# COMPATIBILITY OF THE DIMENSIONAL REDUCTION AND VARIATION PROCEDURES FOR A QUADRATIC CURVATURE MODEL WITH A KALUZA-KLEIN ANSATZ

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**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

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# ABSTRACT

## COMPATIBILITY OF THE DIMENSIONAL REDUCTION AND VARIATION PROCEDURES FOR A QUADRATIC CURVATURE MODEL WITH A KALUZA-KLEIN ANSATZ

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The Kaluza-Klein theory is investigated for a 5D quadratic curvature Lagrangian. In order to obtain the field equations in the actual 4D spacetime, there are two different approaches composed of two successive applications to be implemented to the 5D action. Either one applies the least action principle with respect to the 5D variables and then uses the dimensional reduction mechanism, or changes the order by first reducing the action and then takes the variations with respect to the constituent fields of the theory. In this work, the field equations are obtained from the reduced form of the action and the results are compared with those obtained by the reverse procedure. The differences between these two procedures along with their outcomes are presented in detail, and their origins are explained.

Keywords: Kaluza-Klein Theory, Dimension Reduction, Least Action Principle

## ÖZ

**BOYUT İNDİRGEMESİ VE VARYASYON İŞLEMLERİNİN İKİNCİ DERECEDEN EĞRİLİKLİ VE KALUZA-KLEİN ANZATSLI MODELDE UYUMLULUĞU**

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Kaluza-Klein teorisi, 5D'de, ikinci dereceden eğrilik Lagrange yoğunluğu için araştırıldı. Gerçek 4D uzay-zamanda alan denklemlerini elde etmek için, 5D eylemine tatbik edilecek iki ardışık uygulamadan oluşan iki ayrı yaklaşım vardır. Ya 5D değişkenlerine göre en az eylem ilkesi uygulanır ve ardından boyut indirgeme mekanizması kullanılır, ya da sıra değiştirilir, önce boyut indirgenir ve ardından teoriyi oluşturan alanlara göre varyasyon alınır. Bu çalışmada, eylem fonksiyonelinin indirgenmiş formundan alan denklemleri elde edilmiş ve sonuçlar ters prosedürle elde edilenlerle karşılaştırılmıştır. Bu iki prosedür arasındaki farklılıklar ve bunların sonuçları ayrıntılı olarak sunulmuş, ayrıca sebepleri açıklanmıştır.

Anahtar Kelimeler: Kaluza-Klein Kuramı, Boyut İndirgemesi, En Az Eylem İlkesi

To my family

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## LIST OF ABBREVIATIONS

| | |
|---|---|
| 4D | 4 dimensions |
| 5D | 5 dimensions |
| KK | Kaluza-Klein |
| EH | Einstein-Hilbert |
| BP | Bopp-Podolsky |
| BD | Brans-Dicke |
| EM | electromegnetism |
| QED | quantum electrodynamics |

# CHAPTER 1

# INTRODUCTION

Special relativity theory was published by Einstein in 1905 [1]. According to this theory speed of light is the maximum speed at which all matter and information in the universe can propagate. Researchers verified this fact through experiments. Also, the theory of the special relativity claimed that space and time are inseparably connected. The electromagnetic interaction of charged particles are well represented in special relativity. However, gravitation is not present in this theory.

Einstein completed his famous theory about gravitation which is called the general relativity theory [2]. This theory is known as Einstein's major contribution to science. Prior to Einstein, physicists considered gravity as a force that pulls large objects together. Certainly, this is not wrong. Gravity is a force that pulls us downward towards the Earth. The Earth's orbit is maintained by gravity, which pulls it towards the Sun. This is called the Newtonian view of gravity. Einstein showed that gravity is not only a force, but it is also a geometrical expression of space and time. This brought a very different and highly insightful perspective to physics. The geometry or in other words the curvature of space and time alter due to existence of energy and mass. Through this space objects follow a path according to its curvature.

General relativity had its initial achievement in 1915, when Einstein offered a theoretical explanation for the anomalous precession of Mercury's perihelion. This is a gravitational event that does not agree with Newton's theory. In Newtonian theory, this is due to the fact that the planetary system is an n-body system instead of a two body system, meaning that the motion of one planet is perturbed by the motion of all the others. Mercury's orbit has a short period and a large eccentricity, thus the perihelion position may be accurately calculated by observation. Prior to general relativity, there existed a difference of 43 seconds of arc per century between the classical estimate and the observed shift. Although the difference is small, it is substantial on an astronomical scale. This represents approximately a hundred times the likely error which is observational. Astronomers have been concerned by this inconsistency. In order to justify this, it was proposed that there was another planet which was called Vulcan. Vulcan's orbit was within Mercury's orbit. On the other hand, Vulcan does not exist. General relativity rectifies this inconsistency, as it correctly predicts 42.98 seconds of arc every century in theory [3].

In 1919, the theory has been tested experimentally for the first time. When the solar eclipse occurred on May 29,1919, the estimated deflection of light rays passing near to the Sun was detected as 1.75 seconds of arc by British expeditions [4, 5].

Same year, Einstein also predicted gravitational waves. The Lorentz invariance of general relativity allows for the presence of gravitational waves. Firstly, gravitational waves were observed indirectly in a neutron star pair in 1970 [6]. The decrease in their orbital period was explained by the fact that they were emitting gravitational waves. Almost a century after Einstein's prediction, these waves were discovered in 2015. In 2016, researchers from LIGO, announced the first detection of gravitational waves [7].

Again by Einstein, general theory of relativity was implemented to the structure of the entire universe in 1917 [8]. He realized that the general field equations indicated a dynamic cosmos that might shrink or expand. Because there was no empirical evidence for a dynamic cosmos at the time, Einstein added a new component in his field equations which is called the cosmological constant. The reason for adding such a constant was to secure a static universe. Edwin Hubble discovered that the universe is expanding as a result of his observations [9]. This indicated that such a constant should not exist in the equations of the general relativity. Therefore, Einstein dismissed this constant after a while as he considered it to be theoretically insufficient. Einstein expressed that the cosmological constant was the biggest mistake of his life. However, the studies conducted in the last 20 years have shown significant evidence that Einstein's cosmological constant is not a mistake. Our current understanding of the universe seems to be only possible with the existence of the cosmological term [10].

By assuming that the Einstein-Hilbert action is the correct action for general relativity, Einstein field equations may be obtained. The Einstein-Hilbert action for general relativity was first expressed from a scalar quantity obtained only from the spacetime metric. The Palatini variation is significant because it implies that in general relativity, the Christoffel connection does not need to be inserted as a distinct supposition because the knowledge is already present in the Lagrangian. Einstein suggested that action principle includes metric and affine connection that behave as independent variables. Then by using the variational principle, field equations can be obtained separately in terms of metric and connection. As opposed the Levi-Civita connection, this technique permitted to compute the field equations more general metric affine connection. This method is known as Palatini method [11]. According to Palatini formalism, metric and connection are both basic and independent dynamical variables. Einstein used also this method to study general relativity. However, when he dealt with the Einstein-Hilbert action, both methods provided identical equations of motion, because the independent connection turns out to be the Levi-Civita connection. In fact, this outcome is particular to Einstein-Hilbert action. Furthermore, spacetime can be endowed with torsion, when the Lagrangian is explicitly linked with the connection. Consequently, the Einstein-Palatini formalism provides a strong tool for more general gravitational theories.

Einstein used a variational method to generate some incomplete set of equations. However, this approach was insufficient since Lagrangian was not covariant. Hilbert derived Einstein's equations by applying a metric variational principle with a correct Lagrangian density. His results were presented in the renowned message to the Göttingen Academy on November 20,1915 [12]. This day is considered to be very important because it is known as the birthday of the metric variational principle.

Renormalization is the procedure of absorbing divergent elements of a computation that produce illogical infinite outputs into a few quantifiable variables that provide finite solutions. It is known that the Einstein Hilbert action is not renormalizable. In order to obtain a renormalizable action, some adjustments should be made. These adjustments should be made up of a variety of curvature invariants to secure coordinate invariance. When curvature terms $a\,R^2 + b\,R_{mn}R^{mn}$ are added to the Einstein-Hilbert action, then the action becomes renormalizable [13]. This theory is known as quadratic gravity. This feature is retained if there exist renormalizable coupling terms in quantum field theory. Because of the importance of the quadratic gravity, some researchers investigated whether quadratic gravity is meaningful or not. In quantum field theory ghosts are the non-physical solutions. The majority of the previous research has focused on the ghost problem appearing in quantum mechanical models. Nevertheless, there has been some recent improvements in the problem of ghosts. Therefore, the situation of a relativistic theory like a quantum gravity theory maintain a promising area for further studies.

For a very long time physicists sought for and still are looking for a single framework that unifies and explains the behaviour of all the forces and particles in nature. In particular, unification of the two main relativistic classical fields, namely gravitation and electromagnetism still bears some interesting points to be investigated.

From a historical point of view, it is worthwhile mentioning Weyl's theory which appeared in 1918, suggesting a unification between electromagnetism and gravity [14]. He presented a novel geometrisation of physics, by improving the geometry which is a sort of an extension of Riemann geometry. In Euclidean geometry one may compare vectors at various points in terms of their lengths. However in Riemann geometry, this is not possible. The direction of a vector at a remote location through parallel transport is determined by the path along which the parallel transport occurred. The direction of the vector changes in parallel transportation along the loop which is closed.

Although the direction of the vector changes in Riemann geometry, the length of the vector does not change. According to Weyl, this was a contradiction in the construction of a real field theory. Weyl generalised the idea of parallel transportation. He did this by developing it with a metric independent structure. Therefore, Weyl's approach yields a symmetric linear connection that is not only metric dependent. This structure contributed to a better grasp of mathematical description of gravity. Weyl added a new geometrical item into the theory which he called it the length connection. On the parallel transport, it calculated the length of the vector. Interestingly, Weyl was able to connect the length connection to the electromagnetic vector potential. Therefore, he made a connection between the electromagnetic field and the geometrical frame.

Weyl's theory was established on his statement that world's geometry or in other words the rules of the nature should not alter when there is a rescaling of the metric tensor. He regarded this metric regauging (or metric calibration) as a desirable form of symmetry that should be expressed in the Lagrangian. Weyl continued to create a review version which the related connection term is metrically invariant under the regauging. That observation encouraged Weyl's idea that measure invariance is a symmetry he would use to shape the laws of nature. According to Weyl, Ricci and Riemann tensors were invariant under the gauge transformation. Also the electromagnetic tensor was invariant under the gauge transformation. Weyl started to create a Lagrangian that expressed these symmetries in the light of these discoveries. Although Weyl's theory is very important theory for the unification, it also has some violation with the quantum mechanics. If we want to give an example of this, we can give atomic clocks. The atomic clocks has spectral lines. The frequency of this lines is determined by the position and previous histories of the atom. However, experiments shows that this is not the case. They seem to be unaffected by the history of the atom. Atomic clocks develop time units. Experiments indicate that they are conveyed in an integrated manner. As quantum theory developed, it was realised that Weyl's theory clashed even more fundamentally with experiment, since in quantum theory there was a direct relationship between clock speeds and particle masses. Natural frequency of a particle is determined by the speed of light, Planck constant and rest mass. In Weyl geometry, this means that the clocks will depend on masses of the particles as well as their histories. For instance, if two protons have different histories, their masses are also different. However, this violates the principle of quantum mechanics because according to quantum mechanics, particles of the same type must have the same mass. The theory that Weyl developed was quite complex. Despite the fact that it was logical mathematically, it did not make sense from a physical viewpoint.

Among many other important mathematical contributions to the geometrical framework of the gravity theory, Weyl's introduction of the idea of gauge transformations and gauge invariance is paramount.

A few years later, another suggestion to unify electromagnetism with gravity was made by Kaluza. The majority of the physics community considered Kaluza's opinion weird and unexpected. He suggested a unified field theory in which space and time extended to five dimensions rather than four. Luckily, in a system like Kaluza's there is a way for a fifth dimension to be kept hidden. The fifth dimension differs from the others in its geometrical scheme. As we know, the three dimensional space is very large. According to Kaluza, the fifth dimension is not like the other three spacelike dimensions, rather it coils into a small circle, to become unnoticeable. Kaluza initially told Einstein about this concept in 1919. He was then able to publish his ideas in 1921 [15]. Einstein admired Kaluza's theory. However, after a while he realised that Kaluza's theory has some flaws. Nevertheless, Einstein proceeded to study this theory because he expected it would pioneer something more promising. Kaluza had no intention of providing a self contained theory to explain a new empirical evidence. By using general relativity and electromagnetic theory, the results of Kaluza's theory can already be obtained. One of his major goal was to deduce the behaviour of the geodesic of the charged particle. In order to achieve this goal, the Riemann metric of the manifold was retained. Even if Kaluza's notion is not fundamentally distinct from Einstein's statement, at the time, it was the best unifying mechanism available. Apart from Einstein and Grommer's criticism [16], there were actually no direct responses to Kaluza's work until Klein's article published in 1926. Kaluza's and Klein's ideas appear to be closely connected at a first glance, and many writers accepted that Klein just enlarged Kaluza's theory. However, this was not true. Because many people confused Kaluza and Klein, the ideas of the Kaluza and Klein were also confused. Despite the fact that Kaluza's theory includes a formulation of the Lagrangian, he did not use the least action principle. The least action principle was a significant concept for Klein from the start to establish the connection between the electromagnetic and gravitational fields[17]. Klein's initial thought was to connect a quantity related to mass the momentum on the fifth axis. Klein's action appeared under the unified gravitational field and electromagnetic field by using electron in his action. He recommended that in order to combine general relativity and electromagnetic field, a new framework is to be created which is called as the world geometry. Once this is done, we can not in general make any arbitrary separation of gravitation from the electromagnetism. Although Kaluza-Klein was not a very successful theory just by itself, it inspired other unification theories, including the string theory.

In this thesis, we shall study the Kaluza-Klein unification theory from a variational principle point of view for a quadratic curvature Lagrangian. We are motivated by the results of [18] and [19]. In [18] the Kaluza-Klein reduction of a quadratic curvature model coupled to the Einstein-Hilbert action is presented. Non-minimal couplings among the gravitational, electromagnetic and scalar fields in the actual four dimensional spacetime are exhibited in terms of differential forms. However, there the field equations are not evaluated. In [19] field equations are obtained for this model, by directly taking the variation with respect to the five dimensional metric and the connection using the Palatini approach. The reduction procedure is performed afterwards to obtain the field equations in four dimensions. Now, the question arises as to what would have happened if the variations were taken with respect to the constituent fields after the reduced four dimensional action is obtained. Here, we try to answer this question.

In the second chapter, we will review the Kaluza-Klein theory. After presenting Kaluza's approach and then Klein's improvement, we give the current standard formulation. We investigate the Kaluza-Klein mass tower and explain its importance in physics.

In the third chapter, we obtain field equations using various different procedures and we shall compare the results. The Brans-Dicke theory [20] falls into the category of scalar tensor theories [21], developed as an alternative to Einstein's gravity which is currently raising hopes for a better explanation of some cosmological problems. We will also investigate Brans-Dicke theory within the confines of Kaluza-Klein theory.

In the fourth chapter, we try to give an answer to the main question of this thesis. In most general terms, for a theory having dimensions more than four there are two consecutive applications, namely the variation of the action and the reduction mechanism in order to obtain the corresponding field equations in the actual spacetime. This procedure can be reversed by changing the order of the applications. We are looking whether these two procedures are commutative, through a specific model. More explicitly, will the field equations stay the same or how will they change, if one first applies the reduction mechanism to the Kaluza-Klein quadratic Lagrangian then vary it with respect to the constituent fields of the theory using the Palatini approach. This is the reverse procedure, in the sense that the variation was done first with respect to the five dimensional action before applying the reduction mechanism, which was done earlier in [19]. In short, we compare the results of these two procedures, and see whether they are compatible.

The purpose of the Appendices is two fold. First, we relegate some calculational details. Second, we give some mathematical preliminaries, in order to follow and make the necessary comparisons with the existing the literature.

# CHAPTER 2

# KALUZA-KLEIN THEORY

Although it has been almost a century since the first appearance of the Kaluza-Klein theory, interest revolving around this particular idea has not been diminished, neither as being a part of many other realms of physics nor in its own right. Over several decades, physicists are attempting to enhance the Kaluza-Klein theory, yielding to inspirations for other research areas, such as the string theory.

Kaluza-Klein theory combines electromagnetism and gravity with each other, both being the generators of two principal forces in nature. Kaluza's idea was supported by Einstein's theory of general relativity in five dimensional (5D) spacetime.

In this chapter, we concentrate on the basic ideas presented first by Kaluza (published in 1921) [15], and then by Klein (published in 1926) [17]. We, first investigate Kaluza's approach. After writing Kaluza's 5D metric, we introduce the cylinder condition and note its drawbacks. Then, we interpret Kaluza's geodesic equation and give the field equations.

In Sec. 2, we introduce Klein's approach. We see how coordinate transformations result in gauge transformations for the vector potentials. We note that Klein recovered Einstein's and Maxwell's equations by using the 5D action. We review his account of the linear momentum in 5D. Using the periodicity of the fifth dimension we gather how Klein obtained the expressions for the metric tensor, vector potential and the scalar field in terms of Fourier series.

In Sec. 3, we derive the geodesic equation of motion of a charged test particle, from an action principle in 5D.

Finally, in Sec. 4, we study the Kaluza-Klein mass tower. We introduce the hierarchy problem and we discuss extra large dimensions, namely the ADD model. We show how one can obtain Kaluza-Klein mass by making use of the periodicity condition, which in return enables a Fourier series expansion for scalar fields. This, along with the Lagrange density turns out to be a convenient approach to this end.

## 2.1 Kaluza's Approach

Kaluza began his studies from the vacuum hypothesis, which examines the case, when there is neither charge nor matter, but there is only the gravity itself. Mathematically, this condition is represented as

$$\hat{T}_{ab} = 0 \tag{2.1}$$

where $\hat{T}_{ab}$ is the energy momentum tensor (whose explicit form for a variety of fields and particles will be found in the next section).

From vacuum hypothesis, one can ask the question, as to where the electromagnetic field is located. An immediate answer will be that it can be allocated inside the metric g. This would not be possible if the dimension of spacetime is kept as four, since in 4D theories fields have their self vacuum solutions, that cannot be superposed. Therefore Kaluza introduced the electromagnetic field as a part of the 5D metric and wrote the line element as [15]:

$$d\hat{s}^2 = \hat{g}_{AB} d\hat{x}^A d\hat{x}^B \tag{2.2}$$

where capital Latin letters $A, B = 0, 1, 2, 3, 5$ refer to 5D space.

Kaluza's assumption that the five dimensional universe is void of masses and fields other than gravitation led him to conclude that the curvature tensor and Ricci scalar should vanish

$$\begin{aligned} \hat{R}_{AB} &= 0 \\ \hat{R} &= 0 \end{aligned} \tag{2.3}$$

What is the importance of the 5D vacuum in terms of producing some physical fields? Assumptions along this line give some direct conclusions. For instance, the 5D vacuum solutions also offered the well-known 4D solutions.

The general 5D metric tensor related to the this 5D line element is

$$\hat{g}_{AB} = \left( \begin{array}{c|c} \hat{g}_{ab} & \hat{g}_{a5} \\ \hline \hat{g}_{5b} & \hat{g}_{55} \end{array} \right) \tag{2.4}$$

Kaluza defined the 5D metric elements as

$$\begin{aligned} \hat{g}_{ab} &= g_{ab} \\ \hat{g}_{5b} &= A_b \end{aligned} \tag{2.5}$$

Using matrix notation, Kaluza's 5D metric tensor turns to

$$\hat{g}_{AB} = \left( \begin{array}{c|c} g_{ab} & A_a \\ \hline A_b & \varphi \end{array} \right) \tag{2.6}$$

where $\varphi$ represents the scalar field, $g_{ab}$ represents the four dimensional metric and $A_a$ is the electromagnetic four potential. Therefore, in a 5D metric both electromagnetic four potential and metric of the actual spacetime are combined.

Another assumption by Kaluza is the cylinder condition, which means the physical variables of the 5D metric tensor does not depend on the fifth dimension. Mathematically, this is written as

$$\partial_5 \hat{g}_{ab} = 0 \tag{2.7}$$

meaning that the components of the metric are independent of the fifth coordinate.

Kaluza's cylinder condition have some drawbacks. First of all, he did not provide a clear reasoning as to why the fields are not related to the extra coordinate. Secondly, if we assume that there is a fifth dimension, why we do not observe it. It was Klein who gave the answers to the these questions.

The presence of matter and charge did not interest Kaluza. His aim was to embed the electromagnetic fields into the metric. However, Kaluza realised that in order to test his theory, he needed to have a charged test particle. So he studied the behaviour of the charged particle in 5D space. One can elaborate on the 5D current as

$$J^b = KT^{b5} \tag{2.8}$$

where K is the gravitational constant.

One of Kaluza's main goal is to obtain the geodesic of charge particles. By using Eq.(2.2), Klein obtained the geodesic of charge particles which also contains the Lorentz force law [22]

$$m\left(\frac{d^2x^a}{d\tau^2}+\Gamma^a_{mn}\frac{dx^m}{d\tau}\frac{dx^n}{d\tau}\right)=eF^a{}_b\frac{dx^b}{d\tau}-\frac{1}{2K}\frac{e^2}{m}\partial^a\varphi \tag{2.9}$$

There are two approaches to classify geodesics. One of them is choosing the $\varphi$ as a constant. The other one is $\frac{e}{m}$ is very small. However, this is not appropriate in atomic scale like when proton and electron are considered. This ensures that Kaluza's theory for subatomic particles are not valid. Therefore, we can conclude that electrons do not move on geodesics in the 5D. It is the $\varphi$ that determines the path. Kaluza did not deal with the $\varphi$. However, Klein selected $\varphi$ as a constant in order to observe the solution of the geodesic problem.

Due to property of the translational symmetry of the fifth dimension, it can be seen that the charge is suitable to be considered as a conserved quantity. Kaluza used the vacuum theory in order to define charge. He suggested the following equations

$$R_{ab}=K(T_{ab}-\frac{1}{2}g_{ab}T) \tag{2.10}$$

governing the gravity, while mixed components

$$R_{5a}=KT_{5a} \tag{2.11}$$

describe the electromagnetic field equation. Furthermore, component in the extra fifth dimension

$$R_{55}=K(T_{55}-\frac{1}{2}g_{55}T) \tag{2.12}$$

which is the equation of the scalar field.

Equation $(2.10)$ can also be written as

$$G_{ab}=R_{ab}-\frac{1}{2}g_{ab}R=KT_{ab} \tag{2.13}$$

Kaluza's 5D energy momentum tensor can be shown to take the form

$$\hat{T}_{AB}=\left(\begin{array}{c|c} T_{ab} & J^a \\ \hline J_a & 0 \end{array}\right) \tag{2.14}$$

Along with some other researchers, Einstein also thought about where the source term appeared. Grommer's works showed that Kaluza's vacuum field equations did not give the symmetric solutions in some cases. Moreover, Einstein discussed with Pauli about Kaluza's theory. He expressed his discontent about the representation of particles only by singular solutions of the gravitational field, and this is the reason why Einstein abandoned to study Kaluza's theory after some years.

## 2.2 Klein's Approach

Klein investigated the combining of the electromagnetism and general relativity by searching the wave function on a 5D manifold and the metric field [17]. By renewing Kaluza's cylinder condition idea, more conditions were imposed by Klein on the coordinates. One of the condition is that the first four coordinates are those of the usual spacetime. The other condition is that there is no dependency of the fields on $x^5$. This is known as the cylinder condition. The last condition is the $g_{55} = a$, where a is a any constant.

