

INTRANSITIVE INDIFFERENCE UNDER UNCERTAINTY

A Master's Thesis

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INTRANSITIVE INDIFFERENCE UNDER UNCERTAINTY

Graduate School of Economics and Social Sciences
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by

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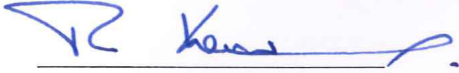
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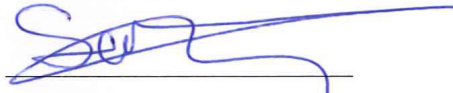
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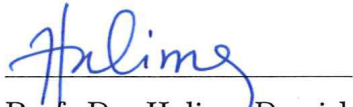
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ABSTRACT

INTRANSITIVE INDIFFERENCE UNDER UNCERTAINTY

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July 2017

We study preferences with intransitive indifference under uncertainty. Our primitive objects are semiorders and we are interested in their Scott-Suppes representations. We obtain a Scott-Suppes representation theorem in the spirit of the celebrated expected utility theorem of von Neumann and Morgenstern (1944).

Keywords: Decision Under Uncertainty, Expected Utility, Intransitive Indifference, Scott-Suppes Representation, Semiorder

ÖZET

BELİRSİZLİK ALTINDA GEÇİŞKEN OLMAYAN KAYITSIZLIKLAR

Dokumacı, Oral Ersoy

Yüksek Lisans, İktisat Bölümü

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Temmuz 2017

Bu çalışmada belirsizlik altında geçişken olmayan kayıtsızlıklar incelenmektedir. Çalışmamızın ana konusunu yarı-sıralamalar ve onların Scott-Suppes gösterimleri oluşturmaktadır. Von Neumann ve Morgenstern'in (1944) beklenen fayda teorimine benzer bir şekilde yarı-sıralamaların Scott-Suppes gösterimleri elde edilmektedir.

Anahtar Kelimeler: Beklenen Fayda, Belirsizlik Altında Karar Alma, Geçişken Olmayan Kayıtsızlıklar, Scott-Suppes Gösterimi, Yarı-sıralamalar

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CHAPTER 1

INTRODUCTION

1.1 Intransitive Indifference and Semiorders

The standard rationality assumption in economic theory argues that the individuals have or should have transitive preferences.¹ A common argument to support the transitivity requirement is that, if the economic agents do not have transitive preferences, then they are subject to money pumps (Fishburn, 1991). Yet, intransitivity of preferences is observed in individual choices through experiments and data (May, 1954; Tversky, 1969).

Intransitive indifference deals with a certain type of intransitivity of preferences. It allows an individual to be indifferent between x and y and also y and z , but not necessarily between x and z .

Formal studies of the idea of intransitive indifference go back to as early as 19th century (Weber, 1834; Fechner, 1860).² The Weber-Fechner law argues that the change in a physical stimulus and the perceived change are related. A very small increase in the actual stimulus, may not result in a change in the perceived stimulus, which suggests intransitivity of perceptual abilities.

A notable example (as translated by Pirlot and Vincke (2013:19)) is given by the famous polymath Jules Henri Poincaré (1905):

Sometimes we are able to make the distinction between two sensations while we cannot distinguish them from a third

¹If an individual thinks that some good x is at least as desirable as good y and y is at least as desirable as some other good z , then x should be at least as desirable as z .

²Although Weber's and Fechner's laws are both on human perception, there are some differences among them. We do not go into details here.

sensation. For example, we can easily make the distinction between a weight of 12 grams and a weight of 10 grams, but we are not able to distinguish each of them from a weight of 11 grams. This fact can symbolically be written: $A = B$, $B = C$, $A < C$.³

Perhaps the most known example of intransitive indifference in economics is due to Luce (1956). Suppose an individual prefers a cup of coffee with one cube of sugar to a cup of coffee with five cubes of sugar. Now, let x be the weight of one cube of sugar in grams. Make four hundred and one cups of coffee, where each labeled with $i = 0, 1, \dots, 400$ and have $(1 + i/100)x$ grams of sugar in the i^{th} cup. Since the increase from one cup to next is so small, this individual is indifferent between any consecutive cups, i and $i + 1$. However, she is not indifferent between the 0^{th} and 400^{th} cups.

Although the mathematical study of intransitive indifference goes back to Wiener (1914), an excellent way to capture the idea of intransitive indifference is introduced by Luce (1956). He coined the term semiorder⁴, which is a binary relation satisfying some conditions that allow for intransitive indifference. Since then, the concept of semiorders has been studied extensively in preference, choice, and utility theory.

1.2 Expected Utility Theory

One of the most fruitful branches of modern economic theory, which has emerged from the seminal work of von Neumann and Morgenstern (1944), has been the decision making under uncertainty. In many fields, including decision theory, microeconomic and macroeconomic theory, game theory, and financial economics, the celebrated expected utility theorem of von Neumann and Morgenstern (1944) has helped in explaining how individuals behave when they face uncertainty.

The properties that a decision maker's preferences have to satisfy in order for the decision maker to act as if having an expected utility function are challenged by many. Some of these properties are often modified or removed in order to explain other types of behavior that are observed frequently in different

³We include the rest of the translated passage in the Appendix.

⁴He originally named such a relation as a "semiordering" but they are usually referred to as semiorders.

economic settings. With similar purposes, in this work we relax the transitivity axiom and try to understand and characterize the behavior of individuals, who have preferences with intransitive indifference, under uncertainty.

1.3 Literature Review

The behavior we are interested in is often discussed in various contexts when modeling bounded rationality such as satisficing behavior and ϵ -equilibrium (Simon, 1956; Radner, 1980). What is unifying in such models is that the decision maker's preferences demonstrate a weaker form of transitivity. In this study we identify a “natural” way of representing preferences with intransitive indifference over the set of lotteries. The intuition behind our representation resembles that of the expected utility theorem of von Neumann and Morgenstern (1944).

Our focus is on a particular representation of semiorders that provides utility representation with a constant threshold as in Scott and Suppes (1958). This representation, which is usually referred to as Scott-Suppes representation, is initially obtained for semiorders that are defined on finite sets (Scott and Suppes, 1958). It is also shown that when semiorders are defined on a countably infinite set, then they admit a Scott-Suppes representation (Manders, 1981). As it is pointed out in Beja and Gilboa (1992), obtaining a Scott-Suppes representation for semiorders defined on uncountable sets proves rather difficult. Recently, both necessary and sufficient conditions for semiorders on uncountable sets to have a Scott-Suppes representation are also acquired (Candeal and Induráin, 2010). This result is relevant for the analytical purposes of our study, as we deal with semiorders defined on the lottery simplex, which is uncountable.

Two papers that contain the most similar analysis to our work is by Vincke (1980) and Nakamura (1988). Vincke (1980) obtains a linear utility function with a non-negative **variable** threshold representation for semiordered mixture spaces, whereas Nakamura (1988) provides a similar analysis for interval orders—a binary relation that also allows for intransitivity of indifference. Our study sharpens the result of Vincke (1980) to linear utility functions with positive **constant** threshold representations, as semiorders are generically associated with such representations (Scott-Suppes representations).

CHAPTER 2

PRELIMINARIES

We begin this section with some observations from the literature and of our own.⁵ We utilize these results in the next section.

2.1 Semiorders

Throughout this work X denotes a non-empty set. We say R is a *binary relation* on X if $R \subseteq X \times X$. Whenever for some $x, y \in X$, we have $(x, y) \in R$, we write $x R y$. Also, if $(x, y) \notin R$, we write $\neg(x R y)$. Below, we define some common properties associated with binary relations.

Definition. A binary relation R on X is

- *reflexive* if for each $x \in X$, we have $x R x$,
- *irreflexive* if for each $x \in X$, we have $\neg(x R x)$,
- *complete* if for each $x, y \in X$, we have $x R y$ or $y R x$,
- *symmetric* if for each $x, y \in X$, $x R y$ implies $y R x$,
- *asymmetric* if for each $x, y \in X$, $x R y$ implies $\neg(y R x)$,
- *transitive* if for each $x, y, z \in X$, $x R y$ and $y R z$ imply $x R z$.

