CHAOS IN BFSS AND ABJM MATRIX MODELS

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ABSTRACT

CHAOS IN BFSS AND ABJM MATRIX MODELS

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In this thesis, we focus on the chaotic dynamics emerging from Banks-Fischler-Shenker-Susskind (BFSS) and Aharony-Bergman-Jafferis-Maldacena (ABJM) models. We first direct our attention on the bosonic part of the BFSS model and introduce mass deformation terms. Using ansatz configurations involving fuzzy-2 and fuzzy-4 spheres with collective time dependence, we find a family of reduced effective Hamiltonians through which we explore the chaotic dynamics emerging from these models. In order to reveal the structure of the chaotic dynamics, we calculate the Lyapunov exponents and plot the Poincaré sections at several different values of the parameters of the system. Secondly, we focus on the bosonic part of the mass deformed ABJM model and obtain a matrix model by reducing it from 2+1 to 0+1 dimensions. Using ansatz configurations consisting of Gomis, Rodriguez-Gomez, Van Raamsdonk and Verlinde (GRVV) matrices, which also describe fuzzy-2 spheres with collective time dependence, we obtain reduced effective Hamiltonians and explore their chaotic dynamics. Relation of our results to the Maldacena-Shenker-Stanford (MSS) bound $\lambda_L \leq 2\pi T$ and the largest Lyapunov exponent is also discussed in detail in both cases.

Keywords: BFSS Matrix Model, ABJM Model, Matrix quantum mechanics, MSS Bound, Chaotic dynamics

ÖZ

BFSS VE ABJM MATRİS MODELLERİNDE KAOS

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Bu tezde BFSS ve ABJM matris modellerinde ortaya çıkan kaotik dinamikler incelenmiştir. İlk bölümde, BFSS matris modelinin bozonik kısmını kapsayan bir çalışma gerçekleştirilmiştir. Fuzzy-2 ve fuzzy-4 kürelerin zamana bağlı fonksiyonlar ile çarpımından meydana gelen konfigürasyonları içeren bir yaklaşım kullanılarak, indirgenmiş etkin Hamiltonianlar elde edilmiştir. Bu Hamiltonianlar vasıtası ile sistemde ortaya çıkan kaotik dinamikler incelenmiştir. Lyapunov üsleri ve Poincaré kesitleri kullanılarak sistemin kaotik yapısı ortaya çıkarılmış, etkin olarak yorumlanmıştır. İkinci bölümde, ABJM matris modelinin bozonik kısmına odaklanılmış ve boyutsal indirgeme ile 2+1 boyuttan 0+1 boyuta indirgenerek elde edilen bir matris modeli incelenmiştir. İlk bölümde uygulanana benzer şekilde, GRVV matrislerinin zamana bağlı fonksiyonlar ile çarpımından oluşan konfigürasyonları içeren yaklaşımlar kullanılarak, indirgenmiş etkin Hamiltonianlar elde edilmiştir. Bu Hamiltonianlar üzerinden sistemin kaotik dinamikleri incelenmiş ve her iki model için de MSS üst sınırı ve en büyük Lyapunov üsteli detaylı bir şekilde incelenmiştir.

Anahtar Kelimeler: BFSS Matris Modeli, ABJM Modeli, Matris kuantum mekaniği, MSS üst sınırı, Kaotik dinamikler

To my family

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2D	2 Dimensional
3D	3 Dimensional
BFSS	Banks-Fischler-Shenker-Susskind
BMN	Berenstein-Maldacena-Nastase
ABJM	Aharony-Bergman-Jafferis-Maldacena
MSS	Maldacena-Shenker-Stanford
GRVV	Gomis-Rodriguez Gomez-Van Raamsdonk-Verlinde
YM	Yang-Mills
CS	Chern-Simons
MLLE	Mean Largest Lyapunov Exponent
LLE	Largest Lyapunov Exponent
DCLQ	Discreate Lightcone Quantization
λ_L	Mean Largest Lyapunov Exponent

CHAPTER 1

INTRODUCTION

Matrix gauge theories associated to string and M-theories have been a very popular and fruitful area of research ever since from their discovery almost twenty years ago [1–3]. During the last decades, these models have been investigated from various perspectives. Among these models, the Banks-Fischler-Shenker-Susskind (BFSS) model deserves partial attention [1]. This matrix model is a supersymmetric $SU(N)$ gauge theory obtained by the dimensional reduction of the $\mathcal{N} = 1$ SUSY Yang-Mills theory in 9+1 dimensions to 0+1 dimensions [4]. The bosonic part of the model consists of nine $N \times N$ Hermitian matrices whose elements are dependend on time only. The BFSS matrix model has a deep connections with the type II-A string theory and appears as the discrete light-cone quantization (DLCQ) of the M-theory in the flat background [4, 6]. BFSS theory may be understood as describing N-coincident $D0$-branes, i.e. point like Drichlet branes in string theory, in the flat backround [4, 7]. The diagonal elements of the matrices describe the positions of the $D0$-branes while the off-diagonal elements desribe states of modes or excitations of the strings stretching between these $D0$-branes. It is well-known that BFSS model has a gravity dual via AdS/CFT correspondence [8]. In the large N and strong coupling/low temperature limit, the $D0$-branes forms a black-brane phase, which is a string theoretical black holes [4, 6, 8].

During the last decade, the chaotic dynamics appears in the Yang-Mills matrix models are well studied in various papers [9–21]. The main motivation for studying chaotic dynamics in Yang-Mills matrix models is the hope of obtaining information regarding the chaotic dynamics and properties of associated to these YM models via holography. To elaborate, with the help of the AdS/CFT correspondence and existing gravity duals of these models, understanding the chaos in the matrix models side may give us a hint about thermalization and evoporation in the black-brane phase [10,13]. Another idea is proposed by Sekino and Sussking called the "fast scrambling conjecture" [9]. These authors suggest that chaos sets in black holes faster than in any other physical system and its rate proportional to the natural logarithm of the number of degrees of freedom of the system.

Another model of interest in this thesis is Aharony-Bergman-Jafferis-Maldacena (ABJM) model [22]. ABJM model is a 2+1 dimensional $\mathcal{N} = 6$ supersymmetric $SU(N) \times SU(N)$ Chern-Simons gauge theory at the Chern-Simons level $(-k, k)$ [23]. The bosonic part of the ABJM matrix model consists of complex scalar field C^I and ${C^I}^\dagger$ where $I : 1, 2, 3, 4$ which are coupled bifundamentally to the $SU(N)$ gauge fields A_μ and $\hat{A}_\mu$. The ABJM model may be understood as describing N coincident $M2$-brane [24]. In fact, since there are eight coordinates transverse to $M2$-brane of M-theory, we have eight scalar fields C^I and ${C^I}^\dagger$. The ABJM model too a gravity dual [24, 25]. This is dual to type IIA string theory on $AdS \times \mathcal{CP}^3$ via AdS/CFT correspondence [24].

In this thesis, we will focus on the bosonic part of the BFSS and ABJM models. Firstly, the BFSS matrix model will be examined. As we discussed earlier, the bosonic part of the BFSS action consist of $N \times N$ Hermitian matrices and $SU(N)$ gauge fields. The action is invariant under the $SU(N)$ gauge transformations and also the $SO(9)$ global symmetry due to the rigid rotation of the matrices. Since the model has large number of degrees of freedoms, finding a general solution to the model does not seems to be computationally doable. For reducing the total number of degrees of freedoms, we may consider matrix models with less number of matrices and a smaller gauge group by considering various degrees of freedom in the BFSS model to be frozen or motivate such matrix models as emerging from dimensionally reducing Yang-Mills theories in d+1-dimensions with $d \leq 9$.

In chapter 3, we focus our attention to the BFSS matrix model with two massive deformation terms to study chaos emerging from the model. In this model, massive deformation terms break the original $SO(9)$ global symmetry into $SO(4) \times SO(5) \times \mathbb{Z}_2$

upon certain ansatz configurations to get the effective action whose chaotic dynamics will be studied. Our ansatz configurations consist of fuzzy two and fuzzy four spheres with collective time dependence. The ansatz configurations studied in [16] inspired us to choice ansatz configurations with larger fuzzy spaces. In this model, we have different reduced effective actions for each matrix level. During our analysis, we consider several different matrix levels to elaborate on our results and obtain a broader perspective. After revealing the chaotic behaviour of the system at each matrix level by using Lyapunov exponents and Poincaré sections, we perform a detailed numerical analysis and show that MLLE increases with energy.Using the virial and equipartition theorems these results enabled us to find an upper bound T_c on the temperatures above which the λ_L values comply with the MSS bound $\lambda_L \leq 2\pi T$ and below which it will eventually be violated. The results of chapter 3 are a part of results obtained in collaboration with O. Oktay and K. Başkan and my supervisor S. Kürkcüoğlu [26].

In chapter 4, we study a Chern-Simons (CS) matrix model that we will obtain by dimensionally reducing the bosonic part of the mass deformed ABJM model in 2+1-dimensions to 0+1-dimensions following standard procedures and subsequently identify the appropriate $'tHooft$ limit [23]. Performing trace over ansatz configurations involving fuzzy two spheres realized in terms of the Gomis-Rodriguez-Gomez-Van Raamsdonk-Verlinde (GRVV) matrices with collective dependence, we obtain a family of reduced effective Lagrangians. Along similar lines to the treatment of the YM matrix model, we inspect how λ_L profiles with respect to energy, and obtain that it either is of the form $\lambda_L \propto (\frac{E}{N^2})^{1/4}$ or $\lambda_L \propto (\frac{E}{N^2} - W)^{1/3}$, where W is a constant determined in terms of the Chern-Simons coupling, mass and the matrix level. Finally, using these results we are able to determine the temperature regimes in which λ_L complies with and violates the MSS bound.

In chapter 5 we summarize our results and discuss some novel directions of research, which may be worthwhile to pursue in the near future.

CHAPTER 2

REVIEW OF THE BFSS MATRIX MODEL

2.1 Bosonic Sector of the BFSS Matrix Model

Banks-Fischler-Shenker-Susskind (BFSS) is a supersymmetric Matrix quantum mechanics model, which describes multiple coincident $D0$ branes. In this thesis, we will only focus on the bosonic part of this model, the reasons for this will become clearer as we proceed. The bosonic part of the BFSS action given as

$$S_{BFSS} = \int \mathrm{d}t L_{BFSS} \,, \tag{2.1a}$$

$$L_{BFSS} = \frac{1}{g^2} \operatorname{Tr}\left(\frac{1}{2}(D_0 X_i)^2 + \frac{1}{4}[X_i, X_j]^2\right), i, j = 1, 2...9. \tag{2.1b}$$

As we will verify shortly, it has $SU(N)$ gauge symmetry and $SO(9)$ global symmetry. X_i's are $N \times N$ Hermitian matrices which transforms in the adjoint representation of the $SU(N)$ gauge group, $X_i \to X_i' = U^\dagger X_i U$ with $U \in SU(N)$. g is the Yang-Mills coupling constant. In units of which $\hbar = 1$ S is dimensionless $\frac{1}{g^2}$ has dimensions of $[Length]^3$. At the classical level, g only changes the energy scale and for this reason, we can scale it to $g = 1$. The covariant derivative D_0 is defined as

$$D_0 X_i = \partial_0 X_i - i[A_0, X_i] \,, \tag{2.2}$$

where A_0 is the $SU(N)$ gauge field, under the $SU(N)$ gauge symmetry it transforms as $A_0 \to A_0' = U^\dagger A_0 U + iU^\dagger \partial_0$. Although g is a constant, it is not a dimensionless quantity. By perfoming dimensional analysis, we can obtain the dimension of g. Let us use the [E] for energy units. The Lagragian must have units of energy and with $\hbar = 1$ action is dimensionless. Therefore, time has dimension $[E]^{-1}$. This implies that D_0 has dimension $[E]$. From (2.1b) it can be seen that $D_0 X_i$ and $[X_i, X_j]$ have same dimension. Similarly, (2.2) fixes the dimensions of X_i and A_0. Therefore D_0,

X_i and A_0 have same dimension $[E] = [Lenght]^{-1}$. Therefore, we have

$$[E] = \frac{1}{g^2}[E]^4 \equiv [L]^{-3/2} \tag{2.3}$$

Thus, g has dimension of $E^{3/2}$.

It is easy to verify $SU(N)$ gauge symmetry of the action by using the given transformations. For the kinetic part of the Lagrangian, using $U^\dagger U = UU^\dagger$ and cyclicity of trace $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$, we have

$$\begin{aligned} \mathrm{Tr}\,\frac{1}{2}(D_0X_i)^2 &= \mathrm{Tr}\,\frac{1}{2}U^\dagger(D_0X_i)UU^\dagger(D_0X_i)U \\ &= \mathrm{Tr}\,\frac{1}{2}(D_0X_i)^2\,, \end{aligned} \tag{2.4}$$

For the potential part we have

$$\begin{aligned} \mathrm{Tr}\big([X_i, X_j]^2\big) &= \mathrm{Tr}\Big(\big[UX_iU^\dagger, UX_jU^\dagger\big]^2\Big) \\ &= \mathrm{Tr}\big(\big[UX_iU^\dagger, UX_jU^\dagger\big]\big[UX_iU^\dagger, UX_jU^\dagger\big]\big) \\ &= \mathrm{Tr}\big(U^\dagger[X_i, X_j]^2U\big) \\ &= \mathrm{Tr}\big([X_i, X_j]^2\big)\,. \end{aligned} \tag{2.5}$$

Therefore, Lagrangian is invariant under $SU(N)$ gauge transformations. As we have stated earlier, the action L_{BFSS} has also an $SO(9)$ global symmetry. This means that it is invariant under the rigid rotations of X_i. Thus we have the following transformation

$$X_i' = R_{ij}X_j\,. \tag{2.6}$$

where $R_{ij} \in SO(9)$. The covariant derivative transforms as

$$D_0(R_{ij}X_j) = R_{ij}(D_0X_j)\,. \tag{2.7}$$

Therefore the kinetic term transforms as

$$\begin{aligned} R_{ik}(D_0X_k)R_{il}(D_0X_l) &= R_{ik}R_{il}(D_0X_k)(D_0X_l) \\ &= \delta_{kl}(D_0X_k)(D_0X_l) \\ &= (D_0X_k)^2 \end{aligned} \tag{2.8}$$

where we have used $RR^T = R^T R = 1, R_{ij}R_{jl} = \delta_{ij}$. The interaction term transforms in a similar manner,

$$\begin{aligned}
[R_{ik}X_k, R_{jl}X_l][R_{im}X_m, R_{jn}X_n] =& R_{ik}R_{jl}R_{im}R_{jm}[X_k, X_l][X_m, X_n] \\
=& \delta_{mk}\delta_{ln}[X_k, X_l][X_m, X_n] \\
=& [X_k, X_l]^2 \\
=& [X_i, X_j]^2 \,.
\end{aligned} \tag{2.9}$$

Since both kinetic and interaction terms are invariant under the $SO(9)$ transformation, we can conclude that S_{BFSS} is invariant under $SO(9)$ transformation.
Let us proceed to write down equations of motion for X_i's and A_0. Starting with A_0, lets apply the following variation

$$A_0 \to A_0 + \delta A_0 \tag{2.10}$$

we observe that A_0 appears only in the kinetic part of the action. Thus, computing the variation of the kinetic term is sufficent to obtain the e.o.m for A_0

$$\begin{aligned}
L_{kinetic} + \delta L_{kinetic} &= \frac{1}{2}\,\mathrm{Tr}\big((D_0X_i - i[\delta A_0, X_i])(D_0X_i - i[\delta A_0, X_i]) + O((\delta A_0)^2)\big)\,, \\
&= \mathrm{Tr}\,\frac{1}{2}(D_0X_i)^2 + \frac{1}{2}\,\mathrm{Tr}(-iD_0X_i[\delta A_0, X_i] - i[\delta A_0, X_i]D_0X_i)\,, \\
&= \mathrm{Tr}\,\frac{1}{2}(D_0X_i)^2 + \mathrm{Tr}(-iD_0X_i[\delta A_0, X_i])\,, \\
&= \mathrm{Tr}\,\frac{1}{2}(D_0X_i)^2 + \mathrm{Tr}(-iD_0X_i(\delta A_0X_i - X_i\delta A_0))\,, \\
&= \mathrm{Tr}\,\frac{1}{2}(D_0X_i)^2 + \mathrm{Tr}(-i\delta A_0X_i(D_0X_i) - i\delta A_0(D_0X_i)X_i)\,, \\
&= \mathrm{Tr}\,\frac{1}{2}(D_0X_i)^2 + \mathrm{Tr}(-i\delta A_0[X_i, D_0X_i])\,,
\end{aligned} \tag{2.11}$$

where we have used the cyclicity of the trace in lines 3 and 4. Therefore, the variation of the action w.r.t. A_0 yields the equation

$$i[X_i, D_0X_i] = 0\,, \tag{2.12a}$$

where sum over $i : 1, ..., 9$ is implied. In fact, without doing any calculation, we can easily see that (2.1b) does not contain any $\partial_0 A_0$ term. Therefore, we can immediately conclude that A_0 is not a dynamical variable and the e.o.m. for A_0 must be nothing but an algebraic equation. This is indeed so for (2.12a) imposes constraint on the system and in the literature, it is called the Gauss Law constaint. Furthermore, in $A_0 = 0$ gauge, the Gauss Law constraint takes the form

$$\left[X_i, \dot{X}_i\right] = 0\,, \tag{2.13}$$

where $\dot{X}_i := \partial_0 X_i$. Nothing that the conjugate momenta associated to X_i's are $P_i = \dot{X}_i$ we may write (2.13) as $[X_i, P_i] = 0$, which reveals the algebraic form of this equation.

Let us proceed with e.o.m. for X_i. By using the following variation

$$X_i \to X_i + \delta X_i\,. \tag{2.14}$$

We obtain

$$S+\delta S = \int \mathrm{d}t\,\mathrm{Tr}\left(\underbrace{\frac{1}{2}(\dot{X}_i + \delta\dot{X}_i)(\dot{X}_i + \delta\dot{X}_i)}_{L_{kin}+\delta L_{kin}}\right) + \int \mathrm{d}t\,\mathrm{Tr}\left(\underbrace{\frac{1}{4}[X_i + \delta X_i, X_j + \delta X_j]^2}_{L_{int}+\delta L_{int}}\right)\,. \tag{2.15}$$

For the kinetic term we find

$$\begin{aligned} L_{kin} + \delta L_{kin} &:= \mathrm{Tr}\left(\frac{1}{2}(\dot{X}_i + \delta\dot{X}_i)(\dot{X}_i + \delta\dot{X}_i)\right) \\ &= \mathrm{Tr}\left(\frac{1}{2}(\dot{X}_i\dot{X}_i + 2\dot{X}_i\delta\dot{X}_i + (\delta\dot{X}_i)^2)\right) \\ &= \mathrm{Tr}\left(\frac{1}{2}\dot{X}_i\dot{X}_i\right) + \cancelto{0}{\mathrm{Tr}\Big(\partial_0(\dot{X}_i\delta X_i)\Big)} - \mathrm{Tr}\Big(\ddot{X}_i\delta X_i\Big) \end{aligned} \tag{2.16}$$

The second term in the last line of (2.16) is a total derivative and it does not contribute to δS since we assume that $\delta S = 0$ initial and final times. To first order in δX_i, the variation of the interaction term in the action (2.15) can be written as follows

$$
\begin{aligned}
L_{int} + \delta L_{int} &= \frac{1}{4}\,\mathrm{Tr}\,[X_i + \delta X_i, X_j + \delta X_j]^2 \\
&= \frac{1}{4}\,\mathrm{Tr}([X_i, X_j] + [X_i, \delta X_j] + [\delta X_i, X_j] + [\delta X_i, \delta X_j])^2
\end{aligned}
$$

$$
\begin{aligned}
\delta L_{int} &= \frac{1}{4}\,\mathrm{Tr}([X_i, X_j]([X_i, \delta X_j] + [\delta X_i, X_j]) + ([X_i, \delta X_j] + [\delta X_i, X_j])[X_i, X_j]) \\
&= \frac{1}{2}\,\mathrm{Tr}\,[X_i, X_j]([X_i, \delta X_j] + [\delta X_i, X_j]) \\
&= \frac{1}{2}\,\mathrm{Tr}(X_iX_j - X_jX_i)(X_i\delta X_j - \delta X_j X_i + \delta X_i X_j - X_j \delta X_i) \\
&= \frac{1}{2}(X_iX_jX_i\delta X_j - X_iX_j\delta X_j X_i + X_iX_j\delta X_i X_j - X_iX_jX_j\delta X_i \\
&- X_jX_iX_i\delta X_j + X_jX_i\delta X_jX_i - X_jX_i\delta X_iX_j + X_jX_iX_j\delta X_i) \\
&= \frac{1}{2}\,\mathrm{Tr}\,\delta X_i(X_jX_iX_j - X_iX_jX_j - X_jX_jX_i + X_jX_iX_j) \\
&+ \frac{1}{2}\,\mathrm{Tr}\,\delta X_j(X_iX_jX_i - X_iX_iX_j - X_jX_iX_i + X_iX_jX_i) \\
&= \mathrm{Tr}\,\delta X_i(X_jX_iX_j - X_iX_jX_j - X_jX_jX_i + X_jX_iX_j) \\
&= \mathrm{Tr}\,\delta X_i(X_j[X_i, X_j] - [X_i, X_j]X_j) \\
&= \mathrm{Tr}\,\delta X_i[X_j, [X_i, X_j]]\,.
\end{aligned}
\tag{2.17}
$$

Thus, combining in results obtained in (2.16) and (2.17), we get the total variation of the action as

$$
\delta S = \int \mathrm{d}t\,\mathrm{Tr}\Big(\delta X_i(-\ddot{X} + [X_j, [X_i, X_j]])\Big) = 0\,. \tag{2.18}
$$

Therefore, the e.o.m. for X_i's read

$$
\ddot{X}_i + [[X_i, X_j], X_j] = 0\,. \tag{2.19}
$$

where the sum over the index $\mathbb{j}$ is implied. Together with the Gauss Law constraint given in (2.13), (2.19) gives the e.o.m. for the BFSS model.

2.2 The BFSS Matrix Model From Dimensional Reduction

In this section we will show how the BFSS matrix model can be obtained by the dimensional reduction of the pure $SU(N)$ Yang-Mils theory in 9+1 dimensions to

0+1 dimensions.

The Yang-Mills action in 9+1 dimension can be written as

$$S = \frac{1}{g^2} \int \mathrm{d}^{10}x \, \mathrm{Tr}\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) , \tag{2.20}$$

where $\mu, \nu = 0, 1, ..., 9$. We are implicitly using the Minkoski metric with the convention $\eta^{\mu\nu} = (-1, 1, 1, ..., 1)$. Corresponding non-abelian field tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \, , \tag{2.21}$$

The gauge field is denoted as $A_\mu = A_\mu(\vec{\mathbf{x}}, t) = (A_0(\vec{\mathbf{x}}, t), A_i(\vec{\mathbf{x}}, t))$ for $i = 1, ..., 9$ and it is valued in the Lie algebra of $SU(N)$ gauge transformations. The gauge field and the field stregth transform as

$$\begin{aligned} A'_\mu &= U^\dagger A_\mu U + iU^\dagger \partial_\mu U \, , \\ F'_{\mu\nu} &= U^\dagger F_{\mu\nu} U \end{aligned} \tag{2.22}$$

where $U = U(\vec{\mathbf{x}}, t) \in SU(N)$.

Dimensional reduction can be defined by assuming that all components of the gauge field and transformations has no spacial depence, i.e., we have

$$A_\mu(\vec{\mathbf{x}}, t) = (A_0(\vec{\mathbf{x}}, t), A_i(\vec{\mathbf{x}}, t)) \to (A_0(t), X_i(t)) \tag{2.23}$$

$$U = U(\vec{\mathbf{x}}, t) \to U(t) \tag{2.24}$$

Under this assumption, the gauge field A_i transforms as

$$A_\mu \to A'_i = U^\dagger A_i U + iU^\dagger \cancelto{0}{\partial_i U} \tag{2.25}$$

$$= U^\dagger A_i U. \tag{2.26}$$

This shows that A_i transform adjointly under $SU(N)$ gauge transformations. That is, just like an adjoint non-abelian scalar field with $SU(N)$ gauge symmetry. Therefore, we may devise the new notation

$$A_\mu = (A_0(t), A_i(t)) \equiv (A_0(t), X_i(t)) \tag{2.27}$$

as the constituent fields. Inserting (2.27) to (2.21) we find for the components of the electromagnetic field tensor

$$F_{ij} = \partial_i X_j - \partial_j X_i - i[X_i, X_j] \tag{2.28a}$$
$$= -i[X_i, X_j]\,, \tag{2.28b}$$
$$\begin{aligned} F_{0i} &= \partial_0 X_i - \partial_i A_0 - i[A_0, X_i] \\ &= \partial_0 X_i - i[A_0, X_i] \\ &= D_0 X_i\,, \end{aligned} \tag{2.28c}$$

where we have used the fact that all spatial derivatives are vanishing. Thus the Lagrangian takes the form

$$\begin{aligned} L = -\frac{1}{4g^2}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) &= \operatorname{Tr}\left(-\frac{1}{4g^2}(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) \\ &= \operatorname{Tr}\left(-\frac{1}{4g^2}(2F_{0i}F^{0i} + F_{ij}F^{ij})\right) \\ &= \operatorname{Tr}\left(-\frac{1}{4g^2}(2(D_0X_i)(-D_0X_i) - [X_i, X_j]^2)\right) \\ &= \frac{1}{g^2}\operatorname{Tr}\left(\frac{1}{2}(D_0X_i)^2 + \frac{1}{4}[X_i, X_j]^2\right) \end{aligned} \tag{2.29}$$

and the action is

$$S = \frac{1}{g^2}\int \mathrm{d}t\, \operatorname{Tr}\left(\frac{1}{2}(D_0X_i)^2 + \frac{1}{4}[X_i, X_j]^2\right), \tag{2.30}$$

which is the BFSS action that we have introduced in equation (2.1). Note that in this process we have redefined the Yang-Mills compling constant $\frac{1}{g^2} \to \frac{V}{g^2} \equiv \frac{1}{g^2}$ where V is the volume of the 9-dimensional space we have integrated over.

2.3 Chaos in BFSS Model

In the past decade or so investigation of chaotic dynamics emerging from quantum mechanical systems has been important field of research in theoretical studies. In particular, there has been significant effort to understand chaos in quantum mechanical matrix models such as the BFSS model [13, 16, 26], as well as SYK(Sachdev, Ye, Kitaev) [27,28] which have known gravity duals via Ads/CFT correspondence [4,24] One of the major problems in black hole physics can be described as information processing and thermalization of black holes and it is expected that understanding chaos in the matrix models can shed some light on these difficult problems. Recent progress in the literature indicates that new interesting conjectures can be obtained by combining quantum information theory and gravity. One such a conjucture is proposed by Susskind and Sekino in 2008 ad goes the name *Fast-Scrambling Conjecture* [9]. Based on ideas from quantum information theory, these authors conjectured that black holes are fast scramblers, meaning that they scramble information faster than any other physical system and become chaotic. In other words, it can also be stated that, physically reasonable systems cannot have a faster scrambling time than a black hole. In equations, one may consider scrambling time τ for a physical system.

$$\tau \geq C(\beta) \log(N) \tag{2.31}$$

where N defined as the total number of degrees of freedom and $C(\beta)$ is a function of β, where β is the inverse Hawking temperature of a black hole [1].

In a paper written by Gur-Ari et al. [12] classical chaotic dynamics of the BFSS matrix model is studied in detail. This matrix model has a well-known gravity dual, dual is described by a "black-brane" phase on the matrix theory side, although the gravity dual is valid at low temperature regime, numerical studies performed so far [4,6,8,26] show no phase transition in the matrix models from low to high temperetures, and therefore authors of [12] proposed that some features of the gravity dual should also survive at the high temperature limit too. This latter high temperature limit is well approximated by the BFSS model at classical level and in [10, 12] it is demonstrated that BFSS model is a fast scrambler.

There are other results in this paper [10, 12] regarding the chaotic dynamics of BFSS model and to prepare for his let us first review the general meaning of Lyapunov exponents which are used to characterize chaotic dynamics.

2.3.1 Lyapunov Exponents

Several equivalent definitions may be stated to describe chaotic behavior in dynamical systems. Suppose we have a dynamical system described by coordinates of the phase space, $y(t) \equiv (q_i^{(+)}, p_i^{(+)})(i : 1...N)$. Also consider that in the phase space, we have two nearby trajectories, denoted as $y(t)$ and $y(t) + \delta y(t)$, the time evolution of $y(t)$

describes trajectories in the phase spae, where $\delta y(t)$ can be understood as the seperation between these two trajectories. Suppose that at $t = 0$, $\delta y(0)$ is sufficiently small so that two initial conditions are only infinitesimally apart. If under time evolution, the system responds in such a way that

$$|\delta y(t)| = \delta y(0) e^{\lambda_i t} \tag{2.32}$$

with at least one $\lambda_i > 0$ than, we may say the distance between two trajectories in phase space grow exponentially apart. λ_i are called Lyapunov exponents. This result can be translated into the general called phrase "high sensitivity to initial conditions" in the context of chaotic dynamics.

2.3.2 Quantum Chaos

Consider the canonical variables $q(t)$ and $p(t)$ of a quantum mechanical system. The out-of-time-order correlator (OTOC) of these may be defined as [29]

$$C(t)_T = -\langle [q(t), p(0)]^2 \rangle \tag{2.33}$$

where the expectation value is to be evaluated in an appopriate state. In the semi-classical limit, we can replace the commuatator $[q(t), p(0)]$ with the Poisson bracket $i\hbar\{q(t), p(0)\}$. In particular, we have

$$\begin{align} \{q(t), p(0)\} &= \frac{\partial q(t)}{\partial q(0)} \underbrace{\frac{\partial p(0)}{\partial p(0)}}_{1} - \frac{\partial q(t)}{\partial p(0)} \cancelto{0}{\frac{\partial p(0)}{\partial q(0)}} \tag{2.34} \\ &= \frac{\partial q(t)}{\partial q(0)}. \tag{2.35} \end{align}$$

Therefore, we have

$$C(t)_T = -\left(i\hbar \frac{\delta q(t)}{\delta q(0)} \right)^2 \tag{2.36}$$

in the semiclassical limit. However, as we discussed in the previous section, from (2.32) we have

$$\frac{\delta q(t)}{\delta q(0)} \propto e^{\lambda_L t}, \tag{2.37}$$

where λ_L is the Lyapunov exponent. Therefore, we have

$$C(t)_T \propto \hbar^2 e^{2\lambda_L t} \tag{2.38}$$

Thus, in the semi-classical limit, the logarithm of OTOC is proportional to the Lyapunov exponent of the system, i.e. $\log(C(t)_T) \propto \lambda_L$.

2.4 Temperature dependence of the Lyapunov exponent

For the BFSS matrix model, we have two parameters. Namely, these are the size of the matrices N and the coupling constant g. At large N and weak coupling BFSS model has the t'Hooft limit, which is characterized by a $1/N$ expansion [12], and governed by the 't Hooft coupling parameter $\lambda_{'tHooft} = g^2 N$, $[\lambda_{'tHooft}] = [Length]$, which is kept finite. At the classical level, temperature T, may be introduced as an additional parameter via the use of the equipartition theorem. Since Lyapunov exponent has the units of $[time]^{-1} = [Length]^{-1}$, we can determine by simple dimensional analysis that,

$$\lambda_L \propto (\lambda_{'tHooft} T)^{1/4} \tag{2.39}$$

Since $\lambda_{'tHooft}$ and T are only dimensionalful parameter of the model. Constant of proportionality can only be determined by the dynamics of the theory and in [12] it is found that $\lambda_T = 0.2924(3)(\lambda_{'tHooft} T)^{1/4}$.

