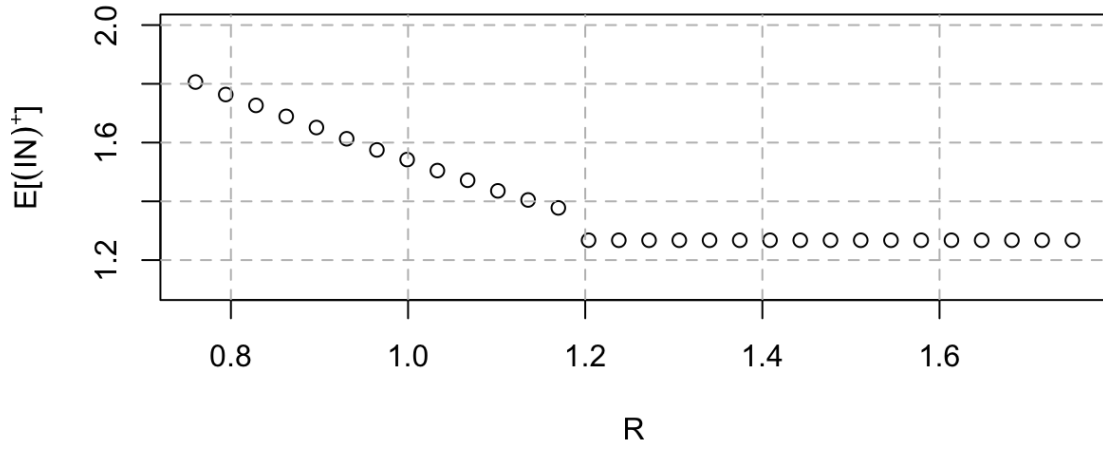
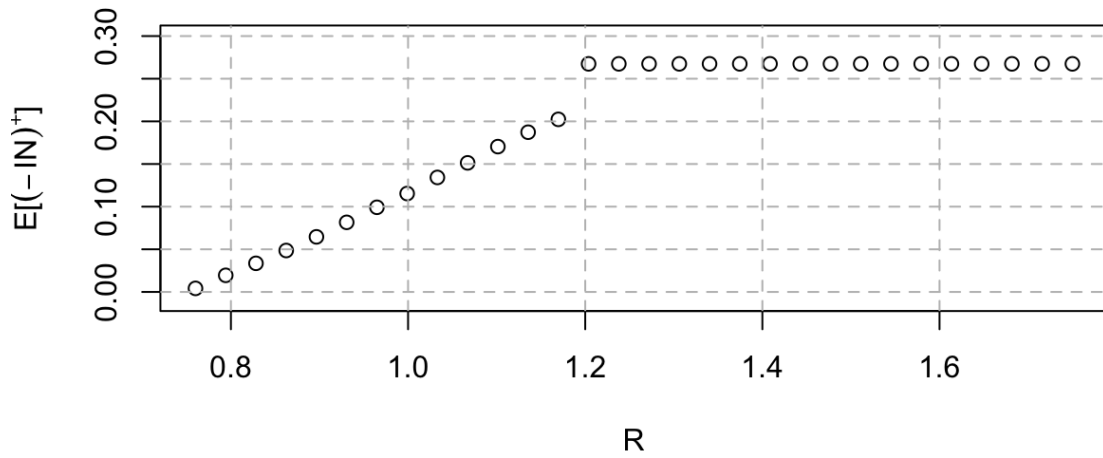


Figure 8.2. R versus Expected Waiting Times.

In Figure 8.2, $\tilde{w}^a$ denotes the expected waiting time of an arriving customer given that the customer arrived at a stock-out instant. We observe that as R increases, the expected waiting times also tend to increase. This behavior is due to the fact that as R increases, the equilibrium arrival rates also increase. Consequently, the probability of an arriving customer observing a stock-out state increases and the expected inter-arrival times between two consequent customers at a stock-out instant decrease. These result in an increase in expected waiting times; however, for $R \geq 1.2$, expected waiting times do not change, since the equilibrium arrival rates are bounded from above by the potential arrival rate, Λ .

Figure 8.3. R versus $\mathbb{E}[(IN)^+]$.Figure 8.4. R versus $\mathbb{E}[(-IN)^+]$.

In Figures 8.3 and 8.4, $\mathbb{E}[(IN)^+]$ and $\mathbb{E}[(-IN)^+]$ represent the expected inventory on hand per unit of time and the expected number of backorders per unit of time,

respectively. Since as R increases, equilibrium arrival rates increase, the stochastic process IN tends to visit states with non-positive values more frequently, which increases the stationary probability mass of these states. Hence, the expected inventory on hand per unit of time decreases, and the expected number of backorders per unit of time increases. For $R \geq 1.2$, since equilibrium rates are at their supremum, the stationary probability mass function of IN does not change.

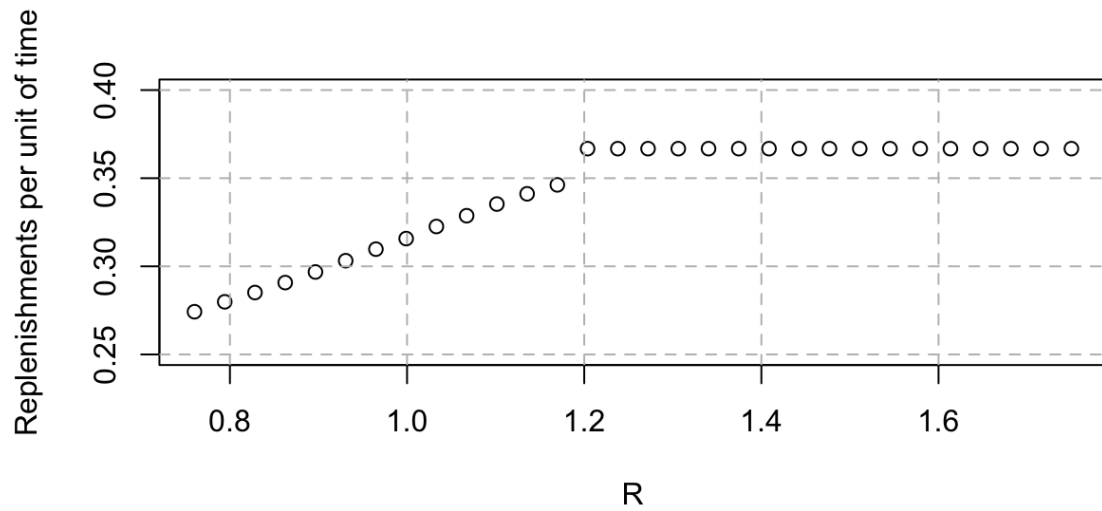


Figure 8.5. R versus $\mathcal{N}$.

In Figure 8.5, as R increases, the equilibrium rates increase; therefore, the probability that an outstanding order is present at any given time increases. The traffic intensity of the system also increases, which increases the density of the system. expected number of replenishments per unit of time.

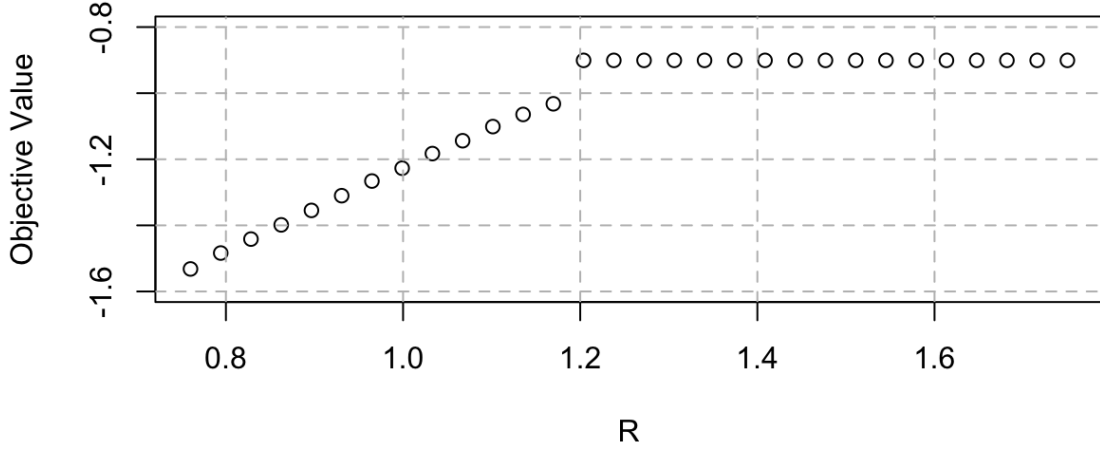


Figure 8.6. R versus Objective Value.

In Figure 8.6, we need to analyze the changes in the three components of the objective function to interpret the behavior. As R increases, Λ_{eff} increases. Although $\mathbb{E}[(IN)^+]$ decreases, $\mathcal{N}$ increases. We note that a decrease in $\mathbb{E}[(IN)^+]$ causes an increase in the objective, whereas an increase in $\mathcal{N}$ causes a decrease in the objective. However, the effects of changes in Λ_{eff} and $\mathbb{E}[(IN)^+]$ on the objective surpass the effect of changes in $\mathcal{N}$ on the objective for the given set of parameters. Therefore, the objective function increases as R increases and does not change for $R \geq 1.2$.

We note that none of the above metrics changes when $R \geq 1.2$. When $R \geq 1.2$, for the given set of parameters, the customer's valuation dominates the effect of waiting.

8.2.2. Effect of Λ on Metrics

In this subsection, we fix the values of the following parameters to $R = 1$, $L = 1$, $p = 0.75$, $h = 1$, $\beta = 1$, $v = 0$, and $K = 2$. Let $\Lambda \in \{2.3i/19 + 0.7 : 0 \leq i \leq 19, i \in \mathbb{N}_0\}$.

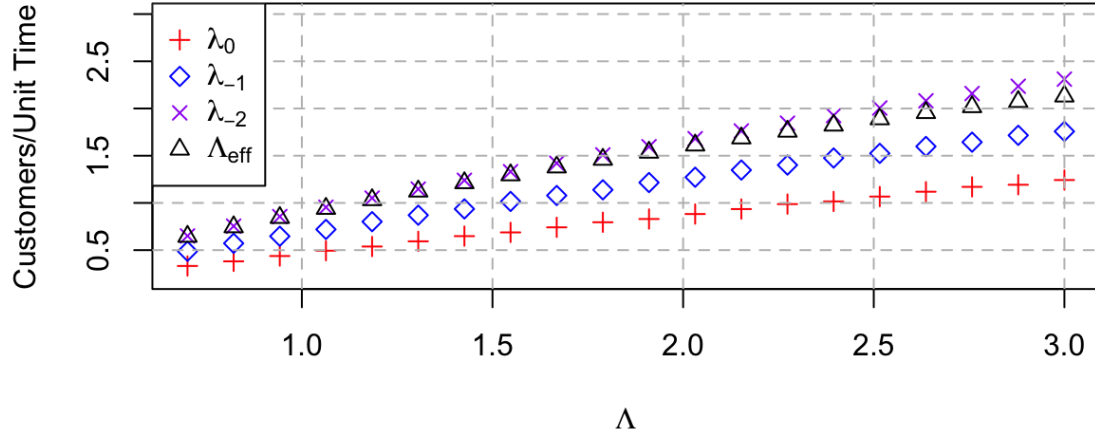


Figure 8.7. Λ versus Equilibrium Arrival Rates.

In Figure 8.7, we observe that as the potential arrival rate increases, the equilibrium arrival rates also increase. Furthermore, as in the previous subsection in Figure 8.1, $\lambda_0 \leq \lambda_{-1} \leq \lambda_{-2}$, which is due to the conditional expected waiting times and the fixed-point equations.

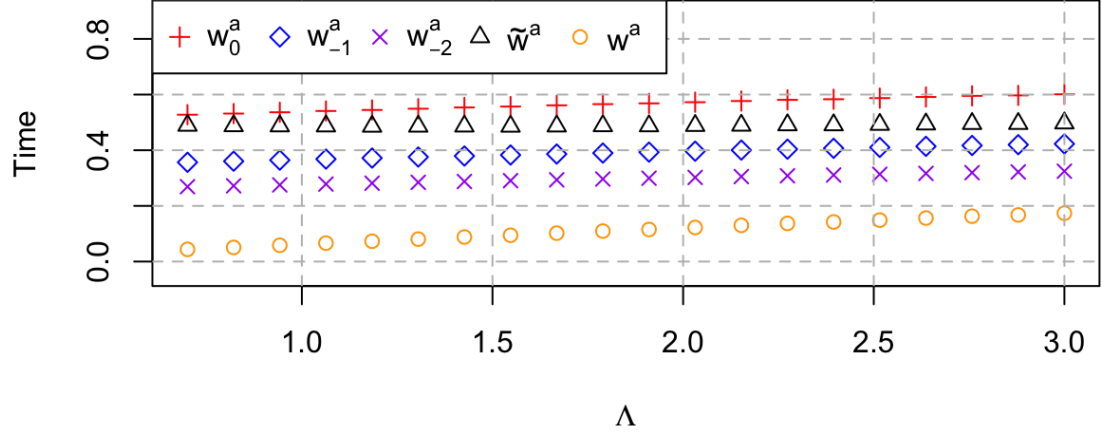


Figure 8.8. Λ versus Expected Waiting Times.

In Figure 8.8, as Λ increases, the expected waiting times also slightly increase. This phenomenon is caused by an increase in the probability that an arriving customer observes $IN \leq 0$, i.e. the system is at a stock-out instant.

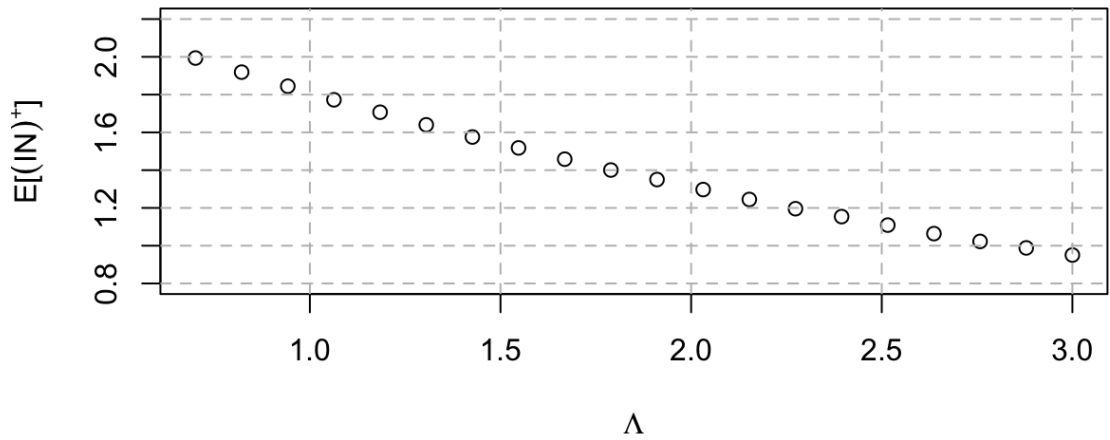


Figure 8.9. Λ versus $E[(IN)^+]$.

In Figure 8.9, we observe that the expected inventory on hand decreases as Λ increases. To see why, we observe that L does not change, yet the expected inter-arrival times between successive customers decrease. Considering the stationary stochastic process that denotes the state of IN , the decrease in the times between arrivals increases the time spent at lower values of IN .

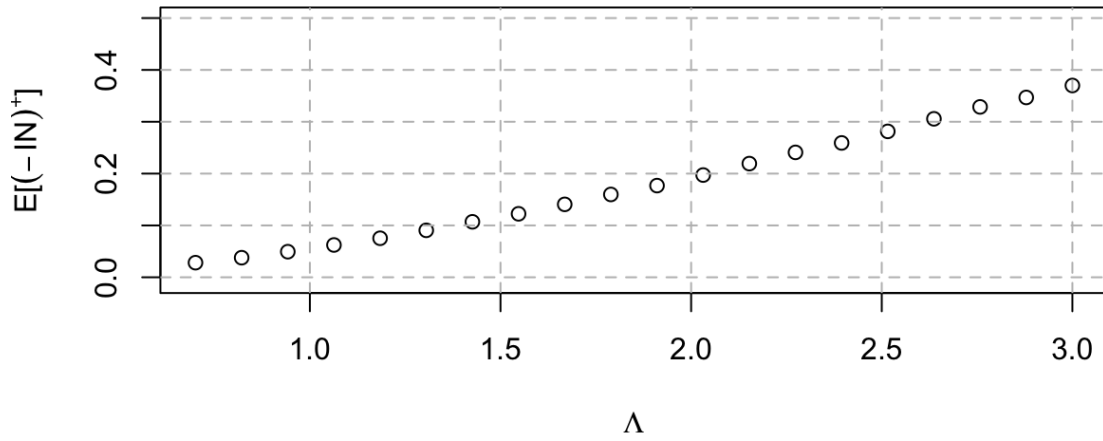


Figure 8.10. Λ versus $\mathbb{E}[(-IN)^+]$.

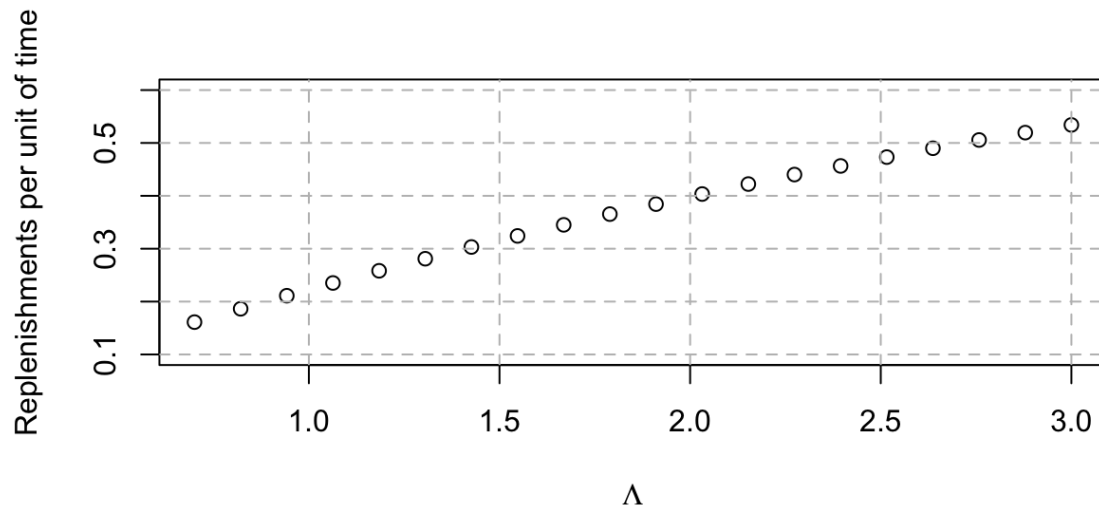


Figure 8.11. Λ versus $\mathcal{N}$.

In Figures 8.10 and 8.11 we observe that the expected time between two orders decreases and the expected number of backorders increases. The reasoning behind both is that the inter-arrival times decrease as Λ increases and the fraction of customers that enter the system at a stock-out instant increases.

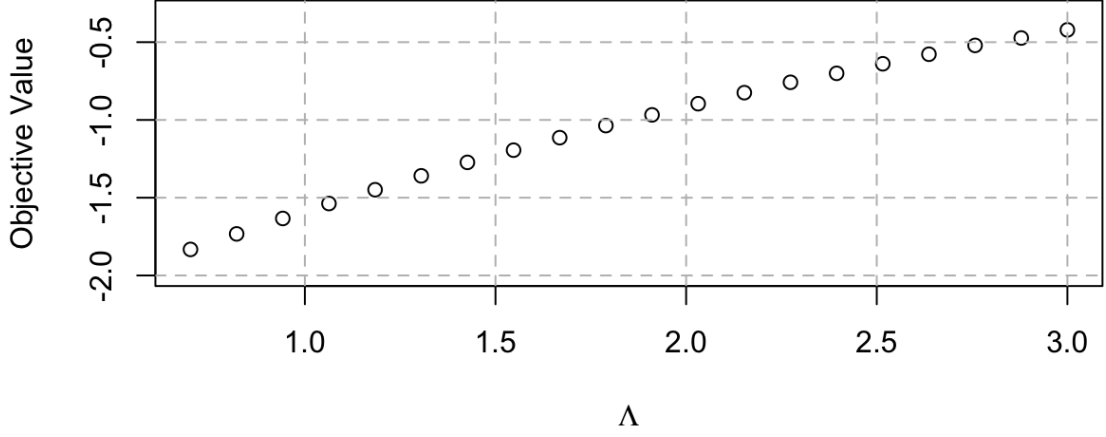


Figure 8.12. Λ versus Objective Value.

Finally, in Figure 8.12, we observe that the objective value increases as Λ increases. Λ_{eff} increases, $\mathbb{E}[(IN)^+]$ decreases, and $\mathcal{N}$ increases. However, the effects of Λ_{eff} and $\mathbb{E}[(IN)^+]$ again dominate the effect of $\mathcal{N}$ on the objective.

8.2.3. Effect of p on Metrics

In this subsection, we fix the values of the following parameters to $R = 1$, $L = 1$, $\Lambda = 1.5$, $h = 1$, $\beta = 1$, $v = 0$, and $K = 2$. Let $p \in \{0.05 + 0.9i/29 : 0 \leq i \leq 29, i \in \mathbb{N}_0\}$.

