MACKEY DECOMPOSITION FOR BRAUER PAIRS

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Utku OKUR
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Mackey Decomposition for Brauer Pairs
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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For a finite group G and an algebraically closed field k of characteristic p, a k-algebra A with a G-action is called a G-algebra. A pair (P, c) such that P is a p-subgroup of G and c is a block idempotent of the G-algebra $A(P)$ is called a Brauer pair. Brauer pairs form a refinement of the G-poset of p-subgroups of a finite group G. We define the ordinary Mackey category $\mathcal{B}$ of Brauer pairs on an interior p-permutation G-algebra A over an algebraically closed field k of characteristic p. We then show that, given a field $\mathbb{K}$ of characteristic zero and a primitive idempotent $f \in A^G$, then the category algebra of $\mathcal{B}_f$ over $\mathbb{K}$ is semisimple.

Keywords: Brauer Pair, Mackey Decomposition.

ÖZET

BRAUER İKİLİLERİ İÇİN MACKEY AYRIŞMASI

Utku OKUR
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Sonlu bir grup G ve cebirsel kapalı bir cisim k verildiğinde, bir k-cebiri A üzerine G-etkisi tanımlanırsa, A'ye G-cebiri ismi verilir. Verilen G'nin bir p-altgrubu P ve $A(P)$'nin bir bloğu c'den oluşan ikili (P, c)'ye Brauer ikilisi denir. Brauer ikilileri, G'nin p-altgruplarının G-etkisi altındaki kısmi sıralı kümesinin bir inceltmesini oluştururlar. Bu tezde, k cebirsel kapalı karakteristiği p olan bir cisim, A bir p-permütasyon içsel G-cebiri olmak üzere, A üzerine tanımlı Brauer ikililerinin olağan Mackey kategorisi $\mathcal{B}$ tanımlanıyor. Ardından, $\mathbb{K}$ karakteristiği sıfır olan bir cisim ve $f \in A^G$ ilkel bir birim güçlü eleman olmak üzere, $\mathcal{B}_f$'in $\mathbb{K}$ üzerine kategori cebirinin neredeyse basit olduğu kanıtlanıyor.

Anahtar sözcükler: Brauer İkilisi, Mackey Ayrışması.

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Contents

Chapter 1

Introduction

Group representation theory concerns itself with interactions between an abstract group and other mathematical objects, generally vector spaces, the observation of which yields an understanding of the nature of the group itself. In particular, modular representation theory uses vector spaces over fields of characteristic p for a prime number p. We shall explore in this thesis, some objects of modular representation theory, called *Brauer pairs* via a category theoretical approach.

Brauer pairs are defined when a finite group G acts on a ring A that also has a compatible k-vector space structure, where k is an algebraically closed field of characteristic p (c.f. G-algebras in Chapter 3.) The action of G is given by a group homomorphism $G \to \mathrm{Aut}_k(A)$ This map shall denoted $a \mapsto {}^g a$ for $g \in G$ and $a \in A$. The action of G on $A = kG$ by conjugation provides a standard example of a G-algebra.

A special kind of a G-algebra, which possesses some properties that a general G-algebra does not, is an *interior* G-algebra, given by a group homomorphism $G \to A^\times$, where $A^\times$ denotes the invertible elements of A. In Chapter 3, we show how interior G-algebras constitute a special class of G-algebras.

For a finite group G, the group algebra $A = kG$ has the property that G constitutes a finite k-basis of A as a vector space. In general, if a G-algebra A has

a k-basis X, where X is also a G-set, then A is called a *permutation G-algebra*. We shall work with p-permutation G-algebras, defined in Chapter 3.

Understanding the primitive idempotents, defined in Chapter 2, of a ring helps understanding the rest of the idempotents, just like a basis of a vector space determines how the rest of the elements behave. The primitive idempotents of the center $Z(A)$ of a finite dimensional algebra A are called the *blocks* of A. If we write $\mathcal{B}(A)$ for the set of blocks of A, then A always has a direct sum decomposition $A = \bigoplus_{b \in \mathcal{B}(A)} Ab$ into subalgebras Ab and these subalgebras are called the *block algebras* of A. Understanding the block algebras Ab amounts to understanding the structure of the whole A.