2.5 Energy Temperature dependence

In order to determine the temperature of the BFSS model at the classical level, we follow the approach given in Gur-Ari et al. [12]. From classical equipartition theorem [30], we already know the average kinetic energy of the system must be proportional to the temperature, and precisely given as,

$$\langle K \rangle = \frac{1}{2} n_{d.o.f} T \tag{2.40}$$

where we work in the units in which the Boltzmann constant, k_B, is set to unity. In (2.40) $n_{d.o.f}$ stands for the number of total degrees of freedom of the system. Since

(2.40) gives temperature dependence of the kinetic energy. We may seek to find a general relation between total energy $E = \langle K \rangle + \langle V \rangle$ and temperature. To do so let us also write down the virial theorem [30, 31] in the form

$$\langle 2K \rangle = \left\langle \sum_j X_j \frac{dV}{dX_j} \right\rangle \tag{2.41}$$

In general if V is a n^{th} order homogenous polynomial of the generalized coordinates, then we may write

$$V = \sum_{l_1,l_2...l_n} c_{l_1,l_2...l_n} X_{l_1} X_{l_2} ... X_{l_n} \tag{2.42}$$

In this case (2.41), reduces into

$$\begin{aligned}
\sum_{k=1}^{n} X_k \frac{\partial V}{\partial X_k} &= \sum_{k=1}^{n} X_k \sum_{l_1,l_2...l_n} c_{l_1,l_2...l_n} X_{l_1} X_{l_2} ... X_{l_{k-1}} X_{l_{k+1}} ... X_{l_n} \delta_{k\, l_k} \\
&= \sum_{k=1}^{n} \sum_{l_1,l_2...l_n} c_{l_1,l_2...l_n} X_{l_1} X_{l_2} ... X_{l_n} \\
&= nV
\end{aligned} \tag{2.43}$$

Therefore $\langle K \rangle = \frac{n}{2} \langle V \rangle$. Combining (2.40) and (2.43) and noting that $\langle E \rangle = \langle K \rangle + \langle V \rangle$ we find

$$\langle E \rangle = \frac{n+2}{2n} n_{d.o.f} T \tag{2.44}$$

What if we have a potential which is not a homogeneous polynomial of the generalized coordinate? To elaborate, suppose that we have a simple Hamiltonian

$$H = P_x^2 + P_y^2 + V(x, y) \tag{2.45}$$

with the potential given as

$$V(x, y) = x^4 + y^4 + x^2y^2 + \alpha_1 x^2 + \alpha_2 y^2 \tag{2.46}$$

Using the virial theorem, we obtain

$$\begin{aligned}
\langle 2K \rangle &= 4x^4 + 4y^4 + 4x^2y^2 + 2\alpha_1 x^2 + 2\alpha_2 y^2 \\
&= 4(x^4 + y^4 + x^2y^2 + \alpha_1 x^2 + \alpha_2 y^2) - 2\alpha_1 x^2 - 2\alpha_2 y^2 \\
&= \langle 4V \rangle - 2\alpha_1 x^2 - 2\alpha_2 y^2
\end{aligned} \tag{2.47}$$

which yields

$$\langle K \rangle = \begin{cases} \geq \langle 2V \rangle & \alpha_1 < 0,\, \alpha_2 < 0 \\ \leq \langle 2V \rangle & \alpha_1 > 0,\, \alpha_2 > 0 \end{cases} \tag{2.48}$$

We may also write, $\langle 2V \rangle = 2(x^4 + y^4 + x^2y^2)$ implying that $\langle K \rangle > \langle V \rangle$ regardless the signs of α_1 and α_2. For a potential of degree n and subleading terms of even order, say $2m$, $\langle K \rangle > \langle V \rangle$ remains valid, while (2.48) generalizes to several branches depending on the signs of the cofficients of the subleading terms.
Using the equipartition theorem in (2.40) and the fact that $\langle K \rangle > \langle V \rangle$, we have the inequality

$$\langle E \rangle = \langle V \rangle + \langle K \rangle \leq n_{d.o.f}T \tag{2.49}$$

However, at $n = 4$ for instance, if the potential in (2.46) is such that $\alpha_1 < 0$, $\alpha_2 < 0$ the first line of (2.40) holds and $\langle V \rangle < \frac{1}{2}\langle K \rangle$, so we may replace (2.49) with the inequality

$$\langle E \rangle \leq \frac{3}{4} n_{d.o.f}T \tag{2.50}$$

and for general case,

$$\langle E \rangle \leq \frac{n+2}{2n} n_{d.o.f}T \tag{2.51}$$

if and only if all the subleading terms of the potential are positive definite. We will make us of both in the result (2.50) and (2.51) in sections 3.2.3 and **??** in chapter 3.

2.6 Maldacena-Shenker-Stanford (MSS) bound

In an article [29] authored Maldacena, Shenker and Stanford (MSS) in 2015, a sharp bound on chaos for thermal quantum mechanical systems with large number of degrees of freedom is conjectured. This bound is temperature dependent in the sense that it implies a upper bound the largest Lyapunov exponent in the form $\lambda_L \leq \frac{2\pi T k_\beta}{\hbar}$. In, what follows we will continue to work in the units in which $k_\beta = 1$ and $\hbar = 1$. In these units this bound is simply expressed as $\lambda_L \leq 2\pi T$. It is suggested in various contexts that chaos in quantum systems can be measured by using the out of time order correlator (OTOC) [32]. We have already defined OTOC for the canonical variables q and p in this section, but now in order to expand the discussion we consider the OTOC of two observables labeled as V and W and follow the discussion in [29]. The OTOC we start with is therefore

$$C(t)_T = -\langle [W(t), V(0)]^2 \rangle \tag{2.52}$$

The *thermal expectation value* used in this expression is defined as $\langle . \rangle = Z^{-1} \operatorname{Tr}\big(e^{-\beta H} .\big)$ where $Z = \operatorname{Tr}\big(e^{-\beta H}\big)$ is the cannonical partition function, $\beta = 1/T$ and H is the Hamiltonian of the system. Note that $e^{-\beta H}$ is the time evolution operator in Euclidean time β so $y^2 \propto \frac{1}{\sqrt{2}} e^{-\frac{1}{2}H}$ corresponds to time evolution of $\beta/2$ and y corresponds to time evolution of $\beta/4$. The square of the commutator generally requires a regularization in quantum field theory [29]. One way of solving this is moving one of the commutators half way around the thermal circle. This gives

$$C(t)_R = -\operatorname{Tr}\big(y^2 [W(t), V(0)] y^2 [W(t), V(0)]\big) \tag{2.53}$$

where $y^4 = \frac{1}{Z} e^{-\beta H}$. After expanding this, we get

$$\begin{aligned} C(t)_R = &= \operatorname{Tr}\big(y^2 W(t) V y^2 V W(t)\big) + \operatorname{Tr}\big(y^2 V W(t) y^2 W(t) V\big) \\ &- \operatorname{Tr}\big(y^2 W(t) V y^2 W(t) V\big) - \operatorname{Tr}\big(y^2 V W(t) y^2 V W(t)\big) \end{aligned} \tag{2.54}$$

Although OTOC has this form, last two terms are related to a function $F(t)$, which may be defined as

$$F(t) = \operatorname{Tr}(y V y W(t) y V y W(t)) \tag{2.55}$$

$F(t)$ corresponds to insertion of the V and W operators at equal spacing around the thermal circle. To be more precise, we consider time as a complex parameter, by letting $t \to t + i\tau$. Here t represents the Lorentzian, and τ represents the Euclidean times, respectively. This new definition of thermal expectation values $\langle . \rangle = Z^{-1} \operatorname{Tr}\big(e^{-\beta H} .\big)$ implies the periodicity in Euclidean time τ, with period β. Thus Euclidean time can be represented by a circle with angles $\frac{2\pi\tau}{\beta}$. Lorentzian time is represented as folds in this diagram to produce $W(t)$. Hence, at complex time $t + i\tau$, $F(t + i\tau)$ is given by rotating $VW(t)$ by angle corresponds to $\tau = \beta/4$, while $F(t)$ corresponds to onset of the operator V and $W(t)$ at equal spacing around the Euclidean time circle. Lets look at the values of $F(t + i\frac{\beta}{4})$ and $F(t - i\frac{\beta}{4})$. These are given as

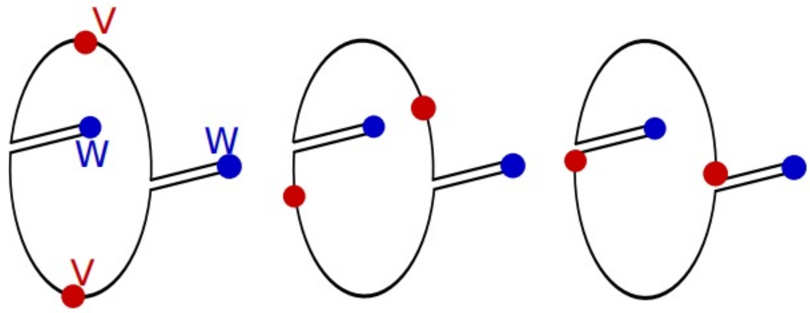


Figure 2.1: Thermal cycle

$$F(t+i\frac{\beta}{4}) = \mathrm{Tr}\big(y^2W(t)Vy^2W(t)V\big) \tag{2.56}$$

$$F(t-i\frac{\beta}{4}) = \mathrm{Tr}\big(y^2VW(t)y^2VW(t)\big) \tag{2.57}$$

They correspond to rotating one of the $VW(t)$ factors by $\tau = \beta/4$ along the Euclidean time circle. Therefore $C(t)_T$ can be written as

$$\begin{aligned} C(t)_R =& \mathrm{Tr}\big(y^2W(t)Vy^2VW(t)\big) + \mathrm{Tr}\big(y^2VW(t)y^2W(t)V\big) \\ &- F(t+i\frac{\beta}{4}) - F(t-i\frac{\beta}{4}) \end{aligned} \tag{2.58}$$

For small t, the $C(t)_R$ is small because in first term of 2.58 and 4th and second term of 2.58 $\&$ 3rd, respectively terms are close to each other. Moreover, we can interpret the terms in the first line of (2.58) as norms of states $yW(t)Vy^{-1}\left|\beta\right\rangle$ where $\left|\beta\right\rangle = Z^{-1/2}\Sigma_n e^{-\beta E_n/2}\left|\bar{n}\right\rangle_L \left|n\right\rangle_R$, so they remain at order one in t when t becomes large. Therefore in order for $C(t)_R$ to increase, $F(t)$ values are expected to decrease. This conclusion can also be drawn by constructing a thermal cycle where the Lorentzian time separation is large. When t value become large, the four point correlation function $VW(t)VW(t)$ decreases due to the large seperation between the pairs. However, for the ordering $VVW(t)W(t)$, it closes to two disconnected two point correlation functions F_d. Therefore for large t, $F(t)$ value decreases. Using ideas emerging from Ads/CFT correspondence and holography authors of [29] gives estimate of $C(t)_R$ and determine a relation between $F(t)$ and F_d. Since explicit calculations in this context are beyond the scope of this thesis, we will only state results of [29], which is given as

$$F(t) = F_d - \frac{f_0}{N^2} e^{\frac{2\pi t}{\beta}} + \mathcal{O}(\frac{1}{N^4}) \tag{2.59}$$

where $F_d = \mathrm{Tr}(y^2 V y^2 V)\,\mathrm{Tr}(y^2 W y^2 W)$ and f_0 is a positive order one constant. After some eleboration this leads to the inequality given as

$$\frac{d}{dt}(F_d - F(t)) \leq \frac{2\pi}{\beta}(F_d - F(t)) \tag{2.60}$$

Recall that in (2.32) we have determined that if the system is chaotic, $\frac{\delta q(t)}{\delta q(0)} \propto e^{\lambda_L t}$, for a system, where λ_L is the largest Lyapunov exponent. We may expect that if the system is chaotic then the value of $F_d - F(t)$ must be proportional to $e^{\lambda_L t}$, i.e.

$$F_d - F(t) \propto \epsilon e^{\lambda_L t} + ... \tag{2.61}$$

combining this with diffential equation in (2.60), we get

$$\lambda_L \leq \frac{2\pi}{\beta} = 2\pi T \tag{2.62}$$

(2.62) is called *the MSS bound*. It is conjectured in [29] that this bound is valid for all quantum mechanical systems with large number of degrees of freedom, and which are holographically dual to gravitational theories. In chapters 3 and 4 we will consider mass deformed forms of BFSS and ABJM models at the classical level, and we will be able to put bounds on temperature T above which the MSS bound is satisfied and how it will eventually be violated as low enough temperatures. The latter is due to the fact that we study these model at classical level, which approaches quantum models only at high temperatures and breaks down eventually at sufficiently low temperatures. Let us also note that, restoring k_β and $\hbar$ we may write, $\lambda_L \leq \frac{2\pi k_\beta T}{\hbar}$ and in the classical limit corresponding to $\hbar \to 0$, no bound on the Lyapunov exponent remains.

2.7 Fuzzy Two Sphere

From semi-classical point of view, fuzzy two sphere can be viewed as a 2-sphere with non-commutative coordinates. One way to obtain fuzzy S^2 is the quantization of the map $S^3 \to S^2$. Suppose we have a coordinate $z \in \mathbb{C}^2 \setminus \{0\}$ which is defined as

$$z = (\alpha_1 + i\beta_1, \alpha_2 + i\beta_2) \tag{2.63}$$

Point z can be mapped onto a sphere via given transformation $\xi = \frac{z}{|z|}$, then we have

$$|\xi|^2 = \alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 = 1\,. \tag{2.64}$$

(2.64) describes a 3-sphere embedded into four dimensional space. Furthermore, the projection map can be defined as $x_i(\xi) = \xi^\dagger \sigma_i \xi$. It is easy to see that under $U(1)$ transformations $\xi \to \xi e^{i\theta}$, projection map is invariant. Similarly, we can notice that $x_i(\xi)$ forms the components of a 3-vector $\vec{\mathbf{x}}(\xi)$. The norm of the vector $\vec{\mathbf{x}}(\xi)$ is

$$\begin{aligned}
\|\vec{\mathbf{x}}(\xi)\|^2 = x_i x_i &= (\xi_a^\dagger (\sigma_\alpha)_{ab} \xi_b)(\xi_c^\dagger (\sigma_\alpha)_{cd} \xi_d) \\
&= \xi_a^\dagger \xi_b \xi_c^\dagger \xi_d (\sigma_\alpha)_{ab} (\sigma_\alpha)_{cd} \\
&= \xi_a^\dagger \xi_b \xi_c^\dagger \xi_d (\delta_{ab}\delta_{cd} - 2\epsilon_{ac}\epsilon_{bd}) \\
&= \xi_a^\dagger \xi_b \xi_c^\dagger \xi_d (-\delta_{ab}\delta_{cd} + 2\delta_{ad}\delta_{bc}) \\
&= -1 + 2 \\
&= 1\,,
\end{aligned} \tag{2.65}$$

where $\alpha = 1, ..., 3$ and $a, b, c, d = 1, 2$. Therefore the components of $\vec{\mathbf{x}}(\xi)$ can be seen as coordinates of S^2 embedded in $\mathbb{R}^3$. This map is called Hopf fibration $U(1) \to S^3 \to S^2$. We can obtain fuzzy S^2 by quantizing the $S^3 \to S^2$ map by changing complex point z and $z^\dagger$ with annihilation creation operators, a and $a^\dagger$. Lets denote new variables as

$$\begin{aligned}
\hat{\xi_\beta} &= \frac{1}{\sqrt{\hat{N}}} a_\beta \\
\hat{\xi_\beta}^\dagger &= \frac{1}{\sqrt{\hat{N}}} a_\beta^\dagger
\end{aligned} \tag{2.66}$$

where $\hat{N} = a_\beta^\dagger a_\beta$. Then we can define quantized projection operator as

$$\begin{aligned}
\widehat{x_i} &= \widehat{\xi}^\dagger \sigma_i \widehat{\xi} \\
&= \frac{1}{\sqrt{\widehat{N}}} a^\dagger \sigma_i a \frac{1}{\sqrt{\widehat{N}}} \\
&= \frac{1}{\widehat{N}} a^\dagger \sigma_i a\,.
\end{aligned} \tag{2.67}$$

The last step follows from the fact that $\left[a_i^\dagger a_j, \frac{1}{\sqrt{N}}\right] = 0$. Since $\left[\widehat{x_i}, \widehat{N}\right] = 0$, $\widehat{x_i}$ can be restricted to act on the $(n+1)$-dimensional subspace of the Fock space spanned by the vectors

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0, 0\rangle\ , \tag{2.68}$$

where $n_1 + n_2 = n$. Then, $\widehat{x}_i$ are linear operators on this finite-dimensional space and act on it as $(n+1) \times (n+1)$ Hermitian matrices.

$SU(2)$ generators L_i in terms of Schwinger construction are given as

$$L_i = \frac{1}{2} a^\dagger \sigma_i a \tag{2.69}$$

and satisfy the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k \, . \tag{2.70}$$

The coordinates of fuzzy S^2 can therefore be expressed in the form

$$\widehat{x}_i = \frac{2}{n} L_i \, , \tag{2.71}$$

where $\frac{2}{n}$ is the noncommutative scaling factor. The commutation relation of the coordinates of fuzzy S^2 is

$$[\widehat{x}_i, \widehat{x}_j] = \frac{2}{n} i\epsilon_{ijk} \widehat{x}_k \, . \tag{2.72}$$

When n goes to infinity, this commutator vanishes identically and the standard S^2 is recovered. The Casimir operator for $SU(2)$ is L_i^2 and has the eigenvalues $\frac{n}{2}\left(\frac{n}{2}+1\right)$ on the spin $j = \frac{n}{2}$ representation of $SU(2)$. We therefore have

$$L_i^2 = \frac{n}{2}(\frac{n}{2} + 1)\mathbb{1}_{n+1} \tag{2.73}$$

on any state of a $SU(2)$ irreducible representations with spin $J = \frac{n}{2}$. By using this equation, radius of fuzzy S^2 is found as

$$\begin{aligned} x_i^2 &= \left(\frac{2}{n}\right)^2 \frac{n}{2}\left(\frac{n}{2}+1\right)\mathbb{1}_{n+1} \\ &= \left(1 + \frac{2}{n}\right)\mathbb{1}_{n+1} \, . \end{aligned} \tag{2.74}$$

2.8 Fuzzy Four Sphere

Fuzzy four-sphere, S_F^4 can be constructed as follows. We introduce the γ-matrices in five dimensions that are associated to the isometry group $SO(5)$ of S^4. They are $4x4$ matrices satisfying the anticommutation relations

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab} 1_4 \, , \tag{2.75}$$

where $a, b = 1, .., 5$.

Generators of the spinor representation $(0, 1)$ of $SO(5)$ are given as

$$G_{ab} = -\frac{i}{4}[\gamma_a \,, \gamma_b] \,, \tag{2.76}$$

and fulfill the $SO(5)$ commutation relations

$$[G_{ab} \,, G_{cd}] = i(\delta_{ac}G_{bd} + \delta_{bd}G_{ac} - \delta_{ad}G_{bc} - \delta_{bc}G_{ad}) \,. \tag{2.77}$$

γ_a act on the four-dimensional spinor space $\mathbb{C}^4$ which is the carrier space of the $(0, 1)$ IRR of $SO(5)$. Let us define the Hilbert space $\mathcal{H}_n$, as the carrier space of the $(0, n)$ IRR of $SO(5)$, which may be formed as the n-fold symmetric tensor product of $\mathbb{C}^4$ as

$$\mathcal{H}_n = \left(\mathbb{C}^4 \otimes \cdots \otimes \mathbb{C}^4\right)_{Sym} \,. \tag{2.78}$$

and has the dimension

$$N = dim(0, n) = \frac{1}{6}(n+1)(n+2)(n+3) \,. \tag{2.79}$$

We may as well form the n-fold symmetric tensor product of the γ-matrices

$$X_a = \frac{1}{2}(\gamma_a \otimes 1_4 \otimes ... \otimes 1_4 + ... + 1_4 \otimes ... \otimes 1_4 \otimes \gamma_a) \tag{2.80}$$

which carry the $(0, n)$ IRR of $SO(5)$ and naturally act on $\mathcal{H}_n$. X_a are $N \times N$ Hermitian matrices, which satisfy the defining relations for the fuzzy four-sphere, which are given as

$$\begin{aligned} X_a X_a &= \frac{1}{4} n(n+4) 1_N \,, \\ \epsilon^{abcde} X_a X_b X_c X_d &= (n+2) X_e \,, \end{aligned} \tag{2.81}$$

where $X_a X_a$ is an $SO(5)$ invariant operator.Commutators of X_a yields

$$[X_a, X_b] = i M_{ab} \,, \tag{2.82}$$

where M_{ab} are the generators of $SO(5)$ in the $(0, n)$ IRR and satisfy the same commutation relations as G_{ab}. We also have that X_a transform as vectors under the $SO(5)$ rotations generated by M_{ab}, i.e.

$$[M_{ab}, X_c] = i(\delta_{ac} X_b - \delta_{bc} X_a) \,. \tag{2.83}$$

From (2.82) we see that, as opposed to the construction of S_F^2, the algebra of X_a's defining the S_F^4 do not close. A more detailed analysis yields that S_F^4 may be understood as a squashed CP_F^3 with S_F^2 fibers In other words, we may state that S_F^4 comes attached with a fuzzy two sphere at its every point. Further details of these features may be found in [4].

CHAPTER 3

YANG-MILLS MATRIX MODELS WITH DOUBLE MASS DEFORMATION

3.1 Massive deformation of BFSS action

The bosonic part of the BFSS action given in (2.1). In this section, we will introduce two distinct mass deformation terms to (2.1). A gauge invariant double mass deformation of (2.1) may be specified as

$$S_{YMM} = N^2 \int dt\, L_{YMM} := N^2 \int dt\, Tr \left(\frac{1}{2}(D_t X_I)^2 + \frac{1}{4}[X_I, X_J]^2 - \frac{1}{2}\mu_1^2 X_a^2 - \frac{1}{2}\mu_2^2 X_i^2 \right) , \tag{3.1}$$

where the indices a and i take on the values $a = 1, .., 5$ and $i = 6, 7, 8$, respectively. Let us also note that (3.1) is written in the 't Hooft limit in which 't Hooft coupling $\lambda_{'tHooft} = g^2 N$ is kept fixed, while N is taken large and g^2 is taken small. $\lambda_{'tHooft}$ has units of (Length)$^{-3}$ and without loss of generality it can be set to unity by scaling all the dimensionful quantities in the action in units of $\lambda^{1/3}$. Also it can restored by scaling $\lambda_{'tHooft}$ set to unity. We can restore $\lambda_{'tHooft}$ by scaling $X_i \to \lambda^{-1/3} X_i$, $A_t \to \lambda^{-1/3} A_t$, $t \to \lambda^{1/3} t$, $\mu_i^2 \to \lambda^{-2/3} \mu_i^2$, if needed. In (3.1) the mass deformation terms ${\mu_1}^2$ and ${\mu_2}^2$ are the quadratic deformations which preserves $U(N)$ gauge symmetry. Moreover they also break the $SO(9)$ down to $SO(5) \times SO(3) \times \mathbb{Z}_2$. The discrete $\mathbb{Z}_2$ factor is due to the $X_9 \to -X_9$ symmetry but during our ansatz choice, we will choice it as $X_9 = 0$. Therefore, this $\mathcal{Z}_\in$ will be insignificant. The equation of motion of action (3.1)for X_I take form in the $A_t = 0$ gauge as

$$\ddot{X}_a + [X_I, [X_I, X_a]] + \mu_1^2 X_a = 0 \,, \tag{3.2a}$$

$$\ddot{X}_i + [X_I, [X_I, X_i]] + \mu_2^2 X_i = 0 \,, \tag{3.2b}$$

$$\ddot{X}_9 + [X_I, [X_I, X_9]] = 0 \,, \tag{3.2c}$$

while form of the Gauss law constraint given in (2.13) remains same. The massive deformation of BFSS model has already known for a while. The BMN matrix model [2] is the massive deformation of the BFSS model which preserves maximal amount of supersymmetry. However, in our present work, we focus on exploring the emerging chaotic dynamics from the Yang-Mills matrix models which could allow for not only fuzzy two-sphere configurations but also higher dimensional fuzzy spheres, and in particular a fuzzy four-sphere. As we mentioned earlier, the two separate mass terms break the $SO(9)$ into $SO(5) \times SO(3) \times \mathbb{Z}_2$ that produces subgroups of equations. The underlying motivation for introducing the specific massive deformation in (3.1) comes from the fact that in two distinct limiting cases, the equations of motion can be solved either with fuzzy two-sphere or fuzzy four-sphere configurations. To be more precise, we have for $X_i = 0 = X_9$, (3.2b) and (3.2c) are satisfied identically, while (3.2a) takes the form

$$\ddot{X}_a + [X_b, [X_b, X_a]] + \mu_1^2 X_a = 0 \,, \tag{3.3}$$

which is satisfied by the fuzzy four sphere configurations $X_a \equiv Y_a$ for $\mu_1^2 = -16$. Here Y_a are $N \times N$ matrices carrying the $(0, n)$ UIR of $SO(5)$ with $N = \frac{1}{6}(n+1)(n+2)(n+3)$ and satisfying the defining properties given in (2.81). Whereas, in the other extreme, one may set $X_a = 0 = X_9$, with the only remaining non-trivial equation of motion

$$\ddot{X}_i + [X_j, [X_j, X_i]] + \mu_2^2 X_i = 0 \,, \tag{3.4}$$

which is solved by fuzzy two sphere configurations $X_i \equiv Z_i$ or their direct sum for $\mu_2^2 = -2$. In this case, X_i are $N \times N$ matrices carrying the spin $j = \frac{N-1}{2}$ UIR of $SO(3) \approx SU(2)$.

By taking all these into account, we determine an ansätze configurations involving fuzzy two- and four- spheres with collective time dependence, which fulfill the Gauss Law constraint given in (2.13). After taking trace over the fuzzy-two and four-sphere configurations, we aim to obtain a reduced models that consist of only four phase space degrees of freedom and we will investigate chaotic behaviour of those reduced models.

3.1.1 Ansatz *I* and the Effective Action

A non trivial but simple configuration can be chosen as multiplying the fuzzy four- and two- sphere matrices with separate collective time-dependent functions. So we have

$$X_a = r(t)\, Y_a\,, \quad X_i = y(t)\, Z_i\,, \quad X_9 = 0\,, \tag{3.5}$$

where $r(t)$ and $y(t)$ are real functions of time. We consider spin-j=$\frac{N-1}{2}$ IRR of $SU(2)$ as the fuzzy S^2 configuration. Furthermore, we form Z_i as a block-diagonal matrix which is formed by direct sum of different IRRs of $SU(2)$. Although Z_i exist at every matrix level, this is not the case for Y_a. Fuzzy four spheres exist at the matrix levels $4, 10, 20...$ as given by dimension $N = \frac{1}{6}(n+1)(n+2)(n+3)$ of the IRR (0,n) of $SO(5)$. Therefore, fuzzy two spheres must be taken at the matrix levels matching the value of N. The mass parameters μ_1^2 and μ_2^2 are unspecified at the beginning. However, in the limitting values for the static solutions of (3.3) and (3.4), the equations are only satisfied in case of $\mu_1^2 = -16$ and $\mu_2^2 = -2$.
After substituting (3.5) configuration in the action (3.1) and taking the trace over the fuzzy four- and two-sphere with matrix levels $N = \frac{1}{6}(n+1)(n+2)(n+3)$ for $n = 1, 2, \cdots, 6$, we obtain the Lagrangian of the reduced models in the form

$$L_n = N^2(c_1\dot{r}^2 + c_2\dot{y}^2 - 8c_1 r^4 - c_2 y^4 - c_1\mu_1^2 r^2 - c_2\mu_2^2 y^2 - c_3 r^2 y^2)\,, \tag{3.6}$$

where $c_\beta = c_\beta(n)$ ($\beta = 1, 2, 3$) depend on n and their values for $n = 1, 2, \cdots, 7$ are listed in the table 3.1 given below. Coefficients given for $n = 7$ in table 3.1 are evaluated by obtaining a polynomial function of n approximating[1] $c_\beta(n)$ for $n = 1, 2, \cdots, 6$ and interpolating this result to $n = 7$.

The corresponding Hamiltonian is easily obtained from (3.6), and it is given by

$$\begin{aligned} H_n(r, y, p_r, p_y) &= \frac{{p_r}^2}{4c_1N^2} + \frac{{p_y}^2}{4c_2N^2} + N^2(8c_1\, r^4 + c_2\, y^4 + c_1\, \mu_1^2 r^2 + c_2\, \mu_2^2 y^2 + c_3\, r^2 y^2)\,, \\ &=: \frac{{p_r}^2}{4c_1N^2} + \frac{{p_y}^2}{4c_2N^2} + N^2 V_n(r, y)\,, \end{aligned} \tag{3.7}$$

[1] These polynomial functions are given as

$$\begin{aligned} c_1(n) &= \frac{1}{2}n(n+4)\,, \\ c_2(n) &= \frac{1}{288}(n^6 + 12n^5 + 58n^4 + 144n^3 + 193n^2 + 132n)\,, \\ c_3(n) &= 0.0093n^7 + 0.20n^6 + 1.35n^5 + 4.22n^4 + 6.99n^3 + 5.44n^2 + 2.43n + 0.36\,. \end{aligned}$$

Table 3.1: Numerical values of coefficients $c_\beta(n)$ for Ansätz *I*.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
c_1	2.5	6	10.5	16	22.5	30	38.5
c_2	1.9	12.4	49.9	153	391.9	881.9	1800
c_3	21	207.7	1080	3970	11691	29493	66345

where $V_n(r,y)$ denotes the potential function. Lets first calculate the Hamilton's equations of motion for the reduced Hamiltonian (3.7),

$$\dot{r} - \frac{p_r}{2c_1N^2} = 0\,, \qquad \dot{y} - \frac{p_y}{2c_2N^2} = 0\,, \tag{3.8a}$$

$$\dot{p_r} + N^2(32c_1r^3 + 2c_1\mu_1^2 r + 2c_3ry^2) = 0\,, \qquad \dot{p_y} + N^2(4c_2y^3 + 2c_2\mu_2^2 y + 2c_3r^2y) = 0\,. \tag{3.8b}$$

Taking the mass parameter values as $\mu_1^2 = -16$ & $\mu_2^2 = -2$, (3.8) become

$$\dot{r} - \frac{p_r}{2c_1N^2} = 0\,, \qquad \dot{y} - \frac{p_y}{2c_2N^2} = 0\,, \tag{3.9a}$$

$$\dot{p_r} + N^2(32c_1r^3 - 32c_1r + 2c_3ry^2) = 0\,, \qquad \dot{p_y} + N^2(4c_2y^3 - 4c_2y + 2c_3r^2y) = 0\,. \tag{3.9b}$$

For a more profound analysis of chaos emerged from these models, we should first calculate the fixed points corresponding to the e.o.m. in (3.9). In the literature, fixed points are defined as stationary points of the phase space, i.e

$$(\dot{r}, \dot{y}, \dot{p_r}, \dot{p_y}) \equiv (0,0,0,0)\,. \tag{3.10}$$

The two of the corresponding equations are trivally solved by $(p_r, p_y) \equiv (0,0)$. That means all the fixed points are confined to $(p_r, p_y) \equiv (0,0)$ plane in the phase space. The rest of the equations are

$$\begin{aligned} -32c_1r^3 + 32c_1r - 2c_3ry^2 &= 0\,, \\ -4c_2y^3 + 4c_2y - 2c_3r^2y &= 0\,, \end{aligned} \tag{3.11}$$

and have the general set of solutions given as

$$(r,y) \equiv \{(0,0)\,, (\pm1,0)\,, (0,\pm1)\,, (\pm h_1, \pm h_2)\,, (\pm h_1, \mp h_2)\}\,, \tag{3.12}$$

Table 3.2: Numerical values of h_1 and h_2.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
h_1	0.26	0.28	0.27	0.26	0.24	0.23
h_2	0.6	0.38	0.25	0.17	0.12	0.093

where h_1 and h_2 are given in terms of c_β as

$$h_1 = -\sqrt{2}i\frac{\sqrt{-c_2c_3+16c_1c_2}}{\sqrt{c_3^2-32c_1c_2}}\,, \quad h_2 = -4i\frac{\sqrt{2c_1c_2-c_1c_3}}{\sqrt{c_3^2-32c_1c_2}}\,. \tag{3.13}$$

For physically reasonable results, we will only consider real solutions. From table 3.1, it is straightforward to compute that both h_1 and h_2 are real except at $n=1$. For $n>1$ the set of fixed points are given as

$$\begin{aligned}(r,y,p_r,p_y) \equiv \{&(0,0,0,0),(\pm1,0,0,0)\,,(0,\pm1,0,0))\,,\\ &(\pm h_1(n),\pm h_2(n),0,0)\,,(\pm h_1(n),\mp h_2(n),0,0)\}\,,\end{aligned} \tag{3.14}$$

where the values of h_1 and h_2 are given in 3.2 below while for the $n=1$ model we only have

$$(r,y,p_r,p_y) \equiv \{(0,0,0,0)\,,(\pm1,0,0,0)\,,(0,\pm1,0,0)\}\,, \tag{3.15}$$

as the fixed points. One may notice that in fact (3.12 are the critical points of the potential V_n. To understand whether the point in (3.11) are local minima or local maximum, we should look into the eigenvalues of the matrix $\frac{\partial^2 V_n}{\partial g_i \partial g_j}$, (where $(g_1,g_2)\equiv(r,y)$). We find that the points $(\pm1,0)$ and $(0\,,\pm1)$ are local minima, $(0,0)$ is a local maximum, while $(\pm h_1(n),\pm h_2(n))$, $(\pm h_1(n),\mp h_2(n))$ are all saddle points of V_n. At the local minma, the value of V_n becomes $V_n(\pm1,0)=-8c_1$ and $V_n(0,\pm1)=-c_2$. From table 3.1, clearly $(\pm1,0)$ is the absolute minimum of V_n for $n=1,2,3$, while $(0,\pm1)$ is the absolute minima for the models with $n=4,5,6$.
Therefore, the fixed point energies are

$$E_F(0,0,0,0)=0\,, \quad E_F(\pm1,0,0,0)=-8N^2c_1\,, \quad E_F(0,\pm1,0,0)=-N^2c_2\,, \tag{3.16}$$

the values of $\frac{E_F(\pm h_1(n),\pm(\mp)h_2(n),0,0)}{N^2}$ and minimum values of the potentials V_n are given in table 3. Note that minimum values of V_n are negative in general. This is expected,

Table 3.3: Numerical values for E_F/N^2 at the critical points $(\pm h_1(n), \pm(\mp)h_2(n), 0, 0)$ and the minimum values of V_n.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$\frac{E_F(\pm h_1(n), \pm(\mp)h_2(n),0,0)}{N^2}$	$-$	-8.6	-13.8	-18.4	-23	-27.7	-32.3
$\frac{V_{(n),min}}{N^2}$	-20	-48	-84	-153	-391.9	-881.9	-1800

due to the presence of the massive deformation terms in the matrix model. Indeed it is already known that, fuzzy sphere vacuum solutions of the BFSS model with qubic and/or quadratic deformation terms are of lower (negative) energy, compared to the zero-energy vacuum solutions of the pure BFSS model with diagonal matrices.

