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# QUASILINEARIZATION TECHNIQUE FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH INITIAL TIME DIFFERENCE

**M. Sc. THESIS**

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## JURY ACCEPTANCE AND APPROVAL

The thesis titled "**Quasilinearization Technique for An Integro-Differential Equation with Initial Time Difference,**" prepared by **Mohammed Ahmed Issa ALHUSSEIN**, was defended on 28/12/2023. It has been accepted unanimously by the jury provided below as a thesis for the degree of Master of Science in Mathematics at the Graduate Education Institute of Tokat Gaziosmanpasa University.

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## DECLARATION

I hereby attest that I have adhered to the prescribed standard thesis format, academic regulations, and ethical principles. Furthermore, I confirm that all materials utilized in this thesis are duly documented in the reference list in accordance with the stipulated regulations and ethical guidelines. I certify that the findings and outcomes presented in this thesis are original and have not been previously obtained elsewhere. Additionally, I affirm that there is no falsification of data within this thesis, and no portion of this work has been submitted as a thesis at this institution or any other university.

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# ABSTRACT

QUASILINEARIZATION TECHNIQUE FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH INITIAL TIME DIFFERENCE

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This thesis uses the Quasilinearization method to address an initial value problem involving a nonlinear integro-differential equation with initial time difference. By transforming the original equation into a sequence of linear differential equations and iteratively solving them, the method produces monotone sequences of lower and upper solutions that uniformly approaches to the unique solution. The impressive quadratic convergence of these sequences resulted in rapid reduction of approximation errors with each iteration.

By performing appropriate computations, it has been successfully demonstrated that the sequences converge to the unique solution of the initial value problem. This achievement has significant implications, as it provides valuable insights into the complex behavior and properties of the nonlinear integro-differential equation with initial time difference. We expect that the study's findings will help us understand and solve some problems in science and engineering. The study is important in that it has new results and shows the effectiveness of the generalized quasilinearization method in solving integro-differential problems given with initial time difference.



**Keywords:** Quasilinearization Method, Integro-Differential Equation, Upper and Lower Solutions.

**ÖZET**

BİR İNTEGRO-DİFERENSİYEL DENKLEM İÇİN BAŞLANGIÇ ZAMAN FARKLI KUASİLİNEERİZASYON METODU


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Bu tezde, başlangıç zaman farkına sahip doğrusal olmayan bir integral diferansiyel denklemi içeren bir başlangıç değer problemini çözmek için kuasilineerizasyon yöntemi kullanılmıştır. Bu yöntemle alt ve üst çözümlerden oluşan monoton diziler lineer diferensiyel denklemlerin çözümleri olmakla birlikte düzgün ve monoton olarak diferensiyel denklemin tek çözümüne yakınsarlar. Ayrıca bu teknikte elde edilen fonksiyon dizileri kuadratik olarak çözüme hızlı bir yakınsama gerçekleştirir.

Uygun işlemler takip edilerek, dizilerin başlangıç değer probleminin tek çözümüne yakınsadığı başarılı bir şekilde gösterildi. Bunun önemli sonuçları vardır, çünkü başlangıç zaman farkı olan doğrusal olmayan intego-diferensiyel denklemin karmaşık davranışı ve özellikleri hakkında önemli sonuçlar ortaya koymaktadır. Çalışmanın bulgularının bilim ve mühendislikteki ortaya çıkan bazı problemleri çözmemizde yardımcı olacağını umuyoruz. Tezde, yeni sonuçlar elde edilmiş olup, başlangıç zaman farkıyla verilen integro-diferensiyel denklemlerin çözümünde genelleştirilmiş kuasilineerizasyon yönteminin etkinliğini göstermesi bakımından önemlidir.

**Anahtar Kelimeler:** Kuasilineerizasyon Metodu, Integro-Diferensiyel Denklemler, Alt ve Üst Çözümler.

**CONTENT**

## PREFACE

First and foremost, I would like to express my gratitude to my esteemed supervisor, Prof. Dr. Ali YAKAR, for his assistance in everything. I would also like to thank Dr. Orhan for his wonderful support, along with all the other professors, since the beginning. Furthermore, I want to thank my family for their financial and moral support. I extend my appreciation to all my dear friends whom I have met since coming to Turkey and who have stood by my side.

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## ABBREVIATIONS

| | |
|---|---|
| QLM | Quasilinearization Method |
| ULS | Upper and Lower Solution |
| MVT | Mean Value Theorem |
| IVP | Initial Value Problem |
| ITD | Initial Time Difference |
| MIT | Monotone Iterative Technique |
| GLB | Greatest Lower Bound |

# 1. INTRODUCTION

The quasilinearization method (QLM) was initially introduced by Kalaba and Bellman [1], and then expanded upon by Lakshmikantham [2,3]. Extensive study has been conducted across multiple fields, resulting in an extensive bibliography and the generation of a substantial historical record [4–21]. For example, this method was used to estimate population growth, while the QLM was applied to make it easier to solve the model of population growth [22]. The resulting matrices are used to convert the partial-temporal population growth model into a nonlinear system of equations in algebra [23] and in many other fields of sciences [24–31]. It has been demonstrated to yield remarkable outcomes when employed to many nonlinear ordinary differential equations in physics, including the Blasius, Lane–Emden, Duffing, and Thomas–Ferm equations. The initial approximations yield highly precise and numerically stable solutions [32–34].

QLM is a constructive and useful approach that provides a clear and analytical representation for solving various types of nonlinear differential equations. This method involves creating monotone sequences, where each iteration corresponds to a solution of related linear differential equations. By demonstrating the uniform and quadratic convergence of these sequences, a unique solution for the given nonlinear differential equations is obtained. The strength of this approach lies in its ability to offer an iterative process for obtaining an approximate solution as well as establishing the existence of unique solution [35].

In the study of nonlinear differential equations, it is customary to modify or perturb the dependent (spatial) variable while maintaining the initial time constant. However, this strategy is considered unrealistic due to the inherent challenge of avoiding errors in the selection of the starting time [36]. The main issue arises from the fact that the solution of an unperturbed dynamical system may begin at a different time compared to that of a disturbed dynamical system.

To address this limitation, a novel concept has emerged, involving the simultaneous modification of both the beginning time and the dependent variable. This innovative approach has yielded several noteworthy results, including advancements in comparison theorems, establishing global existence, understanding stability requirements, employing the method of upper and lower solutions, and applying the monotone iterative approach, among others [37–43].

The primary goal of Bellman and Kalaba's original quasi-linearization technique is to explicitly represent the solution of nonlinear differential equations in an analytical form. This approach allows for obtaining pointwise lower estimates of the solution, particularly when the underlying function is convex. Notably, this method finds significant application in generating a sequence of lower bound solutions for linear differential equations, exhibiting quadratic convergence.

With the passage of time, this technique has gained popularity across various disciplines, especially in the investigation of integro-differential equations. Research in the theory and applications of integro-differential equations has become a novel field, owing to their widespread utilization in mathematically modeling physical phenomena and addressing practical problem [44, 45]. There actually exist multiple iterative methods that can be employed to solve integro-differential equations. Several iterative methods commonly used in mathematical analysis include the monotone iterative technique (MIT), quasilinearization, and their respective generalizations. MIT approach and QLM are iterative methods widely employed in the field of differential equations to establish existence and uniqueness results for a diverse range of equations. Both, in conjunction with the method of upper and lower solutions (ULS), produce monotone iterates that serve as solutions to specific linear differential equations derived from the assumption of the given problem. The generated bounds are proven to converge to the solution of the original problem.

As an initial value problem containing Volterra integro-differential equations with initial time difference, the goal of the current work is to employe the generalized quasi-linearization approach. We take into account the following initial value problem:

$$\frac{dx}{dt} = g\big(t, x(t)\big) + \int_{t_0}^{t} \varphi\,\big(t, s, x(s)\big)ds, \qquad x(t_0) = x_0 \tag{1.1}$$

where $t \in [t_0, t_0 + T]\,, t_0 \geq 0\,, T > 0$ .

In this context, starting with initial functions accepting initial time difference, we construct sequences of upper and lower bounds for the given nonlinear problem which satisfy the corresponding linear equations. Notably, both the lower and upper sequences of approximations have been shown to converge monotonically and quadratically towards the unique solution (1.1) of the problem.

However, the convexity condition required by the quasi-linearization approach has proven to be a barrier to the theory's continued development. Recent advancements in generalization, improvement, and extension have made this approach accessible to a considerably wider range of nonlinear problems by removing the requirement for convexity properties. The approach of generalized quasi-linearization is now generally beneficial in applications due to the investigation of additional possibilities [35].

# 2. BASIC DEFINITIONS AND THEOREMS

## 2.1 Method of Upper and Lower Solutions

Consider the IVP

$$x' = g(t,x), \qquad x(t_0) = x_0 \tag{2.1.1}$$

where $g \in C[J \times R, R]$ and $J = [t_0, T]$. A function $v \in C^1[J, R]$ is said to be a lower solution of (2.1.1) if

$$v' \leq g(t,v), \qquad v(t_0) \leq x_0$$

and an upper solution of (2.1.1) if the inequalities are the opposite.

The method of utilizing upper and lower solutions (ULS) is an intriguing and effective approach for establishing the existence of solutions in nonlinear problems. It centers on simplifying the original problem and then leveraging either the Leray-Schauder topological degree theory or established results regarding the modified problem's existence. This method is combined with the inequality theory, serves to confirm the presence of solutions in the original problem. Moreover, it enables the identification of solutions within a specific range, particularly the structured interval defined by ULS.

In the subsequent section, we will delve deeply into the concept of ULS, which plays a fundamental role in our subsequent discussions. We initially focus on scalar initial value problems (abbreviated as IVP), highlighting the key principles underlying the use of ULS [35].
The following theorem presents a fundamental outcome related to ULS.

**Theorem 2.1.1.** [35] Let $v$ and $m \in C^1[J, R]$. Additionally, $v$ is a lower solution, while $m$ is an upper solution. Under these conditions, suppose that.

$$g(t,x) - g(t,y) \leq L(x - y),$$

Whenever $x \geq y$ for some $L > 0$. Then $v(t_0) \leq m(t_0)$ implies $v(t) \leq m(t)$ on $J$.
Sometimes, it can be advantageous to examine interconnected ULS for the problem (2.1.1), which are defined as

$$v' \leq g(t,m), \;\; v(t_0) \leq x_0$$

$$m' \geq g(t,v), \;\; m(t_0) \geq x_0.$$

In such situations, Additionally, we would require a finding that is consistent with Theorem $(2.1.1)$, In fact, this result is essential for our subsequent discussion.

If we have established the existence of both ULS, denoted as $v$ and $m$ respectively, for the problem $(2.1.1)$, and further confirmed that $v(t) \leq m(t)$ holds over the interval $J$, then we are able to show the following existence result.

**Theorem 2.1.2.** [35] Let $v, m \in C^1\,[J, R]$ be lower and upper solutions of $(2.1.1)$ such that $v(t) \leq m(t)$ on $J$, and let $g \in C[\,\Omega, R]$ where $\Omega = [(t,x) \colon v(t) \leq x \leq m(t), t \in J]$, then there's a solution $x(t)$ of $(2.1.1)$, satisfying $v(t) \leq x(t) \leq m(t)$ on $J$.

**2.2 Quasilinearization**

The quasilinearization technique involves solving the nonlinear differential equation through a recursive process of linear differential equations instead of direct solving. The primary benefit of this method lies in its ability to potentially achieve quadratic convergence towards the solution of the original equation when the process converges. Quadratic convergence implies that the error in the $(n+1)$st iteration becomes asymptotically proportional to the square of the error in the $n$th iteration, resulting in a rapid and faster convergence rate.

In order to derive the linear equations, QLM utilizes the first and second terms of the Taylor series expansion of the original nonlinear equation. This approach can be viewed as a generalized version of the Newton-Raphson formula for functional equations.

Linear differential equations of the boundary-value type with variable coefficients can be efficiently solved on modern computers using the superposition principle. This has led to the development of an efficient recursive formula. However, the technique also faces challenges. The primary challenge emerges when attempting to resolve a set of equations involving algebra while employing the superposition principle. The presence of ill-conditioning in solving this particular set of linear algebraic equations can diminish the effectiveness of the superposition principle.

QLM not only converts the nonlinear equation into a linear equation but also generates a sequence of functions that uniformly converge rapidly towards the solution of the original nonlinear equation. In practical applications, the latter aspect holds greater significance. Starting with a rough initial approximation of the unknown function can eventually lead to the original equation's solution through a series of functions. Generally, for most practical problems, these initial rough approximations can be obtained from engineering experiences and intuitive insights [28].

**Theorem 2.2.1.** [35] Assume that

- **a)** $\alpha_0, \beta_0 \in C^1[J, R]$ are lower and upper solutions of (2.1.1) such that $\alpha_0 \leq \beta_0$ on $J = [0, T]$ ;
- **b)** $g \in C[\Omega, R], g_x(t, x), g_{xx}(t, x) \geq 0$ exists and are continuous on $\Omega$, where $\Omega = \{(t, x) \colon \alpha_0(t) \leq x \leq \beta_0(t), t \in J\}$.

Then, there are monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ that uniformly converge to the unique solution $x(t)$ of (2.1.1) over the interval $J$. Moreover, this convergence is quadratic.