The elements of a G-algebra A fixed by a subgroup $H \leq G$ form a subalgebra of A, denoted A^H. The relationship between A^K and A^H for two subgroups H and K such that $K \leq H$ is investigated via the *restriction* maps $r_K^H : A^H \to A^K$ defined by inclusion and the *transfer* maps $t_K^H : A^K \to A^H$ defined by $a \mapsto \sum_{h \in [H/K]} {}^h a$. One of the properties of the transfer maps t_K^H is that the image $t_K^H(A^K)$ is an ideal in A^H. If we quotient out A^H by the ideal $\sum_{K<H} t_K^H(A^K)$, we get the *Brauer quotient* $A(H) = A^H / \sum_{K<H} t_K^H(A^K)$, which is the final ingredient in the definition of a Brauer pair: For a finite group G and an G-algebra A over a field, an ordered pair (P, c) such that P is a p-subgroup of G and c is a block idempotent of $A(P)$ is called a *Brauer pair*. Investigating the structure of Brauer pairs gives a sense of their importance: In Chapter 3, we define the inclusion relation between Brauer pairs and a G-action that preserves this inclusion relation, so that Brauer pairs form a refinement of the G-poset of p-subgroups of a finite group G.

Chapter 3 is dedicated to some structural theorems concerning Brauer pairs, following the approach of [Cra83] and [AKO11]. In particular, each Brauer pair is associated to a primitive idempotent $f \in A^G$ (cf. Definition 3.23.) and called an f-Brauer pair. The last result of this chapter is that the maximal f-Brauer pairs are related to each other via the conjugation action of G.

A representation of a group G over a ring R is defined to be a group homomorphism $G \to \mathrm{End}_R(M)$ where M is an R-module. In Chapter 4, we define a *representation of a category $\mathcal{C}$ over a ring R* as a functor $F : \mathcal{C} \to {}_R\mathfrak{M}$, where ${}_R\mathfrak{M}$

denotes the category of left R-modules. Substituting the category corresponding to a group, i.e. the category with one object and maps as the elements of a given group, one can see that the definition of a representation of a category generalizes that of a representation of a group. Representations of a category $\mathcal{C}$ over a ring R are in one-to-one correspondence with modules over *category algebras* $R\mathcal{C}$ (cf. Chapter 4 for category algebras.)

In Chapter 5, we give a short summary of the biset category of finite groups, following [Bou90].

We generalize this category to obtain the biset category of Brauer pairs in Chapter 6. An important subcategory of the biset category is the ordinary Mackey biset category, defined in [Bar16], where it is proven that the category algebra of the ordinary Mackey biset category over a field of characteristic 0 is semisimple. We define the ordinary Macket biset category $\mathcal{B}$ of Brauer pairs on an interior p-permutation G-algebra A. In our last result, Theorem 6.4, we consider the full subcategory $\mathcal{B}_f$ of $\mathcal{B}$ with objects restricted to the Brauer pairs associated to a fixed primitive idempotent $f \in A^G$, and we show that the category algebra of $\mathcal{B}_f$ over a field $\mathbb{K}$ of characteristic 0 is semisimple.

Chapter 2

Preliminaries

In the next chapters, we shall frequently deal with rings, modules over rings and algebras over fields. We now cover some necessary facts, propositions and develop the tools that shall become useful later, including results on semisimplicity, the Jacobson radical, primitive idempotents and Morita equivalence.

By an *algebra A over a field* $\mathbb{F}$, we mean that A is both a ring in itself and an $\mathbb{F}$-vector space such that the action of $\mathbb{F}$ is compatible with the multiplication in A: $(\lambda \cdot a)b = a(\lambda \cdot b) = \lambda \cdot (ab)$ for each $\lambda \in \mathbb{F}$ and $a, b \in A$. More generally, given a commutative ring R, an *algebra A over R* is a ring A and a ring homomorphism $\phi : R \to A$ such that the image of this homomorphism lies in the center of A. This definition is designed to allow the *coefficients* coming from R to be able to commute with the multiplication operation of A. Putting $\lambda \cdot \mathrm{a} \coloneqq \phi(\lambda)\mathrm{a}$, one can see that the former definition of an algebra over a field is a special case of the latter.

In what follows, we assume that R is a ring with identity 1_R. Given a ring R, an *R-module* always means a left R-module. If we regard R as an R-module, we use the notation ${}_R R$. Submodules of ${}_R R$ are said to be *left ideals* of R.

Given two rings R and S, a set M is an (R, S)-*bimodule* if M is both a left R-module and a right S-module such that their actions are compatible:

$r(ms) = (rm)s$ for all $r \in R$, $s \in S$ and $m \in M$. Any ring R is canonically an (R, R)-bimodule and we use the notation ${}_RR_R$ if we regard R as an (R, R)-bimodule. Submodules of ${}_RR_R$ are said to be *two-sided ideals* of R. In what follows, an *ideal* shall always mean a two-sided ideal.

A nonzero R-module is said to be *simple* if it has no proper nonzero submodule. Similarly, a nonzero ring R is said to be *simple* if it has no proper nonzero ideal. Simple R-modules and simple rings play the role of building blocks for other R-modules and rings.