Klein's 5D line element is

$$d\hat{s}^2 = \hat{G}_{AB}\, d\hat{x}^A d\hat{x}^B \tag{2.15}$$

Klein also used the cylinder condition. However, he used a different metric from that of Kaluza's. His aim was also to combine general relativity with electromagnetic theory. The technique of Klein was different from that of Kaluza's. He wanted to transform coordinates in 5D among frames using a transformation law. This implies that within 5D frames, the coordinates should transform under the transformation law which is

$$x^A = f^A(x^{b'}) \tag{2.16}$$

The explicit forms of these coordinate transformations are

$$\begin{aligned} x^a &= x^{a'}(x^{0'}, x^{1'}, x^{2'}, x^{3'}) \\ x^5 &= x^{5'} + f^5(x^{0'}, x^{1'}, x^{2'}, x^{3'}) \end{aligned} \tag{2.17}$$

Generally, covariant theory means that when there is a coordinate transformation which is arbitrary differentiable spacetime, the structure of the field equations does not alter. This is also valid in the Kaluza-Klein theory. Generally, a covariant theory also entails gauge freedom. Now, the metric tensor transforms as

$$g'_{a5} = \left(\frac{\partial x^A}{\partial x'^a}\right)\left(\frac{\partial x^B}{\partial x'^5}\right) g_{AB} \tag{2.18}$$

Abelian gauge transformations

$$A'_a = A_a + \partial_a \varphi(x) \tag{2.19}$$

is a result of coordinate transformations to suggest an intrinsic extra dimension. In other words gauge transformation is the extra dimension's spacetime symmetry. Using these transformations and Klein's line element, the metric tensor can be written as

$$\hat{g}_{AB} = \left(\begin{array}{r|r} g_{ab} + A_a A_b & A_a \\ \hline A_b & 1 \end{array}\right) \tag{2.20}$$

where the line element is left unaltered. Also $\hat{g}_{55}$ is invariant under the coordinate transformation. Consequently, Klein described the 5D metric elements as

$$\begin{aligned} \hat{g}_{ab} &= g_{ab} + \varphi^2 A_a A_b \\ \hat{g}_{5b} &= \varphi^2 A_b \end{aligned} \tag{2.21}$$

In the Kaluza-Klein theory the cylinder condition is quite important. It is due to this condition that the reduction of the five dimension to four can be carried out and therefore equations for the constituent fields are obtained in the actual four dimensional spacetime.

Klein's 5D metric tensor in terms of the 4 dimensional components becomes

$$\hat{g}_{MN} = \left(\begin{array}{r|r} g_{ab} + \varphi^2 A_a A_b & \varphi^2 A_a \\ \hline \varphi^2 A_b & \varphi^2 \end{array}\right) \tag{2.22}$$

Therefore, we can conclude that Klein's 5D metric tensor is defined in terms of the 4D metric tensor, scalar potential and vector potential.

Also, Klein's 5D inverse metric tensor can be easily found as

$$\hat{g}^{MN} = \left(\begin{array}{c|r} g^{ab} & -A^a \\ \hline -A^b & \frac{1}{\varphi^2} + A_k A^k \end{array}\right) \tag{2.23}$$

and Klein's line element becomes

$$ds^2 = g_{ab} dx^a dx^b + \varphi^2 (dx^5 + A_a dx^a)^2 \tag{2.24}$$

The field equations of this theory can be found by varying of the 5D counterpart of the Einstein-Hilbert action

$$\mathcal{I} = \frac{1}{16\pi\hat{G}} \int d^5x \sqrt{-\hat{g}}\hat{R} \tag{2.25}$$

where $\hat{R}$ is the Ricci scalar and the extra coordinate is stated as $x^5$.

Using the variational principle and taking $\varphi = 1$, Klein found

$$\begin{aligned} G_{ab} &= KT_{ab} \\ D_a F^{ab} &= 0 \end{aligned} \tag{2.26}$$

as field equations. Therefore, we can conclude that one of the sources that create a curved spacetime are the fields or the particles constituted in the energy momentum tensor. Using the cylinder condition Klein obtained the full field equations. This result is similar to that of Kaluza's.

The approach of weak fields was greatly relaxed by Klein. The slow motion condition was eliminated and electrons were shown to act on geodesics.

To obtain geodesic, Klein used the relation between velocity along the fifth coordinate, and charge of the particle as

$$U^5 = \frac{e}{c}\frac{d\lambda}{d\tau} \tag{2.27}$$

where $\lambda$ represents the geodesic parameter and $\tau$ is proper time in 5D. Now, the Lagrangian can be defined as

$$L = \frac{1}{2}\left(\frac{ds}{d\tau}\right)^2 \tag{2.28}$$

From the assumption of Klein geodesic, and from the form of the Lagrangian, five dimensional momentum can be defined

$$p_i = \frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)} \tag{2.29}$$

Since $L$ does not depend on $x^5$, we conclude that there is a constant momentum which is along the fifth axes. For an electron, the momentum is

$$p_5 = \frac{e\sqrt{a}}{c\sqrt{2K}} \tag{2.30}$$

where $a$ denotes the constant value of the $\varphi$. The value of the $p_5$ is same at any arbitrary point of spacetime under the condition in which $\varphi$ is kept constant.

Klein also assumed that charge $q$ is related to the charge of the electron

$$q = ne \tag{2.31}$$

This allows us to express momentum as

$$p_5 = n\frac{e}{K} \tag{2.32}$$

where n is a integer.

Classical and quantum mechanical momentum has some common features. In quantum mechanics under specific condition quantisation is observed. The concept of compactification was given in the wave analysis in 5D. From the De Broglie hypothesis the fifth component of the linear momentum is quantised

$$p_5 = \frac{ne}{c\sqrt{2K}} = \frac{n\hbar}{\lambda_5} \tag{2.33}$$

where $\lambda_5$ represents the radius of the circle on the compact dimension $x^5$ [22].

Klein intention was to be convinced that the fifth dimension has no physical result. He notes that the fifth dimension has both spacial and circular topology, which explains the reason why the fifth coordinate is not observed. The periodicity of the fifth dimension is expressed as

$$f(x^5) = f(x^5 + 2\pi R) \tag{2.34}$$

Therefore, all constituent fields, i.e., the metric tensor, vector potential and scalar field can expressed as

$$g_{ab}(x^a, x^5) = \sum g_{ab}(x^a)e^{\frac{inx^5}{R}} \tag{2.35}$$

$$A_b(x^a, x^5) = \sum A_b(x^a)e^{\frac{inx^5}{R}} \tag{2.36}$$

$$\varphi(x^a, x^5) = \sum \varphi(x^a)e^{\frac{inx^5}{R}} \tag{2.37}$$

in terms of Fourier series expansion.

## 2.3 Geodesic Equation, Motion of a Charged Test Particle

Both Kaluza and Klein investigated the motion of a test particle, namely the associated geodesic equation to the action written as

$$\mathcal{I} = \frac{1}{2}\int \left(\hat{g}_{AB}\frac{d\hat{x}^A}{d\tau}\frac{d\hat{x}^B}{d\tau}\right)^{\frac{1}{2}} d\tau \tag{2.38}$$

where $\tau$ is an affine parameter. We know that if $\tau$ is an affine parameter and $d\hat{x}^A/d\tau$ is tangent to the geodesic, then

$$\hat{g}_{AB}\hat{K}^A\left(\frac{d\hat{x}^B}{d\tau}\right) = C \tag{2.39}$$

where $\hat{K}^A$ is the Killing vector and $C$ is constant . Here the Killing vector is

$$\hat{K} = \hat{K}^A\frac{\partial}{\partial \hat{x}^A} = \frac{\partial}{\partial x^5} \tag{2.40}$$

with $\hat{K}^A = (0,0,0,0,1)$. Now from (2.39) we have

$$K^5\left(\frac{dx^5}{d\tau}\right) = C \tag{2.41}$$

Here, we are using the components of the metric in Appendix B for an orthogonal coordinate system. The geodesic equation in 5D can be written in the form

$$\frac{d^2\hat{x}^A}{d\tau^2} + \hat{\Gamma}^A_{BC}\frac{d\hat{x}^B}{d\tau}\frac{d\hat{x}^C}{d\tau} = 0 \tag{2.42}$$

For $A = a$ we have

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^a_{b5}\frac{dx^b}{d\tau}\frac{dx^5}{d\tau} + \hat{\Gamma}^a_{5b}\frac{dx^5}{d\tau}\frac{dx^b}{d\tau} = 0 \tag{2.43}$$

Reading off the necessary Christoffel symbols from Appendix (B.16) and (B.21) we have

$$\hat{\Gamma}^a_{5b} = -\frac{1}{2}\varphi F^a{}_b, \qquad \hat{\Gamma}^a_{b5} = -\frac{1}{2}\varphi F^a{}_b \tag{2.44}$$

to be substituted in (2.43), where it becomes

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = C\,\varphi F^a{}_b\frac{dx^b}{d\tau}. \tag{2.45}$$

Now, one can safely identify the constant $C$ with $q/m$, to obtain

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = \frac{q}{m}\varphi F^a{}_b\frac{dx^b}{d\tau} \tag{2.46}$$

describing the motion of a charged particle in an electromagnetic field in curved spacetime, where the right hand side is typically the Lorentz force when $\varphi = 1$ (as was Klein inclined to assume $\varphi$ to be a constant). Otherwise, the right hand side can further be accounted for an interaction between the scalar field and the electromagnetic field.

## 2.4 Kaluza-Klein Mass Tower

Kaluza-Klein theory inspires a variety of models to attain extra dimensions, for the purpose of combining the fundamental forces like electromagnetic force, strong force, weak force and gravity. As an example, we can give the model with large extra dimensions also known as the ADD model. The ADD model compares gravity with other fundamental forces in terms of their strength, and investigates why gravity is quite a weak force when compared with other fundamental forces. This is the so-called hierarchy problem. In this model, the components of the 4D part and the components of the extra dimension of the metric are independent from each other. Therefore the metric takes the form

$$d\hat{s}^2 = \hat{g}_{AB} d\hat{x}^A d\hat{x}^B = g_{ab}(x) dx^a dx^b + g_{cd}(y) dy^c dy^d \tag{2.47}$$

where $A, B = (0, ..., 4+n); a, b = (0, 1, 2, 3)$ and $c, d = (5, ..., 4+n)$. Here $n$ represents number of the extra dimensions.

When the four dimensional spacetime is extended to one or more than one dimensions, separation of the full space gives subspaces. The whole space is called the bulk space. Only, gravity is permitted to propagate through the (4+n) dimensional bulk, while the particles or the fields of the standard model is restricted to reside in our four dimensional universe. That is why gravity is a weak force.

The relation between the Planck mass and (4+n) dimensional scale is formulated as [23]

$$\tilde{M}^2_{Planck} = V_n M^{n+2} \tag{2.48}$$

where M is the fundamental scale which is (4+n) dimensional. Also, $V_n$ represents the $n$ dimensional volume element of the extra dimension. If we suppose all extra dimensions have the same size then $V_n$ can be written as

$$V_n = (2\pi R)^n \tag{2.49}$$

Therefore $(2.48)$ becomes

$$\tilde{M}^2_{Planck} = (2\pi R)^n M^{n+2} \tag{2.50}$$

The assumption that the size of the extra dimensions are the same, comes from the relation between fundamental scale and extra dimensions. The Plank mass is much greater than fundamental scale because the size of the extra dimension is greater than the fundamental scale. Usually $n = 1$ is ignored because $R$ is approximately going to be $10^{13}$ meter. When $n = 2$, $2\pi R$ going to be quite small.

When the dimension is five, ADD scenario is investigated under the condition $n = 1$. The extra dimension is denoted by $z$ which should be space-like. When we deal with scalar particles the following equation can be given as an ansatz in order to get Kaluza-Klein mass

$$\partial^2_z \varphi(z) = -m^2_n \varphi(z) \tag{2.51}$$

This equation is known as the Klein-Gordon equation, from where one can obtain Kaluza-Klein masses, denoted by $m_n$. Moreover, from the boundary conditions $m^2_n$ can be found as the eigenvalues of the mass.

One can investigate the behaviour of the field with one extra dimension where $z$ has a finite range: $[0, 2\pi R]$. Different geometries can be obtained by specifying the range of $z$. If the range of the z is between $(0, 2\pi R)$, the geometry is like a ribbon. If $z = 0 = 2\pi R$ now the geometry is a cylinder. However, the range of the $x^a$ is $[-\infty, \infty]$.

Now, we assume that mass eigenvalues are positive, so $(2.51)$ can be solved by considering

$$\varphi_n(x^a, z) = A_n sin(m_n z) + B_n cos(m_n z) \tag{2.52}$$

where $A_n$ and $B_n$ denote real functions. The best way to specify Kaluza-Klein masses comes from operator diagonalisation. In view of the fact that $m^2$ is a real observable quantity, $p_z$ has to be a symmetric operator. By considering the boundary conditions some Kaluza-Klein states can be obtained. Since the compactified dimension is periodic, one can write the periodicity of the scalar field as

$$\varphi(x, z) = \varphi(x, z + 2\pi R) \tag{2.53}$$

The meaning of the periodicity of $z$ is that Fourier series expansion can be done. By using Fourier series expansion, we can write following equation

$$\Phi(x^a, z) = \sum_{-\infty}^{+\infty} \varphi^n(x^a) e^{\frac{inz}{R}} \tag{2.54}$$

In terms of four dimensional spacetime, $\varphi^n(x^a)$ represents the tower which is composed of four dimensional fields.

Therefore, when compactification is done for the extra dimensions all fields propagating through the bulk have distinct modes. These modes are called as Kaluza-Klein tower of states. Their mode numbers is denoted with $n = (n_1, ..., n_n)$. Then the momentum of the fields can be written as

$$p_n^2 = \frac{n^2}{R^2} \tag{2.55}$$

Here, $n$ is very important in terms of defining the fields. For instance, when $n$ is zero, this means that the fields are homogenous. When $n$ is not equal to zero, in this case Kaluza-Klein modes for spin 0, spin 1 and spin 2 are obtained.

One possible to approach to the issue of Kaluza-Klein masses is considering the Lagrangian density. To this end we write

$$\mathcal{L} = \frac{1}{2} \partial_M \varphi \partial^M \varphi \tag{2.56}$$

where $M = 0, 1, 2, 3, 5$.

By separating $\varphi$ in terms of momentum eigenfunctions, (2.56) can be written as

$$\mathcal{L} = \frac{1}{2} \left( \partial_a \varphi \partial^a \varphi - \partial_z \varphi \partial_z \varphi \right) \tag{2.57}$$

If we substitude $(2.54)$ into $(2.57)$ we obtain

$$\mathcal{L} = \frac{1}{2} \left[ \partial_a (\varphi^n e^{\frac{inz}{R}}) \partial^a (\varphi^m e^{\frac{imz}{R}}) - \partial_z (\varphi^n e^{\frac{inz}{R}}) \partial_z (\varphi^m e^{\frac{imz}{R}}) \right] \tag{2.58}$$

After taking care of the derivatives in the above equation, we get

$$\mathcal{L} = \frac{1}{2} \left( \partial_a \varphi_n \partial^a \varphi_m - \frac{nm}{R^2} \varphi_n \varphi_m \right) e^{\frac{i(n+m)z}{R}} \tag{2.59}$$

When $m = n$, the Lagrangian density becomes:

$$\mathcal{L} = \frac{1}{2}\left(|\partial_a \varphi_n|^2 - \frac{n^2}{R^2}|\varphi_n|^2\right) \tag{2.60}$$

Now, we substitute above Lagrangian density in the action integral

$$\mathcal{I} = \int d^4x \int_0^{2\pi R} \mathcal{L}\, dz = 2\pi R \sum_{-\infty}^{+\infty} \int \frac{1}{2}\left(|\partial_a \varphi_n|^2 - \frac{n^2}{R^2}|\varphi_n|^2\right) d^4x \tag{2.61}$$

which becomes four dimensional. In order to obtain a canonical normalised field, we substitute

$$\hat{\varphi}_n = \frac{1}{\sqrt{2\pi R}}\varphi_n \tag{2.62}$$

into (2.61) to obtain

$$\mathcal{I} = \frac{1}{2}\sum_{-\infty}^{+\infty} \int \left(|\partial_a \hat{\varphi}_n|^2 - \frac{n^2}{R^2}|\hat{\varphi}_n|^2\right) d^4x \tag{2.63}$$

Finally, the Kaluza-Klein masses are deduced to be [24]

$$m_n = \frac{n}{R} \tag{2.64}$$

To conclude this introductory review section about the Kaluza-Klein theory we may say that, apart from being elegantly devised, its importance mostly lies in the fact that it has been and may be still is an inspiration for other unified theories.

# CHAPTER 3

# FIELD EQUATIONS FROM VARIATION PRINCIPLES

In this chapter, we study equations pertinent to the Kaluza-Klein theory from the point of least action principle, i.e., by using $\delta\mathcal{I} = 0$, where $\mathcal{I}$ is the 5D Einstein-Hilbert action.

There are several ways to approach the matter at hand. First, we start with the 5D action and we reduce to it to the actual spacetime; then we vary it with respect to three emerging fields, namely the scalar $\varphi$, $A_k$ and $g^{ab}$, to obtain the field equations. This scheme constitutes a subclass of scalar-tensor theories and is referred as the Jordan frame, in the literature [25], in contrast to the Einstein frame, where in this latter form the curvature appears by itself, i.e., not being multiplied by $\varphi$.

In Sec. 2, the action is varied with respect to the 5D metric $\hat{g}^{AB}$ to obtain the 5D Einstein tensor $\hat{G}_{AB} = 0$. Afterwards we reduce the dimension to obtain three equations from $\hat{G}_{ab} = 0$, $\hat{G}_{a5} = 0$ and $\hat{G}_{55} = 0$. We note that, the order of these two processes, namely variation and reduction, is switched in regard to the Sec. 1. If all three equations turn out to be the same as those of the first section, then we can conclude that these two consecutive operations are commutative in this particular case.

In Sec. 3, we perform conformal rescaling to the action, where reduced the term governing the gravity sector is the same as in the case of 4D. Thus we obtain the field equations in the Einstein frame. It is to be expected that the equations obtained from these two frames although may possess some similarities will yield different results.

In Sec. 4, in order to gain insight how the resulting field equations are affected by the order of operations, the 5D Einstein tensor is conformally transformed, before reducing it to 4D. In the following section we derive the 5D Einstein tensor from the

5D Bianchi identity.

Finally, we discuss the Brans-Dicke (BD) theory [20] and the specific conditions that make the two theories, BD and KK, alike.

### 3.1 Field Equations From the Action

Let us consider the five dimensional action as

$$\mathcal{I} = \frac{1}{16\pi\hat{G}} \int \hat{R}\sqrt{-\hat{g}}\, d^5x \tag{3.1}$$

The Lagrangian density is defined as $\mathcal{L} = L\sqrt{-\hat{g}}$ and here it explicitly is

$$\mathcal{L} = \hat{R}\sqrt{-\hat{g}} \tag{3.2}$$

By making use of the Schur complement, five dimensional $\sqrt{-\hat{g}}$ can be decomposed in the form

$$\sqrt{-\hat{g}} = \varphi\sqrt{-g} \tag{3.3}$$

where the details about the application of the Schur complement [26] is put in the Appendix D. Then, in view of the above decomposition and (B.50), the action (3.1) becomes

$$\mathcal{I} = \frac{2\pi R_c}{16\pi\hat{G}} \int \varphi[R - \frac{1}{4}\varphi^2 F^{ab}F_{ab} - 2\varphi^{-1}D_a\varphi^a]\sqrt{-g}\, d^4x \tag{3.4}$$

where $\varphi^a$ is a shorthand notation: $\varphi^a \equiv D^a\varphi = \partial^a\varphi$. Hence, with the redefinition of the gravitational constant $G = \hat{G}/2\pi R_c$, (3.4) is rewritten as

$$\mathcal{I} = \frac{1}{16\pi G} \int \varphi[R - \frac{1}{4}\varphi^2 F^{ab}F_{ab} - 2\varphi^{-1}D_a\varphi^a]\sqrt{-g}\, d^4x \tag{3.5}$$

The integrands can be separated

$$\mathcal{I} = \frac{1}{16\pi G} \int d^4x(\sqrt{-g})\varphi[R - \frac{1}{4}\varphi^2 F^{ab}F_{ab}] - \frac{1}{8\pi G}\int d^4x(\sqrt{-g})(D_aD^a\varphi) \tag{3.6}$$

for further simplifications.

Here below and in the sequel, we shall frequently be making use of the Stoke's (generalised divergence) theorem

$$\int_\Omega D_kV^k\sqrt{-g}\, d^nx = \int_{\partial\Omega} n_kV^k\sqrt{-g}\, d^{n-1}x \tag{3.7}$$

where $n_k$ is the unit normal to the boundary of manifold $(\partial\Omega)$ and the integrand on the left hand side is a total divergence (times the $n$ dimensional metric volume element) [25].

Therefore, one can drop the second term in (3.6), since it is a total divergence and all the physical fields, as well as this particular one, vanish on the boundary. Finally, we obtain the Kaluza-Klein action

$$\mathcal{I} = \frac{1}{16\pi G}\int d^4x(\sqrt{-g})\varphi[R - \frac{1}{4}\varphi^2 F^{ab}F_{ab}] \tag{3.8}$$

where

$$\mathcal{L} = \sqrt{-g}\varphi R - \frac{1}{4}\sqrt{-g}\varphi^3 F^{ab}F_{ab} \tag{3.9}$$

is the Lagrangian density.

The field equations will be obtained by varying with respect to the fields involved in the action, namely $\varphi$, $A_k$ and the metric $g^{ab}$ .

Now, let us vary $(3.9)$ with respect to $A_k$

$$\delta\mathcal{L} = -\frac{1}{4}\sqrt{-g}\varphi^3\delta(F^{ab}F_{ab}) \tag{3.10}$$

to obtain

$$\delta\mathcal{L} = -\frac{1}{4}\sqrt{-g}\varphi^3(\delta F^{ab}F_{ab} + F^{ab}\delta F_{ab}) \tag{3.11}$$

which can be simplified as

$$\delta\mathcal{L} = -\frac{1}{2}\sqrt{-g}\varphi^3(\delta F_{ab}F^{ab}) \tag{3.12}$$

We know from electromagnetic theory that $F_{ab} = D_a A_b - D_b A_a$. Substituting this definition into above, we get

$$\delta\mathcal{L} = -\frac{1}{2}\sqrt{-g}\varphi^3(D_a\delta A_b - D_b\delta A_a)F^{ab} \tag{3.13}$$

Since $F^{ab}$ is antisymmetric the above equation becomes

$$\delta\mathcal{L} = -\frac{1}{2}\sqrt{-g}\varphi^3(D_a\delta A_b + D_a\delta A_b)F^{ab} \tag{3.14}$$

and then simplifies as

$$\delta\mathcal{L} = -\sqrt{-g}\varphi^3(D_a\delta A_b)F^{ab} \tag{3.15}$$

Consider

$$D_a(\sqrt{-g}\varphi^3 F^{ab}\delta A_b) = D_a(\sqrt{-g}\varphi^3 F^{ab})\delta A_b + \sqrt{-g}\varphi^3 F^{ab} D_a \delta A_b \tag{3.16}$$

then the action

$$\delta\mathcal{I} = -\frac{1}{16\pi G}\int d^4x[D_a(\sqrt{-g}\varphi^3 F^{ab}\delta A_b) - D_a(\sqrt{-g}\varphi^3 F^{ab})\delta A_b] \tag{3.17}$$

Using Stokes's theorem (3.7), the first term of the above equation drops. Then, from $\delta\mathcal{I} = 0$ , we obtain

$$D_a(\sqrt{-g}\varphi^3 F^{ab}) = 0 \tag{3.18}$$

By varying $(3.8)$ with respect to $\varphi$

$$\delta\mathcal{I} = \frac{1}{16\pi G}\int (R - \frac{3}{4}\varphi^2 F^{ab} F_{ab})\delta\varphi\sqrt{-g}d^4x \tag{3.19}$$

and in view of $\delta\mathcal{I} = 0$, we obtain

$$R = \frac{3}{4}\varphi^2 F^{ab} F_{ab} \tag{3.20}$$

which is an algebraic equation, and restricts the scalar field to $R$ and $F_{ab}$. We also have to vary $(3.8)$ with respect to the metric. For this purpose we rewrite $(3.9)$ so that the hidden metric becomes apparent

$$\mathcal{L} = \sqrt{-g}g^{ab}\varphi R_{ab} - \frac{\varphi^3}{4}\sqrt{-g}g^{ma}g^{nb}F_{mn}F_{ab} \tag{3.21}$$

and

$$\begin{aligned}\delta\mathcal{L} &= \delta(\sqrt{-g})g^{ab}\varphi R_{ab} + \sqrt{-g}\delta g^{ab}\varphi R_{ab} \\ &+\varphi g^{ab}\delta R_{ab}\sqrt{-g} - \tfrac{\varphi^3}{4}\delta\sqrt{-g}g^{ma}g^{nb}F_{mn}F_{ab} \\ &-\tfrac{\varphi^3}{4}\sqrt{-g}\delta g^{ma}g^{nb}F_{mn}F_{ab} - \tfrac{\varphi^3}{4}\sqrt{-g}g^{ma}\delta g^{nb}F_{mn}F_{ab}\end{aligned} \tag{3.22}$$