3.1.2 Linear Stability Analysis in the Phase Space

Although extrama of V_n gives us an opportunity to comment on dynamics of the sysmtem, it does not provide sufficient information about stability of the fixed points. For this purpose, we perform a first order stability analysis around the fixed point of H_n given in (3.14) and (3.15). Combining our results with the Lyapunov spectrum and the Poincare sections, we will able to comment on the outset and variation of chaos.
For a more decent calculation, lets introduce a new notation

$$(g_1, g_2, g_3, g_4) \equiv (r, y, p_r, p_y) \,, \tag{3.17}$$

the Jacobian matrix from g_α and $\dot{g}_\alpha$ given as

$$J_{\alpha\beta} \equiv \frac{\partial \dot{g}_\alpha}{\partial g_\beta} \,, \tag{3.18}$$

the eigenvalues of the Jacobian matrix will allow us to find whether the fixed point is stable or not [33–36]. In the matrix form, we have

$$J(r,y) \equiv \begin{pmatrix} 0 & 0 & \frac{1}{2c_1N^2} & 0 \\ 0 & 0 & 0 & \frac{1}{2c_2N^2} \\ J_{31} & -2N^2c_3ry & 0 & 0 \\ -2N^2c_3ry & J_{42} & 0 & 0 \end{pmatrix} \,, \tag{3.19}$$

Table 3.4: Eigenvalues of the Jacobian $J(r, y)$ at the fixed points.

Fixed Points	Eigenvalues of $J(r, y)$
$(0, 0, 0, 0)$	$\pm 4, \pm\sqrt{2}$
$(\pm 1, 0, 0, 0)$	$(\pm 4i\sqrt{2}, \pm i\frac{\sqrt{c_3 c_2 c_1^2 - 2c_2^2 c_1^2}}{c_1 c_2})$
$(0, \pm 1, 0, 0)$	$(\pm 2i, \pm i\frac{\sqrt{c_3 c_1 c_2^2 - 16c_1^2 c_2^2}}{c_1 c_2})$
$(h_1(2), h_2(2), 0, 0)$	$(\pm 2.5, \pm i3.2)$
$(h_1(3), h_2(3), 0, 0)$	$(\pm 2.9, \pm i3.4)$
$(h_1(4), h_2(4), 0, 0)$	$(\pm 3.1, \pm i3.5)$
$(h_1(5), h_2(5), 0, 0)$	$(\pm 3.1, \pm i3.5)$
$(h_1(6), h_2(6), 0, 0)$	$(\pm 3.2, \pm i3.5)$
$(h_1(7), h_2(7), 0, 0)$	$(\pm 3.2, \pm i3.4)$

where J_{31} and J_{42} are

$$\begin{aligned} J_{31} &= N^2(32c_1 - 96c_1 r^2 - 2c_3 y^2)\,, \\ J_{42} &= N^2(4c_2 - 12c_2 y^2 - 2c_3 r^2)\,. \end{aligned} \tag{3.20}$$

Eigenvalues of $J(r, y)$ at the fixed points (3.14) are easily evaluated and listed in the table 3.4 below. A fixed point is stable if all the real eigenvalues of the Jacobian are negative, and unstable if the Jacobian has at least one real positive eigenvalue. Also there may be cases like all the eigenvalues of Jacobian matrix are purely imaginary at fixed point. This case is called the borderline case and it means first order analysis is not sufficient to determine whether the fixed point is stable or unstable. All the eigenvalues of the Jacobian matrix at different fixed point are given in table 3.4. Clearly, we see that $(0, 0, 0, 0)$ and $(h_1(n), h_2(n), 0, 0)$ are all unstable fixed points as the corresponding Jacobians have at least one real positive eigenvalue. The values of $c_\alpha(n)$ are given in table 3.1 and we see that $(\pm 1, 0, 0, 0)$ and $(0, \pm 1, 0, 0)$ are borderline cases. We are not going to explore the structure of these borderline fixed points any further, as we expect that their impact on the chaotic dynamics should be rather small compared to those of the unstable fixed points, which we just identified at the linear level.

3.2 Chaotic Dynamics

3.2.1 Lyapunov Spectrum and Poincaré Sections

As we discussed in the previous sections, our main tool for determining chaos is the Lyapunov exponent and the Poincaré sections. For calculating the Lyapunov spectrum of our models, we run a Matlab code that numerically solves the Hamilton's equations of motion in (3.9). For each values of n, we run the code for 40 times with randomly selected initial conditions. These initial conditions satisfy a given energy and the code calculates the mean of the time series from all runs for each of the Lyapunov exponents at each value of n. To elaborate our initial condition selection process, lets denote a generic set of initial conditions at $t = 0$ by $(r(0), y(0), p_r(0), p_y(0))$. For $H_{n<4}$ we take $y(0) = 0$ and for $H_{n\geq 4}$, $r(0) = 0$ as part of the initial condition and subsequently generate three random numbers ω_i $(i = 1, 2, 3)$ and define $\Omega_i = \frac{\omega_i}{\sqrt{\omega_i^2}}\sqrt{E'}$ for a given energy $E' = E + |V_{(n),min}|$ of the system[2], so that $E' = \Omega_i^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2$. Subsequently, we take positive roots in the expressions[3]

$$p_r(0) = N\sqrt{4c_1\Omega_1^2}\,, \quad p_y(0) = N\sqrt{4c_2\Omega_2^2}\,, \tag{3.21}$$

and the real roots of

$$\begin{aligned} 8c_1r^4(0) - 16c_1r^2(0) + 8c_1 - \Omega_3^2 &= 0 \quad \text{for} \quad H_{n<4} \\ c_2y^4(0) - 2y^2(0) + c_2 - \Omega_3^2 &= 0 \quad \text{for} \quad H_{n\geq 4}\,, \end{aligned} \tag{3.22}$$

During the last step, our code randomly selects from the real solutions of 3.22. In our computations, we use a time step of 0.25 and run the code for a sufficient amount of computer time to clearly see where the Lyapunov exponent stabilizes. The Lyapunov exponent vs time series graphs are given in below for different n and E/N^2 values. From the graphs, we can clearly see that $n = 1$ is a special case . For $n = 1$, with the increase in energy, the Lyapunov exponent converges to zero, i.e (figures 3.1a

[2] Let us note that, via introducing E', we have simply shifted the minimum of the potentials $V_{(n)}$ to zero value. This is convenient for describing and using the process of random selection of initial conditions for the reduced models, as well as, in presenting some of our results. For instance, the time series for Lyapunov exponents and the plots of the Poincaré sections. However, we will continue to use the energy variable E while examining the variation of the largest Lyapunov exponents and discussing the relation of the latter to the temperature.

[3] We have checked that randomly selecting positive and negative roots in (3.21) does not cause any significant impact on our results.

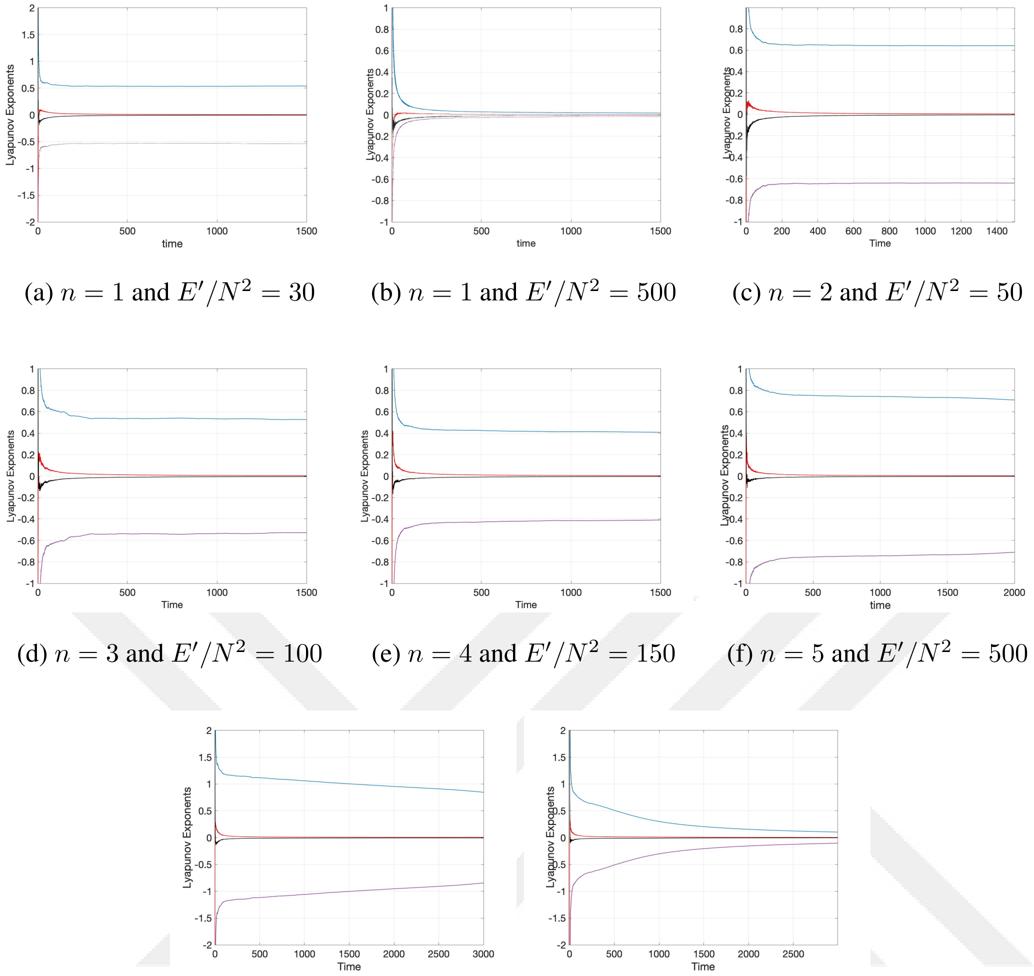
and 3.1b); for $E' \approx 30 \cdot 4^2$ (i.e. $E \approx 10$) there is a positive Lyapunov exponent however for $E' \approx 500 \cdot 4^2$ it convers into zero. This is a direct indication of non-chaotic behaviour. We will also discuss the possible reasons for why such a distinct behaviour happens in $n = 1$ model.

Another method for diagnosing chaotic behavour is looking Poincaré sections at several different values of the energy. In our analysis, we particularly focus on energies above the energy of unstable fixed points $(\pm h_1(n), \pm(\mp)h_2(n), 0, 0)$ to give a visual proof of how the phase space trajectories develop and capture the onset of chaotic dynamics. We use the same initial condition selection procedure for both the Poincaré section and the Lyapunov spectrum. We take $y(0) = 0$ as fixed initial condition for $H_{n\leq 4}$ and $r(0) = 0$ for $H_{n\geq 4}$. Therefore it is convenient to look at the Poincaré sections on the $r - p_r$-plane and the $y - p_y$-plane, respectively in these cases. Our results for the Poincaré sections are given in the figure 3.2 for $n = 1, 2, 4$. The results of $n = 3, 5, 6, 7$ cases are given in the appendix and given in the figures **??**. From the plot, we infer that the phase space is consist of quasi-periodic orbits at energies below E'_F. However for energies around E'_F quasi periodic trajectories and chaotic ones coexist. For energies larger than E_F the phase space is filled by chaotic trajectories.

3.2.2 Dependence of the Largest Lyapunov Exponent on Energy

In order to explore the dependence of the mean largest Lyapunov exponent(MLLE)(λ_n) to energy, we calculate the MLLE at a large interval of energy values. The fixed point energies from the previous section is of vital importance for our analysis. From the figures 3.3 clearly indicate that the value of the MLLE sharply increases once the energy of systems exceeds the fixed point energy E_F of points $h_1(n), \pm(\mp)h_2(n), 0, 0)$. Error bars at each data point is found by evaluating mean square error using MLLE and the LLE values of each of the 40 runs.

From figure 3.3d - 3.3g, there is a significant decrease in the value of MLLE and the mean square errors appear to be larger than the rest of the data points. After a careful analysis, we figure out that less than a quarter of the 40 initial conditions approach to zero. This causes a sharp decrease in the MLLE values and increase in the mean



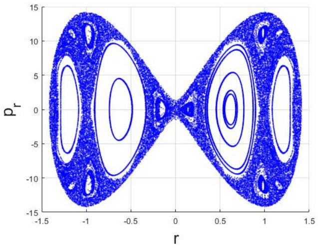
(a) $n = 1$ and $E'/N^2 = 30$ (b) $n = 1$ and $E'/N^2 = 500$ (c) $n = 2$ and $E'/N^2 = 50$

(d) $n = 3$ and $E'/N^2 = 100$ (e) $n = 4$ and $E'/N^2 = 150$ (f) $n = 5$ and $E'/N^2 = 500$

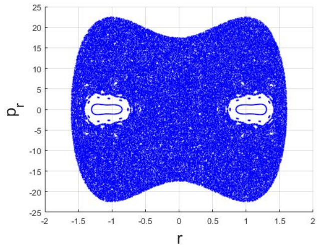
(g) $n = 6$ and $E'/N^2 = 1500$ (h) $n = 7$ and $E'/N^2 = 2000$

Figure 3.1: Time Series for Lyapunov Exponents

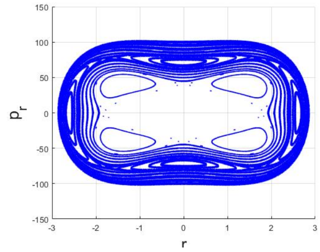
square errors. After excluding these points, we get the expected form of the MLLE vs E/N^2 graphs. Yellow data points on 3.3d - 3.3e illustrate the results without excluding these initial conditions. This kind of phenomena may occur due to the increased presence of either in periodic or quasi-periodic orbits start from these "special" initial conditions. However, since the value of MLLE still quite large, excluding these initial conditions is not going to change any of our results. Excluding these points enables us to get improved best fit curves. For the rest of the n values, we do not apply a similar analysis because they are already distributed in a more scattered fashion and this does not have a significant impact on our curve fitting tools. In the next section, we will discuss our physical motivation to the form of best fitting forms and how it



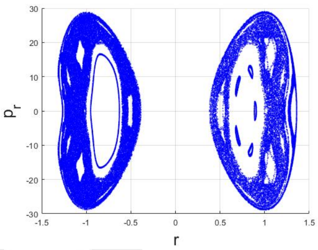
(a) $n = 1$ and $E'/N^2 = 20$



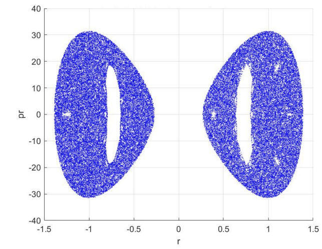
(b) $n = 1$ and $E'/N^2 = 50$



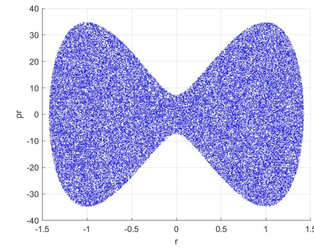
(c) $n = 1$ and $E'/N^2 = 1000$



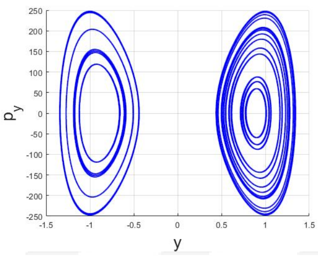
(d) $n = 2$ and $E'/N^2 = 34$



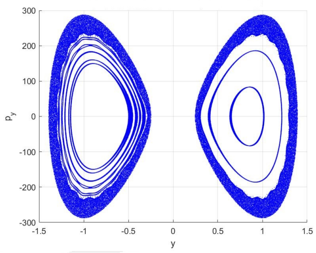
(e) $n = 2$ and $E'/N^2 = 41$



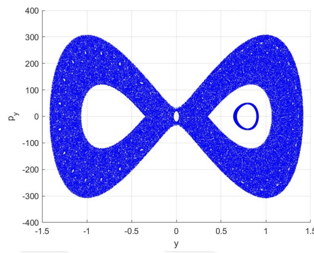
(f) $n = 2$ and $E'/N^2 = 50$



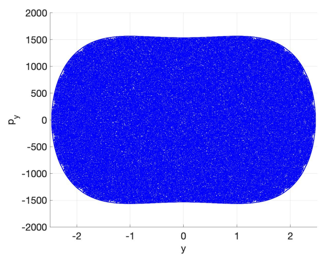
(g) $n = 4$ and $E'/N^2 = 100$



(h) $n = 4$ and $E'/N^2 = 135$



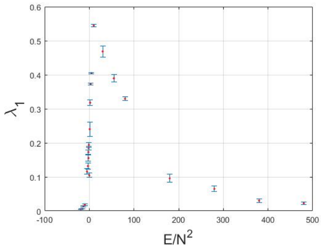
(i) $n = 4$ and $E'/N^2 = 155$



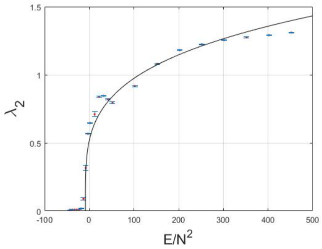
(j) $n = 4$ and $E'/N^2 = 4000$

Figure 3.2: Poincaré Sections (Plots are given w.r.t. the shifted energies $E' = E + |V_{(n),min}|$)

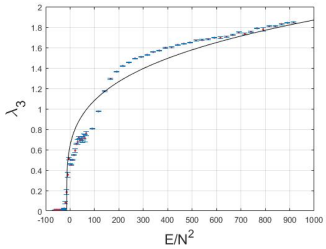
will fit to our data samples.



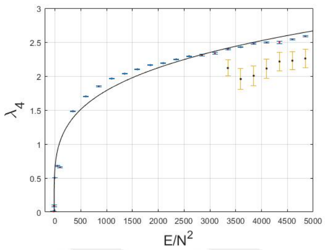
(a) λ_1 vs. E/N^2



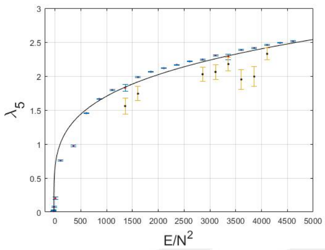
(b) λ_2 vs. E/N^2



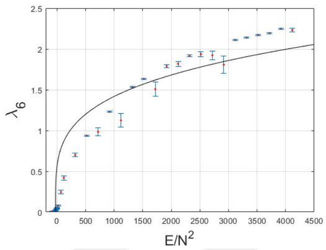
(c) λ_3 vs. E/N^2



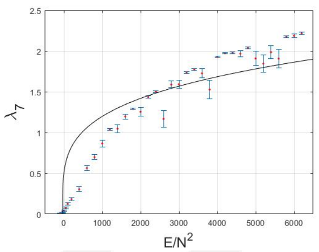
(d) λ_4 vs. E/N^2



(e) λ_5 vs. E/N^2



(f) λ_6 vs. E/N^2



(g) λ_7 vs. E/N^2

Figure 3.3: MLLE vs. E/N^2 for $\mu_1^2 = -16$ and $\mu_2^2 = -2$

3.2.3 Temperature Dependence of the Lyapunov Exponent for the Mass Deformed BFSS Matrix Model

In previous sections, we have discussed the temperature dependence of the Lyapunov exponent for BFSS model and mentioned it was found as $\lambda = 0.2924(3)(\lambda_{'tHooft}T)^{1/4}$ in [12]. Similarly, we have also discussed the energy temperature dependence via equipartition theorem and showed that for a purely quartic potential, we have

$$E = \langle K \rangle + \langle V \rangle = \frac{3}{4} n_{d.o.f} T \tag{3.23}$$

where $n_{d.o.f} = 8(N^2-1)-36$ being the independent degrees of freedom in the BFSS matrix model. Since we set $\lambda_{'tHooft}$ into unity, the relation between the Lyapunov exponent and temperature becomes $\lambda = 0.2924T^{1/4}$. Immediately we can comment on whether our relation satisfies the MSS bound, $2\pi T$. It only violates this bound only at small temperatures, precisely below than $T = (\frac{0.292}{2\pi})^{\frac{4}{3}} = 0.015$ which is consistent because generally the classical theories are useful for describing the systems in the high temperature regime.

After adding the massive deformation terms, the potential is no longer purely quadratic. Therefore, virial theorem is not sufficient to express the kinetic and potential terms in terms of other. From (3.1), we can express the Lagrangian as $L_{YMM} = K - V$. Applying the virial theorem, we find

$$\begin{aligned} 2\langle K\rangle &= 2\langle V\rangle - 2\,Tr\frac{1}{4}[X_I, X_J]^2\,, \\ &= 4\langle V\rangle - 2\,Tr\,\frac{1}{2}\mu_1^2 X_a^2 - 2\,Tr\,\frac{1}{2}\mu_2^2 X_i^2\,. \end{aligned}$$

First equality implies that $\langle V\rangle < \langle K\rangle$, since $Tr[X_I, X_J][X_I, X_J]^\dagger = -Tr[X_I, X_J]^2 \geq 0$ for Hermitian X_I, while the second implies $2\langle V\rangle \leq \langle K\rangle$ for $\mu_1^2 < 0$, $\mu_2^2 < 0$. Since $\langle V\rangle < \langle K\rangle$ is always satisfied and for $\mu_1^2 < 0$, $\mu_2^2 < 0$ that $\langle V\rangle < \langle K\rangle$ is always satisfied and for $\mu_1^2 < 0$, $\mu_2^2 < 0$, we further have that $2\langle V\rangle \leq \langle K\rangle$. Thus, we have $E \leq \frac{3}{4} n_{d.o.f} T$, where now $n_{d.o.f} = 7(N^2 - 1) - 28$ since we set $X_9 = 0$ in our ansatz choice. Therefore, in large N limit, we may take $n_{d.o.f} \approx 7N^2$. Our ansatz choice consist of the fuzzy two- and four- sphere matrices with collective time dependence. This configuration in fact describes a N coincident $D0$-branes and open string states stretching between them. The reduced action that we obtained after tracing over this configuration effectively model their dynamics. If we combine two results, we find an upper bound for energy

$$\frac{E}{N^2} \leq \frac{21}{4} T\,. \tag{3.24}$$

Thus, motivated by the $\frac{E}{N^2}$ dependence of temperature, we find the best fit curve of the MLLE vs E/N^2 as

$$\lambda_n = \alpha_n \left(\frac{E}{N^2} - \frac{(E_n)_F}{N^2} \right)^{1/4}, \tag{3.25}$$

where $(E_n)_F$ is the fixed point energy at level n. $\frac{(E_n)_F}{N^2}$ part exists since the sharp increase in the MLLE values after the energy exceeds $(E_n)_F$ value. The best fitting curves for the given functional form (3.25) are also given in figure 3.3, while the coefficients α_n are listed in table 3.5 below:

Table 3.5: α_n values for the best fit curves (3.25), Upper bounds for T_c and β_n values for the inequality (3.27).

n	α_n	T_c	β_n
2	0.3017	0.0832	0.4567
3	0.3313	0.1029	0.5015
4	0.3168	0.1046	0.4795
5	0.3016	0.1045	0.4565
6	0.2505	0.0911	0.3791
7	0.2113	0.0796	0.3198

From table 3.5, we can easily see that α_n values are very close to each other for $2 \leq n \leq 4$ and the our fitting curves are quite good. However, for $n = 6$ and $n = 7$, we have a scattered data set that cause a decrease in the quality of he fitting curves. As we have mentioned earlier, the quality of the fits have significantly increased after excluding the "special" initial conditions. In fact, the cofficient $c_{(1,2,3)}$ are analytically calculated until $n \leq 6$. For $n = 7$, we obtained the values via extrapolation of our exact results for $n \leq 6$. Therefore, this may be the reason why our fitting is not as good as the rest of the n values.

Lets proceed with the temperature dependence of the MLLEs. Clearly, from (3.24), at $T = 0$, we have $E \leq 0$. Since the MLLE values are non-vanishing until around $(E_n)_F$ or below, this indicates that the MSS bound is violated at zero temperature. In fact, we can analytically find an interval where the MSS bound is violated, i.e $0 \leq T \leq T_c$ where T_c critical temperature. Recall that we have

$$\frac{E}{N^2} \leq \frac{21}{4}T \tag{3.26}$$

Thus, if we use (3.26) with (3.25) and use the MSS bound as a lower bound, we get

$$\alpha_n \left(\frac{21}{4}T - \frac{(E_n)_F}{N^2} \right)^{1/4} \geq 2\pi T \tag{3.27}$$

T_c is the temperature that saturates this inequality. For $n = 2, \cdots, 7$, numerical values of T_c are given in 3.5 and they are roughly an order of magnitude larger than the critical temperature 0.015 determined for the BFSS model in [12, 37]. Moreover, at sufficiently large energies, the fitting curve becomes

$$\lambda_n = \alpha_n \left(1 - \frac{(E_n)_F}{E}\right)^{1/4} \left(\frac{E}{N^2}\right)^{1/4} \approx \alpha_n \left(\frac{E}{N^2}\right)^{1/4} \,, \quad \text{for} \quad E \gg |(E_n)_F| > 0 \,, \tag{3.28}$$

thus we have

$$\lambda_n \leq \beta_n \, T^{1/4} \,, \quad \beta_n := \left(\frac{21}{4}\right)^{1/4} \alpha_n \,, \tag{3.29}$$

The values of β_n are given in 3.5 . Clearly, for high temperature regime, λ_n values are smaller than the MSS bound $2\pi T$.

CHAPTER 4

ABJM MATRIX MODEL

We will start this chapter, by reviewing some of the basic features of the Aharony-Bergman-Jafferis-Maldacena (ABJM) [22] model. ABJM model is a supersymmetric Chern-Simons gauge theory in 2+1 dimensions, which describes the dynamics of coincident (nonabelian) M2-branes. We will confine our discussion to the bosonic sector of this model [22].

4.1 Bosonic Sector of the ABJM Model

Bosonic part of the action for the ABJM model is given by

$$\begin{aligned} S_{ABJM} = \int d^3x \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \operatorname{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \hat{\partial}_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \\ - \operatorname{Tr}\Big(D_\mu C_I^\dagger D^\mu C^I \Big) + \frac{4\pi^2}{3k^2} \operatorname{Tr}\Big(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K \Big) \\ + \operatorname{Tr}\Big(4 C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6 C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger \Big) \end{aligned} \tag{4.1}$$

This action is invariant under $SU(N) \times SU(N)$ gauge transformations. $A_\mu, \hat{A}_\mu$, $(\mu : 0, 1, 2)$ are two gauge field transforming under the left $SU(N)$ and right $SU(N)$, respectively. C^I $(I : 1, 2, 3, 4)$ complex scalar fields and $C^{I\dagger}$ are the Hermitian conjugates of C^I. These complex fields are coupled to $A_\mu, \hat{A}_\mu$ via the bifundamental representation of the $SU(N) \times SU(N)$ gauge group. The covariant derivates are therefore given as

$$D_\mu C^I = \partial_\mu C^I + i A_\mu C^I - i C^I \hat{A}_\mu \,, \tag{4.2a}$$

$$D_\mu C_I^\dagger = \partial_\mu C^{I\dagger} - i C^{I\dagger} A_\mu + i \hat{A}_\mu C^{I\dagger} \tag{4.2b}$$

The theory also has a rigid $SU(4) \approx SO(6)$ global symmetry, which corresponds to the rigid rotation of the scalar fields C^I among themself.