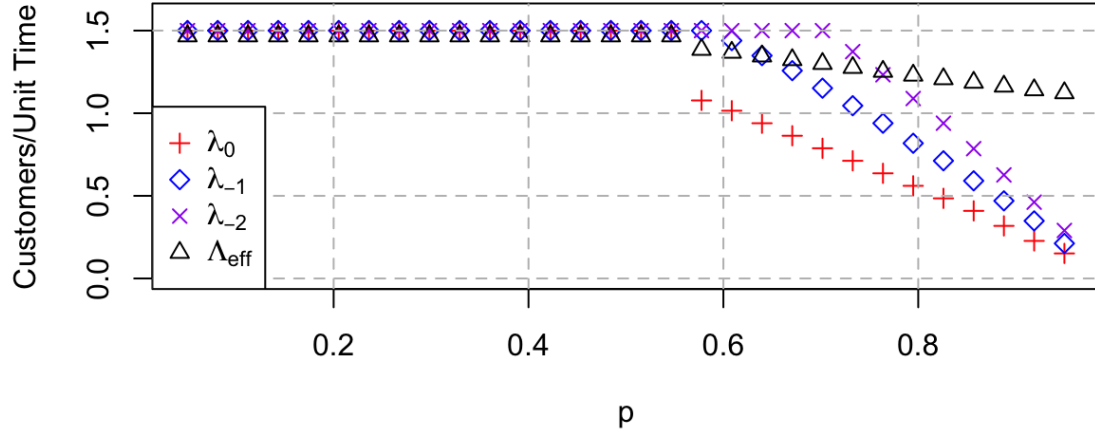


Figure 8.13. p versus Equilibrium Arrival Rates.

In Figure 8.13, we observe that for $p \leq 0.55$, an arriving customer does not balk, even if she does not find any items available. Thus, for $p \leq 0.55$, $\lambda_n = \Lambda$ for $n \geq -2$. However, as p increases, the marginal gain of the product for an arriving customer decreases, i.e. $R - p$.

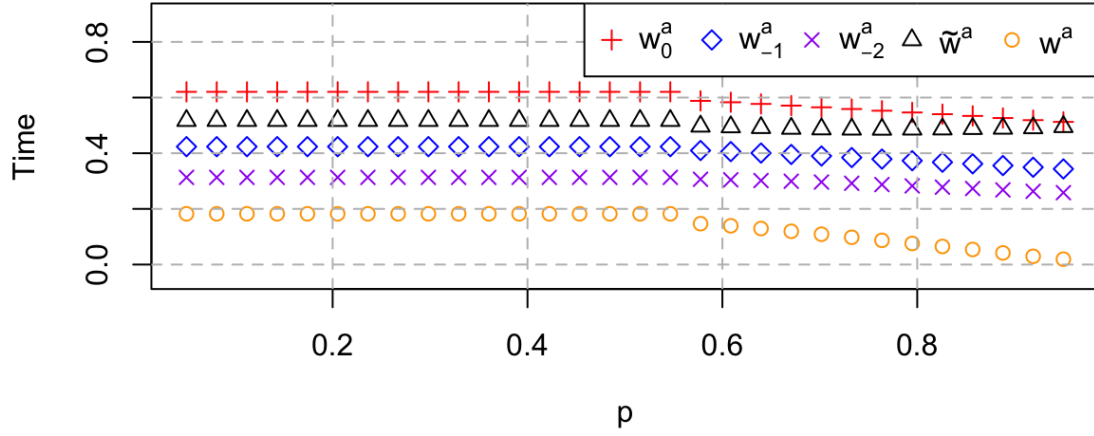


Figure 8.14. p versus Expected Waiting Times.

As p increases, the equilibrium arrival rates decrease, since customers choose not to wait for the product. In Figure 8.14, as p increases, the expected waiting times also decrease due to the increase in the expected inter-arrival times. The arrival of the customer when IN hits 0 triggers the replenishment order, and customers who will wait arrive later than they used to, which decreases their expected waiting times.

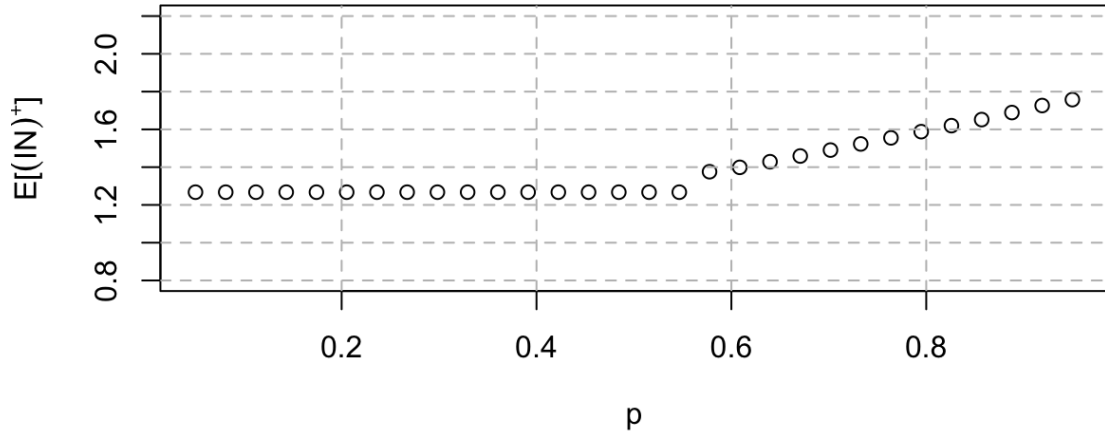


Figure 8.15. p versus $\mathbb{E}[(IN)^+]$.

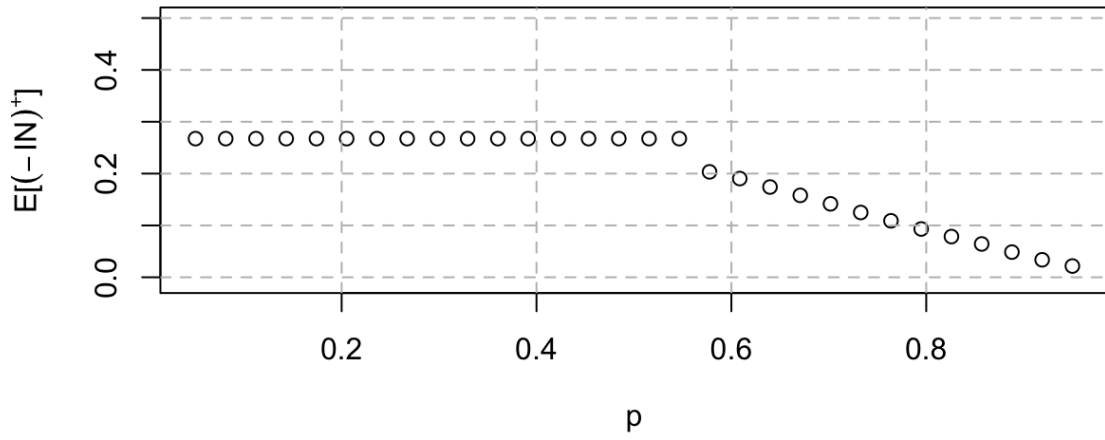


Figure 8.16. p versus $\mathbb{E}[(-IN)^+]$.

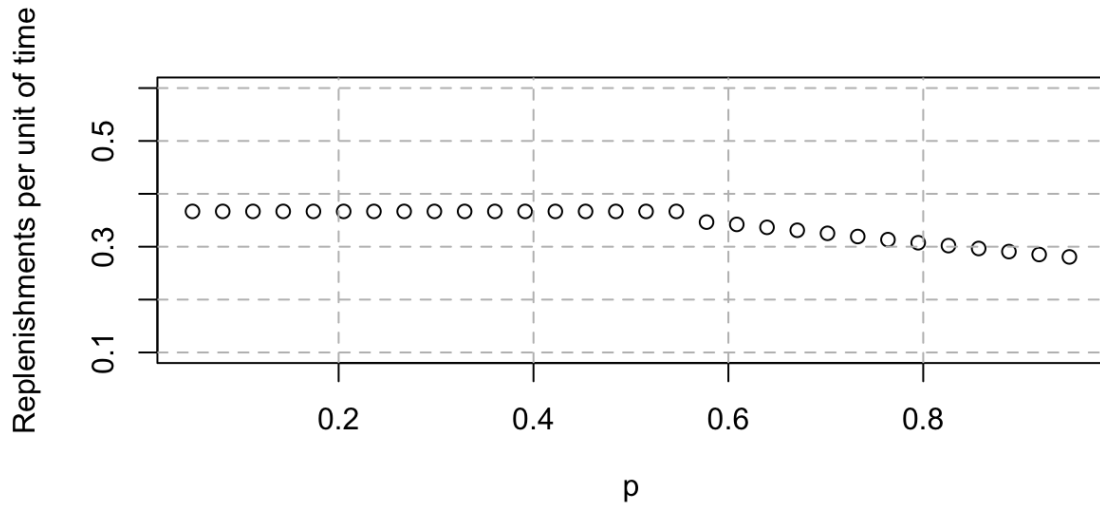


Figure 8.17. p versus $\mathcal{N}$.

We observed that as p increases, customers who observe stock-out states balk more. In Figure 8.15, this phenomenon causes the expected inventory on hand to increase. Furthermore, the same reasoning applies to the decrease in the expected number of backorders and the expected number of replenishments per unit of time in Figures 8.16 and 8.17, respectively.

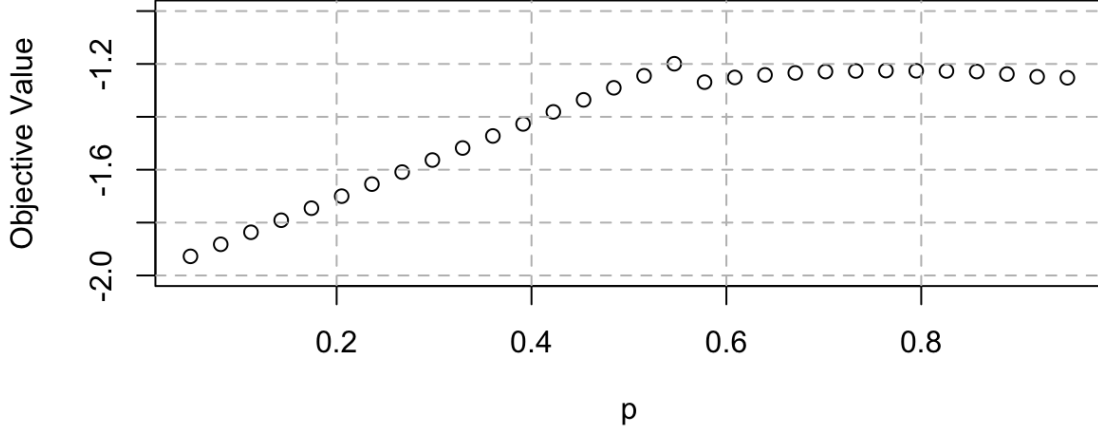


Figure 8.18. p versus Objective Value.

Finally, in Figure 8.18, we observe that for $p \leq 0.55$, the objective value increases even though the stationary distribution or equilibrium rates do not change. This is due to the existence of the term $(p - v)\Lambda_{eff}$ in the objective function. For $p \geq 0.55$, the objective function behaves non-linearly because p increases but Λ_{eff} decreases and the term $(p - v)\Lambda_{eff}$ no longer behaves linearly. In addition, the expected inventory on hand and the expected number of backorders change, as mentioned above.

8.2.4. Effect of L on Metrics

In this subsection, we fix the values of the following parameters to $R = 1$, $p = 0.75$, $\Lambda = 1.5$, $h = 1$, $\beta = 1$, $v = 0$, and $K = 2$. Let $L \in \{0.5 + 1.5i/24 : 0 \leq i \leq 24, i \in \mathbb{N}_0\}$.

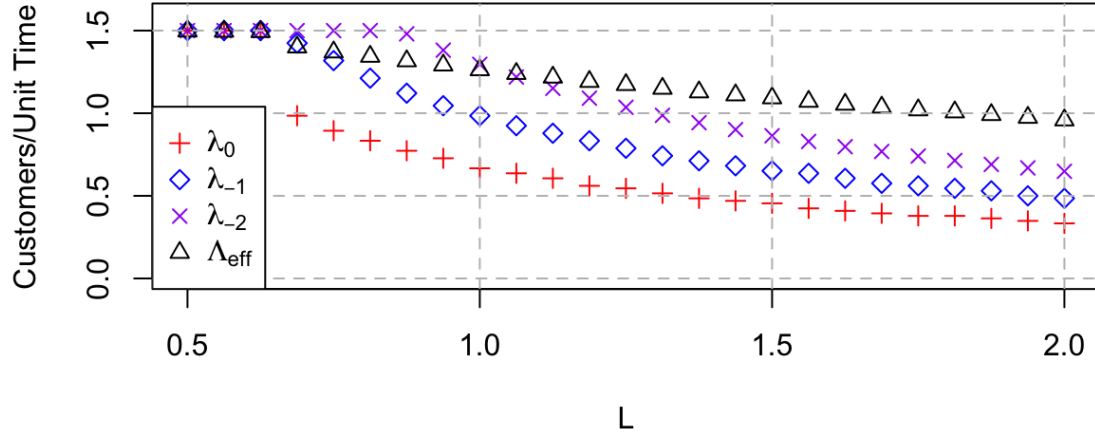


Figure 8.19. L versus Equilibrium Arrival Rates.

In Figure 8.19, we observe that as L increases, the expected waiting times increase, resulting in an increase in the state-dependent balking probabilities of customers who arrive at stock-out instants. Hence, the equilibrium arrival rates at stock-out instants decrease.

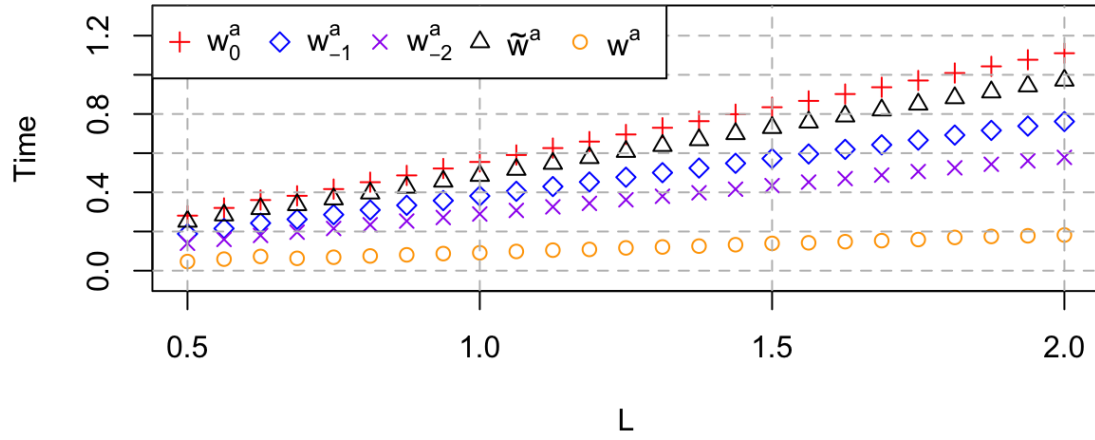


Figure 8.20. L versus Expected Waiting Times.

As L increases, the conditional and stationary expected waiting times increase, as expected.

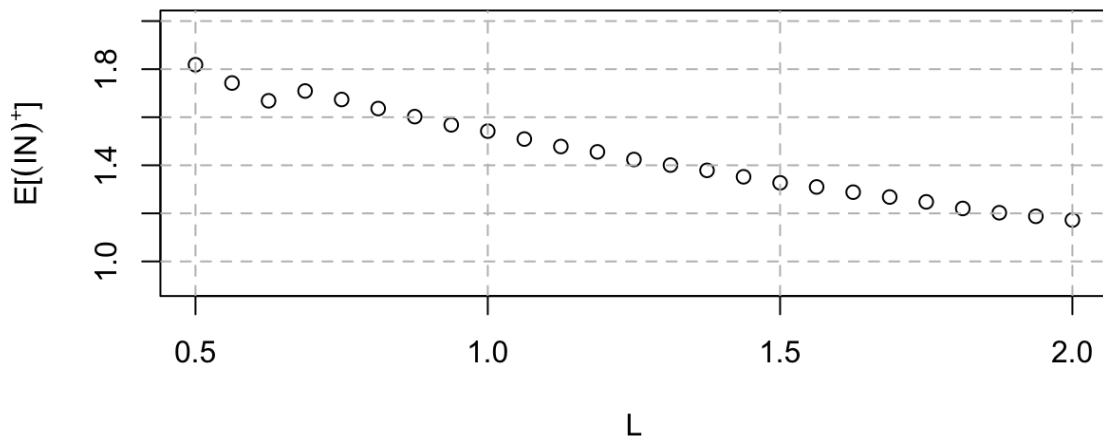


Figure 8.21. L versus $\mathbb{E}[(IN)^+]$.

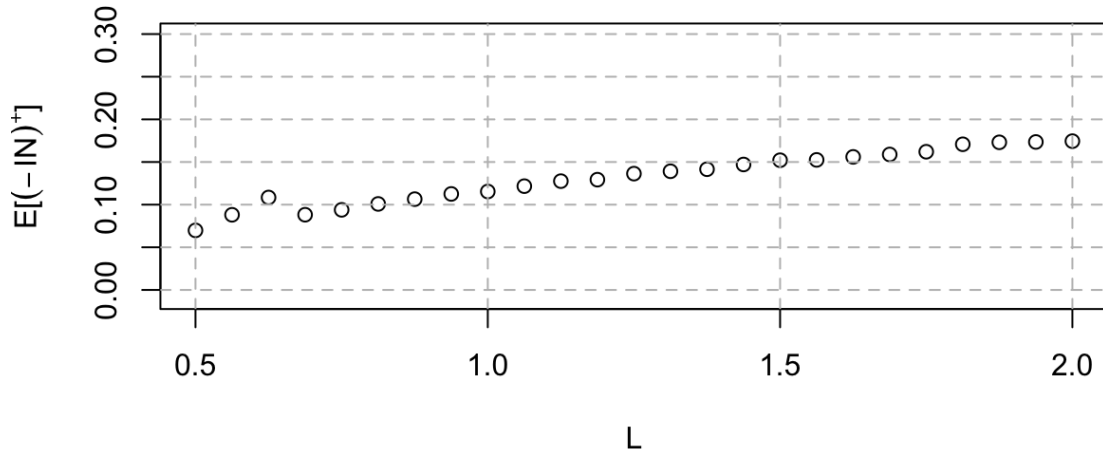


Figure 8.22. L versus $\mathbb{E}[(-IN)^+]$.

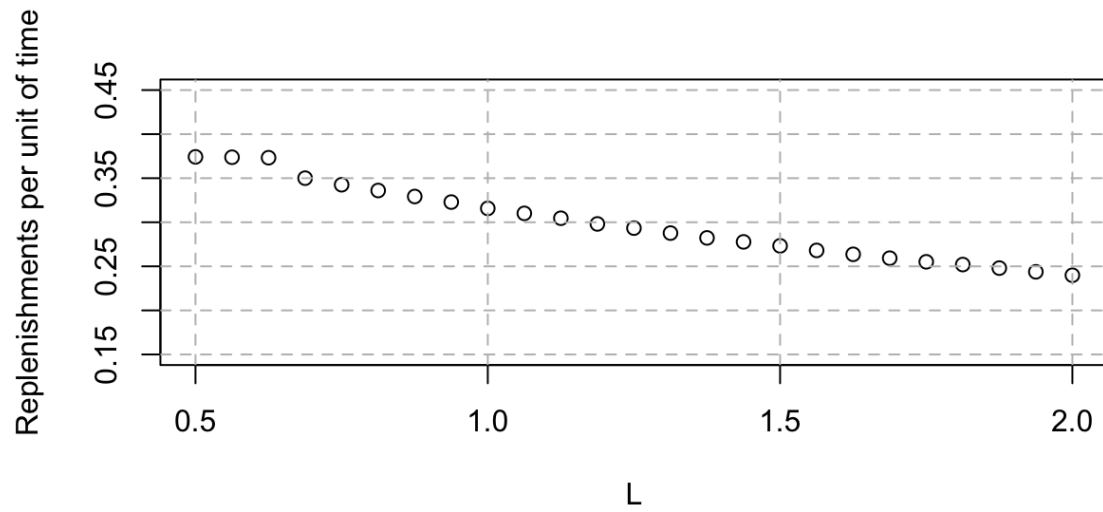


Figure 8.23. L versus $\mathcal{N}$.

We observed that as L increases, equilibrium arrival rates at stock-out instants decrease, and expected waiting times increase. The increase in L causes the stochastic

process IN to spend more time in states where a replenishment order is currently on its way, i.e. $IO > 0$. Hence, the expected inventory on hand decreases, the expected number of backorders increases, and the expected number of replenishments per unit of time decreases. There may be slight fluctuations in general trend, which is caused by the fixed-point equations, the conditional expected waiting times, and the decrease in the equilibrium arrival rates. The respective changes in $\mathbb{E}[(IN)^+]$, $\mathbb{E}[(-IN)^+]$, and $\mathcal{N}$ are available in Figures 8.21, 8.22, and 8.23.