Let R be a reflexive binary relation on X and $x, y \in X$. We define the *strict part of R* , denoted P , as $x P y$ if $x R y$ and $\neg(y R x)$. Whereas, we define the *indifference part of R* , denoted I , as $x I y$ if $x R y$ and $y R x$.

⁵We refer interested readers for further details to the following: Fishburn (1970a), Beja and Gilboa (1992), Candeal and Induráin (2010), Fishburn (1970b), Kreps (1988), Ok (2007), Aleskerov et al. (2007), Pirlot and Vincke (2013).

A binary relation R on X is a *weak order* if it is complete and transitive, and is an *equivalence relation* if it is reflexive, symmetric, and transitive. We state without a proof that if R is a weak order on X , then I is an equivalence relation on X .

Definition (Semiorde). Let P and I be two binary relations on X . The pair (P, I) is a *semiorde* on X if

- I is reflexive (*reflexivity*),
- for each $x, y \in X$, exactly one of $x P y$, $y P x$, or $x I y$ holds (*trichotomy*),
- for each $x, y, z, t \in X$, $x P y$, $y I z$, $z P t$ imply $x P t$ (*strong intervality*),
- for each $x, y, z, t \in X$, $x P y$, $y P z$, $z I t$ imply $x P t$ (*semitransitivity*).

The definition above is slightly different than the definition of semiorde introduced by Luce (1956). Since both definitions are equivalent,⁶ our analysis remains unaffected. Now, we provide some immediate observations for semiorde.

- Claim 1: P is irreflexive. Since I is *reflexive*, by *trichotomy*, for each $x \in X$, we have $\neg(x P x)$.
- Claim 2: I is symmetric. Suppose for some $x, y \in X$, we have $x I y$. By *trichotomy*, we cannot have $y P x$ and $x P y$. Hence, $y I x$.
- Claim 3: P is asymmetric. Suppose for some $x, y \in X$, we have $x P y$. By *trichotomy*, we cannot have $y P x$. Hence, $\neg(y P x)$.
- Claim 4: P is transitive. Suppose for some $x, y, z \in X$, we have $x P y P z$.⁷ Since I is *reflexive*, $x P y I y P z$. By *strong intervality*, we have $x P z$.
- Claim 5: $x I y$ if and only if $\neg(x P y)$ and $\neg(y P x)$. Directly follows from *trichotomy*.
- Claim 6: Every weak order is a semiorde. Directly follows since for every *weak order* R , both P and I are *transitive*.

⁶We show the equivalence of both definitions in the Appendix.

⁷ $x P y P z$ is a shorthand notation for $x P y$ and $y P z$. We use this notational convention throughout.

Hence, it is useful to keep in mind that every result that applies to semiorders directly applies to weak orders as well. Similarly, any (counter) example of a weak order showing that a certain statement about weak orders is not true, also implies that this statement is not true for semiorders either.

Example 1. We give an example of a canonical semiorder.

Let $x, y \in \mathbb{R}$ and define (P, I) on $\mathbb{R}$ as follows:

- $x P y$ if $x > y + 1$,
- $x I y$ if $|x - y| \leq 1$.

It is straightforward to see that I is *reflexive* and (P, I) satisfies *trichotomy*.

Let $x, y, z, t \in \mathbb{R}$. Suppose $x P y I z P t$. This implies $x > y + 1$ and $|y - z| \leq 1$. Hence, $x > z$. Moreover, $z P t$ implies $z > t + 1$. Thus, $x > t + 1$, which implies $x P t$. So, (P, I) satisfies *strong intervality*. It is similar to show that (P, I) is *semitransitive*. Therefore, (P, I) is a *semiorder* on $\mathbb{R}$.

If (P, I) on $\mathbb{R}$ in Example 1 were a weak order, then I would be transitive. Yet, we have $0 I 1$ and $1 I 2$ but $2 P 0$. Therefore, not every semiorder is a weak order.

Looking at the semiorder definition, one might wonder why we do not have the following axiom:

- for each $x, y, z, t \in X$, $x I y, y P z, z P t$ imply $x P t$ (*reverse semitransitivity*).⁸

It turns out that for a pair of binary relations (P, I) on X , if I is reflexive and (P, I) satisfies trichotomy and strong intervality,⁹ then (P, I) is semitransitive if and only if it is reverse semitransitive.

In order to see that, let (P, I) on X be a *semiorder*. Suppose there exist $x, y, z, t \in X$ such that $x I y P z P t$ but $\neg(x P t)$. By *trichotomy*, $x I t$ or $t P x$.

⁸As far as we know, there is not a common name for this axiom. Strong intervality is also referred to as pseudotransitivity (Bridges, 1983). Strong intervality, semitransitivity, and reverse semitransitivity are together referred to as generalized pseudotransitivity in Gensemer (1987). It is also worth noting that strong intervality and semitransitivity axioms are usually given in terms of an irreflexive binary relation P . We again show the equivalence of these axioms stated for P or a pair (P, I) in the Appendix.

⁹Such a relation is called an *interval order*, which is introduced by Fishburn (1970a).

If $x I t$, then $y P z P t I x$, which by *semitransitivity*, imply $y P x$. This contradicts with $x I y$. If, on the other hand, $t P x$, then we have $y P z P t P x$, which imply $y P x$. But we also have $x I y$, again a contradiction. In order to see the reverse, suppose (P, I) on X is *reflexive*, satisfies *trichotomy* and *strong intervality*, and is *reverse semitransitive*. Moreover, suppose there exist $x, y, z, t \in X$ such that $x P y P z I t$ but $\neg(x P t)$. By *trichotomy*, $x I t$ or $t P x$. If $x I t$, then $t I x P y P z$, which by *reverse semitransitivity* imply $t P z$. But $z I t$, which is a contradiction. If, on the other hand, $t P x$, then $t P x P y P z$, which imply $t P z$. This contradicts with $z I t$, which completes our proof.

Definition. Let (P, I) be a couple of binary relations on X that satisfies trichotomy and $x, y \in X$. We define the following binary relations on X :

- $x R y$ if $\neg(y P x)$ (i.e., $x P y$ or $x I y$),
- $x P_0 y$ if there exists $z \in X$ such that $[x P z \text{ and } z R y]$ or $[x R z \text{ and } z P y]$,
- $x R_0 y$ if $\neg(y P_0 x)$,
- $x I_0 y$ if $x R_0 y$ and $y R_0 x$.¹⁰

It follows from the definition of R_0 that we have:

- $x R_0 y$ if and only if for each $z \in X$, $[y R z \text{ implies } x R z]$ and $[z R x \text{ implies } z R y]$,
- $x R_0 y$ if and only if for each $z \in X$, $[y P z \text{ implies } x P z]$ and $[z P x \text{ implies } z P y]$,

In the rest of the study, we refer to a semiorder (P, I) on X simply as a semiorder R on X .

Lemma 1. Let R be a semiorder on X and $x, y, z \in X$. If $x R_0 y P z$ or $x P y R_0 z$, then $x P z$.

Proof. Let R be a semiorder on X and $x, y, z \in X$. Suppose $x R_0 y P z$ but $z R x$. Since $x R_0 y$ and $z R x$, we have $z R y$. This contradicts with $y P z$. Now, suppose $x P y R_0 z$ but $z R x$. Because $y R_0 z$ and $z R x$, we have $y R x$, which contradicts with $x P y$. □

¹⁰We follow the notation of Candeal and Induráin (2010).

We give a slightly modified version of an essential result of Luce (1956).¹¹

Proposition 1. *If R is a semiorder on X , then R_0 is a weak order on X .*

Proof. Let R be a semiorder on X .

- **Claim 1: R_0 is complete.** Suppose on the contrary that there exist $x, y \in X$ such that $\neg(x R_0 y)$ and $\neg(y R_0 x)$. This implies that there exist $z, z' \in X$ such that $[x R z P y \text{ or } x P z R y]$ and $[y R z' P x \text{ or } y P z' R x]$.

Case 1: $x R z P y$ and $y R z' P x$. Since $z P y R z' P x$, by *strong intervality* and the fact that P is *transitive*, we have $z P x$. This contradicts with $x R z$.

Case 2: $x R z P y$ and $y P z' R x$. Since $x R z P y P z'$, by *reverse semitransitivity* and the fact that P is *transitive*, we have $x P z'$. This contradicts with $z' R x$.