4.1.1 Dimensional Reduction of the ABJM Model

We would like to obtain the reduction of this action to $0+1$-dimensions. We will follow the approach we used in section 2.2 for the reduction of the Yang-Mills theory in $9+1$ dimensions to $0+1$ to obtain the bosonic part of the BFSS model. Thus for the Chern-Simons gauge fields we write

$$A_\mu \equiv (A_0, X_i)\,, \quad \hat{A}_\mu \equiv (\hat{A}_0, \hat{X}_i) \tag{4.3a}$$

where A_0, X_i, $\hat{A}_0$, $\hat{X}_i$, $(i : 1, 2)$ are functions of time only. Let us consider first the Chern-Simons terms in the Lagrangian associated to S_{ABJM}. We have

$$\begin{aligned} L_{CS} &:= L_k + L_{-k} \\ L_{CS} &= \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda) \end{aligned} \tag{4.4}$$

Since A_μ, $\hat{A}_\mu$ are given as in (4.3a) and they are functions of time only, we have

$$\begin{aligned} L_k &= \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\Big(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda\Big) \\ &= \frac{k}{4\pi}\,\mathrm{Tr}\Big(-\epsilon^{i0j}X_i\partial_0 X_j + \frac{2}{3}i(\epsilon^{0ij}A_0X_iX_j + \epsilon^{i0j}X_iA_0X_j + \epsilon^{ij0}X_iX_jA_0)\Big) \\ &= \frac{k}{4\pi}\,\mathrm{Tr}\Big(\epsilon^{ij}(X_i\dot{X}_j + 2iA_0X_iX_j)\Big)\,, \end{aligned} \tag{4.5}$$

where $\dot{X}_i = \partial_0 X_i$ as usual. Similarly, we can easily find

$$L_{-k} = -\frac{k}{4\pi}\,\mathrm{Tr}\Big(\epsilon^{ij}(\hat{X}_i\dot{\hat{X}}_j + 2i\hat{A}_0\hat{X}_i\hat{X}_j)\Big)\,, \tag{4.6}$$

Thus the reduction of L_{CS} to 0+1 dimensions yields

$$L_{CS(reduced)} = \frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\Big(X_i\dot{X}_j + 2iA_0X_iX_j - \hat{X}_i\dot{\hat{X}}_j - 2i\hat{A}_0\hat{X}_i\hat{X}_j\Big) \tag{4.7}$$

$$= \frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\Big(X_iD_0X_j + \hat{X}_iD_0\hat{X}_j\Big) \tag{4.8}$$

where $D_i X_i = \partial_0 + i[A_0, X_i]$. After reducing to $0+1$ dimensions, spatial part of the covariant derivatives, take the form.

$$\begin{aligned} D_i C^I &= \cancel{\partial_i C^I} + iX_i C^I - iC^I \hat{X}_i\,, \\ &= iX_i C^I - iC^I \hat{X}_i\,, \\ D_i C^{I\dagger} &= \cancel{\partial_i C^{I\dagger}} - iC^{I\dagger} X_i + i\hat{X}_i C^{I\dagger}\,, \\ &= -iC^{I\dagger} X_i + i\hat{X}_i C^{I\dagger}\,, \end{aligned} \tag{4.9}$$

while the time derivative part of the covariant derivatives become

$$\begin{aligned} D_0 C^I &= \partial_0 C^I + iA_0 C^I - iC^I \hat{A}_0\,, \\ D_0 C^{I\dagger} &= \partial_0 C^{I\dagger} - iC^{I\dagger} A_0 + i\hat{A}_0 C^{I\dagger}. \end{aligned} \tag{4.10}$$

Interaction terms among C^I remain in the same form, except that it is now understood that C^I and $C^{I\dagger}$ are $N \times N$ complex matrices whose entries depend on time only. Reduced action reads

$$\begin{aligned} S_{ABJM(0+1)} = V_2 \int dt \frac{k}{4\pi} \epsilon^{ij} \operatorname{Tr}\Big(X_i \dot{X}_j + 2iA_0 X_i X_j - \hat{X}_i \dot{\hat{X}}_j - 2i\hat{A}_0 \hat{X}_i \hat{X}_j\Big) \\ - \operatorname{Tr}\big(D_i C^{I\dagger} D_i C^I\big) - \operatorname{Tr}\big(D_0 C^{I\dagger} D_0 C^I\big) + \textit{interaction terms among } C^I\textit{'s} \end{aligned} \tag{4.11}$$

Upon integrating over the compact dimensions we obtain the volume factor, say V_2, in $S_{ABJM(0+1)}$ given in 4.11. This serves as the gauge coupling constant $(1/g^2)$ and we may introduce the 't Hooft coupling as

$$\lambda_{'tHooft} = \frac{N}{V_2} \tag{4.12}$$

or we may write $V_2 = \frac{N}{\lambda_{'tHooft}}$. By this scaling we may set $\lambda_{'tHooft}$ to unity and therefore write the S_{ABJ} by replacing V_2 with N. To introduce $\lambda_{'tHooft}$ back we may scale the fields and parameters as $X_i \to \lambda^{-1/2} X_i$, $\hat{X}_i \to \lambda^{-1/2} \hat{X}_i$, $A_0 \to \lambda^{-1/2} A_0$, $\hat{A}_0 \to \lambda^{-1/2} \hat{A}_0$, $Q_\alpha \to \lambda^{-1/4} Q_\alpha$, $R_\alpha \to \lambda^{-1/4} R_\alpha$ and $\mu \to \lambda^{-1/2} \mu$.

We now proceed to obtain the equations of motion for, A_0, $\hat{A}_0$, X, $\hat{X}_i$, C^I, and $C^{I\dagger}$. We start with evaluating variation of $S_{ABJM(0+1)}$ w.r.t. A_0 and $\hat{A}_0$. Letting $A_0 \to A_0 + \delta A_0$ gives

$$\begin{aligned} \delta S_{ABJM(0+1)} = \int dt \frac{k}{4\pi} \epsilon^{ij} \operatorname{Tr}(2i\delta\,, A_0 X_i X_j) - \operatorname{Tr}\big(-iC^{I\dagger} \delta A_0 D_0 C^I\big) \\ - \operatorname{Tr}\big(D_0 C^{I\dagger} i\delta A_0 C^I\big) \end{aligned} \tag{4.13}$$

$$\frac{k}{4\pi}\epsilon^{ij}2i\,\mathrm{Tr}(X_iX_j\delta A_0)+\mathrm{Tr}\big(iD_0C^IC^{I\dagger}\delta A_0\big)-\mathrm{Tr}\big(iC^ID_0C^{I\dagger}\delta A_0\big)=0 \qquad (4.14)$$

where we have used cyclicity of the trace to obtain the 4.14 from 4.13. Making the $A_0=\hat{A}_0=0$ gauge choice, time part of the covariant derivatives in (4.10) simply become $D_0C^I=\partial_0C^I=\dot{C}^I$ and $D_0C^{I\dagger}=\partial_0C^{I\dagger}=\dot{C}^{I\dagger}$ and we have from (4.13)

$$\begin{aligned}\mathrm{Tr}\bigg(\frac{k}{4\pi}\epsilon^{ij}2iX_iX_j+i\dot{C}^IC^{I\dagger}-iC^I\dot{C}^{I\dagger}\bigg)\delta A_0&=0\\ \frac{k}{4\pi}\epsilon^{ij}2iX_iX_j+i\dot{C}^IC^{I\dagger}-iC^I\dot{C}^{I\dagger}&=0\\ \frac{k}{4\pi}[X_1,X_2]+C^I\dot{C}^{I\dagger}-C^I\dot{C}^{I\dagger}&=0\end{aligned} \qquad (4.15)$$

A similar result follows for $\hat{A}_0\to\hat{A}_0+\delta\hat{A}_0$ with $k\to -k$ and $A_0\to\hat{A}_0$ in (4.13). With $\hat{A}_0\to\hat{A}_0+\delta\hat{A}_0$ we find

$$\begin{aligned}-\frac{k}{4\pi}2i\epsilon^{ij}\,\mathrm{Tr}\Big(\delta\hat{A}_0\hat{X}_i\hat{X}_j\Big)-\mathrm{Tr}\Big(i\delta\hat{A}_0C^{I\dagger}D_0C^I\Big)+\mathrm{Tr}\Big(D_0C^{I\dagger}iC^I\delta\hat{A}_0\Big)&=0\\ -\frac{k}{4\pi}2i\epsilon^{ij}\,\mathrm{Tr}\Big(\hat{X}_i\hat{X}_j\delta\hat{A}_0\Big)-\mathrm{Tr}\Big(iC^{I\dagger}\dot{C}^I\delta\hat{A}_0\Big)+\mathrm{Tr}\Big(\dot{C}^{I\dagger}iC^I\delta\hat{A}_0\Big)&=0\\ \mathrm{Tr}\,\big(-\frac{k}{4\pi}2i\epsilon^{ij}(\hat{X}_i\hat{X}_j))-(iC^{I\dagger}\dot{C}^I)+(\dot{C}^{I\dagger}iC^I)\big)\delta\hat{A}_0&=0 \qquad (4.16)\\ -\frac{k}{4\pi}2i\epsilon^{ij}(\hat{X}_i\hat{X}_j))-(iC^{I\dagger}\dot{C}^I)+(i\dot{C}^{I\dagger}iC^I)&=0\\ -\frac{k}{2\pi}[X_1,X_2]-C^{I\dagger}\dot{C}^I+\dot{C}^{I\dagger}C^I&=0\end{aligned}$$

Thus, from (4.15) and (4.16) we read the the Gauss law constraint as a set of two algebraic coupled equations given by

$$\begin{aligned}\frac{k}{4\pi}[X_1,X_2]+P_C^IC^{I\dagger}-C^IP_C^{I\dagger}&=0\,,\\ -\frac{k}{2\pi}[\hat{X}_1,\hat{X}_2]-C^{I\dagger}P_C^I+P_C^{I\dagger}C^I&=0\,,\end{aligned} \qquad (4.17)$$

where $P_C^I=\frac{\partial L}{\partial_0 C^I}$, $P_C^{I\dagger}=\frac{\partial L}{\partial_0 C^{I\dagger}}$ are the corresponding conjugate momenta.

Consider the variation of X_i, $X_i\to X_i+\delta X_i$. In the $A_0=\hat{A}_0=0$ gauge, (4.7) gives

$$\begin{aligned}L_{CS(reduced)}+\delta L_{CS(reduced)}&=\frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\,\big(-(X_i+\delta X_i)(\dot{X}_j+\delta\dot{X}_j)-\hat{X}_i\dot{\hat{X}}_j\big)\\ \delta L_{CSr}&=\frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\,\big((-X_i\delta\dot{X}_j-\delta X_i(\dot{X}_j)\big)\\ &=\frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\,\big((\dot{X}_i\delta X_j-\delta X_i(\dot{X}_j)\big)\\ &=\frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\,\big(\epsilon^{ij}(\dot{X}_i\delta X_j-\epsilon^{ji}\delta X_j(\dot{X}_i)\big)\\ &=\frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\,\big((2\dot{X}_i\delta X_j\big)\end{aligned} \qquad (4.18)$$

Variation of the rest of action w.r.t. to X_i gives

$$
\begin{aligned}
L_{rest} + \delta L_{rest} &= \mathrm{Tr}\left(i(C_I^\dagger(X_i + \delta X_i) - \hat{X}_i C^{I\dagger}) i((X_i + \delta X_i)C^I - C^I \hat{X}_i)\right) \\
\delta L_{rest} &= -\,\mathrm{Tr}\left(C^{I\dagger} X_i \delta X_i C^I + C^{I\dagger}\delta X_i X_i C^I - C^{I\dagger}\delta X_i C^I \hat{X}_i - \hat{X}_i C^{I\dagger}\delta X_i C^I\right) \\
&= -\,\mathrm{Tr}\left(C^I C^{I\dagger} X_i \delta X_i + X_i C^I C^{I\dagger}\delta X_i - C^I \hat{X}_i C^{I\dagger}\delta X_i - C^I \hat{X}_i C^{I\dagger}\delta X_i\right) \\
&= -\,\mathrm{Tr}\left((C^I C^{I\dagger} X_i + X_i C^I C^{I\dagger} - 2C^I \hat{X}_i C^{I\dagger})\delta X_i\right)
\end{aligned}
\tag{4.19}
$$

In order to infer the e.o.m. for X_i we have to set $\delta L_{rest} + \delta L_{CS(reduced} = 0$. From (4.18) and (4.19) we therefore get

$$
\begin{aligned}
\delta L_{total} &= \mathrm{Tr}\left(\frac{k}{4\pi}\epsilon^{ij}(2\dot{X}_i \delta X_j) - C^I C^{I\dagger} X_i - X_i C^I C^{I\dagger} + 2C^I \hat{X}_i C^{I\dagger}\delta X_i\right), \\
&= \mathrm{Tr}\left(-\frac{k}{4\pi}\epsilon^{ij}(2\dot{X}_j) - C^I C^{I\dagger} X_i - X_i C^I C^{I\dagger} + 2C^I \hat{X}_i C^{I\dagger}\right)\delta X_i\,,
\end{aligned}
\tag{4.20}
$$

which yields

$$
\frac{k}{2\pi}\epsilon^{ij}\dot{X}_j + C^I C^{I\dagger} X_i + X_i C^I C^{I\dagger} - 2C^I \hat{X}_i C^{I\dagger} = 0 \tag{4.21}
$$

Similarly we have for the e.o.m. of $\hat{X}_i$.
Consider the variation $C^{I\dagger}$. $C^{I\dagger} \to C^{I\dagger} + \delta C^{I\dagger}$. Let us continue to calculate variation of the Lagrangian in pieces. For the covariant derivative part, we have

$$
\begin{aligned}
L_{cov} + \delta L_{cov} &= -(D_i(C^{I\dagger} + \delta C^{I\dagger})D_i C^I - (D_0(C^{I\dagger} + \delta C^{I\dagger})D_0 C^I) \\
&= -([(-i(C^{I\dagger} + \delta C^{I\dagger})X_i + i\hat{X}_i(C^{I\dagger} + \delta C^{I\dagger})]D_i C^I) - ((\dot{C}^{I\dagger} + \delta\dot{C}^{I\dagger})\dot{C}^I)) \\
\delta L_{cov} &= -\,\mathrm{Tr}\Big(-\delta C^{I\dagger}\ddot{C}^I\Big) - \mathrm{Tr}\Big(\delta C^{I\dagger} X_i D_i C^I - \hat{X}_i \delta C^{I\dagger} D_i C^I\Big) \\
&= -\,\mathrm{Tr}\Big(-\ddot{C}^I \delta C^{I\dagger} + X_i D_i C^I \delta C^{I\dagger} - D_i C^I \hat{X}_i \delta C^{I\dagger}\Big) \\
&= -\,\mathrm{Tr}\left((-\ddot{C}^I + X_i D_i C^I - D_i C^I \hat{X}_i)\delta C^{I\dagger}\right)
\end{aligned}
\tag{4.22}
$$

In the second line of (4.22) we have performed integration by parts. Thus, the variation of the covariant derivative part of the Lagrangian can be written as

$$
\delta L_{cov} = \mathrm{Tr}\Big(\ddot{C}^I - X_i D_i C^I + D_i C^I \hat{X}_i\Big)\delta C^{I\dagger} \tag{4.23}
$$

Lets us now look at the interaction part of the Lagrangian. Since it is rather long, it is useful to split it into four terms:

$$
L_{int} = L_1 + L_2 + L_3 + L_4 \tag{4.24}
$$

where,

$$
\begin{aligned}
L_1 + \delta L_1 &:= \frac{4\pi^2}{3k^2} \operatorname{Tr}\left((C^I(C_I^\dagger + \delta C_{I\dagger})C^J(C_J^\dagger + \delta C_J^\dagger)C^K(C_K^\dagger + \delta C_K^\dagger))\right), \\
\delta L_1 &= \frac{4\pi^2}{3k^2} \operatorname{Tr}\left(C^I \delta C_{I\dagger} C^J C_J^\dagger C^K C_K^\dagger + C^I C_I^\dagger C^J \delta C_J^\dagger C^K C_K^\dagger + C^I C_I^\dagger C^J C_J^\dagger C^K \delta C_K^\dagger\right), \\
&= \frac{4\pi^2}{3k^2} \operatorname{Tr}\left(C^J C_J^\dagger C^K C_K^\dagger C^I \delta C_I^\dagger + C^K C_K^\dagger C^I C_I^\dagger C^J \delta C_J^\dagger + C^I C_I^\dagger C^J C_J^\dagger C^K \delta C_K^\dagger\right), \\
&= \operatorname{Tr} \frac{4\pi^2}{3k^2}\left(C^J C_J^\dagger C^K C_K^\dagger C^I \delta C_I^\dagger + C^K C_K^\dagger C^J C_J^\dagger C^I \delta C_I^\dagger + C^K C_K^\dagger C^J C_J^\dagger C^I \delta C_I^\dagger\right), \\
&= \frac{4\pi^2}{k^2} \operatorname{Tr}\Big(C^K C_K^\dagger C^J C_J^\dagger C^I\Big)\delta C_I^\dagger.
\end{aligned} \tag{4.25}
$$

In 4.25 we have used the cyclicity of trace and relabeled the dummy indeces as needed to obtain the last line.

The second part of the interaction term can be written as

$$
\begin{aligned}
L_2 + \delta L_2 &= \frac{4\pi^2}{3k^2} \operatorname{Tr}\left((C_I^\dagger + \delta C_I^\dagger)C^I(C_J^\dagger + \delta C_J^\dagger)C^J(C_K^\dagger + \delta C_K^\dagger)C^K\right), \\
\delta L_2 &= \frac{4\pi^2}{3k^2} \operatorname{Tr}\left(\delta C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K + C_I^\dagger C^I \delta C_J^\dagger C^J C_K^\dagger C^K + C_I^\dagger C^I C_J^\dagger C^J \delta C_K^\dagger C^K\right), \\
&= \frac{4\pi^2}{3k^2} \operatorname{Tr}\left(C^I C_J^\dagger C^J C_K^\dagger C^K \delta C_I^\dagger + C^J C_K^\dagger C^K C_I^\dagger C^I \delta C_J^\dagger + C^K C_I^\dagger C^I C_J^\dagger C^J \delta C_K^\dagger\right), \\
&= \frac{4\pi^2}{3k^2} \operatorname{Tr}\left(C^I C_J^\dagger C^J C_K^\dagger C^K \delta C_I^\dagger + C^I C_K^\dagger C^K C_J^\dagger C^J \delta C_I^\dagger + C^I C_K^\dagger C^K C_J^\dagger C^J \delta C_I^\dagger\right), \\
&= \frac{4\pi^2}{k^2} \operatorname{Tr}\left(C^I C_K^\dagger C^K C_J^\dagger C^J\right)\delta C_I^\dagger.
\end{aligned} \tag{4.26}
$$

For the 3rd term of the interaction Lagrangian, we have

$$
\begin{aligned}
L_3 + \delta L_3 &= \frac{4\pi^2}{3k^2}(4C^I(C_J^\dagger + \delta C_J^\dagger)C^K(C_I^\dagger + \delta C_I^\dagger)C^J(C_K^\dagger + \delta C_K^\dagger)), \\
\delta L_3 &= \frac{4\pi^2}{3k^2}(4C^I \delta C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger + 4C^I C_J^\dagger C^K \delta C_I^\dagger C^J C_K^\dagger + 4C^I C_J^\dagger C^K C_I^\dagger C^J \delta C_K^\dagger), \\
&= \frac{4\pi^2}{3k^2}(4C^K C_I^\dagger C^J C_K^\dagger C^I \delta C_J^\dagger + 4C^J C_K^\dagger C^I C_J^\dagger C^K \delta C_I^\dagger + 4C^I C_J^\dagger C^K C_I^\dagger C^J \delta C_K^\dagger), \\
&= \frac{4\pi^2}{3k^2}(4C^K C_J^\dagger C^I C_K^\dagger C^J \delta C_I^\dagger + 4C^J C_K^\dagger C^I C_J^\dagger C^K \delta C_I^\dagger + 4C^K C_J^\dagger C^I C_K^\dagger C^J \delta C_I^\dagger), \\
&= \frac{4\pi^2}{k^2} \operatorname{Tr}\Big(4C^K C_J^\dagger C^I C_K^\dagger C^J\Big)\delta C_I^\dagger,
\end{aligned} \tag{4.27}
$$

Finally, the 4rd term of the interaction Lagrangian can be written as

$$\begin{aligned}
L_4+\delta L_4 &= -\frac{4\pi^2}{3k^2}\,\mathrm{Tr}\left(6C^I(C_J^\dagger+\delta C_J^\dagger)C^J(C_I^\dagger+\delta C_I^\dagger)C^K(C_K^\dagger+\delta C_K^\dagger)\right),\\
\delta L_4 &= -\frac{4\pi^2}{3k^2}\,\mathrm{Tr}\left(6C^I\delta C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger+6C^I C_J^\dagger C^J\delta C_I^\dagger C^K C_K^\dagger+6C^I(C_J^\dagger C^J C_I^\dagger C^K\delta C_K^\dagger\right),\\
&= -\frac{4\pi^2}{3k^2}\,\mathrm{Tr}\left(6C^J C_I^\dagger C^K C_K^\dagger C^I\delta C_J^\dagger+6C^K C_K^\dagger C^I C_J^\dagger C^J\delta C_I^\dagger+6C^I C_J^\dagger C^J C_I^\dagger C^K\delta C_K^\dagger\right),\\
&= -\frac{4\pi^2}{3k^2}\,\mathrm{Tr}\left(6C^I C_J^\dagger C^K C_K^\dagger C^J\delta C_I^\dagger+6C^K C_K^\dagger C^I C_J^\dagger C^J\delta C_I^\dagger+6C^K C_J^\dagger C^J C_K^\dagger C^I\delta C_I^\dagger\right),\\
&= -\frac{4\pi^2}{k^2}\,\mathrm{Tr}\left(2C^I C_J^\dagger C^K C_K^\dagger C^J+2C^K C_K^\dagger C^I C_J^\dagger C^J+2C^K C_J^\dagger C^J C_K^\dagger C^I\right)\delta C_I^\dagger,
\end{aligned} \tag{4.28}$$

Combining 4.25-4.28, we can calculate the equation of motion for $C^{I\dagger}$. The total variation of the Lagrangian w.r.t. $C^{I\dagger}\to C^{I\dagger}+\delta C^{I\dagger}$ can be written as

$$\delta L=\delta L_{cov}+\delta L_1+\delta L_2+\delta L_3+\delta L_4 \tag{4.29}$$

$$\begin{aligned}
\delta L &= \mathrm{Tr}\Big(\ddot{C}^I-X_iD_iC^I+D_iC^I\hat{X}_i\Big)\delta C^{I\dagger}+\frac{4\pi^2}{k^2}\,\mathrm{Tr}\Big(C^K C_K^\dagger C^J C_J^\dagger C^I\Big)\delta C_I^\dagger\\
&\quad+\frac{4\pi^2}{k^2}\,\mathrm{Tr}\left(C^I C_K^\dagger C^K C_J^\dagger C^J\right)\delta C_I^\dagger+\frac{4\pi^2}{k^2}\,\mathrm{Tr}\Big(4C^K C_J^\dagger C^I C_K^\dagger C^J\Big)\delta C_I^\dagger\\
&\quad-\frac{4\pi^2}{k^2}\,\mathrm{Tr}\left(2C^I C_J^\dagger C^K C_K^\dagger C^J+2C^K C_K^\dagger C^I C_J^\dagger C^J+2C^K C_J^\dagger C^J C_K^\dagger C^I\right)\delta C_I^\dagger.
\end{aligned} \tag{4.30}$$

The equation of motion for $C^{I\dagger}$ is therefore

$$\begin{aligned}
&\ddot{C}^I-X_iD_iC^I+D_iC^I\hat{X}_i+C^K C_K^\dagger C^J C_J^\dagger C^I+\frac{4\pi^2}{k^2}\left(C^I C_K^\dagger C^K C_J^\dagger C^J+4C^K C_J^\dagger C^I C_K^\dagger C^J\right)\\
&\quad-2(C^I C_J^\dagger C^K C_K^\dagger C^J+C^K C_K^\dagger C^I C_J^\dagger C^J+C^K C_J^\dagger C^J C_K^\dagger C^I\right)=0.
\end{aligned} \tag{4.31}$$

We can obtain the equation of motion for C^I in a similar manner. Since, we are not going to need it directly, we do not include it here.

In order to motivate the ensuring developments, let's us briefly look at some specific type of choices that we may make in order to satisfy the Gauss law constraints given in (4.15). For instance, we may consider that $X_i=\alpha(t)\tilde{X}_i$and $\hat{X}_i=\beta(t)\tilde{\hat{X}}_i$ where $\tilde{X}_i$, $\tilde{\hat{X}}_i$ are diagonal matrices and $\alpha(t)$, $\beta(t)$ are real functions of time. Since diagonal matrices commute, we have $[X_1,X_2]=0=\left[\hat{X}_1,\hat{X}_2\right]$ Therefore, Gauss Law constraint takes the form

$$\begin{aligned} \dot{C}^I C^{I\dagger} - C^I \dot{C}^{I\dagger} &= 0, \\ -C^{I\dagger}\dot{C}^I + \dot{C}^{I\dagger} C^I &= 0, \end{aligned} \tag{4.32}$$

A possible choice for C^I, ${C^I}^\dagger$ would be

$$C^I = \gamma(t)^I \tilde{C}^I, \quad C^{I\dagger} = \gamma(t)^I {\tilde{C^I}}^\dagger, \tag{4.33}$$

where $\gamma(t)^I$ $(I\,:\,1,2,3,4)$ are real functions of time only and there is no sum over I in the r.h.s of (4.33). So we may introduce as many as four real time dependent functions to the model via C^I and ${C^I}^\dagger$ in this manner. If we use our choices for C^I and $C^{I\dagger}$ in (4.32), we get

$$\begin{aligned} \gamma(\dot{t})^I\gamma(t)^I\tilde{C}^I\tilde{C}^{I\dagger} - \gamma(\dot{t})^I\gamma(t)^I\tilde{C}^I\tilde{C}^{I\dagger} &\stackrel{?}{=} 0 \\ = \gamma(\dot{t})^1\gamma(t)^1\cancelto{0}{(\tilde{C}^1\tilde{C}^{1\dagger} - \tilde{C}^1\tilde{C}^{1\dagger})} + \gamma(\dot{t})^2\gamma(t)^2\cancelto{0}{(\tilde{C}^2\tilde{C}^{2\dagger} - \tilde{C}^2\tilde{C}^{2\dagger})}& \\ +\gamma(\dot{t})^3\gamma(t)^3\cancelto{0}{(\tilde{C}^3\tilde{C}^{3\dagger} - \tilde{C}^3\tilde{C}^{3\dagger})} + \gamma(\dot{t})^4\gamma(t)^4\cancelto{0}{(\tilde{C}^4\tilde{C}^{4\dagger} - \tilde{C}^4\tilde{C}^{4\dagger})}& \\ &= 0 \end{aligned} \tag{4.34}$$

Thus, this may be one way to satisfy the Gauss Law Constraint.

4.1.2 Massive Deformation of the ABJM Model

ABJM model has a massive deformation which was introduced in the paper [23]. Contrary to the massive deformation of BFSS model namely the BMN model [2], which preserves part of supersymmetry this [24] massive deformation preserves all the supersymmetry of the ABJM theory. Massive deformation of the ABJM model admit fuzzy spheres as vacuum solutions, which has implications in type II A string theory as well as in AdS/CMT (Condensed Matter Theory) correspondence [23, 24]. As both of the topics are beyond the scope of this thesis, we will not elaborate on them any further here.

Consider the splitting of the complex scalar fields in two sets $C^I \rightarrow (Q^\alpha, R^\alpha)$, $(\alpha : 1, 2)$. This will facilitate to introduce as we will see shortly mass deformation terms. This breaks the original $SU(4) \times U(1)$ global symmetry down to $SU(2) \times SU(2) \times U(1) \times Z_2$. α index on Q^α transforms under the first $SU(2)$ factor while α index on R^α via the second $SU(2)$ factor. We write the bosonic part of the mass deformed action as

$$\begin{aligned} L_{bosonic} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\Big(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\hat{\partial}_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\Big) \\ - \mathrm{Tr}\Big(D_\mu C_I^\dagger D^\mu C^I\Big) - \mathrm{Tr}\,|D_\mu Q^\alpha|^2 - \mathrm{Tr}\,|D_\mu R^\alpha|^2 - V \end{aligned} \tag{4.35}$$

where $V = \mathrm{Tr}(|M^\alpha|^2 + |N^\alpha|^2)$ and

$$M^\alpha = \mu Q^\alpha + \frac{2\pi}{k}(2Q^{[\alpha}Q_\beta^\dagger Q^{\beta]} + R^\beta R_\beta^\dagger Q^\alpha - Q^\alpha R_\beta^\dagger R^\beta + 2Q^\beta R_\beta^\dagger R^\alpha - 2R^\alpha R_\beta^\dagger Q^\beta)\,, \tag{4.36a}$$

$$N^\alpha = -\mu R^\alpha + \frac{2\pi}{k}(2R^{[\alpha}R_\beta^\dagger R^{\beta]} + Q^\beta Q_\beta^\dagger R^\alpha - R^\alpha Q_\beta^\dagger Q^\beta + 2R^\beta Q_\beta^\dagger Q^\alpha - 2Q^\alpha Q_\beta^\dagger R^\beta)\,, \tag{4.36b}$$

In 4.36a and 4.36b, we used the notation

$$Q^{[\alpha}Q_\beta^\dagger Q^{\beta]} := Q^\alpha Q_\beta^\dagger Q^\beta - Q^\beta Q_\beta^\dagger Q^\alpha \tag{4.37a}$$

$$R^{[\alpha}R_\beta^\dagger R^{\beta]} := R^\alpha R_\beta^\dagger R^\beta - R^\beta R_\beta^\dagger R^\alpha \tag{4.37b}$$

Massive deformation terms in the potential breaks the $SU(4) \times U(1)_B$ global symmetry down to $SU(2) \times SU(2) \times U(1)_A \times U(1)_B \times Z_2$. Here the 1^{st} copy of $SU(2)$ acts on Q^α's while the 2^{nd} on R^α's. Q^α, R^α have the charges $+1, -1$ under $U(1)_A$, which they have both $+1$ charge under $U(1)_B$.

Since the potential V is positive definite, the ground states of the mass deformed ABJM model can be found by setting the potential V in (4.35) to zero. To satisfy this requirement, we need to have

$$M^\alpha = 0 = N^\alpha \tag{4.38}$$

There are two immediate solutions to these equations (4.38). These are provided by [38]

$$i) R^\alpha = cG^\alpha \,, Q^\alpha = 0 \,, \tag{4.39a}$$

or

$$ii) R^\alpha = 0 \,, Q^{\alpha\dagger} = cG^\alpha \,, \tag{4.39b}$$

where G^α satisfies

$$G^\alpha = G^\alpha G^\dagger_\beta G^\beta - G^\beta G^\dagger_\beta G^\alpha . \tag{4.40}$$

It is easy to calculate the value of c. Lets us insert (4.39a) into (4.36b). Since $Q^\alpha = 0$, we only have the first two term, summing in (4.36b), these yields

$$\begin{aligned}
N^\alpha &= -\mu R^\alpha + \frac{2\pi}{k}(2R^{[\alpha} R^\dagger_\beta R^{\beta]}) \\
&= -\mu c G^\alpha + \frac{2\pi}{k}(2c^3 G^{[\alpha} G^\dagger_\beta G^{\beta]}) \\
&= -\mu c G^\alpha + \frac{2\pi}{k}(2c^3 (G^\alpha G^\dagger_\beta G^\beta - G^\beta G^\dagger_\beta G^\alpha)) \\
&= -\mu c G^\alpha + \frac{2\pi}{k}(2c^3 G^\alpha) \\
&= (-\mu + \frac{4\pi}{k} c^2) c G^\alpha
\end{aligned} \tag{4.41}$$

Thus, to satisfy (4.38) we have $c = 0$ or $c = \sqrt{\frac{k\mu}{4\pi}}$. If we choose $c = 0$, (4.39) becomes the trivial solution. Thus, we have $c = \sqrt{\frac{k\mu}{4\pi}}$.