This method was employed to compute population growth by employing the quasi-linearization technique to simplify the solution of the population growth model. The matrices obtained are then utilized to convert the partial-temporal population growth model into a set of nonlinear algebraic equations [46].

The nonlinear two-point boundary value problems are solved using a combination of the finite difference method and QLM. Unlike marching integration techniques, this method does not encounter stability issues. Furthermore, a suggested approach seeks to minimize the rapid-access memory demands of digital computers. This technique is employed to address the steady-state equations arising from mass and energy balances in a tubular reactor with axial diffusion. [47]

## 2.3 Fundamental Definitions

**Definition 2.3.1.** [48] The sequence of functions $\{f_m(x)\}$ is considered to uniformly converge to a function $f(x)$ in the interval $[\alpha, \beta]$ if for any positive real number $\epsilon$ there exists an integer $N$ such that for all $m$ greater than or equal to $N$, $|f_m(x) - f(x)| \leq \epsilon$ for all $x$ in $[\alpha, \beta]$.

**Theorem 2.3.1.** [48] If $\{f_m(x)\}$ is a sequence of continuous functions in the interval $[\alpha, \beta]$, which uniformly converges to $f(x)$, then $f(x)$ is continuous in the interval $[\alpha, \beta]$.

**Definition 2.3.2.** [48] A sequence of functions $\{f_m(x)\}$ is called to be equi-continuous in the interval $[\alpha, \beta]$ if for any given positive $\varepsilon$ there exists a positive $\delta$, such that if $x_1,\ x_2 \in [\alpha, \beta]$, $|x_1 - x_2| \leq \delta$ then

$$|f_m(x_1) - f_m(x_2)| \leq \varepsilon \text{ for all } m \in N.$$

**Definition 2.3.3.** [48] A sequence of functions $\{f_m(x)\}$ is said to be uniformly bounded in an interval $[\alpha, \beta]$, if there exists a constant $M$ such that $|f_m(x)| \leq M$ for all $m \in N$.

**Theorem 2.3.2.** [48] **(Ascoli–Arzela Theorem)** If an infinite set $S$ of functions is both uniformly bounded and equi-continuous in an interval $[\alpha, \beta]$, then it necessarily contains a subsequence that converges uniformly in the same interval $[\alpha, \beta]$.

**Theorem 2.3.3.** [48] (**Peano Existence Theorem)** If $g(t, y)$ is bounded and continuous on a domain $R$ containing $(t_0, x_0)$ then there exist at least one solution to the initial value problem (2.1.1)

Note that Peano's existence theorem does not necessitate the inclusion of the Lipschitz condition as an essential component.

Gronwall's integral inequality is important in studying numerous features of ordinary differential equations. We recommend using it specifically to prove the uniqueness of solutions.

**Theorem 2.3.4.** [49] Assume that $\gamma(x)$ and $\mu(x)$ are non-negative continuous functions on $[a, b]$. Let $K > 0$, be a constant. Then the inequality

$$\gamma(x) \leq K + \int_a^x \gamma(s)\mu(s)ds\,, \qquad a \leq x \leq b \tag{2.3.1}$$

implies the inequality

$$\gamma(x) \leq K e^{\int_a^x \mu(s)ds}, \qquad a \leq x \leq b.$$

**Proof**: Let us define

$$h(x) = K + \int_a^x \gamma(s)\mu(s)ds\,,\ \ a \leq x \leq b$$

$$h(a) = K$$

$$h'(x) = \gamma(x)\mu(x).$$

If we multiply (2.3.1) by $\mu(x)$

$$\gamma(x)\mu(x) \leq (K + \int_a^x \gamma\,(s)\mu(s)ds)\mu(x)$$

$$\frac{\gamma(x)\mu(x)}{K + \int_a^x \gamma\,(s)\mu(s)ds} \leq \mu(x),\ \ a \leq x \leq b.$$

Note that $\gamma(x)\mu(x)$ is the derivative of $K + \int_a^x \gamma\,(s)\mu(s)ds$. After integrating from $a$ to $x$, it leads to

$$ln\left(K + \int_a^x \gamma\,(s)\mu(s)ds\right) - ln\,(K)\ \leq \int_a^x \mu\,(s)ds,$$

or, in other words

$$K + \int_{\mathrm{a}}^x \gamma\,(s)\mu(s)ds \leq K e^{\int_{\mathrm{a}}^x \mu(s)ds}.$$

This inequality leads to the desired conclusion

$$\gamma(x) \leq h(x) \leq K e^{\int_a^x \mu(s)ds}.$$

**Definition 2.3.4** [49] A function $g\,(t, x)$ defined in a region $D \subset R \times R$ is said to satisfy uniform Lipschitz condition in the variable $t$ with a Lipschitz constant $L$, if the inequality

$$|g\,(t,\,x_1) - g(t, x_2)| \leq L|x_1 - x_2|$$

holds whenever $(t,\ x_1), (t,\ x_2)$ are in $D$. In such a case we denote $g$ to be a member of the class Lip $(D, L)$.

As a consequence of the definition, a function $g(t, x)$ satisfies Lipschitz condition if and only if there exists a constant $L > 0$ such that

$$\frac{|g(t,\ x_1) - g(t, x_2)|}{|x_1 - x_2|} \leq L, \quad x_1 \neq x_2$$

whenever $(t,\ x_1), (t, x_2) \in D$.

**Theorem 2.3.5** [49] Let $g(t, x)$ be a continuous function defined over rectangle

$R = \{(t, x): |t - t_0| \leq p_1, |x - x_0| \leq p_2\}$. Here $p_1, p_2$ are some positive real numbers. Let $\frac{\partial g}{\partial x}$ be defined and continuous on $R$. Then $g(t, x)$ satisfies the Lipschitz condition in $R$.

**Proof:** Since $\frac{\partial g}{\partial x}$ is continuous on $R$ there exists a positive constant $K$ such that

$$\left|\frac{\partial g(t, x)}{\partial x}\right| \leq K \tag{2.3.2}$$

for all $(t, x)$ in $R$. Let $(t, x_1), (t, x_2)$ be any two points in $R$. By the mean value theorem of differential calculus, there exist a number $s$ which lies between $x_1$ and $x_2$ such that

$$g(t, x_1) - g(t, x_2) = \frac{\partial g}{\partial x}(t, s)(x_1 - x_2),$$

Since the point $(t, s)$ lies in between $x_1$ and $x_2$ then by using (2.3.2) we have

$$|g(t, x_1) - g(t, x_2)| \leq K|x_1 - x_2|$$

whenever $(t, x_1), (t, x_2)$ belong to $R$. The proof is complete.

**Theorem 2.3.6** [48] **(Lipschitz Uniqueness Theorem).** Let $g(t, x)$ be a continuous and satisfy a uniform Lipschitz condition in $\overline{\Pi}: |t - t_0| \leq a_1, |x - x_0| \leq a_2,\ a_1, a_2 > 0$. Then the IVP (2.1.1) has at most one solution $|t - t_0| \leq a_1$.

Now we give the definition of convex functions in the following.

**Definition 2.3.5.** [50] Let $I$ be an interval and $f: I \rightarrow R$.

**(i)** $f$ is said to be convex on $I$ if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $0 \leq \alpha \leq 1$ and for all $x, y \in I$.

**(ii)** $f$ is said to be concave on $I$ if and only if $-f$ is convex on $I$.

Notice that by definition, a function $f$ is convex on an interval $I$ if and only if $f$ is convex on every closed subinterval of $I$.

The definition of quadratic convergence is now presented. The faster form of convergence that we shall cover here is quadratic convergence, which is typically seen as an ideal if possible.

**Definition 2.3.6.** [35] Suppose we have a sequence $\{x_n(t)\}$ of functions defined on $I$ that converge to the function $y(t)$ on $I$. Then the sequence $\{x_n(t)\}$ is called to converge quadratically if the following inequality is satisfied

$$\max_I |x_{n+1} - y(t)| \leq C \max_I |x_n - y(t)|^2, n = 1,2, \dots$$

for some constant $C > 0$.

The Leibniz integral rule offers a method for differentiating a definite integral in cases where the limits of integration depend on the variables of differentiation. This technique is often known as differentiation under the integral sign.

**Definition 2.3.7.** [50] To take the derivative of the integral $\int_{n(\theta)}^{m(\theta)} g\,(\theta, x)dx$ with respect to $\theta$, we commonly utilize the helpful Leibniz rule, which is provided as follows:

$$\frac{\partial}{\partial\theta}\int_{n(\theta)}^{m(\theta)} g(\theta,x)dx = \int_{n(\theta)}^{m(\theta)} \frac{\partial g(\theta,x)}{\partial\theta}dx + g\big(\theta, m(\theta)\big)\frac{dm}{d\theta} - g\big(\theta, n(\theta)\big)\frac{dn}{d\theta},$$

In the $\theta x$-plane, there exists a domain $D$ that encompasses the rectangular region $R$, where $a \leq \theta \leq b$ and $x_0 \leq x \leq x_1$. Within this domain, both $g(\theta, x)$ and $\partial g/\partial\theta$ are continuous functions. Additionally, the functions $n(\theta)$ and $m(\theta)$, with continuous derivatives for $a < \theta < b$, define the limits of integration. It is noteworthy to mention that calculus textbooks frequently include the Leibniz rule, and our attention will be on its practical implementation rather than presenting its theoretical proof.

## 3. MAIN RESULTS

### 3.1 Definition of the problem

Integro-differential equations are commonly encountered as mathematical models in various fields [44, 45]. The study of Volterra integro-differential equations and their practical applications has become a novel area of research. Consider the following initial value problem (IVP)

$$\frac{dx}{dt} = g\big(t, x(t)\big) + \int_{t_0}^{t} \varphi\left(t, s, x(s)\right) ds, \qquad x(t_0) = x_0, \tag{3.1.1}$$

with $t \in J_1 = [t_0, t_0 + T]$ , $t \geq 0$ , $T > 0$, assuming appropriate conditions on $g$ and $\varphi$.

In this section, we will study on the quasilinearization method in order to apply a Volterra integro-differential equation (3.1.1) with ITD.

### 3.2 Application of the QLM with Initial Time Difference (ITD)

It is necessary to articulate the following lemma, which plays a crucial role in establishing the main result.

Let $J_1 = [t_0, t_0 + T]$  $J_2 = [\tau_0, \tau_0 + T]$ and $J_3 = [t_0, \tau_0 + T]$ where $t_0, \tau_0, T$ are positive real number.

**Lemma 3.2.1** Assume that the following conditions hold:

- **(i)** $g \in C[J_3 \times R, R]$, $\varphi \in C[J_3 \times J_3 \times R, R]$, and $\varphi(t, s, u)$ is monotone non-decreasing in $u$ for each fixed $(t, s) \in J_3 \times J_3$;
- **(ii)** $n \in C^1[J_1 \times R], m \in C^1[J_2 \times R], \tau_0 > t_0$ and

$$n'(t) \leq g\big(t, n(t)\big) + \int_{t_0}^{t} \varphi\left(t, s, n(s)\right) ds,$$

for $t_0 \leq t \leq t_0 + T$

$$m'(t) \geq g\big(t, m(t)\big) + \int_{\tau_0}^{\mathrm{t}} \varphi\left(t, s, m(s)\right) ds.$$

For $\tau_0 \leq t \leq \tau_0 + T$;

**(iii)** For $(t,s) \in J_3 \times J_3$ and $x \geq y$, we get

$$g(t,x) - g(t,y) \leq L_1(x-y), \text{ and}$$

$$\varphi(t,s,x) - \varphi(t,s,y) \leq L_2(x-y), t \in J_1$$

where $2{L_1}^2 > L_2 \geq 0$;

**(iv)** $g(t,x)$ is non-decreasing with respect to $t$ for each $x$ and also for any $\varepsilon > 0$,

$$\varphi(t+\varepsilon, s+\varepsilon, x) \geq \varphi(t,s,x)$$

Then we have $n(t) \leq m(t+\eta)$ on $t \in J_1$ or $n(t-\eta) \leq m(t)$ on $t \in J_2$ provided

$$n(t_0) \leq m(\tau_0).$$

**Proof:** Let $m_0(t) = m(t+\eta)$, for $t_0 \leq t \leq t_0 + T$. Clearly $m_0(t_0) = m(\tau_0) \geq n(t_0)$. Also, from (i) & (iv) we write

$$m'_0(t) = m'(t+\eta) \geq g\big(t+\eta, m(t+\eta)\big) + \int_{\tau_0}^{t+\eta} \varphi\left(t+\eta, s, m(s)\right) ds$$

$$\geq \; g\big(t, m_0(t)\big) + \int_{\tau_0}^{t+\eta} \varphi\left(t+\eta, s, m(s)\right) ds$$

Replacing the variable $s = \zeta + \eta \Rightarrow \; \zeta + (\tau_0 - t_0)$ , it follows that

$$m'_0(t) \geq \; g\big(t, m_0(t)\big) + \int_{t_0}^{t} \varphi\left(t+\eta, \zeta+\eta, m(\zeta+\eta)\right) d\zeta$$

or changing $\zeta$ with $s$ and considering assumption (iv)

$$m'_0(t) \geq \; g\big(t, m_0(t)\big) + \int_{t_0}^{t} \varphi\left(t, \zeta, m_0(\zeta)\right) d\zeta$$

$$= \; g\big(t, m_0(t)\big) + \int_{t_0}^{t} \varphi\big(t, s, m_0(s)\big)\, ds$$

for $t_0 \leq t \leq t_0 + T$.