The proofs of the following two lemmas are routine and we skip them.

Lemma 2.1. *Given a ring R, the following hold:*
1. For an R-module N and a submodule M of N, the quotient module N/M is simple if and only if M is a maximal submodule of N.
2. For a left ideal I of R, the R-module R/I is a simple if and only if I is a maximal left ideal.
3. For an ideal I of R, the ring R/I is simple if and only if I is a maximal ideal.

Lemma 2.2 (Modular Law)**.** *Given an R-module M and R-submodules U, V and W, if $W \subseteq U$, then the identity $(U \cap V) + W = U \cap (V + W)$ holds.*

We shall work with direct sums of R-modules with finitely many summands, which we now define:

Definition 2.3. If $M = V + W$ is a sum of R-modules, then M is defined to be a *direct sum* of V and W provided $V \cap W = \{0\}$. In this situation, we write $M = V \bigoplus W$ and say that V and W are *direct summands* of M. If M has no non-zero proper direct summand, it is said to be an *indecomposable* module.

Corollary 2.4. *If $M = V \bigoplus W$ is a direct sum decomposition of R-modules and $W \subseteq U \subseteq M$ is a chain of submodules, then the identity $U = (U \cap V) \bigoplus W$ holds.*

Proof. We have, $U = U \cap M = U \cap (V \bigoplus W) = (U \cap V) + W$, where the last identity is the modular law. By the hypothesis, the intersection $U \cap V \cap W$ of the summands of the last sum is equal to $V \cap W$, which is zero, as $M = V \bigoplus W$ is a direct sum. $\square$

Definition 2.5. If $M = M_1 + \ldots + M_n$ is a sum of R-modules, then M is said to be a *direct sum* of $\{M_i\}$ provided $M_i \cap (M_1 + \ldots + M_{i-1} + M_{i+1} + \ldots + M_n) = 0$ for each i. In this situation, we write $M = M_1 \oplus \ldots \oplus M_n$.

An R-module M that is isomorphic to a direct sum of finitely many simple modules is called *semisimple*. In the literature, a direct sum of infinitely many simple modules is also called semisimple and many of the facts we show below can be adapted to such modules, but as we shall not need that generality, our restrictive definition shall help avoid some repetition.

For an R-module M, a series of inclusions $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$ such that M_{i-1} is maximal in M_i is called a *finite composition series* for M. The simple quotients M_i/M_{i-1} are called *composition factors* of M.

For an R-module M, having a finite composition series is equivalent to being both Artinian and Noetherian: See [Isa94, p. 145] for the definition of Artinian, Noetherian and see [Isa94, Theorem 11.3] for the proof of this proposition. Moreover, if a finite composition series of an R-module exists, then it is unique up to a permutation of the composition factors by the Jordan-Hölder Theorem [Isa94, Theorem 10.5]. This theorem says that if there are two composition series $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$ and $0 = M'_0 \subsetneq M'_1 \subsetneq \ldots \subsetneq M'_m = M$ of M, then $n = m$ and the composition factors of the first series occur with the same number of isomorphic copies in the second series. In this case, the number n is called the *composition length* of M.

A sum of finitely many simple modules is always a direct sum, as the next lemma shows. Before we start the proof, we make a note that if $M = U + V$ such that $U = U_1 \oplus \ldots \oplus U_n$ and $U \cap V = 0$, then we can write $M = U_1 \oplus \ldots \oplus U_n \oplus V$.

Lemma 2.6. *For an R-module M, the following are equivalent:*
i) M is a direct sum of finitely many simple modules (M is semisimple).
ii) M is a sum of finitely many simple modules.
iii) M has a finite composition series and every submodule of M is a direct summand of M.

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Let $M = \sum_{i\in I} S_i$, for $|I| < \infty$, be a finite sum of simple modules. Let N be any submodule of M and consider the collection of subsets J of I such that $W := N \bigoplus (\bigoplus_{j\in J} S_j)$ is a direct sum. This finite collection is non-empty, as $\varnothing$ lies in it. Choose a maximal element J, i.e. J is maximal with respect to inclusion such that $W = N \bigoplus (\bigoplus_{j\in J} S_j)$ is a direct sum. The proof is complete if we can show that $M = W$. Suppose, for a contradiction, that $W \subsetneq M$. This implies that there is some S_i, for $i \in I - J$, such that $S_i \nsubseteq W$. Then, the chain of inclusions $0 \subseteq S_i \cap W \subsetneq S_i$ implies that $S_i \cap W = 0$, by the simplicity of S_i. But then, by the note above, $W \bigoplus S_i = N \bigoplus \left(\bigoplus_{j\in J\cup\{i\}} S_j \right)$ a direct sum, contradicting the maximality property of J. Therefore, we conclude that for each submodule N of M, there is some $J \subseteq I$ such that $M = N \bigoplus_{j\in J} S_j$. Putting $N = 0$, we see that $M = S_1 \bigoplus \ldots \bigoplus S_n$ is semisimple. To define a composition series for M, put $M_0 = 0$ and $M_i = S_1 \bigoplus \ldots \bigoplus S_i$ for $i = 1, \ldots, n$. Then, by the second isomorphism theorem, the quotient M_i/M_{i-1} is isomorphic to the simple module S_i, so M has a finite composition series of length n.