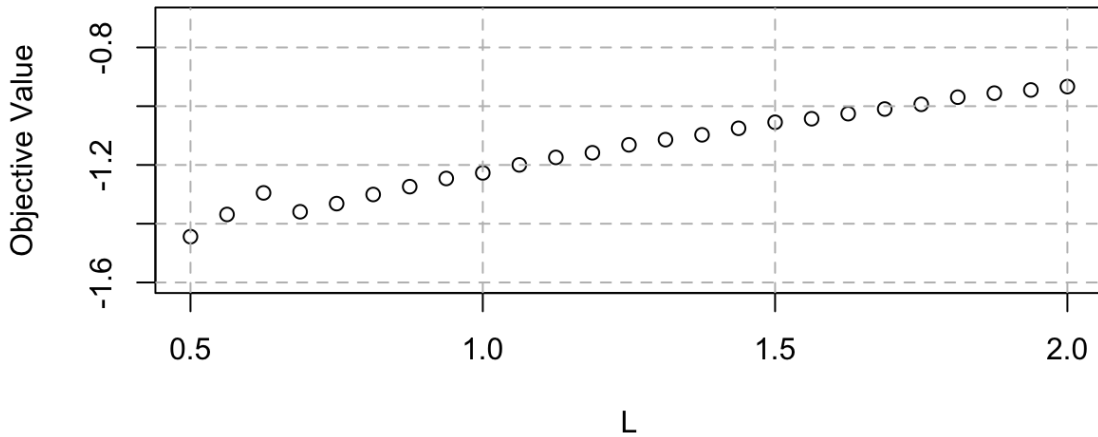


Figure 8.24. L versus Objective Value.

Finally, in Figure 8.24, we observe that the objective function is mainly increasing with increasing values of L . The reason is mainly the sharp decrease in the expected on-hand inventory compared to the slight increase in the expected number of backorders and decrease in the equilibrium effective arrival rate.

8.2.5. Effect of β on Metrics

In this subsection, we fix the values of the following parameters to $R = 1$, $p = 0.75$, $A = 1.5$, $h = 1$, $L = 1$, $v = 0$, and $K = 2$. Let $\beta \in \{0.5 + 2i/29 : 0 \leq i \leq 29, i \in$

$N_0\}$.

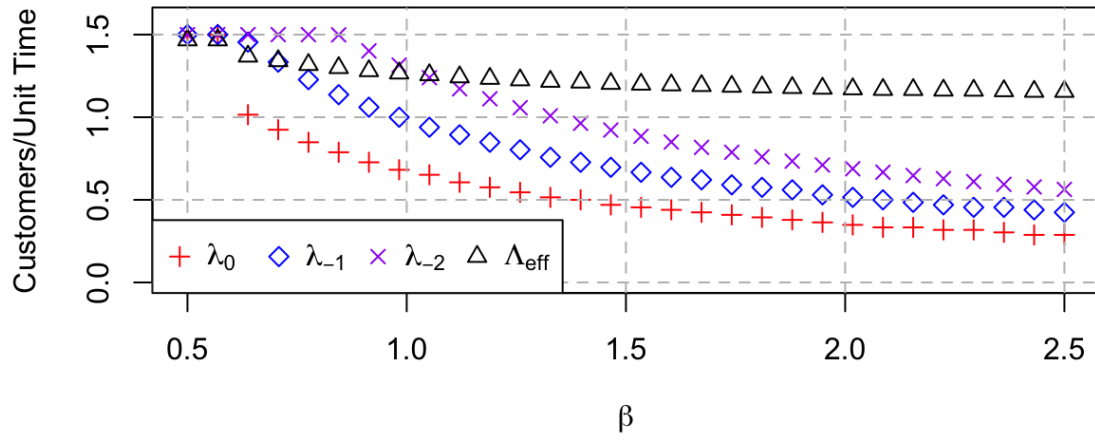


Figure 8.25. β versus Equilibrium Arrival Rates.

We know that as β increases, customers are more sensitive to waiting. Hence, in Figure 8.25, it can be observed that as β increases, the equilibrium arrival rates at the stock-out instants decrease.

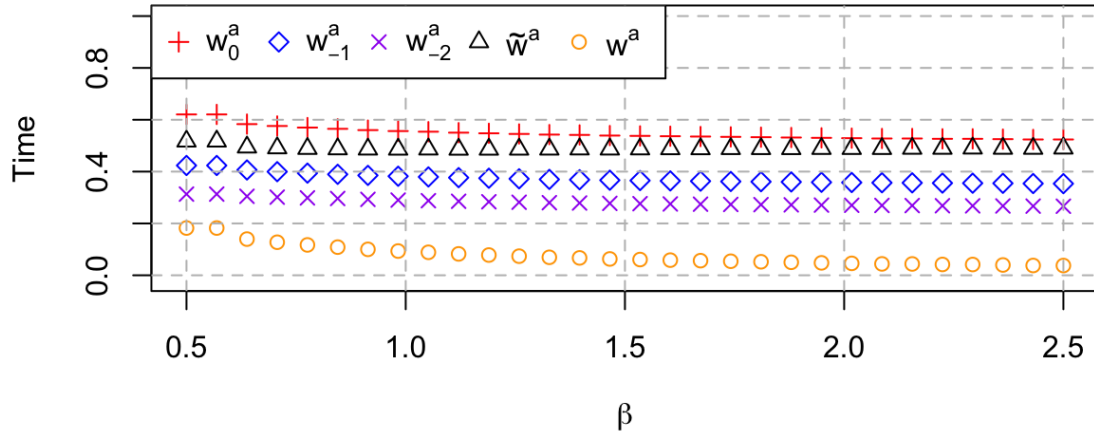


Figure 8.26. β versus Expected Waiting Times.

As β increases, the equilibrium arrival rates decrease and the expected waiting times also decrease. An increase in β can be interpreted as a decrease in the patience of customers.

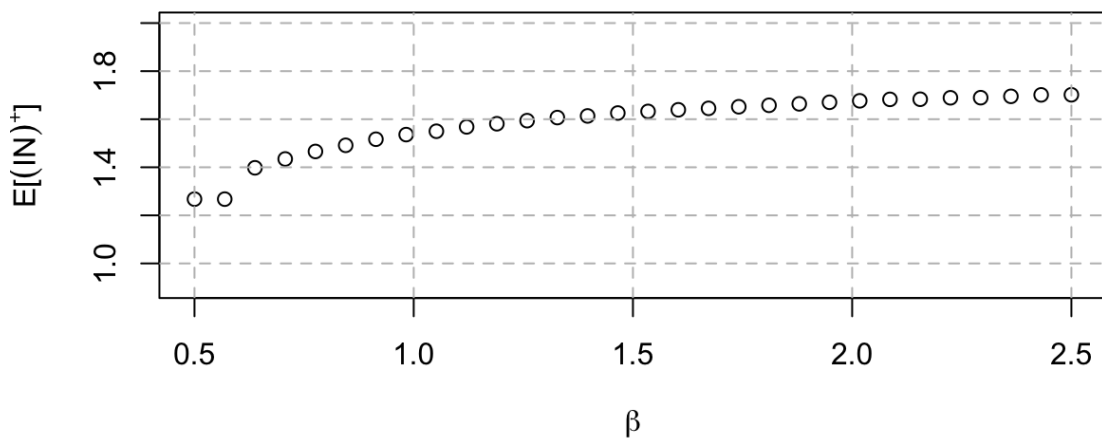


Figure 8.27. β versus $\mathbb{E}[(IN)^+]$.

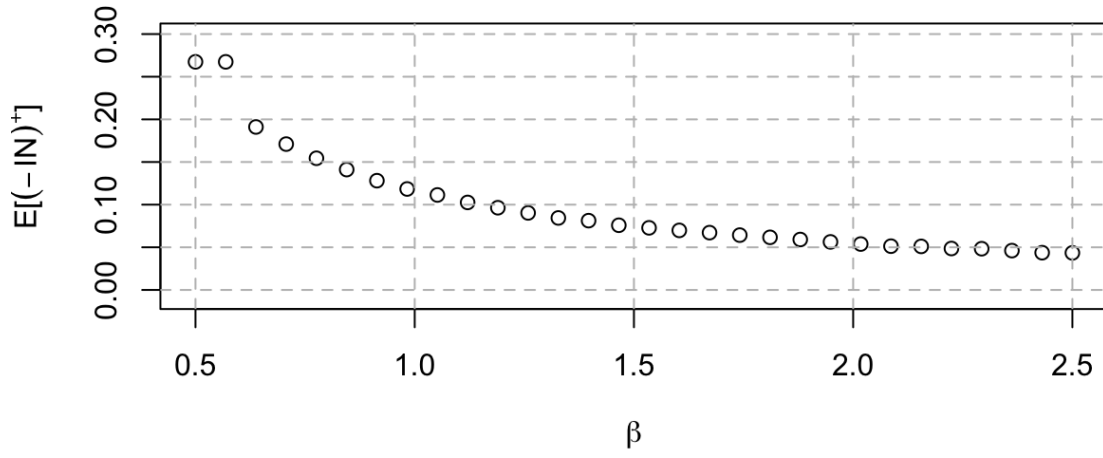


Figure 8.28. β versus $\mathbb{E}[(-IN)^+]$.

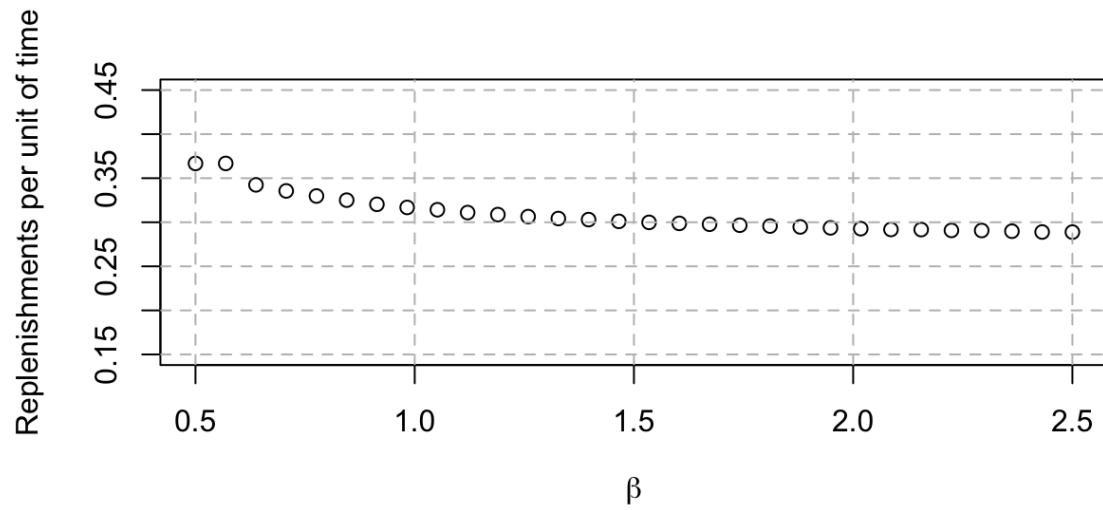


Figure 8.29. β versus $\mathcal{N}$.

We observed that as β increases, the equilibrium arrival rates and the conditional waiting times at stock-out instants increase. Hence, the stochastic process that gov-

erns the state of IN spends more time in positive states. The expected inventory on hand increases, the expected number of backorders decreases, and consequently the expected replenishments per unit of time decreases. The respective changes in the aforementioned metrics are available in Figures 8.27, 8.28, and 8.29.

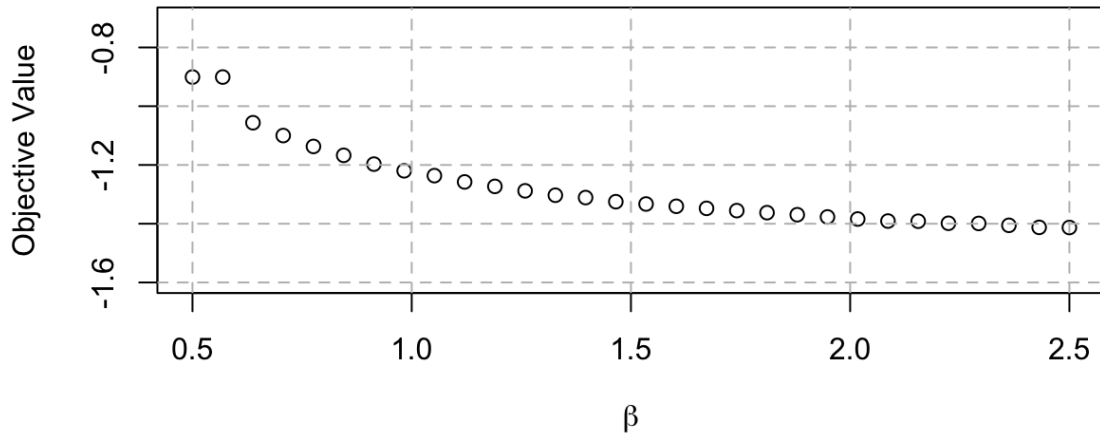


Figure 8.30. β versus Objective Value.

Finally, in Figure 8.30, we observe that the objective function is mainly decreasing with increasing values of β . The reason is mainly the sharp increase in the expected on-hand inventory and the slight decrease in the effective arrival rate per unit of time.

In the subsections above, we have performed one-way sensitivity analysis with respect to the parameters: A, L, β, p, R . We will not conduct a one-way sensitivity analysis for the remaining parameters, i.e. K, v, h , since they are only present in the objective function and a change in any of them would cause a linear respective change on the objective function. In the next section, we provide numerical approximations for some cases where more than one order can be outstanding.

8.3. Numerical Approximation for $(r = 0, Q = 2, \mathcal{B} = 2)$ System

In this section, we present numerical findings on the application of the findings in Subsection 5.2.4. We refer to Corollary 5.8 for the queueing model and the description of the numerical method. For the sake of an example, let $Q = 2, \Lambda = \lambda_0 = \lambda_1 = 2, L = 3, \lambda_2 = 1.7, \lambda_3 = 1.5$, and $\mathcal{C} = 4$. Hence, the system of interest is a bulk queueing model where there are two identical servers serving customers in batches of two. Let $n = 100$. Upon forming the linearization matrix, $\mathbf{M}$, and inverting it, we have the steady-state probability density approximations.

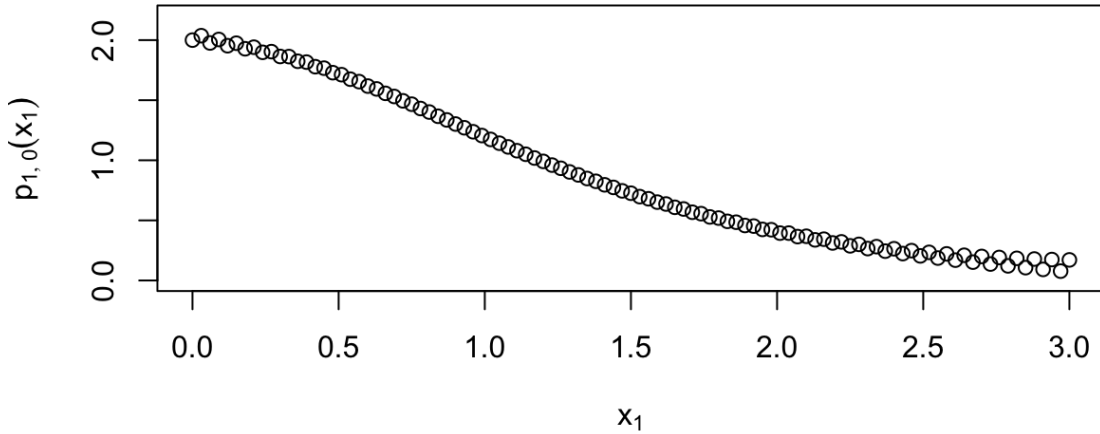
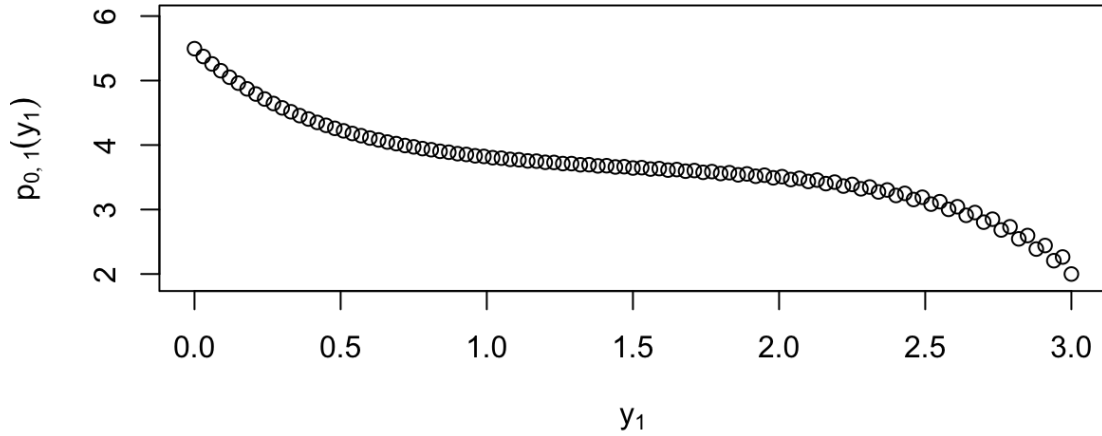


Figure 8.31. $p_{1,0}(x_1)$ for $x_1 \leq 3$.

The numerical approximation of the steady-state probability density function $p_{1,0}(x_1)$ for values of $x_1 \leq 3$ can be viewed in Figure 8.31. We note that $p_{1,0}(x_1)$ is defined for the values $x_1 \geq L$, and the function takes the form: $p_{1,0}(x_1) = p_{1,0}(L)e^{-\Lambda(x_1-L)}$.

Figure 8.32. $p_{1,0}(y_1)$.

In Figure 8.32, the numerical approximation of the steady-state probability density function $p_{0,1}(y_1)$ can be viewed. The numerical approximations for the steady-state probability densities $p_{1,1}(x_1, y_1)$ and $p_{0,2}(y_1, y_2)$ can be viewed in a three-dimensional space, since there are two independent variables. However, as we have noted in Corollary 5.8, both densities can be expressed as transformations of $p_{0,1}(y_1)$. Next, we present steady-state distribution estimates for the number of customers in the queueing system.

Table 8.1. Numerical vs. Simulation Estimates 1.

Estimation Method	π_0	π_1	π_2	π_3	π_4
Numerical	0.0218	0.05879	0.2407	0.2223	0.4564
Simulation	0.0229	0.06108	0.2397	0.2238	0.4525

From Table 8.1, it can be seen that the numerical method provides correct estimates. For the simulation algorithm, a handwritten discrete event simulation algorithm was used. The simulation estimates consist of 40,000 minutes of warm-up and a total

of 400,000 minutes of runtime.

Finally, let us provide estimates of the numerical method for another set of parameters. Let $\Lambda = \lambda_0 = \lambda_1 = 1.6$, $\lambda_2 = 2$, $\lambda_3 = 2.3$, $L = 1.7$, $n = 100$.

Table 8.2. Numerical vs. Simulation Estimates 2.

Estimation Method	π_0	π_1	π_2	π_3	π_4
Numerical	0.05957	0.1338	0.2821	0.1904	0.3341
Simulation	0.0630	0.1384	0.2801	0.1918	0.3267

Although the numerical method mostly provides consistent estimates, there are a few shortcomings of the proposed method. First, the numerical method requires one to invert a large, mostly sparse matrix. The complexity of the numerical method increases exponentially as the capacity or Q increases. The reason is that we are required to do functional approximations for every density, and the grid of points grows exponentially. Furthermore, we have to form the linear system of equations for the generation of the linearization matrix $\mathbf{M}$, which is not straightforward. However, the method does not exist in the literature and we believe is a new approach to provide approximations for complex queueing systems. In the next section, we present experimental results for a grid of parameter combinations and over a subset of policies.

8.4. Experiments for a Set of Parameters

In this section, we will define a set of values for the parameters presented in Chapter 3 and determine the suboptimal quadruplets $(r^*, Q^*, \mathcal{B}^*, Info^*)$ for the retailer. We apply the exact solution methods presented in Chapter 6 to all quadruplets where $2 \leq Q \leq 4$ and at most one outstanding order is allowed. The parametric forms of steady-state behaviors for all 30 quadruplets are presented in Appendix A. For each combination of parameters, we will evaluate the equilibrium behavior, inventory metrics, and objective function. There are 30 quadruplets, and we will present the optimal

quadruplet and the corresponding equilibrium behavior for a total of 64 combinations of parameters. We have specified two levels for 6 of the parameters as follows:

- $p \in \{0.25, 0.5\}$,
- $R \in \{0.75, 1\}$,
- $\beta \in \{1, 1.5\}$,
- $L \in \{1, 1.5\}$,
- $h \in \{0.1, 0.2\}$,
- $K \in \{1, 2\}$.

We fix $\lambda = 1, v = 0$. All possible combinations of the parameters are available in Table B.1. The results of the experiments are available in Table B.2. Including the arrival rate and the retailer's purchase price, we have to analyze an 8 dimensional parameter space mapped into a 4 dimensional space. Therefore, we fix the arrival rate(λ) at 1 and the retailer's purchase price per unit(v) at 0. The former is due to restricting the fixed-ordering size to be smaller than or equal to 4, and the latter is due to reducing the curse of dimensionality. The dimensionality and inter-connectedness of the parameters yields the analysis difficult. It is not straightforward to prescribe optimal policies. Therefore, we will analyze the results of the experiments and comment on the changes in optimal quadruplets.