The third term of the above equation can be calculated

$$\delta R_{ab} = D_k\delta\Gamma^k_{ab} - D_b\delta\Gamma^k_{ak} \tag{3.23}$$

with

$$\delta\Gamma^k_{ab} = -\frac{1}{2}[g_{aj}D_b\delta g^{kj} + g_{bj}D_a\delta g^{kj} - g_{ai}g_{bj}D^k\delta g^{ij}] \tag{3.24}$$

where the details are relegated to Appendix D. We also need the covariant derivative of the connection

$$D_k\delta\Gamma^k_{ab} = -\frac{1}{2}[g_{aj}D_kD_b\delta g^{kj} + g_{bj}D_kD_a\delta g^{kj} - g_{ai}g_{bj}D_kD^k\delta g^{ij}] \tag{3.25}$$

Similarly, we can calculate $D_k\delta\Gamma^k_{ab}$. Then, we are ready to obtain $g^{ab}\delta R_{ab}$

$$\begin{aligned} g^{ab}\delta R_{ab} &= -\tfrac{1}{2}g^{ab}(g_{aj}D_kD_b\delta g^{kj} + g_{bj}D_kD_a\delta g^{kj} - g_{ai}g_{bj}D_kD^k\delta g^{ij} \\ &\quad -g_{aj}D_bD_k\delta g^{jk} - g_{kj}D_bD_a\delta g^{jk} + g_{ai}g_{kj}D_bD^k\delta g^{ij}) \end{aligned} \tag{3.26}$$

After simplifications, we get

$$\varphi g^{ab}\delta R_{ab} = \varphi g_{kj}D_nD^n\delta g^{kj} - \varphi D_kD_j\delta g^{kj} \tag{3.27}$$

Now, using

$$\varphi g_{kj}D_nD^n\delta g^{kj} = D_n(\varphi g_{kj}D^n\delta g^{kj}) - D^n(D_n\varphi g_{kj}\delta g^{kj}) + (D_nD^n\varphi)g_{kj}\delta g^{kj} \tag{3.28}$$

and

$$\varphi D_kD_j\delta g^{kj} = D_k(\varphi D_j\delta g^{kj}) - D_j(D_k\varphi\delta g^{kj}) + (D_jD_k\varphi)\delta g^{kj} \tag{3.29}$$

apart from the total divergences

$$\int d^4x\sqrt{-g}\varphi g^{ab}\delta R_{ab} \tag{3.30}$$

takes the form

$$\int d^4x\sqrt{-g}[(D_nD^n\varphi)g_{ab} - D_aD_b\varphi]\delta g^{ab} \tag{3.31}$$

Then by integrating (3.22), variation of (3.8) with respect to metric turns to

$$\begin{aligned} \delta\mathcal{I} &= \int d^4x\big\{\sqrt{-g}(-\tfrac{1}{2}g_{ab}\varphi R + \varphi R_{ab})\delta g^{ab} + \sqrt{-g}[(D_nD^n\varphi)g_{ab} - D_aD_b\varphi]\delta g^{ab} \\ &\quad -\tfrac{\varphi^3}{4}(-\tfrac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}F^{mn}F_{mn} + \delta g^{ma}F_m{}^bF_{ab}\sqrt{-g} + \delta g^{nb}F^a{}_nF_{ab}\sqrt{-g})\big\} \\ &= \int d^4x\big\{\sqrt{-g}(-\tfrac{1}{2}g_{ab}\varphi R + \varphi R_{ab})\delta g^{ab} + \sqrt{-g}[(D_nD^n\varphi)g_{ab} - D_aD_b\varphi]\delta g^{ab} \\ &\quad -\tfrac{\varphi^3}{2}(F_{am}F_b{}^m - \tfrac{1}{4}g_{ab}F^{mn}F_{mn})\sqrt{-g}\,\delta g^{ab}\big\} \end{aligned} \tag{3.32}$$

where we employed [3]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} \tag{3.33}$$

For $\delta\mathcal{I} = 0$, we get

$$\begin{aligned}&\sqrt{-g}(-\tfrac{1}{2}g_{ab}\varphi R + \varphi R_{ab}) + \sqrt{-g}[(D_n D^n\varphi)g_{ab} - D_a D_b\varphi] \\ &-\tfrac{\varphi^3}{2}(F_{am}F_b{}^m - \tfrac{1}{4}g_{ab}F^{mn}F_{mn})\sqrt{-g} = 0\end{aligned} \tag{3.34}$$

and rearranging we have

$$(R_{ab} - \frac{1}{2}g_{ab}R) = \frac{\varphi^2}{2}(F_{am}F_b{}^m - \frac{1}{4}g_{ab}F^{mn}F_{mn}) + \frac{1}{\varphi}(D_a D_b\varphi - g_{ab}D_n D^n\varphi) \tag{3.35}$$

and can concisely be rewritten as

$$G_{ab} = \frac{1}{2}\varphi^2 T^{(EM)}_{ab} + \varphi^{-1}T^{(\varphi)}_{ab} \tag{3.36}$$

which is of the form of Einstein's equation, where

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R \tag{3.37}$$

is the Einstein's tensor in 4D, $T^{(EM)}_{ab}$ is the electromagnetic energy-momentum tensor

$$T^{(EM)}_{ab} = F_{ak}F_b{}^k - \frac{1}{4}g_{ab}F_{mn}F^{mn} \tag{3.38}$$

and

$$T^{(\varphi)}_{ab} = D_a D_b\varphi - g_{ab}D_n D^n\varphi \tag{3.39}$$

is the energy-momentum tensor for the scalar field. In the next section this equation will take a slightly different form and we will compare (3.36) with equation (3.81) .

## 3.2 A Direct Approach to Field Equations

Another approach to obtain the field equations from the 5D action in Eq.(3.1) is to directly vary it with respect to 5D metric $\hat{g}_{AB}$. Then one can continue to evaluate

$$\delta\int \hat{R}\,\sqrt{-\hat{g}}\,d^5x = 0 \tag{3.40}$$

with a similar procedure as that of the 4D case, to obtain

$$\hat{G}_{AB} = 0 \tag{3.41}$$

with

$$\hat{G}_{AB} \equiv \hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} \tag{3.42}$$

where $\hat{G}_{AB}$ is the Einstein's tensor in 5D.

Now for $\hat{G}_{ab} = 0$ we have

$$G_{ab} = \frac{\varphi^2}{2}(F_{ak}F_b{}^k - \frac{1}{4}g_{ab}F_{mn}F^{mn}) + \varphi^{-1}(D_aD_b\varphi - g_{ab}D_kD^k\varphi) \qquad (3.43)$$

which attains exactly the same form as in (3.36) and as in [27]. There are two more equations to be derived from the fifth components. The first one is from

$$\hat{G}_{a5} \equiv \hat{R}_{a5} - \frac{1}{2}\hat{g}_{5a}\hat{R} = 0 \qquad (3.44)$$

which yields

$$D_kF^k{}_a = 3\varphi^{-1}\varphi^kF_{ak} \qquad (3.45)$$

where, we see that the scalar field and the electromagnetic field itself appearing on the right hand side act as a source term. This can be rewritten to obtain the same equation (3.18), which is obtained when the Kaluza-Klein action (3.8) is varied with respect to $A_k$. Finally from

$$\hat{G}_{55} \equiv \hat{R}_{55} - \frac{1}{2}\hat{g}_{55}\hat{R} = 0 \qquad (3.46)$$

we have after simplification

$$R = \frac{3}{4}\,\varphi^2F_{jk}F^{jk} \qquad (3.47)$$

which is exactly in the same form as in (3.20). Therefore from (3.41) and (B.50) we have the field equations of the standard Kaluza-Klein theory as they frequently appear in the literature [27].

Furthermore, one can be tempted to take the trace of $\hat{G}_{AB}$ in (3.41), which yields $\hat{R} = 0$, and thus we have $\hat{R}_{AB} = 0$. In this case, from Appendix (B.46) we have the same equation as that of (3.45), and from (B.48) we have

$$D^kD_k\varphi = \frac{1}{4}\varphi^3F^{jk}F_{jk} \qquad (3.48)$$

which does not resemble to any of the field equations in the previous section and is somewhat different from (3.70) due to an extra term. We should also notice that equation from $\hat{R}_{ab} = 0$ is

$$R_{ab} = \frac{\varphi^2}{2}\,F_{ak}F_b{}^k + \varphi^{-1}\,D_aD_b\varphi \qquad (3.49)$$

and attains a different form that of (3.43). In fact, one should be faithful to the full-fledged Einstein's equation $\hat{G}_{AB} = 0$. After all it is an expression for the total energy, and in our actual spacetime, due to the presence of the electromagnetic and the scalar fields the energy content of the spacetime is not restricted only to that of the gravitational field. Therefore setting $\hat{R}_{AB} = 0$ does not seem to be a legitimate approach in 5D, as in the case of 4D to obtain field equations.

### 3.3 Conformal Rescaling

In $(3.8)$, we observe that there is a product of Ricci curvature and scalar field. The scalar field $\varphi$ should not appear to obtain the canonical gravitational term. This can be achieved using a conformal rescaling

$$\tilde{g}_{ab} = \Phi^2(x^a) g_{ab} \tag{3.50}$$

on the condition that $\Phi^2 > 0$.

One can write Ricci tensor in a N dimensional spacetime as

$$\begin{aligned} \tilde{R}_{ab} &= R_{ab} - \Phi^{-1} g_{ab} D_m D^m \Phi - (N-2)\Phi^{-1}(D_a D_b \Phi) \\ &\quad -(N-3)\Phi^{-2} g_{ab}(D_m\Phi)(D^m\Phi) + 2(N-2)\Phi^{-2}(D_a\Phi)(D_b\Phi) \end{aligned} \tag{3.51}$$

Also Ricci scalar can be written in N dimensional spacetime as

$$\tilde{R} = \Phi^{-2} R - 2(N-1)\Phi^{-3} D_m D^m \Phi - (N-1)(N-4)\Phi^{-4}(D_m\Phi)(D^m\Phi) \tag{3.52}$$

The transformation of the N dimensional determination of the metric is the following equation

$$\tilde{g} = \Phi^{2N} g \tag{3.53}$$

Therefore, $(3.8)$ alter as

$$\tilde{\mathcal{I}} = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} [\varphi \tilde{R} - \frac{1}{4}\varphi^3 F^{ab} F_{ab}] \tag{3.54}$$

When the N=4, Ricci scalar and determinant of the metric is going to be

$$\begin{aligned} \tilde{g} &= \Phi^8 g \\ \tilde{R} &= \Phi^{-2} R - 6\Phi^{-3} D_m D^m \Phi \end{aligned} \tag{3.55}$$

When we put the Ricci scalar and metric determinant to the (3.54 ), our action becomes

$$\tilde{\mathcal{I}} = \frac{1}{16\pi G}\int d^4x(\sqrt{-g})[\Phi^2\varphi R - 6\varphi\Phi D_m D^m\Phi - \frac{1}{4}\varphi^3 F^{ab}F_{ab}] \tag{3.56}$$

Now, our aim is to get R without their coefficient. For this reason, $\Phi = \varphi^{-\frac{1}{2}}$ should be selected. After putting $\Phi = \varphi^{-\frac{1}{2}}$ to the above equation, and doing some simple arrangements one can get the result as

$$\tilde{\mathcal{I}} = \frac{1}{16\pi G}\int d^4x(\sqrt{-g})[R - \frac{1}{4}\varphi F^{ab}F_{ab} + 3\varphi^{-1}D_m D^m\varphi - \frac{9}{2}\varphi^{-2}\varphi_k\varphi^k] \tag{3.57}$$

Consider

$$3\varphi^{-1}D_m D^m\varphi = D_m(3\varphi^{-1}D^m\varphi) - 3\varphi^{-2}D_k\varphi D^k\varphi \tag{3.58}$$

By using divergence theorem and arranging the terms, we get

$$\tilde{\mathcal{I}} = \frac{1}{16\pi G}\int d^4x(\sqrt{-g})[R - \frac{1}{4}\varphi F^{ab}F_{ab} - \frac{3}{2}\varphi^{-2}\varphi_k\varphi^k] \tag{3.59}$$

Now we can vary the $(3.59)$ with respect to vector potential $A_k$, metric $g^{ab}$ and scalar potential $\varphi$. Then using the $\delta\mathcal{I} = 0$, we will obtain the field equations. Also, we can omit tildes.

The Lagrangian density is

$$\mathcal{L} = (\sqrt{-g})[R - \frac{1}{4}\varphi F^{ab}F_{ab} - \frac{3}{2}\varphi^{-2}\varphi_k\varphi^k] \tag{3.60}$$

Let us vary the Lagrangian density with respect to $A_k$

$$\delta\mathcal{L} = -\frac{1}{4}\sqrt{-g}\varphi\delta(F^{ab}F_{ab}) \tag{3.61}$$

By simplifying, we get

$$\delta\mathcal{L} = -\frac{1}{2}\sqrt{-g}\varphi(\delta F_{ab}F^{ab}) \tag{3.62}$$

Using $F_{ab} = D_aA_b - D_bA_a$ and taking the variation of this identity, we get

$$\delta\mathcal{L} = -\frac{1}{2}\sqrt{-g}\varphi(D_a\delta A_b - D_b\delta A_a)F^{ab} \tag{3.63}$$

Then after simplification, we obtain

$$\delta\mathcal{L} = -\sqrt{-g}\varphi(D_a\delta A_b)F^{ab} \tag{3.64}$$

Consider

$$D_a(\sqrt{-g}\varphi F^{ab}\delta A_b) = D_a(\sqrt{-g}\varphi F^{ab})\delta A_b + \sqrt{-g}\varphi F^{ab} D_a\delta A_b \tag{3.65}$$

Then by using divergence theorem, our action turns to

$$\delta\mathcal{I} = \frac{1}{16\pi G}\int d^4x D_a(\sqrt{-g}\varphi F^{ab})\delta A_b \tag{3.66}$$

using $\delta\mathcal{I} = 0$ , we obtain

$$D_a(\sqrt{-g}\varphi F^{ab}) = 0 \tag{3.67}$$

Variation of the (3.59) with respect to $\varphi$ is

$$\delta\mathcal{I} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[-\frac{1}{4}F^{ab}F_{ab}\delta\varphi + 3\varphi^{-3}D_k\varphi D^k\varphi\delta\varphi - 3\varphi^{-2}(D_k\delta\varphi)D^k\varphi] \tag{3.68}$$

We use

$$D_k(\sqrt{-g}\varphi^{-2}D^k\varphi\delta\varphi) = D_k(\sqrt{-g}\varphi^{-2}D^k\varphi)\delta\varphi + \sqrt{-g}\varphi^{-2}D^k\varphi D_k\delta\varphi \tag{3.69}$$

to deal with the third term in (3.68). The left hand side of the above equation vanishes on the surface due to divergence theorem. Thus, after simplifications, $\delta\mathcal{I} = 0$ renders the equation

$$D_kD^k\varphi - \varphi^{-1}D_k\varphi D^k\varphi = -\frac{\varphi^2}{12}F^{ab}F_{ab} \tag{3.70}$$

Now, let us vary the (3.59) with respect to $g^{ab}$. For convenience, we consider each term separately

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \tag{3.71}$$

where

$$\begin{aligned} &\mathcal{L}_1 = \sqrt{-g}R = \sqrt{-g}g^{ab}R_{ab}, \qquad \mathcal{L}_2 = -\tfrac{\varphi}{4}\sqrt{-g}g^{ma}g^{nb}F_{mn}F_{ab}, \\ &\mathcal{L}_3 = -\tfrac{3}{2}\varphi^{-2}g^{mn}(D_m\varphi)(D_n\varphi)\sqrt{-g}. \end{aligned} \tag{3.72}$$

Then we can write $\delta\mathcal{L}_1$ as

$$\delta\mathcal{L}_1 = (\delta\sqrt{-g})\,R + \sqrt{-g}\delta g^{ab}R_{ab} + \sqrt{-g}g^{ab}\delta R_{ab} \tag{3.73}$$

For the third term we use Palatini equation [3]

$$\delta R^k_{amb} \doteq D_m(\delta\Gamma^k_{ab}) - D_b(\delta\Gamma^k_{am}) \tag{3.74}$$

where $\doteq$ denotes that we are working in geodesic coordinates in which $\Gamma^a_{bc} = 0$. Since $\delta\Gamma^a_{bc}$ in a tensor, we have

$$\delta R_{ab} = D_k(\delta\Gamma^k_{ab}) - D_b(\delta\Gamma^k_{ak}) \tag{3.75}$$

Therefore

$$\int_\Omega g^{ab}\delta R_{ab}\sqrt{-g}\, d^4x = \int_\Omega D_k(g^{ab}\delta\Gamma^k_{ab} - g^{ak}\delta\Gamma^b_{ab})\sqrt{-g}\, d^4x \tag{3.76}$$

which can be converted to surface integral by divergence theorem (3.7) and vanishes since the variations vanish on the surface of $\Omega$. Thus, we can safely omit the third term in (3.73). Then, $\delta\mathcal{L}_1$ simplifies as

$$\delta\mathcal{L}_1 = \sqrt{-g}(-\frac{1}{2}g_{ab}R + R_{ab})\delta g^{ab} \tag{3.77}$$

Now, we can vary $\mathcal{L}_2$

$$\begin{aligned} \delta\mathcal{L}_2 &= -\tfrac{\varphi}{4}(-\tfrac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}F^{mn}F_{mn} + \delta g^{ma}F_m{}^bF_{ab}\sqrt{-g} + \delta g^{nb}F^a{}_nF_{ab}\sqrt{-g}) \\ &= -\tfrac{\varphi}{2}(F_{am}F_b{}^m - \tfrac{1}{4}g_{ab}F^{mn}F_{mn})\sqrt{-g}\,\delta g^{ab} \end{aligned} \tag{3.78}$$

where, again we have used (3.33).

The remaining variation for $\mathcal{L}_3$

$$\begin{aligned} \delta\mathcal{L}_3 &= -\tfrac{3}{2}\varphi^{-2}[\delta g^{ab}(D_a\varphi)(D_b\varphi)\sqrt{-g} - \tfrac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}(D_m\varphi)(D^m\varphi)] \\ &= -\tfrac{3}{2}\varphi^{-2}[(D_a\varphi)(D_b\varphi) - \tfrac{1}{2}g_{ab}(D_m\varphi)(D^m\varphi)]\sqrt{-g}\delta g^{ab} \end{aligned} \tag{3.79}$$

Now taking into account (3.71) to combine the pieces and considering $\delta\mathcal{L} = 0$, we obtain

$$R_{ab} - \frac{1}{2}g_{ab}R - \frac{\varphi}{2}(F_{ak}F_b{}^k - \frac{1}{4}g_{ab}F_{mn}F^{mn}) - \frac{3}{2}\varphi^{-2}(D_a\varphi D_b\varphi - \frac{1}{2}g_{ab}D_k\varphi D^k\varphi) = 0 \tag{3.80}$$

which can be rewritten in the form of an Einstein's equation

$$G_{ab} = \frac{1}{2}\varphi T^{(EM)}_{ab} - \frac{3}{2}\varphi^{-2}T^{(\varphi)}_{ab} \tag{3.81}$$

where $G_{ab}$, is the Einstein's tensor in 4D, $T^{(EM)}_{ab}$ is the electromagnetic energy-momentum tensor and $T^{(\varphi)}_{ab}$ is the energy-momentum tensor of the scalar field $\varphi$

$$T^{(\varphi)}_{ab} = D_a\varphi D_b\varphi - \frac{1}{2}g_{ab}D_k\varphi D^k\varphi \tag{3.82}$$

In the literature Kaluza-Klein field equations obtained by rescaling of the metric is referred as the equations of the Pauli frame (or, interchangeably as of the Einstein frame), whereas Jordan frame is used for the unscaled metric as is carried out in the previous section.

### 3.4 Conformally Transformed Einstein's Tensor

When the conformal transformation is applied to (3.42) we obtain

$$\tilde{\hat{R}}_{AB} - \frac{1}{2}\tilde{\hat{g}}_{AB}\tilde{\hat{R}} = 0 \tag{3.83}$$

where we know that

$$\begin{aligned} \tilde{g}_{ab} &= \Phi^2 g_{ab} \\ \tilde{R}_{ab} &= R_{ab} - \Phi^{-1} g_{ab} D_m D^m \Phi - 2\Phi^{-1}(D_a D_b \Phi) \\ &\quad -\Phi^{-2} g_{ab}(D_m\Phi)(D^m\Phi) + 4\Phi^{-2}(D_a\Phi)(D_b\Phi) \\ \tilde{R} &= \Phi^{-2}R - 6\Phi^{-3}D_m D^m \Phi \end{aligned} \tag{3.84}$$

Substituting (3.84) to (3.83), we have

$$\begin{aligned} & R_{ab} - \Phi^{-1} g_{ab} D_m D^m \Phi - 2\Phi^{-1}(D_a D_b \Phi) \\ & -\Phi^{-2} g_{ab}(D_m\Phi)(D^m\Phi) + 4\Phi^{-2}(D_a\Phi)(D_b\Phi) \\ & -\tfrac{1}{2}\Phi^2 g_{ab}(\Phi^{-2}R - 6\Phi^{-3}D_m D^m\Phi) = 0 \end{aligned} \tag{3.85}$$

Let $\Phi = \varphi^{-\frac{1}{2}}$ , taking the derivative of the above equation and arranging terms give us

$$\begin{aligned} & R_{ab} - \tfrac{1}{2} g_{ab} R + \tfrac{5}{4}\varphi^{-2} g_{ab} D_m\varphi D^m\varphi - \varphi^{-1} g_{ab} D_m D^m \varphi \\ & -\tfrac{1}{2}\varphi^{-2} D_a\varphi D_b\varphi + \varphi^{-1} D_a D_b \varphi = 0 \end{aligned} \tag{3.86}$$

### 3.5 Bianchi Identities in 5D

5D first Bianchi identity can be represented implicitly

$$\hat{R}^{A}{}_{[BCD]} = 0 \tag{3.87}$$

When we consider the spacetime components of $(3.87)$ we obtain

$$\begin{aligned}\hat{R}^a{}_{[bmn]} &= \hat{R}^a{}_{bmn} + \hat{R}^a{}_{mnb} + \hat{R}^a{}_{nbm} = 0 \\ R^a{}_{[bmn]} &= 0\end{aligned} \tag{3.88}$$

This is called as 4D first Bianchi identity. If we consider $\hat{R}^5{}_{[bmn]}$ of (3.87), our 5D first Bianchi identity is going to be

$$\begin{aligned}\hat{R}^5{}_{[bmn]} &= \hat{R}^5{}_{bmn} + \hat{R}^5{}_{mnb} + \hat{R}^5{}_{nbm} = 0 \\ &= D_b F_{mn} + D_m F_{nb} + D_n F_{bm} = 0\end{aligned} \tag{3.89}$$

What we got is that the homogenous Maxwell equations

$$D_{[b} F_{mn]} = 0 \tag{3.90}$$

For the second Bianchi identity in 5D, we have

$$\hat{D}_{[K} \hat{R}^A{}_{B|CD]} = 0 \tag{3.91}$$

Therefore, we can obtain the following equations

$$\hat{D}_{[k} \hat{R}^a{}_{b|mn]} = 0 \tag{3.92}$$

$$\hat{D}_{[5} \hat{R}^a{}_{b|mn]} = 0 \tag{3.93}$$

$$\hat{D}_{[5} \hat{R}^a{}_{b|5n]} = 0 \tag{3.94}$$

Equation (3.93) becomes

$$\hat{D}_{[5} \hat{R}^a{}_{b|mn]} = \hat{D}_5 \hat{R}^a{}_{bmn} + \hat{D}_m \hat{R}^a{}_{bn5} + \hat{D}_n \hat{R}^a{}_{b5m} = 0 \tag{3.95}$$

Finding covariant derivatives of Riemann tensors and substituting (3.95), 5D second Bianchi identities turns to

$$\hat{D}_{[5} \hat{R}^a{}_{b|mn]} = \frac{1}{2}\varphi(-F^a{}_k R^k{}_{bmn} + F^k{}_b R^a{}_{kmn}) + \frac{1}{2}\varphi(D_n D_m - D_m D_n) F^a{}_b \tag{3.96}$$

We consider the identity

$$(D_n D_m - D_m D_n) F^a{}_b = -R^a{}_{kmn} F^k{}_b + R^k{}_{bmn} F^a{}_k \tag{3.97}$$

and substitute it to $(3.96)$, to obtain

$$\hat{D}_{[5} \hat{R}^a{}_{b|mn]} = \frac{1}{2}\varphi(-F^a{}_k R^k{}_{bmn} + F^k{}_b R^a{}_{kmn} + F^a{}_k R^k{}_{bmn} - F^k{}_b R^a{}_{kmn}) \equiv 0 \tag{3.98}$$

as expected.