Firstly, we will examine the outcome for the strict case, that is, $n(t) < m_0(t) = m(t + \eta)$ on $J_1$. Let us suppose that the first inequality in (ii) is strict. Now, assume that the conclusion of the theorem is incorrect, then there must exist a value $t_1$ such that.

$$n(t_1) = m_0(t_1), \;\; n(t) < m_0(t), \quad t_0 \le t < t_1 \qquad (3.2.1)$$

That means $n(t) \le m_0(t)$ on $J_1$ is false, and $t_1$ is greatest lower bound (GLB) of the set

$$A = \{t \in J_1 : n(t) > m_0(t)\}$$

Since $n(t) < m_0(t)$ for $t_0 \le t < t_1$ and $\varphi$ is monotone non-decreasing in third variable, we get $\varphi(t, s, n(s)) \le \varphi(t, s, m_0(s))$ for $t_0 \le s \le t_1$.

Now for $t_0 \le t < t_1$,

$$n(t) < m_0(t)$$

$$n(t) - n(t_1) < m_0(t) - m_0(t_1) \qquad (3.2.2)$$

divided equation (3.2.2) by $t - t_1$ yields

$$\frac{n(t) - n(t_1)}{t - t_1} > \frac{m_0(t) - m_0(t_1)}{t - t_1} \qquad (3.2.3)$$

now by taking limit for (3.2.3) we have

$$\frac{n(t) - n(t_1)}{t - t_1} \ge \frac{m_0(t) - m_0(t_1)}{t - t_1}$$

$$n'(t_1) \ge {m_0}'(t_1).$$

Recalling first inequalities in (ii) to be strict and nondecreasing property of $\varphi(t, s, u)$ in $u$ then we have

$$n'(t_1) \ge {m_0}'(t_1)$$

$$\ge g(t_1, m_0(t_1)) + \int_{t_0}^{t_1} \varphi\,(t_1, s, m_0(s))ds$$

$$\ge g(t_1, n(t_1)) + \int_{t_0}^{t_1} \varphi\,(t_1, s, n(s))ds$$

$$> n'(t_1)$$

which is contradiction.

Hence it follows that $n(t) < m_0(t)$ $on\ J_1$ .

To prove the claim of Lemma 3.2.1 for the non-strict case, we set $m_\epsilon(t) = m_0(t) + \epsilon e^{2L_1 t}$, where $\epsilon > 0$. It is obvious that $n(t_0) \leq m_0(t_0) < m_\epsilon(t_0)$. Taking derivatives of both sides and utilizing (ii), we get

$$m'_\epsilon(t) = m_0{}'(t) + 2L_1\epsilon e^{2L_1 t}$$

$$m'_\epsilon(t) \geq g\big(t, m_0(t)\big) + \int_{t_0}^{t} \varphi\,\big(t, s, m_0(s)\big)ds + 2L_1\epsilon e^{2L_1 t}$$

On account of $m_\epsilon(t) > m_0(t)$ on $J_1$ together with assumption (iii), we write

$$g(t, m_\epsilon) - g(t, m_0) \leq L_1(m_\epsilon - m_0)$$

$$g(t, m_0) \geq g(t, m_\epsilon) - L_1(m_\epsilon - m_0)$$

and

$$\varphi\big(t, s, m_\epsilon(s)\big) - \varphi\big(t, s, m_0(s)\big) \leq L_2(m_\epsilon - m_0)$$

$$\varphi\big(t, s, m_0(s)\big) \geq \varphi(t, s, m_\epsilon(s)) - L_2(m_\epsilon - m_0)$$

If we continue the process where we left off

$$m'_\epsilon(t) \geq g\big(t, m_\epsilon(t)\big) - L_1\big(m_\epsilon(t) - m_0(t)\big)$$

$$+ \int_{t_0}^{t} \varphi\,((t, s, m_\epsilon(s)) - L_2(m_\epsilon(s) - m_0(s))ds + 2L_1\epsilon e^{2L_1 t}$$

$$= g\big(t, m_\epsilon(t)\big) + \int_{t_0}^{t_1} \varphi\big(t, s, m_\epsilon(s)\big)\,ds - \left[L_1\epsilon e^{2L_1 t} + \int_{t_0}^{t_1} L_2\epsilon e^{2L_1 s}\,ds\right] + 2L_1\epsilon e^{2L_1 t}$$

$$= g\big(t, m_\epsilon(t)\big) + \int_{t_0}^{t_1} \varphi\big(t, s, m_\epsilon(s)\big)\,ds + L_1\epsilon e^{2L_1 t} - \frac{L_2\epsilon}{2L_1}[e^{2L_1 t} - e^{2L_1 t_0}]$$

$$= g\big(t, m_\epsilon(t)\big) + \int_{t_0}^{t_1} \varphi\big(t, s, m_\epsilon(s)\big)\,ds + \left[L_1 - \frac{L_2}{2L_1}\right]\epsilon e^{2L_1 t} + \frac{L_2\epsilon}{2}\,e^{2L_1 t_0}$$

$$= g(t, m_\epsilon(t)) + \int_{t_0}^{t} \varphi\,(t, s, m_\epsilon(s))ds + \left[\frac{2{L_1}^2 - L_2}{2L_1}\right]\epsilon e^{2L_1 t} + \frac{L_2\epsilon}{2} e^{2L_1 t_0}.$$

Since we know that $\left[\frac{2{L_1}^2 - L_2}{2L_1}\right]\epsilon e^{2L_1 t} + \frac{L_2\epsilon}{2} e^{2L_1 t_0} > 0$, we get

$$m'_\epsilon(t) > g(t, m_\epsilon(t)) + \int_{t_0}^{t} \varphi\,(t, s, m_\epsilon(s))ds.$$

From the reasons we discussed above for the strict case, it follows that

$$n(t) < m_\epsilon(t) \quad \text{ on } J_1.$$

Now taking limit as $\epsilon \to 0$ we find $n(t) \le m_0(t) = m(t + \eta)$ on $J_1$. ■

For the case $t_0 > \tau_0$, the lemma can be represented in a parallel way. The following lemma is given in this context.

**Lemma 3.2.2.** Let the conditions (i) – (iii) in Lemma 3.2.1 hold with $t_0 > \tau_0$. Moreover, we assume the following

**(iv)** $g(t, x)$ is non-increasing with respect to $t$ for each $x$ and also for any $\varepsilon > 0$,

$$\varphi(t + \varepsilon, s + \varepsilon, x) \le \varphi(t, s, x)$$

Then, the validity of the result of Theorem Lemma 3.2.1 remains unchanged.

**Proof:** The proof is similar to previous Lemma 3.2.1. Hence, we omit the details.

Let $\alpha_0 \in C^1[J_1, R]$ , $\beta_0 \in C^1[J_2, R]$ with $\alpha_0(t) \le \beta_0(t + \eta_1)$ on $J_1$. Let us define the following sets

$$\Lambda_1 = [(t, x):\ \alpha_0(t) \le x \le \beta_0(t + \eta_1), t \in J_1\,]$$

$$\Lambda_2 = [(t, s, x):\ \alpha_0(t) \le x \le \beta_0(t + \eta_1), t, s \in J_1\,]$$

Observe that each set is compact.

We are now in position to provide the main result.

**Theorem 3.2.1.** Assume that the following conditions listed below hold:

**(I)** $g \in C[J_3 \times R, R]$, $\varphi \in C[J_3 \times J_3 \times R, R]$ and $\varphi(t, s, x)$ is monotone non-decreasing in $x$ for each fixed $(t, s)$;

**(II)** $\alpha_0 \in C^1[J_1, R]$ , $\beta_0 \in C^1[J_2, R]$ such that $\alpha_0(t) \le \beta_0(t+\eta_1)$

$$\alpha'_0(t) \le g\big(t, \alpha_0(t)\big) + \int_{t_0}^{t} \varphi\left(t, s, \alpha_0(s)\right) ds, \qquad t \in J_1$$

$$\beta'_0(t) \ge g\left(t, \beta_0(t)\right) + \int_{\tau_0}^{t} \varphi\left(t, s, \beta_0(s)\right) ds, \qquad t \in J_2$$

with $\alpha_0(t_0) \le x(\sigma_0) \le \beta_0(\tau_0)$ where $t_0 < \sigma_0 < \tau_0$ and $\eta_1 = \tau_0 - t_0$;

**(III)** $g_x \in C[\Lambda_1, R]$ and $g_x(t,x)$ is nondecreasing in $x$ for each $t$ and satisfies for

$$g(t,x) \ge g(t,y) + g_x(t,y)(x-y) \ , x \ge y,$$

$$|g_x(t,x) - g_x(t,y)| \le N_1|x-y| \ N_1 > 0,$$

Furthermore, let $\varphi_x \in C[\Lambda_2, R]$ and be nondecreasing in $x$ for each $(t,s)$ with

$$\varphi(t,s,x) \ge \varphi(t,s,y) + \varphi_x(t,s,y)(x-y), \qquad x \ge y,$$

$$|\varphi_x(t,s,x) - \varphi_x(t,s,y)| \le N_2|x-y|, N_2 \ge 0,$$

with $2{N_1}^2 \ge N_2$;

**(IV)** $g(t,x)$ are non-decreasing with respect to $t$ and for any $\varepsilon > 0$,

$$\varphi(t+\varepsilon, s+\varepsilon, x) \ge \varphi(t,s,x).$$

Then, there exist monotone sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ that uniformly and quadratically converge to the unique solution of

$$x'(t) = g\big(t, x(t)\big) + \int_{\sigma_0}^{t} \varphi\left(t, s, x(s)\right) ds, \qquad t \in [\sigma_0, \sigma_0 + T] \tag{3.2.4}$$

with $x(\sigma_0) = x_0$.

**Proof**:

We set $\tilde{\alpha}_0(t) = \alpha_0(t)$ and $\tilde{\beta}_0(t) = \beta_0(t+\eta_1)$ where $\eta_1 = \tau_0 - t_0$. Then

$$\begin{aligned}\tilde{\alpha}'_0(t) &= \alpha'_0(t)\\ &\le g\big(t, \alpha_0(t)\big) + \varphi\big(t, s, \alpha_0(s)\big) ds\\ &\le g\big(t, \tilde{\alpha}_0(t)\big) + \int_{t_0}^{t} \varphi\left(t, s, \tilde{\alpha}_0(s)\right) ds\end{aligned}$$

where $\tilde{\alpha}_0(t_0) \leq x_0 \leq \tilde{\beta}_0(t_0)$.

This gives that $\tilde{\alpha}_0(t)$ is the lower solution of the problem (3.2.4). Meanwhile,

$$\tilde{\beta}'_0(t) = \beta'_0(t + \eta_1)$$

$$\geq g\big(t + \eta_1, \beta_0(t + \eta_1)\big) + \int_{\tau_0}^{t+\eta_1} \varphi\left(t + \eta_1, s, \beta_0(s)\right) ds$$

And by assumption **(IV)**, we have the ability to carry out a modification in the following manner

$$g(t + \eta_1, \beta_0(t + \eta_1)) \geq g(t, \beta_0(t + \eta_1))$$

and we have $\tilde{\beta}_0(t) = \beta_0(t + \eta_1)$ so,

$$\tilde{\beta}'_0(t) \geq g\left(t, \tilde{\beta}_0(t)\right) + \int_{\tau_0}^{t+\eta_1} \varphi\left(t + \eta_1, s, \beta_0(s)\right) ds$$

modifying the variable $s = \zeta + \eta_1 = \zeta + (\tau_0 - t_0)$ and $ds = d\zeta$, Then

$$\tilde{\beta}_0'(t) \geq g\left(t, \tilde{\beta}_0(t)\right) + \int_{t_0}^{t} \varphi\left(t + \eta_1, \zeta + \eta_1, \beta_0(\zeta + \eta_1)\right) d\zeta$$

again, by assumption **(IV)**

$$\tilde{\beta}_0'(t) \geq g\left(t, \tilde{\beta}_0(t)\right) + \int_{t_0}^{t} \varphi\left(t, \zeta, \tilde{\beta}_0(\zeta)\right) d\zeta$$

exchanging the variable again

$$\tilde{\beta}'_0(t) \geq g\left(t, \tilde{\beta}_0(t)\right) + \int_{t_0}^{t} \varphi\left(t, s, \tilde{\beta}_0(s)\right) ds$$

which shows that $\tilde{\beta}_0(t)$ is an upper solution of the problem (3.2.4).

We know from assumption **(III)** that for $u \geq v$

$$g(t, u) \geq g(t, v) + g_x(t, v)(u - v)$$

$$\varphi(t, s, u) \geq \varphi(t, s, v) + \varphi_x(t, s, v)(u - v)$$

and since $g_x$ and $\varphi_x$ are continuous on the compact sets $\Lambda_1$ and $\Lambda_2$ respectively, we conclude that

$$g(t,u) - g(t,v) \leq L(u-v)$$

and

$$\varphi(t,s,u) - \varphi(t,s,v) \leq N(u-v) \tag{3.2.5}$$

for $t \in J_1, (t,s) \in J_1 \times J_1,\ L \geq 0, N > 0.$

In view of both $g$ an $\varphi$ d satisfy the one-sided Lipschitz condition in the problem, we can derive through the successive utilization of Lemma 3.2.1, that the IVP (3.1.1) has a unique solution.