We note that in some cases it may be equally beneficial to provide full or partial or no information. However, if providing full information leads all of the state-dependent balking probabilities to be 0, then the equilibrium behavior is exactly the same as providing no information. Depending on the set of parameters, providing partial information may also lead to the same equilibrium behavior as in the full information and no information; however, partial information can also be dominated by the other information policies. For the cases where at most one order may be outstanding, in Section 8.2 we have presented that conditional expected waiting times at arrival epochs decrease as IN decreases. If the fixed-point equation of λ_0 , corresponding to the full information case, produces a state-dependent balking probability 0 in equilibrium, then

other fixed-point equations corresponding to $\lambda_{-1}, \dots, \lambda_{1-B}$ must also provide the same result. Intuitively, the reasoning is that in the full information case, an arriving customer that observes $IN = 0$ makes the worst possible conditional waiting time estimate compared to other customers who arrive in the system at a stock-out instant. The partial information case may be dominated by the other two cases, and it is because, in the partial information case, the expected waiting time estimate of a customer that arrives the system at a stock-out instant is inflated by the steady-state probability of arriving the system at a stock-out instant.

We observe that in setup 2 it becomes more beneficial for the retailer to provide no information, compared to the adjacent setups. The reason is that as β increases, customers do not tolerate waiting and providing information deteriorates arrival rates at stock-out instants. In fact, in setup 2, providing complete information causes the system to have an effective arrival rate of 0.92.

In setup 9, Q^* reduces to 3 for the first time. We note that although the expected number of replenishments per unit of time increases and the effective arrival rate decreases, the expected inventory on-hand decreases. Due to the values of K, p, h the retailer prefers to lose customers and order more rather than hold more stock. Additionally, partial information appears to be the most profitable information type. The reason is that providing full information causes $\lambda_0 \approx 0.87$ yet $\lambda_{-1} = 1$. Hence, the effective arrival rate for providing complete information is around 0.94. Additionally, providing no information to customers causes $\lambda_{NI} \approx 0.99$, which leads to an effective arrival rate of around 0.952. Therefore, the maximum effective arrival rate is obtained by supplying partial information to customers.

In setup 15, the suboptimal fixed-order-quantity turns out to be 3. Decreasing Q leads to an increase in the expected number of replenishments per unit of time and a decrease in the effective arrival rate. However, the effect of the decrease in the expected inventory on-hand dominates the negative effect of the former changes on the objective function. Again, customers are supplied with partial information. For the

triplet $(r = 0, Q = 3, \mathcal{B} = 2)$, the partial information case produces a higher effective arrival rate compared to the full or no information cases.

In setup 22, the suboptimal backorder limit is equal to 1. In fact, the second best policy turns out to be the $(r = 0, Q = 4, \mathcal{B} = 2)$ model without information where the effective arrival rate is approximately 0.769. The third best policy is the $(r = 0, Q = 4, \mathcal{B} = 3)$ model without information, and the effective arrival rate is around 0.768. The expected inventory on-hand and replenishments per unit time metrics are ± 0.001 with respect to the suboptimal policy, i.e. no information $(r^* = 0, Q^* = 4, \mathcal{B}^* = 1)$. The setting may be summarized as high waiting time, low perceived value of service, high price item, and low waiting tolerance. The profit of the system increases by providing no information and decreasing the maximum number of allowed backorders.

In the next chapter, we present our conclusions and future work regarding the study.

9. CONCLUSION

Our motivation was to provide equilibrium behavior of limited-backorder continuous review (r, Q) models experiencing heterogeneous customers that are sensitive to waiting. We investigated the effect of providing customers with three separate levels of information: No information, stock-out information, and full information. We studied exponential and deterministic procurement lead-time settings. The heterogeneity of customers and information concepts require us to study (r, Q) models that experience state-dependent arrivals. We demonstrate the literature in Chapter 2, our notation, and the mathematical model in Chapter 3.

In Chapter 4, we derive the conditional waiting time distribution function for a continuous review, unlimited backorder (r, Q) model. The derivation methodology and conditional distribution functions appear to be new contributions to the literature.

In the beginning of Chapter 5, we were able to formally generalize any continuous review $(r, Q, \mathcal{B})$ policy with arbitrary arrival and procurement processes as an analogous bulk queueing system. Hence, for the deterministic lead-time setting, we determine the analogous queueing system. We derive the Forward Kolmogorov equations corresponding to that system and provide exact solutions where at most one server is operational. For the cases where capacity limit is exceeded, we invented a finite-difference-based numerical method to numerically solve the Forward Kolmogorov equations and approximate the steady-state behavior. We note that the capacity setting in the queueing system, i.e. at most one server is operational, corresponds to at most one order outstanding case in the $(r, Q, \mathcal{B})$ model. Therefore, for an arbitrary value of Q , continuous review, state-dependent Poisson arrivals, deterministic procurement lead-time and at most one order outstanding case settings, we provide complete and exact steady-state behavior. Backorders are allowed; however, the sum of the number of backorders and the value of the safety stock, i.e. r , must be smaller than or equal to $Q - 1$. We provide the exact stationary distribution of any inventory state variable

at arrival, departure, or arbitrary epochs. Additionally, we provide exact conditional or unconditional waiting time distribution functions at arrival or arbitrary epochs. For the cases where more than order may be outstanding, we approximate the steady-state variable through the finite-difference-based technique that we invented. In Chapter 6, we propose the methodology to find the exact equilibrium behavior of the continuous review, at most one outstanding order $(r, Q, \mathcal{B})$ model for each proposed information scenario combined with heterogeneous customers with respect to waiting sensitivity. For the cases where more than one outstanding order is allowed, we utilize the numerical method we have derived. All of the findings appear to be new contributions to the literature.

The exponential procurement lead-time setting's analysis is completely different compared to the deterministic lead-time setting. In Chapter 7, we utilize continuous time Markov Chain methods and propose a new combinatorial approach to characterize the steady-state distribution of inventory state variables and distribution functions of customer waiting times. For information availability cases and heterogeneous customers, we derive the exact equilibrium behavior for at most one outstanding order case, and develop a combinatorial method to determine exact equilibrium behavior for the cases where more than one outstanding order is allowed. We have presented an approach to characterize exact conditional or unconditional customer waiting time distributions. To our knowledge, the findings of this chapter also appear to be new contributions to the literature on (r, Q) models.

In the beginning of Chapter 8, we present how some performance metrics are evaluated. Then, we perform one-way sensitivity analysis for most of the parameters presented in Chapter 3. We conclude Chapter 8 with suboptimal scenarios on a variety of parameter combinations, we state the suboptimal quadruplets $(r^*, Q^*, \mathcal{B}^*, Info^*)$ and comment on how the system behaves.

Regarding future work, the exact steady-state distribution of the number of customers present in the bulk queueing system with state-dependent Poisson arrivals,

deterministic service times, and more than one server operational is still not available. We have presented the Forward Kolmogorov equations to express that system; however, they are difficult to solve. We presented numerical techniques that provide satisfactory approximations in most cases. However, the numerical method is not straightforward to implement and its complexity grows exponentially. Exact or approximate methods to produce steady-state results could be studied.



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APPENDIX A: EXACT STEADY-STATE SOLUTIONS

In this chapter, we present the derivations of the analytic results demonstrated in the previous section. We compute the steady-state metrics with respect to the analogous state-dependent bulk queueing system. We note that if $\tilde{\lambda}_i$ denotes the arrival rate per unit of time when $IN = i$, then $\tilde{\lambda}_i = \lambda_{r+Q-i}$; where λ_{r+Q-i} denotes the arrival rate per unit of time in the analogous queueing system when the number of customers present in the system is $r + Q - i$. Similarly, we let the vector of $\tilde{\pi}_i$ s denote the steady-state probability corresponding to the inventory system; on the other hand, the vector of probability measures $\{\pi_i\}$ denotes the steady-state probability corresponding to the analogous queueing system. As mentioned above, $\tilde{\pi}_i = \pi_{r+Q-i}$. Finally, we note that the steady-state probability vector $\{\pi^d\}$ has to be normalized to sum up to 1, since they form a probability measure.

A.1. Full/Partial Information ($r = 0, Q = 2, \mathcal{B} = 1$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_2 y_1} e^{\lambda_2 L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \lambda_2 \pi_0 e^{-\lambda_2 y_1} e^{\lambda_2 L} e^{\lambda_2 x_1}.$$

$$\begin{aligned} \pi_1 \Lambda &= p_{0,1}(0) \\ &= \Lambda \pi_0 e^{\lambda_2 L}. \end{aligned}$$

$$\pi_1 = \pi_0 e^{\lambda_2 L}.$$

$$\begin{aligned}\pi_2 &= \int_0^L p_{0,1}(y_1) dy_1 \\ &= \frac{\Lambda \pi_0 e^{\lambda_2 L} (1 - e^{-\lambda_2 L})}{\lambda_2}.\end{aligned}$$

$$\begin{aligned}\pi_3 &= \int_0^L \int_0^{y_1} p_{1,1}(x_1, y_1) dx_1 dy_1 \\ &= \Lambda \pi_0 e^{\lambda_2 L} L - \frac{\Lambda \pi_0 e^{\lambda_2 L} (1 - e^{-\lambda_2 L})}{\lambda_2}.\end{aligned}$$

$$\begin{aligned}\mathbb{E}[W_0^a] &= \int_0^L \frac{(L - y_1) p_{1,1}(0, y_1)}{\pi_2 \lambda_2} dy_1 \\ &= L + \frac{L \Lambda \pi_0}{\pi_2 \lambda_2} - \frac{\Lambda \pi_0 e^{\lambda_2 L} (1 - e^{-\lambda_2 L})}{\lambda_2^2 \pi_2}.\end{aligned}$$

Then we are required to solve the fixed-point equation:

$$\lambda_2 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right),$$

to obtain the equilibrium arrival rate λ_2 when a stock-out occurs. We note that:

$$\mathbb{E}[W^a] = \pi_2^a \mathbb{E}[W_0^a],$$

where $\pi_2^a = \pi_2 \lambda_2 / \lambda_{eff}$, and $\lambda_{eff} = \sum_{i=0}^2 \lambda_i \pi_i$. Finally,

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda \pi_0 (e^{\lambda_2 L} - 1).$$

$$\mathcal{C} = L + \pi_0^d \frac{2}{\Lambda} + \pi_1^d \frac{1}{\Lambda}.$$

A.2. No Information ($r = 0, Q = 2, \mathcal{B} = 1$)

The steady-state behavior for the case of no information is exactly the same with the previous section, however; now $\lambda_i = \lambda$ and the fixed-point equation to be solved is:

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right).$$

A.3. ($r = 1, Q = 2, \mathcal{B} = 0$)

The solutions are as presented in Section A.2, where we set $\lambda_n = \Lambda$. We note that no fixed-point equation is required to be solved, since customers are not allowed to wait.

A.4. Full Information ($r = 0, Q = 3, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_3 y_1} e^{\lambda_3 L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \lambda_3 \pi_0 e^{-\lambda_3 y_1} e^{\lambda_3 L} e^{-(\lambda_4 - \lambda_3)x_1}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda \lambda_3 \lambda_4 \pi_0 e^{-\lambda_3 y_1} e^{\lambda_3 L} e^{-(\lambda_4 - \lambda_3)x_2} e^{\lambda_4 x_1}.$$

For $x_1 \leq L$, it can be shown that:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left(\Lambda \pi_0 + \frac{\Lambda \pi_0 \lambda_3}{(\lambda_4 - \lambda_3 - \Lambda)} [1 - e^{-(\lambda_4 - \lambda_3 - \Lambda)x_1}] \right).$$

Then,

$$\begin{aligned} \pi_1 &= \pi_0(1 - e^{-\Lambda L}) + \frac{\pi_0 \lambda_3 (1 - e^{-\Lambda L})}{\lambda_4 - \lambda_3 - \Lambda} \\ &\quad - \frac{\Lambda \lambda_3 \pi_0 (1 - e^{-(\lambda_4 - \lambda_3)L})}{(\lambda_4 - \lambda_3 - \Lambda)(\lambda_4 - \lambda_3)} + \pi_0 e^{-\Lambda L} \\ &\quad + \frac{\lambda_3 \pi_0 (e^{-\Lambda L} - e^{-(\lambda_4 - \lambda_3)L})}{\lambda_4 - \lambda_3 - \Lambda}. \end{aligned}$$

$$\begin{aligned} \pi_2 &= \frac{p_{0,1}(0)}{\Lambda} \\ &= \pi_0 e^{\lambda_3 L}. \end{aligned}$$

$$\begin{aligned} \pi_3 &= \int_0^L p_{0,1}(y_1) dy_1 \\ &= \frac{\lambda_0 \pi_0 e^{\lambda_3 L} (1 - e^{-\lambda_3 L})}{\lambda_3}. \end{aligned}$$

$$\begin{aligned} \pi_4 &= \int_0^L \int_0^{y_1} p_{1,1}(x_1, y_1) dx_1 dy_1 \\ &= \Lambda \lambda_3 \pi_0 e^{\lambda_3 L} \int_0^L \int_0^{y_1} e^{-(\lambda_4 - \lambda_3)x_1} e^{-\lambda_3 y_1} dx_1 dy_1 \\ &= \frac{\Lambda \pi_0 e^{\lambda_3 L} (1 - e^{-\lambda_3 L})}{\lambda_4 - \lambda_3} - \frac{\Lambda \lambda_3 \pi_0 e^{\lambda_3 L} (1 - e^{-\lambda_4 L})}{(\lambda_4 - \lambda_3) \lambda_4}. \end{aligned}$$

$$\begin{aligned} \pi_5 &= \int_0^L \int_0^{y_1} \int_0^{x_2} p_{2,1}(x_1, x_2, y_1) dx_1 dx_2 dy_1 \\ &= \Lambda \lambda_3 \lambda_4 \pi_0 e^{\lambda_3 L} \int_0^L \int_0^{y_1} \int_0^{x_2} e^{-(\lambda_4 - \lambda_3)x_2} e^{-\lambda_3 y_1} e^{\lambda_4 x_2} dx_1 dx_2 dy_1 \end{aligned}$$

$$= \Lambda \lambda_3 \pi_0 e^{\lambda_3 L} \left[\frac{L}{\lambda_3} - \frac{1 - e^{-\lambda_3 L}}{\lambda_3^2} - \frac{1 - e^{-\lambda_3 L}}{\lambda_3(\lambda_4 - \lambda_3)} + \frac{1 - e^{-\lambda_4 L}}{\lambda_4(\lambda_4 - \lambda_3)} \right].$$

$$\begin{aligned} \mathbb{E}[W_0^a] &= \int_0^L \frac{(L - y_1) p_{1,1}(0, y_1)}{\lambda_3 \pi_3} dy_1 \\ &= \int_0^L \frac{(L - y_1) \Lambda \pi_0 \lambda_3 e^{-\lambda_3 y_1} e^{\lambda_3 L}}{\lambda_3 \pi_3} dy_1 \\ &= L + \frac{\Lambda \pi_0 e^{\lambda_3 L}}{\pi_3} \left[\frac{L e^{-\lambda_3 L}}{\lambda_3} - \frac{1 - e^{-\lambda_3 L}}{\lambda_3} - \frac{1 - e^{-\lambda_3 L}}{\lambda_3^2} \right]. \end{aligned}$$

$$\begin{aligned} \mathbb{E}[W_{-1}^a] &= \int_0^L \int_0^{y_1} \frac{(L - y_1) p_{2,1}(0, x_2, y_1)}{\lambda_4 \pi_4} dx_2 dy_1 \\ &= \int_0^L \int_0^{y_1} \frac{(L - y_1) \Lambda \pi_0 \lambda_3 \lambda_4 e^{-(\lambda_4 - \lambda_3)x_2} e^{-\lambda_3 y_1} e^{\lambda_3 L}}{\lambda_4 \pi_4} dx_2 dy_1 \\ &= L - \frac{\Lambda \pi_0 \lambda_3 e^{\lambda_3 L}}{\pi_4(\lambda_4 - \lambda_3)} \left[-\frac{L e^{-\lambda_3 L}}{\lambda_3} + \frac{1 - e^{-\lambda_3 L}}{\lambda_3^2} + \frac{L e^{-\lambda_4 L}}{\lambda_4} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} \right]. \end{aligned}$$

There are two fixed-point equations that must be solved simultaneously. The first fixed-point equation is:

$$\lambda_3 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right).$$

Equivalently, the second fixed-point equation is:

$$\lambda_4 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_{-1}^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \frac{\Lambda \pi_0 \lambda_3 (1 - e^{-(\lambda_4 - \lambda_3)L})}{\lambda_4 - \lambda_3}.$$

$$\pi_2^d = \Lambda \pi_0 \lambda_3 \left\{ \frac{e^{\lambda_3 L} - 1}{\lambda_3} - \frac{1 - e^{-(\lambda_4 - \lambda_3)L}}{\lambda_4 - \lambda_3} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{3}{\Lambda} + \pi_1^d \frac{2}{\Lambda} + \pi_2^d \frac{1}{\Lambda}.$$

A.5. Partial Information ($r = 0, Q = 3, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_e y_1} e^{\lambda_e L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \lambda_e \pi_0 e^{-\lambda_e y_1} e^{\lambda_e L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda \lambda_e^2 \pi_0 e^{-\lambda_e y_1} e^{\lambda_e L} e^{\lambda_e x_1}.$$

For $x_1 \leq L$, it can be shown that:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left[\Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda x_1} - 1) \right].$$

Then,

$$\pi_1 = \pi_0 (1 - e^{-\Lambda L}) + \pi_0 \lambda_e L - \frac{\pi_0 \lambda_e (1 - e^{-\Lambda L})}{\Lambda} + e^{-\Lambda L} \pi_0 + \frac{\pi_0 \lambda_e}{\Lambda} - \frac{\pi_0 \lambda_e e^{-\Lambda L}}{\Lambda}.$$

$$\pi_2 = \pi_0 e^{\lambda_e L}.$$

$$\begin{aligned}
\pi_3 &= \int_0^L p_{0,1}(y_1) dy_1 \\
&= \frac{\pi_0 \Lambda e^{\lambda_e L} (1 - e^{-\lambda_e L})}{\lambda_e}.
\end{aligned}$$

$$\begin{aligned}
\pi_4 &= \int_0^L \int_0^{y_1} p_{1,1}(x_1, y_1) dx_1 dy_1 \\
&= \Lambda \pi_0 \lambda_e e^{\lambda_e L} \left[-\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right].
\end{aligned}$$

$$\begin{aligned}
\pi_5 &= \int_0^L \int_0^{y_1} \int_0^{x_2} p_{2,1}(x_1, x_2, y_1) dx_1 dx_2 dy_1 \\
&= \Lambda \pi_0 \lambda_e e^{\lambda_e L} \left[\frac{L}{\lambda_e} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} + \frac{L e^{-\lambda_e L}}{\lambda_e} - \frac{1 - e^{\lambda_e L}}{\lambda_e^2} \right].
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[W_0^a] &= \int_0^L \frac{(L - y_1) p_{1,1}(0, y_1)}{\lambda_e \pi_3} dy_1 \\
&= L + \frac{\Lambda \pi_0 \lambda_e e^{\lambda_e L}}{\pi_3 \lambda_e} \left[\frac{L e^{-\lambda_e L}}{\lambda_e} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right].
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[W_{-1}^a] &= \int_0^L \int_0^{y_1} \frac{(L - y_1) p_{2,1}(0, x_2, y_1)}{\lambda_e \pi_4} dx_2 dy_1 \\
&= L - \frac{\Lambda \pi_0 \lambda_e^2 e^{\lambda_e L}}{\pi_4 \lambda_e} \left[-\frac{L^2 e^{-\lambda_e L}}{\lambda_e} - \frac{2L e^{-\lambda_e L}}{\lambda_e^2} + \frac{2}{\lambda_e^3} (1 - e^{-\lambda_e L}) \right].
\end{aligned}$$

We must solve one fixed-point equation to determine the arrival rate in the stock-out instants, i.e. λ_e , namely:

$$\lambda_e = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[\tilde{W}^a]} \right),$$

where $\mathbb{E}[\tilde{W}^a] = \mathbb{E}[W_0^a] \pi_3^a / (\pi_3^a + \pi_4^a) + \mathbb{E}[W_{-1}^a] \pi_4^a / (\pi_3^a + \pi_4^a)$ denotes the expected waiting

time incurred by an arriving customer, given that the customer arrived at a stock-out instant.