### 3.6 Einstein Tensor in 5D From Bianchi Identity

Bianchi identity in 5D can be exploited to extract Einstein tensor, as a conserved quantity. For this purpose let us consider (3.91) and contract two indices

$$\hat{D}_K \hat{R}^K{}_{BMN} - \hat{D}_M \hat{R}_{BN} + \hat{D}_N \hat{R}_{BM} = 0 \tag{3.99}$$

which can be rewritten as

$$\hat{D}_K \hat{R}^K{}_{BMN} - \hat{\delta}^K{}_M \hat{D}_K \hat{R}_{BN} + \hat{\delta}^K{}_N \hat{D}_K \hat{R}_{BM} = 0 \tag{3.100}$$

to obtain a conserved quantity

$$\hat{D}_K (\hat{R}^K{}_{BMN} - \hat{\delta}^K{}_M \hat{R}_{BN} + \hat{\delta}^K{}_N \hat{R}_{BM}) = 0 \tag{3.101}$$

Again contracting two indices we have

$$\hat{D}_K (2\hat{R}^K{}_M - \hat{\delta}^K{}_M \hat{R}) = 0 \tag{3.102}$$

or in a more familiar form

$$\hat{D}_K (\hat{R}^K{}_M - \frac{1}{2}\hat{\delta}^K{}_M \hat{R}) = 0 \tag{3.103}$$

where the quantity inside the parenthesis is manifestly conserved and can be recognised easily as the Einstein tensor $\hat{G}^K{}_M$ in 5D.

### 3.7 The Brans-Dicke Theory

The Brans-Dicke (BD) (or the Jordan-Thiry-Brans-Dicke) theory, like among many others, is an attempt to describe gravity [20]. It can be given as an example of a scalar tensor theory, where the physical metric differs by a conformal factor which can be some scalar field depending on the coordinates. In BD theory the gravitation constant, denoted by $G$, is not considered to be a constant, but $1/G$ is replaced with a scalar field $\varphi$, which alters $G$ from one point to another. Apart from their differences, Einstein's relativity and the Brans-Dicke theory, have some similarities, like both are metric theories and are derivable from the least-action principle.

If we go back to equation (2.9), on the right hand side there is a scalar field, that required an explanation at the time it was first introduced by Kaluza. With the BD theory, it received a possible interpretation. In that particular case, it should represent a short range field in order not to violate the principle of universal free fall [28].

Unlike that of the Kaluza-Klein case, the Brans-Dicke action is already in 4D, where the scalar field is put by hand in the action

$$\mathcal{I} = \frac{1}{16\pi}\int d^4x\sqrt{-g}(\varphi R - w\frac{\varphi_a\varphi^a}{\varphi} + L_M) \tag{3.104}$$

where $w$ is a parameter which is known as the coupling constant of Brans-Dicke. All contributing fields other than the gravitation and the scalar field are included in the Lagrangian density $\sqrt{-g}L_M$ .

Now, we consider the BD Lagrangian density

$$\mathcal{L} = \sqrt{-g}\varphi R - \sqrt{-g}w\frac{\varphi_a\varphi^a}{\varphi} + \sqrt{-g}L_M \tag{3.105}$$

and take the variation with respect to $\varphi$

$$\delta\mathcal{L} = \sqrt{-g}R\delta\varphi + \sqrt{-g}w\varphi^{-2}\delta\varphi\varphi_a\varphi^a - \sqrt{-g}2w\varphi^{-1}D_a\delta\varphi\varphi^a \tag{3.106}$$

We use

$$D_a(\sqrt{-g}\varphi^{-1}\varphi^a\delta\varphi) = D_a(\sqrt{-g}\varphi^{-1}\varphi^a)\delta\varphi + \sqrt{-g}\varphi^{-1}\varphi^a D_a\delta\varphi \tag{3.107}$$

and the Gauss law to obtain

$$\delta\mathcal{I} = \frac{1}{16\pi}\int d^4x\sqrt{-g}[R + w\varphi^{-2}\varphi_a\varphi^a + 2wD_a(\varphi^{-1}\varphi^a)]\delta\varphi \tag{3.108}$$

Then we have

$$R + w\varphi^{-2}\varphi_a\varphi^a + 2wD_a(\varphi^{-1}\varphi^a) = 0 \tag{3.109}$$

Taking the derivative of the third term and rearranging terms results in

$$\varphi R + 2wD_nD^n\varphi - w\frac{\varphi_a\varphi^a}{\varphi} = 0 \tag{3.110}$$

Now variate (3.105) with respect to metric $g^{ab}$. For this purpose we write the Lagrangian density with its hidden metrics

$$\mathcal{L} = \sqrt{-g}(\varphi g^{ab}R_{ab} - \frac{w}{\varphi}g^{mn}\varphi_m\varphi_n + L_M) \tag{3.111}$$

For convenience let us consider each term separately as : $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$. Here

$$\mathcal{L}_1 = \sqrt{-g}\varphi g^{ab} R_{ab} \tag{3.112}$$

whose variation has been given in detail in part 3.1, where we have obtained the following result

$$\delta\mathcal{L}_1 = \sqrt{-g}\varphi(R_{ab} - \frac{1}{2}g_{ab}R)\delta g^{ab} - (D_a D_b\varphi - g_{ab}D_n D^n\varphi)\delta g^{ab} \tag{3.113}$$

For the second term we have

$$\mathcal{L}_2 = -\sqrt{-g}w\varphi^{-1}g^{mn}\varphi_m\varphi_n \tag{3.114}$$

and we take its variation with respect to $g^{ab}$

$$\delta\mathcal{L}_2 = -\delta(\sqrt{-g})w\varphi^{-1}g^{mn}\varphi_m\varphi_n - \sqrt{-g}w\varphi^{-1}\delta g^{mn}\varphi_m\varphi_n \tag{3.115}$$

After simplifications we have

$$\delta\mathcal{L}_2 = \sqrt{-g}\frac{w}{\varphi}(-\varphi_a\varphi_b + \frac{1}{2}g_{ab}\varphi_m\varphi^m)\delta g^{ab} \tag{3.116}$$

Now the third term is

$$\mathcal{L}_3 = \sqrt{-g}L_M \tag{3.117}$$

Variation of $\mathcal{L}_3$ with respect to $g^{ab}$ produces

$$\delta\mathcal{L}_3 = -\sqrt{-g}8\pi T_{ab}\delta g^{ab} \tag{3.118}$$

Taking into account (3.113), (3.116) and (3.118), gives

$$\begin{aligned}\delta\mathcal{I} &= \tfrac{1}{16\pi}\int d^4x\sqrt{-g}\big\{\varphi(R_{ab} - \tfrac{1}{2}g_{ab}R) - (D_a D_b\varphi - g_{ab}D_n D^n\varphi)\\ &+w\,\varphi^{-1}(-\varphi_a\varphi_b + \tfrac{1}{2}g_{ab}\varphi_m\varphi^m) - 8\pi T_{ab}\big\}\delta g^{ab}\end{aligned} \tag{3.119}$$

So, from $\delta\mathcal{I} = 0$, we have

$$\varphi(R_{ab} - \frac{1}{2}g_{ab}R) - (D_a D_b\varphi - g_{ab}D_n D^n\varphi) + \frac{w}{\varphi}(-\varphi_a\varphi_b + \frac{1}{2}g_{ab}\varphi_m\varphi^m) - 8\pi T_{ab} = 0 \tag{3.120}$$

Rearranging above, we obtain the field equation

$$G_{ab} = \frac{1}{\varphi}(D_a D_b\varphi - g_{ab}D_k D^k\varphi) + \frac{w}{\varphi^2}(D_a\varphi\, D_b\varphi - \frac{1}{2}g_{ab}\, D_k\varphi D^k\varphi) + 8\pi T_{ab} \tag{3.121}$$

The trace of (3.120) is

$$-\varphi R + 3D_nD^n\varphi + w\frac{\varphi_a\varphi^a}{\varphi} = 8\pi T \tag{3.122}$$

From (3.110) we write

$$\varphi R = -2wD_nD^n\varphi + w\frac{\varphi_a\varphi^a}{\varphi} \tag{3.123}$$

Substitute (3.123) to (3.122). Then we get the expression

$$(2w+3)D_nD^n\varphi = 8\pi T \tag{3.124}$$

which is the wave equation of Brans-Dicke [20].

### 3.7.1 Equivalence of KK and BD

In order to see the equivalence of the KK and the BD theories, we go back to the action in (3.6) from where the KK equations have been derived. Since the first part of the action is already dealt with, now we consider the second term

$$\mathcal{I}_D = -\frac{1}{8\pi G}\int d^4x\sqrt{-g}D_aD^a\varphi \tag{3.125}$$

whose Lagrangian density is

$$\mathcal{L}_D = -\sqrt{-g}D_aD^a\varphi = -\sqrt{-g}g^{ab}D_aD_b\varphi \tag{3.126}$$

Let us take into account

$$g^{ab}\varphi D_aD_b\varphi = D_a(g^{ab}\varphi D_b\varphi) - g^{ab}D_a\varphi D_b\varphi \tag{3.127}$$

Letting the total divergence term to be zero we have

$$\sqrt{-g}g^{ab}D_aD_b\varphi = -\sqrt{-g}g^{ab}\varphi^{-1}D_a\varphi D_b\varphi \tag{3.128}$$

Then, our lagrangian density becomes

$$\mathcal{L}_D = \sqrt{-g}g^{ab}\varphi^{-1}D_a\varphi D_b\varphi \tag{3.129}$$

Taking the variation of above with respect to $g^{ab}$

$$\delta\mathcal{L}_D = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}\varphi^{-1}D_k\varphi D^k\varphi + \sqrt{-g}\delta g^{ab}\varphi^{-1}D_a\varphi D_b\varphi \tag{3.130}$$

and rearranging, we obtain

$$\delta \mathcal{L}_D = \frac{1}{\varphi}(D_a\varphi D_b\varphi - \frac{1}{2}g_{ab}D_k\varphi D^k\varphi)\sqrt{-g}\delta g^{ab} \tag{3.131}$$

Adding this result to (3.34), and taking $F_{mn} = 0$, we have

$$(R_{ab} - \frac{1}{2}g_{ab}R) = \frac{1}{\varphi}(D_aD_b\varphi - g_{ab}D_kD^k\varphi) - \frac{1}{\varphi^2}(D_a\varphi D_b\varphi - \frac{1}{2}g_{ab}D_k\varphi D^k\varphi) \tag{3.132}$$

Finally, we observe that, this result is the same as (3.121) for $w = -1$ and $T_{ab} = 0$. Therefore, the KK and BD produce the same equations, when the value of $w$ is restricted.

# CHAPTER 4

# KALUZA-KLEIN REDUCTION FOR A QUADRATIC CURVATURE

Theories of gravitation involving quadratic terms of the Riemann tensor has a long history. We may refer to many references that contain studies along this line [29, 30, 31, 32]. Therefore, one can immediately generalise such a quadratic action to its Kaluza-Klein form. As to our knowledge, this has first appeared in [18], where the dimensional reduction is carried out using the language of differential forms. Later on, in [19] the field equations and the energy-momentum tensor is provided by taking the variations with respect to the 5D connection $\hat{\Gamma}^{A}_{BC}$ and the 5D metric $\hat{g}^{AB}$, independently. Considering the connection and the metric as independent dynamical variables is known as the Palatini approach [3]. Here, we shall use the same method, i.e., the Palatini approach, however we will reverse the order of taking the variations first by the 5D variables and then applying the reduction procedure to the results of the variational principle. In other words, we shall first reduce the 5D action to its 4D form, and then take the independent variations with respect to 4D fields. After the reduction procedure there should appear more fields to be considered. Namely, in addition to the 4D metric and the connection, there will be the electromagnetic field tensor $F_{ab}$ and the scalar field $\varphi$. Therefore, one has to take variations with respect to those fields, which were hidden before the reduction. Our main question is how does altering the order of these two consecutive procedures yield a change in the resulting equations. This chapter is dedicated to answer this question. After observing the differences in the equations, it may be possible to identify the source of this issue. To this end, we will take the variations with respect the 4D dynamical variables of each term appearing in the reduced action.

In this chapter, we will also refer to the literature for comparison, if those terms have been somehow analysed, either on their own, or in connection with some other fields. Finally, we will present our observations as a whole.

### 4.1 The Quadratic Action and Its Reduced Form

The five dimensional gravitation action

$$\mathcal{I} = \frac{1}{16\pi\hat{G}} \int d^5x \hat{\mathcal{L}} \tag{4.1}$$

explicitly reads as

$$\mathcal{I} = \frac{1}{16\pi\hat{G}} \int d^5x \sqrt{-\hat{g}} \hat{R}_{JKMN} \hat{R}^{JKMN} \tag{4.2}$$

when the Lagrangian is formed from a 5D quadratic curvature.

We start by expressing the 5D invariant by a natural splitting the Riemann tensor in terms of its spacetime, fifth and the mixed components, also keeping in mind that

$$\sqrt{-\hat{g}} = \sqrt{-g}\varphi \tag{4.3}$$

Then, the action becomes

$$\mathcal{I} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\varphi(\hat{R}_{jkmn}\hat{R}^{jkmn} + 4\hat{R}_{5kmn}\hat{R}^{5kmn} + 4\hat{R}_{j5m5}\hat{R}^{j5m5}) \tag{4.4}$$

We define $L = \varphi L_q$, where

$$L_q = \hat{R}_{jkmn}\hat{R}^{jkmn} + 4\hat{R}_{5kmn}\hat{R}^{5kmn} + 4\hat{R}_{j5m5}\hat{R}^{j5m5} \tag{4.5}$$

When we substitute for the above Riemanns from Appendix B, we have

$$\begin{aligned} L_q \ &= R_{jkmn}R^{jkmn} - \tfrac{3}{2}\varphi^2 R^j{}_{kmn} F_j{}^k F^{mn} \\ &+\tfrac{1}{8}\varphi^4[3F_{jk}F^{jk}F_{mn}F^{mn} + 5(F_{jm}F^{mn}F_{ni}F^{ij})] + \varphi^2 D_k F_{mn} D^k F^{mn} \\ &+4\varphi^{-2} D_m\varphi_n D^m\varphi^n - 2\varphi F^{mk}F^n{}_k D_m(D_n\varphi) \\ &+8\varphi(D_kF_{mn}\varphi^m F^{kn} + D_kF_{mn}\varphi^k F^{mn}) + 6\varphi_k\varphi^k F^{mn}F_{mn} + 6\varphi_m\varphi_n F^{mk}F^n{}_k \end{aligned} \tag{4.6}$$

Finally, our action becomes

$$\begin{aligned}\mathcal{I} \;&= \tfrac{1}{16\pi G}\int d^4x\sqrt{-g}\,\Big\{\varphi R_{jkmn}R^{jkmn} - \tfrac{3}{2}\varphi^3 R^j{}_{kmn}F_j{}^kF^{mn}\\
&+\tfrac{1}{8}\varphi^5[3F_{jk}F^{jk}F_{mn}F^{mn} + 5(F_{jm}F^{mn}F_{ni}F^{ij})] + \varphi^3 D_kF_{mn}D^kF^{mn}\\
&+4\varphi^{-1}D_m\varphi_nD^m\varphi^n - 2\varphi^2F^{mk}F^n{}_kD_m(D_n\varphi)\\
&+8\varphi^2(D_kF_{mn}\varphi^mF^{kn} + D_kF_{mn}\varphi^kF^{mn})\\
&+6\varphi(\varphi_k\varphi^kF^{mn}F_{mn} + \varphi_m\varphi_nF^{mk}F^n{}_k)\Big\}\end{aligned} \tag{4.7}$$

## 4.2 Field Equations From the Reduced Action

In this section we shall treat each term in (4.7) one by one and collect them at the end. So we define

$$\mathcal{I}_i = \int d^4x\,\mathcal{L}_i = \int d^4x\,\sqrt{-g}\,L_i \tag{4.8}$$

where the subscript $i$ is representing each term.

• $\mathcal{L}_1$

The action for the first term is

$$\mathcal{I}_1 = \int d^4x\sqrt{-g}\,\varphi R_{jkmn}R^{jkmn} \tag{4.9}$$

By treating the gravitational field $R_{jkmn}$ as a gauge field [33], dynamics resulting from a quadratic curvature Lagrangian has been studied extensively [30, 32, 34] (and the references therein). An immediate generalization would be the construction of this formalism for the KK ansatz [18, 19]. In fact, it is these latter developments that paved the way for this current work.

Now, consider

$$\delta\mathcal{L}_1 = 2\varphi R_{jkmn}\delta R^{jkmn}\sqrt{-g} \tag{4.10}$$

to vary with respect to the connection. Let us take the Palatini equation into account [3]:

$$\delta R^j{}_{kmn} = D_m(\delta\Gamma^j_{kn}) - D_n(\delta\Gamma^j_{km}) \tag{4.11}$$

and

$$D_m(\sqrt{-g}\varphi R_j{}^{kmn}\delta\Gamma^j_{kn}) = D_m(\sqrt{-g}\varphi R_j{}^{kmn})\delta\Gamma^j_{kn}+\sqrt{-g}\varphi R_j{}^{kmn}D_m\delta\Gamma^j_{kn} \quad (4.12)$$

Substituting (4.11)

$$\delta\mathcal{I}_1 = 2\int d^4x\sqrt{-g}\varphi R_j{}^{kmn}\{D_m(\delta\Gamma^j_{kn}) - D_n(\delta\Gamma^j_{km})\} \quad (4.13)$$

and using (4.12) we have

$$\begin{aligned}\delta\mathcal{I}_1 &= 2\int d^4x\{[D_m(\sqrt{-g}\varphi R_j{}^{kmn}\delta\Gamma^j_{kn}) - D_m(\sqrt{-g}\varphi R_j{}^{kmn})\delta\Gamma^j_{kn}] \\ &-[D_n(\sqrt{-g}\varphi R_j{}^{kmn}\delta\Gamma^j_{km}) - D_n(\sqrt{-g}\varphi R_j{}^{kmn})\delta\Gamma^j_{km}\}\end{aligned} \quad (4.14)$$

where the first and third terms are tensor densities of weight +1. By divergence theorem these terms in (4.14) convert to a surface integral where the integrand vanishes on the boundary. Therefore we have

$$\delta\mathcal{I}_1 = 4\int d^4x D_n(\sqrt{-g}\varphi R_j{}^{kmn})\delta\Gamma^j_{km} \quad (4.15)$$

and the expression resulting from this term

$$\gamma_1 := -4D_m(\sqrt{-g}\varphi R_j{}^{kmn}) \quad (4.16)$$

since from [3] we know that $D_m(\sqrt{-g}) = 0$. One can also write

$$\gamma_1 := -4\sqrt{-g}D_m(\varphi R_j{}^{kmn}) \quad (4.17)$$

Variation of $\mathcal{I}_1$ with respect to $\varphi$ is

$$\delta\mathcal{I}_1 = \int d^4x\sqrt{-g}\delta\varphi R_{jkmn}R^{jkmn} \quad (4.18)$$

Using above, we obtain

$$\Phi_1 := \sqrt{-g}R_{jkmn}R^{jkmn} \quad (4.19)$$

In order to vary with respect to $g^{ab}$, we rewrite our Lagrangian in terms of the hidden metrics

$$\mathcal{L}_1 = \varphi g^{mi}g^{np}R_{jkmn}R^{jk}{}_{ip}\sqrt{-g} \quad (4.20)$$

Varying the above Lagrangian, we have

$$\begin{aligned}\delta\mathcal{L}_1 = \;& (\delta^m_a\delta^i_b g^{np}\varphi R_{jkmn}R^{jk}{}_{ip}\sqrt{-g} + g^{mi}\delta^n_a\delta^p_b\varphi R_{jkmn}R^{jk}{}_{ip}\sqrt{-g})\delta g^{ab} \\ & -\tfrac{1}{2}g_{ab}\varphi R_{jkmn}R^{jkmn}\sqrt{-g})\delta g^{ab}\end{aligned} \quad (4.21)$$

where we used

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} \tag{4.22}$$

Using the symmetry properties of the Riemann tensor and simplifying we have

$$\delta\mathcal{L}_1 = \varphi(2R_{anjk}R_b{}^{njk} - \frac{1}{2}g_{ab}R_{mnjk}R^{mnjk})\sqrt{-g}\delta g^{ab} \tag{4.23}$$

We define the energy-momentum tensor $T_{ab}$ as the coefficient of $\delta g^{ab}$ such that

$$T_{ab}\delta g^{ab} = \frac{1}{2}\frac{1}{\sqrt{-g}}\delta\mathcal{L} \tag{4.24}$$

Therefore, we obtain

$$T_{ab} = \varphi(R_{anjk}R_b{}^{njk} - \frac{1}{4}g_{ab}R_{mnjk}R^{mnjk}) \tag{4.25}$$

• $\mathcal{L}_2$

For the second term we have

$$\mathcal{I}_2 = \int d^4x\sqrt{-g}\varphi^3 R^j{}_{kmn}F_j{}^k F^{mn} \tag{4.26}$$

Early works dealing with Lagrangians given in the above form are mostly formal mathematical studies in essence [35, 36, 37]. Later on the motivations for the inclusion of $RF^2$ became diverse[38]. An exhausted list of $R_xF^2$ non-minimal couplings are presented in [39], where $R_x$ refers to an element of the set $R_x = \{R, R_{ij}, R_{ijnm}\}$ and the couplings of $F^2$ are linear with respect to $R_x$. These authors argue that such couplings depict drastic conditions where the gravitational and electromagnetic fields are intense and the speed of the gravitational-electromagnetic wave differs from that of light in vacuum. We see that one such particular coupling as in (4.26) singles out naturally through the reduction procedure of the model we have, and excludes all other forms that might be added arbitrarily by hand.