Let $\tilde{\alpha}_1, \tilde{\beta}_1$ be the solution of the IVPs:

$$\tilde{\alpha}'_1(t) = g\big(t+\eta_2, \tilde{\alpha}_0(t)\big) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\big(\tilde{\alpha}_1(t) - \tilde{\alpha}_0(t)\big)$$

$$+ \int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\alpha}_1(s) - \tilde{\alpha}_0(s))\big]\, ds$$

$$\tilde{\alpha}_1(t_0) = x_0, \tag{3.2.6}$$

And

$$\tilde{\beta}'_1(t) = g\left(t+\eta_2, \tilde{\beta}_0(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\left(\tilde{\beta}_1(t) - \tilde{\beta}_0(t)\right)$$

$$+ \int_{t_0}^{t} \left[\varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_0(s)\right) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\beta}_1(s) - \tilde{\beta}_0(s))\right] ds$$

$$\tilde{\beta}_1(t_0) = x_0 \tag{3.2.7}$$

where $\eta_2 = \sigma_0 - t_0$.

For problems possesses linear equations, it follows immediately that both (3.2.6) and (3.2.7) have unique solutions. Our first goal is to demonstrate that

$$\tilde{\alpha}_0(t) \leq \tilde{\alpha}_1(t) \leq \tilde{\beta}_1(t) \leq \tilde{\beta}_0(t) \tag{3.2.8}$$

on $t \in [t_0, t_0 + T]$.

To carry out that, we set $p(t) = \tilde{\alpha}_0(t) - \tilde{\alpha}_1(t)$. Clearly $p(t_0) \leq 0$, then $p'(t) = \tilde{\alpha}'_0(t) - \tilde{\alpha}'_1(t)$

$$\leq g\big(t+\eta_2, \tilde{\alpha}_0(t)\big) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\right)ds - g\big(t+\eta_2, \tilde{\alpha}_0(t)\big)$$

$$- g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\big(\tilde{\alpha}_1(t) - \tilde{\alpha}_0(t)\big)$$

$$- \int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\alpha}_1(s)$$

$$- \tilde{\alpha}_0(s))\big]\, ds$$

$$p'(t) \leq g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)p(t) + \int_{t_0}^{t} \big[\varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)\big]\, p(s)ds$$

By employing Lemma 3.2.1 yields $p(t) \leq 0, t \geq t_0$, which in turn proves $\tilde{\alpha}_0(t) \leq \tilde{\alpha}_1(t), t \in J_1$. Similarly, we set

$q(t) = \tilde{\beta}_1(t) - \tilde{\beta}_0(t)$. Obviously, $q(t_0) \geq 0$ and

$$q'(t) = \tilde{\beta}'_1(t) - \tilde{\beta}'_0(t),$$

$$\leq g\left(t+\eta_2, \tilde{\beta}_0(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\left(\tilde{\beta}_1(t) - \tilde{\beta}_0(t)\right)$$

$$+ \int_{t_0}^{t} \left[\varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_0(s)\right) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\beta}_1(s)\right.$$

$$\left. - \tilde{\beta}_0(s))\right] ds - g\left(t+\eta_2, \tilde{\beta}_0(t)\right) - \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_0(s)\right) ds$$

$$q'(t) \leq g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)q(t) + \int_{t_0}^{t} \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)q(s)ds$$

By Lemma 3.2.1 we reach that $q(t) \geq 0,\ t \geq t_0$, which in turn reveals that

$$\tilde{\beta}_1(t) \leq \tilde{\beta}_0(t), t \in J_1.$$

It is remains to show that $\tilde{\alpha}_1(t) \leq \tilde{\beta}_1(t),\ t \in J_1$. By the fact that $\tilde{\alpha}_0(t) \leq \tilde{\alpha}_1(t)$ and $\tilde{\beta}_1(t) \leq \tilde{\beta}_0(t)$, and using the inequalities in (III) implies

$$\tilde{\alpha}'_1(t) = g\big(t+\eta_2, \tilde{\alpha}_0(t)\big) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\big(\tilde{\alpha}_1(t) - \tilde{\alpha}_0(t)\big)$$
$$+ \int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\alpha}_1(s)$$
$$- \tilde{\alpha}_0(s))\big]\, ds$$

$$\tilde{\alpha}'_1(t) \le g\big(t+\eta_2, \tilde{\alpha}_1(t)\big) + \int_{t_0}^{t} \varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_1(s)\big)\, ds$$

and

$$\tilde{\beta}'_1(t) = g\left(t+\eta_2, \tilde{\beta}_0(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\left(\tilde{\beta}_1(t) - \tilde{\beta}_0(t)\right)$$
$$+ \int_{t_0}^{t} \left[\varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_0(s)\right) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\beta}_1(s)\right.$$
$$\left.- \tilde{\beta}_0(s))\right]\, ds$$

$$\tilde{\beta}'_1(t) \ge g\left(t+\eta_2, \tilde{\beta}_1(t)\right) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_1(s)\right) ds$$

also note that $\tilde{\alpha}_1(t_0) = \tilde{\beta}_1(t_0) = x_0$. Simply, immediate application of Lemma 3.2.1. yields $\tilde{\alpha}_1(t) \le \tilde{\beta}_1(t), t \in J_1$, which justifies that the inequality (3.2.8) is correct.

Assume that for some $n > 1$,

$$\tilde{\alpha}'_n(t) \le g\big(t+\eta_2, \tilde{\alpha}_n(t)\big) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\right) ds$$

$$\tilde{\beta}'_n(t) \ge g\left(t+\eta_2, \tilde{\beta}_n(t)\right) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_n(s)\right) ds$$

and

$$\tilde{\alpha}_0(t) \le \tilde{\alpha}_n(t) \le \tilde{\beta}_n(t) \le \tilde{\beta}_0(t), \qquad t \in J_1.$$

Now for $n+1$, we have to prove that

$$\tilde{\alpha}_n(t) \le \tilde{\alpha}_{n+1}(t) \le \tilde{\beta}_{n+1}(t) \le \tilde{\beta}_n(t), \qquad t \in J_1.$$

where $\tilde{\alpha}_{n+1}(t)$ and $\tilde{\beta}_{n+1}(t)$ are the solutions of the following relations

$$\tilde{\alpha}'_{n+1}(t) = g\big(t+\eta_2, \tilde{\alpha}_n(t)\big) + g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\big(\tilde{\alpha}_{n+1}(t) - \tilde{\alpha}_n(t)\big) + \int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)(\tilde{\alpha}_{n+1}(s) - \tilde{\alpha}_n(s))\big]\, ds$$

$$\tilde{\alpha}_{n+1}(t_0) = x_0$$

$$\tilde{\beta}'_{n+1}(t) = g\left(t+\eta_2, \tilde{\beta}_n(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\left(\tilde{\beta}_{n+1}(t) - \tilde{\beta}_n(t)\right) + \int_{t_0}^{t} \left[\varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_n(s)\right) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)(\tilde{\beta}_{n+1}(s) - \tilde{\beta}_n(s))\right] ds$$

$$\tilde{\beta}_{n+1}(t_0) = x_0.$$

In order to accomplish this, we establish the function $p(t) = \tilde{\alpha}_n(t) - \tilde{\alpha}_{n+1}(t)$ so that $p(t_0) = 0$.

If we proceed in a similar way as we discussed above

$$p'(t) = \tilde{\alpha}'_n(t) - \tilde{\alpha}'_{n+1}(t),$$

$$\leq g\big(t+\eta_2, \tilde{\alpha}_n(t)\big) + \int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)\big]\, ds - g\big(t+\eta_2, \tilde{\alpha}_n(t)\big) - g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\big(\tilde{\alpha}_{n+1}(t) - \tilde{\alpha}_n(t)\big) - \int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)(\tilde{\alpha}_{n+1}(s) - \tilde{\alpha}_n(s))\big]\, ds$$

$$p'(t) \leq g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\big(p(t)\big) + \int_{t_0}^{t} \big[\varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)\big(p(s)\big)\big]\, ds$$

Employing Lemma 3.2.1, we get $p(t) \leq 0$, implying that $\tilde{\alpha}_n(t) \leq \tilde{\alpha}_{n+1}(t)$ , $t \in J_1$.

Analogously, building $q(t) = \tilde{\beta}_{n+1}(t) - \tilde{\beta}_n(t)$ yields $q(t_0) = 0$, and

$$q'(t) = \widetilde{\beta}'_{n+1}(t) - \widetilde{\beta}'_n(t)$$

$$\leq g\left(t+\eta_2,\tilde{\beta}_n(t)\right)+g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)\left(\tilde{\beta}_{n+1}(t)-\tilde{\beta}_n(t)\right)$$
$$+\int_{t_0}^{t}\left[\varphi\left(t+\eta_2,s+\eta_2,\tilde{\beta}_n(s)\right)+\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)(\tilde{\beta}_{n+1}(s)\right.$$
$$\left.-\tilde{\beta}_n(s))\right]ds-\ g\left(t+\eta_2,\tilde{\beta}_n(t)\right)-\int_{t_0}^{t}\varphi\left(t+\eta_2,s+\eta_2,\tilde{\beta}_n(s)\right)ds$$

$$q'(t)\leq g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)q(t)+\int_{t_0}^{t}\big[\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)q(s)\big]\,ds$$

Since $q(t_0)=0$ , by Lemma 3.2.1 we get $q(t)\geq 0$, and this implies $\tilde{\beta}_{n+1}(t)\leq\tilde{\beta}_n(t)$, $t\in J_1$.

By making use of the inequalities in (3.2.5), we acquire

$$\tilde{\alpha}'_{n+1}(t)=g\big(t+\eta_2,\tilde{\alpha}_n(t)\big)+g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)\big(\tilde{\alpha}_{n+1}(t)-\tilde{\alpha}_n(t)\big)$$
$$+\int_{t_0}^{t}\big[\varphi\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)+\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)(\tilde{\alpha}_{n+1}(s)$$
$$-\tilde{\alpha}_n(s))\big]\,ds$$

$$\tilde{\alpha}'_{n+1}(t)\leq g\big(t+\eta_2,\tilde{\alpha}_{n+1}(t)\big)+\int_{t_0}^{t}\big[\varphi\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_{n+1}(s)\big)\big]\,ds$$

and

$$\tilde{\beta}'_{n+1}(t)$$
$$=g\left(t+\eta_2,\tilde{\beta}_n(t)\right)+g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)\left(\tilde{\beta}_{n+1}(t)-\tilde{\beta}_n(t)\right)$$
$$+\int_{t_0}^{t}\left[\varphi\left(t+\eta_2,s+\eta_2,\tilde{\beta}_n(s)\right)+\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)(\tilde{\beta}_{n+1}(s)\right.$$
$$\left.-\tilde{\beta}_n(s))\right]ds$$

$$\tilde{\beta}'_{n+1}(t)\geq g\left(t+\eta_2,\tilde{\beta}_{n+1}(t)\right)+\int_{t_0}^{t}\varphi\left(t+\eta_2,s+\eta_2,\tilde{\beta}_{n+1}(s)\right)ds$$

Also note that $\tilde{\alpha}_{n+1}(t_0)=\tilde{\beta}_{n+1}(t_0)=x_0$. The direct application of Lemma 3.2.1 yields $\tilde{\alpha}_{n+1}(t)\leq\tilde{\beta}_{n+1}(t),\ t\in J_1$.

Now we have shown the inequalities $\tilde{\alpha}_n(t) \leq \tilde{\alpha}_{n+1}(t)$, $\tilde{\beta}_{n+1}(t) \leq \tilde{\beta}_n(t)$ and $\tilde{\alpha}_{n+1}(t) \leq \tilde{\beta}_{n+1}(t)$. Combining them, we get

$$\tilde{\alpha}_n(t) \leq \tilde{\alpha}_{n+1}(t) \leq \tilde{\beta}_{n+1}(t) \leq \tilde{\beta}_n(t)\,, \quad t \in J_1.$$

Hence, by induction, for all $n$ we arrive at

$$\tilde{\alpha}_0(t) \leq \tilde{\alpha}_1(t) \leq \ldots . \leq \tilde{\alpha}_n(t) \leq \tilde{\beta}_n(t) \leq \ldots . \leq \tilde{\beta}_1(t) \leq \tilde{\beta}_0(t), \quad t \in J_1.$$

Now by the standard arguments [51], we are entitled to demonstrate that the sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ converge monotonically and uniformly to the unique solution $\tilde{x}(t)$ on $J_1$ of the problem

$$\tilde{x}'(t) = g\big(t+\eta_2, \tilde{x}(t)\big) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{x}(s)\right) ds, \qquad t \in [t_0, t_0+T]$$

$$\tilde{x}(t_0) = x_0,$$

If we change the variables in the equation as $x(t+\eta_2) = \tilde{x}(t)$ and $\sigma = t + \eta_2$, we can write down

$$x'(\sigma) = g\big(\sigma, x(\sigma)\big) + \int_{\sigma_0}^{\sigma} \varphi\left(\sigma, s, x(\xi)\right) d\xi, \qquad \sigma \in [\sigma_0, \sigma_0+T].$$

with $x(\sigma_0) = x_0$.

It should be noted that we make the additional modification, setting $\xi = s + \eta_2$.

Actually, the last equation reduces to the problem (3.2.4), i.e.,

$$x'(t) = g\big(t, x(t)\big) + \int_{\sigma_0}^{t} \varphi\left(t, s, x(s)\right) ds, \qquad t \in [\sigma_0, \sigma_0+T]$$

$$x(\sigma_0) = x_0$$

Now we will demonstrate that the convergence is quadratic.