(iii) $\Rightarrow$ (i): We proceed by induction on the length of a composition series of M, which is a well-defined number by the Jordan-Hölder Theorem. Our claim is that for all $n \geq 1$, if M has composition length n such that every submodule of M is a direct summand, then M is semisimple. Basis step: If $n = 1$, then M is itself simple and we are done. Inductive hypothesis: Suppose that all modules of composition length $n - 1$ are semisimple. Inductive step: Let M be any module with composition length n. This means, M has a composition series $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$. By the hypothesis, the submodule M_{n-1} is a direct summand of M, in other words, $M = M_{n-1} \bigoplus W$ for some submodule W. By the inductive hypothesis, M_{n-1} is semisimple. Then, using the second isomorphism theorem, $W \cong W/0 = W/(M_{n-1} \cap W) \cong (M_{n-1} + W)/M_{n-1} = M/M_{n-1}$ is a simple composition factor, i.e. W is simple and $M = M_{n-1} \bigoplus W$ is semisimple. $\square$

Corollary 2.7. *Submodules and quotient modules of a semisimple R-module are also semisimple R-modules.*

Proof. Let M be a semisimple R-module. First, let us show the statement for the quotient modules of M. Let N be a submodule of M and consider the canonical

map, $f : M \to M/N$. For a simple submodule S of M, using the first isomorphism Theorem, $f(S) \cong S/\ker(f|_S)$ is either simple or zero, hence $M/N = f(M) = f(S_1 \bigoplus \ldots \bigoplus S_n) = f(S_1) + \ldots + f(S_n)$ is a sum of finitely many simple or zero modules and we are done by the previous lemma. Now let N be a submodule of M. By the previous lemma, $M = N \bigoplus N_1$ for some submodule N_1. Then, by the second isomorphism theorem, $N \cong N/0 = N/(N \cap N_1) \cong (N \bigoplus N_1)/N = M/N_1$ and by the first part of the proof, N is semisimple. □

An ideal I of a product of finitely many rings $\prod_{i=1}^{n} R_i$ must be in the form $\prod_{i=1}^{n} I_i$ for some ideals I_i of R_i: To prove this, we start with the basis step, $n = 2$. Let I be an ideal of $R \times S$ for rings R and S. Let $J := \{r \in R : (r, 0) \in I\}$ and let $K \subseteq S$ be defined symmetrically. We shall show that J and K are ideals of R and S, respectively, and that $I = J \times K$. To prove that J is an ideal in R, let $x, y \in J$ and let $(r, s) \in R \times S$. Then, by the definition of J, we know that $(x, 0), (y, 0) \in I$. Then, as I is an ideal, $(x, 0) + (y, 0) = (x + y, 0) \in I$, in particular, $x + y \in J$. Also, $(r, s).(x, 0) = (rx, 0) \in I$, in particular, $rx \in J$. The subset K is similarly an ideal of S. For the equality in question, first take an element $(j, k) \in J \times K$. Then, we have $(j, 0), (0, k) \in I$, from which it follows that $(j, k) = (j, 0) + (0, k) \in I$, as I is an ideal. Conversely, let $(x, y) \in I$. Then, $(1, 0).(x, y) = (x, 0) \in I$, from which, it follows that $x \in J$. Similarly, $y \in K$ and we get $I = J \times K$. In the inductive step, let I be an ideal of $R_1 \times \ldots \times R_n$. By the case $n = 2$ applied to the rings $R_1 \times \ldots \times R_{n-1}$ and R_n, the ideal I is in the form $J \times I_n$ where, J is an ideal of $R_1 \times \ldots \times R_{n-1}$ and I_n is an ideal of R_n. Then, by the inductive hypothesis, J is in the form $I_1 \times \ldots \times I_{n-1}$ for some ideals I_i of R_i, so that $I = J \times I_n = (I_1 \times \ldots \times I_{n-1}) \times I_n$. One consequence of this fact is that if R is a direct product $\prod_i R_i$ of finitely many simple rings, then, $R/I = (\prod_i R_i)/(\prod_i I_i) \cong \prod_i R_i/I_i$, where the last isomorphism is given by $(r_1, \ldots, r_n) + (\prod_i I_i) \mapsto (r_1 + I_1, \ldots, r_n + I_n)$, is a direct product of simple rings or zero rings (by definition, a zero ring is not simple), which, in turn, is isomorphic to a direct product of simple rings. Hence, as a counterpart of Corollary 2.7, we note the following: A quotient of a direct product of simple rings is also a direct product of simple rings. However, subrings of simple rings might not be simple: $\mathbb{Z}$ is a subring of the field $\mathbb{C}$, and it is evidently not simple.