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda \pi_0 \lambda_e L.$$

$$\pi_2^d = \Lambda \pi_0 \lambda_e \left\{ \frac{e^{\lambda_e L} - 1}{\lambda_e} - L \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{3}{\Lambda} + \pi_1^d \frac{2}{\Lambda} + \pi_2^d \frac{1}{\Lambda}.$$

A.6. No Information ($r = 0, Q = 3, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \lambda \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{1,1}(x_1, y_1) = \lambda^2 \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \lambda^3 \pi_0 e^{-\lambda y_1} e^{\lambda L} e^{\lambda x_1}.$$

For $x_1 \leq L$, it can be shown that:

$$p_{1,0}(x_1) = \lambda \pi_0.$$

Hence,

$$\pi_1 = \pi_0(1 + \lambda L).$$

$$\begin{aligned}\pi_2 &= p_{0,1}(0)/\lambda \\ &= \pi_0 e^{\lambda L}.\end{aligned}$$

$$\pi_3 = \pi_0 e^{\lambda L} (1 - e^{-\lambda L}).$$

$$\begin{aligned}\pi_4 &= \int_0^L \int_0^{y_1} p_{1,1}(x_1, y_1) dx_1 dy_1 \\ &= \int_0^L \int_0^{y_1} \lambda^2 \pi_0 y_1 e^{-\lambda y_1} e^{\lambda L} dx_1 dy_1 \\ &= \lambda^2 \pi_0 e^{\lambda L} \left[-\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right].\end{aligned}$$

$$\begin{aligned}\pi_5 &= \int_0^L \int_0^{y_1} \int_0^{x_2} p_{2,1}(x_1, x_2, y_1) dx_1 dx_2 dy_1 \\ &= \lambda^2 \pi_0 e^{\lambda L} \left[\frac{L}{\lambda} - \frac{1 - e^{-\lambda L}}{\lambda^2} + \frac{L e^{-\lambda L}}{\lambda} - \frac{1 - e^{-\lambda L}}{\lambda^2} \right].\end{aligned}$$

$$\begin{aligned}\mathbb{E}[W_0^a] &= \int_0^L \frac{(L - y_1) p_{1,1}(0, y_1)}{\lambda \pi_3} dy_1 \\ &= L - \frac{\lambda^2 \pi_0 e^{\lambda L}}{\pi_3 \lambda} \left[-\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right].\end{aligned}$$

$$\mathbb{E}[W_{-1}^a] = \int_0^L \int_0^{y_1} \frac{(L - y_1) p_{2,1}(0, x_2, y_1)}{\lambda \pi_4} dx_2 dy_1$$

$$= L - \frac{\lambda^3 \pi_0 e^{\lambda L}}{\pi_4 \lambda} \left[-\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left(-\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right) \right].$$

One fixed-point equation has to be solved for obtaining the equilibrium arrival rate λ , namely:

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right),$$

where $\mathbb{E}[W^a] = \mathbb{E}[W_0^a] \pi_3^a + \mathbb{E}[W_{-1}^a] \pi_4^a$.

$$\pi_0^d = \lambda \pi_0.$$

$$\pi_1^d = \lambda^2 \pi_0 L.$$

$$\pi_2^d = \lambda^2 \pi_0 \left\{ \frac{e^{\lambda L} - 1}{\lambda} - L \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{3}{\lambda} + \pi_1^d \frac{2}{\lambda} + \pi_2^d \frac{1}{\lambda}.$$

A.7. ($r = 2, Q = 3, \mathcal{B} = 0$)

The solution procedures are exactly the same as in Section A.6, while now customers are not allowed to wait. Therefore, $\lambda_n = \Lambda$.

A.8. Full/Partial Information ($r = 1, Q = 3, \mathcal{B} = 1$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\Lambda y_1} e^{\Lambda L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda^2 \pi_0 e^{-(\lambda_e - \Lambda)x_1} e^{-\Lambda y_1} e^{\Lambda L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda^2 \lambda_e \pi_0 e^{-(\lambda_e - \Lambda)x_2} e^{\lambda_e x_1} e^{-\Lambda y_1} e^{\Lambda L}.$$

For $x_1 \leq L$, it can be shown that:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left[\Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_e - 2\Lambda)x_1})}{\lambda_e - 2\Lambda} \right].$$

Therefore,

$$\begin{aligned} \pi_1 &= \pi_0(1 - e^{-\Lambda L}) + \frac{\Lambda \pi_0}{\lambda_e - 2\Lambda}(1 - e^{-\Lambda L}) - \frac{\Lambda^2 \pi_0}{(\lambda_e - 2\Lambda)(\lambda_e - \Lambda)}(1 - e^{-(\lambda_e - \Lambda)L}) \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left[\Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_e - 2\Lambda)L})}{\lambda_e - 2\Lambda} \right]. \end{aligned}$$

$$\pi_2 = \pi_0 e^{\Lambda L}.$$

$$\pi_3 = \pi_0 e^{\Lambda L} (1 - e^{-\Lambda L}).$$

$$\pi_4 = \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{\lambda_e - \Lambda} \left[\frac{1 - e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{\lambda_e L}}{\lambda_e} \right].$$

$$\pi_5 = \Lambda^2 \pi_0 e^{\Lambda L} \left[\frac{L}{\Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda^2} - \frac{1 - e^{-\Lambda L}}{(\lambda_e - \Lambda)\Lambda} + \frac{1 - e^{-\lambda_e L}}{\lambda_e(\lambda_e - \Lambda)} \right].$$

$$\begin{aligned} \mathbb{E}[W_0^a] &= \int_0^L \int_0^{y_1} \frac{(L - y_1)p_{2,1}(0, x_2, y_1)}{\pi_4 \lambda_e} dx_2 dy_1 \\ &= L - \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{\pi_4(\lambda_e - \Lambda)} \left[-\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{L e^{-\lambda_e L}}{\lambda_e} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right]. \end{aligned}$$

Then, to evaluate the equilibrium arrival rate λ_e at stock-out instants, we must solve the fixed-point equation:

$$\lambda_e = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda^2 \pi_0 \frac{1 - e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda}.$$

$$\pi_2^d = \Lambda^2 \pi_0 \left\{ \frac{e^{\Lambda L} - 1}{\Lambda} - \frac{1 - e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{3}{\Lambda} + \pi_1^d \frac{2}{\Lambda} + \pi_2^d \frac{1}{\Lambda}.$$

A.9. No Information ($r = 1, Q = 3, \mathcal{B} = 1$)

$$p_{0,1}(y_1) = \lambda \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{1,1}(x_1, y_1) = \lambda^2 \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \lambda^3 \pi_0 e^{\lambda x_1} e^{-\lambda y_1} e^{\lambda L}.$$

It can be shown that:

$$p_{1,0}(x_1) = \lambda \pi_0.$$

Hence,

$$\pi_1 = \lambda \pi_0 L + \pi_0.$$

$$\pi_2 = \pi_0 e^{\lambda L}.$$

$$\pi_3 = \pi_0 (e^{\lambda L} - 1).$$

$$\pi_4 = \lambda^2 \pi_0 e^{\lambda L} \left(-\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{\lambda L}}{\lambda^2} \right).$$

$$\pi_5 = \lambda^2 \pi_0 e^{\lambda L} \left(\frac{L}{\lambda} - \frac{1 - e^{\lambda L}}{\lambda^2} + \frac{L e^{\lambda L}}{\lambda} - \frac{1 - e^{\lambda L}}{\lambda^2} \right).$$

$$\mathbb{E}[W_0^a] = L - \frac{\lambda^2 \pi_0 e^{\lambda L}}{\pi_4} \left[-\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left\{ -\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\} \right].$$

Hence, we are required to solve one fixed-point equation to obtain the equilibrium arrival rate λ :

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right),$$

where $\mathbb{E}[W^a] = \mathbb{E}[W_0^a] \pi_4^a$. The expected cycle-time evaluation is exactly the same as in the $(r = 0, Q = 3, \mathcal{B} = 2)$ case.

A.10. Full/Partial Information $(r = 0, Q = 3, \mathcal{B} = 1)$

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_3 y_1} e^{\lambda_3 L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \pi_0 \lambda_3 e^{\lambda_3 x_1} e^{-\lambda_3 y_1} e^{\lambda_3 L}.$$

It can be shown that for $x_1 \leq L$:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left[\Lambda \pi_0 + \frac{\Lambda \pi_0 \lambda_3 (e^{(\lambda_3 + \Lambda)x_1} - 1)}{\lambda_3 + \Lambda} \right].$$

Then,

$$\pi_1 = \pi_0 (1 - e^{-\Lambda L}) + \frac{\Lambda \pi_0}{\lambda_3 + \Lambda} (e^{\lambda_3 L} - 1) - \frac{\lambda_3 \pi_0 (1 - e^{-\Lambda L})}{\lambda_3 + \Lambda}$$

$$+ \frac{e^{-\Lambda L}}{\Lambda} \left[\Lambda \pi_0 + \lambda_3 \Lambda \pi_0 \frac{e^{(\lambda_3 + \Lambda)L} - 1}{\lambda_3 + \Lambda} \right].$$

$$\pi_2 = \pi_0 e^{\lambda_3 L}.$$

$$\pi_3 = \frac{\pi_0 \Lambda e^{\lambda_3 L}}{\lambda_3} (1 - e^{-\lambda_3 L}).$$

$$\pi_4 = \Lambda \pi_0 e^{\lambda_3 L} \left[L - \frac{1 - e^{-\lambda_3 L}}{\lambda_3} \right].$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda \pi_0 e^{\lambda_3 L}}{\pi_3} \left[-\frac{L e^{-\lambda_3 L}}{\lambda_3} + \frac{1 - e^{-\lambda_3 L}}{\lambda_3^2} \right].$$

Similarly, to obtain the equilibrium arrival rate λ_3 , we are required to solve the single fixed-point equation:

$$\lambda_3 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda \pi_0 (e^{\lambda_3 L} - 1).$$

$$\mathcal{C} = L + \pi_0^d \frac{3}{\Lambda} + \pi_1^d \frac{2}{\Lambda}.$$

A.11. No Information ($r = 0, Q = 3, \mathcal{B} = 1$)

The equilibrium behavior is exactly the same as in Section A.10; yet, we must set $\lambda_i = \lambda$ for $i \leq 3$. The fixed-point equation takes the form:

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right),$$

where $\mathbb{E}[W^a] = \mathbb{E}[W_0^a] \pi_3^a$.

A.12. ($r = 1, Q = 3, \mathcal{B} = 0$)

The solution procedures are exactly the same as in Section A.10, while now customers are not allowed to wait. Therefore, $\lambda_n = \Lambda$.

A.13. Full Information ($r = 0, Q = 4, \mathcal{B} = 3$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_4 y_1} e^{\lambda_4 L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \lambda_4 \pi_0 e^{-(\lambda_5 - \lambda_4)x_1} e^{-\lambda_4 y_1} e^{\lambda_4 L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda \lambda_4 \lambda_5 \pi_0 e^{-(\lambda_6 - \lambda_5)x_1} e^{-(\lambda_5 - \lambda_4)x_2} e^{-\lambda_4 y_1} e^{\lambda_4 L}.$$

$$p_{3,1}(x_1, x_2, x_3, y_1) = \Lambda \lambda_4 \lambda_5 \lambda_6 \pi_0 e^{\lambda_6 x_1} e^{-(\lambda_6 - \lambda_5)x_2} e^{-(\lambda_5 - \lambda_4)x_3} e^{-\lambda_4 y_1} e^{\lambda_4 L}.$$

It can be shown that for $x_1 \leq L$,

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left[\Lambda \pi_0 + \frac{\Lambda \pi_0 \lambda_4}{\lambda_5 - \lambda_4 - \Lambda} (1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)x_1}) \right].$$

Then,

$$\begin{aligned}\pi_1 &= \pi_0(1 - e^{-\Lambda L}) + \frac{\lambda_4\pi_0(1 - e^{-\Lambda L})}{\lambda_5 - \lambda_4 - \Lambda} - \frac{\Lambda\pi_0\lambda_4(1 - e^{-(\lambda_5 - \lambda_4)L})}{(\lambda_5 - \lambda_4 - \Lambda)(\lambda_5 - \lambda_4)} \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left[\Lambda\pi_0 + \frac{\Lambda\pi_0\lambda_4(1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)L})}{\lambda_5 - \lambda_4 - \Lambda} \right].\end{aligned}$$

It can be shown that for $x_2 \leq L$,

$$\begin{aligned}p_{2,0}(x_1, x_2) &= e^{-\Lambda x_1} \left[\Lambda e^{-\Lambda(x_2 - x_1)} \left\{ \Lambda\pi_0 + \frac{\Lambda\pi_0\lambda_4}{\lambda_5 - \lambda_4 - \Lambda} [1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)(x_2 - x_1)}] \right\} \right. \\ &+ \left. \frac{\Lambda\pi_0\lambda_4\lambda_5 e^{-(\lambda_5 - \lambda_4)x_2} e^{-(\lambda_4 - \lambda_5)x_1} [1 - e^{-(\lambda_6 - \lambda_4 - \Lambda)x_1}]}{\lambda_6 - \lambda_4 - \Lambda} \right].\end{aligned}\quad (\text{A.1})$$

Furthermore, let us partition the domain of $p_{2,0}(x_1, x_2)$ into two disjoint sets, i.e. $\mathcal{I} \equiv \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2 \leq L\}$ and $\mathcal{II} \equiv \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2, L \leq x_2\}$. Then, we know that:

$$\pi_2 = \int_0^L \int_0^{x_2} p_{2,0}(x_1, x_2) dx_1 dx_2 + \int_L^\infty \int_0^{x_2} p_{2,0}(x_1, x_2) dx_1 dx_2.$$

We know how to compute the integral over the domain $\mathcal{I}$ by virtue of Equation (A.1). Let us focus on the domain $\mathcal{II}$. By the Integral-Differential equation corresponding to the non-boundary behavior of $p_{2,0}(x_1, x_2)$ for $x_2 \geq L$:

$$\frac{\partial p_{2,0}(x_1, x_2)}{\partial x_1} + \frac{\partial p_{2,0}(x_1, x_2)}{\partial x_2} = -\Lambda p_{2,0}(x_1, x_2).$$

Upon integrating both sides on the two dimensional plane $\mathcal{II}$, and employing the Fundamental Theorem of Calculus in multi-dimensions:

$$- \int_L^\infty p_{2,0}(0, x_2) dx_2 - \int_0^L p_{2,0}(x_1, L) dx_1 = -\Lambda \int_L^\infty \int_0^{x_2} \Lambda p_{2,0}(x_1, x_2) dx_1 dx_2,$$

which we know how to evaluate. Then,

$$\pi_2 = \left\{ \Lambda^2 \pi_0 + \frac{\Lambda^2 \pi_4 \lambda_4}{\lambda_5 - \lambda_4 - \Lambda} \right\} \left\{ -\frac{Le^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\}$$

$$\begin{aligned}
& + \frac{\Lambda^2 \pi_0 \lambda_4}{(\lambda_5 - \lambda_4 - \Lambda)^2} \left\{ \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\
& + \frac{\Lambda \pi_0 \lambda_4 \lambda_5}{(\lambda_6 - \lambda_4 - \Lambda)(\lambda_4 + \Lambda - \lambda_5)} \left\{ \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\
& - \frac{\Lambda \pi_0 \lambda_4 \lambda_5}{(\lambda_6 - \lambda_4 - \Lambda)(\lambda_6 - \lambda_4)} \left\{ \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} - \frac{1 - e^{-(\lambda_6 + \lambda_5 - 2\lambda_4)L}}{\lambda_6 + \lambda_5 - 2\lambda_4} \right\} \\
& + \frac{e^{-\Lambda L}}{\Lambda} \left[\Lambda \pi_0 + \frac{\Lambda \pi_0 \lambda_4 (1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)L})}{\lambda_5 - \lambda_4 - \Lambda} \right] \\
& + e^{-\Lambda L} \Lambda \pi_0 L + \frac{\Lambda \pi_0 \lambda_4 e^{-\Lambda L}}{\lambda_5 - \lambda_4 - \Lambda} \left[L - \frac{e^{-(\lambda_5 - \lambda_4 - \Lambda)L} (1 - e^{-(\Lambda + \lambda_4 - \lambda_5)L})}{\Lambda + \lambda_4 - \lambda_5} \right] \\
& + \frac{\pi_0 \lambda_4 \lambda_5 e^{-(\lambda_5 - \lambda_4)L} (e^{(\lambda_5 - \Lambda - \lambda_4)L} - 1)}{(\lambda_6 - \lambda_4 - \Lambda)(\lambda_5 - \lambda_4 - \Lambda)} \\
& - \frac{\pi_0 \lambda_4 \lambda_5 e^{-(\lambda_5 - \lambda_4)L} (1 - e^{-(\lambda_6 - \lambda_4)L})}{(\lambda_6 - \lambda_4 - \Lambda)(\lambda_6 - \lambda_4)}.
\end{aligned}$$

$$\pi_3 = \pi_0 e^{\lambda_4 L}.$$

$$\pi_4 = \frac{\Lambda \pi_0 e^{\lambda_4 L} (1 - e^{-\lambda_4 L})}{\lambda_4}.$$

$$\pi_5 = \frac{\Lambda \pi_0 \lambda_4 e^{\lambda_4 L}}{\lambda_5 - \lambda_4} \left[\frac{1 - e^{-\lambda_4 L}}{\lambda_4} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5} \right].$$

$$\pi_6 = \frac{\Lambda \pi_0 \lambda_4 \lambda_5 e^{\lambda_4 L}}{\lambda_6 - \lambda_5} \left[\frac{1 - e^{-\lambda_4 L}}{\lambda_4 (\lambda_5 - \lambda_4)} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5 (\lambda_5 - \lambda_4)} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4 (\lambda_6 - \lambda_4)} + \frac{1 - e^{-\lambda_6 L}}{\lambda_6 (\lambda_6 - \lambda_4)} \right].$$

$$\begin{aligned}
\pi_7 = & \Lambda \pi_0 \lambda_4 \lambda_5 e^{\lambda_4 L} \left\{ \frac{L}{\lambda_4 \lambda_5} - \frac{1 - e^{-\lambda_4 L}}{\lambda_5 \lambda_4^2} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4 \lambda_5 (\lambda_5 - \lambda_4)} + \frac{1 - e^{-\lambda_5 L}}{\lambda_5^2 (\lambda_5 - \lambda_4)} \right. \\
& - \frac{1 - e^{-\lambda_4 L}}{\lambda_4 (\lambda_6 - \lambda_5) (\lambda_5 - \lambda_4)} + \frac{1 - e^{-\lambda_5 L}}{\lambda_5 (\lambda_6 - \lambda_5) (\lambda_5 - \lambda_4)} \\
& \left. + \frac{1 - e^{-\lambda_4 L}}{\lambda_4 (\lambda_6 - \lambda_4) (\lambda_6 - \lambda_5)} - \frac{1 - e^{-\lambda_6 L}}{\lambda_6 (\lambda_6 - \lambda_4) (\lambda_6 - \lambda_5)} \right\}.
\end{aligned}$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda\pi_0\lambda_4 e^{\lambda_4 L}}{\pi_4\lambda_4} \left\{ -\frac{Le^{-\lambda_4 L}}{\lambda_4} + \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} \right\}.$$

$$\mathbb{E}[W_{-1}^a] = L - \frac{\Lambda\pi_0\lambda_4\lambda_5 e^{\lambda_4 L}}{\lambda_5\pi_5(\lambda_5 - \lambda_4)} \left\{ -\frac{Le^{-\lambda_4 L}}{\lambda_4} + \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} + \frac{Le^{-\lambda_5 L}}{\lambda_5} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5^2} \right\}.$$

$$\begin{aligned} \mathbb{E}[W_{-2}^a] &= L - \frac{\Lambda\pi_0\lambda_4\lambda_5\lambda_6 e^{\lambda_4 L}}{\pi_6\lambda_6(\lambda_6 - \lambda_5)(\lambda_5 - \lambda_4)} \\ &\quad \cdot \left\{ -\frac{Le^{-\lambda_4 L}}{\lambda_4} + \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} + \frac{Le^{-\lambda_5 L}}{\lambda_5} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5^2} \right\} \\ &\quad + \frac{\Lambda\pi_0\lambda_4\lambda_5\lambda_6 e^{\lambda_4 L}}{\pi_6\lambda_6(\lambda_6 - \lambda_5)(\lambda_6 - \lambda_4)} \\ &\quad \cdot \left\{ -\frac{Le^{-\lambda_4 L}}{\lambda_4} + \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} + \frac{Le^{-\lambda_6 L}}{\lambda_6} - \frac{1 - e^{-\lambda_6 L}}{\lambda_6^2} \right\}. \end{aligned}$$

We are obliged to solve three fixed-point equations simultaneously to determine the unknown equilibrium arrival rates $\lambda_4, \lambda_5, \lambda_6$. Namely, these equations are:

$$\lambda_4 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right),$$

$$\lambda_5 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_{-1}^a]} \right),$$

$$\lambda_6 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_{-2}^a]} \right).$$

$$\pi_0^d = \Lambda\pi_0.$$

$$\pi_1^d = \frac{\Lambda\pi_0\lambda_4(1 - e^{-(\lambda_5 - \lambda_4)L})}{\lambda_5 - \lambda_4}.$$

$$\pi_2^d = \frac{\Lambda\pi_0\lambda_4\lambda_5}{\lambda_6 - \lambda_5} \left\{ \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} - \frac{1 - e^{-(\lambda_6 - \lambda_4)L}}{\lambda_6 - \lambda_4} \right\}.$$

$$\begin{aligned} \pi_3^d &= \Lambda\pi_0(e^{\lambda_4 L} - 1) - \frac{\Lambda\pi_0\lambda_4(1 - e^{-(\lambda_5 - \lambda_4)L})}{\lambda_5 - \lambda_4} \\ &\quad - \frac{\Lambda\pi_0\lambda_4\lambda_5}{\lambda_6 - \lambda_5} \left\{ \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} - \frac{1 - e^{-(\lambda_6 - \lambda_4)L}}{\lambda_6 - \lambda_4} \right\}. \end{aligned}$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda} + \pi_3^d \frac{1}{\Lambda}.$$