We first start by varying (4.26) with respect to $A_i$

$$\delta\mathcal{L}_2 = \sqrt{-g}\varphi^3\left\{R^{jk}{}_{mn}(D_j\delta A_k - D_k\delta A_j)F^{mn} + R_{jk}{}^{mn}F^{jk}(D_m\delta A_n - D_n\delta A_m)\right\} \tag{4.27}$$

then

$$\begin{aligned}\delta\mathcal{L}_2 &= \sqrt{-g}\varphi^3\left\{R^{jk}{}_{mn}(\delta^i_k D_j\delta A_i - \delta^i_j D_k\delta A_i)F^{mn}\right. \\ &\left.+R_{jk}{}^{mn}F^{jk}(\delta^i_n D_m\delta A_i - \delta^i_m D_n\delta A_i)\right\}\end{aligned} \tag{4.28}$$

Using

$$D_j(\varphi^3 R^{ji}{}_{mn}F^{mn}\delta A_i) = D_j(\varphi^3 R^{ji}{}_{mn}F^{mn})\delta A_i + \varphi^3 R^{ji}{}_{mn}F^{mn}D_j\delta A_i \tag{4.29}$$

we obtain

$$\begin{aligned}\delta\mathcal{L}_2 &= \sqrt{-g}\varphi^3\big\{-D_j(R^{ji}{}_{mn}F^{mn})\delta A_i + D_k(R^{ik}{}_{mn}F^{mn})\delta A_i \\ &-D_m(R_{jk}{}^{mi}F^{jk})\delta A_i + D_n(R_{jk}{}^{in}F^{jk})\delta A_i\big\} \\ &= \sqrt{-g}\varphi^3\left\{-D_j[(R^{ji}{}_{mn}F^{mn} - R^{ij}{}_{mn}F^{mn} + R_{mk}{}^{ji}F^{mk} + R_{nk}{}^{ij}F^{nk})]\delta A_i\right\} \\ &= (-4)\sqrt{-g}\varphi^3 D_j R^{ji}{}_{mn}F^{mn}\delta A_i\end{aligned} \tag{4.30}$$

so variation yields

$$a_2 := -4D_k(\sqrt{-g}\varphi^3 R^{kj}{}_{mn}F^{mn}) \tag{4.31}$$

or

$$a_2 := -4\sqrt{-g}D_k(\varphi^3 R^{kj}{}_{mn}F^{mn}) \tag{4.32}$$

Now, we vary with respect to the connection $\Gamma^j_{nk}$

$$\begin{aligned}\delta\mathcal{I}_2 &= \int d^4x\sqrt{-g}\varphi^3(\delta R^j{}_{kmn})F_j{}^kF^{mn} \\ &= \int d^4x\sqrt{-g}\varphi^3(D_m\delta\Gamma^j_{nk} - D_n\delta\Gamma^j_{mk})F_j{}^kF^{mn} \\ &= \int d^4x\sqrt{-g}\varphi^3(D_m\delta\Gamma^j_{nk} + D_m\delta\Gamma^j_{nk})F_j{}^kF^{mn}\end{aligned} \tag{4.33}$$

Consider

$$\begin{aligned}&D_m(\sqrt{-g}\varphi^3 F_j{}^kF^{mn}\delta\Gamma^j_{nk}) \\ &= D_m(\sqrt{-g}\varphi^3 F_j{}^kF^{mn})\delta\Gamma^j_{nk} + (\sqrt{-g}\varphi^3 F_j{}^kF^{mn})D_m\delta\Gamma^j_{nk}\end{aligned} \tag{4.34}$$

and also

$$\int_V d^4x D_m(\sqrt{-g}\varphi^3 F_j{}^kF^{mn}\delta\Gamma^j_{nk}) = \int_{\partial V}\sqrt{-g}\varphi^3 F_j{}^kF^{mn}\delta\Gamma^j_{nk}dS_m \tag{4.35}$$

Since variations vanish on the boundary of the volume, we obtain

$$\delta\mathcal{I}_2 = \int d^4x D_m(\sqrt{-g}\varphi^3 F_j{}^kF^{mn})\,\delta\Gamma^j_{nk} \tag{4.36}$$

so

$$\gamma_2 := D_m(\sqrt{-g}\varphi^3 F_j{}^k F^{mn}) \tag{4.37}$$

Varying with respect to $\varphi$

$$\delta\mathcal{I}_2 = \int d^4x\sqrt{-g}R^j{}_{kmn}F_j{}^k F^{mn} 3\varphi^2\delta\varphi \tag{4.38}$$

using $\delta\mathcal{I}_2$ from above, we end up with the expression

$$\Phi_2 := 3\sqrt{-g}\varphi^2 R^j{}_{kmn}F_j{}^k F^{mn} \tag{4.39}$$

Varying with respect to the metric $g^{ab}$

$$\begin{aligned}\delta\mathcal{I}_2 &= \int d^4x\,\varphi^3 R^{ikmn}F^{ab}F^{cd}\left(\delta g_{ia}g_{kb}g_{mc}g_{nd}\sqrt{-g} + g_{ia}\delta g_{kb}g_{mc}g_{nd}\sqrt{-g}\right.\\ &\quad \left.+\, g_{ia}g_{kb}\delta g_{mc}g_{nd}\sqrt{-g} + g_{ia}g_{kb}g_{mc}\delta g_{nd}\sqrt{-g} + g_{ia}g_{kb}g_{mc}g_{nd}\delta\sqrt{-g}\,\right)\end{aligned} \tag{4.40}$$

performing the necessary contractions

$$\begin{aligned}\delta\mathcal{I}_2 &= \int d^4x\,\varphi^3(R_{akmn}F_b{}^k F^{mn} + R_{jamn}F^j{}_b F^{mn} + R_{jkan}F^{jk}F_b{}^n\\ &\quad +R_{jkma}F^{jk}F^m{}_b + \tfrac{1}{2}g_{ab}R_{jkmn}F^{jk}F^{mn})\sqrt{-g}\delta g^{ab}\end{aligned} \tag{4.41}$$

we obtain

$$\begin{aligned}\delta\mathcal{I}_2 &= \int d^4x\,\varphi^3(2R_{akmn}F_b{}^k F^{mn}\\ &\quad +2R_{akmn}F_b{}^k F^{mn} + \tfrac{1}{2}g_{ab}R_{jkmn}F^{jk}F^{mn})\sqrt{-g}\delta g^{ab}\end{aligned} \tag{4.42}$$

Referring to Appendix B to symmetrize the term $(R_{akmn}F_b{}^k F^{mn})$ and considering (4.24) we have

$$\begin{aligned}T_{ab} &= \varphi^3(2R_{anjk}F_b{}^n F^{jk} - \tfrac{1}{4}g_{ab}R_{jkmn}F^{jk}F^{mn})\\ &= \varphi^3(R_{anjk}F_b{}^n F^{jk} + R_{bnjk}F_a{}^n F^{jk} - \tfrac{1}{4}g_{ab}R_{jkmn}F^{jk}F^{mn})\\ &= \varphi^3(R_{anjk}F_b{}^n + R_{bnjk}F_a{}^n - \tfrac{1}{4}g_{ab}R_{jkmn}F^{mn})\,F^{jk}\end{aligned} \tag{4.43}$$

• $\mathcal{L}_3$

The action for the third term is

$$\mathcal{I}_3 = \int d^4x\sqrt{-g}\,\varphi^5 F_{jk}F^{jk}F_{mn}F^{mn} \tag{4.44}$$

Such type of invariants in Lagrangians appear in various contexts. For instance, when the square of Born-Infeld Lagrangian is considered [40], or as one may expect in the reduction of 5D Gauss-Bonnet invariant [41]. Apart from that $F^n$ invariants were also considered to be worthwhile in their own right, at least from a mathematical point of view [42]. The most interesting results arising from $F^4$ terms has been found the context of QED in Minkowskian space, when photons interact with a strong magnetic field [43].

Specifically, one can evaluate following two quadratic invariants in terms of the electric $E_i$ and magnetic $B_i$ field vectors

$$\begin{aligned} F_{jk}F^{jk}F_{mn}F^{mn} &= 4(E^2 - B^2)^2 \\ F^{jk}F_{km}F^{mn}F_{nj} &= 2(E^2 - B^2)^2 + 4(E \cdot B)^2 \end{aligned} \tag{4.45}$$

to grasp the strength of the fields as compared to the two fundamental quadratic invariants $F_{jk}F^{jk} = 2(E^2 - B^2)$ and $F_{jk}{}^*F^{jk} = (E \cdot B)$.

Now, we take the variation of $\mathcal{L}_3$ with respect to $A_n$

$$\delta\mathcal{L}_3 = 8\sqrt{-g}\,\varphi^5 F^{ij}F_{ij}F^{mn}D_m\delta A_n \tag{4.46}$$

and consider

$$\begin{aligned} &D_k(\sqrt{-g}\varphi^5 F^{ij}F_{ij}F^{mn}\delta A_n) \\ &= D_k(\sqrt{-g}\varphi^5 F^{ij}F_{ij}F^{mn})\delta A_n + \sqrt{-g}\varphi^5 F^{ij}F_{ij}F^{mn}D_k\delta A_n \end{aligned} \tag{4.47}$$

to obtain

$$\delta\mathcal{I}_3 = 8\int d^4x\,\left\{D_m(\sqrt{-g}\varphi^5 F^{ij}F_{ij}F^{mn}\delta A_n) - D_m(\sqrt{-g}\varphi^5 F^{ij}F_{ij}F^{mn})\,\delta A_n\right\} \tag{4.48}$$

Using the divergence theorem and $\delta\mathcal{I}_3$, we have the following expression

$$a_3 := -8D_k(\sqrt{-g}\varphi^5 F_{mn}F^{mn}F^{kj}) \tag{4.49}$$

Variation of $\mathcal{L}_3$ with respect to $\varphi$

$$\delta\mathcal{L}_3 = 5\varphi^4 F_{jk}F^{jk}F_{mn}F^{mn}\sqrt{-g}\,\delta\varphi \tag{4.50}$$

gives

$$\delta\mathcal{I}_3 = 5\int d^4x\sqrt{-g}\,\varphi^4 F_{jk}F^{jk}F_{mn}F^{mn}\,\delta\varphi \tag{4.51}$$

From above, we conclude that

$$\Phi_3 := 5\sqrt{-g}\varphi^4 F_{jk}F^{jk}F_{mn}F^{mn} \tag{4.52}$$

We rewrite $\mathcal{L}_3$ so that the metric tensor is easily seen

$$\mathcal{L}_3 = \varphi^5 g^{jp}g^{kq}g^{rm}g^{sn}F_{jk}F_{pq}F_{mn}F_{rs}\sqrt{-g} \tag{4.53}$$

Then the variation of $\mathcal{L}_3$ with respect to $g^{ab}$ becomes

$$\delta\mathcal{L}_3 = \varphi^5 F_{mn}F^{mn}\left(F_{aj}F_b{}^j + F^j{}_aF_{jb} + F_b{}^jF_{aj} + F^j{}_bF_{ja} - \frac{1}{2}g_{ab}F_{ij}F^{ij}\right)\sqrt{-g}\,\delta g^{ab} \tag{4.54}$$

After simplifications, the energy-momentum tensor takes the form

$$T_{ab} = \varphi^5 F_{mn}F^{mn}(2F_{ak}F_b{}^k - \frac{1}{4}g_{ab}F_{ij}F^{ij}) \tag{4.55}$$

• $\mathcal{L}_4$

The action for the fourth term is

$$\mathcal{I}_4 = \int d^4x\sqrt{-g}\,\varphi^5 F_{jm}F^{mn}F_{ni}F^{ij} \tag{4.56}$$

Let us first consider the variation with respect to $F^{ij}$

$$\begin{aligned}\delta\mathcal{L}_4 \;&= \sqrt{-g}\varphi^5(\delta F_{jm}F^{mn}F_{ni}F^{ij} + F_{jm}\delta F^{mn}F_{ni}F^{ij}\\ &+F_{jm}F^{mn}\delta F_{ni}F^{ij} + F_{jm}F^{mn}F_{ni}\delta F^{ij})\end{aligned} \tag{4.57}$$

To vary $\mathcal{L}_4$ with respect to $A_k$, it can be observed that all of the four terms above can be treated similarly. Therefore it is sufficient to perform the variation only on one of them. For instance we may consider the third term. Now for this term, the variation is

$$\delta\mathcal{L}_{43} = \sqrt{-g}\varphi^5\left\{F_{jm}F^{mn}(\delta_i^k D_n\delta A_k - \delta_n^k D_i\delta A_k)F^{ij}\right\} \tag{4.58}$$

Making necessary calculations and using the divergence theorem the third term becomes

$$\delta\mathcal{L}_{43} = -2D_i(\sqrt{-g}\,\varphi^5 F^{im}F_{mj}F^{jk})\,\delta A_k \tag{4.59}$$

After treating the remaining terms in a similar way, adding up all of them and using $\delta\mathcal{I}_4$, this expression simplifies as

$$a_4 := -8D_j(\sqrt{-g}\,\varphi^5 F^{jm}F_{mi}F^{ik}) \tag{4.60}$$

Variation of $\mathcal{L}_4$ with respect to $\varphi$ is

$$\delta\mathcal{L}_4 = 5\varphi^4 F_{jm}F^{mn}F_{ni}F^{ij}\sqrt{-g}\,\delta\varphi \tag{4.61}$$

Thus, $\delta\mathcal{I}_4$ yields

$$\Phi_4 := 5\varphi^4 F_{jm}F^{mn}F_{ni}F^{ij}\sqrt{-g} \tag{4.62}$$

Rewriting $\mathcal{L}_4$ to take the variation with respect to metric $g^{ab}$, we have

$$\mathcal{L}_4 = \sqrt{-g}\varphi^5 g^{jk}g^{mp}g^{ns}g^{iq}F_{jm}F_{ps}F_{ni}F_{qk} \tag{4.63}$$

After the variation we obtain the energy-momentum tensor

$$\begin{aligned} T_{ab} &= \varphi^5(2F_{am}F^{mn}F_{ni}F^i{}_b - \tfrac{1}{4}g_{ab}F_{jm}F^{mn}F_{ni}F^{ij}) \\ &= \varphi^5(2F_{am}F^i{}_b - \tfrac{1}{4}g_{ab}F_{jm}F^{ij})\,F^{mn}F_{ni} \end{aligned} \tag{4.64}$$

• $\mathcal{L}_5$

The action for the fifth term is

$$\mathcal{I}_5 = \int d^4x\sqrt{-g}\,\varphi^3 D_kF_{mn}D^kF^{mn} \tag{4.65}$$

Lagrangians in the form of $(DF)^2$ has been of interest since long, and is referred to Bopp-Podolsky electrodynamics [44, 45]. There can be three distinguishable forms by their contracted indices, that are non-trivially equivalent

$$\mathcal{L}^{(1)} = D_kF^{mk}D_nF_m{}^n \tag{4.66}$$

$$\mathcal{L}^{(2)} = D_kF^{mn}D_nF_m{}^k \tag{4.67}$$

$$\mathcal{L}^{(3)} = D_kF^{mn}D^kF_{mn} \tag{4.68}$$

The relation between $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ can be found by using total divergences

$$D_kF_m{}^kD_nF^{mn} = D_k(F_m{}^kD_nF^{mn}) - F_m{}^kD_kD_nF^{mn} \tag{4.69}$$

and

$$D_n F_m{}^k D_k F^{mn} = D_n(F_m{}^k D_k F^{mn}) - F_m{}^k D_n D_k F^{mn} \tag{4.70}$$

Then $\mathcal{L}^{(2)} - \mathcal{L}^{(1)}$ becomes [46]

$$\mathcal{L}^{(2)} - \mathcal{L}^{(1)} = D_n(F_m{}^k D_k F^{mn}) - D_k(F_m{}^k D_n F^{mn}) + F_m{}^k(D_k D_n - D_n D_k)F^{mn} \tag{4.71}$$

By using

$$(D_k D_n - D_n D_k)F^{mn} = R^m{}_{lkn} F^{ln} - R^{ln}{}_{kn} F^m{}_l \tag{4.72}$$

and arranging the indices, (4.71) becomes

$$\mathcal{L}^{(2)} = D_n(F_m{}^k D_k F^{mn}) - D_k(F_m{}^k D_n F^{mn}) + R_{lk} F^{lm} F_m{}^k + R_{mlkn} F^{ln} F^{mk} + \mathcal{L}^{(1)} \tag{4.73}$$

Now to vary with respect to $A_n$, we first consider

$$\begin{aligned} \delta\mathcal{I}_5 &= \int d^4x\sqrt{-g}\varphi^3(D_k\delta F_{mn}D^kF^{mn} + D_kF_{mn}D^k\delta F^{mn}) \\ &= \int d^4x\sqrt{-g}\varphi^3(2D_k\delta F_{mn}D^kF^{mn}) \\ &= \int d^4x\sqrt{-g}\varphi^3[2D_k\delta(D_mA_n - D_nA_m)D^kF^{mn}] \end{aligned} \tag{4.74}$$

After some arrangements, $\delta\mathcal{I}_5$ attains the form

$$\delta\mathcal{I}_5 = 4\int d^4x\sqrt{-g}\varphi^3 D_k D_m \delta A_n D^k F^{mn} \tag{4.75}$$

We shall make use of the following relation

$$\begin{aligned} D_k\left[\varphi^3 D^k F^{mn}\left(D_m\delta A_n\right)\right] &= D_m\left[\varphi^3 D_k D^k F^{mn}\,\delta A_n\right] + D_m(\varphi^3 D_k D^k F^{mn})\,\delta A_n \\ &\quad + \varphi^3 D^k F^{mn}\left(D_k D_m \delta A_n\right) \end{aligned} \tag{4.76}$$

to obtain

$$\begin{aligned} \delta\mathcal{I}_5 &= 4\int dx^4\sqrt{-g}\big\{D_k\left[\varphi^3 D^k F^{mn}\left(D_m\delta A_n\right)\right] \\ &\big\{ - D_m\left[\varphi^3 D_k D^k F^{mn}\,\delta A_n\right] - D_m(\varphi^3 D_k D^k F^{mn})\delta A_n\big\} \end{aligned} \tag{4.77}$$

From the divergence theorem, first and second terms goes to zero. Then, we have

$$\delta\mathcal{I}_5 = -4\int d^4x D_m(\sqrt{-g}\,\varphi^3 D_k D^k F^{mn})\,\delta A_n \tag{4.78}$$

Thus, we find the expression

$$a_5 := -4D_m(\sqrt{-g}\varphi^3 D_k D^k F^{mn}) \tag{4.79}$$

For $\Gamma^a_{bc}$, we consider again

$$\delta\mathcal{I}_5 = \int d^4x\sqrt{-g}\varphi^3[2\delta(D_kF_{mn})D^kF^{mn}] \tag{4.80}$$

where

$$D_kF_{mn} = \partial_kF_{mn} - \Gamma^l_{mk}F_{ln} - \Gamma^l_{nk}F_{ml} \tag{4.81}$$

and

$$\delta(D_kF_{mn}) = -\delta\Gamma^l_{mk}F_{ln} - \delta\Gamma^l_{nk}F_{ml} \tag{4.82}$$

Then

$$\delta\mathcal{I}_5 = \int d^4x\sqrt{-g}2\varphi^3(-\delta^l_a\delta^b_m\delta^c_kF_{ln}D^kF^{mn} - \delta^l_a\delta^b_n\delta^c_kF_{ml}D^kF^{mn})\,\delta\Gamma^a_{bc} \tag{4.83}$$

After carrying out the necessary but simple calculations, we obtain

$$\delta\mathcal{I}_5 = -4\int d^4x\sqrt{-g}\varphi^3F_{na}D^cF^{nb}\delta\Gamma^a_{bc} \tag{4.84}$$

Thus from above, we find

$$\gamma_5 := -4\sqrt{-g}\varphi^3F_{na}D^cF^{nb} \tag{4.85}$$

Variation of $\mathcal{I}_5$ with respect to $\varphi$ is

$$\delta\mathcal{I}_5 = 3\int d^4x\sqrt{-g}\,\varphi^2D_kF_{mn}D^kF^{mn}\delta\varphi \tag{4.86}$$

So, we get

$$\Phi_5 := 3\sqrt{-g}\varphi^2D_kF_{mn}D^kF^{mn} \tag{4.87}$$

Now, we make the metrics visible in $\mathcal{L}_5$

$$\mathcal{L}_5 = g^{kl}g^{mi}g^{nj}\varphi^3D_lF_{mn}D_kF_{ij}\sqrt{-g} \tag{4.88}$$

so that the variation with respect to $g^{ab}$ can be handled to give

$$\begin{aligned}\delta\mathcal{L}_5 &= \left\{\delta g^{kl}g^{mi}g^{nj} + g^{kl}\delta g^{mi}g^{nj} + g^{kl}g^{mi}\delta g^{nj}\right\}\varphi^3D_lF_{mn}D_kF_{ij}\sqrt{-g}\\ &+\varphi^3D^kF^{ij}D_kF_{ij}(-\tfrac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab})\end{aligned} \tag{4.89}$$

which can be rewritten as

$$\begin{aligned}\delta\mathcal{L}_5 &= (\delta^k_a\delta^l_b g^{mi}g^{nj} + g^{kl}\delta^m_a\delta^i_b g^{nj} + g^{kl}g^{mi}\delta^n_a\delta^j_b)\delta g^{ab}\varphi^3 D_l F_{mn} D_k F_{ij}\sqrt{-g} \\ &\quad +\varphi^3 D^k F^{ij} D_k F_{ij}(-\tfrac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab})\end{aligned} \tag{4.90}$$

Then, the energy-momentum tensor takes the form

$$T_{ab} = \frac{1}{2}\varphi^3(D_a F_{ij} D_b F^{ij} + 2D^k F_a{}^j D_k F_{bj} - \frac{1}{2}g_{ab}D^k F^{ij} D_k F_{ij}) \tag{4.91}$$

• $\mathcal{L}_6$

For the sixth term we have the action

$$\mathcal{I}_6 = \int d^4x\sqrt{-g}\,\varphi^{-1}(D_m\varphi_n)(D^m\varphi^n) \tag{4.92}$$

and we vary $\delta\mathcal{L}_6$ with respect to $\varphi$

$$\delta\mathcal{L}_6 = \big\{-\varphi^{-2}(D_m D_n\varphi D^m D^n\varphi)\delta\varphi + 2\varphi^{-1}(D^m D^n\varphi)D_m D_n\delta\varphi\big\}\sqrt{-g} \tag{4.93}$$

We know from D'Inverno [3] that

$$D_a(\sqrt{-g}K^a) = \partial_a(\sqrt{-g}K^a) \tag{4.94}$$

where $K^a$ is an arbitrary vector. So one can write

$$\partial_k\delta\varphi = D_k\delta\varphi \tag{4.95}$$

To evaluate the second term in (4.93) we consider

$$\begin{aligned}&\varphi^{-1}(D^m D^n\varphi)D_m D_n\delta\varphi = D_m(\varphi^{-1}D^m D^n\varphi D_n\delta\varphi) \\ &-D_n[\,D_m(\varphi^{-1}D^m D^n\varphi)\delta\varphi\,] + D_n D_m(\varphi^{-1}D^m D^n\varphi)\delta\varphi\end{aligned} \tag{4.96}$$

where the second and the third terms above on the right hand side are total divergences. Therefore, we have

$$\begin{aligned}&\int d^4x\sqrt{-g}\varphi^{-1}(D^m D^n\varphi)D_m D_n\delta\varphi \\ &= \int d^4x\big\{\sqrt{-g}D_n D_m(\varphi^{-1}D^m D^n\varphi)\big\}\delta\varphi\end{aligned} \tag{4.97}$$

Then the expression is obtained as

$$\Phi_6 := \sqrt{-g}\varphi^{-2}[(D_m D_n\varphi)D^m D^n\varphi] + 2\sqrt{-g}D_n D_m(\varphi^{-1}D^m D^n\varphi) \tag{4.98}$$

The same term

$$\mathcal{L}_6 = \sqrt{-g}\varphi^{-1}(D_m\varphi_n)(D^m\varphi^n) \tag{4.99}$$

is to be varied with respect to connection $\Gamma^a_{bc}$. So, we expand the covariant derivative

$$\begin{aligned}
\delta\mathcal{L}_6 &= \sqrt{-g}\varphi^{-1}\big\{\delta(D_m\varphi_n)D^m\varphi^n + D_m\varphi_n\delta(D^m\varphi^n)\big\} \\
&= 2\sqrt{-g}\varphi^{-1}\delta(D_m\varphi_n)D^m\varphi^n \\
&= 2\sqrt{-g}\varphi^{-1}\delta(\partial_m\varphi_n - \Gamma^k_{nm}\varphi_k))D^m\varphi^n \\
&= 2\sqrt{-g}\varphi^{-1}(-\delta\Gamma^k_{nm}\varphi_k)(D^m\varphi^n) \\
&= -2\sqrt{-g}\varphi^{-1}\varphi_a D^c\varphi^b\delta\Gamma^a_{bc}
\end{aligned} \tag{4.100}$$

Then

$$\delta\mathcal{I}_6 = -2\int d^4x\,\sqrt{-g}\,\varphi^{-1}(\varphi_a D^c\varphi^b)\delta\Gamma^a_{bc} \tag{4.101}$$

and finally we have

$$\gamma_6 := -2\sqrt{-g}\varphi^{-1}(\varphi_a D^c\varphi^b) \tag{4.102}$$

In order to vary with respect to $g^{ab}$ , we write explicitly the metric tensor involved in the contractions

$$\mathcal{L}_6 = \varphi^{-1}g^{mj}g^{nk}D_m\varphi_n D_j\varphi_k\sqrt{-g} \tag{4.103}$$

Now we vary with respect to $g^{ab}$

$$\begin{aligned}
\delta\mathcal{L}_6 &= \varphi^{-1}\big\{(\delta^m_a\delta^j_b g^{nk}\delta g^{ab}D_m\varphi_n D_j\varphi_k + g^{mj}\delta^n_a\delta^k_b\delta g^{ab}D_m\varphi_n D_j\varphi_k)\sqrt{-g} \\
&-\tfrac{1}{2}D_m\varphi_n D^m\varphi^n\sqrt{-g}g_{ab}\delta g^{ab}\big\}
\end{aligned} \tag{4.104}$$

After simplifications we have

$$\delta\mathcal{L}_6 = \varphi^{-1}\big\{D_a\varphi_n D_b\varphi^n + D_m\varphi_a D_m\varphi^b - \frac{1}{2}g_{ab}D_m\varphi_n D^m\varphi^n\big\}\sqrt{-g}\delta g^{ab} \tag{4.105}$$

Now consider

$$D_j(\partial_k\varphi) = \partial_j\partial_k - \Gamma^m_{kj}\partial_m\varphi \tag{4.106}$$

and

$$D_k(\partial_j\varphi) = \partial_k\partial_j - \Gamma^m_{jk}\partial_m\varphi \tag{4.107}$$

Therefore, we conclude that

$$D_j\partial_k = D_k\partial_j \tag{4.108}$$

So, using

$$D_n\varphi_a = D_a\varphi_n \tag{4.109}$$

we obtain

$$T_{ab} = \varphi^{-1}(D_a\varphi_n D_b\varphi^n - \frac{1}{4}g_{ab}D_m\varphi_n D^m\varphi^n) \tag{4.110}$$

• $\mathcal{L}_7$

For the seventh term we have

$$\mathcal{I}_7 = \int d^4x\sqrt{-g}\,\varphi^2 F^{mk}F^n{}_k D_m(D_n\varphi)\sqrt{-g} \tag{4.111}$$