Case 3: $x P z R y$ and $y R z' P x$. Since $z' P x P z R y$, by *semitransitivity* and the fact that P is *transitive*, we have $z' P y$. This contradicts with $y R z'$.

Case 4: $x P z R y$ and $y P z' R x$. Since $x P z R y P z'$, by *strong intervality* and the fact that P is *transitive*, we have $x P z'$. This contradicts with $z' R x$.

- **Claim 2: R_0 is transitive.** Let $x, y, z, t \in X$. Suppose $x R_0 y R_0 z$. Since $x R_0 y$ and $t R x$, we have $t R y$. Similarly, since $y R_0 z$ and $t R y$, we have $t R z$. Moreover, if $z R t$, then by $y R_0 z$, we have $y R t$. Similarly, since $x R_0 y$ and $y R t$, we have $x R t$. Hence, $x R_0 z$.

Therefore, R_0 is a weak order on X . □

Corollary. If R is a semiorder on X , then I_0 is an equivalence relation on X .

In Proposition 1 to prove that R_0 is complete, we invoke semitransitivity (and reverse semitransitivity). But can we do better, that is, given a binary relation R on X that is reflexive and that satisfies trichotomy and strong intervality, is the associated R_0 on X complete? The answer is negative, as we show with an example.¹²

¹¹See Theorem 1 in Luce (1956).

¹²In fact, this is an example of an interval order.

Example 2. Define R on $[0, 1]$ such that:

- for each $x, y \in (0, 1]$, we have $x P y$ if $x > y$ and $x I y$ if $x = y$,
- for each $x \in [0, 1]$, we have $0 I x$.

It is easy to show that R is *reflexive* and satisfies *trichotomy*. Let $x, y, z, t \in [0, 1]$. We claim if $x P y I z P t$, then $x P t$. Suppose $x P y I z P t$. Since for each $w \in [0, 1]$, we have $0 I w$, it follows $y \neq 0$ and $t \neq 0$. Moreover, if $x = 0$ or $z = 0$, then the claim vacuously holds. Hence, suppose $x, y, z, t \in (0, 1]$. This implies $x > y = z > t$. Thus, $x P t$. So, R satisfies *strong intervality*. Finally, since $0.5 P 0.1 I 0$ and $0 I 1 P 0.5$, we have $0.5 P_0 0$ and $0 P_0 0.5$. Hence, $\neg(0 R_0 0.5)$ and $\neg(0.5 R_0 0)$. Therefore, R_0 is not *complete*.

2.2 Continuity

From this point on, $X = \{x_1, x_2, \dots, x_n\}$ denotes a set with $n \in \mathbb{N}$ alternatives. A *lottery* on X is a list $p = (p_1, p_2, \dots, p_n)$ such that $\sum p_i = 1$ and for each $i \in \{1, 2, \dots, n\}$, we have $p_i \geq 0$, where x_i occurs with probability p_i . We denote the set of all lotteries on X as L . It is easy to see that for each lottery $p, q \in L$ and $\alpha \in (0, 1)$, we have $\alpha p + (1 - \alpha)q \in L$. We endow L with the standard metric of $\mathbb{R}^n$.

Definition. A reflexive binary relation R on L is

- *continuous* if for each $q \in L$, the sets

$$UC(q) := \{p \in L : p R q\} \text{ and } LC(q) := \{p \in L : q R p\}$$

are closed,

- *mixture-continuous* if for each $p, q, r \in L$, the sets

$$UMC(q; p, r) := \{\alpha \in [0, 1] : [\alpha p + (1 - \alpha)r] R q\}$$

and

$$LMC(q; p, r) := \{\alpha \in [0, 1] : q R [\alpha p + (1 - \alpha)r]\}$$

are closed.

Lemma 2. If a semiorder R on L is continuous, then it is mixture-continuous.

Proof. Let R be a *continuous semiorder* on L . Suppose for each $n \in \mathbb{N}$, we have $\alpha_n \in UMC(q; p, r)$ and $(\alpha_n) \rightarrow \alpha$. Since $[0, 1]$ is closed, we have $\alpha \in [0, 1]$. Moreover, because for each $n \in \mathbb{N}$, we have that $[\alpha_n p + (1 - \alpha_n)r] \in UC(q)$ and $UC(q)$ is closed, it follows $[\alpha p + (1 - \alpha)r] \in UC(q)$. Hence, $UMC(q; p, r)$ is closed. It is similar to show that $LMC(q; p, r)$ is closed. $\square$

Next, we investigate the relationship between R on L and the associated weak order R_0 on L in terms of continuity and mixture-continuity.

Example 3. We first provide an example of a continuous semiorder where its associated weak order is **not** mixture-continuous.

Define R on $[0, 1]$ such that:

- for each $p \in [0, 1]$, we have $0.5 I p$,
- for each $p, p' \in (0.5, 1]$ and $q, q' \in [0, 0.5)$, we have $p I p'$, $p P q$, and $q I q'$.

It is straightforward to show that R is *reflexive* and satisfies *trichotomy*.

Moreover, since there do not exist $p, q, r, s \in [0, 1]$ such that $p P q I r P s$ or $p P q P r I s$, R vacuously satisfies *strong intervality* and *semitransitivity*.

Hence, R is a *semiorder*. Furthermore, $UC(0.5) = LC(0.5) = [0, 1]$ and for each $p \in (0.5, 1], q \in [0, 0.5)$, we have $UC(p) = [0.5, 1]$, $LC(p) = [0, 1]$,

$UC(q) = [0, 1]$, $LC(q) = [0, 0.5]$. Thus, R is *continuous*. Finally, let

$p, p' \in (0.5, 1], q \in [0, 0.5)$. Since $p P q$, we have $p P_0 q$. Also $p I_0 p'$. Moreover, since $p P q I 0.5$, we have $p P_0 0.5$. So,

$$UMC_0(1; 0, 1) := \{\alpha \in [0, 1] : [\alpha 0 + (1 - \alpha)1] R_0 1\} = [0, 0.5),$$

which is not closed. Therefore, R_0 is not *mixture-continuous*.

Example 4. We now give an example of a semiorder where its associated weak order is continuous but the semiorder itself is **not** mixture-continuous.

Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$, $p, q \in L$, and $\epsilon \in (0, 0.5]$. We define R on L such that:

- $p P q$ if $p_1 \geq q_1 + \epsilon$,
- $p I q$ if $\neg(p P q)$ and $\neg(q P p)$.

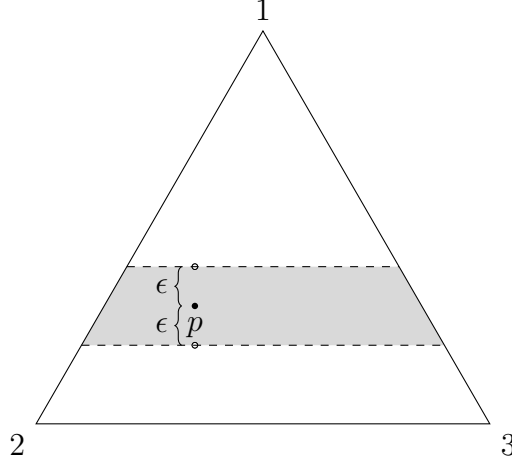


Figure 1: Example 4
Lottery p is indifferent to every lottery in the gray area.

Since $\epsilon > 0$, for any $p, q \in L$, both of $p P q$ and $q P p$ cannot hold. This together with the definition of I give *reflexivity* and *trichotomy*. Moreover, let $p, q, r, s \in L$. Suppose $p P q I r P s$. This implies $p_1 \geq q_1 + \epsilon$ and $|q_1 - r_1| < \epsilon$. Hence, $p_1 > r_1$. Furthermore, $r P s$ implies $r_1 \geq s_1 + \epsilon$. Thus, $p_1 > s_1 + \epsilon$, which in turn implies $p P s$. So, R satisfies *strong intervality*. It is similar to show that R is *semitransitive*. Therefore, R is a *semiorder* on L . Moreover, it is straightforward to show that for each $p, q \in L$, we have $p R_0 q$ if and only if $p_1 \geq q_1$. Hence, R_0 is *continuous*. Finally, consider $UMC((1, 0, 0); (1 - \epsilon, \epsilon/2, \epsilon/2), (1, 0, 0))$. It is easy to see that

$$UMC((1, 0, 0); (1 - \epsilon, \epsilon/2, \epsilon/2), (1, 0, 0)) = (0, 1],$$

which is not closed. Therefore, R is not *mixture-continuous*.