To achieve this goal, allow $p_{n+1}(t) = \tilde{x}(t) - \tilde{\alpha}_{n+1}(t)$, clearly $p_{n+1}(t_0) = 0$.

Then

$$p'_{n+1}(t) = \tilde{x}'(t) - \tilde{\alpha}'_{n+1}(t),$$

$$= g\big(t+\eta_2,\tilde{x}(t)\big) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{x}(s)\right)ds - g\big(t+\eta_2,\tilde{\alpha}_n(t)\big)$$

$$-g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)\big(\tilde{\alpha}_{n+1}(t)-\tilde{\alpha}_n(t)\big)$$

$$-\int_{t_0}^{t}\big[\varphi\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)+\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)(\tilde{\alpha}_{n+1}(s)$$

$$-\tilde{\alpha}_n(s))\big]\,ds$$

$$= g_x\big(t+\eta_2,\xi_1(t)\big)\big(\tilde{x}(t)-\tilde{\alpha}_n(t)\big)$$

$$+\int_{t_0}^{t}\big[\varphi_x\big(t+\eta_2,s+\eta_2,\xi_2(s)\big)(\tilde{x}(s)-\tilde{\alpha}_n(s))\big]\,ds$$

$$-g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)\big(\tilde{\alpha}_{n+1}(t)-\tilde{x}(t)+\tilde{x}(t)-\tilde{\alpha}_n(t)\big)$$

$$-\int_{t_0}^{t}\big[\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)(\tilde{\alpha}_{n+1}(s)-\tilde{x}(s)+\tilde{x}(s)-\tilde{\alpha}_n(s))\big]\,ds$$

where we have used the mean value theorem with the fact $\tilde{\alpha}_n \le \xi_1, \xi_2 \le \tilde{x}$. Thus

$$p'_{n+1}(t) = g_x\big(t+\eta_2,\xi_1(t)\big)p_n(t) - g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)p_n(t)$$

$$+ g_x\big(t+\eta_2,\tilde{\alpha}_n(t)\big)\big(p_{n+1}(t)\big)$$

$$+\int_{t_0}^{t}\big[\varphi_x\big(t+\eta_2,s+\eta_2,\xi_2(s)\big)\big(p_n(s)\big)\big]\,ds$$

$$-\int_{t_0}^{t}\big[\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)\big(p_n(s)\big)\big]\,ds$$

$$+\int_{t_0}^{t}\big[\varphi_x\big(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\big)(p_{n+1}(s))\big]\,ds$$

$$\le N_1\left(\xi_1(t)-\tilde{\alpha}_n(t)\right)p_n(t) + g_x\left(t+\eta_2,\tilde{\alpha}_n(t)\right)p_{n+1}(t)$$

$$+\int_{t_0}^{t} N_2\left(\xi_2(s)-\tilde{\alpha}_n(s)\right)p_n(s)ds$$

$$+\int_{t_0}^{t}\left[\varphi_x\left(t+\eta_2,s+\eta_2,\tilde{\alpha}_n(s)\right)(p_{n+1}(s))\right]ds$$

$$p'_{n+1}(t) \le N_1 p^2{}_n(t) + Lp_{n+1}(t) + N_2\int_{t_0}^{t} p^2{}_n\,(s)ds + L_1\int_{t_0}^{t} p_{n+1}\,(s)ds$$

where $g_x\big(t+\eta_2, \tilde{x}(t)\big) \le L,\ \ \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{x}(s)\big) \le L_1$ .

By Lemma 3.2.1, we get $p_{n+1}(t) \le r(t),\ t \in J_1$. Here, $r(t)$ corresponds to the solution of the linear integro-differential equation:

$$r'(t) = Lr(t) + L_1 \int_{t_0}^{t} r(s)ds + F(t), \qquad p_{n+1}(t_0) \le r(t_0), \tag{3.2.9}$$

where

$$F(t) = N_1 {p^2}_n(t) + N_2 \int_{t_0}^{t} {p^2}_n(s)ds.$$

We note that

$$F(t) \le (N_1 + N_2 T) \max_{t \in J_1} {p^2}_n(t)\,, \quad t \in J_1 \tag{3.2.10}$$

To find an estimate for $r(t)$ given by (3.2.9), Let us describe

$$z(t) = \int_{t_0}^{t} r(s)\, ds \tag{3.2.11}$$

Then, the equation (3.2.9) becomes

$$z''(t) - Lz'(t) - L_1 z(t) = F(t);\ \ z'(t_0) = 0,\ \ z(t_0) = 0$$

This equation is a second-order nonhomogeneous linear differential equation so that we can solve it using the variation of parameters method.

To find general solution we need complementary solution $z_c(t)$ and a particular solution $z_p(t)$.

$z_c(t)$ is the general solution of homogeneous differential equation

$$z''(t) - Lz'(t) - L_1 z(t) = 0.$$

The corresponding characteristic equation is

$$\lambda^2 - L\lambda - L_1 = 0.$$

By using quadratic formula, we find roots of characteristic equation as

$$\lambda_{1,2} = \frac{L \pm \sqrt{L^2 + 4L_1}}{2}$$

where $L^2 + 4L_1 = \Delta > 0$, that indicates the real and different $\lambda_1 \neq \lambda_2$ roots. Then

$$z_c(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},$$

where $C_1$ and $C_2$ are arbitrary constants.

For particular solution $z_p(t)$, $z_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where $y_1(t) = e^{\lambda_1 t}$, $y_2(t) = e^{\lambda_2 t}$ and $v_1(t), v_2(t)$ are unknown function of $t$ which still to be determined.

To find $v_1(t), v_2(t)$ first solve the following linear systems of equations simultaneously for $v'_1(t), v'_2(t)$,

$$v'_1(t)y_1(t) + v'_2(t)y_2(t) = 0$$

$$v'_1(t)y'_1(t) + v'_2(t)y'_2(t) = F(t) \tag{3.2.12}$$

Since $y_1(t), y_2(t)$ are two linear independent solution of the same equation, their Wronskian is not zero that means the system (3.2.12) has a nonzero determinant and can be solved uniquely for $v'_1(t), v'_2(t)$.

$$v'_1(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ F(t) & y'_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 0 & e^{\lambda_2 t} \\ F(t) & \lambda_2 e^{\lambda_2 t} \end{vmatrix}}{\begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_1 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_1 t} \end{vmatrix}}$$

$$= \frac{-e^{\lambda_2 t}F(t)}{(\lambda_2 - \lambda_1)e^{(\lambda_1 + \lambda_2)t}}$$

$$v'_1(t) = \frac{-F(t)}{(\lambda_2 - \lambda_1)e^{\lambda_1 t}} \ . \tag{3.2.13}$$

$$v'_2(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ {y'}_1(t) & F(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ {y'}_1(t) & {y'}_2(t) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} e^{\lambda_1 t} & 0 \\ \lambda_1 e^{\lambda_1 t} & F(t) \end{vmatrix}}{\begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_1 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_1 t} \end{vmatrix}}$$

$$= \frac{e^{\lambda_1 t} F(t)}{(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)t}}$$

$$v'_2(t) = \frac{F(t)}{(\lambda_2 - \lambda_1) e^{\lambda_2 t}}\,. \tag{3.2.14}$$

Integrating both sides of equations (3.2.13) and (3.2.14) to compute $v_1(t), v_2(t)$

$$v_1(t) = \int_{t_0}^{t} \frac{-F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_1 s}} ds,$$

$$v_2(t) = \int_{t_0}^{t} \frac{F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_2 s}} ds$$

$$z_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$$

$$= y_1(t) \int_{t_0}^{t} \frac{-F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_1 s}} ds + y_2(t) \int_{t_0}^{t} \frac{F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_2 s}} ds$$

so, the general solution is

$$z(t) = z_c(t) + z_p(t)$$

Substituting $t_0$

$$z(t_0) = z_c(t_0) + z_p(t_0)$$

$z(t_0) = C_1 e^{\lambda_1 t_0} + C_2 e^{\lambda_2 t_0} + 0$, where $z_p(t_0)$ is equal to zero, that means $z(t_0) = z_c(t_0)$. Recalling that the complementary solution takes zero at $t = t_0$, that is mean $z_c(t_0) = 0$, then

$$C_1 e^{\lambda_1 t_0} + C_2 e^{\lambda_2 t_0} = 0 \tag{3.2.15}$$

also

$$z'(t_0) = z'_c(t_0) + z'_p(t_0)$$

$$z(t_0) = \lambda_1 C_1 e^{\lambda_1 t_0} + \lambda_2 C_2 e^{\lambda_2 t_0} + 0$$

$$\lambda_1 C_1 e^{\lambda_1 t_0} + \lambda_2 C_2 e^{\lambda_2 t_0} = 0 \tag{3.2.16}$$

From (3.2.15) and (3.2.16), we get

$$C_1 e^{\lambda_1 t_0} + C_2 e^{\lambda_2 t_0} = 0$$

$$\lambda_1 C_1 e^{\lambda_1 t_0} + \lambda_2 C_2 e^{\lambda_2 t_0} = 0$$

In equation (3.2.15) multiply both sides by ($e^{-\lambda_1 t_0}$)

we get

$$C_1 = -C_2 e^{(\lambda_2 - \lambda_1) t_0}$$

Now substitute ($C_1$) into equation (3.2.17)

$$\lambda_2 C_2 e^{\lambda_2 t_0} - \lambda_1 C_2 e^{\lambda_2 t_0} = 0$$

$$(\lambda_2 C_2 - \lambda_1 C_2) e^{\lambda_2 t_0} = 0$$

$$C_2(\lambda_2 - \lambda_1) = 0,$$

$\because \lambda_2 \neq \lambda$, then $C_1 = 0$, $C_2 = 0$. This result in $z_c(t) = 0$.

Then

$$z(t) = y_1(t) \int_{t_0}^{t} \frac{-F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_1 s}} ds + y_2(t) \int_{t_0}^{t} \frac{F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_2 s}} ds$$

Taking derivative of both sides with applying Leibniz rule, we get

$$z'(t) = e^{\lambda_1 t} \left( \frac{-F(t)}{(\lambda_2 - \lambda_1) e^{\lambda_1 t}} \right) + \lambda_1 e^{\lambda_1 t} \int_{t_0}^{t} \frac{-F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_1 s}} ds + e^{\lambda_2 t} \left( \frac{F(t)}{(\lambda_2 - \lambda_1) e^{\lambda_2 t}} \right) + \lambda_2 e^{\lambda_2 t} \int_{t_0}^{t} \frac{F(s)}{(\lambda_2 - \lambda_1) e^{\lambda_2 s}} ds$$

$$z'(t) = \lambda_1 e^{\lambda_1 t} \int_{t_0}^{t} \frac{-F(s)}{(\lambda_2 - \lambda_1)e^{\lambda_1 s}} ds + \lambda_2 e^{\lambda_2 t} \int_{t_0}^{t} \frac{F(s)}{(\lambda_2 - \lambda_1)e^{\lambda_2 s}} ds$$

where $\lambda_1 = \frac{L - \sqrt{L^2 + 4L_1}}{2}$ , $\lambda_2 = \frac{L + \sqrt{L^2 + 4L_1}}{2}$

so, $\lambda_2 - \lambda_1 = \sqrt{L^2 + 4L_1}$

$$z'(t) = \frac{\lambda_1 e^{\lambda_1 t}}{\sqrt{L^2 + 4L_1}} \int_{t_0}^{t} - F(s)\left(e^{-\lambda_1 s}\right) ds + \frac{\lambda_2 e^{\lambda_2 t}}{\sqrt{L^2 + 4L_1}} \int_{t_0}^{t} F\,(s)\left(e^{-\lambda_2 s}\right) ds$$

By maximizing the function, we can take $F(s)$ out of the integral

$$z'(t) \leq \frac{-\lambda_1}{\sqrt{L^2 + 4L_1}} \max_{t \in J_1} F(t) \int_{t_0}^{t} \left(e^{(t-s)\lambda_1}\right) ds + \frac{\lambda_2}{\sqrt{L^2 + 4L_1}} \max_{t \in J_1} F(t) \int_{t_0}^{t} \left(e^{(t-s)\lambda_2}\right) ds$$

$$= \frac{\max_{t \in J_1} F(t)}{\sqrt{L^2 + 4L_1}} \left( \left[ \left(e^{(t-t)\lambda_1}\right) - \left(e^{(t-t_0)\lambda_1}\right) \right] - \left[ \left(e^{(t-t)\lambda_2}\right) - \left(e^{(t-t_0)\lambda_2}\right) \right] \right)$$

$$= \frac{\max_{t \in J_1} F(t)}{\sqrt{L^2 + 4L_1}} \left[ 1 - \left(e^{(t-t_0)\lambda_1}\right) - 1 + \left(e^{(t-t_0)\lambda_2}\right) \right]$$

$$= \frac{\max_{t \in J_1} F(t)}{\sqrt{L^2 + 4L_1}} \left[ \left(e^{(t-t_0)\lambda_2}\right) - \left(e^{(t-t_0)\lambda_1}\right) \right]$$

$$\leq \frac{\max_{t \in J_1} F(t)}{\sqrt{L^2 + 4L_1}} \left[ \left(e^{(t-t_0)\lambda_2}\right) + \left(e^{(t-t_0)\lambda_1}\right) \right]$$

where $\lambda_1 \leq L,\ \lambda_2 \leq L$ and $t - t_0 \leq T$ so that

$$z'(t) \leq \frac{\max_{t \in J_1} F(t)}{\sqrt{L^2 + 4L_1}} \left(2e^{LT}\right)$$

where $z'(t) = r(t)$, and $r(t) = p_{n+1}(t)$.