Variation of $\mathcal{L}_7$ with respect to $A_j$ is

$$\delta\mathcal{L}_7 = \varphi^2\big\{g^{mi}g^{ks}g^{np}[(D_i\delta A_s - D_s\delta A_i)F_{pk} + (D_p\delta A_k - D_k\delta A_p)F_{is}](D_m\varphi_n)\big\} \tag{4.112}$$

Consider

$$D_i(\varphi^2 Z^{nji}{}_n\delta A_j) = D_i(\varphi^2 Z^{nji}{}_n)\delta A_j + \varphi^2 Z^{nji}{}_n D_i\delta A_j \tag{4.113}$$

where we have defined

$$Z_{ikmn} := F_{ik}(D_m\varphi_n) \tag{4.114}$$

After taking care of total divergences, we arrive at

$$a_7 := 2\sqrt{-g}D_i\big\{\varphi^2(F^{ki}D_j\varphi_k + F_j{}^k D_i\varphi_k)\big\} \tag{4.115}$$

Using (4.108) Variation of $\mathcal{L}_7$ with respect to connection $\Gamma^a_{bc}$ is

$$\delta\mathcal{L}_7 = \varphi^2 F^{mk}F^n{}_k(-\delta\Gamma^j_{nm})D_j\varphi\sqrt{-g} \tag{4.116}$$

which becomes

$$\delta\mathcal{L}_7 = -\varphi^2 F^{ck}F^b{}_k(D_a\varphi)\sqrt{-g}\,\delta\Gamma^a_{bc} \tag{4.117}$$

after simplifications. From $\delta\mathcal{I}_7$, we have the expression

$$\gamma_7 := -\varphi^2 F^{ck}F^b{}_k(D_a\varphi)\sqrt{-g} \tag{4.118}$$

Variation with respect to $\varphi$ is

$$\delta\mathcal{L}_7 = \big\{\delta\varphi(2\varphi F^{mk}F^n{}_k D_m\varphi_n) + \varphi^2 F^{mk}F^n{}_k D_m D_n\delta\varphi\big\}\sqrt{-g} \tag{4.119}$$

Consider

$$D_m(\varphi^2 F^{mk}F^n{}_k D_n\delta\varphi) \;=\varphi^2 F^{mk}F^n{}_k D_m D_n\delta\varphi + \big\{D_n[D_m(\varphi^2 F^{mk}F^n{}_k)\delta\varphi] - D_n D_m(\varphi^2 F^{mk}F^n{}_k)\delta\varphi\big\} \tag{4.120}$$

Apart from the total divergences we obtain

$$\delta\mathcal{I}_7 = \int d^4x\sqrt{-g}\big\{D_n D_m(\varphi^2 F^{mk}F^n{}_k) + 2\varphi F^{mk}F^n{}_k D_m\varphi_n\big\}\,\delta\varphi \tag{4.121}$$

From $\delta\mathcal{I}_7$, we obtain

$$\Phi_7 := \sqrt{-g}D_n D_m(\varphi^2 F^{mk}F^n{}_k) + 2\sqrt{-g}\varphi F^{mk}F^n{}_k D_m\varphi_n \tag{4.122}$$

Rewriting $\mathcal{L}_7$ with the metric being made visible

$$\mathcal{L}_7 = (\sqrt{-g}\,\varphi^2 g^{ik}g^{mn}g^{pj}F_{km}F_{pn}D_iD_j\varphi) \tag{4.123}$$

then variation of $\mathcal{L}_7$ with respect to metric $g^{ab}$ is

$$\begin{aligned}\delta\mathcal{L}_7 \;&= \varphi^2\big\{\delta^i_a\delta^k_b g^{mn}g^{pj}F_{km}F_{pn}D_iD_j\varphi + g^{ik}\delta^m_a\delta^n_b g^{pj}F_{km}F_{pn}D_iD_j\varphi \\ &+ g^{ik}g^{mn}\delta^p_a\delta^j_b F_{km}F_{pn}D_iD_j\varphi - \tfrac{1}{2}g_{ab}g^{ik}g^{mn}g^{pj}F_{km}F_{pn}D_iD_j\varphi\big\}\sqrt{-g}\,\delta g^{ab}\end{aligned} \tag{4.124}$$

After necessary calculations and contractions, the expression above yields to

$$\begin{aligned}\delta\mathcal{L}_7 = \;&\varphi^2\big\{F_a{}^kF^i{}_kD_iD_b\varphi + F_b{}^kF^i{}_kD_iD_a\varphi + F_a{}^iF^b{}_jD_iD_j\varphi \\ &-\tfrac{1}{2}g_{ab}F^{ik}F^j{}_kD_iD_j\varphi\big\}\sqrt{-g}\delta g^{ab}\end{aligned} \tag{4.125}$$

From (4.24) we have

$$T_{ab} = \frac{1}{2}\varphi^2\big\{F_a{}^kF^i{}_kD_iD_b\varphi + F_b{}^kF^i{}_kD_iD_a\varphi + F_a{}^iF_b{}^jD_iD_j\varphi - \frac{1}{2}g_{ab}F^{ik}F^j{}_kD_iD_j\varphi\big\} \tag{4.126}$$

• $\mathcal{L}_8$

The eighth term is in the form

$$\mathcal{I}_8 = \frac{1}{2}\int d^4x\sqrt{-g}\,\varphi^2(D_kF_{mn}\varphi^mF^{kn} + D_kF_{mn}\varphi^kF^{mn}) \tag{4.127}$$

To our knowledge we have not encountered a Lagrangian containing such a term, apart from this particular KK reduction. For convenience, we take into account

$$\delta\mathcal{L}_8 = \delta(\sqrt{-g}L_8) \tag{4.128}$$

where

$$L_8 = \frac{1}{2}\varphi^2(D_k F_{mn}\varphi^m F^{kn} + D_k F_{mn}\varphi^k F^{mn}) \tag{4.129}$$

We start by varying with respect to the potentials $A_k$. As before we rewrite $L_8$ so that the potentials can be seen clearly. So we have

$$\begin{aligned} \delta L_8 &= \tfrac{1}{2}\varphi^2\big\{(D_k D_m \delta A_n - D_k D_n \delta A_m)\varphi^m F^{kn} + D^k F_m{}^n \varphi^m (D_k \delta A_n - D_n \delta A_k) \\ &\quad +(D_k D_m \delta A_n - D_k D_n \delta A_m)\varphi^k F^{mn} + D_k F^{mn}\varphi^k (D_m \delta A_n - D_n \delta A_m)\big\} \end{aligned} \tag{4.130}$$

In order to carry out terms involving $D_m D_k\, \delta A_n$, we consider

$$D_k(\varphi^m F^{kn} D_m \delta A_n) = D_k(\varphi^m F^{kn}) D_m \delta A_n + \varphi^m F^{kn}(D_m D_k \delta A_n) \tag{4.131}$$

and

$$D_m(D_k(\varphi^m F^{kn})\delta A_n) = D_m D_k(\varphi^m F^{kn})\delta A_n + D_k(\varphi^m F^{kn}) D_m \delta A_n \tag{4.132}$$

to obtain

$$\begin{aligned} D_k(\varphi^m F^{kn} D_m \delta A_n) &= D_m\left[D_k(\varphi^m F^{kn})\,\delta A_n\right] - D_m D_k(\varphi^m F^{kn})\,\delta A_n \\ &\quad +\varphi^m F^{kn}(D_m D_k \delta A_n) \end{aligned} \tag{4.133}$$

In view of the divergence theorem (3.7), terms with total divergences in (4.133) vanish. Then terms involving double total derivatives of $\delta A_k$ can be replaced by

$$\varphi^m F^{kn} D_k D_m \delta A_n = D_m D_k(\varphi^m F^{kn})\delta A_n \tag{4.134}$$

Simplifying yields

$$\begin{aligned} \delta L_8 &= \tfrac{1}{2}\varphi^2\big\{D_m D_k(\varphi^m F^{kn}) + D_m D_k(\varphi^n F^{mk}) \\ &\quad +2D_m D_k(\varphi^k F^{mn}) \\ &\quad +D_k[(D^k F^n{}_m)\varphi^m] + D_k[(D^n F_m{}^k)\varphi^m] \\ &\quad +2D_m[(D_k F^{nm})\varphi^k]\big\}\delta A_n \end{aligned} \tag{4.135}$$

Therefore, we obtain

$$\begin{aligned} a_8 := \;& \tfrac{1}{2}\sqrt{-g}\varphi^2\big\{D_m D_k[F^{kn}\varphi^m + F^{mk}\varphi^n + 2F^{mn}\varphi^k \\ & +D_k[(D^k F^n{}_m)\varphi^m + (D^n F_m{}^k)\varphi^m] + 2D_m[(D_k F^{nm})\varphi^k]\big\} \end{aligned} \tag{4.136}$$

In order to vary with respect to $\Gamma^a_{bc}$ we rewrite $L_8$ so that the connections become apparent

$$L_8 = \frac{1}{2}\varphi^2\{(\partial_k F_{mn} - \Gamma^j_{mk}F_{jn} - \Gamma^j_{nk}F_{mj}[\varphi^m F^{kn} + \varphi^k F^{mn}]\} \tag{4.137}$$

Then we have

$$\begin{aligned} \delta L_8 &= \tfrac{1}{2}\varphi^2\left\{(\partial_k F_{mn} - \delta\Gamma^j_{mk}F_{jn} - \delta\Gamma^j_{nk}F_{mj}[\varphi^{(m}F^{k)n}]\right\} \\ &= \tfrac{1}{2}\varphi^2(-\delta^j_a\delta^b_m\delta^c_k F_{jn} - \delta^j_a\delta^b_n\delta^c_k F_{mj})[\varphi^m F^{kn} + \varphi^k F^{mn}]\delta\Gamma^a_{bc} \end{aligned} \tag{4.138}$$

After simplification, this becomes

$$\delta L_8 = \frac{1}{2}\varphi^2(\varphi^b F_{an}F^{cn} - 2\varphi^c F_{aj}F^{bj} - \varphi^n F_{an}F^{cb})\delta\Gamma^a_{bc} \tag{4.139}$$

Finally the coefficient of $\delta\Gamma^a_{bc}$ gives

$$\gamma_8 := \frac{1}{2}\sqrt{-g}\varphi^2(\varphi^b F_{aj}F^{cj} - 2\varphi^c F_{aj}F^{bj}) \tag{4.140}$$

Variation with respect to $\varphi$ is

$$\delta L_8 = \frac{1}{2}(2\varphi^{-1}\delta\varphi L_8 + \varphi^2 D_k F^m{}_n D_m\delta\varphi F^{kn} + \varphi^2 D^k F_{mn}D_k\delta\varphi F^{mn}) \tag{4.141}$$

Using

$$D_m[\varphi^2 D^m F_{kn}\delta\varphi F^{kn}] = D_m[\varphi^2(D^m F_{kn})F^{kn}]\delta\varphi + \varphi^2(D^m F_{kn})F^{kn}D_m\delta\varphi \tag{4.142}$$

we have

$$\delta L_8 = \frac{1}{2}\{(2D_k F_{mn}\varphi^{(m}F^{k)n}) - D_m[\varphi^2(D_k F_{mn})F^{kn} + \varphi^2(D^m F_{kn})F^{kn}]\}\delta\varphi \tag{4.143}$$

Then the expression becomes

$$\Phi_8 := \frac{1}{2}\sqrt{-g}\{(2D_k F_{mn}\varphi^{(m}F^{k)n}) - D_m[\varphi^2(D_k F_{mn})F^{kn} + \varphi^2(D^m F_{kn})F^{kn}]\} \tag{4.144}$$

To take the variation with respect to $g^{ab}$ we rewrite $\mathcal{L}_8$ as

$$\mathcal{L}_8 = \frac{1}{2}\varphi^2\{g^{kj}g^{mi}g^{pn}D_j F_{mn}\varphi_i F_{kp} + g^{kj}g^{mi}g^{np}D_j F_{mn}\varphi_k F_{ip}\}(\sqrt{-g}) \tag{4.145}$$

Then we have

$$\begin{aligned} \delta\mathcal{L}_8 &= \tfrac{1}{2}\sqrt{-g}\,\varphi^2\left\{[(\delta^k_a\delta^j_b g^{mi}g^{pn} + g^{kj}\delta^m_a\delta^i_b g^{pn} + g^{kj}g^{mi}\delta^p_a\delta^n_b)D_j F_{mn}\varphi_i F_{kp}\right. \\ &\quad (\delta^k_a\delta^j_b g^{mi}g^{np} + g^{kj}\delta^m_a\delta^i_b g^{np} + g^{kj}g^{mi}\delta^n_a\delta^p_b)D_j F_{mn}\varphi_k F_{ip}]\,\delta g^{ab} \\ &\quad \left. -\tfrac{1}{2}g_{ab}\sqrt{-g}\delta g^{ab}L_8\right\} \end{aligned} \tag{4.146}$$

Simplifications yield

$$\begin{aligned}\delta\mathcal{L}_8 = \;& \tfrac{1}{2}\varphi^2\big\{D_b F^m{}_n \varphi_m F_a{}^n + D_j F_{an}\varphi_b F^{jn} + D_j F^i{}_b \varphi_i F^j{}_a \\ & D_b F^i{}_n \varphi_a F_i{}^n + D_j F_{an}\varphi^j F_b{}^n + D_j F_{ma}\varphi_j F^m{}_b - \tfrac{1}{2} g_{ab} L_8\big\}\sqrt{-g}\,\delta g^{ab}\end{aligned} \tag{4.147}$$

Therefore, we obtain

$$\begin{aligned}T_{ab} \;&= \tfrac{1}{4}\varphi^2\big\{(D_a F^{mn}\varphi_m F_{bn} + D_k F_{an}\varphi_b F^{kn} + D_k F_{am}\varphi^m F_b{}^k \\ & D_a F_{mn}\varphi_b F^{mn} + 2D_k F_{am}\varphi^k F_b{}^m) - \tfrac{1}{2} g_{ab} D_k F_{mn}\varphi^{(m}F^{k)n}\big\}\end{aligned} \tag{4.148}$$

- $\mathcal{L}_9$

The nineth term is

$$\mathcal{I}_9 = \int d^4x\sqrt{-g}\,\varphi\varphi_k\varphi^k F^{mn}F_{mn} \tag{4.149}$$

We vary $\mathcal{L}_9$ with respect to $A_n$

$$\delta\mathcal{L}_9 = 2\sqrt{-g}\varphi\varphi_k\varphi^k F^{mn}(D_m\delta A_n) \tag{4.150}$$

Considering

$$D_m(\sqrt{-g}\varphi\varphi_k\varphi^k F^{mn}\delta A_n) = D_m(\sqrt{-g}\varphi\varphi_k\varphi^k F^{mn})\delta A_n + \sqrt{-g}\varphi\varphi_k\varphi^k F^{mn} D_m\delta A_n \tag{4.151}$$

and using the Stoke's theorem, we obtain

$$\delta\mathcal{I}_9 = -2\int d^4x D_m(\sqrt{-g}\,\varphi\varphi_k\varphi^k F^{mn})\delta A_n \tag{4.152}$$

From above, the expression becomes

$$a_9 := -2D_m(\sqrt{-g}\varphi\varphi_k\varphi^k F^{mn}) \tag{4.153}$$

or since from [3] $D_m\sqrt{-g} = 0$, equally well as

$$a_9 := -2\sqrt{-g}D_m(\varphi\varphi_k\varphi^k F^{mn}) \tag{4.154}$$

Variation of $\mathcal{L}_9$ with respect to $\varphi$ is

$$\begin{aligned}&\delta\varphi\varphi_k\varphi^k F^{mn}F_{mn}\sqrt{-g} + (\varphi F^{mn}F_{mn}D_k\delta\varphi\varphi^k + \varphi F^{mn}F_{mn}\varphi^k D_k\delta\varphi)\sqrt{-g} \\ &= \delta\varphi\varphi_k\varphi^k F^{mn}F_{mn}\sqrt{-g} + 2(\varphi F^{mn}F_{mn}\varphi^k D_k\delta\varphi)\sqrt{-g}\end{aligned} \tag{4.155}$$

We consider (4.94) and (4.95)

$$D_k(\sqrt{-g}\varphi F^{mn}F_{mn}\varphi^k\delta\varphi) = D_k(\sqrt{-g}\varphi F^{mn}F_{mn}\varphi^k)\delta\varphi + \sqrt{-g}\varphi F^{mn}F_{mn}\varphi^k D_k\delta\varphi \tag{4.156}$$

to obtain

$$\delta\mathcal{I}_9 = \int d^4x\Big\{-2D_k(\sqrt{-g}\varphi F^{mn}F_{mn}\varphi^k) + \varphi_k\varphi^k F^{mn}F_{mn}\sqrt{-g}\Big\}\delta\varphi \tag{4.157}$$

So, we have

$$\Phi_9 := -2D_k(\sqrt{-g}\varphi F^{mn}F_{mn}\varphi^k) + \varphi_k\varphi^k F^{mn}F_{mn}\sqrt{-g} = 0 \tag{4.158}$$

or

$$\Phi_9 := -2\sqrt{-g}D_k(\varphi F^{mn}F_{mn}\varphi^k) + \varphi_k\varphi^k F^{mn}F_{mn}\sqrt{-g} = 0 \tag{4.159}$$

We rewrite $\mathcal{L}_9$ with noticeable metrics as

$$\mathcal{L}_9 = \sqrt{-g}\varphi g^{kj}\varphi_k\varphi_j g^{mi}g^{np}F_{mn}F_{ip} \tag{4.160}$$

After necessary simplifications, the variation of $\mathcal{L}_9$ with respect to $g^{ab}$ becomes

$$\begin{aligned}\delta\mathcal{L}_9 = \ &\Big\{\varphi_a\varphi_b F_{mn}F^{mn} + \varphi_k\varphi^k F_{an}F_b{}^n + \varphi_k\varphi^k F^i{}_a F_{ib} \\ &-\tfrac{1}{2}g_{ab}\varphi_k\varphi^k F_{mn}F^{mn}\Big\}\varphi\sqrt{-g}\,\delta g^{ab}\end{aligned} \tag{4.161}$$

So, the energy-momentum tensor takes the form

$$T_{ab} = \varphi(\varphi_k\varphi^k F_{ai}F^{bi} + \frac{1}{2}\varphi_a\varphi_b F_{mn}F^{mn} - \frac{1}{4}g_{ab}\varphi_k\varphi^k F_{mn}F^{mn}) \tag{4.162}$$

• $\mathcal{L}_{10}$

The tenth and the last term is

$$\mathcal{I}_{10} = \int d^4x\sqrt{-g}\varphi\varphi_m\varphi_n F^{mk}F^n{}_k \tag{4.163}$$

Varying $\mathcal{L}_{10}$ with respect to $A_k$, we have

$$\begin{aligned}\delta\mathcal{L}_{10} \ &= \Big\{-D_m(\varphi\varphi^m\varphi^n F_n{}^k) + D_m(\varphi\varphi^k\varphi^n F_n{}^m) \\ &-D_n(\varphi\varphi^m\varphi^n F_m{}^k) + D_n(\varphi\varphi^m\varphi^k F_m{}^n)\Big\}\sqrt{-g}\,\delta A_k\end{aligned} \tag{4.164}$$

Therefore, we have

$$a_{10} := 2D_m(\varphi\varphi^k\varphi^n F_n{}^m - \varphi\varphi^m\varphi^n F_n{}^k)\sqrt{-g} \tag{4.165}$$

Varying $\mathcal{I}_{10}$ with respect to $\varphi$

$$\delta\mathcal{L}_{10} = \sqrt{-g}\delta\varphi\varphi_m\varphi_n F^{mk}F^n{}_k + 2D_m\delta\varphi(\varphi\varphi_n F^{mk}F^n{}_k)\sqrt{-g} \tag{4.166}$$

and using

$$D_m(\sqrt{-g}\varphi\varphi_n F^{mk}F^n{}_k\delta\varphi) = D_m(\sqrt{-g}\varphi\varphi_n F^{mk}F^n{}_k)\delta\varphi + \sqrt{-g}\varphi\varphi_n F^{mk}F^n{}_k D_m\delta\varphi \tag{4.167}$$

we obtain the expression

$$\Phi_{10} := \sqrt{-g}\varphi_m\varphi_n F^{mk}F^n{}_k - 2D_m(\sqrt{-g}\varphi\varphi_n F^{mk}F^n{}_k) \tag{4.168}$$

Variation with respect to $g^{ab}$

$$\begin{aligned} \delta\mathcal{L}_{10} &= \big\{\delta_a^m\delta_b^i g^{nj}g^{kp}\varphi\varphi_m\varphi_n F_{ip}F_{jk} + g^{mi}\delta_a^n\delta_b^j g^{kp}\varphi\varphi_m\varphi_n F_{ip}F_{jk} \\ &\quad + g^{mi}g^{nj}\delta_a^k\delta_b^p\varphi\varphi_m\varphi_n F_{ip}F_{jk} - \tfrac{1}{2}g_{ab}\varphi\varphi_m\varphi_n F^{mk}F^n{}_k\big\}\sqrt{-g}\,\delta g^{ab} \end{aligned} \tag{4.169}$$

In view of (4.24), we have the energy-momentum tensor

$$T_{ab} = \varphi(\frac{1}{2}\varphi^i\varphi^j F_a{}^i F_b{}^j + \varphi_a\varphi_i F_{bk}F^{ik} + \varphi_b\varphi_i F_{ak}F^{ik} - \frac{1}{4}g_{ab}\varphi_i\varphi_j F^{ik}F^j{}_k) \tag{4.170}$$

## 4.3 Comparisons

In order to compare our results with those given in article [19], in particular the energy-momentum tensors we have obtained, we list five of them below. This is because in that article, there are only five expressions to be compared. Here, we use the notation $T_{ab}(^i)$, where the superscript $i$ denotes the tensor pertinent to each term in (4.7). So

$$T'^{(1)}_{ab} = \varphi T^{(1)}_{ab} \tag{4.171}$$

$$T^{(1)}_{ab} = R_{anjk}R_b{}^{njk} - \frac{1}{4}g_{ab}R_{mnjk}R^{mnjk} \tag{4.172}$$

$$T'^{(2)}_{ab} = \varphi^3 T^{(2)}_{ab} \tag{4.173}$$

$$T^{(2)}_{ab} = R_{anjk}F_b{}^nF^{jk} + R_{bnjk}F_a{}^nF^{jk} - \frac{1}{4}g_{ab}R_{jkmn}F^{jk}F^{mn} \tag{4.174}$$

$$T'^{(3)}_{ab} = \varphi^5 T^{(3)}_{ab} \tag{4.175}$$

$$T^{(3)}_{ab} = F_{mn}F^{mn}(2F_{ak}F_b{}^k - \frac{1}{4}g_{ab}F_{ij}F^{ij}) \tag{4.176}$$

$$T'^{(4)}_{ab} = \varphi^5 T^{(4)}_{ab} \tag{4.177}$$

$$\begin{aligned} T^{(4)}_{ab} &= 2F_{am}F^{mn}F_{ni}F^i{}_b - \frac{1}{4}g_{ab}F_{jm}F^{mn}F_{ni}F^{ij} \\ &= (2F_{am}F^i{}_b - \frac{1}{4}g_{ab}F_{jm}F^{ij})\,F^{mn}F_{ni} \end{aligned} \tag{4.178}$$

$$T'^{(5)}_{ab} = \frac{1}{2}\varphi^3 T'^{(5)}_{ab} \tag{4.179}$$

$$T^{(5)}_{ab} = D_aF_{ij}D_bF^{ij} + 2D^kF_a{}^jD_kF_{bj} - \frac{1}{2}g_{ab}D^kF^{ij}D_kF_{ij} \tag{4.180}$$

Now, we retrieve equations (specifically 46,53,47,50 and 51, respectively) from [19]. Here, we use the notation ${}^{(i)}T'_{ab}$ to denote the tensors in the above reference. This list reads as

$${}^{(1)}T'_{ab} ={}^{(1)} T_{ab} \tag{4.181}$$

$${}^{(1)}T_{ab} = R_{anjk}R_b{}^{njk} - \frac{1}{4}g_{ab}R_{mnjk}R^{mnjk} \tag{4.182}$$

$${}^{(2)}T'_{ab} = \varphi^2\ {}^{(2)}T_{ab} \tag{4.183}$$

$$\begin{aligned} {}^{(2)}T_{ab} &= \frac{1}{2}(R_{anjk}F_b{}^nF^{jk} + R_{bnjk}F_a{}^nF^{jk}) - \frac{1}{4}g_{ab}R_{jkmn}F^{jk}F^{mn} \\ &= F^{jk}\left[\frac{1}{2}(R_{anjk}F_b{}^n + R_{bnjk}F_a{}^n) - \frac{1}{4}g_{ab}R_{jkmn}F^{mn}\right] \end{aligned} \tag{4.184}$$

$$^{(3)}T'_{ab} = \varphi^4 \; ^{(3)}T_{ab} \tag{4.185}$$

$$^{(3)}T_{ab} = F_{mn}F^{mn}(F_{ak}F_b{}^k - \frac{1}{4}g_{ab}F_{ij}F^{ij}) \tag{4.186}$$

$$^{(4)}T'_{ab} = \varphi^4 \; ^{(4)}T_{ab} \tag{4.187}$$

$$\begin{aligned} ^{(4)}T_{ab} &= F_{am}F^{mn}F_{ni}F^i{}_b - \frac{1}{4}g_{ab}F_{jm}F^{mn}F_{ni}F^{ij} \\ &= (F_{am}F^i{}_b - \frac{1}{4}g_{ab}F_{jm}F^{ij})\, F^{mn}F_{ni} \end{aligned} \tag{4.188}$$

$$^{(5)}T'_{ab} = \varphi^2 \; ^{(5)}T_{ab} \tag{4.189}$$

$$^{(5)}T_{ab} = D_aF_{ij}D_bF^{ij} - \frac{1}{4}g_{ab}D^kF^{ij}D_kF_{ij} \tag{4.190}$$

### 4.4 Concluding Remarks

In this chapter we derived the governing field equations from the reduced quadratic curvature Lagrangian. In order to compare our results with those in reference [19], we shall simplify them by considering a constant scalar field. We note that even in this simplified case, the non-minimal couplings between the gravitational field and the electromagnetic (EM) field are considerably intricate. There are also terms with the fourth power of the EM field. It turns out that, this simplification is sufficient to derive the necessary conclusions in regard to the main statement of our problem.