2.3 Independence Axioms

Definition. A reflexive binary relation R on L satisfies the

- *independence axiom* if for each $p, q, r \in L$ and $\alpha \in (0, 1)$, $p R q$ if and only if $[\alpha p + (1 - \alpha)r] R [\alpha q + (1 - \alpha)r]$,
- *midpoint indifference axiom* if for each $p, q, r \in L$, $p I q$ implies $[1/2p + 1/2r] I [1/2q + 1/2r]$.

It is easy to show that if a weak order R on L satisfies the independence axiom,

then it satisfies the midpoint indifference axiom, and for each $p, q, r \in L$ and $\alpha \in (0, 1)$ we have

- $p P q$ if and only if $[\alpha p + (1 - \alpha)r] P [\alpha q + (1 - \alpha)r]$,
- $p I q$ if and only if $[\alpha p + (1 - \alpha)r] I [\alpha q + (1 - \alpha)r]$.

Below, we show that a semiorder satisfying the independence axiom cannot have intransitive indifference.¹³ Hence, the study of semiorders satisfying the independence axiom is equivalent to the study of weak orders satisfying the independence axiom.

Proposition 2. *Let R be a semiorder on L . If R satisfies the independence axiom, then I is transitive.*

Proof. Let R be a semiorder on L that satisfies the independence axiom.

Suppose there exist $p, q, r \in L$ such that $p I q I r$ but $p P r$. The independence axiom and $p P r$ together imply that for each $\alpha \in (0, 1)$, we have $p P [\alpha p + (1 - \alpha)r] P r$. Since $p P [\alpha p + (1 - \alpha)r] P r I q$, by semitransitivity, we have $p P q$. This contradicts with $p I q$. $\square$

2.4 Utility Representations

Let P be an irreflexive binary relation on X . We say $u : X \rightarrow \mathbb{R}$ is a *utility representation* of P if for each $x, y \in X$, $x P y$ if and only if $u(x) > u(y)$. As semiorders allow for intransitivity of indifference, a convenient representation that reflects this fact is as follows.

Definition (Scott-Suppes Representation). Let P be an irreflexive binary relation on X , $u : X \rightarrow \mathbb{R}$ be a function, and $k \in \mathbb{R}_{++}$. We say (u, k) is a *Scott-Suppes representation* of P if for each $x, y \in X$, $x P y$ if and only if $u(x) > u(y) + k$.

Here k acts as a threshold of utility discrimination, that is, if the absolute value of the utility difference of two alternatives is less than or equal to k , then it is as if the decision maker cannot consider these two alternatives to be significantly different from each other. Equivalently, one can think that for the

¹³See also Fishburn (1968).

decision maker to strictly prefer one alternative over the other, there is a certain utility threshold to be exceeded.

Let $u : L \longrightarrow \mathbb{R}$ be a function. We say u is *linear* if for each $p, q \in L$ and for each $\alpha \in [0, 1]$, we have $u(\alpha p + (1 - \alpha)q) = \alpha u(p) + (1 - \alpha)u(q)$.

We now state an important theorem that we utilize in our main result.

Theorem 1 (Vincke (1980)). *Let (P, I) be a pair of binary relations on L .*

- (P, I) is a *semiorder*,
- R_0 is *mixture-continuous* and satisfies the *midpoint indifference axiom*,
and
- for each $p, q \in L$, if $p P q$, then there exists $p' \in L$ such that $p I p'$ and for each $r \in L$, we have $p' P_0 r$ implies $p P r$

if and only if there exist a linear function $u : L \longrightarrow \mathbb{R}$ and a non-negative real function $\sigma : L \longrightarrow \mathbb{R}_+$ such that for each $p, q \in L$, we have

- $p P q$ if and only if $u(p) > u(q) + \sigma(q)$,
- $p I q$ if and only if $u(p) + \sigma(p) \geq u(q)$ and $u(q) + \sigma(q) \geq u(p)$,
- $p I_0 q$ if and only if $u(p) = u(q)$,
- $u(p) > u(q)$ implies $u(p) + \sigma(p) \geq u(q) + \sigma(q)$,
- $u(p) = u(q)$ implies $\sigma(p) = \sigma(q)$.

CHAPTER 3

A REPRESENTATION THEOREM

A reflexive binary relation R on X is *non-trivial* if there exist $x, y \in X$ such that $x P y$. We say $x \in X$ is *maximal* with respect to R if for each $y \in X$ we have $x R y$. We denote the set of all maximal elements of X with respect to R as M_R .

Definition. A binary relation R on L is *regular* if there are no $p, q \in L$ and no sequences $(p_n), (q_n) \in L^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $p R p_n$ and $p_{n+1} R p_n$ or for each $n \in \mathbb{N}$, we have $q_n R q$ and $q_n R q_{n+1}$.

Regularity (of the strict part of a semiorder) is a necessary condition for Scott-Suppes representation as shown in Candeal and Induráin (2010). We utilize regularity and non-triviality to obtain utility representation with positive threshold instead of non-negative threshold.

Definition. A reflexive binary relation R on L is *mixture-symmetric* if for each $p, q \in L$ and $\alpha \in (0, 1)$, $p I [\alpha p + (1 - \alpha)q]$ implies $q I [\alpha q + (1 - \alpha)p]$.

This axiom is first introduced in Nakamura (1988) to obtain a utility representation with a constant threshold for interval orders. We utilize this axiom for semiorders with similar purposes.

3.1 The Theorem

Theorem 2 (Expected Scott-Suppes Utility Representation). *Let R be a non-trivial semiorder on L .*

- *R is mixture-symmetric,*

- P is regular,
- R_0 is mixture-continuous and satisfies the midpoint indifference axiom, and
- for each $p, q \in L$, if $p P q$, then there exists $p' \in L$ such that $p I p'$ and for each $r \in L$, we have $p' P_0 r$ implies $p P r$

if and only if there exists a linear function $u : L \longrightarrow \mathbb{R}$ and $k \in \mathbb{R}_{++}$ such that (u, k) is a Scott-Suppes representation of P .

Proof. We first show that the axioms imply expected Scott-Suppes utility representation.

Since all of the hypotheses of Theorem 1 are satisfied, there exist a *linear* function $u : L \longrightarrow \mathbb{R}$ and a non-negative real function $\sigma : L \longrightarrow \mathbb{R}_+$ such that for each $p, q \in L$ we have:

- $p P q$ if and only if $u(p) > u(q) + \sigma(q)$,
- $p I q$ if and only if $u(p) + \sigma(p) \geq u(q)$ and $u(q) + \sigma(q) \geq u(p)$,
- $p I_0 q$ if and only if $u(p) = u(q)$,
- $u(p) > u(q)$ implies $u(p) + \sigma(p) \geq u(q) + \sigma(q)$,
- $u(p) = u(q)$ implies $\sigma(p) = \sigma(q)$.

Our initial aim is to show that for each $p, q \in L \setminus M_R$, we have $\sigma(p) = \sigma(q) > 0$. Since R is *non-trivial*, the set of all *non-maximal* elements of X with respect to R is non-empty.

- **Claim 1:** For each $p \in L \setminus M_R$, $\sigma(p) > 0$.