A.14. Partial Information ($r = 0, Q = 4, \mathcal{B} = 3$)

$$p_{0,1}(y_1) = \Lambda\pi_0 e^{-\lambda_e y_1} e^{\lambda_e L}.$$

$$p_{1,1}(x_1, y_1) = \lambda\pi_0\lambda_e e^{-\lambda_e y_1} e^{\lambda_e L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \lambda\pi_0\lambda_e^2 e^{-\lambda_e y_1} e^{\lambda_e L}.$$

$$p_{3,1}(x_1, x_2, x_3, y_1) = \lambda\pi_0\lambda_e^3 e^{\lambda_e x_1} e^{-\lambda_e y_1} e^{\lambda_e L}.$$

It can be shown easily that for $x_1 \leq L$:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left\{ \Lambda\pi_0 + \pi_0\lambda_e(e^{\Lambda x_1} - 1) \right\},$$

implying that:

$$\pi_1 = \pi_0(1 - e^{-\Lambda L}) + \pi_0 \lambda_e L - \frac{\pi_0 \lambda_e}{\Lambda} (1 - e^{-\Lambda L}) + \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda L} - 1) \right\}.$$

$$p_{2,0}(x_1, x_2) = e^{-\Lambda x_2} \Lambda \left\{ \Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda(x_2 - x_1)} - 1) \right\} + \pi_0 \lambda_e^2 (1 - e^{-\Lambda x_1}).$$

Implying that:

$$\begin{aligned} \pi_2 &= \Lambda^2 \pi_0 \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} + \Lambda \pi_0 \lambda_e \left\{ \frac{L}{\Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{L e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\ &+ \pi_0 \lambda_e^2 \left\{ \frac{L^2}{2} - \frac{L}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} + \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda L} - 1) \right\} + \Lambda \pi_0 e^{-\Lambda L} L \\ &+ \frac{\pi_0 \lambda_e}{\Lambda} (1 - e^{-\Lambda L}) - \pi_0 \lambda_e e^{-\Lambda L} L + \frac{\pi_0 \lambda_e^2 L}{\Lambda} - \frac{\pi_0 \lambda_e^2}{\Lambda} (1 - e^{-\Lambda L}). \end{aligned}$$

$$\pi_3 = \pi_0 e^{\lambda_e L}.$$

$$\pi_4 = \frac{\Lambda \pi_0 e^{\lambda_e L} (1 - e^{-\lambda_e L})}{\lambda_e}.$$

$$\pi_5 = \Lambda \pi_0 \lambda_e e^{\lambda_e L} \left\{ -\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.$$

$$\pi_6 = \frac{\Lambda \pi_0 \lambda_e^2 e^{\lambda_e L}}{2} \left\{ -\frac{L^2 e^{-\lambda_e L}}{\lambda_e} + \frac{2}{\lambda_e} \left[-\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right] \right\}.$$

$$\begin{aligned} \pi_7 &= \Lambda \pi_0 \lambda_e^2 e^{\lambda_e L} \left\{ \frac{L}{\lambda_e^2} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^3} - \frac{1}{\lambda_e} \left[-\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right] \right. \\ &\quad \left. - \frac{1}{2} \left[-\frac{L^2 e^{-\lambda_e L}}{\lambda_e} + \frac{2}{\lambda_e} \left(-\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right) \right] \right\}. \end{aligned}$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda\pi_0 e^{\lambda_e L}}{\pi_4} \left\{ -\frac{Le^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.$$

$$\mathbb{E}[W_{-1}^a] = L - \frac{\Lambda\pi_0\lambda_e e^{\lambda_e L}}{\pi_5} \left\{ -\frac{L^2 e^{-\lambda_e L}}{\lambda_e} + \frac{2}{\lambda_e} \left[-\frac{Le^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right] \right\}.$$

$$\mathbb{E}[W_{-2}^a] = L - \frac{\Lambda\pi_0\lambda_e^2 e^{\lambda_e L}}{2\pi_6} \left\{ -\frac{L^3 e^{-\lambda_e L}}{\lambda_e} - \frac{3L^2 e^{-\lambda_e L}}{\lambda_e^2} + \frac{6}{\lambda_e} \left[-\frac{Le^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right] \right\}.$$

We are required to solve one fixed-point equation to obtain the unknown rate parameter λ_e :

$$\lambda_e = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[\tilde{W}^a]} \right),$$

where

$$\mathbb{E}[\tilde{W}^a] = \frac{\mathbb{E}[W_0^a]\pi_4^a}{\pi_4^a + \pi_5^a + \pi_6^a} + \frac{\mathbb{E}[W_{-1}^a]\pi_5^a}{\pi_4^a + \pi_5^a + \pi_6^a} + \frac{\mathbb{E}[W_{-2}^a]\pi_6^a}{\pi_4^a + \pi_5^a + \pi_6^a}.$$

$$\pi_0^d = \Lambda\pi_0.$$

$$\pi_1^d = \Lambda\pi_0\lambda_e L.$$

$$\pi_2^d = \frac{\Lambda\pi_0\lambda_e^2 L^2}{2}.$$

$$\pi_3^d = \Lambda \pi_0 \lambda_e^2 \left\{ \frac{e^{\lambda_e L} - 1}{\lambda_e} - \frac{L}{\lambda_e} - \frac{L^2}{2} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda} + \pi_3^d \frac{1}{\Lambda}.$$

A.15. No Information ($r = 0, Q = 4, \mathcal{B} = 3$)

$$p_{0,1}(y_1) = \lambda \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{1,1}(x_1, y_1) = \lambda^2 \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \lambda^3 \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{3,1}(x_1, x_2, x_3, y_1) = \lambda^4 \pi_0 e^{\lambda x_1} e^{-\lambda y_1} e^{\lambda L}.$$

It can be shown that for $x_1 \leq L$:

$$p_{1,0}(x_1) = \pi_0 \lambda.$$

Therefore,

$$\pi_1 = \pi_0 \lambda L + \pi_0.$$

For $x_2 \leq L$, $p_{2,0}(x_1, x_2)$ takes the form:

$$p_{2,0}(x_1, x_2) = \lambda^2 \pi_0.$$

$$\pi_2 = \frac{\pi_0 \lambda^2 L^2}{2} + \pi_0 + \pi_0 \lambda L.$$

$$\pi_3 = \pi_0 e^{\lambda L}.$$

$$\pi_4 = \pi_0 e^{\lambda L} (1 - e^{-\lambda L}).$$

$$\pi_5 = \lambda^2 \pi_0 e^{\lambda L} \left\{ -\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\}$$

$$\pi_6 = \frac{\lambda^3 \pi_0 e^{\lambda L}}{2} \left\{ -\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left[-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

$$\begin{aligned} \pi_7 = & \lambda^3 \pi_0 e^{\lambda L} \left\{ \frac{L}{\lambda^2} - \frac{1 - e^{-\lambda L}}{\lambda^3} + \frac{Le^{-\lambda L}}{\lambda^2} - \frac{1 - e^{-\lambda L}}{\lambda^3} \right. \\ & \left. - \frac{\lambda^3 \pi_0 e^{\lambda L}}{2} \left[-\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left(-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right) \right] \right\}. \end{aligned}$$

$$\mathbb{E}[W_0^a] = L - \frac{\lambda \pi_0 e^{\lambda L}}{\pi_4} \left\{ -\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\}.$$

$$\mathbb{E}[W_{-1}^a] = L - \frac{\lambda^2 \pi_0 e^{\lambda L}}{\pi_5} \left\{ -\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left[-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

$$\mathbb{E}[W_{-2}^a] = L - \frac{\lambda^3 \pi_0 e^{\lambda L}}{2\pi_6} \left\{ -\frac{L^3 e^{-\lambda L}}{\lambda} - \frac{3L^2 e^{-\lambda L}}{\lambda^2} + \frac{6}{\lambda^2} \left[-\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

Then to determine the unknown intensity parameter λ , we solve a single fixed-point equation:

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right),$$

where $\mathbb{E}[W^a] = \mathbb{E}[W_0^a] \pi_4^a + \mathbb{E}[W_{-1}^a] \pi_5^a + \mathbb{E}[W_{-2}^a] \pi_6^a$.

$$\pi_0^d = \lambda \pi_0.$$

$$\pi_1^d = \lambda^2 \pi_0 L.$$

$$\pi_2^d = \frac{\lambda^3 \pi_0 L^2}{2}.$$

$$\pi_3^d = \lambda^3 \pi_0 \left\{ \frac{e^{\lambda L} - 1}{\lambda^2} - \frac{L}{\lambda} - \frac{L^2}{2} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\lambda} + \pi_1^d \frac{3}{\lambda} + \pi_2^d \frac{2}{\lambda} + \pi_3^d \frac{1}{\lambda}.$$

A.16. ($r = 3, Q = 4, \mathcal{B} = 0$)

The solution procedures are exactly the same as in Section A.15, while now customers are not allowed to wait. Therefore, $\lambda_n = \Lambda$.

A.17. Full Information ($r = 1, Q = 4, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\Lambda y_1} e^{\Lambda L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda^2 \pi_0 e^{\Lambda L} e^{-\Lambda y_1} e^{-(\lambda_5 - \Lambda)x_1}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda^2 \pi_0 \lambda_5 e^{\Lambda L} e^{-\Lambda y_1} e^{-(\lambda_5 - \Lambda)x_2} e^{-(\lambda_6 - \lambda_5)x_1}.$$

$$p_{3,1}(x_1, x_2, x_3, y_1) = \Lambda^2 \pi_0 \lambda_5 \lambda_6 e^{\Lambda L} e^{-\Lambda y_1} e^{-(\lambda_5 - \Lambda)x_3} e^{-(\lambda_6 - \lambda_5)x_2} e^{\lambda_6 x_1}.$$

For $x_1 \leq L$, it can be shown that:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_5 - 2\Lambda)x_1})}{\lambda_5 - 2\Lambda} \right\}.$$

$$\begin{aligned} \pi_1 &= \pi_0 (1 - e^{-\Lambda L}) + \frac{\Lambda \pi_0 (1 - e^{-\Lambda L})}{\lambda_5 - 2\Lambda} - \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_5 - \Lambda)L})}{(\lambda_5 - 2\Lambda)(\lambda_5 - \Lambda)} \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_5 - 2\Lambda)L})}{\lambda_5 - 2\Lambda} \right\}. \end{aligned}$$

For $x_2 \leq L$:

$$\begin{aligned} p_{2,0}(x_1, x_2) &= e^{-\Lambda x_2} \Lambda \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_5 - 2\Lambda)(x_2 - x_1)})}{\lambda_5 - 2\Lambda} \right\} \\ &+ \frac{\Lambda^2 \pi_0 \lambda_5}{\lambda_6 - 2\Lambda} \left\{ e^{-(\lambda_5 - \Lambda)x_2} e^{-(2\Lambda - \lambda_5)x_1} - e^{-(\lambda_5 - \Lambda)x_2} e^{-(\lambda_6 - \lambda_5)x_1} \right\}. \end{aligned}$$

$$\begin{aligned}
\pi_2 = & \left\{ \Lambda^2 \pi_0 + \frac{\Lambda^3 \pi_0}{\lambda_5 - 2\Lambda} \right\} \left\{ -\frac{Le^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\
& + \frac{\Lambda^3 \pi_0}{(\lambda_5 - 2\Lambda)^2} \left\{ \frac{1 - e^{-(\lambda_5 - \Lambda)L}}{\lambda_5 - \Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\
& + \frac{\Lambda^2 \pi_0 \lambda_5}{(\lambda_6 - \Lambda)(2\Lambda - \lambda_5)} \left\{ \frac{1 - e^{-(\lambda_5 - \Lambda)L}}{\lambda_5 - \Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\
& - \frac{\Lambda^2 \pi_0 \lambda_5}{(\lambda_6 - \Lambda)(\Lambda + \lambda_6 - \lambda_5)} \left\{ \frac{1 - e^{-(\lambda_5 - \Lambda)L}}{\lambda_5 - \Lambda} - \frac{1 - e^{-\lambda_6 L}}{\lambda_6} \right\} \\
& + \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_5 - 2\Lambda)L})}{\lambda_5 - 2\Lambda} \right\} \\
& + e^{-\Lambda L} \left\{ \Lambda \pi_0 L + \frac{\Lambda^2 \pi_0 L}{\lambda_5 - 2\Lambda} + \frac{\Lambda^2 e^{-(\lambda_5 - 2\Lambda)L} (1 - e^{-(\lambda_5 - 2\Lambda)L})}{(\lambda_5 - 2\Lambda)^2} \right\} \\
& + \frac{\Lambda^2 \pi_0 \lambda_5 e^{-(\lambda_5 - \Lambda)L}}{\lambda_6 - \Lambda} \left\{ \frac{1 - e^{-(2\Lambda - \lambda_5)L}}{2\Lambda - \lambda_5} - \frac{1 - e^{-(\lambda_6 - \lambda_5 - \Lambda)L}}{\lambda_6 - \lambda_5 - \Lambda} \right\}.
\end{aligned}$$

$$\pi_3 = \pi_0 e^{\Lambda L}.$$

$$\pi_4 = \pi_0 e^{\Lambda L} (1 - e^{-\Lambda L}).$$

$$\pi_5 = \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{(\lambda_5 - \Lambda)} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5} \right\}.$$

$$\pi_6 = \frac{\Lambda^2 \pi_0 \lambda_5 e^{\Lambda L}}{\lambda_6 - \lambda_5} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda(\lambda_5 - \Lambda)} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5(\lambda_5 - \Lambda)} - \frac{1 - e^{-\Lambda L}}{\Lambda(\lambda_6 - \Lambda)} + \frac{1 - e^{-\lambda_6 L}}{\lambda_6(\lambda_6 - \Lambda)} \right\}.$$

$$\begin{aligned}
\pi_7 = & \Lambda^2 \pi_0 e^{\Lambda L} \left\{ \frac{L}{\Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda^2} - \frac{1 - e^{-\Lambda L}}{\Lambda(\lambda_5 - \Lambda)} + \frac{1 - e^{-\lambda_5 L}}{\lambda_5(\lambda_5 - \Lambda)} \right\} \\
& - \frac{\Lambda^2 \pi_0 e^{\Lambda L} \lambda_5}{\lambda_6 - \lambda_5} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda(\lambda_5 - \Lambda)} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5(\lambda_5 - \Lambda)} - \frac{1 - e^{-\Lambda L}}{\Lambda(\lambda_6 - \Lambda)} + \frac{1 - e^{-\lambda_6 L}}{\lambda_6(\lambda_6 - \Lambda)} \right\}.
\end{aligned}$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda^2 \pi_0 \lambda_5 e^{\Lambda L}}{\lambda_5 \pi_5 (\lambda_5 - \Lambda)} \left\{ -\frac{Le^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{Le^{-\lambda_5 L}}{\lambda_5} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5^2} \right\}.$$

$$\begin{aligned} \mathbb{E}[W_{-1}^a] &= L \\ &- \frac{\Lambda^2 \pi_0 \lambda_5 \lambda_6 e^{\Lambda L}}{\lambda_6 \pi_6 (\lambda_6 - \lambda_5) (\lambda_5 - \Lambda)} \left\{ -\frac{Le^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{Le^{-\lambda_5 L}}{\lambda_5} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5^2} \right\} \\ &+ \frac{\Lambda^2 \pi_0 \lambda_5 \lambda_6 e^{\Lambda L}}{\lambda_6 \pi_6 (\lambda_6 - \lambda_5) (\lambda_6 - \Lambda)} \left\{ -\frac{Le^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{Le^{-\lambda_6 L}}{\lambda_6} - \frac{1 - e^{-\lambda_6 L}}{\lambda_6^2} \right\}. \end{aligned}$$

We must solve two fixed-point equations simultaneously to find equilibrium rates λ_5 and λ_6 :

$$\lambda_5 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right),$$

$$\lambda_6 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_{-1}^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_5 - \Lambda)L})}{\lambda_5 - \Lambda}.$$

$$\pi_2^d = \frac{\Lambda^2 \pi_0 \lambda_5}{\lambda_6 - \lambda_5} \left\{ \frac{1 - e^{-(\lambda_5 - \Lambda)L}}{\lambda_5 - \Lambda} - \frac{1 - e^{-(\lambda_6 - \Lambda)L}}{\lambda_6 - \Lambda} \right\}.$$

$$\pi_3^d = \Lambda^2 \pi_0 \lambda_5 \left\{ \frac{e^{\Lambda L} - 1}{\Lambda \lambda_5} - \frac{1 - e^{-(\lambda_5 - \Lambda)L}}{(\lambda_5 - \Lambda) \lambda_5} - \frac{1 - e^{-(\lambda_5 - \Lambda)L}}{(\lambda_6 - \lambda_5) (\lambda_5 - \Lambda)} + \frac{1 - e^{-(\lambda_6 - \Lambda)L}}{(\lambda_6 - \lambda_5) (\lambda_6 - \Lambda)} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda} + \pi_3^d \frac{1}{\Lambda}.$$

A.18. Partial Information ($r = 1, Q = 4, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\Lambda y_1} e^{\Lambda L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda^2 \pi_0 e^{\Lambda L} e^{-\Lambda y_1} e^{-(\lambda_e - \Lambda)x_1}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda^2 \pi_0 \lambda_e e^{\Lambda L} e^{-\Lambda y_1} e^{-(\lambda_e - \Lambda)x_2}.$$

$$p_{3,1}(x_1, x_2, x_3, y_1) = \Lambda^2 \pi_0 \lambda_e^2 e^{\lambda_e x_1} e^{\Lambda L} e^{-\Lambda y_1} e^{-(\lambda_e - \Lambda)x_3}.$$

It can be shown that for $x_1 \leq L$:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_e - 2\Lambda)x_1})}{\lambda_e - 2\Lambda} \right\}.$$

Then,

$$\begin{aligned} \pi_1 &= \pi_0 (1 - e^{-\Lambda L}) + \frac{\Lambda \pi_0 (1 - e^{-\Lambda L})}{\lambda_e - 2\Lambda} - \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_e - \Lambda)L})}{(\lambda_e - 2\Lambda)(\lambda_e - \Lambda)} \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_e - 2\Lambda)L})}{\lambda_e - 2\Lambda} \right\} \end{aligned}$$

It can be shown that for $x_2 \leq L$:

$$\begin{aligned}
p_{2,0}(x_1, x_2) &= \Lambda e^{-\Lambda x_2} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0}{\lambda_e - 2\Lambda} [1 - e^{-(\lambda_e - 2\Lambda)x_2} e^{-(2\Lambda - \lambda_e)x_1}] \right\} \\
&+ \frac{\Lambda^2 \pi_0 \lambda_e e^{-(\lambda_e - \Lambda)x_2}}{\lambda_e - 2\Lambda} [e^{-(2\Lambda - \lambda_e)x_1} - 1]
\end{aligned}$$

Hence,

$$\begin{aligned}
\pi_2 &= \left\{ \Lambda^2 \pi_0 + \frac{\Lambda^3 \pi_0}{\lambda_e - 2\Lambda} \right\} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\
&- \frac{\Lambda^3 \pi_0}{\lambda_e - 2\Lambda} \left\{ \frac{1 - e^{-(\lambda_e - \Lambda)L}}{(2\Lambda - \lambda_e)(\lambda_e - \Lambda)} - \frac{1 - e^{-\Lambda L}}{(2\Lambda - \lambda_e)\Lambda} \right\} \\
&+ \frac{\Lambda^2 \pi_0 \lambda_e (1 - e^{-(\lambda_e - \Lambda)L})}{(\lambda_e - 2\Lambda)(2\Lambda - \lambda_e)(\lambda_e - \Lambda)} + \frac{\Lambda^2 \pi_0 \lambda_e (1 - e^{-\Lambda L})}{(\lambda_e - 2\Lambda)^2 \Lambda} \\
&- \frac{\Lambda^2 \pi_0 \lambda_e}{\lambda_e - 2\Lambda} \left\{ -\frac{L e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda} + \frac{1 - e^{-(\lambda_e - \Lambda)L}}{(\lambda_e - \Lambda)^2} \right\} \\
&+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0 (1 - e^{-(\lambda_e - 2\Lambda)L})}{\lambda_e - 2\Lambda} \right\} + e^{-\Lambda L} \Lambda \pi_0 L \\
&+ \frac{\Lambda^2 \pi_0 e^{-\Lambda L}}{\lambda_e - 2\Lambda} \left\{ L - e^{-(\lambda_e - 2\Lambda)L} \frac{[1 - e^{-(2\Lambda - \lambda_e)L}]}{2\Lambda - \lambda_e} \right\} \\
&+ \frac{\Lambda \pi_0 \lambda_e}{\lambda_e - 2\Lambda} \left\{ \frac{e^{-(\lambda_e - \Lambda)L} (1 - e^{-(2\Lambda - \lambda_e)L})}{2\Lambda - \lambda_e} - e^{-(\lambda_e - \Lambda)L} \right\}.
\end{aligned}$$

$$\pi_3 = \pi_0 e^{\Lambda L}.$$

$$\pi_4 = \pi_0 e^{\Lambda L} (1 - e^{-\Lambda L}).$$

$$\pi_5 = \frac{\Lambda \pi_0 e^{\Lambda L} (1 - e^{-\Lambda L})}{\lambda_e - \Lambda} - \frac{\Lambda^2 \pi_0 e^{\Lambda L} (1 - e^{-\lambda_e L})}{(\lambda_e - \Lambda) \lambda_e}.$$

$$\begin{aligned}
\pi_6 &= -\frac{\Lambda^2 \pi_0 \lambda_e e^{\Lambda L}}{\lambda_e - \Lambda} \left\{ -\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\} + \frac{\Lambda \pi_0 \lambda_e e^{\Lambda L} (1 - e^{-\Lambda L})}{(\lambda_e - \Lambda)^2} \\
&- \frac{\Lambda^2 \pi_0 e^{\Lambda L} (1 - e^{-\lambda_e L})}{(\lambda_e - \Lambda)^2}.
\end{aligned}$$

$$\begin{aligned}
\pi_7 &= \Lambda\pi_0 e^{\Lambda L} \left\{ L - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} - \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{\lambda_e - \Lambda} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{-\lambda_e L}}{\lambda_e} \right\} \\
&+ \frac{\Lambda^2 \pi_0 \lambda_e e^{\Lambda L}}{\lambda_e - \Lambda} \left\{ -\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\} \\
&- \frac{\Lambda^2 \pi_0 \lambda_e e^{\Lambda L}}{(\lambda_e - \Lambda)^2} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{-\lambda_e L}}{\lambda_e} \right\}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[W_0^a] &= L - \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{\pi_5(\lambda_e - \Lambda)} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\
&- \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{\pi_5(\lambda_e - \Lambda)} \left\{ -\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[W_{-1}^a] &= L - \frac{\Lambda^2 \pi_0 \lambda_e e^{\Lambda L}}{\pi_6(\lambda_e - \Lambda)} \left\{ -\frac{L^2 e^{-\lambda_e L}}{\lambda_e} + \frac{2}{\lambda_e} \left[-\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right] \right\} \\
&- \frac{\Lambda^2 \pi_0 \lambda_e e^{\Lambda L}}{\pi_6(\lambda_e - \Lambda)^2} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{L e^{-\lambda_e L}}{\lambda_e} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.
\end{aligned}$$

To obtain the equilibrium rate λ_e we solve a single fixed-point equation:

$$\lambda_e = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[\tilde{W}^a]} \right),$$

where $\mathbb{E}[\tilde{W}^a] = \mathbb{E}[W_0^a] \pi_5^a / (\pi_5^a + \pi_6^a) + \mathbb{E}[W_{-1}^a] \pi_6^a / (\pi_5^a + \pi_6^a)$.