By taking account of the calculations done for each term in the Lagrangian density (4.7), the field equation that yields from the variation of connection is

$$D_kR^k{}_{jmn} = \varphi^2 \left( \frac{3}{8}D_k(F^k{}_mF_{nj}) + F^k{}_nD_jF_{km} \right) \tag{4.191}$$

Here, we chose $\varphi$ not to depend on $x^A$. The terms on the right hand side can be interpreted as a source for a Stephenson-Kilmister-Yang type of field equation. They come from the $RF^2$ and the BP terms, respectively.

Similarly, the overall energy-momentum tensor (again with $\varphi$ not depending on $x^A$)

becomes

$$
\begin{aligned}
T_{ab} \;&= \varphi \left\{ (R_{anjk} R_b{}^{njk} - \tfrac{1}{4} g_{ab} R_{mnjk} R^{mnjk}) \right. \\
&- \tfrac{3}{2} \varphi^2 (R_{anjk} F_b{}^n + R_{bnjk} F_a{}^n - \tfrac{1}{4} g_{ab} R_{jkmn} F^{mn})\, F^{jk} \\
&+ \tfrac{3}{8} \varphi^4 (2 F_{ak} F_b{}^k - \tfrac{1}{4} g_{ab} F_{ij} F^{ij}) F_{mn} F^{mn} \\
&+ \tfrac{5}{8} \varphi^4 (2 F_{am} F^i{}_b - \tfrac{1}{4} g_{ab} F_{jm} F^{ij})\, F^{mn} F_{ni} \\
&\left. + \tfrac{1}{2} \varphi^2 (D_a F_{ij} D_b F^{ij} + 2 D^k F_a{}^j D_k F_{bj} - \tfrac{1}{2} g_{ab} D^k F^{ij} D_k F_{ij}) \right\}
\end{aligned}
\tag{4.192}
$$

One can immediately expect that when the Lagrangian density is reduced, there will be an additional $\varphi$ that multiplies all terms coming from the 4D form of the 5D $\sqrt{-\hat{g}}$. Therefore, when the variation is taken with respect to the 4D metric, there is an extra $\varphi$ appearing in $T_{ab}$, as we see in (4.192) compared to $\hat{T}_{ab}$, the reduced form of $\hat{T}_{AB}$ to the actual spacetime.

Totally, we have four equations coming from the variation of each variable. On the other hand there are seven equations governing the fields, coming from the splitting of the 4D, mixed and the fifth components when the variation with respect to 5D metric $\hat{g}_{AB}$ and the 5D connection $\hat{\Gamma}^A_{BC}$, is performed first in [19].

Even by taking into account these simple considerations, it can easily be noted that the two successive procedures, namely reductions and variations, that we used to obtain the field equations, when their order is reversed do not give the same results. Further discussions in regard to differences and their origins will be given in the next chapter.

# CHAPTER 5

# CONCLUSION

In this thesis, using reduction mechanism on the 5D Lagrangian and variational principle for the reduced 4D action, the field equations are obtained for a quadratic curvature model within the framework of the Kaluza-Klein theory. Then, the resulting field equations are compared with those obtained by the least action principle that are performed straightaway to the 5D action. The origin of the differences in the equations are demonstrated clearly.

In Chapter 2, we made a general review of the standard KK theory. We started by introducing Kaluza's approach, then we explained how Klein elaborated the theory. In this context, we studied the cylinder condition, the geodesic equation and the meaning of the fifth component of linear momentum. By using the periodicity of the fifth dimension, we showed how Klein expressed the metric tensor, vector potentials and scalar fields in terms of Fourier series. Pertinent to those ideas, we discussed the Kaluza-Klein mass tower and showed how one can obtain the Kaluza-Klein mass.

In Chapter 3, the field equations are derived from the variational principle for 5D Einstein-Hilbert action. First, this action is reduced into the real 4D spacetime. Then, we varied the reduced action with respect to $g^{ab}$, $A_k$ and $\varphi$, by the least action principle, to obtain three field equations.

There is an alternative way to approach the problem, which is to vary the 5D Einstein-Hilbert action with respect to the 5D metric $\hat{g}^{AB}$ to obtain $\hat{G}_{AB} = 0$. Then, the dimension is reduced in order to obtain field equations from the splitting of the components as: $\hat{G}_{ab} = 0, \quad \hat{G}_{a5} = 0, \quad \hat{G}_{55} = 0$, again culminating in three equations. We note that, we end up with the same number of equations. Comparing these two sets of equations, we see that they give the same results, irrespective of the order of the usage of the reduction and variation procedures. Thus these two applications are commutative in regard to the case of Einstein-Hilbert action in 5D.

We have also studied conformal rescaling and obtained the field equations in the Einstein frame. Furthermore, we derived the 5D Einstein tensor from the 5D Bianchi identities.

Our exploration on the standard KK theory concludes with the Brans-Dicke equations obtained by using the least action principle. Thus, we have also found the conditions under which the Kaluza-Klein and Brans-Dicke field equations are equivalent.

Chapter 4 is devoted to find an answer to the main question of this thesis, namely whether the variation and reduction procedures yield the same results when their order of implementation is interchanged. Earlier, in reference [19], the approach was to take variation of

$$\hat{\mathcal{L}} = \sqrt{-\hat{g}}\hat{R}_{JKMN}\hat{R}^{RJKMN} \tag{5.1}$$

directly with respect to the 5D metric $\hat{g}^{AB}$

$$\hat{T}_{AB} = \frac{1}{2\sqrt{-g}}\frac{\delta\hat{\mathcal{L}}}{\delta\hat{g}^{AB}} \tag{5.2}$$

in order to obtain the energy-momentum tensor

$$\hat{T}_{AB} = \hat{R}_{AKMN}\hat{R}_{B}{}^{KMN} - \frac{1}{4}\hat{g}_{AB}R_{JKMN}R^{JKMN} \tag{5.3}$$

whose spacetime components are

$$\hat{T}_{ab} = \hat{R}_{aKMN}\hat{R}_{b}{}^{KMN} - \frac{1}{4}\hat{g}_{ab}R_{JKMN}R^{JKMN} \tag{5.4}$$

This expression is reduced to become

$$
\begin{aligned}
\hat{T}_{ab} &= \big\{ (R_{anjk}R_b{}^{njk} - \tfrac{1}{4}g_{ab}R_{mnjk}R^{mnjk}) \\
&\quad -\tfrac{3}{2}\varphi^2(\tfrac{1}{2}R_{anjk}F_b{}^n + \tfrac{1}{2}R_{bnjk}F_a{}^n - \tfrac{1}{4}g_{ab}R_{jkmn}F^{mn})\, F^{jk} \\
&\quad +\tfrac{3}{8}\varphi^4(F_{ak}F_b{}^k - \tfrac{1}{4}g_{ab}F_{ij}F^{ij})F_{mn}F^{mn} + \tfrac{5}{8}\varphi^4(F_{am}F^i{}_b - \tfrac{1}{4}g_{ab}F_{jm}F^{ij})\, F^{mn}F_{ni} \\
&\quad +\tfrac{1}{2}\varphi^2(\tfrac{1}{2}D_aF_{ij}D_bF^{ij} + D^kF_a{}^jD_kF_{bj} - \tfrac{1}{2}g_{ab}D^kF^{ij}D_kF_{ij}) \big\}
\end{aligned}
\tag{5.5}
$$

where we considered the case with $\varphi = constant$. This is the optimal amount of simplification, for comparative purposes. In addition to the pure gravitational part, it includes non-minimal couplings and non-linear electrodynamics. Besides, we have the advantage of ignoring the difference between Einstein and Jordan frames.

In this chapter, we have also used the Palatini method; however we first reduced the action to the actual spacetime as in (4.7), where the constituent fields of the KK theory are explicit in the Lagrangian. This being the case, we took variations with respect to metric, connection, four potentials $A_k$ and $\varphi$.

We observe that, in our energy-momentum tensor, (4.192), apart from the terms coming from the variation of $\sqrt{-g}$, there is a factor of 2. Except for that, these two expressions are formally similar when compared with the results given in [19]. The factor 2 comes from the fact that, when first reduced, otherwise hidden $4D$ metrics come into play to form invariants from the constituent fields.

In Chapter 3 we saw that in the standard KK theory, there are three sets of equations, irrespective of the order of the successive procedures. They come from the variations with respect to three fields or from the splitting of $\hat{G}_{AB}$ into its two pure and one mixed components. On the other hand the usage of the Palatini method for the quadratic curvature model in [19], yield seven equations, four coming from the splitting of $D_K\hat{R}^K{}_{JMN} = 0$ and three from (5.3). In our case, we have totally four equations, one for each field. We have also included the variations with respect to $\Gamma^a_{bc}$ and it's outcome as a field equations has its restriction on the system. This is another point in terms of the differences originating from order switching.

Perhaps, one might have expected that, unlike the standard KK theory, these two approaches would yield different results due to the fact that, the Lagrangian in (5.1) is quadratic while the Einstein-Hilbert Lagrangian is linear in terms of the curvature. Nevertheless, the explicit forms of the expressions and the exact cause of the differences would not be clear until one carries out the calculations. Finally, we conclude that although both approaches have some similar results, it cannot be said that these two successive operations are commutative.

To the best of our knowledge such a comparison has not been investigated elsewhere. No matter how small it may seem, we believe we have filled a gap in this regard.

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# APPENDIX A

# DIFFERENTIAL FORMS

Within the scientific areas of differential geometry and tensor calculus, the employment of differential forms is a versatile tool for multivariable calculus, giving a bound together approach to characterize coordinates over curves, surfaces, volumes, and higher-dimensional manifolds. For a full fledged account of the subject one can consult to [47, 48].

The space of all totally antisymmetric tensors of rank p over the manifold $M$ is denoted by $\Lambda^p(M)$ and its elements are called p-forms.

One simplest example is the object that occurs under the integral sign

$$\int P\, dx + Q\, dy + R\, dz \tag{A.1}$$

which is a 1-form.
A p-form $A$ in a coordinate chart $x^\mu$ is expanded as

$$A = \frac{1}{p!} A_{\mu_1 \dots \mu_p}(x)\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{A.2}$$

For a p-form A $\epsilon\, \Lambda^p(M)$ a and q-form B $\epsilon\, \Lambda^q(M)$ a wedge product ($\wedge$-product) is defined as

$$A \wedge B = \frac{1}{p!q!} A_{\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_{p+q}} \tag{A.3}$$

to yield a $(p+q)$-form, which is subject to

$$A \wedge B = (-1)^{pq} B \wedge A \tag{A.4}$$

and is called the exterior algebra. We denote $D$ as the dimension of $M$ and note that $\Lambda^p(M) = \{0\}$ if $p > D$.

For instance the wedge product of two 1-forms is

$$A \wedge B = \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)\, dx^\mu \wedge dx^\nu \tag{A.5}$$

There are many operators acting on p-forms. These are usually defined by maps from a form space to another form space. In the sequel we shall introduce the most common operations.

• The Hodge dual operation $^*$ is a linear map from the space of $\Lambda^p(M)$ to the space of $\Lambda^{D-p}(M)$

$$^* : \Lambda^p(M) \longrightarrow \Lambda^{D-p}(M) \tag{A.6}$$

where to a p-form as in (A.2) it assigns a (D-p)-form as

$$^*A = \frac{1}{p!(D-p)!}\varepsilon^{\mu_1 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_D}\, A_{\mu_1 \cdots \mu_p}(x)\, dx^{\mu_{p+1}} \wedge ... \wedge dx^{\mu_D} \tag{A.7}$$

and depends on the metric defined on $M$. Here the relation between antisymmetric tensor density $\varepsilon$ and Levi-civita antisymmetric symbol $\epsilon$ is given as

$$\varepsilon_{\mu_1 \ldots \mu_D} = \sqrt{|g|}\epsilon_{\mu_1 \ldots \mu_D} \tag{A.8}$$

When the Hodge star is applied twice on a p-form, that results in

$$^*(^*A) = (-1)^{p(D-p)+s}A \tag{A.9}$$

where $s$ is the number of negative eigenvalues of the metric. Here $s = 1$, since we work with the Minkowskian metric $\eta_{\mu\nu}$ =diag$(-1, 1, ..., 1)$.

The invariant volume element

$$\begin{aligned} ^*1 \ &= \tfrac{1}{D!}\varepsilon_{\mu_1 \ldots \mu_D}\, dx^{\mu_1} \wedge ... \wedge dx^{\mu_D} \\ &= \varepsilon_{0123 \ldots D-1}\, dx^0 \wedge dx^1 .... \wedge dx^{D-1} \\ &= (-1)^{(D-1)}\sqrt{|g|}\, d^D x \end{aligned} \tag{A.10}$$

• The exterior derivative d is a linear map

$$d : \Lambda^p(M) \longrightarrow \Lambda^{p+1}(M) \tag{A.11}$$

such that

$$dA = \frac{1}{(p+1)!}\partial_\mu(A_{\mu_1 \ldots \mu_p})dx^\mu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \tag{A.12}$$

For instance the exterior derivative of a scalar function $\phi$ is

$$d\phi = \partial_\mu \phi \, dx^\mu \tag{A.13}$$

and the exterior derivative of a 1-form $A = A_\mu dx^\mu$ is

$$dA = \frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu) \, dx^\nu \wedge dx^\mu \tag{A.14}$$

When a second exterior derivative of (A.12) is taken and taking account of the continuity of the partial derivatives (i.e.,$\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$) it is seen that

$$d(dA) = 0 \tag{A.15}$$

which is called the Poincaré lemma

$$d^2 = 0 \tag{A.16}$$

The exterior derivative operator is an anti-derivation in the sense that for $A \epsilon \Lambda^p(M)$ and $B \epsilon \Lambda^q(M)$ we have

$$d(A \wedge B) = d(A) \wedge B + (-1)^p A \wedge dB \tag{A.17}$$

• The interior derivative operator $i_X$ on the elements of $\Lambda^p(M)$, with respect to a vector field $X = X^\mu \, \partial_\mu$ is a map

$$i_X : \Lambda^p(M) \longrightarrow \Lambda^{p-1}(M) \tag{A.18}$$

such that $i_{\partial_\mu} dx^\nu = \delta^\nu{}_\mu$. It is defined by

$$i_X A = \frac{1}{(p-1)} X^{\mu_1} A_{\mu_1 \ldots \mu_p}(x) \, dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \tag{A.19}$$

with the anti-derivative property

$$i_X(A \wedge B) = i_X(A) \wedge B + (-1)^p A \wedge i_X B \tag{A.20}$$

It can easily be shown that $i_X^2 = 0$.

• The Laplace-Beltrami operator is a map

$$\Delta : \Lambda^p(M) \longrightarrow \Lambda^p(M) \tag{A.21}$$

whose operation on a p-form A is defined as

$$\Delta A \equiv^* d^* d \, A + d^* d^* \, A \tag{A.22}$$

## A.1 Electromagnetic Theory

By using electromagnetic four potential A which is the 1 form $A = A_\mu dx^\mu$, we can write electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.23}$$

The $F_{\mu\nu}$ can be written as a two form

$$F = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu \tag{A.24}$$

In exact form, F is expressed as

$$F = dA \tag{A.25}$$

The exterior derivative of F becomes

$$dF = ddA = 0 \tag{A.26}$$

The expression above gives the homogenous part of the Maxwell's equation in tensor notation

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \tag{A.27}$$

The Hodge dual of F is

$$^*F = \frac{1}{4}\varepsilon_{\alpha\beta\mu\nu} F^{\alpha\beta} dx^\mu \wedge dx^\nu \tag{A.28}$$

and is responsible for the non-homogenous part of the Maxwell's equation expressed in differential forms

$$d\,^*F = \,^*J \tag{A.29}$$

where $J = J_\mu\, dx^\mu$ is defined as a electromagnetic source 1-form.

## A.2 Cartan Structure Equations

We define metric as a symmetric second rank tensor

$$\begin{aligned} g &= g_{\mu\nu}\, dx^\mu \otimes dx^\nu \\ &= \eta_{ab}\, e^a \otimes e^b \end{aligned} \tag{A.30}$$

such that

$$e^a = h^a{}_\mu \, dx^\mu \tag{A.31}$$

Here, $\eta_{ab} = diag(-1, 1, 1, 1)$ is the Minkowskian metric, and $h^a{}_\mu$ are the vier-beins. With respect to $e^a$ there is a natural vector field $X = X^a \, \partial_a$, such that

$$X_b(e^a) = \delta^a{}_b \tag{A.32}$$

Cartan's structure equations are

$$\tau^a = D\, e^a \equiv de^a + w^a{}_c \wedge e^c \tag{A.33}$$

and

$$R^a{}_b = dw^a{}_b + w^a{}_c \wedge w^c{}_b \tag{A.34}$$

where $w^a{}_c$ is the connection 1-form, $\tau^a$ is the torsion 2-form, and $R^a{}_b$ is the curvature 2-form.

The following expressions

$$D\tau^a = d\tau^a + w^a{}_c \wedge \tau^c \tag{A.35}$$

$$DR^a{}_b = dR^a{}_b + w^a{}_c \wedge R^c{}_b - w^c{}_b \wedge R^a{}_c = 0 \tag{A.36}$$

are the first and the second Bianchi identities, respectively. The definition of the covariant exterior derivative $D$ acting on tensor valued forms is clear from the above equations.

The connection 1-form $w^a{}_c$ is antisymmetric with respect to its two indices, which can be seen from

$$D\eta^{ab} = d\eta^{ab} + w^a{}_c \, \eta^{cb} + w^b{}_c \, \eta^{ac} = 0 \tag{A.37}$$

The spin coefficients can be read off from

$$w^a{}_b = w^a{}_{b\,c} \, e^c \tag{A.38}$$

In order to show the second Bianchi identity, we substitute (A.34) in (A.36)

$$\begin{aligned} dw^a{}_c \wedge w^c{}_b - w^a{}_c \wedge dw^c{}_b + w^a{}_c \wedge (dw^c{}_b + w^c{}_k \wedge w^k{}_b) \\ -w^c{}_b \wedge (dw^a{}_c + w^a{}_k \wedge w^k{}_c) = 0 \end{aligned} \tag{A.39}$$

where we have also rearranged and used the following

$$
\begin{aligned}
&w^c{}_b \wedge dw^a{}_c = -dw^a{}_c \wedge w^c{}_b, \\
&w^a{}_c \wedge w^c{}_k \wedge w^k{}_b = w^a{}_k \wedge w^k{}_c \wedge w^c{}_b = w^c{}_b \wedge w^a{}_k \wedge w^k{}_c
\end{aligned}
\tag{A.40}
$$

One can read off the orthonormal components of the Riemann tensor from

$$R^a{}_b = \frac{1}{2} R^a{}_{b\,cd}\, e^c \wedge e^d \tag{A.41}$$

## A.3 Electromagnetic Theory in Curved Space

In terms of the potentials the electromagnetic field tensor $F$ is expressed as

$$F_{ab} = D_a A_b - D_b A_a = \partial_a A_b - \partial_b A_a \tag{A.42}$$

with $D = e^a D_a$.

In terms of orthonormal basis 1-forms $e^a$, $F$ is represented as a 2-form

$$F = \frac{1}{2} F_{ab}\, e^a \wedge e^b \tag{A.43}$$

It can be seen that exterior covariant derivative of $F$ is

$$DF = 0 \tag{A.44}$$

and when expressed in terms of the conventional tensor notation becomes

$$D_a F_{bc} + D_b F_{ca} + D_c F_{ab} = 0 \tag{A.45}$$

giving the homogenous Maxwell equations. The non-homogenous Maxwell's equation can be written as

$$D^* F \;=\; {}^* J \tag{A.46}$$

where $J = J_a\, e^a$ is the source one-form. In terms of tensor notation, above equation is expressed as

$$D_a F^{ab} = J^b \tag{A.47}$$

# APPENDIX B

# CALCULATION OF THE RIEMANN TENSOR

We have given the components of the Riemann tensor in Chapter 2. This appendix provides the explicit calculations of the components of the Riemann tensor in the fünf-bein basis, using Cartan's structure equations.

The five dimensional coordinate system is

$$\hat{x}^A = (x^a, x^5) \tag{B.1}$$

where the indices with capital Latin letters take values $A = 0, 1, 2, 3, 5$, whereas all the indices with lowercase Latin letters take values $a = 0, 1, 2, 3$. The Kaluza-Klein metric in five dimensions can conveniently be written as

$$\hat{G}(x^j, x^5) = G(x^j) + \varphi(x^j)[A(x^j) + dx^5] \otimes \varphi(x^j)[A(x^j) + dx^5] \tag{B.2}$$

where

$$G(x^j) = \eta_{ab}\, e^a \otimes e^b \tag{B.3}$$

With the following choice of the orthogonal basis 1-forms

$$\hat{E}^a(x^j, x^5) = e^a(x^j) \qquad \text{and} \qquad \hat{E}^5 = \varphi(x^j)(A(x^j) + dx^5) \tag{B.4}$$

one can write the Kaluza-Klein metric succinctly

$$\hat{G}(x^j, x^5) = \eta_{AB} \hat{E}^A \otimes \hat{E}^B \tag{B.5}$$

where $\hat{E}^A = (\hat{E}^a, \hat{E}^5)$ satisfying $\iota_{X_B}(\hat{E}^A) = \delta^A_B$ . In terms of the coordinate basis, fünf-beins can be expressed as $\hat{E}^A = h^A{}_\mu dx^\mu$, where $\mu = 0, 1, 2, 3, 5$ .

The electromagnetic potential $A(x^j)$ is a 1-form, such that

$$A(x^j) = A_a(x^j)\, e^a \tag{B.6}$$

and the electromagnetic field

$$F(x^j) = dA(x^j) \tag{B.7}$$

is a 2-form with

$$F(x^j) = \frac{1}{2} F_{ab}\, e^a \wedge e^b \tag{B.8}$$

Also note that $\varphi(x^j)$ is a scalar field.

Cartan structure equations are

$$\hat{T}^A = d\hat{E}^A + \hat{\Omega}^A{}_B \wedge \hat{E}^B \tag{B.9}$$

and

$$\hat{R}^A{}_B = d\hat{\Omega}^A{}_B + \hat{\Omega}^A{}_M \wedge \hat{\Omega}^M{}_B \tag{B.10}$$

Above $\hat{T}^A$ are the torsion 2-forms, $\hat{\Omega}^A{}_M$ are the connection 1-forms and $\hat{R}^A{}_B$ are the curvature 2-forms. Here we shall work in a tortion-free space, thus take $\hat{T}^A = 0$.