Suppose on the contrary that there exists $p \in L \setminus M_R$ such that $\sigma(p) = 0$. Since p is *non-maximal*, there exists $q \in L$ such that $q P p$. We first show that for each $\alpha \in (0, 1)$, we have $\sigma(\alpha p + (1 - \alpha)q) = 0$. Suppose not, that is, there exists $\alpha \in (0, 1)$ such that $\sigma(\alpha p + (1 - \alpha)q) > 0$. If $u(\alpha p + (1 - \alpha)q) + \sigma(\alpha p + (1 - \alpha)q) \geq u(q)$, then because $u(q) > u(\alpha p + (1 - \alpha)q)$, we have $[\alpha p + (1 - \alpha)q] I q$. Hence, *mixture symmetry* implies $p I [\alpha q + (1 - \alpha)p]$. This contradicts with $u(p) + \sigma(p) = u(p) < u(\alpha q + (1 - \alpha)p)$. Thus, suppose

$u(\alpha p + (1 - \alpha)q) + \sigma(\alpha p + (1 - \alpha)q) < u(q)$. This together with the facts that $u(q) > u(p)$ and u being *linear* imply that $0 < \sigma(\alpha p + (1 - \alpha)q)/[u(q) - u(p)] < \alpha$. So,

$$0 < \beta := \alpha - \frac{\sigma(\alpha p + (1 - \alpha)q)}{u(q) - u(p)} < \alpha.$$

Again the *linearity* of u implies

$u(\beta p + (1 - \beta)q) = u(\alpha p + (1 - \alpha)q) + \sigma(\alpha p + (1 - \alpha)q)$. Since $u(\beta p + (1 - \beta)q) + \sigma(\beta p + (1 - \beta)q) \geq u(\alpha p + (1 - \alpha)q)$ and $u(\alpha p + (1 - \alpha)q) + \sigma(\alpha p + (1 - \alpha)q) \geq u(\beta p + (1 - \beta)q)$, we have $[\alpha p + (1 - \alpha)q] I [\beta p + (1 - \beta)q]$. Because

$$[\alpha p + (1 - \alpha)q] I \left[\left(\frac{\beta}{\alpha} \right) (\alpha p + (1 - \alpha)q) + \left(\frac{\alpha - \beta}{\alpha} \right) q \right],$$

mixture symmetry implies

$$q I \left[\left(\frac{\beta}{\alpha} \right) q + \left(\frac{\alpha - \beta}{\alpha} \right) (\alpha p + (1 - \alpha)q) \right] = [(1 + \beta - \alpha)q + (\alpha - \beta)p].$$

Therefore, *mixture symmetry* implies $p I [(1 + \beta - \alpha)p + (\alpha - \beta)q]$. Since $\sigma(p) = 0$ and $u((1 + \beta - \alpha)p + (\alpha - \beta)q) > u(p)$, we also have $[(1 + \beta - \alpha)p + (\alpha - \beta)q] P p$. This contradicts with *trichotomy*. Hence, for each $\alpha \in (0, 1)$, we have $\sigma(\alpha p + (1 - \alpha)q) = 0$. Now, for each $n \in \mathbb{N}$ set $\alpha_n = 1/(n + 1)$. Because, for each $\alpha \in (0, 1)$, $\sigma(\alpha p + (1 - \alpha)q) = 0$, we have $q P \cdots P [\alpha_{n+1}p + (1 - \alpha_{n+1})q] P [\alpha_n p + (1 - \alpha_n)q] P \cdots P [\alpha_1 p + (1 - \alpha_1)q]$. This contradicts with the *regularity* of P . Thus, for each $p \in L \setminus M_R$, we have $\sigma(p) > 0$.

- **Claim 2:** For each $p, q \in L \setminus M_R$, $\sigma(p) = \sigma(q)$.

Suppose on the contrary that there exist $p, q \in L \setminus M_R$ such that $\sigma(p) \neq \sigma(q)$. Since $p I_0 q$ if and only if $u(p) = u(q)$ and $u(p) = u(q)$ implies $\sigma(p) = \sigma(q)$, we have $\neg(p I_0 q)$. Hence, without loss of generality, suppose $q P_0 p$. Because q is *non-maximal*, there exists $r \in L$ such that $r P q$. Since $r P q$ and $q P_0 p$, we have $r P p$. Hence, $u(p) + \sigma(p) < u(r)$. This together with $u(r) > u(p)$ imply $0 < \sigma(p)/[u(r) - u(p)] < 1$. Thus,

$$0 < \alpha := 1 - \frac{\sigma(p)}{u(r) - u(p)} < 1.$$

The *linearity* of u implies $u(\alpha p + (1 - \alpha)r) = u(p) + \sigma(p)$. Because

$u(r) > u(q) > u(p)$, we have

$$0 < \beta := \frac{u(r) - u(q)}{u(r) - u(p)} < 1.$$

Again the *linearity* of u implies $u(\beta p + (1 - \beta)r) = u(q)$. So, $r P [\beta p + (1 - \beta)r]$. This implies $\alpha > (1 - \beta)$. Otherwise, we would have $r I [\beta p + (1 - \beta)r]$ as well, which would contradict with *trichotomy*. Therefore,

$$0 < \gamma := \frac{(1 - \alpha)[u(r) - u(p)]}{\beta[u(r) - u(q)]} < 1.$$

Case 1: $\sigma(p) > \sigma(q)$. Suppose $\alpha > \beta$. Since $\alpha > \beta > 1 - \alpha$ and u is *linear*, $u(\gamma q + (1 - \gamma)r) = u(\alpha r + (1 - \alpha)p)$. Because $p I [\alpha p + (1 - \alpha)r]$, *mixture symmetry* implies $r I [\alpha r + (1 - \alpha)p]$. Since $u(\gamma q + (1 - \gamma)r) = u(\alpha r + (1 - \alpha)p)$, we also have $r I [\gamma q + (1 - \gamma)r]$. Again *mixture symmetry* implies $q I [\gamma r + (1 - \gamma)q]$. Because $\sigma(p) > \sigma(q)$ and u is *linear*, we have $u(\gamma r + (1 - \gamma)q) > u(q) + \sigma(q)$. This further implies $[\gamma r + (1 - \gamma)q] P q$, which contradicts with *trichotomy*. A similar argument yields a contradiction for the case $\alpha \leq \beta$ as well.

Case 2: $\sigma(p) < \sigma(q)$. This case is similar to Case 1.

Hence for each $p, q \in L \setminus M_R$, we have $\sigma(p) = \sigma(q)$.

Now, for each $p \in L \setminus M_R$, let $k = \sigma(p)$. Since u is a continuous function on a compact domain, there exists $\bar{p} \in L$ such that for each $q \in L$, $u(\bar{p}) \geq u(q)$. Because R is *non-trivial*, there exists $q \in L$ such that $\bar{p} P q$. Otherwise, for each $p, q \in L$, we would have $p I q$, which would contradict with *non-triviality*. Hence, there exists $p' \in L$ such that $\bar{p} I p'$ and for each $r \in L$, we have $p' P_0 r$ implies $\bar{p} P r$. Thus, for each $p \in M_R$, we have $u(p') \leq u(p)$. This together with *mixture symmetry* imply that for each $p, q \in M_R$, we have $|u(p) - u(q)| \leq k$. Therefore, (u, k) is a *Scott-Suppes representation* of P .

Next, we show that expected Scott-Suppes utility representation implies the axioms.¹⁴

Suppose there exists a *linear* function $u : L \rightarrow \mathbb{R}$ and $k \in \mathbb{R}_{++}$ such that (u, k) is a *Scott-Suppes representation* of P .

It is easy to show that since $k \in \mathbb{R}_{++}$, I is *reflexive* and R satisfies *trichotomy*.

¹⁴See also Vincke (1980) and Candeal and Induráin (2010) for this part of the proof.

Let $p, q, r, s \in L$. Suppose $p P q I r P s$. This implies $p > q + k$ and $|q - r| \leq k$. Hence, $p > r$. Moreover, $r P s$ implies $r > s + k$. Thus, $p > s + k$, which implies $p P s$. So, R satisfies *strong intervality*. It is similar to show that R is *semitransitive*. Therefore, R is a *semiorder* on L .

Let $p, q \in L$ and $\alpha \in (0, 1)$. Suppose $p I \alpha p + (1 - \alpha)q$. This implies $|u(p) - u(\alpha p + (1 - \alpha)q)| \leq k$. Since u is *linear*, $|u(p) - [\alpha u(p) + (1 - \alpha)u(q)]| \leq k$. Rearranging the terms gives $|[\alpha u(q) + (1 - \alpha)u(p)] - u(q)| \leq k$. Hence, $q I \alpha q + (1 - \alpha)p$. Thus, R is *mixture-symmetric*.

Suppose P is not *regular*, that is, there exist $p \in L$ and $(p_n) \in L^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $p P p_n$ and $p_{n+1} P p_n$ or there exist $q \in L$ and $(q_n) \in L^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $q_n P q$ and $q_n P q_{n+1}$. First, suppose that there exist $p \in L$ and $(p_n) \in L^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $p P p_n$ and $p_{n+1} P p_n$. Let $d = u(p) - u(p_1)$ and $i = \lceil d/k \rceil$. We have

$$u(p_{i+1}) > u(p_1) + ik > u(p_1) + d = u(p_1) + u(p) - u(p_1) = u(p).$$

This contradicts with $p P p_{i+1}$. For the other case, a similar argument yields the desired conclusion. Hence, P is *regular*.