Note that, we should also write an explicit solution for the equation defining the matrices G^α (4.40) A potential solution of (4.40) is known as the GRVV-algebra due to Gomis, Rodriguez-Gomez, Van Raamsdonk and Verlinde [38] and is given as follows

$$\begin{aligned}
(G^1)_{mn} &= \sqrt{m-1}\,\delta_{m,n} \,, \\
(G^2)_{mn} &= \sqrt{N-m}\,\delta_{m+1,n} \,, \\
(G^\dagger_1)_{mn} &= \sqrt{m-1}\,\delta_{m,n} \,, \\
(G^\dagger_2)_{mn} &= \sqrt{N-n}\,\delta_{n+1,m} \,,
\end{aligned} \tag{4.42}$$

(4.42) is a possible solution of (4.40). It is a representation of the G^α matrices known as the GRVV algebra. Note that all G^α, $G^\dagger_\alpha$ are $N \times N$-matrices. To check that (4.42) satisfies (4.40), we can perform an easy calculation. Lets check (4.40) with G^1 from (4.42). We have

$$
\begin{aligned}
(G^1)_{mn} &\stackrel{?}{=} (G^1(G_1^\dagger G^1 + G_2^\dagger G^2))_{mn} - ((G^1 G_1^\dagger + G^2 G_2^\dagger)G^1)_{mn}\,, \\
&= (G^1)_{mk}(G_1^\dagger G^1 + G_2^\dagger G^2)_{kn} - (G^1 G_1^\dagger + G^2 G_2^\dagger)_{ml} G^1_{ln}\,, \\
&= (G^1)_{mk}(G^\dagger_{1kp} G^1_{pn} + G^\dagger_{2kq} G^2_{qn}) - (G^1_{m\lambda} G^\dagger_{1\lambda l} + G^2_{m\alpha} G^\dagger_{2\alpha l}) G^1_{ln}\,, \\
&= \sqrt{m-1}\,\delta_{m,k}(\sqrt{k-1}\delta_{k,p}\sqrt{p-1}\delta_{p,n} + \sqrt{N-q}\,\delta_{q+1,k}\sqrt{N-q}\,\delta_{q+1,n})\,, \\
&- (\sqrt{m-1}\,\delta_{m,\lambda}\sqrt{\lambda-1}\,\delta_{\lambda,l} + \sqrt{N-m}\,\delta_{m+1,\alpha}\sqrt{N-l}\,\delta_{l+1,\alpha})\sqrt{l-1}\,\delta_{l,n}\,, \\
&= \sqrt{m-1}\,\delta_{m,k}(\sqrt{k-1}\delta_{k,n}\sqrt{n-1} + \sqrt{N-k+1}\,\sqrt{N-k+1}\,\delta_{k,n})\,, \\
&- (\sqrt{m-1}\,\delta_{m,l}\sqrt{l-1} + \sqrt{N-m}\,\delta_{m+1,l+1}\sqrt{N-l})\sqrt{l-1}\,\delta_{l,n}\,, \\
&= \sqrt{m-1}(\delta_{m,n}\sqrt{n-1}\sqrt{n-1} + (N-n+1)\delta_{m,n}\,, \\
&- (\sqrt{m-1}\,\delta_{m,n}\sqrt{n-1} + \sqrt{N-m}\,\delta_{m+1,n+1}\sqrt{N-n})\sqrt{n-1}\,, \\
&= \sqrt{m-1}(\delta_{m,n}(n-1) + (N-n+1)\delta_{m,n}\,, \\
&- ((m-1)\delta_{m,n} + (N-m+1)\,\delta_{m,n})\sqrt{n-1}\,, \\
&= \sqrt{m-1}(N-1)\delta_{m,n} - (N)\sqrt{m-1}\delta_{m,n}\,, \\
&= \sqrt{m-1}\delta_{m,n}\,, \\
&= (G^1)_{mn}.
\end{aligned}
\tag{4.43}
$$

One can check that the rest of the equations are also satisfied by using a similar approach.

We may also note that since Q^α and R^α are bifundamental complex scalar fields w.r.t to the $SU(N) \times SU(N)$ gauge symmetry, we may expect that $G^\alpha G^\dagger_\beta$ is in the adjoint of the first $SU(N)$ gauge group while $G^\dagger_\alpha G^\beta$ is in the adjoint of the second $SU(N)$ gauge group. To see how this may works, we look at

i) $GG^\dagger$ bilinears:

$$
\begin{aligned}
(G^1 G_1^\dagger)_{mn} &= (m-1)\,\delta_{m,n} \\
(G^2 G_2^\dagger)_{mn} &= (N-m)\,\delta_{m,n} \\
(G^1 G_2^\dagger)_{mn} &= \sqrt{(m-1)\,(N-m+1)}\,\delta_{m,n+1} \\
(G^2 G_1^\dagger)_{mn} &= \sqrt{(N-m)m}\,\delta_{m+1,n}
\end{aligned}
\tag{4.44}
$$

and we also have

$$\begin{aligned}(G^\alpha G^\dagger_\alpha)_{mn} &= (G^1G^\dagger_1 + G^2G^\dagger_2)_{mm} \\ &= (G^1G^\dagger_1)_{mn} + (G^2G^\dagger_2)_{mm} \\ &= (m-1)\delta_{m,n} + (N-m)\delta_{m,n} \\ &= (N-1)\delta_{m,n}\end{aligned} \tag{4.45}$$

Defining $J^\alpha{}_\beta = G^\alpha G_\beta{}^\dagger$ we can find

$$[J^\alpha{}_\beta, J^\gamma{}_\zeta] = \delta^\gamma{}_\beta J^\alpha{}_\zeta - \delta^\alpha{}_\zeta J^\gamma{}_\beta \tag{4.46}$$

(4.46) are the commutation relations for $U(2)$. The traceless part of $J^\alpha{}_\beta$ is

$$\hat{J}^\alpha_\beta = J^\alpha{}_\beta - \frac{1}{2}J\delta^\alpha_\beta, \tag{4.47}$$

where $J = J^\alpha{}_\alpha$. To see that (4.47) yields the conventional $SU(2)$ algebra we may write

$$J_i = (\sigma_i^T)^\alpha{}_\beta J_\alpha{}^\beta = (\sigma_i)_\beta{}^\alpha J_\alpha{}^\beta \tag{4.48}$$

This gives

$$[J_i, J_j] = \left[(\sigma_i)_\beta{}^\alpha J_\alpha{}^\beta, (\sigma_j)_\zeta{}^\gamma J_\gamma{}^\zeta\right] = 2i\epsilon_{ijk}J_k. \tag{4.49}$$

We may also note that

$$J = J^\alpha{}_\alpha = (N-1)1_N \tag{4.50}$$

is a trivial representation of $U(1)$ which commutes with all J^α_β i.e $[J, J_i] = 0$, so J is in the center of $U(2)$. We may indicate this $U(1)$ as $U(1) \simeq U(2)/SU(2)$.

ii) $G^\dagger G$ bilinears:

$$\begin{aligned}(G^\dagger_1G^1)_{mn} &= (m-1)\,\delta_{m,n} \\ (G^\dagger_2G^2)_{mn} &= (N-m+1)\,\delta_{m,n} - N\,\delta_{m,1}\delta_{n,1} \\ (G^\dagger_1G^2)_{mn} &= \sqrt{(m-1)(N-m)}\,\delta_{m+1,n} \\ (G^\dagger_2G^1)_{mn} &= \sqrt{(m-2)(N-m+1)}\,\delta_{m,n+1} \\ (G^\dagger_\alpha G^\alpha)_{mn} &= N\delta_{mn} - N\delta_{m,1}\delta_{n,1}\end{aligned} \tag{4.51}$$

We may define the bilinear $\bar{J}_i{}^\beta = G^\dagger_\alpha G^\beta$. Performing a calculation similar to that in (4.46), this gives

$$[\bar{J}_\beta{}^\alpha, \bar{J}_\nu{}^\mu] = \delta^\mu_\beta \bar{J}_\nu{}^\alpha - \delta^\alpha_\nu \bar{J}_\beta{}^\mu \tag{4.52}$$

The traceless part is $\hat{\bar{J}}^{\alpha}_{\beta} = \bar{J}_{\beta}{}^{\alpha} - \frac{1}{2}\bar{J}\delta^{\alpha}_{\beta}$, where $\bar{J} = \bar{J}_{\alpha}{}^{\alpha}$. With

$$\bar{J}_i = (\sigma_i^T)^{\alpha}_{\beta}\bar{J}_{\alpha}{}^{\beta} = (\sigma_i)^{\alpha}_{\beta}\bar{J}^{\beta}_{\alpha} \tag{4.53}$$

we find, the $SU(2)$ commutation relations,

$$[\bar{J}_i, \bar{J}_j] = 2i\epsilon_{ijk}\bar{J}_k \tag{4.54}$$

Note that the trace part, $\bar{J}$ does carry the trivial representation of $U(1)$ in this case. Instead, we have

$$(\bar{J}_{\alpha}{}^{\alpha})_{mn} = (\bar{J})_{mn} = N\delta_{mn} - N\delta_{m,1}\delta_{n,1} \tag{4.55}$$

We may note that $[\bar{J}, \bar{J}_i] = 0$, however $\bar{J}$ is not in the center of $U(2)$ since $\bar{J}$ does not commute with J^1_2 and J^2_1.These results have some consequences which are not going to relevant for the discussions of the rest of this thesis.

(4.48) and (4.53) show that GRVV algebra has two copies of $SU(2)$ i.e $SO(4) \simeq SU(2) \times SU(2)$ however it turns out that J_i and $\bar{J}_i$ are not independent and what we have is essentially a single $SU(2)$ and therefore a fuzzy sphere. Further details has to presented to prove this , which are beyond the scope for our purposes in this thesis. Details may be found in [23].

4.1.3 Dimensional reduction of the Mass Deformed ABJM Model

The procedure for calculating the dimensional reduction of the mass deformed ABJM matrix model is similar to the case without the mass deformation. From $S_{reduced}$ in(4.11) and using $C^I \to (R^{\alpha}, Q^{\alpha})$, we may write the reduced action

$$\begin{aligned} S_{reduced} = N \int dt \frac{k}{4\pi}\epsilon^{ij}\,\mathrm{Tr}\Big(-X_i\dot{X}_j + \hat{X}_i\dot{\hat{X}}_j\Big) - \mathrm{Tr}\big(D_iQ^{\alpha\dagger}D_iQ^{\alpha}\big) - \mathrm{Tr}\big(D_iR^{\alpha\dagger}D_iR^{\alpha}\big) \\ + \mathrm{Tr}\big(D_0Q^{\alpha\dagger}D_0Q^{\alpha})\big) + \mathrm{Tr}\big(D_0R^{\alpha\dagger}D_0R^{\alpha}\big) - \mathrm{Tr}\big(\big|M^2\big| + \big|N^2\big|\big) \end{aligned} \tag{4.56}$$

where $A_0 = 0 = \hat{A}_0$ gauge is already fixed and the fields have solely time dependence after dimensional reduction. What happens to the Guass law constraints? We have from (4.17) that

$$\begin{aligned} \frac{k}{4\pi}[X_1, X_2] + \dot{Q^{\alpha}}Q^{\alpha\dagger} - Q^{\alpha}\dot{Q^{\alpha\dagger}} + \dot{R^{\alpha}}R^{\alpha\dagger} - R^{\alpha}\dot{R^{\alpha\dagger}} = 0\,, \\ -\frac{k}{2\pi}[\hat{X}_1, \hat{X}_2] - Q^{\alpha\dagger}\dot{Q^{\alpha}} + \dot{Q^{\alpha\dagger}}Q^{\alpha} - R^{\alpha\dagger}\dot{R^{\alpha}} + \dot{R^{\alpha\dagger}}R^{\alpha} = 0\,, \end{aligned} \tag{4.57}$$

We now proceed to calculate the Hamiltonian and the Hamilton equation of motions for dimensionally reduced Mass Deformed ABJM matrix model.

4.1.4 Hamiltonian of the Dimensionally reduced Mass Deformed ABJM Model

The action of the reduced system is given in (4.56). The associated Lagrangian density can be read off by nothing that $S_{reduced} = \int dt \, \mathrm{Tr}\, \mathcal{L}$, where $L_{reduced} = \mathrm{Tr}\, \mathcal{L}$ is the Lagrangian.
Ingeneral we may express the corresponding Hamiltonian density

$$\mathcal{H} = \sum_i P_{q_i} \dot{q}_i - \mathcal{L} \tag{4.58}$$

and the Hamiltonian as $H := \mathrm{Tr}\, \mathcal{H} = \mathrm{Tr}\, p_i \dot{q}_i - \mathrm{Tr}\, \mathcal{L}$, where q_i's are the generalized coordinates and p_i are the associated conjugate momenta.
For the mass deformed ABJM model, (4.58) gives the Hamiltonian density,

$$\mathcal{H} = P_{Q^\alpha} \dot{Q}^\alpha + P_{Q^{\alpha\dagger}} \dot{Q}^{\alpha\dagger} + P_{R^\alpha} \dot{R}^\alpha + P_{R^{\alpha\dagger}} \dot{R}^{\alpha\dagger} + P_{X_i} \dot{X}_i + P_{\hat{X}_i} \dot{\hat{X}}_i - \mathcal{L} \tag{4.59}$$

The conjugate momenta's are given as

$$\begin{aligned}
P_{X_i} &= \frac{\partial L}{\partial \dot{X}_i} = -\frac{k}{4\pi} \epsilon^{ij} X_j \\
P_{\hat{X}_i} &= \frac{\partial L}{\partial \dot{\hat{X}}_i} = \frac{k}{4\pi} \epsilon^{ij} \hat{X}_j \\
P_{Q^\alpha} &= \frac{\partial L}{\partial \dot{Q}^\alpha} = \dot{Q}^{\alpha\dagger} \\
P_{Q^{\alpha\dagger}} &= \frac{\partial L}{\partial \dot{Q}^{\alpha\dagger}} = \dot{Q}^\alpha \\
P_{R^\alpha} &= \frac{\partial L}{\partial \dot{R}^\alpha} = \dot{R}^{\alpha\dagger} \\
P_{R^{\alpha\dagger}} &= \frac{\partial L}{\partial \dot{R}^{\alpha\dagger}} = \dot{R}^\alpha
\end{aligned} \tag{4.60}$$

If we combine 4.60 and 4.59 with the reduced Lagrangian of the system , we obtain

$$
\begin{aligned}
\mathcal{H} &= P_{Q^\alpha}\dot{Q}^\alpha + P_{Q^{\alpha\dagger}}\dot{Q}^{\alpha\dagger} + P_{R^\alpha}\dot{R}^\alpha + P_{R^{\alpha\dagger}}\dot{R}^{\alpha\dagger} + P_{X_i}\dot{X}_i + P_{\hat{X}_i}\dot{\hat{X}}_i - L\,,\\
&= P_{Q^\alpha}P_{Q^{\alpha\dagger}} + P_{Q^{\alpha\dagger}}P_{Q^\alpha} + P_{R^\alpha}P_{R^{\alpha\dagger}} + P_{R^{\alpha\dagger}}P_{R^\alpha} - \frac{k}{4\pi}\epsilon^{ij}X_j\dot{X}_i + \frac{k}{4\pi}\epsilon^{ij}\hat{X}_j\dot{\hat{X}}_i - L\,,\\
&= P_{Q^\alpha}P_{Q^{\alpha\dagger}} + P_{Q^{\alpha\dagger}}P_{Q^\alpha} + P_{R^\alpha}P_{R^{\alpha\dagger}} + P_{R^{\alpha\dagger}}P_{R^\alpha} - \cancel{\frac{k}{4\pi}\epsilon^{ij}X_j\dot{X}_i}\,,\\
&+ \cancel{\frac{k}{4\pi}\epsilon^{ij}\hat{X}_j\dot{\hat{X}}_i} + \cancel{\frac{k}{4\pi}\epsilon^{ij}X_i\dot{X}_j} - \cancel{\frac{k}{4\pi}\epsilon^{ij}\hat{X}_i\dot{\hat{X}}_j}\,,\\
&- \dot{Q}^{\alpha\dagger}\dot{Q}^\alpha - \dot{R}^{\alpha\dagger}\dot{R}^\alpha + D_iQ^{\alpha\dagger}D_iQ^\alpha + D_iR^{\alpha\dagger}D_iR^\alpha\,,\\
&+ (|M^\alpha|^2 + |N^\alpha|^2)\,,\\
&= \cancel{P_{Q^\alpha}P_{Q^{\alpha\dagger}}} + P_{Q^{\alpha\dagger}}P_{Q^\alpha} + \cancel{P_{R^\alpha}P_{R^{\alpha\dagger}}} + P_{R^{\alpha\dagger}}P_{R^\alpha} - \cancel{P_{Q^\alpha}P_{Q^{\alpha\dagger}}} - \cancel{P_{R^\alpha}P_{R^{\alpha\dagger}}}\,,\\
&+ D_iQ^{\alpha\dagger}D_iQ^\alpha + D_iR^{\alpha\dagger}D_iR^\alpha + (|M^\alpha|^2 + |N^\alpha|^2)\,,\\
&= P_{Q^{\alpha\dagger}}P_{Q^\alpha} + P_{R^{\alpha\dagger}}P_{R^\alpha} + D_iQ^{\alpha\dagger}D_iQ^\alpha + D_iR^{\alpha\dagger}D_iR^\alpha + M^\alpha M^{\alpha\dagger} + N^\alpha N^{\alpha\dagger}
\end{aligned}
\tag{4.61}
$$

Note that $\mathcal{H}$ does not have any X_i, $\hat{X}_i$ or P_{X_i}, $P_{\hat{X}_i}$ as the Hamiltonian for the Chern-Simons part of the action is vanishing [39].

4.1.5 Hamilton Equations of Motion

In (4.61), we calculated the Hamiltonian of the system. Since it is somewhat non-trivial to obtain the associated Hamilton's equations of motion and although we will not use them directly, we give a derivation of them here, with details relocated to the appendices A. They are defined as

$$\dot{X}_i = \frac{\partial H}{\partial P_{X_i}} = 0\,, \qquad \dot{\hat{X}}_i = \frac{\partial H}{\partial P_{\hat{X}_i}} = 0 \tag{4.62a}$$

$$\dot{Q}_\alpha = \frac{\partial H}{\partial P_{Q_\alpha}} = P_{Q^\dagger_\alpha}\,, \quad \dot{Q}^\dagger_\alpha = \frac{\partial H}{\partial P_{Q^\dagger_\alpha}} = P_{Q_\alpha} \tag{4.62b}$$

$$\dot{R}_\alpha = \frac{\partial H}{\partial P_{R_\alpha}} = P_{R^\dagger_\alpha}\,, \quad \dot{R}^\dagger_\alpha = \frac{\partial H}{\partial P_{R^\dagger_\alpha}} = P_{R_\alpha} \tag{4.62c}$$

The rest of the equations given by

$$\dot{P}_{X_i} = -\frac{\partial H}{\partial X_i}\,, \quad \dot{P}_{\hat{X}_i} = -\frac{\partial H}{\partial \hat{X}_i} \tag{4.63a}$$

$$\dot{P}_{Q_i} = -\frac{\partial H}{\partial Q_i}\,, \quad \dot{P}_{Q_i^\dagger} = -\frac{\partial H}{\partial Q_i^\dagger} \tag{4.63b}$$

$$\dot{P}_{R_i} = -\frac{\partial H}{\partial R_i}\,, \quad \dot{P}_{R_i^\dagger} = -\frac{\partial H}{\partial R_i^\dagger} \tag{4.63c}$$

The equations given in (4.63) are not trivial like in (4.62). The explicit calculations are given in appendix A, the final form of the equations are

$$\dot{P}_{X_i} = i\left((D_iQ^\alpha){Q^\dagger}_\alpha - Q^\alpha(D_i{Q^\dagger}_\alpha) + (D_iR^\alpha){R^\dagger}_\alpha - R^\alpha(D_i{R^\dagger}_\alpha)\right)^T \tag{4.64a}$$

$$\dot{P}_{\hat{X}_i} = i\left(-{Q^\dagger}_\alpha(D_iQ^\alpha) + (D_i{Q^\dagger}_\alpha)Q^\alpha - {R^\dagger}_\alpha(D_iR^\alpha) + (D_i{R^\dagger}_\alpha)R^\alpha\right)^T \tag{4.64b}$$

$$\begin{aligned}\dot{P}_{Q_\gamma} = -\Bigg(& i[(D_i{Q^\dagger}_\gamma)X_i - \hat{X}_i(D_i{Q^\dagger}_\gamma)] + \mu {M^\dagger}_\gamma + \frac{2\pi}{k}[2{Q^\dagger}_\alpha M^\alpha {Q^\dagger}_\gamma \\ & - 2{Q^\dagger}_\gamma M^\alpha {Q^\dagger}_\alpha + 2{Q^\dagger}_\beta Q^\beta {M^\dagger}_\gamma + 2M_\alpha Q^\alpha {Q^\dagger}_\gamma - 2{Q^\dagger}_\gamma Q^\alpha {M_\alpha}^\dagger \\ & - 2{M^\dagger}_\gamma Q^\beta {Q^\dagger}_\beta + {M^\dagger}_\gamma R^\beta {R^\dagger}_\beta - {R^\dagger}_\beta R^\beta M^\dagger_\gamma + 2{R^\dagger}_\gamma Q^\alpha {M^\dagger}_\alpha - 2{M^\dagger}_\alpha R^\alpha {R^\dagger}_\gamma \\ & + {Q^\dagger}_\gamma R^\alpha {N^\dagger}_\alpha - {N^\dagger}_\alpha R^\alpha {Q^\dagger}_\gamma + 2{N^\dagger}_\gamma R^\beta {Q^\dagger}_\beta - 2{Q^\dagger}_\beta R^\beta {N^\dagger}_\gamma \\ & + {Q^\dagger}_\gamma N^\alpha {R^\dagger}_\alpha - {R^\dagger}_\alpha N^\alpha {Q^\dagger}_\gamma + 2{R^\dagger}_\gamma N^\alpha {Q^\dagger}_\alpha - 2{Q^\dagger}_\alpha N^\alpha {R^\dagger}_\gamma]\Bigg)^T\end{aligned} \tag{4.64c}$$

$$\begin{aligned}\dot{P}_{{Q^\dagger}_\gamma} = -\Bigg(& i[(D_iQ_\gamma)\hat{X}_i - X_i(D_iQ_\gamma)] + \mu M_\gamma + \frac{2\pi}{k}[2Q^\gamma {Q^\dagger}_\alpha M^\alpha \\ & + 2M^\gamma {Q^\dagger}_\beta Q_\beta - 2Q^\beta {Q^\dagger}_\beta M_\gamma - 2M^\alpha {Q^\dagger}_\alpha Q^\gamma + 2Q_\gamma {M_\alpha}^\dagger Q^\alpha \\ & - 2Q^\alpha {M^\dagger}_\alpha Q^\gamma + R^\beta {R^\dagger}_\beta M_\gamma - M^\gamma {R^\dagger}_\beta R^\beta + 2M_\alpha {R^\dagger}_\alpha R^\gamma - 2R^\gamma {R^\dagger}_\alpha M^\alpha \\ & + R^\alpha {N^\dagger}_\alpha Q^\gamma - Q^\gamma {N^\dagger}_\alpha R^\alpha + 2Q^\gamma {N^\dagger}_\alpha R^\gamma - 2R^\gamma {N^\dagger}_\alpha Q^\alpha \\ & + N^\alpha {R^\dagger}_\alpha Q^\gamma - Q^\gamma {R^\dagger}_\alpha N^\alpha + 2Q^\beta {R^\dagger}_\beta N^\gamma - 2N^\gamma {R^\dagger}_\beta Q^\beta]\Bigg)^T\end{aligned}$$

$$\begin{aligned}\dot{P}_{R_\gamma} = -\Bigg(& i[(D_i{R^\dagger}_\gamma)X_i - \hat{X}_i(D_i{R^\dagger}_\gamma)] - \mu {N^\dagger}_\gamma + \frac{2\pi}{k}[2{Q^\dagger}_\gamma M^\alpha {R^\dagger}_\alpha \\ & - 2{R^\dagger}_\alpha M^\alpha {Q^\dagger}_\gamma + 2{M^\dagger}_\gamma Q^\beta {R^\dagger}_\beta - 2{R^\dagger}_\beta Q^\beta {M_\gamma}^\dagger + {R^\dagger}_\gamma M^\alpha {Q^\dagger}_\alpha \\ & - {Q^\dagger}_\alpha M^\alpha {R^\dagger}_\gamma + {R^\dagger}_\gamma Q^\alpha {M^\dagger}_\alpha - {M^\dagger}_\alpha Q^\alpha {R^\dagger}_\gamma + {R^\dagger}_\beta R^\beta {N^\dagger}_\gamma + {N^\dagger}_\gamma R^\gamma {R^\dagger}_\beta \\ & - {R^\dagger}_\gamma R^\alpha {N^\dagger}_\alpha - {N^\dagger}_\gamma R^\beta {R^\dagger}_\beta + {N^\dagger}_\gamma Q^\beta {Q^\dagger}_\beta - {Q^\dagger}_\beta Q^\beta {N^\dagger}_\gamma + 2{Q^\dagger}_\gamma Q^\alpha {N^\dagger}_\alpha \\ & - 2{N^\dagger}_\alpha Q^\alpha {Q^\dagger}_\gamma + {R^\dagger}_\alpha N^\alpha {R^\dagger}_\gamma - {R^\dagger}_\gamma N^\alpha {R^\dagger}_\alpha]\Bigg)^T\end{aligned} \tag{4.64d}$$

$$\dot{P}_{R^{\dagger}{}_{\gamma}} = -\Big(i[(D_i R_\gamma)\hat{X}_i - X_i(D_i R_\gamma)] - \mu N_\gamma + \frac{2\pi}{k}[M^\alpha Q^\dagger{}_\alpha R^\dagger{}_\gamma \tag{4.64e}$$
$$- R^\beta Q^\dagger{}_\beta M^\gamma + 2R^\gamma Q^\dagger{}_\alpha M^\alpha - 2R^\beta Q^\dagger{}_\beta M^\gamma + Q^\alpha M^\dagger{}_\alpha R^\gamma$$
$$- R^\gamma M^\dagger{}_\alpha Q^\alpha + 2R^\alpha M^\dagger{}_\alpha Q^\gamma - 2Q^\gamma M^\dagger{}_\alpha R^\alpha + R^\gamma N^\dagger{}_\gamma R^\alpha - R^\alpha N^\dagger{}_\alpha R^\gamma$$
$$+ N^\gamma R^\dagger{}_\beta R^\beta - R^\beta R^\dagger{}_\beta N^\gamma - N^\alpha R^\dagger{}_\alpha R^\gamma + Q^\beta Q^\dagger{}_\beta N^\gamma$$
$$- N^\gamma Q^\dagger{}_\beta Q^\beta + 2N^\alpha Q^\dagger{}_\alpha Q^\alpha - 2Q^\gamma Q^\dagger{}_\alpha N^\alpha + R^\alpha R^\dagger{}_\alpha N^\alpha]\Big)^T$$

4.2 Ansatz I and the Effective action

In this chapter, we will consider two different ansatz configurations involving the GRVV matrices and satisfying the Gauss law constraints. Both of these ansatz configurations involve collective time-dependence which are introduced via real functions of time. The first configurations we study is given as

$$\begin{aligned} X_i &= \alpha(t)\text{diag}((a_i)_1, (a_i)_2, ..., (a_i)_N) \\ \hat{X}_i &= \beta(t)\text{diag}((b_i)_1, (b_i)_2, ..., (b_i)_N) \end{aligned} \tag{4.65}$$

where $(a_i)_K$,$(b_i)_K$ for $(i = 1, 2)$ and $(K = 1, 2, ..., N)$ are constants. Thus X_i and $\hat{X}_i$ are taken as diagonal matrcies Q_i and R_i are given as

$$Q_\alpha = \phi_\alpha(t) G_\alpha, \;\; R_\alpha = 0 \tag{4.66}$$

where no sum over on the repeated index α is implied in (4.65) and (4.66) all. $\phi_\alpha(t)$, $\alpha(t)$, $\beta(t)$ are real functions of time. After calculating the e.o.m. for $\alpha(t)$ and $\beta(t)$, we find that the coupled equation have only one possible solution which is the trivial solution, i.e. $\alpha(t) = \beta(t) = 0$. This result is provided in the appendix B. Therefore setting X_i and $\hat{X}_i$ to zero and writting (4.66) in the action (4.56) and performing the traces over the GRVV matrices at the level of $N \times N$ matrices, we obtain the reduced system's Lagrangian

$$L_N = N^2(N-1)\big(\frac{1}{2}\dot{\phi_2}^2 + \frac{1}{2}\dot{\phi_1}^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{8\pi\mu}{k}\phi_2^2\phi_1^2 - \frac{8\pi^2}{k^2}\phi_2^4\phi_1^2 - \frac{8\pi^2}{k^2}\phi_2^2\phi_1^4\big) \tag{4.67}$$

The corresponding Hamiltonian is

$$\begin{aligned} H_N(\phi_1,\phi_2,p_{\phi_1},p_{\phi_2}) &= \frac{p_{\phi_1}^2}{2N^2(N-1)} + \frac{p_{\phi_2}^2}{2N^2(N-1)} \\ &+ N^2(N-1)\left(\frac{1}{2}\mu^2(\phi_1^2+\phi_2^2) + \frac{8\pi\mu}{k}\phi_2^2\phi_1^2 + \frac{8\pi^2}{k^2}\phi_2^4\phi_1^2 + \frac{8\pi^2}{k^2}\phi_2^2\phi_1^4\right) \\ &=: \frac{p_{\phi_1}^2}{2N^2(N-1)} + \frac{p_{\phi_2}^2}{2N^2(N-1)} + V_N(\phi_1,\phi_2) \end{aligned} \tag{4.68}$$

where $V_N(\phi_1,\phi_2)$ denotes the potential of this reduced system. For $k>0$ we also see that $V_N(\phi_1,\phi_2)$ is positive definite, while for $k<0$ and the maximum value of $V_N(\phi_1,\phi_2)$ is negative. To explore the dynamics of the model, we calculate the Hamiltonian's equations of motion. These take the form

$$\dot{\phi}_1 - \frac{p_{\phi_1}}{N^2(N-1)} = 0\,, \tag{4.69a}$$

$$\dot{\phi}_2 - \frac{p_{\phi_2}}{N^2(N-1)} = 0\,, \tag{4.69b}$$

$$\dot{p}_{\phi_1} + N^2(N-1)\left(\mu^2\phi_1 + \frac{16\pi\phi_1\phi_2^2}{k} + \frac{16\pi^2\phi_1\phi_2^4}{k^2} + \frac{32\pi^2\phi_1^3\phi_2^2}{k^2}\right) = 0\,, \tag{4.69c}$$

$$\dot{p}_{\phi_2} + N^2(N-1)\left(\mu^2\phi_2 + \frac{16\pi\phi_1^2\phi_2}{k} + \frac{16\pi^2\phi_1^4\phi_2}{k^2} + \frac{32\pi^2\phi_1^2\phi_2^3}{k^2}\right) = 0. \tag{4.69d}$$

For concreteness, we will first consider the configuration with $\mu^2 = 1$ and $k = 1$. We will continue to keep $\mu^2 = 1$ but remark upon the the impact of other k values of these parameters on the dynamics later on.

In order to gain some immediate information on the system, it is useful to explore its fixed points and also investigate the stability of the system around these points at the linear level. Fixed points of a Hamiltonian system are defined as stationary points in the phase space, i.e. [26]

$$(\dot{\phi_1},\dot{\phi_2},\dot{p_{\phi_1}},\dot{p_{\phi_2}}) = (0,0,0,0) \tag{4.70}$$

If we combine (4.70) and (4.69), it leads to four algebraic equations, two of which are trivially satisfied by taking $(p_{\phi_1},p_{\phi_2}) \equiv (0,0)$. The remaining two gives us two coupled algebraic equations which are

$$N^2(N-1)\big(\phi_2 + 16\pi\phi_2\phi_1^2 + 16\pi^2\phi_2\phi_1^4 + 32\pi^2\phi_2^3\phi_1^2\big) = 0 \tag{4.71a}$$

$$N^2(N-1)\big(\phi_1 + 16\pi\phi_2^2\phi_1 + 16\pi^2\phi_2^4\phi_1k^2 + 32\pi^2\phi_2^2\phi_1^3\big) = 0 \tag{4.71b}$$

The only real solution of this system is the trivial solution,

$$(\phi_1,\phi_2) \equiv (0,0) \tag{4.72}$$

Thus the only fixed point of this Hamiltonian system is given as

$$(\phi_1, \phi_2, p_{\phi_1}, p_{\phi_2}) \equiv (0, 0, 0, 0). \tag{4.73}$$

Clearly the corresponding fixed point energy of the system is 0, i.e. we have $E_F(0, 0, 0, 0) = 0$.

To continue our analysis, let us inspect linear stability of the system around this fixed point in order to determine whether it is a stable or unstalbe fixed point.