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda^2 \pi_0 \frac{1 - e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda}.$$

$$\pi_2^d = \Lambda^2 \pi_0 \lambda_e \left\{ -\frac{L e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda} + \frac{1 - e^{-(\lambda_e - \Lambda)L}}{(\lambda_e - \Lambda)^2} \right\}.$$

$$\pi_3^d = \Lambda^2 \pi_0 \lambda_e \left\{ \frac{e^{\Lambda L} - 1}{\Lambda \lambda_e} - \frac{1 - e^{-(\lambda_e - \Lambda)L}}{(\lambda_e - \Lambda) \lambda_e} + \frac{L e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda} - \frac{1 - e^{-(\lambda_e - \Lambda)L}}{(\lambda_e - \Lambda)^2} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda} + \pi_3^d \frac{1}{\Lambda}.$$

A.19. No Information ($r = 1, Q = 4, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \lambda \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{1,1}(x_1, y_1) = \lambda^2 \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \lambda^3 \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{3,1}(x_1, x_2, x_3, y_1) = \lambda^4 \pi_0 e^{\lambda x_1} e^{-\lambda y_1} e^{\lambda L}.$$

$$\pi_1 = \lambda \pi_0 L + \pi_0.$$

$$\pi_2 = \frac{\lambda^2 L^2 \pi_0}{2} - \pi_0 + \pi_0 \lambda L.$$

$$\pi_3 = \pi_0 e^{\lambda L}.$$

$$\pi_4 = \pi_0 e^{\lambda L} (1 - e^{-\lambda L}).$$

$$\pi_5 = \lambda^2 \pi_0 e^{\lambda L} \left\{ -\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\}.$$

$$\pi_6 = \frac{\lambda^3 \pi_0 e^{\lambda L}}{2} \left\{ -\frac{Le^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left[-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

$$\begin{aligned} \pi_7 &= \lambda^3 \pi_0 e^{\lambda L} \left\{ \frac{L}{\lambda^2} - \frac{1 - e^{-\lambda L}}{\lambda^3} + \frac{Le^{-\lambda L}}{\lambda} - \frac{1 - e^{-\lambda L}}{\lambda^2} \right\} \\ &\quad - \frac{\lambda^3 \pi_0 e^{\lambda L}}{2} \left\{ -\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left[-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}. \end{aligned}$$

$$\mathbb{E}[W_0^a] = L - \frac{\lambda^2 \pi_0 e^{\lambda L}}{\pi_5} \left\{ -\frac{L^2 e^{\lambda L}}{\lambda} + \frac{2}{\lambda} \left[-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

$$\mathbb{E}[W_{-1}^a] = L - \frac{\lambda^3 \pi_0 e^{\lambda L}}{2\pi_6} \left\{ -\frac{L^3 e^{-\lambda L}}{\lambda} - \frac{3L^2 e^{-\lambda L}}{\lambda^2} + \frac{6}{\lambda^2} \left[-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

Then we must solve a single fixed-point equation to determine the value of λ :

$$\lambda = A \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right).$$

$$\pi_0^d = \lambda \pi_0.$$

$$\pi_1^d = \lambda^2 \pi_0 L.$$

$$\pi_2^d = \lambda^3 \pi_0 L^2 / 2.$$

$$\pi_3^d = \lambda^3 \pi_0 \left\{ \frac{e^{\lambda L} - 1}{\lambda^2} - \frac{L}{\lambda} - \frac{L^2}{2} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\lambda} + \pi_1^d \frac{3}{\lambda} + \pi_2^d \frac{2}{\lambda} + \pi_3^d \frac{1}{\lambda}.$$

A.20. Full/Partial Information ($r = 2, Q = 4, \mathcal{B} = 1$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\Lambda y_1} e^{\Lambda L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda^2 \pi_0 e^{-\Lambda y_1} e^{\Lambda L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda^3 \pi_0 e^{-(\lambda_6 - \Lambda)x_1} e^{-\Lambda y_1} e^{\Lambda L}.$$

$$p_{3,1}(x_1, x_2, x_3, y_1) = \Lambda^2 \lambda_6 \pi_0 e^{\lambda_6 x_1} e^{-(\lambda_6 - \Lambda)x_2} e^{-\Lambda y_1} e^{\Lambda L}.$$

$$\pi_1 = \pi_0(\Lambda L + 1).$$

$$\begin{aligned} \pi_2 &= \Lambda \pi_0 \left\{ L - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} + \frac{\Lambda^2 \pi_0}{\lambda_6 - 2\Lambda} \left\{ L - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\ &- \frac{\Lambda^3 \pi_0}{(\lambda_6 - 2\Lambda)(\lambda_6 - \Lambda)} \left\{ L - \frac{1 - e^{-(\lambda_6 - \Lambda)L}}{\lambda_6 - \Lambda} \right\} + \pi_0 \\ &+ \frac{1}{\Lambda} \left\{ \Lambda \pi_0 (1 - e^{-\Lambda L}) + \frac{\Lambda^2 \pi_0 (1 - e^{-\Lambda L})}{\lambda_6 - 2\Lambda} - \frac{\Lambda^3 \pi_0 (1 - e^{-(\lambda_6 - \Lambda)L})}{(\lambda_6 - 2\Lambda)(\lambda_6 - \Lambda)} \right\}. \end{aligned}$$

$$\pi_3 = \pi_0 e^{\Lambda L}.$$

$$\pi_4 = \pi_0 (e^{\Lambda L} - 1).$$

$$\pi_5 = \pi_0 \Lambda^2 e^{\Lambda L} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\}.$$

$$\pi_6 = \frac{\Lambda^3 e^{\Lambda L} \pi_0}{\lambda_6 - \Lambda} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\}.$$

$$\pi_6 = \frac{\Lambda^3 e^{\Lambda L} \pi_0}{\lambda_6 - \Lambda} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} - \frac{\pi_0 \Lambda^3 e^{\Lambda L}}{(\lambda_6 - 2\Lambda)^2} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{-\lambda_6 L}}{\lambda_6} \right\}.$$

$$\begin{aligned} \pi_7 &= \Lambda^3 e^{\Lambda L} \left\{ \frac{L}{\Lambda^2} - \frac{1 - e^{-\Lambda L}}{\Lambda^3} \right\} - \Lambda^2 e^{\Lambda L} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\ &\quad - \frac{\Lambda^3 e^{\Lambda L}}{\lambda_6 - \Lambda} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} + \frac{\Lambda^3 e^{\Lambda L}}{(\lambda_6 - \Lambda)^2} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\lambda_6 L}}{\lambda_6} \right\}. \end{aligned}$$

$$\begin{aligned} \mathbb{E}[W_0^a] &= L - \frac{\pi_0 \Lambda^3 e^{\Lambda L}}{(\lambda_6 - \Lambda) \pi_6} \left\{ -\frac{L^2 e^{-\Lambda L}}{\Lambda} + \frac{2}{\Lambda} \left[-\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right] \right\} \\ &\quad + \frac{\pi_0 \Lambda^3 e^{\Lambda L}}{(\lambda_6 - \Lambda)^2 \pi_6} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{L e^{-\lambda_6 L}}{\lambda_6} - \frac{1 - e^{-\lambda_6 L}}{\lambda_6^2} \right\}. \end{aligned}$$

To compute λ_6 we solve the fixed-point equation:

$$\lambda_6 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda^2 \pi_0 L.$$

$$\pi_2^d = \frac{\Lambda^3 \pi_0}{\lambda_6 - \Lambda} \left\{ L - \frac{1 - e^{-(\lambda_6 - \Lambda)L}}{\lambda_6 - \Lambda} \right\}.$$

$$\pi_3^d = \Lambda^3 \pi_0 \left\{ \frac{e^{\Lambda L} - 1}{\Lambda^2} - \frac{L}{\Lambda} - \frac{L}{\lambda_6 - \Lambda} + \frac{1 - e^{-(\lambda_6 - \Lambda)L}}{(\lambda_6 - \Lambda)^2} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda} + \pi_3^d \frac{1}{\Lambda}.$$

A.21. No Information ($r = 2, Q = 4, \mathcal{B} = 1$)

The steady-state probabilities are exactly the same as in Section A.15. However, now only one customer is allowed to wait at any instant. Hence,

$$\mathbb{E}[W_0^a] = L - \frac{\lambda^3 \pi_0 e^{\lambda L}}{2\pi_6} \left\{ -\frac{L^3 e^{-\lambda L}}{\lambda} - \frac{3L^2 e^{-\lambda L}}{\lambda^2} + \frac{6}{\lambda^2} \left[-\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

We must solve one fixed-point equation to determine the value of λ :

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right),$$

where $\mathbb{E}[W^a] = \mathbb{E}[W_0^a] \pi_6^a$.

$$\pi_0^d = \lambda \pi_0.$$

$$\pi_1^d = \lambda^2 \pi_0 L.$$

$$\pi_2^d = \frac{\lambda^3 \pi_0 L^2}{2}.$$

$$\pi_3^d = \lambda^3 \pi_0 \left\{ \frac{e^{\lambda L} - 1}{\lambda^2} - \frac{L}{\lambda} - \frac{L^2}{2} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\lambda} + \pi_1^d \frac{3}{\lambda} + \pi_2^d \frac{2}{\lambda} + \pi_3^d \frac{1}{\lambda}.$$

A.22. Full Information ($r = 0, Q = 4, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_4 y_1} e^{\lambda_4 L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \lambda_4 \pi_0 e^{\lambda_4 L} e^{-\lambda_4 y_1} e^{-(\lambda_5 - \lambda_4)x_1}.$$

$$p_{2,1}(x_1, x_2, y_1) = \lambda_5 \lambda_4 \Lambda \pi_0 e^{\lambda_5 x_1} e^{-(\lambda_5 - \lambda_4)x_1} e^{-\lambda_4 y_1} e^{\lambda_4 L}.$$

It can be shown that for $x_1 \leq L$:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left\{ \Lambda \pi_0 + \frac{\Lambda \lambda_4 \pi_0 (1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)x_1})}{\lambda_5 - \lambda_4 - \Lambda} \right\}.$$

Hence,

$$\begin{aligned} \pi_1 &= \pi_0 (1 - e^{-\Lambda L}) + \frac{\lambda_4 \pi_0 (1 - e^{-\Lambda L})}{\lambda_5 - \lambda_4 - \Lambda} - \frac{\Lambda \lambda_4 \pi_0 (1 - e^{-(\lambda_5 - \lambda_4)L})}{(\lambda_5 - \lambda_4 - \Lambda)(\lambda_5 - \lambda_4)} \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda \lambda_4 \pi_0 (1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)L})}{\lambda_5 - \lambda_4 - \Lambda} \right\}. \end{aligned}$$

We can show that for $x_2 \leq L$:

$$\begin{aligned} p_{2,0}(x_1, x_2) &= e^{-\Lambda x_2} \Lambda^2 \pi_0 + \frac{e^{-\Lambda x_2} \Lambda^2 \lambda_4 \pi_0}{\lambda_5 - \lambda_4 - \Lambda} \left[1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)(x_2 - x_1)} \right] \\ &+ \frac{e^{-\Lambda x_1} \lambda_5 \lambda_4 \Lambda \pi_0 e^{-(\lambda_5 - \lambda_4)x_2} e^{-(\lambda_4 - \lambda_5)x_1}}{\lambda_4 + \Lambda} \left[e^{(\lambda_4 + \Lambda)x_1} - 1 \right]. \end{aligned}$$

Hence,

$$\begin{aligned} \pi_2 &= \left\{ \Lambda^2 \pi_0 + \frac{\Lambda^2 \lambda_4 \pi_0}{\lambda_5 - \lambda_4 - \Lambda} \right\} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\ &+ \frac{\Lambda^2 \lambda_4 \pi_0}{(\lambda_5 - \lambda_4 - \Lambda)^2} \left\{ \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\ &+ \frac{\lambda_5 \lambda_4 \pi_0 \Lambda}{(\lambda_4 + \Lambda) \lambda_5} \left\{ \frac{e^{\lambda_4 L} - 1}{\lambda_4} - \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_3} \right\} \\ &- \frac{\lambda_5 \lambda_4 \Lambda \pi_0}{(\lambda_4 + \Lambda)(\lambda_5 - \lambda_4 - \Lambda)} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} \right\} \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda \lambda_4 \pi_0 (1 - e^{-(\lambda_5 - \lambda_4 - \Lambda)L})}{\lambda_5 - \lambda_4 - \Lambda} \right\} + e^{-\Lambda L} \Lambda \pi_0 L \\ &+ \frac{\Lambda e^{-\Lambda L} \lambda_4 \pi_0}{\lambda_5 - \lambda_4 - \Lambda} \left\{ L - \frac{e^{-(\lambda_5 - \lambda_4 - \Lambda)L} (1 - e^{-(\Lambda + \lambda_4 - \lambda_5)L})}{\Lambda + \lambda_4 - \lambda_5} \right\} \\ &+ \frac{\lambda_5 \lambda_4 \pi_0 e^{-(\lambda_5 - \lambda_4)L}}{\lambda_4 + \Lambda} \left\{ \frac{e^{\lambda_5 L} - 1}{\lambda_5} - \frac{1 - e^{-(\lambda_4 + \Lambda - \lambda_5)L}}{\lambda_5 + \Lambda - \lambda_5} \right\}. \end{aligned}$$

$$\pi_3 = \pi_0 e^{\lambda_4 L}.$$

$$\pi_4 = \frac{\pi_0 \Lambda e^{\lambda_4 L} (1 - e^{-\lambda_4 L})}{\lambda_4}.$$

$$\pi_5 = \frac{\Lambda \pi_0 \lambda_4 e^{\lambda_4 L}}{\lambda_5 - \lambda_4} \left\{ \frac{1 - e^{-\lambda_4 L}}{\lambda_4} - \frac{1 - e^{-\lambda_5 L}}{\lambda_5} \right\}.$$

$$\pi_6 = \lambda_4 \Lambda \pi_0 e^{\lambda_4 L} \left\{ \frac{L}{\lambda_4} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} - \frac{1 - e^{-\lambda_4 L}}{(\lambda_5 - \lambda_4) \lambda_4} + \frac{1 - e^{-\lambda_5 L}}{\lambda_5 (\lambda_5 - \lambda_4)} \right\}.$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda\pi_0 e^{\lambda_4 L}}{\pi_4} \left\{ -\frac{Le^{-\lambda_4 L}}{\lambda_4} + \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} \right\}.$$

$$\mathbb{E}[W_{-1}^a] = L - \frac{\lambda_4 \Lambda \pi_0 e^{\lambda_4 L}}{\pi_5(\lambda_5 - \lambda_4)} \left\{ +\frac{Le^{-\lambda_4 L}}{\lambda_4} - \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} - \frac{Le^{-\lambda_5 L}}{\lambda_5} + \frac{1 - e^{-\lambda_5 L}}{\lambda_5^2} \right\}.$$

We must solve two fixed-point equations simultaneously to find the unknown rate parameters λ_4 and λ_5 , which are:

$$\lambda_4 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right),$$

$$\lambda_5 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_{-1}^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \frac{\Lambda \lambda_4 \pi_0 (1 - e^{-(\lambda_5 - \lambda_4)L})}{\lambda_5 - \lambda_4}.$$

$$\pi_2^d = \lambda_4 \Lambda \pi_0 \left\{ \frac{e^{\lambda_4 L} - 1}{\lambda_4} - \frac{1 - e^{-(\lambda_5 - \lambda_4)L}}{\lambda_5 - \lambda_4} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda}.$$

A.23. Partial Information ($r = 0, Q = 4, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_e y_1} e^{\lambda_e L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \lambda_e \pi_0 e^{\lambda_e L} e^{-\lambda_e y_1}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda \lambda_e^2 \pi_0 e^{\lambda_e L} e^{\lambda_e x_1} e^{-\lambda_e y_1}.$$

It can be shown that for $x_1 \leq L$:

$$p_{1,0}(x_1) = e^{-\Lambda x_1} \left\{ \Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda x_1} - 1) \right\}.$$

Hence,

$$\pi_1 = \pi_0(1 - e^{-\Lambda L}) + \pi_0 \lambda_e L - \frac{\pi_0 \lambda_e (1 - e^{-\Lambda L})}{\Lambda} + \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda L} - 1) \right\}.$$

Similarly, it can be shown that for $x_2 \leq L$:

$$p_{2,0}(x_1, x_2) = e^{-\Lambda x_2} \Lambda \left\{ \Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda(x_2 - x_1)} - 1) \right\} + \frac{\lambda_e^2 \Lambda \pi_0 e^{\lambda_e x_1}}{\lambda_e + \Lambda} - \frac{\lambda_e^2 \Lambda \pi_0 e^{-\Lambda x_1}}{\lambda_e + \Lambda}.$$

Hence,

$$\begin{aligned} \pi_2 &= \left\{ \Lambda^2 \pi_0 + \Lambda \pi_0 \lambda_e \right\} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} + \pi_0 \lambda_e L - \frac{\pi_0 \lambda_e (1 - e^{-\Lambda L})}{\Lambda} \\ &+ \frac{\lambda_e^2 \Lambda \pi_0}{\lambda_e + \Lambda} \left\{ \frac{e^{\lambda_e L} - 1}{\lambda_e^2} - \frac{L}{\lambda_e} - \frac{L}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} + \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \pi_0 \lambda_e (e^{\Lambda L} - 1) \right\} \\ &+ \Lambda \pi_0 e^{-\Lambda L} L + \frac{\pi_0 \lambda_e (1 - e^{-\Lambda L})}{\Lambda} - \pi_0 \lambda_e e^{-\Lambda L} L + \frac{\lambda_e^2 \pi_0}{\lambda_e + \Lambda} \left\{ \frac{e^{\lambda_e L} - 1}{\lambda_e} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\}. \end{aligned}$$

$$\pi_3 = \pi_0 e^{\lambda_e L}.$$

$$\pi_4 = \frac{\Lambda \pi_0 e^{\lambda_e L} (1 - e^{-\lambda_e L})}{\lambda_e}.$$

$$\pi_5 = \Lambda \lambda_e \pi_0 e^{\lambda_e L} \left\{ -\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.$$

$$\pi_6 = \lambda_e \Lambda \pi_0 e^{\lambda_e L} \left\{ \frac{L}{\lambda_e} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} + \frac{L e^{-\lambda_e L}}{\lambda_e} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda \pi_0 e^{\lambda_e L}}{\pi_4} \left\{ -\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.$$

$$\mathbb{E}[W_{-1}^a] = L - \frac{\lambda_e \Lambda \pi_0 e^{\lambda_e L}}{\pi_5} \left\{ -\frac{L^2 e^{-\lambda_e L}}{\lambda_e} + \frac{2}{\lambda_e} \left[-\frac{L e^{-\lambda_e L}}{\lambda_e} + \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right] \right\}.$$

Then, we must solve one fixed-point equation to determine the value of λ_e :

$$\lambda_e = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[\tilde{W}^a]} \right),$$

where $\mathbb{E}[\tilde{W}^a] = \mathbb{E}[W_0^a] \pi_4^a / (\pi_4^a + \pi_5^a) + \mathbb{E}[W_{-1}^a] \pi_5^a / (\pi_4^a + \pi_5^a)$.