Hence

$$\max_{t \in J_1} p_{n+1}(t) \leq \frac{\left(2e^{LT}\right)}{\sqrt{L^2 + 4L_1}} (N_1 + N_2 T) \max_{t \in J_1} {p^2}_n(t) \ , \qquad t \in J_1.$$

This reveals that $\{\tilde{\alpha}_n(t)\}$ converges quadratically to $\tilde{x}(t)$.

We still need to demonstrate that $\{\tilde{\beta}_n(t)\}$ converges with quadratic convergence to $\tilde{x}(t)$,

let $q_{n+1}(t) = \tilde{\beta}_{n+1}(t) - \tilde{x}(t)$, clearly $q_{n+1}(t_0) = 0$, and $q_{n+1}(t) \geq 0$ , $t \in J_1$.

Then

$$q'_{n+1}(t) = \tilde{\beta}'_{n+1}(t) - \tilde{x}'(t)$$

$$\begin{aligned}
&= g\left(t+\eta_2, \tilde{\beta}_n(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\left(\tilde{\beta}_{n+1}(t) - \tilde{\beta}_n(t)\right) \\
&\quad + \int_{t_0}^{t} \Big[\varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_n(s)\right) \\
&\quad + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)\left(\tilde{\beta}_{n+1}(s) - \tilde{\beta}_n(s)\right)\Big] ds - g\big(t+\eta_2, \tilde{x}(t)\big) \\
&\quad - \int_{t_0}^{t} \varphi\,(t+\eta_2, s+\eta_2, \tilde{x}(s)) ds \\
&= g_x\big(t+\eta_2, \xi_1(t)\big)\left(\tilde{\beta}_n(t) - \tilde{x}(t)\right) \\
&\quad + g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\left(\tilde{\beta}_{n+1}(t) - \tilde{x}(t) + \tilde{x}(t) - \tilde{\beta}_n(t)\right) \\
&\quad + \int_{t_0}^{t} \Big[\varphi_x\big(t+\eta_2, s+\eta_2, \xi_2(s)\big)\left(\tilde{\beta}_n(s) - \tilde{x}(s)\right)\Big] ds \\
&\quad + \int_{t_0}^{t} \Big[\varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)\left(\tilde{\beta}_{n+1}(s) - \tilde{x}(s) + \tilde{x}(s) - \tilde{\beta}_n(s)\right)\Big] ds
\end{aligned}$$

by mean value theorem with $\tilde{x} \leq \xi_1, \xi_2 \leq \tilde{\beta}$. Thus

$$\begin{aligned}
q'_{n+1}(t) &= g_x\big(t+\eta_2, \xi_1(t)\big)\left(\tilde{\beta}_n(t) - \tilde{x}(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\left(\tilde{\beta}_{n+1}(t) - \tilde{x}(t)\right) \\
&\quad - g_x\big(t+\eta_2, \tilde{\alpha}_n(t)\big)\left(\tilde{\beta}_n(t) - \tilde{x}(t)\right) \\
&\quad + \int_{t_0}^{t} \Big[\varphi_x\big(t+\eta_2, s+\eta_2, \xi_2(s)\big)\left(\tilde{\beta}_n(s) - \tilde{x}(s)\right)\Big] ds \\
&\quad + \int_{t_0}^{t} \Big[\varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)\left(\tilde{\beta}_{n+1}(s) - \tilde{x}(s)\right)\Big] ds \\
&\quad - \int_{t_0}^{t} \Big[\varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_n(s)\big)\left(\tilde{\beta}_n(s) - \tilde{x}(s)\right)\Big] ds
\end{aligned}$$

$$= N_1\big(\xi_1(t) - \tilde{\alpha}_n(t)\big)q_n(t) + g_x\big(t + \eta_2, \tilde{\alpha}_n(t)\big)q_{n+1}(t)$$

$$+ \int_{t_0}^{t} N_2\left(\xi_2(s) - \tilde{\alpha}_n(s)\right)q_n(s)ds$$

$$+ \int_{t_0}^{t} \left[\varphi_x\big(t + \eta_2, s + \eta_2, \tilde{\alpha}_n(s)\big)q_{n+1}(s)\right] ds$$

$$q'_{n+1}(t) \leq N_1\left(\tilde{\beta}_n(t) - \tilde{\alpha}_n(t)\right)q_n(t) + g_x\big(t + \eta_2, \tilde{\alpha}_n(t)\big)q_{n+1}(t)$$

$$+ \int_{t_0}^{t} N_2\left(\tilde{\beta}_n(s) - \tilde{\alpha}_n(s)\right)q_n(s)ds$$

$$+ \int_{t_0}^{t} \left[K_x\big(t + \eta_2, s + \eta_2, \tilde{\alpha}_n(s)\big)q_{n+1}(s)\right] ds$$

$$\leq N_1\left(\tilde{\beta}_n(t) - \tilde{x}(t) + \tilde{x}(t) - \tilde{\alpha}_n(t)\right)q_n(t) + g_x\left(t + \eta_2, \tilde{\alpha}_n(t)\right)q_{n+1}(t)$$

$$+ \int_{t_0}^{t} N_2\left(\tilde{\beta}_n(s) - \tilde{x}(s) + \tilde{x}(s) - \tilde{\alpha}_n(s)\right)q_n(s)ds$$

$$+ \int_{t_0}^{t} \left[\varphi_x\left(t + \eta_2, s + \eta_2, \tilde{\alpha}_n(s)\right)q_{n+1}(s)\right] ds$$

$$= N_1\big(q_n{}^2(t) + p_n(t)q_n(t)\big) + g_x\big(t + \eta_2, \tilde{\alpha}_n(t)\big)q_{n+1}(t)$$

$$+ \int_{t_0}^{t} N_2\left(q_n{}^2(s) + p_n(s)q_n(s)\right)ds$$

$$+ \int_{t_0}^{t} \left[\varphi_x\big(t + \eta_2, s + \eta_2, \tilde{\alpha}_n(s)\big)q_{n+1}(s)\right] ds$$

$$\leq N_1\left(\frac{3}{2}q_n{}^2(t) + \frac{p_n{}^2(t)}{2}\right) + g_x\left(t + \eta_2, \tilde{\alpha}_n(t)\right)q_{n+1}(t) + \int_{t_0}^{t} N_2\left(\frac{3}{2}q_n{}^2(s) + \frac{p_n{}^2(s)}{2}\right)ds$$

$$+ \int_{t_0}^{t} \varphi_x\left(t + \eta_2, s + \eta_2, \tilde{\alpha}_n(s)\right)q_{n+1}(s)\, ds$$

$$\leq N_1\left(\frac{3}{2}q_n{}^2(t) + \frac{p_n{}^2(t)}{2}\right) + Lq_{n+1}(t) + \int_{t_0}^{t} N_2\left(\frac{3}{2}q_n{}^2(s) + \frac{p_n{}^2(s)}{2}\right)ds$$

$$+ \int_{t_0}^{t} L_1 q_{n+1}(s)\, ds$$

where $g_x\big(t + \eta_2, \tilde{x}(t)\big) \leq L$ , $\varphi_x\big(t + \eta_2, s + \eta_2, \tilde{x}(s)\big) \leq L_1$ .

By Lemma 3.2.1, we have that $q_{n+1}(t) \leq r(t),\ t \in J_1$. Here $r(t)$ is the solution of the related linear integro-differential equation.

$$r'(t) = Lr(t) + L_1 \int_{t_0}^{t} r\,(s)ds + G(t), \qquad q_{n+1}(t_0) = r(t_0) \Rightarrow r(t_0) = 0 \qquad (3.2.19)$$

where

$$G(t) = N_1\left(\frac{3}{2}{q_n}^2(t) + \frac{{p_n}^2(t)}{2}\right) + \int_{t_0}^{t} N_2\left(\frac{3}{2}{q_n}^2(s) + \frac{{p_n}^2(s)}{2}\right)ds$$

we note that

$$G(t) \leq (N_1 + N_2T)\max_{t\in J_1}\left(\frac{3}{2}{q_n}^2(t) + \frac{{p_n}^2(t)}{2}\right), \qquad t \in J_1. \qquad (3.2.20)$$

We desire to find an estimate for $r(t)$ given by (3.2.19).

Let

$$z(t) = \int_{t_0}^{t} r\,(s)ds \qquad (3.2.21)$$

then the equation (3.2.19) transforms to $z''(t) - Lz'(t) - L_1z(t) = G(t),\ z'(t_0) = 0,$ $z(t_0) = 0$. This is a differential equation of the second order, linear, and nonhomogeneous, with constant coefficients. We can solve it with pursuing the similar process that we used above. Therefore, the complementary solution $z_c(t)$ vanishes, i.e., $z_c(t) = 0$ on $J_1$. For the particular solution $z_p(t)$, we employe the method of variation of parameters.

$z_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where $y_1(t) = e^{\lambda_1 t}$, $y_2(t) = e^{\lambda_2 t}$ and

$$v_1(t) = \int_{t_0}^{t} \frac{-G(s)}{(\lambda_2 - \lambda_1)e^{\lambda_1 s}}ds$$

$$v_2(t) = \int_{t_0}^{t} \frac{G(s)}{(\lambda_2 - \lambda_1)e^{\lambda_2 s}}ds.$$

The particular solution $z_p(t)$ is

$$z_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

$$z_p(t) = y_1(t)\int_{t_0}^{t} \frac{-G(s)}{(\lambda_2-\lambda_1)e^{\lambda_1 s}}\,ds + y_2(t)\int_{t_0}^{t} \frac{G(s)}{(\lambda_2-\lambda_1)e^{\lambda_2 s}}\,ds.$$

so, the general solution is $z(t) = z_p(t)$,

$$z(t) = y_1(t)\int_{t_0}^{t} \frac{-G(s)}{(\lambda_2-\lambda_1)e^{\lambda_1 s}}\,ds + y_2(t)\int_{t_0}^{t} \frac{G(s)}{(\lambda_2-\lambda_1)e^{\lambda_2 s}}\,ds$$

We now aim to find $z'(t)$ with the help of Leibniz rule

$$\begin{aligned} z'(t) = {} & e^{\lambda_1 t}\left(\frac{-G(t)}{(\lambda_2-\lambda_1)e^{\lambda_1 t}}\right) + \lambda_1 e^{\lambda_1 t}\int_{t_0}^{t} \frac{-G(s)}{(\lambda_2-\lambda_1)e^{\lambda_1 s}}\,ds + e^{\lambda_2 t}\left(\frac{G(t)}{(\lambda_2-\lambda_1)e^{\lambda_2 t}}\right) \\ & + \lambda_2 e^{\lambda_2 t}\int_{t_0}^{t} \frac{G(s)}{(\lambda_2-\lambda_1)e^{\lambda_2 s}}\,ds \end{aligned}$$

$$= r_1 e^{r_1 t}\int_{t_0}^{t} \frac{-G(s)}{(\lambda_2-\lambda_1)e^{\lambda_1 s}}\,ds + \lambda_2 e^{\lambda_2 t}\int_{t_0}^{t} \frac{G(s)}{(\lambda_2-\lambda_1)e^{\lambda_2 s}}\,ds$$

$$= \frac{\lambda_1 e^{\lambda_1 t}}{\sqrt{L^2+4L_1}}\int_{t_0}^{t} -G(s)\left(e^{-\lambda_1 s}\right)ds + \frac{\lambda_2 e^{\lambda_2 t}}{\sqrt{L^2+4L_1}}\int_{t_0}^{t} G(s)\left(e^{-\lambda_2 s}\right)ds$$

$$\leq \frac{-\lambda_1}{\sqrt{L^2+4L_1}}\max_{t\in J_1} G(t)\int_{t_0}^{t}\left(e^{(t-s)\lambda_1}\right)ds + \frac{\lambda_2}{\sqrt{L^2+4L_1}}\max_{t\in J_1} G(t)\int_{t_0}^{t}\left(e^{(t-s)\lambda_2}\right)ds$$

$$= \frac{\max_{t\in J_1} G(t)}{\sqrt{L^2-4L_1}}\left(\left[\left(e^{(t-t)\lambda_1}\right) - \left(e^{(t-t_0)\lambda_1}\right)\right] - \left[\left(e^{(t-t)\lambda_2}\right) - \left(e^{(t-t_0)\lambda_2}\right)\right]\right)$$

$$= \frac{\max_{t\in J_1} G(t)}{\sqrt{L^2-4L_1}}\left[1 - \left(e^{(t-t_0)\lambda_1}\right) - 1 + \left(e^{(t-t_0)\lambda_2}\right)\right]$$

$$= \frac{\max_{t\in J_1} G(t)}{\sqrt{L^2-4L_1}}\left[\left(e^{(t-t_0)\lambda_2}\right) - \left(e^{(t-t_0)\lambda_1}\right)\right]$$

$$= \frac{\max_{t\in J_1} G(t)}{\sqrt{L^2-4L_1}}\left[\left(e^{(t-t_0)\lambda_2}\right) + \left(e^{(t-t_0)\lambda_1}\right)\right]$$

where $\lambda_1 \leq L$, $\lambda_2 \leq L$ and $t - t_0 \leq T$

Accordingly,

$$z'(t) \leq \frac{\max\limits_{t\in J_1} G(t)}{\sqrt{L^2 + 4L_1}}(2e^{LT})$$

where $z'(t) = r(t)$ , and $r(t) = q_{n+1}(t)$.