A similar such as analysis was performed for the mass deformed BFSS model in section 3.1.1 and we follow the same procedure outlined there in what follows. For simplicity, let us introduce the notation

$$(q_1, q_2, q_3, q_4) \equiv (\phi_1, \phi_2, p_{\phi_1}, p_{\phi_2}) \tag{4.74}$$

From q_α and $\dot{q}_\alpha$, we may form the Jacobian matrix

$$J \equiv [J]_{\alpha\beta} = \frac{\partial \dot{q}_\alpha}{\partial q_\beta} \tag{4.75}$$

whose eigenvalues will allow us to determine structure at the fixed point $(0, 0, 0, 0)$. The eigenvalues of $J(0, 0, 0, 0)$ are given as $\{0, 0, -i\sqrt{2}, i\sqrt{2}\}$. As we discussed in [26], a fixed point is unstable if all the real eigenvalues are negative. Similarly, all the eigenvalues can be purely imaginary. In that case, the stability analysis is inconclusive. Since the eigenvalues are non-negative and purely imaginary, our stability analysis is inconclusive. The fixed point is a borderline type at the certain level. We will not perform a more detailed analysis however.

4.3 Chaotic Dynamics

4.3.1 Lyapunov spectrum

We have already introduced meaning the Lyapunov exponent at the beginning of this thesis. Similarly to the Yang-Mills matrix model, treated in chapter 3 in order to obtain the Lyapunov spectrum for our models we run a Matlab code, which numerically solves the Hamilton's equations of motion given in (4.69) as different matrix levels. In particular we compute H_N at (N : $5, 10, 15, 20, 25$) at several different values of the energy. We run the code 40 times with randomly selected initial conditions satisfying a given energy and calculate the mean of the time series from all runs for each of the Lyapunov exponents at each value of N. In order to give certain effectiveness to the random initial condition selection process we developed a simple approach which we briefly explain next. Let us denote a generic set of initial conditions at $t = 0$ by $(\phi_1(0), \phi_2(0), p_{\phi_1}(0), p_{\phi_2}(0))$. First of all, we generate four random numbers and denote three of them as ω_i $(i = 1, 2, 3)$ and define $\Omega_i = \frac{\omega_i}{\sqrt{\omega_i^2}}\sqrt{E}$ where E is the energy of the system. Lets denote the last random number as ω_4. Clearly, we have $\sum_{i=1}^{3} \Omega_i^2 = E$. With the help of this relation and the energy functiional i.e. Hamiltonian given in (4.68), in fact, we can construct an algorithm for the selection of the initial conditions. Recall that we have

$$E_N(\phi_1, \phi_2, p_{\phi_1}, p_{\phi_2}) = \frac{p_{\phi_1}^2}{2N^2(N-1)} + \frac{p_{\phi_2}^2}{2N^2(N-1)} + V_N \tag{4.76}$$

Taking for instance, $\Omega_1^2 = \frac{p_{\phi_1}^2}{2N^2(N-1)}$, $\Omega_2^2 = \frac{p_{\phi_2}^2}{2N^2(N-1)}$ and $\Omega_3^2 = V_N$. We select the initial conditions for $p_{\phi_1}(0)$ and $p_{\phi_2}(0)$ as

$$p_{\phi_1}(0) = \pm N\sqrt{(N-1)\Omega_1^2} \quad p_{\phi_2}(0) = \pm N\sqrt{(N-1)\Omega_2^2} \tag{4.77}$$

Finally we need to select either $\phi_1(0)$ or $\phi_2(0)$ as ω_4. Since V_N is invariant under $\phi_1 \leftrightarrow \phi_2$ exchange, which pne we select randomly is immaterial. In our calculations, we take $\phi_2(0) = \omega_4$. Then $\phi_1(0)$ will be the solution of

$$V_N\left(\phi_1(0), \omega_4\right) - \Omega_3^2 = 0 \tag{4.78}$$

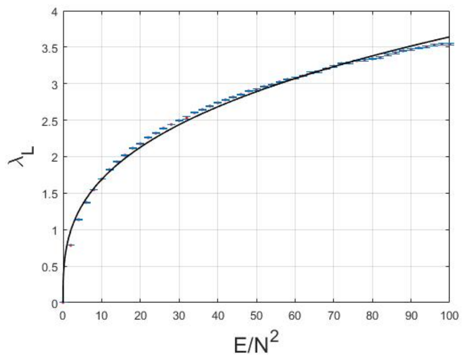
For some values of ω_4, (4.78) may not have a real solution. However the code runs until it finds a non-emty solution set.

4.3.2 Dependence of the Largest Lyapunov Exponent on Energy

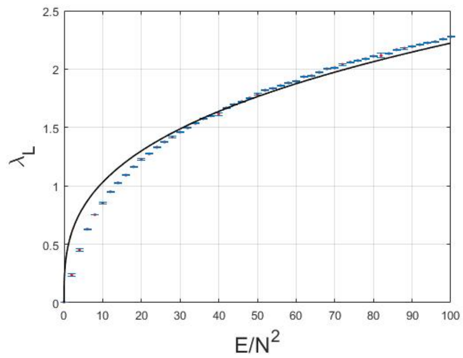
In order to see the dependence of the mean largest Lyapunov exponent(MLLE), (denoted as λ_L in the figures and the tables) to energy, we look at a broad range of energies and see if this is consistent with our expectations. Since we are already working in the 't Hooft limit, it is useful to consider λ_L versus E/N^2 to catch the main features of chaotic dynamics.

Case i) $k \geq 1$ and $\mu = 1$, we find that E/N^2 range as $0 : 2 : 100$ and $0 : 10 : 500$

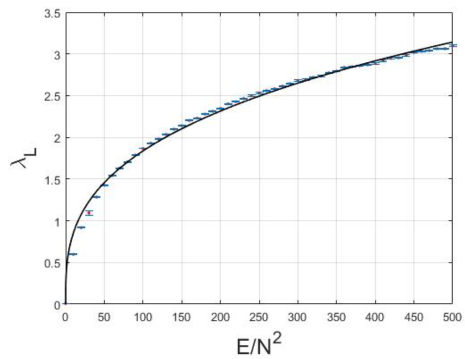
depending on the value of N and k. Clearly, from figure:(4.1),(4.2), we can say that our fitting curve almost perfectly passes through all the data points when E/N^2 range is equal to $0:2:100$. We also find that when $N = 25$, $0:10:500$ scale gives much better results. Furthermore, we also show the possible effects of changing the value of k at fixed value of N. In figure 4.3, we take $N = 10$ as fixed and changed k as $k = 1, 5, 10$. When $k = 1$, and E/N^2 range is $0:2:100$ scale is enough to see the pattern. However for $k = 5, 10$, we have to increase the energy scale for seeing a better picture.



(a) $N = 5$

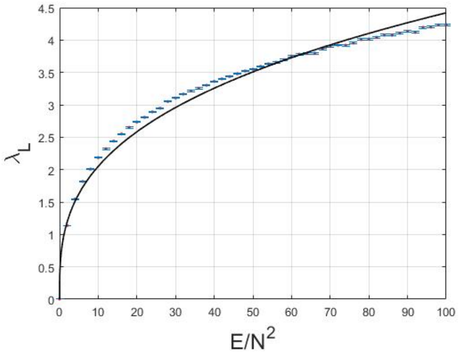


(b) $N = 15$

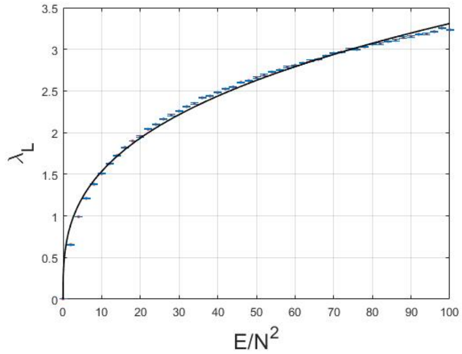


(c) $N = 25$

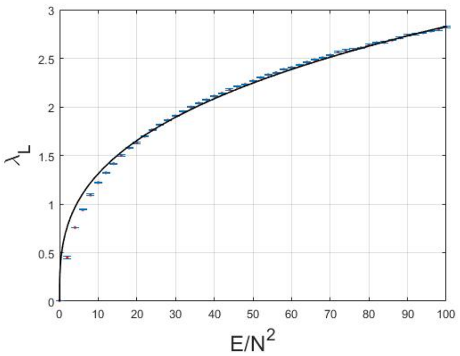
Figure 4.1: Lyapunov spectra and best fitting curves in the form $\lambda_L = \alpha_N(\frac{E}{N^2})^{1/3}$(see the discussion in section 4.3.3) at $k = 2$ and $\mu = 1$ for ansatz I



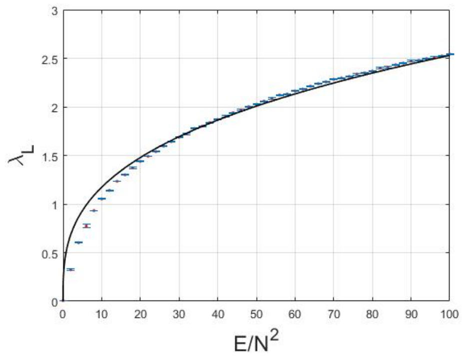
(a) $N = 5$



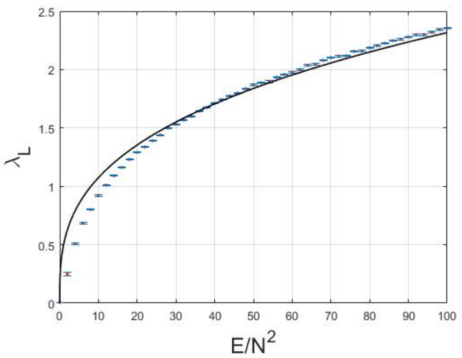
(b) $N = 10$



(c) $N = 15$

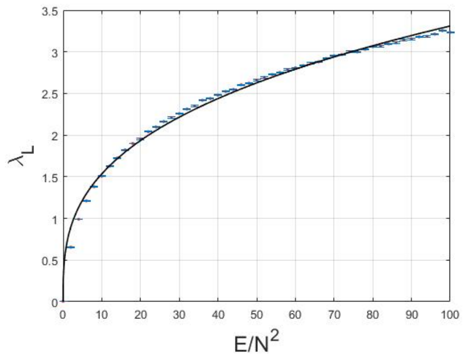


(d) $N = 20$

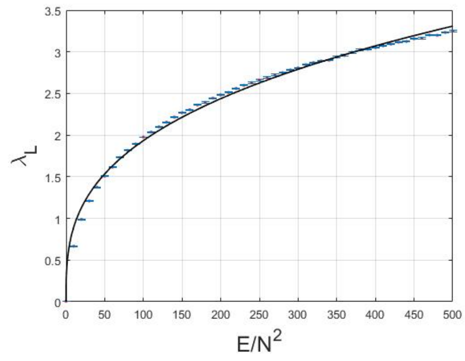


(e) $N = 25$

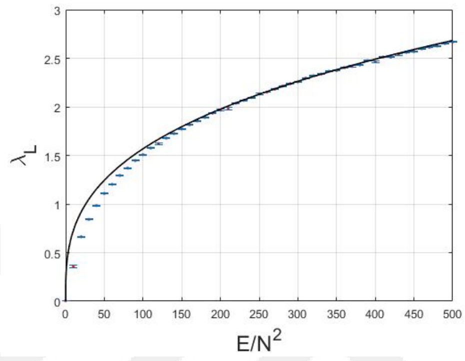
Figure 4.2: Lyapunov spectra and best fitting curves in the form $\lambda_L = \alpha_N(\frac{E}{N^2})^{1/3}$(see the discussion in section 4.3.3) at $k = 1$ and $\mu = 1$ for ansatz I



(a) $k = 1$



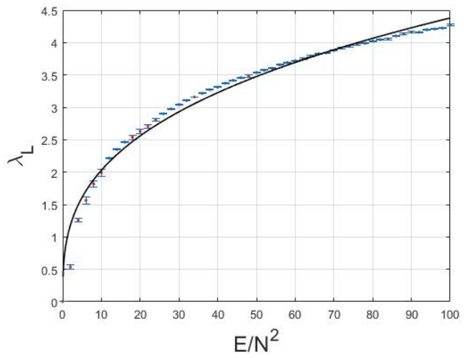
(b) $k = 5$



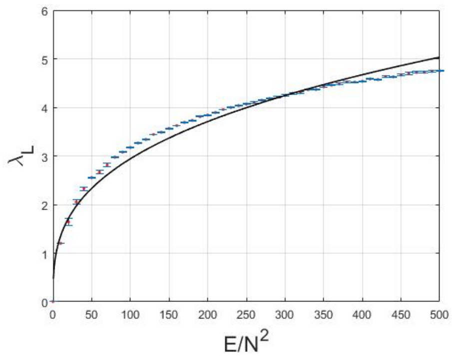
(c) $k = 10$

Figure 4.3: Lyapunov spectra and best fitting curves in the form $\lambda_L = \alpha_N(\frac{E}{N^2})^{1/3}$(see the discussion in section 4.3.3) for $N = 10$ at $k = 1, 5, 10$

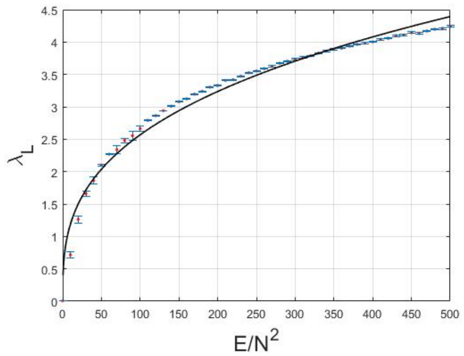
Case ii) $k \leq -1$ and $\mu = 1$, in this case, we find that if we take E/N^2 range as $0 : 10 : 500$, it works perfect almost for all N values. Clearly, from figure:(4.4),(4.5), we can say that our fitting curve almost perfectly passes through all the data points when E/N^2 range is equal to $0 : 10 : 500$.



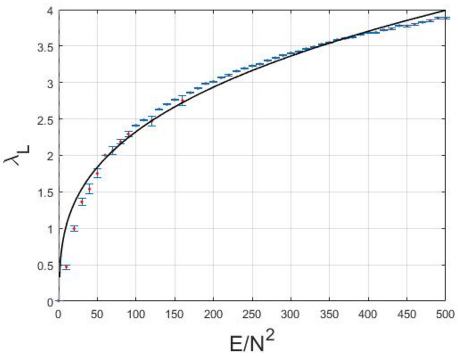
(a) $N = 5$



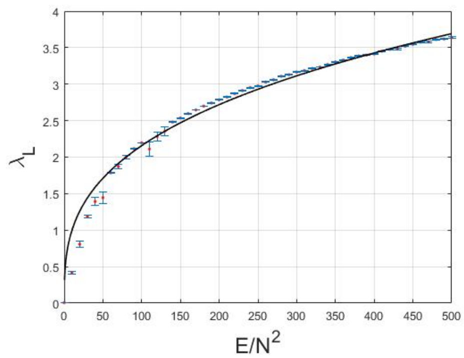
(b) $N = 10$



(c) $N = 15$

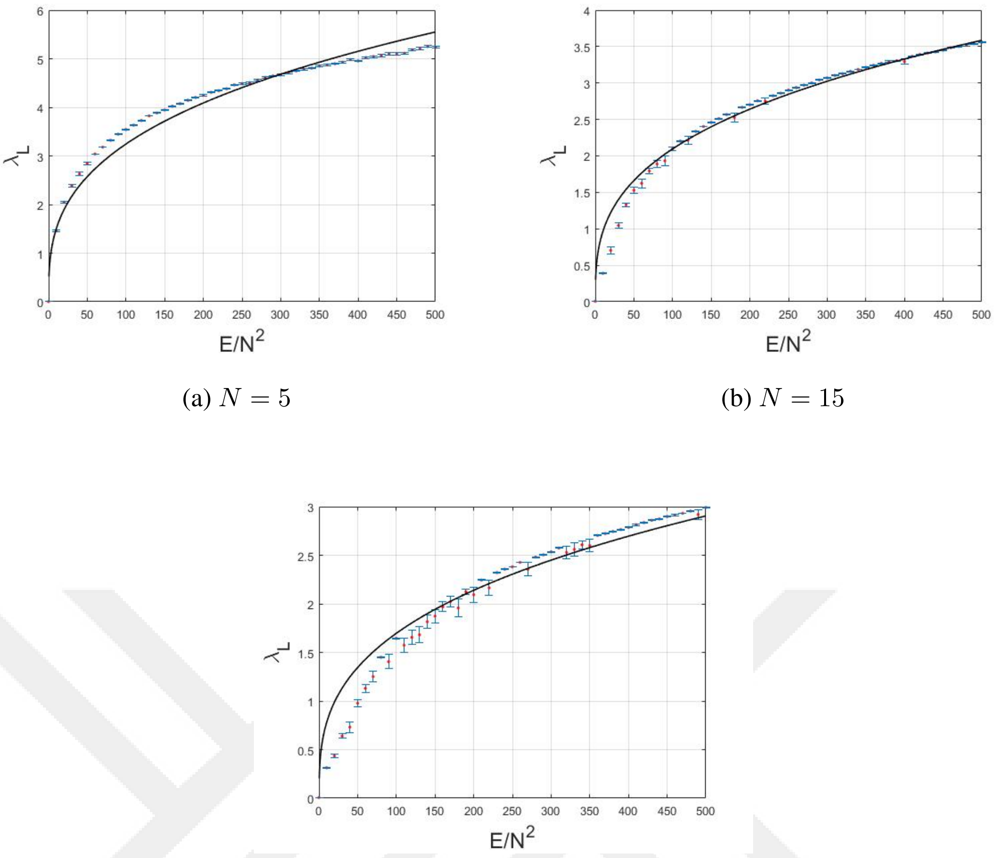


(d) $N = 20$



(e) $N = 25$

Figure 4.4: Lyapunov spectra and best fitting curves in the form $\lambda_L = \alpha_N(\frac{E}{N^2} - \gamma_N)^{1/3}$(see the discussion in section 4.3.3) at $k = -1$ and $\mu = 1$ for ansatz I



(a) $N = 5$

(b) $N = 15$

(c) $N = 25$

Figure 4.5: Lyapunov spectra and best fitting curves in the form $\lambda_L = \alpha_N(\frac{E}{N^2} - \gamma_N)^{1/3}$(see the discussion in section 4.3.3) at $k = -2$ and $\mu = 1$ for ansatz I

4.3.3 Temperature dependence of the Lyapunov Exponent

In [12] temperature dependence of the Largest Lyapunov exponent of the BFSS model in the 't Hooft limit was determined using dimensional analysis. In [26] this approach was also followed for the mass deformed BFSS model. These results we discussed in chapter 3 to this thesis. For the present matrix model which we have obtained by dimensionally reducing the mass deformed ABJM model to $0+1$ dimensions, a similar analysis can be performed.

Firstly, we may expect that $\lambda_L \propto (\lambda_{t'Hooft} T)^{\alpha}$ where α is a constant that needs to be determined. Next, note that in section 4.1.1, we have introduced the $\lambda_{'tHooft}$ for this

ABJM matrix model as $\lambda_{'tHooft} = \frac{N}{V_2}$ where V_2 is the volume of the 2-dimensional space we have integrated over in going from 2+1 to 0+1 dimensions. Clearly, $\lambda_{t'Hooft}$ has a dimension of $[Length]^{-2}$, we already know that each of λ_L and T have a dimension of $[Length]^{-1}$. Putting these fact together $[L]^{-1} = [L^{-2}L^{-1}]^{\alpha} = [L^{-3}]^{\alpha}$ and therefore we find $\alpha = 1/3$.

Like in 4.3.3, we use of the virial and equipartition theorems to introduce the temperature depence. First, let us start with counting the total number of degrees of freedom of dimensionally reduced mass deformed ABJM matrix model to proceed, we count the number of degrees of freedom (d.o.f.) of the involved complex valued matrices. Recall that each $N \times N$ complex matrix has $2N^2$ degrees of freedom. In case of Hermitian matrices, we have N^2 degrees of freedom; due to the Hermiticity requirement the number of d.o.f. is cut by half. In (4.56), X_i and $\hat{X}_i$ are $N \times N$ hermitian matrices and $i = 1, 2$. This is $4N^2$ d.o.f in total. Furthermore, Q^{α} and R^{α} are $N \times N$ complex matrices and $\alpha = 1, 2$. Since the Hermitian conjugates do not contribute any additional degrees of freedom, we have four independent complex valued $N \times N$ matrices and contributing $8N^2$ d.o.f.. Therefore, the number of degrees of freedom of the system without any constraints is $4N^2 + 8N^2 = 12N^2$. However, we have already discussed that the mass deformed ABJM matrix model has $SU(2) \times SU(2) \times U(1) \times U(1) \times Z_2$ symmetry. Each $SU(2)$ contributes to $n_{d.o.f.} = 3$ and each $U(1)$ to $n_{d.o.f.} = 3$ and $n_{d.o.f.} = 1$, respectively. Moreover, the Gauss law constraint given in (4.17) imposes $2N^2$ independent conditions. Therefore the independent degrees of freedom drops down to $12N^2 - 2N^2 - 8 = 10N^2 - 8$.

For ansatz I, since X_i and $\hat{X}_i$ are affectively zero (See the discussion in appendix B) and due to the fact that $R_{\alpha} = 0$ in this ansatz, we need to substract another $4N^2$ d.o.f. due to X_i, $\hat{X}_i$ and another $4N^2$ for $R_{\alpha} = 0$. Nevertheless the Gauss law gives only N^2 conditions since R_{α} are clearly zero. Therefore the total number of d.o.f. is $10N^2 - 8 - 8N^2 + N^2 = 3N^2 - 8$. Therefore we have

$$\langle K \rangle = (3N^2 - 8)T \tag{4.79}$$

Table 4.1: Numerical values of coefficients α_N for $AnsatzI$ for $k = 1$ and $\mu = 1$.

	$N = 5$	$N = 10$	$N = 15$	$N = 20$	$n = 25$
α_N	0.9522	0.713	0.6092	0.5448	0.499
T_c	0.1022	0.0662	0.0523	0.0442	0.0388

Table 4.2: Numerical values of coefficients α_N and T_c values for $AnsatzI$ for $k = 2$ and $\mu = 1$.

	$N = 5$	$N = 15$	$N = 25$
α_N	0.7845	0.4788	0.3958
T_c	0.0764	0.0364	0.0274

For large N, we may take $n_{d.o.f.} \approx 3N^2$. Lets apply the virial theorem to V_N in (4.68), we have

$$\begin{aligned}\langle 2K \rangle &= \langle 2V \rangle + 2N^2(N-1)\left(\frac{8\pi\mu\phi_2^2\phi_1^2}{k} + \frac{16\pi^2\phi_2^4\phi_1^2}{k^2} + \frac{16\pi^2\phi_2^2\phi_1^4}{k^2}\right) \\ &=: \langle 2V \rangle + \tilde{V}(\phi_1, \phi_2)_N \end{aligned} \tag{4.80}$$

where the first line involves the definition of $\tilde{V}(\phi_1, \phi_2)_N$.
The minimum of the $\tilde{V}(\phi_1, \phi_2)_N$ given as

$$Min\left(V'(\phi_1, \phi_2)_N\right) = \begin{cases} 0 & \text{if } k \text{ and } \mu \text{ have the same signs} \\ \frac{N^2(N-1)k\mu^3}{27\pi} & \text{if } k \text{ and } \mu \text{ have a different signs} \end{cases} \tag{4.81}$$

Case i) $k \geq 1$ and $\mu = 1$, we can say that $K \geq V$. This implies that we have $\langle E \rangle \leq n_{d.o.f}T$. In the large N limit using (4.79), this reduces to $\langle E \rangle \leq 3N^2T$. Therefore, we may write

$$\frac{E}{N^2} \leq 3T\,. \tag{4.82}$$

In view of (4.82), we examine the best fit curves of the form

$$\lambda_L = \alpha_N \left(\frac{E}{N^2}\right)^{1/3} \tag{4.83}$$

to the λ_L versus $\frac{E}{N^2}$ data given in figure 4.2 and figure 4.1 for $N = 5, 10, 15, 20, 25$ for $k = 1$ and $k = 2$ cases. First discuss the MSS bound for this case $k \geq 1$ and $\mu = 1$.

Table 4.3: Numerical values of coefficients α_N and T_c values for $AnsatzI$ for $N = 10$ at $k = 1, 5, 10$.

	$k = 1$	$k = 5$	$k = 10$
α_N	0.6453	0.4168	0.3338
T_c	0.0570	0.0296	0.0212

α_N values obtained from the fitting curves are given in the tables 4.1 and 4.2. We are now in a position to compare and relate our result to the MSS bound $\lambda_L \leq 2\pi T$ on the Largest Lyapunov exponent for quantum chaos. This band was discussed briefly in chapter 2 in section 2.3.2 and is conjectued to be satisfied in systems which are holographically dual to gravity. ABJM theory has a gravity dual too, and this is discussed in some detail fo instance in [24]. Therefore, we may expect that MSS bound should hold for quantum chaotic dynamics in ABJM model too.Nevertheless, on treatment of the ABJM matrix model in the present chapter is classical and this can only approximate the quantum dynamics at sufficiently high temperatures. In other words, we should eventually be expect that the MSS bound violated for the present model at low enough temperatures. Indeed using (4.82) and (4.83) we find that there is a critical temperature which we may denote as T_c which is given by solving the equation

$$\alpha_N (3T)^{1/3} = 2\pi T_c \tag{4.84}$$

which yields

$$T_c = \sqrt{3} (\frac{\alpha_N}{2\pi})^{3/2}. \tag{4.85}$$

We understand that for $T \geq T_c$ on present model is in agreement with the MSS bound on λ_L, which for $T \leq T_c$, there is a temperature at and below which MSS bound will be violated. Thus, we may say that T_c is an upper bound for the critical temperature at or below which MSS bound will eventually be violated by our model. Critical values of the temperature for the models at different matrix levels are given in the tables 4.1, 4.2 and 4.3. We observe from these tables that with increasing matrix size T_c values tend to decrease. This is agreed with the fact that 't Hooft limit is better approaches with increasing matrix size.

Table 4.4: Numerical values of coefficients α_N for $AnsatzI$ for $k = -1$ and $\mu = 1$.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$
α_N	0.9435	0.634	0.5536	0.5022	0.4652
γ_N	0.0236	0.0531	0.0825	0.1120	0.1415
T_c	0.0805	0.0555	0.0453	0.0391	0.0349

Table 4.5: Numerical values of coefficients α_N for $AnsatzI$ for $k = -2$ and $\mu = 1$.

	$n = 5$	$n = 15$	$n = 25$
α_N	0.6994	0.4519	0.365
γ_N	0.0472	0.165	0.283
T_c	0.0643	0.0334	0.0244

Case ii) $k \leq -1$ and $\mu = 1$. In this case the potential $\tilde{V}(\phi_1, \phi_2)_N$ is not positive definite as already noted. Furthermore we just saw that $Min\left(\tilde{V}_N(\phi_1, \phi_2)\right) = \frac{N^2(N-1)k\mu^3}{27\pi} \leq 0$, since μ and k have different signs. Adding and subtracting $|Min(V_2')|$ to the (4.80), we get

$$\langle 2K \rangle = \langle 2V \rangle + \underbrace{\tilde{V}_2(\phi_1, \phi_2) + \left|Min(\tilde{V}_2)\right|}_{\geq 0} - \left|Min(\tilde{V}_2)\right|, \tag{4.86}$$

which implies that

$$\langle K \rangle \geq \langle V \rangle - \frac{1}{2}|Min(V_2')| \,, \tag{4.87}$$

Since we habe $E = K + V$, so we may write

$$E = \underbrace{K + V - \frac{1}{2}|Min(V_2')|}_{\leq n_{d.o.f.}T} + \frac{1}{2}|Min(V_2')| \tag{4.88}$$

and conclude using $K = \frac{1}{2}n_{d.o.f.}T$ and infering 4.87 that $K + V - \frac{1}{2}|Min(V_2')| \leq n_{d.o.f.}T$. Thus we have,

$$\frac{E - \frac{1}{2}|Min(V_2')|}{N^2} \leq \frac{n_{d.o.f.}T}{N^2} \tag{4.89}$$

In Ansatz I, for sufficiently large N, we have $n_{d.o.f.} \approx 3N^2$. This gives

$$\frac{E - \frac{1}{2}|Min(V_2')|}{N^2} \leq 3T \tag{4.90}$$

$$\frac{E}{N^2} - \gamma_N \leq 3T\,, \quad \gamma_N = \frac{|Min(V_2')|}{2N^2} \tag{4.91}$$

Take this results into consideration, we conjecture that the best form of the fit curve may be picked in this form

$$\lambda_N = \alpha_N \left(\frac{E}{N^2} - \gamma_N\right)^{1/3} \tag{4.92}$$

Using this for the fitting to $\lambda_{_N}$ versus $\frac{E}{N^2}$ data given figure (4.4) for $N = 5, 10, 15, 20, 25$ and figure (4.5) for $N = 5, 15, 25$, we obtained the curves in these figures. By the same line of reasoning as in the previous case we find from 4.91 and 4.92 that the critical temperature is given as $T_c = \sqrt{3}(\frac{\alpha_n}{2\pi})^{3/2}$. The corresponding values are given in table 4.4 and 4.5.
Finally it is also interesting to note that in both case *i)* and case *ii)* T_c values decrease not only with increasing values of N but also with increasing values of $|k|$ too.

4.4 Ansatz II

We would like to introduce another ansatz configuration with non-zero R_α and Q_α matrices. Again we will use GRVV matrices as a core element of the Ansatz II, and multiply them with two different collective time-dependent functions. Our choice is given as

$$Q_1 = q(t)G_1\,, R_1 = r(t)G_1\,, Q_2 = q(t)G_2\,, R_2 = r(t)G_2 \tag{4.93}$$

We continue to use X_i and $\hat{X}_i$ arbitrary diagonal matrices in the form (4.17). In a manner quite similar to that outlined for ansatz I in section 4.2, calculating the e.o.m. for $\alpha(t)$ and $\beta(t)$, we find that only solution is $\alpha(t) = \beta(t) = 0$. Thus we simply have $X_i = 0 = \hat{X}_i$.
Substituting the matrix configuration 4.93 to the action (4.56), we perform the trace over the GRVV matrices, we obtain the effective Lagrangian in terms of $q(t)$ and $r(t)$

is given as

$$L(N) = N^2(N-1)\left(\dot{q}(t)^2 + \dot{r}(t)^2 - \mu^2 q(t)^2 - \mu^2 r(t)^2 - \frac{8\pi\mu}{k}q(t)^4 + \frac{8\pi\mu}{k}r(t)^4 \right.$$
$$\left. + \frac{12\pi^2}{k^2}q(t)^4 r(t)^2 + \frac{12\pi^2}{k^2}q(t)^2 r(t)^4 - \frac{16\pi^2}{k^2}q(t)^6 - \frac{16\pi^2}{k^2}r(t)^6\right) \tag{4.94}$$

The corresponding Hamiltonian is

$$H(N) = \frac{p_q(t)^2}{4N^2(N-1)} + \frac{p_r(t)^2}{4N^2(N-1)} + N^2(N-1)\left(\mu^2 q(t)^2 + \mu^2 r(t)^2 + \frac{8\pi\mu}{k}q(t)^4 \right.$$
$$\left. - \frac{8\pi\mu}{k}r(t)^4 - \frac{12\pi^2}{k^2}q(t)^4 r(t)^2 - \frac{12\pi^2}{k^2}q(t)^2 r(t)^4 + \frac{16\pi^2}{k^2}q(t)^6 + \frac{16\pi^2}{k^2}r(t)^6\right)$$
$$:= \frac{p_q^2}{4N^2(N-1)} + \frac{p_r^2}{4N^2(N-1)} + V_N(q,r)\,, \tag{4.95}$$

where $V_N(q,r)$ is the effective potential defined by the relevant terms in the first line of 4.95.

Let us first note that the potential is not positive definite for either $k \geq 0$ or $k \leq 0$ and the minimum value of the potential is negative. Furthermore we see that $V_N(q,r)$ is symmetric under the exchage of q and r. From the form of the potential we observe that for $k - k$, two terms which are proportional to $\frac{1}{k}$ change sign. But this can be compensated by exchanging q and r. Thus, we conclude that the dynamics due to this potential will be independent from the sign of k.