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda \lambda_e L \pi_0.$$

$$\pi_2^d = \Lambda \pi_0 \lambda_e \left\{ \frac{e^{\lambda_e L} - 1}{\lambda_e} - L \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda}.$$

A.24. No Information ($r = 0, Q = 4, \mathcal{B} = 2$)

$$p_{0,1}(y_1) = \lambda \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{1,1}(x_1, y_1) = \lambda^2 \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{2,1}(x_1, x_2, y_1) = \lambda^3 \pi_0 e^{\lambda x_1} e^{-\lambda y_1} e^{\lambda L}.$$

$$\pi_1 = \lambda \pi_0 L + \pi_0.$$

$$\pi_2 = \pi_0 e^{\lambda L}.$$

$$\pi_3 = \pi_0 e^{\lambda L}.$$

$$\pi_4 = \pi_0 e^{\lambda L} (1 - e^{-\lambda L}).$$

$$\pi_5 = \lambda^2 \pi_0 e^{\lambda L} \left\{ -\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\}.$$

$$\pi_6 = \lambda^2 \pi_0 e^{\lambda L} \left\{ \frac{L}{\lambda} - \frac{1 - e^{-\lambda L}}{\lambda^2} + \frac{Le^{-\lambda L}}{\lambda} - \frac{1 - e^{-\lambda L}}{\lambda^2} \right\}.$$

$$\mathbb{E}[W_0^a] = L - \frac{\lambda \pi_0 e^{\lambda L}}{\pi_4} \left\{ -\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\}.$$

$$\mathbb{E}[W_{-1}^a] = L - \frac{\lambda^2 \pi_0 e^{\lambda L}}{\pi_5} \left\{ -\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left[-\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

One fixed-point equation must be solved:

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right).$$

$$\pi_0^d = \lambda \pi_0.$$

$$\pi_1^d = \lambda^2 L \pi_0.$$

$$\pi_2^d = \lambda^2 \pi_0 \left\{ \frac{e^{\lambda L} - 1}{\lambda} - L \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\lambda} + \pi_1^d \frac{3}{\lambda} + \pi_2^d \frac{2}{\lambda}.$$

A.25. ($r = 2, Q = 4, \mathcal{B} = 0$)

The solution procedures are exactly the same as in Section A.2, while now customers are not allowed to wait. Therefore, $\lambda_n = \Lambda$.

A.26. Full/Partial Information ($r = 1, Q = 4, \mathcal{B} = 1$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{\Lambda L} e^{-\Lambda y_1}.$$

$$p_{1,1}(x_1, y_1) = \Lambda^2 \pi_0 e^{\Lambda L} e^{-(\lambda_e - \Lambda)x_1} e^{-\Lambda y_1}.$$

$$p_{2,1}(x_1, x_2, y_1) = \Lambda^2 \lambda_e \pi_0 e^{\lambda_e x_1} e^{\Lambda L} e^{-(\lambda_e - \Lambda)x_2} e^{-\Lambda y_1}.$$

$$\begin{aligned} \pi_1 &= \pi_0(1 - e^{-\Lambda L}) + \frac{\Lambda \pi_0(1 - e^{-\Lambda L})}{\lambda_e - 2\Lambda} - \frac{\Lambda^2 \pi_0(1 - e^{-(\lambda_e - \Lambda)L})}{(\lambda_e - 2\Lambda)(\lambda_e - \Lambda)} \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0(1 - e^{-(\lambda_e - 2\Lambda)L})}{\lambda_e - 2\Lambda} \right\}. \end{aligned}$$

$$\begin{aligned} \pi_2 &= \left\{ \Lambda^2 \pi_0 + \frac{\Lambda^3 \pi_0}{\lambda_e - 2\Lambda} \right\} \left\{ -\frac{L e^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\ &+ \frac{\Lambda^2 \pi_0}{(\lambda_e - 2\Lambda)^2} \left\{ \frac{1 - e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\ &+ \frac{\lambda_e \Lambda^2 \pi_0}{2\Lambda} \left\{ \frac{e^{\Lambda L} - 1}{\Lambda \lambda_e} - \frac{1 - e^{-(\lambda_e - \Lambda)L}}{\lambda_e(\lambda_e - \Lambda)} - \frac{1 - e^{-(\lambda_e - \Lambda)L}}{(2\Lambda - \lambda_e)(\lambda_e - \Lambda)} \right\} \\ &+ \frac{\lambda_e \Lambda^2 \pi_0}{2\Lambda(2\Lambda - \lambda_e)} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} + \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda^2 \pi_0(1 - e^{-(\lambda_e - 2\Lambda)L})}{\lambda_e - 2\Lambda} \right\} \\ &+ e^{-\Lambda L} \left\{ \Lambda \pi_0 L + \frac{\Lambda^2 \pi_0 L}{\lambda_e - 2\Lambda} \right\} + \frac{e^{-\Lambda L} \Lambda^2 \pi_0 e^{-(\lambda_e - 2\Lambda)L} (1 - e^{-(2\Lambda - \lambda_e)L})}{(\lambda_e - 2\Lambda)^2} \end{aligned}$$

$$+ \frac{\lambda_e \Lambda \pi_0 e^{-(\lambda_e - \Lambda)L}}{2\Lambda} \left\{ \frac{e^{\lambda_e L} - 1}{\lambda_e} - \frac{1 - e^{-(2\Lambda - \lambda_e)L}}{2\Lambda - \lambda_e} \right\}.$$

$$\pi_3 = \pi_0 e^{\Lambda L}.$$

$$\pi_4 = \pi_0 e^{\Lambda L} (1 - e^{-\Lambda L}).$$

$$\pi_5 = \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{\lambda_e - \Lambda} \left\{ \frac{1 - e^{-\Lambda L}}{\Lambda} - \frac{1 - e^{-\lambda_e L}}{\lambda_e} \right\}.$$

$$\pi_6 = \Lambda^2 \pi_0 e^{\Lambda L} \left\{ \frac{L}{\Lambda} - \frac{1 - e^{-\Lambda L}}{\Lambda^2} - \frac{1 - e^{-\Lambda L}}{\Lambda(\lambda_e - \Lambda)} + \frac{1 - e^{-\lambda_e L}}{\lambda_e(\lambda_e - \Lambda)} \right\}.$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda^2 \pi_0 e^{\Lambda L}}{\pi_5(\lambda_e - \Lambda)} \left\{ -\frac{Le^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} + \frac{Le^{-\lambda_e L}}{\lambda_e} - \frac{1 - e^{-\lambda_e L}}{\lambda_e^2} \right\}.$$

We must solve one fixed-point equation to obtain the equilibrium arrival rate λ_e at stock-out instants:

$$\lambda_e = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda^2 \pi_0 \frac{1 - e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda}.$$

$$\pi_2^d = \Lambda^2 \pi_0 \left\{ \frac{e^{\Lambda L} - 1}{\Lambda} - \frac{1 - e^{-(\lambda_e - \Lambda)L}}{\lambda_e - \Lambda} \right\}.$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda} + \pi_2^d \frac{2}{\Lambda}.$$

A.27. No Information ($r = 1, Q = 4, \mathcal{B} = 1$)

The steady-state behavior is exactly the same as in Section A.24, and:

$$\mathbb{E}[W_0^a] = L - \frac{\lambda^2 \pi_0 e^{\lambda L}}{\pi_5} \left\{ -\frac{L^2 e^{-\lambda L}}{\lambda} + \frac{2}{\lambda} \left[-\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right] \right\}.$$

One fixed-point equation must be solved:

$$\lambda = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W^a]} \right).$$

A.28. Full/Partial Information ($r = 0, Q = 4, \mathcal{B} = 1$)

$$p_{0,1}(y_1) = \Lambda \pi_0 e^{-\lambda_4 y_1} e^{\lambda_4 L}.$$

$$p_{1,1}(x_1, y_1) = \Lambda \lambda_4 \pi_0 e^{\lambda_4 L} e^{-\lambda_4 y_1} e^{\lambda_4 x_1}.$$

$$\begin{aligned} \pi_1 &= \pi_0 (1 - e^{-\Lambda L}) + \frac{\Lambda \pi_0 (e^{\lambda_4 L} - 1)}{\lambda_4 + \Lambda} - \frac{\lambda_4 \pi_0 (1 - e^{-\Lambda L})}{\lambda_4 + \Lambda} \\ &+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda \lambda_4 \pi_0 (e^{(\lambda_4 + \Lambda)L} - 1)}{\lambda_4 + \Lambda} \right\}. \end{aligned}$$

$$\begin{aligned}
\pi_2 &= \left\{ \Lambda^2 \pi_0 - \frac{\Lambda^2 \lambda_4 \pi_0}{\lambda_4 + \Lambda} \right\} \left\{ -\frac{Le^{-\Lambda L}}{\Lambda} + \frac{1 - e^{-\Lambda L}}{\Lambda^2} \right\} \\
&+ \frac{\Lambda^2 \lambda_4 \pi_0}{(\lambda_4 + \Lambda)^2} \left\{ \frac{e^{\lambda_4 L} - 1}{\lambda_4} - \frac{1 - e^{-\Lambda L}}{\Lambda} \right\} \\
&+ \frac{e^{-\Lambda L}}{\Lambda} \left\{ \Lambda \pi_0 + \frac{\Lambda \lambda_4 \pi_0 (e^{(\lambda_4 + \Lambda)L} - 1)}{\lambda_4 + \Lambda} \right\} + \Lambda \pi_0 e^{-\Lambda L} L \\
&+ \frac{\Lambda \lambda_4 \pi_0 e^{\lambda_4 L} (1 - e^{-(\lambda_4 + \Lambda)L})}{(\lambda_4 + \Lambda)^2} - \frac{\Lambda \lambda_4 \pi_0 e^{-\Lambda L} L}{\lambda_4 + \Lambda}.
\end{aligned}$$

$$\pi_3 = \pi_0 e^{\lambda_4 L}.$$

$$\pi_4 = \frac{\Lambda \pi_0 e^{\lambda_4 L} (1 - e^{-\lambda_4 L})}{\lambda_4}.$$

$$\pi_5 = \Lambda \pi_0 e^{\lambda_4 L} \left\{ L - \frac{1 - e^{-\lambda_4 L}}{\lambda_4} \right\}.$$

$$\mathbb{E}[W_0^a] = L - \frac{\Lambda \pi_0 e^{\lambda_4 L}}{\pi_4} \left\{ -\frac{Le^{-\lambda_4 L}}{\lambda_4} + \frac{1 - e^{-\lambda_4 L}}{\lambda_4^2} \right\}.$$

To determine the arrival rate λ_4 in equilibrium, we solve the fixed-point equation:

$$\lambda_4 = \Lambda \left(\frac{R - p}{\beta \mathbb{E}[W_0^a]} \right).$$

$$\pi_0^d = \Lambda \pi_0.$$

$$\pi_1^d = \Lambda \pi_0 (e^{\lambda_4 L} - 1).$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\Lambda} + \pi_1^d \frac{3}{\Lambda}.$$

A.29. No Information ($r = 0, Q = 4, \mathcal{B} = 1$)

$$p_{0,1}(y_1) = \lambda \pi_0 e^{-\lambda y_1} e^{\lambda L}.$$

$$p_{1,1}(x_1, y_1) = \lambda^2 \pi_0 e^{\lambda x_1} e^{-\lambda y_1} e^{\lambda L}.$$

$$\pi_1 = \pi_0 e^{\lambda L}.$$

$$\begin{aligned} \pi_2 &= \frac{\lambda \pi_0}{2} \left\{ -\frac{L e^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\} + \frac{\lambda \pi_0}{4} \left\{ \frac{e^{\lambda L} - 1}{\lambda} - \frac{1 - e^{-\lambda L}}{\lambda} \right\} \\ &+ \frac{\pi_0 (e^{\lambda L} + e^{-\lambda L})}{2} + \frac{\lambda \pi_0}{2} \left\{ e^{-\lambda L} L + \frac{e^{\lambda L} (1 - e^{-2\lambda L})}{2\lambda} \right\}. \end{aligned}$$

$$\pi_3 = \pi_0 e^{\lambda L}.$$

$$\pi_4 = \pi_0 e^{\lambda L} (1 - e^{-\lambda L}).$$

$$\pi_5 = \lambda \pi_0 e^{\lambda L} \left\{ L - \frac{1 - e^{-\lambda L}}{\lambda} \right\}.$$

$$\mathbb{E}[W_0^a] = L - \frac{\lambda\pi_0 e^{\lambda L}}{\pi_4} \left\{ -\frac{Le^{-\lambda L}}{\lambda} + \frac{1 - e^{-\lambda L}}{\lambda^2} \right\}.$$

$$\pi_0^d = \lambda\pi_0.$$

$$\pi_1^d = \lambda\pi_0(e^{\lambda L} - 1).$$

$$\mathcal{C} = L + \pi_0^d \frac{4}{\lambda} + \pi_1^d \frac{3}{\lambda}.$$

A.30. ($r = 1, Q = 4, \mathcal{B} = 0$)

The solution procedures are exactly the same as in Section A.29, while now customers are not allowed to wait. Therefore, $\lambda_n = \Lambda$.

APPENDIX B: EXPERIMENTS

Table B.1. Experiment Setups.

Setup	β	R	p	h	L	K
1	1	0.75	0.25	0.1	1	1
2	1.5	0.75	0.25	0.1	1	1
3	1	1	0.25	0.1	1	1
4	1.5	1	0.25	0.1	1	1
5	1	0.75	0.5	0.1	1	1
6	1.5	0.75	0.5	0.1	1	1
7	1	1	0.5	0.1	1	1
8	1.5	1	0.5	0.1	1	1
9	1	0.75	0.25	0.2	1	1
10	1.5	0.75	0.25	0.2	1	1
11	1	1	0.25	0.2	1	1
12	1.5	1	0.25	0.2	1	1
13	1	0.75	0.5	0.2	1	1
14	1.5	0.75	0.5	0.2	1	1
15	1	1	0.5	0.2	1	1
16	1.5	1	0.5	0.2	1	1
17	1	0.75	0.25	0.1	1.5	1
18	1.5	0.75	0.25	0.1	1.5	1
19	1	1	0.25	0.1	1.5	1
20	1.5	1	0.25	0.1	1.5	1
21	1	0.75	0.5	0.1	1.5	1
22	1.5	0.75	0.5	0.1	1.5	1
23	1	1	0.5	0.1	1.5	1
24	1.5	1	0.5	0.1	1.5	1
25	1	0.75	0.25	0.2	1.5	1

Table B.1. Experiment Setups (cont.)

Setup	β	R	p	h	L	K
26	1.5	0.75	0.25	0.2	1.5	1
27	1	1	0.25	0.2	1.5	1
28	1.5	1	0.25	0.2	1.5	1
29	1	0.75	0.5	0.2	1.5	1
30	1.5	0.75	0.5	0.2	1.5	1
31	1	1	0.5	0.2	1.5	1
32	1.5	1	0.5	0.2	1.5	1
33	1	0.75	0.25	0.1	1	2
34	1.5	0.75	0.25	0.1	1	2
35	1	1	0.25	0.1	1	2
36	1.5	1	0.25	0.1	1	2
37	1	0.75	0.5	0.1	1	2
38	1.5	0.75	0.5	0.1	1	2
39	1	1	0.5	0.1	1	2
40	1.5	1	0.5	0.1	1	2
41	1	0.75	0.25	0.2	1	2
42	1.5	0.75	0.25	0.2	1	2
43	1	1	0.25	0.2	1	2
44	1.5	1	0.25	0.2	1	2
45	1	0.75	0.5	0.2	1	2
46	1.5	0.75	0.5	0.2	1	2
47	1	1	0.5	0.2	1	2
48	1.5	1	0.5	0.2	1	2
49	1	0.75	0.25	0.1	1.5	2
50	1.5	0.75	0.25	0.1	1.5	2
51	1	1	0.25	0.1	1.5	2
52	1.5	1	0.25	0.1	1.5	2

Table B.1. Experiment Setups (cont.)

Setup	β	R	p	h	L	K
53	1	0.75	0.5	0.1	1.5	2
54	1.5	0.75	0.5	0.1	1.5	2
55	1	1	0.5	0.1	1.5	2
56	1.5	1	0.5	0.1	1.5	2
57	1	0.75	0.25	0.2	1.5	2
58	1.5	0.75	0.25	0.2	1.5	2
59	1	1	0.25	0.2	1.5	2
60	1.5	1	0.25	0.2	1.5	2
61	1	0.75	0.5	0.2	1.5	2
62	1.5	0.75	0.5	0.2	1.5	2
63	1	1	0.5	0.2	1.5	2
64	1.5	1	0.5	0.2	1.5	2

Table B.2. Experiment Results.

Setup	Q^*	r^*	$\mathcal{B}^*$	$Info^*$	Λ_{eff}	$\mathbb{E}[(IN)^+]$	$\mathbb{E}[-(IN)^+]$	$\mathcal{N}$
1	4	0	3	F/N	0.994	1.62	0.123	0.249
2	4	0	3	N	0.994	1.62	0.123	0.249
3	4	0	3	F/P/N	0.994	1.62	0.123	0.249
4	4	0	3	F/N	0.994	1.62	0.123	0.249
5	4	0	3	N	0.994	1.62	0.123	0.249
6	4	0	3	N	0.994	1.62	0.123	0.249
7	4	0	3	F/N	0.994	1.62	0.123	0.249
8	4	0	3	N	0.994	1.62	0.123	0.249
9	3	0	2	P	0.956	1.167	0.144	0.319
10	3	0	1	N	0.891	1.219	0.109	0.297
11	3	0	2	F/P/N	0.967	1.152	0.152	0.322

Table B.2. Experiment Results (cont.)

Setup	Q^*	r^*	$\mathcal{B}^*$	$Info^*$	Λ_{eff}	$\mathbb{E}[(IN)^+]$	$\mathbb{E}[-(IN)^+]$	$\mathcal{N}$
12	3	0	2	P	0.956	1.167	0.144	0.319
13	4	0	3	N	0.994	1.62	0.123	0.249
14	4	0	3	N	0.994	1.62	0.123	0.249
15	3	0	2	P	0.956	1.167	0.144	0.319
16	4	0	3	N	0.994	1.62	0.123	0.249
17	4	0	3	N	0.978	1.268	0.267	0.244
18	4	0	3	N	0.978	1.268	0.267	0.244
19	4	0	3	N	0.978	1.268	0.267	0.245
20	4	0	3	N	0.978	1.268	0.267	0.244
21	4	0	3	N	0.937	1.312	0.245	0.234
22	4	0	1	N	0.815	1.492	0.143	0.204
23	4	0	3	N	0.978	1.268	0.267	0.244
24	4	0	3	N	0.978	1.268	0.267	0.244
25	4	0	3	N	0.978	1.268	0.267	0.244
26	4	0	3	N	0.978	1.268	0.267	0.244
27	3	0	2	P	0.896	0.881	0.28	0.299
28	4	0	3	N	0.978	1.268	0.268	0.244
29	4	0	2	N	0.933	1.302	0.234	0.233
30	4	0	1	N	0.815	1.492	0.143	0.204
31	4	0	3	N	0.978	1.268	0.267	0.244
32	4	0	3	N	0.978	1.268	0.267	0.244
33	4	0	1	N	0.916	1.711	0.084	0.229
34	4	0	1	N	0.916	1.711	0.084	0.229
35	4	0	1	N	0.916	1.711	0.084	0.229
36	4	0	1	N	0.916	1.711	0.084	0.229
37	4	0	3	N	0.994	1.620	0.123	0.249
38	4	0	3	N	0.994	1.620	0.123	0.249

Table B.2. Experiment Results (cont.)

Setup	Q^*	r^*	$\mathcal{B}^*$	$Info^*$	Λ_{eff}	$\mathbb{E}[(IN)^+]$	$\mathbb{E}[(-IN)^+]$	$\mathcal{N}$
39	4	0	3	F/N	0.994	1.620	0.123	0.249
40	4	0	3	N	0.994	1.620	0.123	0.249
41	4	0	2	N	0.975	1.628	0.115	0.244
42	4	0	2	N	0.975	1.628	0.115	0.244
43	4	0	2	N	0.975	1.628	0.115	0.244
44	4	0	2	N	0.975	1.628	0.115	0.244
45	4	0	3	N	0.994	1.62	0.123	0.249
46	4	0	3	N	0.994	1.62	0.123	0.249
47	4	0	3	F/P/N	0.994	1.62	0.123	0.249
48	4	0	3	N	0.994	1.62	0.123	0.249
49	4	0	1	N	0.847	1.459	0.153	0.212
50	4	0	1	N	0.847	1.459	0.153	0.212
51	4	0	1	N	0.847	1.459	0.153	0.212
52	4	0	1	N	0.847	1.459	0.153	0.212
53	4	0	2	N	0.933	1.302	0.234	0.233
54	4	0	2	N	0.769	1.486	0.16	0.192
55	4	0	3	N	0.978	1.268	0.267	0.244
56	4	0	3	N	0.978	1.268	0.267	0.244
57	4	0	2	N	0.934	1.300	0.235	0.233
58	4	0	2	N	0.934	1.300	0.235	0.233
59	4	0	2	N	0.934	1.300	0.235	0.233
60	4	0	2	N	0.934	1.300	0.235	0.233
61	4	0	2	N	0.933	1.302	0.234	0.233
62	4	0	2	N	0.769	1.486	0.160	0.192
63	4	0	3	N	0.978	1.268	0.267	0.244
64	4	0	3	N	0.978	1.268	0.267	0.244