Hence

$$\max_{t\in J_1} q_{n+1}(t) \leq \frac{(2e^{LT})}{\sqrt{L^2 + 4L_1}}(N_1 + N_2 T) \max_{t\in J_1}\left(\frac{3}{2}{q_n}^2(t) + \frac{{p_n}^2(t)}{2}\right), \qquad t \in J_1.$$

Thus $\{\tilde{\beta}_n(t)\}$ converges quadratically to $\tilde{x}(t)$, the unique solution of (3.2.4). ■

We will discuss an important integro-differential inequality in the following lemma because it is required for the subsequent theorem.

**Lemma 3.2.3** [35] Let $\lambda \in C[J_1, R], \omega \in C[J_1 \times J_1, R_+]$. Consider the integro-differential inequality

$$p'(t) \leq \lambda(t)p(t) - \int_{t_0}^{t} \omega(t,s)p(s)ds\,, \quad t \in J_1, \qquad p(t_0) \leq 0. \tag{3.2.29}$$

Assume that

$$\int_{t_0}^{t}\left[e^{-\int_s^t \lambda(\tau)d\tau}\omega(t,s)\right]ds \;\leq \frac{1}{T}, \tag{3.2.30}$$

then $p(t) \leq 0, t \in J_1$.

**Proof**:

Let $m(t) = p(t)e^{-\int_{t_0}^{t}\lambda(s)ds},\; t \in J_1$. Then it follows from (3.2.29) that

$$m'(t) = p'(t)e^{-\int_{t_0}^{t}\lambda(s)ds} + p(t)(-\lambda(t))e^{-\int_{t_0}^{t}\lambda(s)ds}$$

$$m'(t) \leq \left[\lambda(t)p(t) - \int_{t_0}^{t}\omega(t,s)p(s)ds\right]e^{-\int_{t_0}^{t}\lambda(s)ds} - \lambda(t)p(t)e^{-\int_{t_0}^{t}\lambda(s)ds}$$

$$\leq -\int_{t_0}^{t}\omega(t,s)p(s)ds\left[e^{-\int_{t_0}^{t}\lambda(s)ds}\right]$$

$$\leq -\int_{t_0}^{t} \omega(t,s)p(s)e^{-\int_{t_0}^{s}\lambda(\tau)d\tau}e^{-\int_{s}^{t}\lambda(\tau)d\tau}\,ds$$

$$\leq -\int_{t_0}^{t} \omega(t,s)m(s)e^{-\int_{s}^{t}\lambda(\tau)d\tau}\,ds \tag{3.2.31}$$

proving that $m(t) \leq 0$ is enough, where $t \in J_1$.

Assume that this statement is false, meaning that $m(t)$ is greater than zero. Next, we evaluate two potential situations:

(a) $m(t_0) = 0$ ,
(b) $m(t_0) < 0$.

We first think the possibility of (a). In this case, let us claim for a moment that there would exist a $t_1 \in (t_0, T]$ such that $m(t_1) > 0$ and that $m(t) \geq 0$ for $t \in [t_0, t_1]$. Thus, from (3.2.31), we obtain $m'(t) \leq 0$ for $t \in [t_0, t_1]$, which result in that $m(t)$ is non-increasing in $[t_0, t_1]$. Then $m(t_1) \leq m(t_0) = 0$. This leads to contradiction.

On the other hand, when $m(t_0) < 0$, then it can be found a $t_2 \in [t_0, t_1]$ such that

$$m(t_2) = m(t) = -\mu, \quad \mu > 0.$$

By employing the mean value theorem (MVT) over $[t_2, t_1]$, we get

$$m'(t^*) = \frac{m(t_1) - m(t_2)}{t_1 - t_2}$$

$$= \frac{m(t_1) + \mu}{t_1 - t_2}$$

$$> \frac{\mu}{T} \tag{3.2.32}$$

where $t_2 \leq t^* \leq t_1$.
Additionally, it is evident from equations (3.2.31) to (3.2.32) that

$$m'(t^*) \leq -\int_{t_0}^{t^*} e^{-\int_{s}^{t^*}\lambda(\tau)d\tau}\omega(t^*,s)m(s)\,ds$$

$$\leq -m(t_2)\int_{t_0}^{t^*} e^{-\int_{s}^{t^*}\lambda(\tau)d\tau}\omega(t^*,s)\,ds$$

$$= \mu \int_{t_0}^{t^*} \left[ e^{-\int_s^{t^*} \lambda(\tau) d\tau} \omega(t^*, s) \right] ds$$

$$\leq \frac{\mu}{T} \ . \tag{3.2.33}$$

We arrive at a contradiction as a result of the inequalities (3.2.32) and (3.2.33).

Notice that, Theorem 3.2.1 is proven under the fact that $\varphi(t, s, x)$ is nondecreasing wrt third variable. In case the kernel $\varphi(t, s, x)$ is nonincreasing in $x$, we must modify Theorem 3.2.1 in a suitable form. In this contex, we shall need the foregoing lemma to reach the similar conclusion proved in theorem 3.2.1.

Next theorem, which broadens the application of the previous theorem, will now be taken into consideration.

**Theorem 3.2.2.** Assume that the following conditions listed below hold:

**(A1)** $g \in C[J_3 \times R, R]$, $\varphi \in C[J_3 \times J_3 \times R, R]$ and $\varphi(t, s, x) \leq 0$ and is monotone non-increasing in $x$ for each fixed $(t, s)$;

**(A2)** $\alpha_0 \in C^1[J_1, R]$ , $\beta_0 \in C^1[J_2, R]$ such that $\alpha_0(t) \leq \beta_0(t + \eta_1)$

$$\alpha'_0(t) \leq g\big(t, \alpha_0(t)\big) + \int_{t_0}^{t} \varphi\left(t, s, \alpha_0(s)\right) ds, \qquad t \in J_1$$

$$\beta'_0(t) \geq g\left(t, \beta_0(t)\right) + \int_{\tau_0}^{t} \varphi\left(t, s, \beta_0(s)\right) ds, \qquad t \in J_2$$

with $\alpha_0(t_0) \leq x(\sigma_0) \leq \beta_0(\tau_0)$ where $t_0 < \sigma_0 < \tau_0$ and $\eta_1 = \tau_0 - t_0$;

**(A3)** $g_x \in C[\Lambda_1, R]$ and $g_x(t, x)$ is nondecreasing in $x$ for each $t$ and satisfies for

$$g(t, x) \geq g(t, y) + g_x(t, y)(x - y) \ , x \geq y,$$

$$|g_x(t, x) - g_x(t, y)| \leq N_1 |x - y| \text{ with } N_1 \geq 0,$$

Furthermore, let $\varphi_x \in C[\Lambda_2, R]$ and be nondecreasing in $x$ for each $(t, s)$ with

$$\varphi(t, s, x) \geq \varphi(t, s, y) + \varphi_x(t, s, y)(x - y), \qquad x \geq y,$$

$$|\varphi_x(t, s, x) - \varphi_x(t, s, y)| \leq N_2 |x - y| \text{ with } N_2 \geq 0;$$

**(A4)** $1 + T \int_{t_0}^{t} e^{-\int_s^t g_x(\tau, v(\tau)) d\tau} \varphi_x\big(t, s, v(s)\big) ds \ \geq 0$, $\alpha_0(t_0) \leq v_0 \leq \beta_0(\tau_0)$;

**(A5)** $f(t, x)$ are non-decreasing with respect to $t$ and for any $\varepsilon > 0$,

$$\varphi(t+\varepsilon, s+\varepsilon, x) \geq \varphi(t, s, x).$$

Then, there exist monotone sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ that uniformly and quadratically converge to the unique solution of

$$x'(t) = g\big(t, x(t)\big) + \int_{\sigma_0}^{t} \varphi\big(t, s, x(s)\big)\, ds, t \in [\sigma_0, \sigma_0 + T] \quad (3.2.34)$$

with $x(\sigma_0) = x_0$.

**Proof:** We set up the functions $\tilde{\alpha}_0(t) = \alpha_0(t)$ and $\tilde{\beta}_0(t) = \beta_0(t+\eta_1)$

which can be shown that that $\tilde{\alpha}_0(t)$ and $\tilde{\beta}_0(t)$ on $J_1$ are lower and upper solutions of the problem (3.2.1), respectively.

In light of the fact that $g$ and $\varphi$ satisfy the Lipschitz condition, our problem has a unique solution.

We describe $\tilde{\alpha}_1$, $\tilde{\beta}_1$ as the solution of the following IVPs:

$$\begin{aligned}\tilde{\alpha}'_1(t) = {} & g\big(t+\eta_2, \tilde{\alpha}_0(t)\big) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\big(\tilde{\alpha}_1(t) - \tilde{\alpha}_0(t)\big) \\ & + \int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\alpha}_1(s) \\ & - \tilde{\alpha}_0(s))\big]\, ds\end{aligned}$$

$$\tilde{\alpha}_1(t_0) = x_0, \quad (3.2.36)$$

and

$$\begin{aligned}\tilde{\beta}'_1(t) = {} & g\left(t+\eta_2, \tilde{\beta}_0(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\left(\tilde{\beta}_1(t) - \tilde{\beta}_0(t)\right) \\ & + \int_{t_0}^{t} \left[\varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_0(s)\right) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\beta}_1(s) \right. \\ & \left. - \tilde{\beta}_0(s))\right] ds\end{aligned}$$

$$\tilde{\beta}_1(t_0) = x_0. \quad (3.2.37)$$

We then construct monotone increasing sequence $\{\tilde{\alpha}_n(t)\}$ and decreasing sequence $\{\tilde{\beta}_n(t)\}$. To reach this purpose, we initially prove

$$\tilde{\alpha}_0(t) \le \tilde{\alpha}_1(t) \le \tilde{\beta}_1(t) \le \tilde{\beta}_0(t),\ \ t \in J_1. \tag{3.2.38}$$

Let $p(t) = \tilde{\alpha}_0(t) - \tilde{\alpha}_1(t)$. Note that $p(t_0) \le 0$. After taking derivatives

$$p'(t) = \tilde{\alpha}'_0(t) - \tilde{\alpha}'_1(t)$$

$$\le g\big(t+\eta_2, \alpha_0(t)\big) + \int_{t_0}^{t} \varphi\big(t+\eta_2, s+\eta_2, \alpha_0(s)\big)\, ds$$

$$-g\big(t+\eta_2, \tilde{\alpha}_0(t)\big) - g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\big(\tilde{\alpha}_1(t) - \tilde{\alpha}_0(t)\big)$$

$$-\int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\alpha}_1(s) - \tilde{\alpha}_0(s))\big]\, ds$$

$$p'(t) \le g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)p(t) + \int_{t_0}^{t} \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)p(s)\, ds.$$

Upon recalling Lemma 3.2.3, we can treat $\lambda(t) = g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)$,

$\omega(t,s) = -\varphi_x(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s))$ and $v(t) = \tilde{\alpha}_0(t)$.

Combining this fact with assumptions (A1) and (A2) leads to the inequality $p(t) \le 0$, $t \ge t_0$, which then reveals $\tilde{\alpha}_0(t) \le \tilde{\alpha}_1(t)$, on $J_1$.

Similarly, one can show that $\tilde{\beta}_1(t) \le \tilde{\beta}_0(t)$, on $J_1$.

It is remains to show that $\tilde{\alpha}_1(t) \le \tilde{\beta}_1(t),\ t \in J_1$. Recalling the facts $\tilde{\alpha}_0(t) \le \tilde{\alpha}_1(t)$ and $\tilde{\beta}_1(t) \le \tilde{\beta}_0(t)$, and using the inequalities in $(A3)$, we get

$$\tilde{\alpha}'_1(t) = g\big(t+\eta_2, \tilde{\alpha}_0(t)\big) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\big(\tilde{\alpha}_1(t) - \tilde{\alpha}_0(t)\big)$$

$$+\int_{t_0}^{t} \big[\varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\alpha}_1(s) - \tilde{\alpha}_0(s))\big]\, ds$$

$$\tilde{\alpha}'_1(t) \le g\big(t+\eta_2, \tilde{\alpha}_1(t)\big) + \int_{t_0}^{t} \varphi\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_1(s)\big)\, ds$$

and

$$\tilde{\beta}'_1(t) = g\left(t+\eta_2, \tilde{\beta}_0(t)\right) + g_x\big(t+\eta_2, \tilde{\alpha}_0(t)\big)\left(\tilde{\beta}_1(t) - \tilde{\beta}_0(t)\right)$$

$$+ \int_{t_0}^{t} \left[\varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_0(s)\right) + \varphi_x\big(t+\eta_2, s+\eta_2, \tilde{\alpha}_0(s)\big)(\tilde{\beta}_1(s) - \tilde{\beta}_0(s))\right] ds$$

$$\tilde{\beta}'_1(t) \geq g\left(t+\eta_2, \tilde{\beta}_1(t)\right) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{\beta}_1(s)\right) ds$$

also note that $\tilde{\alpha}_1(t_0) = \tilde{\beta}_1(t_0) = x_0$ the direct application of Lemma 3.2.1 yields

$$\tilde{\alpha}_1(t) \leq \tilde{\beta}_1(t), t \in J_1$$

For the general case, we use the Mathematical Induction as we have done before to find for all $n$

$$\tilde{\alpha}_0(t) \leq \tilde{\alpha}_1(t) \leq \ldots . \leq \tilde{\alpha}_n(t) \leq \tilde{\beta}_n(t) \leq \ldots . \leq \tilde{\beta}_1(t) \leq \tilde{\beta}_0(t), t \in J_1.$$

In accordance with the standard techniques [51], it is possible to demonstrate that the sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ converge uniformly and monotonically to $\tilde{x}(t)$, the unique solution of

$$\tilde{x}'(t) = g\big(t+\eta_2, \tilde{x}(t)\big) + \int_{t_0}^{t} \varphi\left(t+\eta_2, s+\eta_2, \tilde{x}(s)\right) ds, \qquad t \in J_1 = [t_0, t_0+T]$$

$$\tilde{x}(t_0) = x_0,$$

By substituting the variables in the equation with $\sigma = t+\eta_2,\ \xi = s+\eta_2$ and $x(t+\eta_2) = \tilde{x}(t)$, we express the equation as follows:

$$x'(\sigma) = g\big(\sigma, x(\sigma)\big) + \int_{\sigma_0}^{\sigma} \varphi\left(\sigma, \xi, x(\xi)\right) d\xi, \qquad \sigma \in [\sigma_0, \sigma_0+T]$$

with $x(\sigma_0) = x_0$.