Definition 2.8. A ring R is said to be *semisimple* if the R-module ${}_R R$ is semisimple.

Quite a bit is known on the structure of semisimple rings. Firstly, consider the case that ${}_RR$ is a simple R-module. Then, it follows easily that R is simple as a ring, but the converse need not hold, a counter-example being $R = M_2(\mathbb{R})$: By Lemma 2.32 below, $M_2(\mathbb{R})$ is a simple ring. However, the matrices with all entries zero except for a single column form a nonzero and proper left ideal of $M_2(\mathbb{R})$.

More generally, if ${}_RR$ semisimple, then R is a direct product of simple rings: The Artin-Wedderburn Theorem states that a semisimple ring R is isomorphic to a direct product of matrix rings over some suitable division rings, the dimensions of which are the multiplicities of the simple R-modules in the direct sum decomposition of ${}_RR$ [Lam91, Theorem 3.5]. We shall not prove this theorem in full but use a version of it for finite dimensional algebras.

For an R-module M, the elements $r \in R$ such that $rm = 0$ for all $m \in M$ is said to *annihilate* M and the set of annihilators of M is denoted $\operatorname{ann}_R(M)$. This set is easily seen to be an ideal of R.

If $f : R \to S$ is a ring epimorphism and M is an S-module, then a module action of R on M can always be defined via f, such that a subset N of M is an S-submodule if and only if N is an R-submodule. Hence, submodule-related properties of the S-module ${}_SM$ are carried over to the R-module ${}_RM$, including the semisimplicity of ${}_SM$ as an S-module. On the other hand, if $f : R \to S$ is a ring epimorphism and M is an R-module, a well-defined action of S on M can be defined via f provided $\ker(f) \subseteq \operatorname{ann}_R(M)$: Given $s \in S$ and $m \in M$, we define $s \cdot m := rm$ where $r \in R$ is any element such that $f(r) = s$. Again, in this case, a subset N of M is an S-submodule if and only if N is an R-submodule. Hence, submodule-related properties of the R-module ${}_RM$ are carried over to the S-module ${}_SM$.

Corollary 2.9. *Given a ring R, an ideal I of R and an R-module M such that $I \subseteq \operatorname{ann}_R(M)$, then ${}_RM$ is semisimple if and only if ${}_{R/I}M$ semisimple. In particular, if R is a semisimple ring, then the ring R/I is also semisimple.*

Proof. We apply the above remark to the epimorphism $f : R \to R/I$ and the first sentence is proven.

For the second sentence, assume that R is semisimple. Then, R/I is a quotient of R as an R-module, so that by Corollary 2.7, we have that R/I is semisimple as an R-module. Now, as we have $\ker(f) = I = \text{ann}_R(R/I)$, we can apply the first part to obtain that R/I is also semisimple as an R/I-module, in other words, R/I is a semisimple ring. □

Lemma 2.10. *Given a ring R with identity, ${}_RR$ is semisimple if and only if all finitely generated R-modules are semisimple.*

Proof.
($\Leftarrow$): As ${}_RR$ is generated by 1_R, the hypothesis applies to ${}_RR$ itself.
($\Rightarrow$): Let M be an R-module with generators, $m_1, \ldots, m_n$. Then the mapping $f : R \times \ldots \times R \to M$ given by $(r_1, \ldots, r_n) \mapsto (r_1 m_1, \ldots, r_n m_n)$ is an R-module epimorphism, and so $M \cong R/\ker(f)$ is a semisimple R-module, by Corollary 2.7. □

The intersection of maximal left ideals of a ring R is called the *Jacobson radical* of R and it shall be denoted $J(R)$. The Jacobson radical measures, in a sense, how close a ring is to being semisimple, as we shall soon explore.

Let us collect some of the properties of $J(R)$. The first property in our list is that the Jacobson radical has an alternative expression in terms of annihilators.