It is easy to see that for each $p, q \in L$, we have $p R_0 q$ if and only if $u(p) \geq u(q)$. Since u is a continuous function, preimage of a closed set is closed. Hence, R_0 is *continuous*. This in turn implies that R_0 is *mixture-continuous*.

Let $p, q \in L$ with $p I_0 q$. This implies $u(p) = u(q)$. Hence, for any $r \in L$, we have $1/2u(p) + 1/2u(r) = 1/2u(q) + 1/2u(r)$. The *linearity* of u implies $u(1/2p + 1/2r) = u(1/2q + 1/2r)$. Thus, $[1/2p + 1/2r] I_0 [1/2q + 1/2r]$. So, R_0 satisfies the *midpoint indifference axiom*.

Finally, suppose $p, q \in L$ with $p P q$. Since $p P q$, we have $u(p) > u(q) + k$. Hence,

$$0 < \alpha := \frac{k}{u(p) - u(q)} < 1.$$

Because u is *linear*, we have $u(\alpha q + (1 - \alpha)p) = u(p) - k$. Hence, $p I [\alpha q + (1 - \alpha)p]$. Moreover, suppose for some $r \in L$, we have $[\alpha q + (1 - \alpha)p] P_0 r$. This implies $u(\alpha q + (1 - \alpha)p) = u(p) - k > u(r)$. Thus, $p P r$.

□

Let (u, k) be an expected Scott-Suppes utility representation of a semiorder R on L and $a, b \in \mathbb{R}_{++}$. Define $\tilde{u} : L \rightarrow \mathbb{R}$ as for each $p \in L$, $\tilde{u}(p) = au(p) + b$. It is straightforward to show that $(\tilde{u}, ak)$ is another expected Scott-Suppes utility representation of R . Let $p, q \in L$. We have $p P q$ if and only if $u(p) > u(q) + k$ if and only if $a(u(p)) + b > a(u(q) + k) + b$ if and only if $\tilde{u}(p) > \tilde{u}(q) + ak$. Hence, $(\tilde{u}, ak)$ is a Scott-Suppes representation of R . Moreover, since $\tilde{u}$ is an affine transformation of a linear function, it is linear. Therefore, $(\tilde{u}, ak)$ is another expected Scott-Suppes utility representation of R .

3.2 Independence of the Axioms

Let R be a non-trivial semiorder on L . We consider the following axioms.

- A1. R is mixture-symmetric.
- A2. P is regular.
- A3. R_0 is mixture-continuous.
- A4. R_0 satisfies the midpoint indifference axiom.
- A5. For each $p, q \in L$, if $p P q$, then there exists $p' \in L$ such that $p I p'$ and for each $r \in L$, we have $p' P_0 r$ implies $p P r$.

We first provide an example where A1-A5 hold.

Example 5. Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$, $p, q \in L$, and $\epsilon \in (0, 0.5]$. We define R on L such that:

- $p P q$ if $p_1 > q_1 + \epsilon$,
- $p I q$ if $\neg(p P q)$ and $\neg(q P p)$.

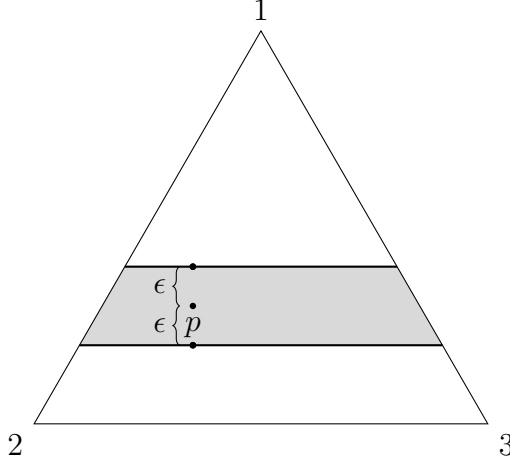


Figure 2: Example 5

Lottery p is indifferent to every lottery in the gray area.

It is straightforward to show that R is a *non-trivial semiorder*. Let $p, q \in L$ and $\alpha \in (0, 1)$. Suppose $p I [\alpha p + (1 - \alpha)q]$. This implies

$$|p_1 - \alpha p_1 - q_1 + \alpha q_1| \leq \epsilon.$$

Rearranging the terms gives

$$|\alpha q_1 + (1 - \alpha)p_1 - q_1| \leq \epsilon.$$

Hence, $q I [\alpha q + (1 - \alpha)p]$. Thus, A1 holds. Since $\epsilon > 0$, it is easy to show that P is *regular*. So, A2 holds as well. Moreover, it is easy to see that for each $p, q \in L$, we have $p R_0 q$ if and only if $p_1 \geq q_1$. Hence, R_0 is *continuous*, which in turn implies that it is *mixture-continuous*. Therefore, A3 holds. Let $r \in L$.

Suppose for some $p, q \in L$, we have $p I_0 q$. Because for each $p, q \in L$, $p I_0 q$ if and only if $p_1 = q_1$, we have $p_1 = q_1$. Hence, $1/2p_1 + 1/2r_1 = 1/2q_1 + 1/2r_1$. Thus, $[1/2p + 1/2r] I_0 [1/2q + 1/2r]$. So, A4 holds. Finally, let $p, q \in L$. Suppose $p P q$. This implies $p_1 > \epsilon$. Define $p' = (p_1 - \epsilon, 1 - (p_1 - \epsilon), 0)$. Since $p_1 > \epsilon$, we have $p' \in L$. Moreover, $p I p'$. Let $r \in L$. Suppose $p' P_0 r$. Because for each $r \in L$, $p' P_0 r$ if and only if $p'_1 > r_1$, we have $p P r$. Therefore, A5 holds.

We now show that the axioms in Theorem 2 are mutually independent by providing an example for each axiom.

Example 6 (A1, A2, A3, A4 $\Rightarrow$ A5). Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$, $p, q \in L$, and $\epsilon \in (0, 0.5]$. We define R on L such that:

- $p P q$ if $p_1 \geq q_1 + \epsilon$,
- $p I q$ if $\neg(p P q)$ and $\neg(q P p)$.

Since $\epsilon \in (0, 1)$, R is *non-trivial*. In Example 4 we show that R is a *semiorder* and R_0 is *continuous*. Thus, A3 holds. Let $p, q \in L$ and $\alpha \in (0, 1)$. Suppose $p I [\alpha p + (1 - \alpha)q]$. This implies

$$|p_1 - \alpha p_1 - q_1 + \alpha q_1| < \epsilon.$$

Rearranging the terms gives

$$|\alpha q_1 + (1 - \alpha)p_1 - q_1| < \epsilon.$$

So, $q I [\alpha q + (1 - \alpha)p]$. Therefore, A1 holds. Since ϵ is positive, it is easy to show that P is *regular*. Hence, A2 holds as well. It is straightforward to show that for each $p, q \in L$, we have $p I_0 q$ if and only if $p_1 = q_1$. Let $r \in L$. Suppose for some $p, q \in L$, we have $p I_0 q$. This implies $p_1 = q_1$. Thus,

$$1/2p_1 + 1/2r_1 = 1/2q_1 + 1/2r_1,$$

which in turn implies

$$[1/2p + 1/2r] I_0 [1/2q + 1/2r].$$

So, A4 holds. Finally, let $p = (1, 0, 0)$. We have $p P (0, 1, 0)$. Suppose for some $p' \in L$, we have $p I p'$. Define $r = ([p'_1 - (1 - \epsilon)]/2, 1 - \{[p'_1 - (1 - \epsilon)]/2\}, 0)$. Because $p I p'$, we have $p_1 - p'_1 < \epsilon$. Therefore, $r \in L$. It is easy to show that $p' P_0 r$ but $p I r$. Hence, A5 does not hold.

Example 7 (A1, A2, A3, A5 $\Rightarrow$ A4). Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p P q$ if $p_1 > q_1 + 0.6$,
- $p I q$ if $\neg(p P q)$ and $\neg(q P p)$.