A closer look reveals that the final problem is identical to

$$x'(t) = g\big(t, x(t)\big) + \int_{t_0}^{t} \varphi\left(t, s, x(s)\right) ds, \qquad t \in [\sigma_0, \sigma_0+T]$$

$$x(\sigma_0) = x_0$$

The final challenge is to arrive at quadratic convergence. To accomplish this, we proceed in a manner that is quite similar to the process that has been debated in greater detail previously. We therefore leave out the details of it. ■

# 4. CONCLUSION

QLM was employed to tackle an IVP that includes a nonlinear integro-differential equation with an ITD. This method involved transforms the original equation into a series of linear differential equations, which were then solved iteratively. The process produced sequences of upper and lower solutions that uniformly approaches the unique solution of the main problems. The remarkable quadratic convergence of these sequences led to a rapid reduction in approximation errors with each iteration.

## REFERENCES


[1] R.E. Bellman, R.E. Kalaba, *Quasilinearization and nonlinear boundary-value problems*, American Elsevier, 1965.

[2] V. Lakshmikantham, *An extension of the method of quasilinearization*, J Optim Theory Appl, **82** (1994), no. 2, 315–321.

[3] V. Lakshmikantham, *Further improvement of generalized quasilinearization method*, Nonlinear Anal Theory Methods Appl, **27** (1996), 223–227.

[4] J. Nieto, *Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions*, Proceedings of the American Mathematical Society, **125** (1997), no. 9, 2599–2604.

[5] A. Cabada, J.J. Nieto, *Quasilinearization and rate of convergence for higher-order nonlinear periodic boundary-value problems*, J Optim Theory Appl, **108** (2001), 97–107.

[6] B. Ahmad, J.J. Nieto, N. Shahzad, *The Bellman–Kalaba–Lakshmikantham quasilinearization method for neumann problems*, J Math Anal Appl, **257** (2001), 356–363.

[7] B. Ahmad, J.J. Nieto, N. Shahzad, *Generalized quasilinearization method for mixed boundary value problems*, Appl Math Comput, **133** (2002), 423–429.

[8] M.A. El-Gebeily and D. ORegan, *Existence and quasilinearization for a class of nonlinear elliptic second order partial differential equations*, Dynamic Systems and Applications, **17** (2008), 445-458.

[9] B. Hu, Z. Wang, M. Xu, D. Wang, *Quasilinearization method for an impulsive integro-differential system with delay*, Mathematical Biosciences and Engineering, **19** (2022), 612-623. https://doi.org/10.3934/mbe.2022027.

[10] T. Jankowski, *The generalized quasilinearization for integro-differential equations of volterra type on time scales*, Rocky Mt J Math, **37** (2007), no. 3. https://doi.org/10.1216/rmjm/1183990677.

[11] F.M. Atici, P.W. Eloe, B. Kaymakçalan, *The quasilinearization method for boundary value problems on time scales*, J Math Anal Appl, **276** (2002), 357–372. https://doi.org/10.1016/S0022-247X(02)00466-3.

[12] C. Yakar, M. Arslan, *Quasilinearization method for causal terminal value problems involving riemann-liouville fractional derivatives*, Electronic Journal of Differential Equations, **2019** (2019), no. 11, 1–11.

[13] P. Eloe, J. Jonnalagadda, *Quasilinearization applied to boundary value problems at resonance for riemann-liouville fractional differential equations*, Discrete and

Continuous Dynamical Systems - Series S, **13** (2020). https://doi.org/10.3934/dcdss.2020220.

[14] G. Su, L. Lu, B. Tang, Z. Liu, *Quasilinearization technique for solving nonlinear Riemann-Liouville fractional-order problems*, Appl Math Comput, **378** (2020). https://doi.org/10.1016/j.amc.2020.125199.

[15] M. Delkhosh, K. Parand, *A hybrid numerical method to solve nonlinear parabolic partial differential equations of time-arbitrary order*, Computational and Applied Mathematics, **38** (2019), no. 76. https://doi.org/10.1007/s40314-019-0840-6.

[16] P. Wang, C. Li, J. Zhang, T. Li, *Quasilinearization method for first-order impulsive integro-differential equations*, Electronic Journal of Differential Equations, **2019** (2019), no. 46, 1–14.

[17] S. Torkaman, M. Heydari, G.B. Loghmani, *A combination of the quasilinearization method and linear barycentric rational interpolation to solve nonlinear multi-dimensional Volterra integral equations*, Math Comput Simul, **208** (2023), 366-397 https://doi.org/10.1016/j.matcom.2023.01.039.

[18] E. Najafi, *Nyström-quasilinearization method and smoothing transformation for the numerical solution of nonlinear weakly singular Fredholm integral equations*, J Comput Appl Math, **368** (2020), 1-13. https://doi.org/10.1016/j.cam.2019.112538.

[19] Z. Denton, J.D. Ramírez, *Quasilinearization method for finite systems of nonlinear RL fractional differential equations*, Opuscula Mathematica, **40** (2020), no. 6, 667–683. https://doi.org/10.7494/OpMath.2020.40.6.667.

[20] V.K. Sinha, P. Maroju, *New Development of Variational Iteration Method Using Quasilinearization Method for Solving Nonlinear Problems*, Mathematics, **11**(2023), no. 4, 935. https://doi.org/10.3390/math11040935.

[21] S. Mt Aznam, N.A. Che Ghani, M.S.H. Chowdhury, *A numerical solution for nonlinear heat transfer of fin problems using the Haar wavelet quasilinearization method*, Results Phys, **14** (2019), 1-8. https://doi.org/10.1016/j.rinp.2019.102393.

[22] H.M. Srivastava, F.A. Shah, M. Irfan, *Generalized wavelet quasilinearization method for solving population growth model of fractional order*, Math Methods Appl Sci, **43** (2020), 8753–8762.

[23] F.A. Shah, M. Irfan, K.S. Nisar, *Gegenbauer wavelet quasi-linearization method for solving fractional population growth model in a closed system*, Math Methods Appl Sci, **45** (2022), 3605–3623.

[24] V.B. Mandelzweig, F. Tabakin, *Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs*, Comput Phys Commun, **141** (2001), 268-281. https://doi.org/10.1016/S0010-4655(01)00415-5.

[25] M.N. Koleva, L.G. Vulkov, *Two-grid quasilinearization approach to ODEs with applications to model problems in physics and mechanics*, Comput Phys Commun, **181** (2010), 663–670. https://doi.org/10.1016/j.cpc.2009.11.015.

[26] M. Razavy, V.J. Cote, *Quasilinearization method applied to multidimensional quantum tunneling*, Phys Rev A (Coll Park), **49**(1994), 2266. https://doi.org/10.1103/PhysRevA.49.2266.

[27] V.B. Mandelzweig, *Quasilinearization method and its verification on exactly solvable models in quantum mechanics*, J Math Phys, **40** (1999), 6266–6291. https://doi.org/10.1063/1.533092.

[28] E.S. Lee, *Quasilinearization and invariant imbedding: with applications to chemical engineering and adaptive control*, Elsevier, **41** (2016).

[29] Ş. Yüzbaşı, M. Izadi, *Bessel-quasilinearization technique to solve the fractional-order HIV-1 infection of CD4+ T-cells considering the impact of antiviral drug treatment*, Appl Math Comput, **431** (2022), 1-14. https://doi.org/10.1016/j.amc.2022.127319.

[30] M. Magodora, H. Mondal, P. Sibanda, *Dual solutions of a micropolar nanofluid flow with radiative heat mass transfer over stretching/shrinking sheet using spectral quasilinearization method*, Multidiscipline Modeling in Materials and Structures, **16**(2020), 238-255. https://doi.org/10.1108/MMMS-01-2019-0028.

[31] E.S. Lee, K.M. Wang, *Quasilinearization in biological systems modeling*, in: Modeling and Control of Systems, **121** (2006), 413-421. https://doi.org/10.1007/bfb0041208.

[32] B. Ahmad, A. Alsaedi, B.S. Alghamdi, *Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions*, Nonlinear Anal Real World Appl, **9** (2008), 1727-1740. https://doi.org/10.1016/j.nonrwa.2007.05.005.

[33] K. Parand, H. Yousefi, M. Delkhosh, A. Ghaderi, *A novel numerical technique to obtain an accurate solution to the Thomas-Fermi equation*, Eur Phys J Plus, **131** (2016), 1-16. https://doi.org/10.1140/epjp/i2016-16228-x.

[34] M. Izadi, H.M. Srivastava, *Generalized bessel quasilinearization technique applied to bratu and lane–emden-type equations of arbitrary order*, Fractal and Fractional, **5** (2021). 179. https://doi.org/10.3390/fractalfract5040179.

[35] V. Lakshmikantham, A.S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, Kluwer Academic Publisher, Dordrecht, 1998. https://doi.org/10.1007/978-1-4757-2874-3.

[36] A.A. Martynyuk, *Stability and Control: Theory, Methods*, and Applications, International Applied Mechanics, **58** (2022), no. 2, 123–146.

[37] V. Lakshmikantham, S. Leela, J.V. Devi, *Stability criteria for solutions of differential equations relative to initial time difference*, Nonlinear Analysis and Appl.(to Appear), (1999).

[38] V. Lakshmikantham, A.S. Vatsala, *Differential inequalities with initial time difference and applications*, J Inequal Appl, **3** (1999), 233-244. 586479.

[39] V. Lakshmikantham, A.S. Vatsala, *Theory of differential and integral inequalities with initial time difference and applications*, Analytic and Geometric Inequalities and Applications, (1999), 191–203.

[40] A. Yakar, *Initial time difference quasilinearization for Caputo Fractional Differential Equations*, Adv Differ Equ, **2012** (2012), no. 92. https://doi.org/10.1186/1687-1847-2012-92.

[41] C. Yakar, A. Yakar, *An Extension of the Quasilinearization Method with Initial Time Difference*, *Dynamics of continuous, discrete and impulsive systems. Ser A Math Anal*, **14** (2007), 275-279.

[42] C. Yakar, A. Yakar, *Monotone iterative technique with initial time difference for fractional differential equations*, Hacettepe Journal of Mathematics and Statistics, **40** (2011), no. 2, 331 – 340.

[43] A. Yakar, H. Kutlay, *Monotone iterative technique via initial time different coupled lower and upper solutions for fractional differential equations*, Filomat, **31** (2017), no. 4, 1031-1039. https://doi.org/10.2298/FIL1704031Y.

[44] A. M. Wazwaz, *First Course in Integral Equations*, World Scientific Publishing Company, Chicago, 2015.

[45] R. Precup, *Methods in Nonlinear Integral Equations*, Springer Science & Business Media, Cluj-Napoca, 2002. https://doi.org/10.1007/978-94-015-9986-3.

[46] E. Fathizadeh, R. Ezzati, K. Maleknejad, *Hybrid rational Haar wavelet and block pulse functions method for solving population growth model and Abel integral equations*, Mathematical Problems in Engineering, **2017** (2017), 1-7. https://doi.org/10.1155/2017/2465

[47] E.S. Lee, *Quasilinearization, difference approximation, and nonlinear boundary value problems*, AIChE Journal, **14** (1968), no. 3, 490-496. https://doi.org/10.1002/aic.690140327.

[48] R.P. Agarwal, D. O'Regan, *Preliminaries to Existence and Uniqueness of Solutions*, An Introduction to Ordinary Differential Equations, Springer, New York, (2008). https://doi.org/10.1007/978-0-387-71276-5_7

[49] S.G. Deo, V. Raghavendra, R. Kar, V. Lakshmikantham, *Textbook of ordinary differential equations*, McGraw-Hill Education, New Delhi, 1997.

[50] W. R. Wade, *Introduction to Analysis, Global Edition*, Pearson Higher Ed, New Jersey, 2021.

[51] A. Yakar, *Quaslinearization method for nonlinear equations*, Master Thesis, Gebze Institute of Technology, 2004.