Lemma 2.11. *Given a ring R, the Jacobson radical $J(R)$ is equal to the intersection of the annihilators of simple R-modules.*

Proof. For a simple R-module S and for nonzero $s \in S$, the function $f_s : R \to S$ defined by $r \mapsto rs$ is a surjective R-module homomorphism. So, $R/\ker(f_s)$ is isomorphic to the simple module S, hence $\ker(f_s)$ must be a maximal left ideal of R by Lemma 2.1. Then, $J(R)$ is contained in $\ker(f_s)$. Hence, $J(R) \subseteq \bigcap_{s \in S} ker(f_s) = \text{ann}_R(S)$. Since this is true for all simple S, it follows that $J(R) \subseteq \bigcap_{S \text{ simple}} \text{ann}_R(S)$.

Conversely, let $\mathcal{L}$ be a maximal left ideal of R. By Lemma 2.1, $R/\mathcal{L}$ is a simple R-module, therefore, $\bigcap_{S \text{ simple}} \text{ann}_R(S) \subseteq \text{ann}_R(R/\mathcal{L}) = \mathcal{L}$. Since this is true for all

maximal left ideals $\mathcal{L}$ of R, it follows that $\bigcap_{S \text{ simple}} \operatorname{ann}_R(S) \subseteq \bigcap_{\mathcal{L} \text{ max left ideal}} \mathcal{L} = J(R)$. □

Note that this characterization of the Jacobson radical shows that it is a two-sided ideal.

Lemma 2.12. *For a unital ring R, and an element $y \in R$, the following are equivalent:*
i) $y \in J(R)$
ii) For all $x \in R$, the element $1 - xy$ has a left inverse.

Proof. (i) ⇒ (ii): If $1-xy$ has no left inverse, then the principal left ideal $R(1-xy)$ is a proper ideal. So, $R(1-xy)$ is contained in a maximal left ideal $\mathcal{L}$ by Zorn's Lemma. But then, $y \in J(R) \subseteq \mathcal{L}$ implies that $xy \in \mathcal{L}$ as well, which implies that $1 = 1 - xy + xy \in \mathcal{L}$, a contradiction.

(ii) ⇒ (i) Given a maximal left ideal $\mathcal{L}$ of R, then suppose, for a contradiction, that Ry is not contained in $\mathcal{L}$. Then, the inclusions $\mathcal{L} \subsetneq Ry + \mathcal{L} \subseteq R$ show that $Ry + \mathcal{L} = R$, which means $1_R \in Ry + \mathcal{L}$, i.e. there are some $x \in R$ and $z \in \mathcal{L}$ such that $1_R = xy + z$. By hypothesis, $1 - xy = z$ has a left inverse u, so that $1_R = u(1_R - xy) = uz \in \mathcal{L}$, a contradiction. Therefore, Ry is contained in all maximal left ideals in R. In particular, Ry is contained in the Jacobson radical. □

In a ring R, the elements r of finite multiplicative order are called *nilpotent*. An ideal I consisting of only nilpotent elements is said to be *nil*. If an ideal I satisfies $I^n = 0$ for a finite $n \in \mathbb{N}$, then I is said to be *nilpotent*. Note that a nilpotent ideal is necessarily nil, but the converse may not hold. As a counterexample, we put $R = \mathbb{Z}[x_1, x_2, x_3 \ldots]/\langle x_1^2, x_2^3, x_3^4, \ldots \rangle$, where x_i are infinitely many commuting indeterminates and $I = \langle \overline{x_1}, \overline{x_2}, \overline{x_3}, \ldots \rangle$ [Lam91, p. 56]. We observe that the generators $\overline{x_i}$ of I have finite order, so that any element of the ideal I of the commutative ring R is of finite order. On the other hand, if I itself had finite order, i.e. $I^n = 0$, then we would have $x_n^n \in \langle x_1^2, x_2^3, x_3^4, \ldots, x_n^{n+1}, \ldots \rangle$, a contradiction. Therefore, I is a nil ideal that is not nilpotent.

Now, we show that the Jacobson radical contains all nil ideals of R.

Lemma 2.13. *For a unital ring R, its Jacobson radical $J(R)$ contains all nil ideals.*

Proof. Let I be a nil ideal. Given $y \in I$, for each $x \in R$, the product $xy \in I$ is nilpotent, say with order n. Then, a straightforward computation shows that $\sum_{i=0}^{n-1}(xy)^i$ is a left inverse of $1 - xy$. By the previous lemma, $y \in J(R)$. □

Before the next lemma, note that for a finite dimensional algebra A, it is easy to find a maximal left ideal of A: Let $\mathcal{L}$ be a proper left ideal such that $\dim_k(\mathcal{L})$ is maximal. By repeating this process until we get a simple A-module or the zero module, we obtain a composition series for ${}_AA$. Now, we show that the Jacobson radical is the largest nilpotent ideal, in the following sense:

Lemma 2.14. *For a finite dimensional algebra A, the Jacobson radical $J(A)$ is a nilpotent ideal of A and it contains all nilpotent ideals of A.*

Proof. Picking a composition series $0 = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_n = {}_AA$, we have that $J(A).(A_i/A_{i-1}) = 0$ by Lemma 2.11, hence $J(A).A_i \subseteq A_{i-1}$. By induction, $J(A)^r.A \subseteq A_{n-r}$ for $r = 1, \ldots, n$. In particular, $J(A)^n = J(A)^n.A \subseteq A_0 = 0$, so $J(A)$ is indeed nilpotent. Now, let I be a nilpotent ideal of A, say with $I^m = 0$. Let S be any simple A-module and suppose, for a contradiction, that the submodule IS of S is non-zero. Then, by the simplicity of S, we have the equality $IS = S$. Then, by an inductive argument, $0 = 0.S = I^mS = S$, a contradiction. Hence, IS is zero for any simple R-module S. By Lemma 2.11, I is contained in the Jacobson radical $J(A)$. □

One consequence of this lemma is that, in contrast to general rings, the distinction between nilpotent and nil ideals is not needed for finite dimensional algebras: An ideal I of a finite dimensional algebra A is nil if and only if it is nilpotent.

A ring R is said to be *semiprimitive* or *J-semisimple* if $J(R) = 0$. It is proven in [Lam91, Theorem 4.14] that R is semisimple if and only if it is Artinian and $J(R) = 0$. Instead of this proposition, we shall limit ourselves to finite dimensional algebras and show that $\overline{A} = A/J(A)$ is semisimple.

The Jacobson radical of a finite dimensional algebra can be expressed as an intersection of finitely many maximal left ideals. To show this, we define a descending sequence of left ideals $\{I_i\}$ that must eventually stabilize or stop: Pick a maximal left ideal $\mathcal{L}_0$ of R and put $I_0 \coloneqq \mathcal{L}_0$. Then, put $I_{i+1} = I_i \cap \mathcal{L}$ if there is a maximal left ideal $\mathcal{L}$ that has not appeared in previous steps. If there is no such $\mathcal{L}$, stop. By the hypothesis of finite dimensions, this chain eventually stabilizes or stops, say at the nth step. By the construction of $\{I_i\}$, we have $J(R) = I_n$, which is an intersection of finitely many maximal left ideals.

Lemma 2.15. *For a finite dimensional algebra A, its Jacobson radical $J(A)$ is the smallest left ideal ℓ of A such that A/ℓ is a semisimple A-module.*

Proof. Writing $J(A) = \mathcal{L}_1 \cap \ldots \cap \mathcal{L}_n$ as an intersection of finitely many maximal left ideals $\mathcal{L}_i$, consider the A-module homomorphism $f : A \to A/\mathcal{L}_1 \times \ldots \times A/\mathcal{L}_n$ given by $a \mapsto (a + \mathcal{L}_1, \ldots, a + \mathcal{L}_n)$. The kernel of f is $J(A)$, so we have $A/J(A) = A/\ker(f) \cong f(A) \subseteq A/\mathcal{L}_1 \times \ldots \times A/\mathcal{L}_n$. We have successfully embedded $A/J(A)$ into a semisimple A-module. Then, by Corollary 2.7, $A/J(A)$ is a semisimple A-module itself. Now, let A/ℓ be a semisimple A-module, i.e., let A/ℓ be isomorphic to a direct sum of finitely many simple A-modules. Then, as the Jacobson radical $J(A)$ annihilates simple A-modules, $J(A)$ annihilates the whole A/ℓ, in particular, we obtain $J(A) \subseteq \ell$. □

Corollary 2.16. *A is semisimple if and only if $J(A) = 0$.*

Lemma 2.17. *Given a unital ring R, the radical $J(R)$ is contained in the intersection of maximal ideals of R.*