The Hamilton e.o.m. are given as

$$\dot{q} - \frac{p_q}{N^2(N-1)} = 0 \tag{4.96a}$$
$$\dot{r} - \frac{p_r}{N^2(N-1)} = 0 \tag{4.96b}$$
$$\dot{p_q} + N^2(N-1)\left(2\mu^2 q(t) + \frac{32\pi\mu}{k}q(t)^3 - \frac{48\pi^2}{k^2}q(t)^3 r(t)^2 \right. \tag{4.96c}$$
$$\left. - \frac{24\pi^2}{k^2}q(t)r(t)^4 + \frac{96\pi^2}{k^2}q(t)^5\right) = 0$$
$$\dot{p_r} + N^2(N-1)\left(2\mu^2 r(t) - \frac{32\pi\mu}{k}r(t)^3 - \frac{24\pi^2}{k^2}q(t)^4 r(t) \right. \tag{4.96d}$$
$$\left. - \frac{48\pi^2}{k^2}q(t)^2 r(t)^3 + \frac{96\pi^2}{k^2}r(t)^5\right) = 0$$

We can calculate the fixed points of this system by using (4.70). The first two equations are trivially satisfied by $p_q(t) = 0$ and $p_r(t) = 0$. The other two fixed points are given by the solution of the coupled equations

$$N^2(N-1)\left(2\mu^2 q(t)+\frac{32\pi\mu}{k}q(t)^3-\frac{48\pi^2}{k^2}q(t)^3r(t)^2\right. \tag{4.97a}$$
$$\left.-\frac{24\pi^2}{k^2}q(t)r(t)^4+\frac{96\pi^2}{k^2}q(t)^5\right)=0$$
$$N^2(N-1)\left(2\mu^2 r(t)-\frac{32\pi\mu}{k}r(t)^3-\frac{48\pi^2}{k^2}q(t)^2r(t)^3\right. \tag{4.97b}$$
$$\left.-\frac{24\pi^2}{k^2}q(t)^4r(t)+\frac{96\pi^2}{k^2}r(t)^5\right)=0$$

As an example we consider the case $\mu = 1$ and $k = 1$ case. At the values of the parameter only real solutions of the system (4.97) is given as

$$(q, r) \equiv (0, 0.317)\,, \tag{4.98}$$

therefore the only fixed point is given as

$$(q, r, p_q, p_r) \equiv (0, 0.317, 0, 0). \tag{4.99}$$

The energy corresponding to this fixed point is

$$E(0, 0.317, 0, 0) \approx 0.007N^2(N-1) \tag{4.100}$$

The eigenvalues of the Jacobian matrix at the fixed point is calculated to be $\{-0.4427, 0.4427, -3.12i, 3.12i\}$. Since these include a real positive eigenvalue, we conclude that $(0, 0.317, 0, 0)$ is an unstable fixed point

4.4.1 Dependence λ_L on Energy and Temperature

In order to obtain λ_L at various different energy values, as we did for ansatz I, we solve the Hamilton's equation and evaluate λ_L for 40 randomly selected initial condition and take the average values. The only difference in Ansatz II is that we take $q(0) = 0$ initially and take three randomly generated numbers. Then after following a same steps as in section 4.3.1, we similar procedure, we calculate the initial values of $r(0)$, $p_r(0)$ and $p_q(0)$. Using $\omega_i = \frac{\omega_i}{\sqrt{\omega_i{}^2}}\sqrt{E}$, where ω_i are randomly selected, we choose

$$p_\gamma(0) = \pm N\sqrt{(N-1)\Omega_1^2} \quad p_\phi(0) = \pm N\sqrt{(N-1)\Omega_2^2} \tag{4.101}$$

and then real roots of

$$N^2(N-1)\left(\mu^2 r(0)^2 - \frac{8\pi\mu}{k} r(0)^4 + \frac{16\pi^2}{k^2} r(0)^6\right) - \Omega_3^2 = 0 \tag{4.102}$$

are used to select $r(t)$ at $t = 0$, i.e. $r(0)$ value.

Applying the virial theorem we find that

$$\begin{aligned}\langle 2K\rangle &= \langle 2V\rangle + N^2(N-1)\left(\frac{16\pi\mu}{k} q(t)^4 - \frac{16\pi\mu}{k} r(t)^4 - \frac{48\pi^2}{k^2} q(t)^4 r(t)^2\right. \qquad (4.103)\\ &\quad \left. - \frac{48\pi^2}{k^2} q(t)^2 r(t)^4 + \frac{64\pi^2}{k^2} q(t)^6 + \frac{64\pi^2}{k^2} r(t)^6\right)\\ &:= \langle 2V\rangle + \tilde{V}_2(q,r) \qquad (4.104)\end{aligned}$$

where $\tilde{V}_2(q,r)$ is defined by the 1^{st} line of (4.103). Evaluating the minimum of $\tilde{V}_2(q,r)$ we find that it is given as

$$Min(\tilde{V}_2(q,r)) = -\frac{N^2(N-1)64|k\mu^2|}{135\sqrt{5}\pi} \tag{4.105}$$

Following the same line of development and steps as in section 4.3.3, we have

$$\frac{E}{N^2} - \gamma_N \leq \frac{n_{d.o.f.}T}{N^2} \tag{4.106}$$

where $\gamma_N = \frac{Min(\tilde{V}_2(q,r))}{2N^2}$.

These considerations and the dimensional analysis suggests that we may consider fitting curves of he form

$$\lambda_L = \beta_N\left(\frac{E}{N^2} - \gamma_N\right)^{1/3} \tag{4.107}$$

Case i) $k = \pm 1$ and $\mu = 1$

For these parameter values it appears sufficient to use E/N^2 in the range $0:2:100$. The data points and the fitting curves are depicted in the Figure 4.6 for the matrix levels $N = 5, 10, 15, 20, 25$. We see that there is indeed very good argument between the data and the fitting curves, with α_N values given in the table 4.6.

Case ii) $k = \pm 2$, $\mu = 1$

In this case too it is sufficient to focus on the range $0:2:100$ for E/N^2. Data points and fitting curves are given in the figure 4.7 corresponding α_N values are listed in table 4.7.

Case iii) $k = \pm 5, \pm 10$ and $\mu = 1$

These cases are given in the figure 4.8 and α_N values are given in table 4.8.

4.4.2 Temperature dependence of λ_L and comparison with the MSS boumd

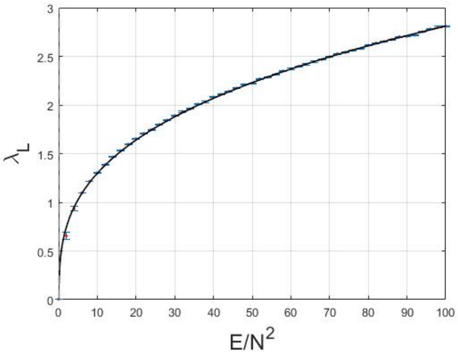
Let us first note that in Ansatz II given explicitly by 4.93 we have contrast Ansatz I in equation 3.5, R_α are no longer zero. Each R_α contributes $2N^2$ degrees of freedom in this case is given as $n_{d.o.f.} = 7N^2 - 8$ which in the large N limit is given as $n_{d.o.f.} \approx 7N^2$. Therefore, we may in this case write using 4.106 that

$$\frac{E}{N^2} - \gamma_N \leq 7T \tag{4.108}$$

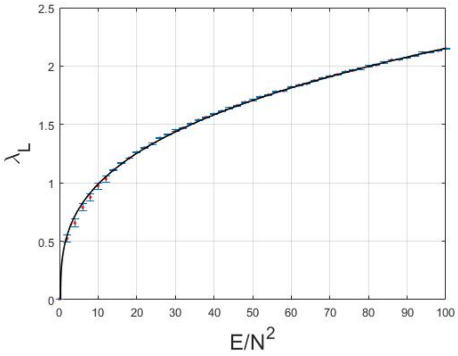
Comparing this result and 4.107 with the MSS bound $\lambda_L \leq 2\pi T$ we find hat the critical temperature is obtained by solving

$$\alpha_N (7T)^{1/3} = 2\pi T \tag{4.109}$$

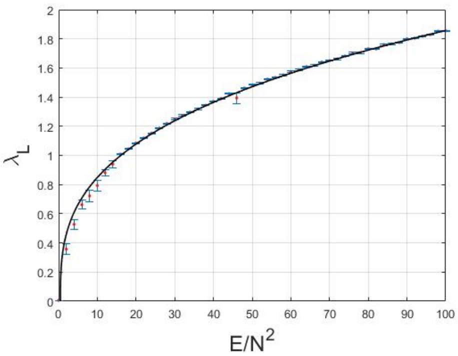
where yields $T_c = \sqrt{7}(\frac{\alpha_N}{2\pi})^{3/2}$. Like in the previous ansatz, we understand that for $T \geq T_c$ on present model is in agreement with the MSS bound on λ_L, which for $T \leq T_c$, there is a temperature at and below which MSS bound will be violated. The critical temperatures are given in table:4.6, 4.7 and 4.8. Again, we observe from these tables that with increasing matrix size T_c values tend to decrease. This is agreed with the fact that 't Hooft limit is better approaches with increasing matrix size.



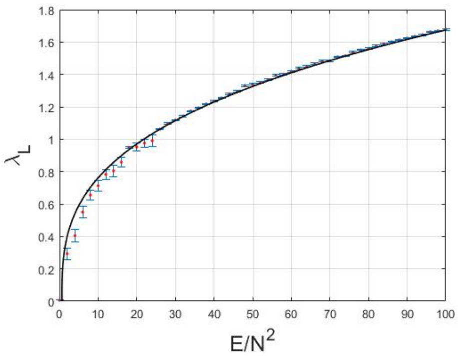
(a) $N = 5$



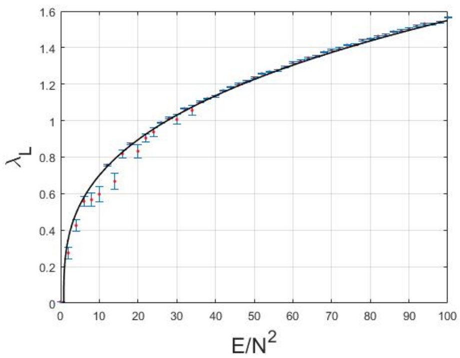
(b) $N = 10$



(c) $N = 10$

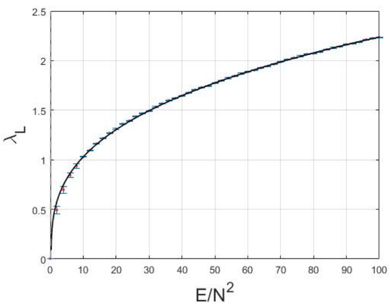


(d) $N = 20$

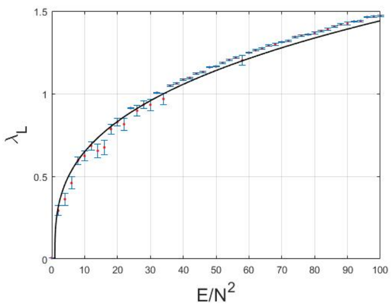


(e) $N = 25$

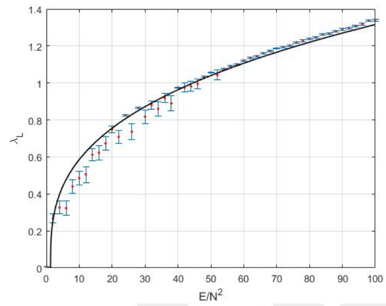
Figure 4.6: Lyapunov spectra and best fitting curves in the form $\lambda_L = \beta_N(\frac{E}{N^2} - \gamma_N)^{1/3}$(see the discussion in section 4.4.1) at $k = \pm 1$ and $\mu = 1$ for ansatz II



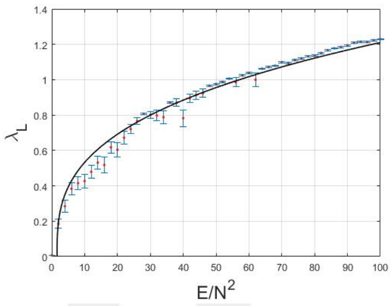
(a) $N = 5$



(b) $N = 15$

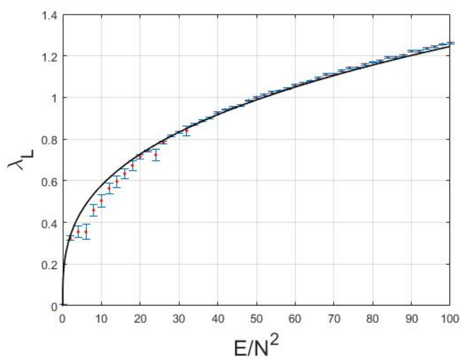


(c) $N = 20$

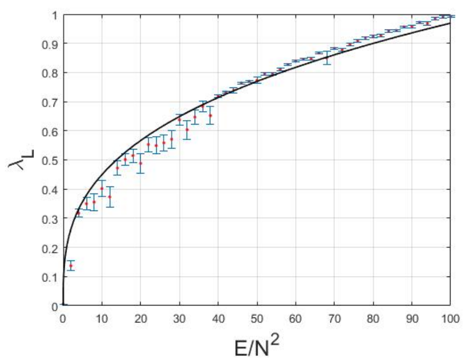


(d) $N = 25$

Figure 4.7: Lyapunov spectra and best fitting curves in the form $\lambda_L = \beta_N(\frac{E}{N^2} - \gamma_N)^{1/3}$(see the discussion in section 4.4.1) at $k = \pm 2$ and $\mu = 1$ for ansatz II



(a) $k = 5$



(b) $k = 10$

Figure 4.8: Lyapunov spectra and best fitting curves in the form $\lambda_L = \beta_N(\frac{E}{N^2} - \gamma_N)^{1/3}$(see the discussion in section 4.4.1) for $n = 10$ at $k = 5, 10$ for ansatz II

Table 4.6: Numerical values of coefficients β_N, γ_N and T_c for $Ansatz II$ for $k = \pm 1$ and $\mu = 1$.

	$N = 5$	$N = 10$	$N = 15$	$N = 20$	$N = 25$
β_N	0.6057	0.464	0.4006	0.3615	0.3346
γ_N	0.135	0.304	0.4725	0.64	0.81
T_c	0.0792	0.0531	0.0426	0.0365	0.0325

Table 4.7: Numerical values of coefficients β_N, γ_N and T_c for $Ansatz II$ for $k = \pm 2$ and $\mu = 1$.

	$N = 5$	$N = 15$	$N = 20$	$N = 25$
β_N	0.4826	0.3164	0.2836	0.2622
γ_N	0.270	0.945	1.2822	1.62
T_c	0.0563	0.0299	0.0254	0.0226

Table 4.8: Numerical values of coefficients β_N, γ_N for $Ansatz II$ for $N = 10$ at $k = 1, 5, 10$.

	$k = 1$	$k = 5$	$k = 10$
β_N	0.4515	0.2716	0.2142
γ_N	0.304	1.5184	3.0369
T_c	0.051	0.0238	0.0167

CHAPTER 5

CONCLUSIONS

In this thesis, we have presented the results of two orijinal research projects. The main purpose of this work have been the exploration of chaotic dynamics emerging from matrix gauge theories. In the first part, we gave a review of the bosonic part of the BFSS matrix model including its derivation from the Yang-Mills model in $9+1$ dimensions by dimensional reduction. We also discussed the important features of fuzzy two- and fuzzy four- spheres.

Chapter 3 was devoted to the examination of the BFSS model with double mass deformation terms. We introduced ansatz configurations including fuzzy-two and fuzzy-two spheres as backgrounds and assuming collective time dependence of matrices. After tracing over the fuzzy sphere at matrix level $N = \frac{1}{2}(n+1)(n+2)(n+3)$, for $n = 1, ..., 7$, we obtained the family of effective models. We numerically solved the Hamilton's equations of motion for these effective models and calculated the Lyapunov specturum at different matrix levels. Furthermore, we also sketched the Ponicaré sections to broaden our perspective. After a very detailed analysis, we revealed that the transition to chaos happends after the unstable fixed point energy is exceeded. With the use of best fit curves of form 3.25 for λ_n versus $\frac{E}{N^2}$ data and the inequality 3.24, we have analytically calculated the critical temperatures where the MSS bound is violated.

In chapter 4, we gave a review of the bosonic part of the ABJM model. We first dimensionally reduced the ABJM model to $0+1$-dimensions and identified the appropriate $'tHooft$ limit. We introduced massive deformation with mass parameter μ, which breaks the original symmetry down to $SU(2) \times SU(2) \times U(1)_A \times U(1)_B \times \mathbb{Z}_2$ by splitting the original complex scalars C^I into (R_α, Q_α). We chose two different ansatz configurations. By tracing over ansatz configurations involving fuzzy two spheres realized in terms of the GRVV matrices with collective dependence, we obtained a family of reduced effective Hamiltonians. We numerically solved the Hamilton's equations of motions of these reduced systems and determined their Lyapunov spectrum. Also we analyzed in detail how the largest Lyapunov exponent, λ_L, changes as a function of E/N^2. With the help of virial and equipartition theorems, we obtained the best fit curves of form $\lambda_L \propto (E/N^2)^{1/3}$ or $\lambda_L \propto (E/N^2 - W)^{1/3}$ depending on the structure of the effective potential. W is a constant determined in terms of the Chern-Simons coupling, mass and the matrix level. During our analysis, we showed that our fitting curve is very consistent with the λ_L vs E/N^2 data. We showed the temperature dependence of the λ_L and gave upper bounds on the temperature above which LLE values comply with the Maldacena-Shenker-Stanford (MSS) bound $2\pi T$.

For further discussion, we may point out the importance of thermalization processes in the Yang-Mills matrix models. Recently, in [21], thermalization in massive deformations of Yang-Mills matrix models has been discussed. The next step of our research might be a quantum mechanical realization of chaotic behavior in the Yang-Mills models.

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APPENDIX A

CALCULATIONS OF THE HAMILTON EQUATIONS OF MOTION

A.1 Hamilton Equations of Motion

$$H = \mathrm{Tr}\,\mathcal{H} = \mathrm{Tr}\left(P_{Q^{\alpha\dagger}}P_{Q^{\alpha}} + P_{R^{\alpha\dagger}}P_{R^{\alpha}} + D_iQ^{\alpha\dagger}D_iQ^{\alpha} \right. \tag{A.1}$$

$$\left. + D_iR^{\alpha\dagger}D_iR^{\alpha} + M^{\alpha}M^{\alpha\dagger} + N^{\alpha}N^{\alpha\dagger}\right) \tag{A.2}$$

Let us start with $\frac{\partial H}{\partial X_j}$. Since only the covarient derivatives contain X_i and $X_i^\dagger$, the contribution from the rest of the Hamiltonian is zero.

$$H_{cov} = \mathrm{Tr}\left(D_iQ^{\alpha\dagger}D_iQ^{\alpha} + D_iR^{\alpha\dagger}D_iR^{\alpha}\right) \tag{A.3}$$

Opening trace with dummy incides aa and derivative with cd, we get

$$\frac{\partial H_{cov}}{\partial X_j} = \frac{\partial\left(D_iQ^{\alpha\dagger}D_iQ^{\alpha} + D_iR^{\alpha\dagger}D_iR^{\alpha}\right)_{aa}}{(\partial X_\gamma)_{cd}} \tag{A.4}$$

After appliying chain rule, we have

$$\frac{\partial(D_iQ^{\alpha\dagger}D_iQ^{\alpha})_{aa}}{(\partial X_j)_{cd}} = \frac{\partial(D_iQ^{\alpha\dagger})_{ab}}{\partial(X_j)_{cd}}(D_iQ^{\alpha})_{ba} + (D_iQ^{\alpha\dagger})_{ab}\frac{\partial(D_iQ^{\alpha})_{ba}}{\partial(X_j)_{cd}} \tag{A.5a}$$

$$\frac{\partial(D_iR^{\alpha\dagger}D_iR^{\alpha})_{aa}}{(\partial X_j)_{cd}} = \frac{\partial(D_iR^{\alpha\dagger})_{ab}}{\partial(X_j)_{cd}}(D_iR^{\alpha})_{ba} + (D_iR^{\alpha\dagger})_{ab}\frac{\partial(D_iR^{\alpha})_{ba}}{\partial(X_j)_{cd}} \tag{A.5b}$$

First of all, lets proceed with (A.5a), recall that covariant derivatives defined in (4.2).

If we put (4.2) inside (A.5a), we will get

$$\begin{aligned}\frac{\partial(D_iQ^{\alpha\dagger})_{ab}}{\partial(X_j)_{cd}} &= \frac{\partial(-iQ^{\alpha\dagger}X_i + i\hat{X}_iQ^{\alpha\dagger})_{ab}}{\partial(X_j)_{cd}} \\ &= \frac{\partial(-iQ^{\alpha\dagger}X_i)_{ab}}{\partial(X_j)_{cd}} \\ &= \frac{\partial(-i(Q^{\alpha\dagger})_{al}(X_i)_{lb})}{\partial(X_j)_{cd}} \\ &= -i\delta_{ij}(Q^{\alpha\dagger})_{al}\delta_{lc}\delta_{bd} \\ &= -i\delta_{ij}(Q^{\alpha\dagger})_{ac}\delta_{bd}\end{aligned} \tag{A.6}$$

From the second part of (A.5a), we get

$$\begin{aligned}\frac{\partial(D_iQ^{\alpha})_{ba}}{\partial(X_j)_{cd}} &= \frac{\partial(iX_iQ^{\alpha} - iQ^{\alpha}\hat{X}_i)_{ba}}{\partial(X_j)_{cd}} \\ &= \frac{\partial(iX_iQ^{\alpha})_{ba}}{\partial(X_j)_{cd}} \\ &= \frac{\partial(i(X_i)_{bl}(Q^{\alpha})_{la})}{\partial(X_j)_{cd}} \\ &= i\delta_{ij}(Q^{\alpha})_{la}\delta_{bc}\delta_{ld} \\ &= i\delta_{ij}(Q^{\alpha})_{da}\delta_{bc}\end{aligned} \tag{A.7}$$

Thus, (A.5a) becomes

$$\frac{\partial(D_iQ^{\alpha\dagger}D_iQ^{\alpha})_{aa}}{(\partial X_j)_{cd}} = -i\delta_{ij}(Q^{\alpha\dagger})_{ac}\delta_{bd}(D_iQ^{\alpha})_{ba} + i\delta_{ij}(Q^{\alpha})_{da}\delta_{bc}(D_iQ^{\alpha\dagger})_{ab} \tag{A.8}$$

$$= -i(D_jQ^{\alpha})_{da}(Q^{\alpha\dagger})_{ac} + i(Q^{\alpha})_{da}(D_iQ^{\alpha\dagger})_{ac} \tag{A.9}$$

$$= -i\big((D_jQ^{\alpha})Q^{\alpha\dagger} - Q^{\alpha}(D_jQ^{\alpha\dagger})\big)_{dc} \tag{A.10}$$

$$= -i\big((D_jQ^{\alpha})Q^{\alpha\dagger} - Q^{\alpha}(D_jQ^{\alpha\dagger})\big)^T \tag{A.11}$$

Since (A.5a) and (A.5b) are symmetric, we already calculated the answer for (A.5b).Just changing $Q^{\alpha} \to R^{\alpha}$, we get

$$\frac{\partial(D_iR^{\alpha\dagger}D_iR^{\alpha})_{aa}}{(\partial X_j)_{cd}} = -i\big((D_jR^{\alpha})R^{\alpha\dagger} - R^{\alpha}(D_jR^{\alpha\dagger})\big)_{dc} \tag{A.12}$$

$$= -i\big((D_jR^{\alpha})R^{\alpha\dagger} - R^{\alpha}(D_jR^{\alpha\dagger})\big)^T \tag{A.13}$$

Thus, we have

$$\begin{aligned}\dot{P_{X_i}} = -\frac{\partial H}{\partial X_i} = i\Bigg(&(D_jQ_{\alpha})Q^{\alpha\dagger} - Q_{\alpha}(D_jQ^{\alpha\dagger}) \\ &+(D_jR^{\alpha})R^{\dagger}_{\alpha} - R_{\alpha}(D_jR^{\alpha\dagger})\Bigg)^T\end{aligned} \tag{A.14}$$

We will follow the same prosedure for $\hat{X}_j$ derivative. Let us start with the first part

$$\begin{aligned}
\frac{\partial(D_iQ^{\alpha\dagger})_{ab}}{\partial(\hat{X}_j)_{cd}} &= \frac{\partial(-iQ^{\alpha\dagger}X_i + i\hat{X}_iQ^{\alpha\dagger})_{ab}}{\partial(\hat{X}_j)_{cd}} \\
&= \frac{\partial(i\hat{X}_iQ^{\alpha\dagger})_{ab}}{\partial(\hat{X}_j)_{cd}} \\
&= \frac{\partial(i(\hat{X}_i)_{al}(Q^{\alpha\dagger})_{lb})}{\partial(\hat{X}_j)_{cd}} \\
&= i\delta_{ij}(Q^{\alpha\dagger})_{lb}\delta_{ca}\delta_{dl} \\
&= i\delta_{ij}(Q^{\alpha\dagger})_{db}\delta_{ca}
\end{aligned} \tag{A.15}$$

The second derivative is

$$\begin{aligned}
\frac{\partial(D_iQ^{\alpha})_{ba}}{\partial(\hat{X}_j)_{cd}} &= \frac{\partial(iX_iQ^{\alpha} - iQ^{\alpha}\hat{X}_i)_{ba}}{\partial(\hat{X}_j)_{cd}} \\
&= \frac{\partial(-iQ^{\alpha}X_i)_{ba}}{\partial(\hat{X}_j)_{cd}} \\
&= \frac{\partial(-i(Q^{\alpha})_{bl}(\hat{X}_i)_{la}}{\partial(\hat{X}_j)_{cd}} \\
&= -i\delta_{ij}(Q^{\alpha})_{bl}\delta_{lc}\delta_{ad} \\
&= -i\delta_{ij}(Q^{\alpha})_{bc}\delta_{ad}
\end{aligned} \tag{A.16}$$

Finally, we have

$$\begin{aligned}
\frac{\partial(D_iQ^{\alpha\dagger}D_iQ^{\alpha})_{aa}}{(\partial\hat{X}_j)_{cd}} &= i\delta_{ij}(Q^{\alpha\dagger})_{db}\delta_{ca}(D_iQ^{\alpha})_{ba} - i(D_iQ^{\alpha\dagger})_{ab}\delta_{ij}(Q^{\alpha})_{bc}\delta_{ad} \\
&= i(Q^{\alpha\dagger})_{db}(D_jQ^{\alpha})_{bc} - i(D_jQ^{\alpha\dagger})_{db}(Q^{\alpha})_{bc} \\
&= i\big(Q^{\alpha\dagger}(D_jQ^{\alpha}) - i(D_jQ^{\alpha\dagger})Q^{\alpha}\big)_{dc} \\
&= i\big(Q^{\alpha\dagger}(D_jQ^{\alpha}) - i(D_jQ^{\alpha\dagger})Q^{\alpha}\big)^T
\end{aligned} \tag{A.17}$$

Again, similarly with the X_i derivative, we can calculate R_α part by just changing Q_α with R_α.

$$\begin{aligned}
\frac{\partial(D_iR^{\alpha\dagger}D_iR^{\alpha})_{aa}}{(\partial\hat{R}_j)_{cd}} &= i\big(R^{\alpha\dagger}(D_jR^{\alpha}) - i(D_jR^{\alpha\dagger})R^{\alpha}\big)_{dc} \\
&= i\big(R^{\alpha\dagger}(D_jR^{\alpha}) - i(D_jR^{\alpha\dagger})R^{\alpha}\big)^T
\end{aligned} \tag{A.18}$$

Thus, $\hat{X}_i$ derivative becomes

$$\begin{aligned}
\dot{P}_{\hat{X}_i} = -\frac{\partial H}{\partial\hat{X}_i} = i\bigg(& -Q^{\alpha\dagger}(D_jQ^{\alpha}) + (D_jQ^{\alpha\dagger})Q^{\alpha} \\
& -R^{\alpha\dagger}(D_jR^{\alpha}) + (D_jR^{\alpha\dagger})R^{\alpha}\bigg)^T
\end{aligned} \tag{A.19}$$

The rest of the (4.63) are similar but more trickier than X_i and $\hat{X}_i$ parts. So lets continue with (4.63b).

$$\begin{aligned} \dot{P_{Q_\gamma}} &= -\frac{\partial H}{\partial Q_\gamma} = -\frac{\partial \operatorname{Tr}\left(D_i Q^{\alpha\dagger} D_i Q^\alpha + M^\alpha M^{\alpha\dagger} + N^\alpha N^{\alpha\dagger}\right)}{\partial Q_\gamma} \\ &= \frac{\partial \operatorname{Tr}\left(D_i Q^{\alpha\dagger} D_i Q^\alpha\right)}{\partial Q_\gamma} + \frac{\partial \operatorname{Tr}\left(M^\alpha M^{\alpha\dagger}\right)}{\partial Q_\gamma} + \frac{\partial \operatorname{Tr}\left(N^\alpha N^{\alpha\dagger}\right)}{\partial Q_\gamma} \end{aligned} \tag{A.20}$$

Again the covariant derivative part is easy, but the potential term is way too complicated. Since we already know how to calculate the covariant derivative part from the previous calculations, lets start with it.

$$\frac{\partial (D_i Q^{\alpha\dagger} D_i Q^\alpha)_{aa}}{(\partial Q_\gamma)_{cd}} = (D_i Q^{\alpha\dagger})_{ab} \frac{\partial (D_i Q^\alpha)_{ba}}{\partial (Q_\gamma)_{cd}} \tag{A.21}$$

$$\begin{aligned} \frac{\partial (D_i Q^\alpha)_{ba}}{\partial (Q_\gamma)_{cd}} &= \frac{\partial (i X_i Q^\alpha - i Q^\alpha \hat{X}_i)_{ba}}{\partial (Q_\gamma)_{cd}} \\ &= \frac{\partial (i X_i Q^\alpha)_{ba}}{\partial (Q_\gamma)_{cd}} - \frac{\partial (i Q^\alpha \hat{X}_i)_{ba}}{\partial (Q_\gamma)_{cd}} \\ &= \frac{\partial (i (X_i)_{bl} (Q^\alpha)_{la})}{\partial (Q_\gamma)_{cd}} - \frac{\partial (i Q^\alpha)_{bl} (\hat{X}_i)_{la}}{\partial (Q_\gamma)_{cd}} \\ &= i\delta_{\alpha\gamma} (X_i)_{bl} \delta_{lc} \delta_{da} - i\delta_{\alpha\gamma} (\hat{X}_i)_{la} \delta_{bc} \delta_{dl} \\ &= i\delta_{\alpha\gamma} (X_i)_{bc} \delta_{da} - i\delta_{\alpha\gamma} (\hat{X}_i)_{da} \delta_{bc} \end{aligned} \tag{A.22}$$

Thus, it follows

$$\begin{aligned} \frac{\partial (D_i Q^{\alpha\dagger} D_i Q^\alpha)_{aa}}{(\partial Q_\gamma)_{cd}} &= i (D_i Q^{\alpha\dagger})_{ab} \delta_{\alpha\gamma} (X_i)_{bc} \delta_{da} - i (D_i Q^{\alpha\dagger})_{ab} \delta_{\alpha\gamma} (\hat{X}_i)_{da} \delta_{bc} \\ &= i\big((D_i Q^{\gamma\dagger}) X_i - \hat{X}_i (D_i Q^{\gamma\dagger})\big)_{dc} \\ &= i\big((D_i Q^{\gamma\dagger}) X_i - \hat{X}_i (D_i Q^{\gamma\dagger})\big)^T \end{aligned} \tag{A.23}$$