In Example 5, we show that A1-A5 hold. In this example, it is exactly the same with Example 5 to show that A1, A2, and A5 hold. However, unlike Example 5, we do not anymore have for each $p, q \in L$, $p R_0 q$ if and only if $p_1 \geq q_1$ and $p I_0 q$ if and only if $p_1 = q_1$. This is because of the fact that for each $p, q \in L$ if

$p_1, q_1 \in [0.4, 0.6]$, then $p I_0 q$. On the other hand, for each $p, q \in L$ if $p_1 \in [0, 0.4) \cup (0.6, 1]$ and $q_1 \in [0, 1]$, then we still have $p R_0 q$ if and only if $p_1 \geq q_1$ and $p I_0 q$ if and only if $p_1 = q_1$. Hence, it is straightforward to show that A3 holds. In order to see that A4 does not hold, consider the following. We have $(0.6, 0.4) I_0 (0.4, 0.6)$ but

$$1/2(0.6, 0.4) + 1/2(1, 0) = (0.8, 0.2) P_0 (0.7, 0.3) = 1/2(0.4, 0.6) + 1/2(1, 0).$$

Thus, A4 does not hold.

Example 8 (A1, A2, A4, A5 $\Rightarrow$ A3). Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p P q$ if $p_1 = 1$ and $q_1 = 0$,
- $p I q$ if $\neg(p P q)$ and $\neg(q P p)$.

Since $(1, 0) P (0, 1)$, R is *non-trivial*. Moreover, it is easy to see that R is *reflexive* and satisfies *trichotomy*. The only case in which for some $p, q, r, s \in L$ we have $p P q I r P s$ is $(1, 0) P (0, 1) I (1, 0) P (0, 1)$. Because $(1, 0) P (0, 1)$, R satisfies *strong intervality*. Furthermore, R vacuously satisfies *semitransitivity*. Hence, R is a *semiorder*. Let $p \in L$. Suppose $p_1 \in (0, 1)$. Then, for each $q \in L$ we have $p I q$. Moreover, for each $\alpha \in (0, 1)$, we have

$$(0, 1) I \alpha(0, 1) + (1 - \alpha)(1, 0) I (1, 0).$$

Thus, A1 holds. Since, the only occurrence of a strict preference is $(1, 0) P (0, 1)$, P is trivially *regular*. So, A2 holds as well. For each $p \in L$, we have $p I_0 p$. Let $p \in L$. Suppose $p_1 \in (0, 1)$. We have $(1, 0) P_0 p P_0 (0, 1)$. Therefore, it is routine to check that A4 holds. Because $(1, 0) I (0.5, 0.5) P_0 (0, 1)$ and $(1, 0) P (0, 1)$, A5 holds. Finally, $UMC_0((0.5, 0.5); (0, 1), (1, 0)) :=$

$$\{\alpha \in [0, 1] : [\alpha(0, 1) + (1 - \alpha)(1, 0)] R_0 (0.5, 0.5)\} = [0, 1),$$

which is not closed. Hence, A3 does not hold.

Example 9 (A1, A3, A4, A5 $\Rightarrow$ A2). Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p P q$ if $p_1 > q_1$,

- $p I q$ if $p_1 = q_1$.

Since $(1, 0) P (0, 1)$, R is *non-trivial*. Moreover, because R is a *weak order*, it is a *semiorder*. Let $p, q \in L$ and $\alpha \in (0, 1)$. If $p I [\alpha p + (1 - \alpha)q]$, then we have $p_1 = \alpha p_1 + (1 - \alpha)q_1$. This implies that $q_1 = \alpha q_1 + (1 - \alpha)p_1$. Hence, $q I [\alpha q + (1 - \alpha)p]$. Thus, A1 holds. It is straightforward to show that for each $p, q \in L$, we have $p R q$ if and only if $p R_0 q$ if and only if $p_1 \geq q_1$. So, R_0 is *continuous*, which in turn implies that it is *mixture-continuous*. Therefore, A3 holds. It is also easy to see that for each $p, q \in L$, we have $p I q$ if and only if $p I_0 q$ if and only if $p_1 = q_1$. Let $r \in L$. Suppose for some $p, q \in L$, we have $p I_0 q$. This implies $p_1 = q_1$. Hence, $1/2p_1 + 1/2r_1 = 1/2q_1 + 1/2r_1$. Thus, $[1/2p + 1/2r] I_0 [1/2q + 1/2r]$. So, A4 holds. Let $p, q \in L$ with $p P q$. This implies that $p_1 > q_1$. Moreover, we have $p I p$. Let $r \in L$. Suppose $p P_0 r$. Since for each $r \in L$, $p P_0 r$ if and only if $p_1 > r_1$, we have $p P r$. Therefore, A5 holds. Finally, it is easy to show that since for each $p, q \in L$, $p P q$ if and only if $p_1 > q_1$, P is not *regular*. Hence, A2 does not hold.

Example 10 (A2, A3, A4, A5 $\nRightarrow$ A1). Let L be the set of lotteries on $X := \{x_1, x_2\}$ and $p, q \in L$. We define R on L such that:

- $p P q$ if $2p_1 > 3q_1 + 1$,
- $p I q$ if $\neg(p P q)$ and $\neg(q P p)$.

Since $(1, 0) P (0, 1)$, R is *non-trivial*. Let $p \in L$. Define $u : L \rightarrow \mathbb{R}$ as $u(p) = \ln(p_1 + 1)$. It is straightforward to show that $(u, \ln(3/2))$ is a *Scott-Suppes representation* of P . Hence, R is a *semiorder* and P is *regular*. Thus, A2 holds. Moreover, it is easy to show that for each $p, q \in L$, we have $p R_0 q$ if and only if $p_1 \geq q_1$. So, R_0 is *continuous*, which in turn implies that it is *mixture-continuous*. Therefore, A3 holds. It is also easy to see that for each $p, q \in L$, we have $p I_0 q$ if and only if $p_1 = q_1$. Let $r \in L$. Suppose for some $p, q \in L$, we have $p I_0 q$. This implies $p_1 = q_1$. Hence,

$$1/2p_1 + 1/2r_1 = 1/2q_1 + 1/2r_1.$$

Thus,

$$[1/2p + 1/2r] I_0 [1/2q + 1/2r],$$

which implies that A5 holds. Let $p, q \in L$. Suppose $p P q$. This implies $p_1 > 1/2$. Define $p' = ((2p_1 - 1)/3, 1 - (2p_1 - 1))$. Because $p_1 > 1/2$, we have

$p' \in L$. Furthermore, we have $p I p'$. Since for each $r \in L$, $p' P_0 r$ if and only if $p'_1 > r_1$, we have $p P r$. So, A6 holds. Finally, we have

$$(1, 0) I (1/3, 2/3) = 1/3(1, 0) + 2/3(0, 1).$$

But, $\neg((0, 1) I [1/3(0, 1) + 2/3(1, 0)])$. Hence, R_0 is not *mixture-symmetric*. Thus, A1 does not hold.



CHAPTER 4

CONCLUSION

In this work we study intransitive indifference under uncertainty. More specifically, we identify necessary and sufficient conditions for a binary relation over the set of lotteries to have a Scott-Suppes representation via a linear utility function. Our main result (Theorem 2) includes all of the axioms that are introduced in Vincke's (1980) theorem. On top of those properties of Vincke (1980), we utilize an axiom of Nakamura (1988) and another from Candeal and Induráin (2010). These two additional conditions help us to convert the non-negative threshold function of Vincke's representation into a positive constant threshold.

Our theorem is parallel to the celebrated expected utility theorem of von Neumann and Morgenstern (1944) in the sense that the utility function we obtain is linear. Yet, unlike their theorem, in order for two lotteries to be distinguishable in comparison, the difference in utilities of the two lotteries must exceed a certain positive threshold that is constant.

We believe that the internal structure of the lottery space makes our analysis plausible, that is, given two lotteries, if the probabilities of the alternatives occurring are very close to each other in both lotteries, then the decision maker may not be able to think of these lotteries to be significantly different in comparison. Just like in Luce's coffee with sugar example, we can convert a lottery bit by bit to another lottery by increasing and decreasing probabilities in small steps. What is even better is that we can do this for every lottery, irrespective of the particular goods in the set of alternatives.