Curvature 2-forms can be written

$$\hat{R}^A{}_B = \frac{1}{2} \hat{R}^A{}_{BMN} \hat{E}^M \wedge \hat{E}^N \tag{B.11}$$

in terms of Riemann curvature tensors and basis 1-forms. Furthermore, one can show that

$$d\hat{R}^A{}_B + \hat{\Omega}^A{}_C \wedge \hat{R}^C{}_B - \hat{\Omega}^C{}_B \wedge \hat{R}^A{}_C = 0 \tag{B.12}$$

which is in fact an equivalent of the Bianchi identity $D_{[A}\hat{R}_{BC|MN]} = 0$.

In order to find the curvature 2-forms, first we have to find the connection 1-forms, where they can be calculated using the first of Cartan's equations (B.9) . So we consider

$$d\hat{E}^5 + \hat{\Omega}^5{}_k \wedge e^k + \hat{\Omega}^5{}_5 \wedge \hat{E}^5 = 0 \tag{B.13}$$

where the last term vanishes, since $\hat{\Omega}^5{}_5 = 0$. Tackling the remaining terms we have

$$-\hat{\Omega}^5{}_k \wedge e^k = \frac{d\varphi}{\varphi} \wedge \hat{E}^5 + \varphi(dA) \tag{B.14}$$

The right hand side can be rearranged in terms of the basis forms

$$-\hat{\Omega}^5{}_k \wedge e^k = \frac{1}{\varphi} \varphi_k e^k \wedge \hat{E}^5 + \frac{1}{2} \varphi F_{jk} e^j \wedge e^k \tag{B.15}$$

Therefore, combining all the necessary terms, we obtain one of the connection 1-form as

$$\hat{\Omega}^5{}_k = \varphi^{-1}\varphi_k \hat{E}^5 + \frac{1}{2}\varphi F_{kj} e^j \tag{B.16}$$

The remaining connection 1-form can be found easily again by using Cartan's equation (B.9). This time we consider

$$d\hat{E}^a = -\hat{\Omega}^a{}_k \wedge e^k + \hat{\Omega}^5{}_a \wedge \hat{E}^5 \tag{B.17}$$

where we used the property that $\hat{\Omega}_{5a} = -\hat{\Omega}_{a5}$ . We also have

$$de^a = -\omega^a{}_k \wedge e^k \tag{B.18}$$

since $\hat{E}^a = e^a$. Equating (B.17) and (B.18) give us

$$-\omega^a{}_k \wedge e^k = -\hat{\Omega}^a{}_k \wedge e^k + \hat{\Omega}^5{}_a \wedge \hat{E}^5 \tag{B.19}$$

Substituting (B.16) into the (B.19)

$$-\omega^a{}_k \wedge e^k + \hat{\Omega}^a{}_k \wedge e^k - (\varphi^{-1}\varphi^a \hat{E}^5 + \frac{1}{2}\varphi F^a{}_k e^k) \wedge \hat{E}^5 = 0 \tag{B.20}$$

Therefore, we obtain the remaining connection 1-form as

$$\hat{\Omega}^a{}_k = \omega^a{}_k - \frac{1}{2}\varphi F^a{}_k \hat{E}^5 \tag{B.21}$$

Now we can calculate the components of the Riemann tensor from curvature 2-forms, using the second Cartan structure equation (B.10). To this end we consider

$$\hat{R}^a{}_b = d\hat{\Omega}^a{}_b + \hat{\Omega}^a{}_k \wedge \hat{\Omega}^k{}_b + \hat{\Omega}^a{}_5 \wedge \hat{\Omega}^5{}_b \tag{B.22}$$

First we need the derivative of the connection one form

$$d\hat{\Omega}^a{}_b = d(\omega^a{}_b - \frac{1}{2}\varphi F^a{}_b \hat{E}^5) \tag{B.23}$$

which becomes

$$\begin{aligned} d\hat{\Omega}^a{}_b = \;& d\omega^a{}_b - \tfrac{1}{2}\varphi_k F^a{}_b e^k \wedge \hat{E}^5 \\ & -\tfrac{1}{2}\varphi F^a{}_b(-\tfrac{1}{2}\varphi F_{kn} e^n \wedge e^k + \varphi^{-1}\varphi_k e^k \wedge \hat{E}^5 - \tfrac{1}{2}\varphi dF^a{}_b \wedge \hat{E}^5) \end{aligned} \tag{B.24}$$

For the other terms we need to calculate: $\hat{\Omega}^5{}_k \wedge \hat{\Omega}^k{}_a$ and $\hat{\Omega}^a{}_5 \wedge \hat{\Omega}^5{}_b$, resulting in

$$\hat{\Omega}^a{}_k \wedge \hat{\Omega}^k{}_b = (\omega^a{}_k \wedge \omega^k{}_b) - \frac{1}{2}\varphi \omega^a{}_k F^k{}_b \hat{E}^5 + \frac{1}{2}\varphi \omega^k{}_b F^a{}_k \hat{E}^5 \tag{B.25}$$

and

$$\hat{\Omega}^a{}_5 \wedge \hat{\Omega}^5{}_b = -\frac{1}{4}\varphi^2 F^a{}_k F_{bj} e^k \wedge e^j - \frac{1}{2}\varphi_b F^a{}_k e^k \wedge \hat{E}^5 + \frac{1}{2}\varphi^a F_{bj} e^j \wedge \hat{E}^5 \quad \text{(B.26)}$$

Substituting (B.24), (B.25) and (B.26) to the (B.22), and making some simple organisation gives the following equation

$$\begin{aligned}\hat{R}^a{}_b = \quad & R^a{}_b - \tfrac{1}{2}\varphi^2 F^a{}_b F - \tfrac{1}{4}\varphi^2 F^a{}_k F_{bj} e^k \wedge e^j + [-\tfrac{1}{2}\varphi D F^a{}_b - F^a{}_b d\varphi \\ & + \tfrac{1}{2}(\varphi^a F_{bk} - \varphi_b F^a{}_k) e^k] \wedge \hat{E}^5\end{aligned} \quad \text{(B.27)}$$

Manipulating the indices, the above equation can be rearranged

$$\begin{aligned}\hat{R}^a{}_b \; &= \tfrac{1}{2} R^a{}_{bmn} e^m \wedge e^n - \tfrac{1}{2}(\tfrac{1}{2}\varphi^2 F^a{}_b F_{mn}) e^m \wedge e^n - \tfrac{1}{4}\varphi^2 F^a{}_m F_{bn} e^m \wedge e^n \\ & + [-\tfrac{1}{2}\varphi D_m F^a{}_b e^m - F^a{}_b \varphi_m e^m + \tfrac{1}{2}(\varphi^a F_{bm} - \varphi_b F^a{}_m) e^m] \wedge \hat{E}^5\end{aligned} \quad \text{(B.28)}$$

Now from (B.11) we have

$$\begin{aligned}\hat{R}^a{}_b \; &= \tfrac{1}{2}\left(\hat{R}^a{}_{bmn} e^m \wedge e^n + \hat{R}^a{}_{bm5}\, e^m \wedge \hat{E}^5 + \hat{R}^a{}_{b5n}\, \hat{E}^5 \wedge e^m\right) \\ &= \tfrac{1}{2}\hat{R}^a{}_{bmn} e^m \wedge e^n + \hat{R}^a{}_{bm5} e^m \wedge \hat{E}^5\end{aligned} \quad \text{(B.29)}$$

The coefficient of $\frac{1}{2}e^m \wedge e^n$ gives the Riemann curvature tensor $\hat{R}^a{}_{bmn}$

$$\hat{R}_{abmn} = R_{abmn} - \frac{1}{2}\varphi^2 F_{ab} F_{mn} - \frac{1}{2}\varphi^2 F_{am} F_{bn} \quad \text{(B.30)}$$

The last term of the above equation should be antisymmetrised eventually to become

$$\hat{R}_{abmn} = R_{abmn} - \frac{1}{4}\varphi^2 (2F_{ab}F_{mn} + F_{am}F_{bn} - F_{an}F_{bm}) \quad \text{(B.31)}$$

Also, the coefficient of $e^m \wedge \hat{E}^5$ gives $\hat{R}^a{}_{bm5}$. Therefore, this component can be read off from (B.28) easily

$$\hat{R}_{ab5m} = \frac{1}{2}\varphi D_m F_{ab} + \frac{1}{2}(2\varphi_m F_{ab} + \varphi_b F_{am} - \varphi_a F_{bm}) \quad \text{(B.32)}$$

To obtain the remaining components again we rewrite (B.10) as

$$\hat{R}^5{}_a = d\hat{\Omega}^5{}_a + \hat{\Omega}^5{}_k \wedge \hat{\Omega}^k{}_a \quad \text{(B.33)}$$

since $\hat{\Omega}^5{}_5 = 0$. To calculate the first term on the right hand side we take the exterior derivative

$$d\hat{\Omega}^5{}_a = \frac{1}{2}d(\varphi F_{aj} e^j) + d(\varphi^{-1}\varphi_a) \wedge \hat{E}^5 + \varphi^{-1}\varphi_a d\hat{E}^5 \quad \text{(B.34)}$$

which becomes

$$d\hat{\Omega}^5{}_a = \frac{1}{2}d(\varphi F_{aj}e^j)+d(\varphi^{-1}\varphi_a)\wedge\hat{E}^5+\varphi^{-1}\varphi_a[-\frac{1}{2}\varphi F_{km}e^m-\varphi^{-1}\varphi_k\hat{E}^5]\wedge e^k \quad \text{(B.35)}$$

Also we need to find $\hat{\Omega}^5{}_k \wedge \hat{\Omega}^k{}_a$ . So

$$\hat{\Omega}^5{}_k \wedge \hat{\Omega}^k{}_a = -\frac{1}{2}\varphi\omega^k{}_a \wedge F_{kj}e^j - \frac{1}{4}\varphi^2 F_{kj}F^k{}_a e^j \wedge \hat{E}^5 - \varphi^{-1}\omega^k{}_a\varphi_k\hat{E}^5 \quad \text{(B.36)}$$

Substituting (B.35) and (B.36) into (B.33) and through some rearrangements we have

$$\hat{R}^5{}_a = \frac{1}{2}D(\varphi F_{aj}e^j) + \varphi_a F + (\varphi^{-1}D\varphi_a + \frac{1}{4}\varphi^2 F_{ak}F^k{}_j e^j) \wedge \hat{E}^5 \quad \text{(B.37)}$$

to be compared with the relevant components of (B.11)

$$\hat{R}^5{}_a = \frac{1}{2}\hat{R}^5{}_{amn}\, e^m \wedge e^n + \hat{R}^5{}_{a5n}\,\hat{E}^5 \wedge e^n \quad \text{(B.38)}$$

finally to obtain

$$\hat{R}_{5a5m} = -\varphi^{-1}D_m\varphi_a - \frac{1}{4}\varphi^2 F_{aj}F^j{}_m \quad \text{(B.39)}$$

Therefore, thanks to Cartan's structure equations, we have obtained the components of the Riemann curvature tensor easily.

### B.0.1 The Reduced Forms of the Riemann, Ricci Tensors and Ricci Scalar

Let us list for convenience the expressions of the reduced 5D Riemann curvature tensors to the actual 4D spacetime that are obtained previously. They are

$$\hat{R}_{abmn} = R_{abmn} - \frac{1}{4}\varphi^2(2F_{ab}F_{mn} + F_{am}F_{bn} - F_{an}F_{bm}) \quad \text{(B.40)}$$

$$\hat{R}_{ab5m} = \frac{1}{2}\varphi D_m F_{ab} + \frac{1}{2}(2\varphi_m F_{ab} + \varphi_b F_{am} - \varphi_a F_{bm}) \quad \text{(B.41)}$$

$$\hat{R}_{5a5m} = -\varphi^{-1}D_m\varphi_a - \frac{1}{4}\varphi^2 F_{aj}F^j{}_m \quad \text{(B.42)}$$

From the above Riemann tensors we evaluate the Ricci tensors. From

$$\hat{R}_{ab} = \hat{R}^k{}_{akb} + \hat{R}^5{}_{a5b} \quad \text{(B.43)}$$

and using (B.40) and (B.42) we have

$$\hat{R}_{ab} = R_{ab} - \frac{1}{2}\varphi^2 F_{ak}F_b{}^k - \varphi^{-1}D_a\varphi_b \tag{B.44}$$

and from

$$\hat{R}_{5a} = \hat{R}^k{}_{5ka} + \hat{R}^5{}_{55a} \tag{B.45}$$

we have

$$\hat{R}_{5a} = -\frac{1}{2}\varphi D_k F^k{}_a - \frac{3}{2}\varphi^k F_{ka} \tag{B.46}$$

and finally for

$$\hat{R}_{55} = \hat{R}^k{}_{5k5} \tag{B.47}$$

we have

$$\hat{R}_{55} = -\varphi^{-1}D_k\varphi^k - \frac{1}{4}\varphi^2 F^k{}_m F^m{}_k \tag{B.48}$$

where, here and above, we have particularly distinguished the fifth components. Again taking into account the five dimensional Riemann curvature tensors, the 5D Ricci scalar

$$\hat{R} = \hat{g}^{AM}\hat{g}^{BN}\hat{R}_{ABMN} = R + 2\hat{R}_5{}^k{}_{5k} \tag{B.49}$$

is reduced to take the form

$$\hat{R} = R - \frac{1}{4}\varphi^2 F^{ab}F_{ab} - 2\varphi^{-1}D_a\varphi^a \tag{B.50}$$

where $R$ is the Ricci scalar in 4D.

# APPENDIX C

# EINSTEIN-HILBERT ACTION

## C.1 Einstein-Hilbert Action in 4D

Hilbert suggested that the simplest possible choice for a Lagrangian can be constructed from the contractions of the Riemann tensor, that is the Ricci scalar, where it includes only the first and second derivatives of the metric. Referring to formulations given in Appendix A, one can write the action in terms of orthonormal basis 1-forms $e^a$

$$\mathcal{I} = \frac{1}{2} \int R_{ab} \wedge {}^*(e^a \wedge e^b) = \int \mathcal{L}\, d^4x \tag{C.1}$$

The above equation can be written as

$$\begin{aligned} \mathcal{I} &= \tfrac{1}{4} \int R_{abmn}(e^m \wedge e^n) \wedge {}^*(e^a \wedge e^b) \\ &= \tfrac{1}{4} \int R_{abmn}(-g^{ma} g^{nb} + g^{mb} g^{na})^* 1 \end{aligned} \tag{C.2}$$

Then we obtain the Eintein-Hilbert action

$$\mathcal{I} = -\frac{1}{2} \int R^* 1 = \frac{1}{2} \int R \sqrt{-g}\, d^4x \tag{C.3}$$

where in view of (A.10) , we used

$${}^*1 = -\sqrt{-g}\, d^4x \tag{C.4}$$

## C.2 Einstein-Hilbert Action in 5D

In a similar manner as is done in the previous section, the Einstein-Hilbert action in 5D can be expressed in the form

$$\mathcal{I} = \frac{1}{16\pi \hat{G}} \int \hat{R}_{AB} \wedge {}^{\#}(\hat{E}^A \wedge \hat{E}^B) \tag{C.5}$$

where the integrand can be decomposed as

$$\hat{\mathcal{L}} = \hat{R}_{AB} \wedge {}^{\#}(\hat{E}^A \wedge \hat{E}^B) = \hat{R}_{ab} \wedge {}^{\#}(\hat{E}^a \wedge \hat{E}^b) + 2\hat{R}_{a5} \wedge {}^{\#}(\hat{E}^a \wedge \hat{E}^5) \tag{C.6}$$

This can be rewritten in terms of the 5D Riemann tensor

$$\begin{aligned} \hat{R}_{AB} \wedge {}^{\#}(\hat{E}^A \wedge \hat{E}^B) &= \left\{\tfrac{1}{2}\hat{R}_{abmn}(e^m \wedge e^n) \wedge {}^{*}(e^a \wedge e^b) \wedge \hat{E}^5\right\} \\ &\quad + (\hat{R}^5{}_{amn} e^m \wedge e^n + 2\hat{R}^5{}_{a5n}\hat{E}^5 \wedge e^n) \wedge {}^{*}e^a \end{aligned} \tag{C.7}$$

Now we recall that

$$\hat{E}^5 = \varphi(A + dx^5) \tag{C.8}$$

where $A = A_k\, e^k$, is one-form. We use

$$(e^m \wedge e^n) \wedge {}^{*}(e^a \wedge e^b) = (-1)(g^{ma}g^{nb} - g^{mb}g^{na})^{*}1 = (g^{ma}g^{nb} - g^{mb}g^{na})\sqrt{-g}\, d^4x \tag{C.9}$$

and

$$(e^n \wedge {}^{*}e^a) \wedge \hat{E}^5 = (-1)g^{na\,*}1 \wedge \hat{E}^5 = g^{na}\varphi\sqrt{-g}\, d^4x\, dx^5 \tag{C.10}$$

to rewite (C.7) as

$$\hat{R}_{AB} \wedge {}^{\#}(\hat{E}^A \wedge \hat{E}^B) = \left\{\frac{1}{2}\hat{R}_{abmn}(g^{ma}g^{nb} - g^{mb}g^{na}) + 2\hat{R}^5{}_{a5n}g^{na}\right\}\varphi\sqrt{-g}\, d^4x\, dx^5 \tag{C.11}$$

since terms like

$$\begin{aligned} &\hat{R}^a{}_{bm5}\, e^m \wedge \hat{E}^5 \wedge \hat{E}^5 \\ &\hat{R}^5{}_{amn}\, e^m \wedge e^n \wedge {}^{*}e^a \end{aligned} \tag{C.12}$$

vanish.

Now, the contraction of the $\hat{R}_{abmn}$ yields

$$\begin{aligned} &\hat{R}_{abmn}(g^{ma}g^{nb} - g^{mb}g^{na}) \\ &= R_{abmn} - \tfrac{1}{4}\varphi^2(2F_{ab}F_{mn} + F_{am}F_{bn} - F_{an}F_{bm})].(g^{ma}g^{nb} - g^{mb}g^{na}) \\ &= 2R - \tfrac{1}{4}\varphi^2(2F^{mn}F_{mn} - 2F^{nm}F_{mn} - F^n{}_m F^m{}_n - F^m{}_n F^n{}_m) \\ &= 2R - \tfrac{3}{2}\varphi^2 F_{mn}F^{mn} \end{aligned} \tag{C.13}$$

Similarly, we contract $\hat{R}^5{}_{a5n}$ to obtain

$$\hat{R}^5{}_{a5n}g^{na} = -\varphi^{-1}D_m\varphi^m + \frac{1}{4}\varphi^2 F_{kj}F^{kj} \tag{C.14}$$

Collecting the terms, $\hat{\mathcal{L}}$ becomes

$$\hat{\mathcal{L}} = \{\frac{1}{2}(2R - \frac{3}{2}\varphi^2 F_{jk}F^{jk}) + 2(-\varphi^{-1}D_k\varphi^k + \frac{1}{4}\varphi^2 F_{jk}F^{jk})\}\varphi\sqrt{-g}d^4xdx^5 \quad \text{(C.15)}$$

Finally, we rearrange the above equation, we obtain the following result

$$\hat{\mathcal{L}} = \{R - \frac{1}{4}\varphi^2 F_{jk}F^{jk} - 2\varphi^{-1}D_k\varphi^k\}\varphi\sqrt{-g}\, d^4x\, dx^5 \quad \text{(C.16)}$$

to be the used in Chapter 3.

# APPENDIX D

# FURTHER PROPERTIES ON THE METRIC

In this Appendix we give some details about calculations involving the metric tensor, that might be useful for the steps withing the main text. The first subsection deals with the variation of the Christoffel symbols with respect to the metric. The second one gives a general procedure to calculate the determinant of 5D metric, even if it is not diagonal.

## D.1 Variation of the Christoffel Symbol With Respect to the Metric

In this section we show how to obtain the variation of the Christoffel symbol in terms of the metric. We start by the usual expression

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{D.1}$$

and take its variation with respect to metric

$$\delta\Gamma^a_{bc} = \frac{1}{2}\delta g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) + \frac{1}{2} g^{ad}(\partial_b \delta g_{dc} + \partial_c \delta g_{bd} - \partial_d \delta g_{bc}) \tag{D.2}$$

We shall make use of the relation

$$\delta g^{ad} = -g^{ak} g^{dl} \delta g_{kl} \tag{D.3}$$

which can be derived as below

$$\begin{aligned} \delta(g^{ak} g_{kl}) &= \delta(\delta^a_l) = 0 \\ &= \delta g^{ak} g_{kl} + g^{ak} \delta g_{kl} \\ &= \delta g^{ak} g_{kl} g^{ld} + g^{ak} \delta g_{kl} g^{ld} \\ &= \delta g^{ad} + g^{ak} \delta g_{kl} g^{ld} \end{aligned} \tag{D.4}$$

Then (D.2) becomes

$$\begin{aligned}\delta\Gamma^a_{bc} &= -\tfrac{1}{2}g^{ak}g^{dl}\delta g_{kl}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) + \tfrac{1}{2}g^{ad}(\partial_b\delta g_{dc} + \partial_c\delta g_{bd} - \partial_d\delta g_{bc}) \\ &= -g^{ad}\delta g_{dl}\Gamma^l_{bc} + \tfrac{1}{2}g^{ad}(\partial_b\delta g_{dc} + \partial_c\delta g_{bd} - \partial_d\delta g_{bc})\end{aligned} \tag{D.5}$$

that can be recast in the form

$$\delta\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b\delta g_{dc} + \partial_c\delta g_{bd} - \partial_d\delta g_{bc} - 2\delta g_{dl}\Gamma^l_{bc}) \tag{D.6}$$

Now, we consider

$$\partial_b\delta g_{dc} = D_b\delta g_{dc} + \Gamma^l_{db}\delta g_{lc} + \Gamma^l_{cb}\delta g_{dl} \tag{D.7}$$

and other similar terms to be substituted in (D.6). Then all terms with Christoffel symbols cancelled out, and so we have

$$\delta\Gamma^a_{bc} = \frac{1}{2}g^{ad}(D_b\delta g_{dc} + D_c\delta g_{bd} - D_d\delta g_{bc}) \tag{D.8}$$

which, in view of (D.3) can be rewritten as

$$\delta\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left[-D_b\left(g_{dk}g_{cj}\delta g^{kj}\right) - D_c\left(g_{bk}g_{dj}\delta g^{kj}\right) + D_d\left(g_{bk}g_{cj}\delta g^{kj}\right)\right] \tag{D.9}$$

Rearranging the terms and indices, finally we obtain

$$\delta\Gamma^a_{bc} = -\frac{1}{2}(g_{cj}D_b\delta g^{aj} + g_{bj}D_c\delta g^{aj} - g_{bi}g_{cj}D^a\delta g^{ij}) \tag{D.10}$$

as the desired result.

## D.2 Decomposition of the 5D Metric Determinant

In this section, we obtain the decomposition of the 5D metric using Schur complement. We have the five dimensional metric as

$$\hat{g}_{MN} = \left(\begin{array}{c|c} g_{ab} + \varphi^2 A_a A_b & \varphi^2 A_a \\ \hline \varphi^2 A_b & \varphi^2 \end{array}\right) \tag{D.11}$$

and its inverse is

$$\hat{g}^{MN} = \left(\begin{array}{c|c} g^{ab} & -A^a \\ \hline -A^b & \frac{1}{\varphi^2} + A_k A^k \end{array}\right) \equiv \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) = A \tag{D.12}$$

Note that, while $g_{ab} + \varphi^2 A_a A_b$ has no inverse, $A_{11} = g^{ab}$ has an inverse. We also have the property

$$det(\hat{g}_{MN}) = [det(\hat{g}^{MN})]^{-1} \tag{D.13}$$

We can write $detA$ as

$$detA = det(A_{11})\, det\left(\frac{A}{A_{11}}\right) \tag{D.14}$$

where

$$\frac{A}{A_{11}} = A_{22} - A_{21}(A_{11})^{-1}A_{12} \tag{D.15}$$

in terms of the Schur complement [26].

Now, by using (D.14) we can easily obtain $det(\hat{g}^{MN})$ and $det(\hat{g}_{MN})$.
Let us calculate $det(\hat{g}^{MN})$

$$\begin{aligned} det(\hat{g}^{MN}) &= det(g^{ab})det(A_k A_k + \tfrac{1}{\varphi^2} - g_{ab}A^a A^b) \\ &= det(g_{ab})\,\varphi^{-2} \end{aligned} \tag{D.16}$$

Then, $det(\hat{g}_{MN})$ becomes

$$det(\hat{g}_{MN}) = [det(\hat{g}^{MN})]^{-1} = \varphi^2 det(g_{ab}) = \varphi^2 g \tag{D.17}$$

Therefore, we obtain

$$\sqrt{-\hat{g}} = \varphi\sqrt{-g} \tag{D.18}$$