Lets proceed with $|M^2|$ part,

$$\begin{aligned} \frac{\partial \operatorname{Tr}\big(M^\alpha M^{\alpha\dagger}\big)}{\partial Q_\gamma} &= \frac{\partial (M^\alpha M^{\alpha\dagger})_{aa}}{(\partial Q_\gamma)_{cd}} \\ &= \frac{\partial (M^\alpha)_{ab}}{\partial (Q_\gamma)_{cd}} (M^{\alpha\dagger})_{ba} + (M^\alpha)_{ab} \frac{\partial (M^{\alpha\dagger})_{ba}}{(\partial Q_\gamma)_{cd}} \end{aligned} \tag{A.24}$$

Recall that M^α defined as

$$M^\alpha = \mu Q^\alpha + \frac{2\pi}{k} (2 Q^{[\alpha} Q^\dagger_\beta Q^{\beta]} + R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta + 2 Q^\beta R^\dagger_\beta R^\alpha - 2 R^\alpha R^\dagger_\beta Q^\beta) \tag{A.25}$$

where $Q^{[\alpha}Q^\dagger_\beta Q^{\beta]} = Q^\alpha Q^\dagger_\beta Q^\beta - Q^\beta Q^\dagger_\beta Q^\alpha$.

$$\frac{\partial(M^\alpha)_{ab}}{\partial(Q_\gamma)_{cd}} = \frac{\partial(\mu Q^\alpha)_{ab}}{\partial(Q_\gamma)_{cd}} + \frac{2\pi}{k}\frac{\partial(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]})_{ab}}{\partial(Q_\gamma)_{cd}} + \frac{2\pi}{k}\frac{\partial(R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta)_{ab}}{\partial(Q_\gamma)_{cd}} + \frac{2\pi}{k}\frac{\partial(2Q^\beta R^\dagger_\beta R^\alpha - 2R^\alpha R^\dagger_\beta Q^\beta)_{ab}}{\partial(Q_\gamma)_{cd}} \tag{A.26}$$

Again, lets calculate term by term, the first term is easy

$$\frac{\partial(\mu Q^\alpha)_{ab}}{\partial(Q_\gamma)_{cd}} = \mu\delta_{\alpha\gamma}\delta_{bc}\delta_{da} \tag{A.27}$$

The second one is given as

$$\begin{aligned}\frac{\partial(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]})_{ab}}{\partial(Q_\gamma)_{cd}} &= \frac{\partial\big(2(Q^\alpha)_{al}(Q^\dagger_\beta)_{lk}(Q^\beta)_{kb} - (Q^\beta)_{al}(Q^\dagger_\beta)_{lk}(Q^\alpha)_{kb}\big)}{\partial(Q_\gamma)_{cd}} \\ &= 2(\delta_{\gamma\alpha}\delta_{ac}(Q^\dagger_\beta)_{dk}(Q^\beta)_{kb} + \delta_{\gamma\beta}\delta_{bd}(Q^\alpha)_{al}(Q^\dagger_\beta)_{lc} \\ &- \delta_{\beta\gamma}\delta_{ac}(Q^\dagger_\beta)_{dk}(Q^\alpha)_{kb} - \delta_{\alpha\gamma}\delta_{bd}(Q^\beta)_{al}(Q^\dagger_\beta)_{lc})\end{aligned} \tag{A.28}$$

The third term is

$$\frac{\partial(R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta)_{ab}}{\partial(Q_\gamma)_{cd}} = R^\beta_{bl}(R^\dagger_\beta)_{lc}\delta^{\alpha\gamma}\delta_{da} - \delta^{\alpha\gamma}\delta_{bc}(R^\dagger_\beta)_{dl}R^\beta_{la} \tag{A.29}$$

The last terms is

$$\frac{\partial(2Q^\beta R^\dagger_\beta R^\alpha - 2R^\alpha R^\dagger_\beta Q^\beta)_{ab}}{\partial(Q_\gamma)_{cd}} = \delta_{bc}(R^\dagger_\gamma)_{dl}(R^\alpha)_{la} - (R^\alpha)_{bl}(R^\dagger_\gamma)_{lc}(R^\alpha)_{da} \tag{A.30}$$

Thus we have

$$\begin{aligned}\frac{\partial(M^\alpha)_{ab}}{\partial(Q_\gamma)_{cd}} &= \mu\delta^{\alpha\gamma}\delta_{bc}\delta_{da} + \frac{2\pi}{k}\big(2\delta^{\alpha\gamma}\delta_{bc}(Q^\dagger_\beta)_{dk}\alpha(Q^\beta)_{ka} + 2(Q^\alpha)_{bk}(Q^\dagger_\gamma)_{kc}\delta_{da} \\ &- 2\delta_{bc}(Q^\dagger_\gamma)_{dk}(Q^\alpha)_{ka} - 2(Q^\beta)_{bl}(Q^\dagger_\beta)_{lc}\delta^{\alpha\gamma}\delta_{da} + (R^\beta)_{bl}(R^\dagger_\beta)_{lc}\delta^{\alpha\gamma}\delta_{da} \\ &- \delta^{\alpha\gamma}\delta_{bc}(R^\dagger_\beta)_{dl}(R^\beta)_{la} + 2\delta_{bc}(R^\dagger_\gamma)_{dl}(R^\alpha)_{la} - 2(R^\alpha)_{bl}(R^\dagger_\gamma)_{lc}\delta_{da}\big)\end{aligned} \tag{A.31}$$

For the $M^\dagger_\alpha$ part, we have

$$\frac{\partial(M^\dagger_\alpha)_{ab}}{\partial(Q^\gamma)_{cd}} = \frac{2\pi}{k}(Q^\dagger_\gamma)_{ac}(Q^\dagger_\alpha)_{db} - 2(Q^\dagger_\alpha)_{db} - 2(Q^\dagger_\alpha)_{ac}(Q^\dagger_\beta)_{db} \tag{A.32}$$

The final form of derivative of M^2 is obtained as

$$\frac{\partial(|M^\alpha|^2)}{\partial(Q^\gamma)} = \Big(\mu M^\dagger_\gamma + \frac{2\pi}{k}\big(2Q^\dagger_\alpha M_\alpha Q^\dagger_\gamma - 2Q^\dagger_\beta M_\alpha Q^\dagger_\alpha + 2Q^\dagger_\beta Q^\beta M^\dagger_\gamma + 2M_\alpha Q^\alpha Q^\dagger_\gamma \tag{A.33}$$

$$- 2Q^\dagger_\gamma Q^\alpha M^\dagger_\alpha - 2M^\dagger_\gamma Q^\beta Q^\dagger_\beta + M^\dagger_\gamma R^\beta R^\dagger_\beta - R^\dagger_\beta R^\beta M^\dagger_\gamma + 2R^\dagger_\gamma R^\alpha M^\dagger_\alpha - 2M^\dagger_\alpha R^\alpha R^\dagger_\gamma\big)\Big)^T$$

The derivative of N^2 can be calculated similarly. After applying similar procedures, we find that

$$\frac{\partial(|M^\alpha|^2)}{\partial(Q^\gamma)} = \frac{2\pi}{k}\big(Q^\dagger_\gamma R^\alpha N^\dagger_\alpha - N^\dagger_\alpha R^\alpha Q^\dagger_\gamma + 2N^{\gamma\dagger} R^\beta Q^\dagger_\beta - 2Q^\dagger_\beta R^\beta N^\dagger_\gamma \tag{A.34}$$

$$+ Q^\dagger_\gamma N^\alpha R^\dagger_\alpha - R^\dagger_\alpha N^\alpha Q^\dagger_\gamma + 2R^\dagger_\gamma N^\alpha Q^\dagger_\alpha - 2Q^\dagger_\alpha N^\alpha R^\dagger_\gamma\big)^T \tag{A.35}$$

After combining all of these results, we find that

$$\dot{P}_{Q_\gamma} = -\Bigg(i[(D_i Q^\dagger{}_\gamma)X_i - \hat{X}_i(D_i Q^\dagger{}_\gamma)] + \mu M^\dagger{}_\gamma + \frac{2\pi}{k}[2Q^\dagger{}_\alpha M^\alpha Q^\dagger{}_\gamma \tag{A.36}$$

$$- 2Q^\dagger{}_\gamma M^\alpha Q^\dagger{}_\alpha + 2Q^\dagger{}_\beta Q^\beta M^\dagger{}_\gamma + 2M_\alpha Q^\alpha Q^\dagger{}_\gamma - 2Q^\dagger{}_\gamma Q^\alpha M_\alpha{}^\dagger$$

$$- 2M^\dagger{}_\gamma Q^\beta Q^\dagger{}_\beta + M^\dagger{}_\gamma R^\beta R^\dagger{}_\beta - R^\dagger{}_\beta R^\beta M^\dagger_\gamma + 2R^\dagger{}_\gamma Q^\alpha M^\dagger{}_\alpha - 2M^\dagger{}_\alpha R^\alpha R^\dagger{}_\gamma$$

$$+ Q^\dagger{}_\gamma R^\alpha N^\dagger{}_\alpha - N^\dagger{}_\alpha R^\alpha Q^\dagger{}_\gamma + 2N^\dagger{}_\gamma R^\beta Q^\dagger{}_\beta - 2Q^\dagger{}_\beta R^\beta N^\dagger{}_\gamma$$

$$+ Q^\dagger{}_\gamma N^\alpha R^\dagger{}_\alpha - R^\dagger{}_\alpha N^\alpha Q^\dagger{}_\gamma + 2R^\dagger{}_\gamma N^\alpha Q^\dagger{}_\alpha - 2Q^\dagger{}_\alpha N^\alpha R^\dagger{}_\gamma]\Bigg)^T$$

The rest of the equations also can be calculated in a similar manner. Let us proceed with the $Q^\dagger_\gamma$, we have

$$\begin{aligned}\dot{P}_{Q^\dagger_\gamma} &= -\frac{\partial H}{\partial Q^\dagger_\gamma} = -\frac{\partial\,\mathrm{Tr}\left(D_i Q^{\alpha\dagger} D_i Q^\alpha + M^\alpha M^{\alpha\dagger} + N^\alpha N^{\alpha\dagger}\right)}{\partial Q^\dagger_\gamma} \\ &= \frac{\partial\,\mathrm{Tr}\left(D_i Q^{\alpha\dagger} D_i Q^\alpha\right)}{\partial Q^\dagger_\gamma} + \frac{\partial\,\mathrm{Tr}\left(M^\alpha M^{\alpha\dagger}\right)}{\partial Q^\dagger_\gamma} + \frac{\partial \mathrm{Tr}\left(N^\alpha N^{\alpha\dagger}\right)}{\partial Q^\dagger_\gamma}\end{aligned} \tag{A.37}$$

We find that

$$\frac{\partial\,\mathrm{Tr}\left(D_i Q^{\alpha\dagger} D_i Q^\alpha\right)}{\partial Q^\dagger_\gamma} = i\left((D_i Q^\dagger_\gamma)X_i - \hat{X}_i(D_i Q^\dagger_\gamma)\right) \tag{A.38}$$

The $|M^\alpha|^2$ is given as

$$\frac{\partial\,\mathrm{Tr}\left(M^\alpha M^{\alpha\dagger}\right)}{\partial Q^\dagger_\gamma} = \Bigg(\mu M_\alpha + \frac{2\pi}{k}\big(2Q^\gamma Q^\dagger_\alpha M_\alpha + 2M_\gamma Q^\dagger_\beta Q^\beta - 2Q^\beta Q^\dagger_\beta M^\gamma - 2M^\alpha Q^\dagger_\alpha Q^\gamma \tag{A.39}$$

$$+ R^\beta R^\dagger_\beta M^\gamma - M^\gamma R^\dagger_\beta R^\beta + 2M^\alpha R^\dagger_\alpha R^\gamma - 2R^\gamma R^\dagger_\alpha M^\alpha + 2Q_\gamma M_\alpha{}^\dagger Q^\alpha - 2Q^\alpha M^\dagger{}_\alpha Q^\gamma\big)\Bigg)^T$$

$$\frac{\partial\,\mathrm{Tr}\left(N^\alpha N^{\alpha\dagger}\right)}{\partial Q^\dagger_\gamma} = \frac{2\pi}{k}\Bigg(R^\alpha N^\dagger_\alpha Q^\gamma - Q^\gamma N^\dagger_\alpha R^\alpha + 2Q^\alpha N^\dagger_\alpha R^\gamma - 2R^\gamma N^\dagger_\alpha Q^\alpha \tag{A.40}$$

$$+ N^\alpha R^\dagger_\alpha Q^\gamma - Q^\gamma R^\dagger_\alpha N^\alpha + 2Q^\beta R^\dagger_\beta N^\gamma - 2N^\gamma R^\dagger_\beta Q^\beta\Bigg)^T$$

Then we have

$$\dot{P}_{Q^{\dagger}{}_{\gamma}} = -\Big(i[(D_iQ_\gamma)\hat{X}_i - X_i(D_iQ_\gamma)] + \mu M_\gamma + \frac{2\pi}{k}[2Q^\gamma Q^\dagger{}_\alpha M^\alpha \tag{A.41}$$

$$+ 2M^\gamma Q^\dagger{}_\beta Q_\beta - 2Q^\beta Q^\dagger{}_\beta M_\gamma - 2M^\alpha Q^\dagger{}_\alpha Q^\gamma + 2Q_\gamma M_\alpha{}^\dagger Q^\alpha$$

$$- 2Q^\alpha M^\dagger{}_\alpha Q^\gamma + R^\beta R^\dagger{}_\beta M_\gamma - M^\gamma R^\dagger{}_\beta R^\beta + 2M_\alpha R^\dagger{}_\alpha R^\gamma - 2R^\gamma R^\dagger{}_\alpha M^\alpha$$

$$+ R^\alpha N^\dagger{}_\alpha Q^\gamma - Q^\gamma N^\dagger{}_\alpha R^\alpha + 2Q^\gamma N^\dagger{}_\alpha R^\gamma - 2R^\gamma N^\dagger{}_\alpha Q^\alpha$$

$$+ N^\alpha R^\dagger{}_\alpha Q^\gamma - Q^\gamma R^\dagger{}_\alpha N^\alpha + 2Q^\beta R^\dagger{}_\beta N^\gamma - 2N^\gamma R^\dagger{}_\beta Q^\beta]\Big)^T$$

For the $R^\dagger_\gamma$ part, we have

$$\begin{aligned} \dot{P}_{R^\dagger_\gamma} &= -\frac{\partial H}{\partial R^\dagger_\gamma} = -\frac{\partial\,\mathrm{Tr}\left(D_iR^{\alpha\dagger}D_iR^\alpha + M^\alpha M^{\alpha\dagger} + N^\alpha N^{\alpha\dagger}\right)}{\partial R^\dagger_\gamma} \\ &= \frac{\partial\,\mathrm{Tr}\left(D_iR^{\alpha\dagger}D_iR^\alpha\right)}{\partial R^\dagger_\gamma} + \frac{\partial\,\mathrm{Tr}\left(M^\alpha M^{\alpha\dagger}\right)}{\partial R^\dagger_\gamma} + \frac{\partial\mathrm{Tr}\left(N^\alpha N^{\alpha\dagger}\right)}{\partial R^\dagger_\gamma} \end{aligned} \tag{A.42}$$

The covariant derivative part is given as

$$\frac{\partial\,\mathrm{Tr}\left(D_iR^{\alpha\dagger}D_iR^\alpha\right)}{\partial R^\dagger_\gamma} = i\big((D_iR^\gamma)\hat{X}_i - X_i(D_iR^\gamma)\big)^T \tag{A.43}$$

The $|M^\alpha|^2$ part is

$$\begin{aligned} \frac{\partial\,\mathrm{Tr}\left(M^\alpha M^{\alpha\dagger}\right)}{\partial R^\dagger_\gamma} = \frac{2\pi}{k}\big(&M_\alpha Q^\dagger_\alpha R^\dagger_\gamma - R^\beta Q^\dagger_\beta M_\gamma + 2R^\gamma Q^\alpha_\alpha M_\alpha - 2R^\beta Q^\dagger_\beta M_\gamma \\ &+ Q^\alpha M^\dagger_\alpha R^\gamma - R^\gamma M^\dagger_\alpha Q^\alpha + 2R^\alpha M^\dagger_\alpha Q^\gamma - 2Q^\gamma M^\dagger_\alpha R^\alpha\big)^T \end{aligned} \tag{A.44}$$

The $|N^\alpha|^2$ part is

$$\frac{\partial\,\mathrm{Tr}\left(N^\alpha N^{\alpha\dagger}\right)}{\partial R^\dagger_\gamma} = \Big(- \mu N^\alpha + \frac{2\pi}{k}\big(R^\gamma N^\dagger_\alpha R^\alpha - R^\alpha N^\dagger_\alpha R^\gamma + N^\gamma R^\dagger_\beta R^\beta - R^\beta R^\dagger_\beta N^\gamma$$
$$- N^\alpha R^\dagger_\alpha R^\gamma + Q^\beta Q^\dagger_\beta N^\gamma - N^\gamma Q^\dagger_\beta Q^\beta + 2N^\alpha Q^\dagger_\alpha Q^\gamma - 2Q^\gamma Q^\dagger_\alpha N^\alpha + R^\gamma R^\dagger{}_\alpha N^\alpha\big)\Big)^T \tag{A.45}$$

Therefore we have

$$\dot{P}_{R^\dagger_\gamma} = -\Big(i[(D_iR_\gamma)\hat{X}_i - X_i(D_iR_\gamma)] - \mu N_\gamma + \frac{2\pi}{k}[M^\alpha Q^\dagger{}_\alpha R^\dagger{}_\gamma \tag{A.46}$$

$$- R^\beta Q^\dagger{}_\beta M^\gamma + 2R^\gamma Q^\dagger{}_\alpha M^\alpha - 2R^\beta Q^\dagger{}_\beta M^\gamma + Q^\alpha M^\dagger{}_\alpha R^\gamma$$

$$- R^\gamma M^\dagger{}_\alpha Q^\alpha + 2R^\alpha M^\dagger{}_\alpha Q^\gamma - 2Q^\gamma M^\dagger{}_\alpha R^\alpha + R^\gamma N^\dagger{}_\gamma R^\alpha - R^\alpha N^\dagger{}_\alpha R^\gamma$$

$$+ N^\gamma R^\dagger{}_\beta R^\beta - R^\beta R^\dagger{}_\beta N^\gamma - N^\alpha R^\dagger{}_\alpha R^\gamma + Q^\beta Q^\dagger{}_\beta N^\gamma$$

$$- N^\gamma Q^\dagger{}_\beta Q^\beta + 2N^\alpha Q^\dagger{}_\alpha Q^\alpha - 2Q^\gamma Q^\dagger{}_\alpha N^\alpha + R^\gamma R^\dagger{}_\alpha N^\alpha]\Big)^T$$

The R_γ part is

$$\dot{P}_{R_\gamma} = -\frac{\partial H}{\partial R_\gamma} = -\frac{\partial\, \mathrm{Tr}\left(D_i R^{\alpha\dagger} D_i R^\alpha + M^\alpha M^{\alpha\dagger} + N^\alpha N^{\alpha\dagger}\right)}{\partial R_\gamma}$$
$$= \frac{\partial\, \mathrm{Tr}\left(D_i R^{\alpha\dagger} D_i R^\alpha\right)}{\partial R_\gamma} + \frac{\partial\, \mathrm{Tr}\left(M^\alpha M^{\alpha\dagger}\right)}{\partial R_\gamma} + \frac{\partial \mathrm{Tr}\left(N^\alpha N^{\alpha\dagger}\right)}{\partial R_\gamma} \tag{A.47}$$

The covariant derivative part is

$$\frac{\partial\, \mathrm{Tr}\left(D_i R^{\alpha\dagger} D_i R^\alpha\right)}{\partial R_\gamma} = i\big((D_i R^\dagger{}_\gamma) X_i - \hat{X}_i (D_i R^\dagger{}_\gamma)\big)^T \tag{A.48}$$

The $|M^\alpha|^2$ part is

$$\frac{\partial\, \mathrm{Tr}\left(M^\alpha M^{\alpha\dagger}\right)}{\partial R_\gamma} = \frac{2\pi}{k}\big(2Q^\dagger{}_\gamma M^\alpha R^\dagger{}_\alpha - 2R^\dagger{}_\alpha M^\alpha Q^\dagger{}_\gamma + 2M^\dagger{}_\gamma Q^\beta R^\dagger{}_\beta - 2R^\dagger{}_\beta Q^\beta M_\gamma{}^\dagger$$
$$+ R^\dagger{}_\gamma M^\alpha Q^\dagger{}_\alpha - Q^\dagger{}_\alpha M^\alpha R^\dagger{}_\gamma + R^\dagger{}_\gamma Q^\alpha M^\dagger{}_\alpha - M^\dagger{}_\alpha Q^\alpha R^\dagger{}_\gamma\big)^T \tag{A.49}$$

The $|N^\alpha|^2$ part is given as

$$\frac{\partial\, \mathrm{Tr}\left(N^\alpha N^{\alpha\dagger}\right)}{\partial R^\dagger_\gamma} = \Bigg(-\mu N^\dagger{}_\gamma + \frac{2\pi}{k}\big(R^\dagger{}_\beta R^\beta N^\dagger{}_\gamma + N^\dagger{}_\gamma R^\gamma R^\dagger{}_\beta - R^\dagger{}_\gamma R^\alpha N^\dagger{}_\alpha - N^\dagger{}_\gamma R^\beta R^\dagger{}_\beta \tag{A.50}$$

$$+ N^\dagger{}_\gamma Q^\beta Q^\dagger{}_\beta - Q^\dagger{}_\beta Q^\beta N^\dagger{}_\gamma + 2Q^\dagger{}_\gamma Q^\alpha N^\dagger{}_\alpha - 2N^\dagger{}_\alpha Q^\alpha Q^\dagger{}_\gamma + R^\dagger{}_\alpha N^\alpha R^\dagger{}_\gamma - R^\dagger{}_\gamma N^\alpha R^\dagger{}_\alpha\big)\Bigg)^T \tag{A.51}$$

Then finally we have

$$\dot{P}_{R_\gamma} = -\Bigg(i[(D_i R^\dagger{}_\gamma) X_i - \hat{X}_i (D_i R^\dagger{}_\gamma)] - \mu N^\dagger{}_\gamma + \frac{2\pi}{k}[2Q^\dagger{}_\gamma M^\alpha R^\dagger{}_\alpha \tag{A.52}$$
$$- 2R^\dagger{}_\alpha M^\alpha Q^\dagger{}_\gamma + 2M^\dagger{}_\gamma Q^\beta R^\dagger{}_\beta - 2R^\dagger{}_\beta Q^\beta M_\gamma{}^\dagger + R^\dagger{}_\gamma M^\alpha Q^\dagger{}_\alpha$$
$$- Q^\dagger{}_\alpha M^\alpha R^\dagger{}_\gamma + R^\dagger{}_\gamma Q^\alpha M^\dagger{}_\alpha - M^\dagger{}_\alpha Q^\alpha R^\dagger{}_\gamma + R^\dagger{}_\beta R^\beta N^\dagger{}_\gamma + N^\dagger{}_\gamma R^\gamma R^\dagger{}_\beta$$
$$- R^\dagger{}_\gamma R^\alpha N^\dagger{}_\alpha - N^\dagger{}_\gamma R^\beta R^\dagger{}_\beta + N^\dagger{}_\gamma Q^\beta Q^\dagger{}_\beta - Q^\dagger{}_\beta Q^\beta N^\dagger{}_\gamma + 2Q^\dagger{}_\gamma Q^\alpha N^\dagger{}_\alpha$$
$$- 2N^\dagger{}_\alpha Q^\alpha Q^\dagger{}_\gamma + R^\dagger{}_\alpha N^\alpha R^\dagger{}_\gamma - R^\dagger{}_\gamma N^\alpha R^\dagger{}_\alpha]\Bigg)^T$$

APPENDIX B

AUXILARY FIELDS VANISH

The covariant derivative of Q_α is given as

$$(D_iQ_\alpha)_{ab} = i(X_iQ_\alpha)_{ab} - i(Q_\alpha\hat{X}_i)_{ab} \tag{B.1}$$

$$= i(X_i)_{ac}(Q_\alpha)_{cb} - i(Q_\alpha)_{ac}(Q_\alpha)_{cb} \tag{B.2}$$

If we open the α index, we get

$$(D_iQ_1)_{ab} = i(X_iQ_1)_{ab} - i(Q_1\hat{X}_i)_{ab} \tag{B.3}$$

$$= i(X_i)_{ac}(\phi_1G_1)_{cb} - i(\phi_1G_1)_{ac}(\hat{X}_i)_{cb} \tag{B.4}$$

$$= i\alpha(t)\phi_1(t)(\tilde{X}_i)_{ac}\sqrt{c-1}\delta_{cb} - i\beta(t)\phi_1(t)\sqrt{a-1}\delta_{ac}(\tilde{\hat{X}}_i)_{cb} \tag{B.5}$$

$$= i\alpha\phi_1\sqrt{b-1}(\tilde{X}_i)_{ab} - i\beta\phi_1\sqrt{a-1}(\tilde{\hat{X}}_i)_{ab} \tag{B.6}$$

$$= i\phi_1(\sqrt{b-1}X_i - \sqrt{a-1}\hat{X}_i)_{ab} \tag{B.7}$$

For the $Q^\dagger$ part, we have

$$(D_iQ_\alpha)^\dagger{}_{ab} = -i\phi_1(\sqrt{b-1}X_i - \sqrt{a-1}\hat{X}_i)_{ba} \tag{B.8}$$

Then the kinetic term becomes

$$\mathrm{Tr}\,|D_iQ_1|^2 = Tr(D_iQ_1)^\dagger(D_iQ_1) = (D_iQ_1)^\dagger{}_{ab}(D_iQ)_{ba} \tag{B.9}$$

$$= \phi_1^2\Big((b-1)(X_i^2)_{bb} + (a-1)(\hat{X_i}^2)_{bb} - \sqrt{a-1}\sqrt{b-1}(X_i\hat{X}_i)_{bb} \tag{B.10}$$

$$-\sqrt{a-1}\sqrt{b-1}(\hat{X}_iX_i)_{bb}\Big)$$

$$= {\phi_1}^2\Big((b-1)\,\mathrm{Tr}\,X_i^2 + (a-1)\,\mathrm{Tr}\,\hat{X_i}^2 - \sqrt{a-1}\sqrt{b-1}(\mathrm{Tr}\,X_i\hat{X}_i + \mathrm{Tr}\,\hat{X}_iX_i)\Big) \tag{B.11}$$

$$= {\phi_1}^2\Big((b-1)\,\mathrm{Tr}\,X_i^2 + (a-1)\,\mathrm{Tr}\,\hat{X_i}^2 - 2\sqrt{a-1}\sqrt{b-1}\,\mathrm{Tr}\,X_i\hat{X}_i\Big) \tag{B.12}$$

$$= {\phi_1}^2\Big((b-1)\,\mathrm{Tr}\,X_i^2 + (a-1)\,\mathrm{Tr}\,\hat{X_i}^2 - 2\sqrt{a-1}\sqrt{b-1}\,\mathrm{Tr}\,X_i\hat{X}_i\Big) \tag{B.13}$$

$$= {\phi_1}^2\left(\alpha^2(b-1)\sum_{\substack{\alpha=1\\ i=1,2}}^{N} {a^i_\alpha}^2 + \beta^2(a-1)\sum_{\alpha=1}^{N} {b^i_\alpha}^2 - 2\alpha\beta\sqrt{a-1}\sqrt{b-1}\sum_{\substack{\alpha=1\\ i=1,2}}^{N} a^i_\alpha b^i_\alpha\right) \tag{B.14}$$

For Q_2 part, we have

$$(D_iQ_2)_{ab} = i(X_i)_{ac}\phi_2(G_2)_{cb} - i(\phi_2G_2)_{ac}(\hat{X}_i)_{cb} \tag{B.15}$$

$$= i\phi_2\sqrt{N-c}\,(X_i)_{ac}\delta_{c+1,b} - i\phi_2\sqrt{N-a}\,\delta_{a+1,c}(\hat{X}_i)_{cb} \tag{B.16}$$

$$= i\phi_2(\sqrt{N-b+1}\,(X_i)_{a,b-1} - \sqrt{N-a}\,(\hat{X}_i)_{a+1,b} \tag{B.17}$$

The $\phi_2^\dagger$ part becomes

$$(D_iQ_2)^\dagger_{ab} = -i\phi_2(\sqrt{N-b+1}(X_i)_{b-1,a} - \sqrt{N-a}(\hat{X}_i)_{b,a+1} \tag{B.18}$$

Then kinetic part becomes

$$\mathrm{Tr}\,|D_iQ_2|^2 = \phi_2^2\Big(\sqrt{N-b+1}\,(X_i^2)_{b-1,b-1} + (N-a)(\hat{X}_i)_{a+1,a+1}$$

$$-\sqrt{N-b+1}\,\sqrt{N-a}\,(X_i)_{a,b-1}(\hat{X}_i)_{b,a+1} - \sqrt{N-b+1}\,\sqrt{N-a}\,(\hat{X}_i)_{a+1,b}(X_i)_{a,b-1} \tag{B.19}$$

If we explicitly open term by term, we get

$$\mathrm{Tr}\,|D_iQ_2|^2 = \phi_2^2\Big((N-b+1)\big(\mathrm{Tr}\,X_i^2 - (X_i^2)_{NN}\big) + (N-a)\Big(\mathrm{Tr}\Big(\hat{X_i}^2 - (\hat{X_i}^2)_{11}\Big)\Big)$$

$$-\sqrt{N-b+1}\,\sqrt{N-a}\,\alpha\,\beta\sum_{\gamma=1}^{N}\sum_{i=1}^{2}(b^i)_{\gamma+1} - \sqrt{N-b+1}\,\sqrt{N-a}\,\alpha\,\beta\sum_{r=1}^{N}\sum_{i=1}^{2}(a^i)_\gamma(b^i)_{\gamma+1}\Big) \tag{B.20}$$

Then the kinetic term becomes,

$$
\begin{aligned}
&\operatorname{Tr}|D_iQ_1|^2 + \operatorname{Tr}|D_iQ_2|^2 = \\
&\alpha^2 \left(\underbrace{\phi_1^2(b-1)\sum_{\gamma=1}^{N}\sum_{i=1}^{2}(a_\gamma^i)^2 + \phi_2^2(N-b+1)\sum_{\gamma=1}^{N-1}\sum_{i=1}^{N-1}(a_\gamma^i)^2}_{A} \right) \\
&+\beta^2 \left(\underbrace{\phi_1^2(a-1)\sum_{\gamma=1}^{N}\sum_{i=1}^{2}(b_\gamma^i)^2 + \phi_2^2(N-a)\sum_{\gamma=1}^{N}\sum_{i=1}^{2}(b_\gamma^i)^2}_{B} \right) \\
&-2\alpha\beta \left(\underbrace{\phi_1^2\sqrt{a-1}\sqrt{b-1}\sum_{\gamma=1}^{N}\sum_{i=1}^{2}a_\gamma^i b_\gamma^i + \phi_2^2\sqrt{N-b+1}\sqrt{N-a}\sum_{\gamma=1}^{N}\sum_{i=1}^{2}(a^i)_\gamma(b^i)_{\gamma+1}}_{C} \right)
\end{aligned}
\tag{B.21}
$$

We can write it as

$$\operatorname{Tr}|D_iQ_1|^2 + \operatorname{Tr}|D_iQ_2|^2 = \alpha^2 A + \beta^2 B - 2\alpha\beta C \tag{B.22}$$

Now e.o.m. for α and β is

$$\alpha A - \beta C = 0 \tag{B.23a}$$

$$\beta B - \alpha C = 0 \tag{B.23b}$$

With adding and subtracting, we get

$$
\begin{aligned}
B/\ & A\alpha - C\beta = 0 \\
C/\ & -C\alpha + B\beta = 0
\end{aligned}
\tag{B.24}
$$

Then finally we have

$$(BA - C^2)\alpha = 0 \tag{B.25}$$

But $BA \neq C^2$ as one can see clearly from the values of A, B, C. Therefore, the only solution is $\alpha = \beta = 0$.