One shortcoming of our work that might potentially be subject to future research is the fact that we do not identify any decision making criteria or

behaviors that satisfy the axioms of our representation theorem. For instance, modified notions of stochastic dominance with probability thresholds and their expected Scott-Suppes utility representations can be obtained.



BIBLIOGRAPHY

- Aleskerov, F., Bouyssou, D., and Monjardet, B. 2007. *Utility maximization, choice and preference* (Vol. 16). Springer Science & Business Media.
- Beja, A., and Gilboa, I. 1992. Numerical representations of imperfectly ordered preferences (a unified geometric exposition). *Journal of Mathematical Psychology*, 36(3), 426–449.
- Bridges, D. S. 1983. A numerical representation of preferences with intransitive indifference. *Journal of Mathematical Economics*, 11(1), 25–42.
- Candeal, J. C., and Induráin, E. 2010. Semiorders and thresholds of utility discrimination: Solving the scott–suppes representability problem. *Journal of Mathematical Psychology*, 54(6), 485–490.
- Fechner, G. 1860. *Elemente der psychophysik* (2 vols). breitzopf and hartel. vol. 1 trans, by he adler (1966). *Elements of psychophysics*.
- Fishburn, P. C. 1968. Semiorders and risky choices. *Journal of Mathematical Psychology*, 5(2), 358–361.
- Fishburn, P. C. 1970a. Intransitive indifference with unequal indifference intervals. *Journal of Mathematical Psychology*, 7(1), 144–149.
- Fishburn, P. C. 1970b. *Utility theory for decision making* (Tech. Rep.). Research Analysis Corp Mclean va.
- Fishburn, P. C. 1991. Nontransitive preferences in decision theory. *Journal of Risk and Uncertainty*, 4(2), 113–134.
- Gensemer, S. H. 1987. Continuous semiorder representations. *Journal of Mathematical Economics*, 16(3), 275–289.
- Kreps, D. 1988. *Notes on the theory of choice*. Colorado: Westview press.
- Luce, R. D. 1956. Semiorders and a theory of utility discrimination. *Econometrica, Journal of the Econometric Society*, 178–191.
- Manders, K. L. 1981. On jnd representations of semiorders. *Journal of Mathematical Psychology*, 24(3), 224–248.

- May, K. O. 1954. Intransitivity, utility, and the aggregation of preference patterns. *Econometrica: Journal of the Econometric Society*, 1–13.
- Nakamura, Y. 1988. Expected utility with an interval ordered structure. *Journal of Mathematical Psychology*, 32(3), 298–312.
- Ok, E. A. 2007. *Real analysis with economic applications* (Vol. 10). Princeton University Press.
- Pirlot, M., and Vincke, P. 2013. *Semiorders: properties, representations, applications* (Vol. 36). Springer Science & Business Media.
- Poincaré, H. 1905. La valeur de la science, flammation, paris. *Rouge A.(2000) Introduction à la physique subatomique, Editions de l'Ecole polytechnique*.
- Radner, R. 1980. Collusive behavior in noncooperative epsilon-equilibria of oligopolies with long but finite lives. *Journal of Economic Theory*, 22(2), 136–154.
- Scott, D., and Suppes, P. 1958. Foundational aspects of theories of measurement. *The Journal of Symbolic Logic*, 23(2), 113–128.
- Simon, H. A. 1956. Rational choice and the structure of the environment. *Psychological Review*, 63(2), 129.
- Tversky, A. 1969. Intransitivity of preferences. *Psychological Review*, 76(1), 31.
- Vincke, P. 1980. Linear utility functions on semiordered mixture spaces. *Econometrica: Journal of the Econometric Society*, 771–775.
- von Neumann, J., and Morgenstern, O. 1944. *Theory of games and economic behavior*. New Jersey: Princeton University Press.
- Weber, E. 1834. De tactu. *Leipzig, Koehler*.
- Wiener, N. 1914. A contribution to the theory of relative position. In *Proceedings of the cambridge philosophical society* (Vol. 17, pp. 441–449).

APPENDIX

We give the full translation of the text from Poincaré (1905) as in Pirlot and Vincke (2013:19,20).

Sometimes we are able to make the distinction between two sensations, while we cannot distinguish them from a third sensation. For example, we can easily make the distinction between a weight of 12 grams and a weight of 10 grams, but we are not able to distinguish each of them from a weight of 11 grams. This fact can symbolically be written $A = B$, $B = C$, $A < C$. This could be considered as a characterization of the physical continuum, as given by observation and experiments; this “contradiction” has been solved by the introduction of the mathematical continuum. The latter is a scale with an infinite number of levels, which do not overlap each other, as do the elements of the physical continuum. The physical continuum is like a nebula whose elements cannot be perceived, even with the most sophisticated instruments; of course, with a good balance (instead of human sensation), it would be possible to distinguish 11 grams from 10 and 12 grams, so that we could write $A < B$, $B < C$, $A < C$. But one could always find other elements D and E such that $A = D$, $D = B$, $A < B$, $B = E$, $E = C$, $B < C$, and the difficulty would be the same; only the mind can resolve it and the answer is the mathematical continuum.

Next, we provide the equivalence of semiorder definitions and axioms.

Definition (Semiordeering, Luce (1956)). Let (P, I) be two binary relations on X . We say (P, I) is a *semiordeering* of X if

- L1. I is reflexive,
- L2. (P, I) satisfies trichotomy,
- L3. (P, I) satisfies strong intervality,

L4. for each $x, y, z, t \in X$, $x P y, y P z, y I t$ imply not both $x I t$ and $z I t$.

We show that under L1, L2, and L3, semitransitivity and L4 are equivalent.

Assume that L1-L4 hold. Let $x, y, z, t \in X$. Suppose $x P y, y P z, z I t$ but $t R x$. If $t P x$, then by L3, $y P z I t P x$ imply $y P x$. This contradicts with $x P y$. On the other hand, if $x I t$, then by L4, $\neg(y I t)$. If $y P t$, then $x P t$, which contradicts with $x I t$. If $t P y$, then $t P z$. This contradicts with $z I t$. Hence, R is *semitransitive*. Now, assume that L1-L3 and *semitransitivity* hold. Let $x, y, z, t \in X$. Suppose $x P y, y P z, y I t$ but $x I t$ and $z I t$. By *semitransitivity*, $x P y P z I t$ imply $x P t$. This contradicts with $x I t$. Thus, L4 holds. This completes our proof.

Instead of defining semiorder with a pair of binary relations, another way of doing it is introduced by Scott and Suppes (1958) as follows.

Definition (Semiorder, Scott and Suppes (1958)). A binary relation P on X is a *semiorder* if

- S1. P is irreflexive,
- S2. for each $x, y, z, t \in X$, $x P y$ and $z P t$ imply $x P t$ or $z P y$,
- S3. for each $x, y, z, t \in X$, $x P y$ and $y P z$ imply $x P t$ or $t P z$.

Let P be a *semiorder* on X according to the definition of Scott and Suppes (1958). For each $x, y \in X$, define I on X as $x I y$ if $\neg(x P y)$ and $\neg(y P x)$. We first show that (P, I) is a *semiordering* of X according to the definition of Luce (1956). It is easy to show that (P, I) satisfies L1 and L2. Let $x, y, z, t \in X$. Suppose $x P y, y I z$, and $z P t$. By S2, we have $x P t$ or $z P y$. Since $y I z$, we have $x P t$. Hence, (P, I) satisfies *strong intervality*. Suppose we have $x P y, y P z$, and $y I t$. By S3, we have $x P t$ or $t P z$. Thus, we have not both $x I t$ and $z I t$. So, (P, I) satisfies L4. Now, let (P, I) be a *semiordering* of X according to the definition of Luce (1956). We show that P is a *semiorder* on X according to the definition of Scott and Suppes (1958). It is easy to see that P is *irreflexive*. Let $x, y, z, t \in X$. Suppose $x P y$ and $z P t$ but $t R x$ and $y R z$. Since P is transitive and (P, I) satisfies *strong intervality*, $z P t R x P y$ imply $z P y$. This contradicts with $y R z$. Hence, P satisfies S2. Suppose we have $x P y$ and $y P z$ but $t R x$ and $z R t$. Because P is transitive and (P, I) satisfies *semitransitivity*, $x P y P z R t$ imply $x P t$. This contradicts with $t R x$. Thus, P satisfies S3, which completes our proof.