Proof. Let $\mathfrak{m}$ be a maximal ideal. We shall show $\mathfrak{m}$ to be the annihilator of a simple R-module. By Zorn's Lemma, $\mathfrak{m}$ is contained in a maximal left ideal $\mathcal{L}$, i.e. there is a chain of left ideals, $\mathfrak{m} \subseteq \mathcal{L} \subsetneq R$. By Lemma 2.1, $R/\mathcal{L}$ is a simple module and $\mathfrak{m} \subseteq \operatorname{ann}_R(R/\mathcal{L})$ because for any $m \in \mathfrak{m}$ and $r + \mathcal{L}$, we have that $m(r + \mathcal{L}) = mr + \mathcal{L} = \mathcal{L}$ since $mr \in \mathfrak{m} \subseteq \mathcal{L}$. Therefore, $\mathfrak{m} \subseteq \operatorname{ann}_R(R/\mathcal{L}) \subsetneq R$, where the last inequality holds because $1_R \notin \operatorname{ann}_R(R/\mathcal{L})$. Now, by the maximality of $\mathfrak{m}$, we obtain $\mathfrak{m} = \operatorname{ann}_R(R/\mathcal{L})$. Therefore, $J(R) \subseteq \operatorname{ann}_R(R/\mathcal{L}) = \mathfrak{m}$ by Lemma 2.11. Since $\mathfrak{m}$ was arbitrarily selected, the claim follows. □

Two ideals I and J of a ring R are said to be *comaximal* or *coprime* if $I + J = R$ holds.

Lemma 2.18 (Chinese Remainder Theorem for Non-Commutative Rings)**.** *Given pairwise comaximal ideals $\{I_i\}_{i=1}^n$ of a ring R, we have a ring isomorphism:*

$$R/(I_1 \cap \ldots \cap I_n) \cong R/I_1 \times \ldots \times R/I_n$$

Proof. To apply induction, let us cover the case $n = 2$: Define a ring homomorphism $f : R \to R/I \times R/J$ by $r \mapsto (r + I, r + J)$. To show that f is surjective, pick an element $(r + I, t + J)$ of $R/I \times R/J$. By the comaximality of I and J, there are some $i \in I$ and $j \in J$ such that $1_R = i + j$. Then,

$$\begin{aligned} f(rj + ti) &= (rj + ti + I, rj + ti + J) \\ &= (rj + I, ti + J) \\ &= (r(1 - i) + I, t(1 - j) + I) \\ &= (r - ri + I, t - tj + J) \\ &= (r + I, t + J) \end{aligned}$$

showing that f is surjective. Furthermore, the kernel of f is clearly $I \cap J$, therefore, by the first isomorphism theorem, $R/(I \cap J) \cong R/I \times R/J$.

For the inductive step, assume that the claim holds for $k - 1$. To show that I_1 and $I_2 \cap \ldots \cap I_k$ are comaximal, we use the hypothesis that I_1 and I_m are comaximal for each $m = 2, \ldots, k$ and write $1_R = x_m + y_m$ such that $x_m \in I_1$ and $y_m \in I_m$. This implies that $1_R = (x_2 + y_2)(x_3 + y_3) \ldots (x_k + y_k) \in I_1 + (I_2 \cap \ldots \cap I_k)$ because for each term in the expression on the right-hand side, if any x_m appears in that term, then the whole term falls inside I_1 and if none of $x_2, \ldots, x_k$ appears in it, then that term could only be $y_2 \ldots y_k$, which is in $I_2 \ldots I_k$. Now, using the case $n = 2$,

$$\begin{aligned} R/(I_1 \cap \ldots \cap I_k) &= R/(I_1 \cap (I_2 \cap \ldots \cap \cap I_k)) \\ &\cong R/I_1 \times R/(I_2 \cap \ldots \cap I_k) && \text{by the case } n = 2 \\ &\cong R/I_1 \times \ldots \times R/I_k && \text{by the inductive hypothesis} \end{aligned}$$

□

Lemma 2.19. *Given a finite dimensional algebra A, its Jacobson radical $J(A)$ is the smallest ideal I of A such that A/I is a product of simple algebras.*

Proof. By an argument similar to the one above for left ideals, we write $J(A)$ as an intersection of finitely many maximal ideals. Then, noting that maximal ideals are always coprime, $A/J(A) = A/(\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n) \cong A/\mathfrak{m}_1 \times \ldots \times A/\mathfrak{m}_n$ by the previous lemma, where the latter expression is a product of simple algebras. Therefore, $A/J(A)$ is indeed a product of simple algebras.

Given an ideal I such that A/I is a product of simple rings, let $A/I \cong \prod_i T_i$ for simple algebras T_i. Then we have a chain of algebra homomorphisms, $A \xrightarrow{f} A/I \xrightarrow{\pi_i} T_i$, where f is the canonical projection $r \mapsto r + I$ and $\{\pi_i\}$ are componentwise projections into the ith coordinate T_i. Then, for each i, $A/\ker(\pi_i \circ f) \cong T_i$ being simple implies that $\ker(\pi_i \circ f)$ is a maximal ideal, and so, $J(A) \subseteq \bigcap_{\mathfrak{m} \text{ max ideal}} \mathfrak{m} \subseteq \bigcap_i \ker(\pi_i \circ f) = I$. □

Lemma 2.20. *Given a finite dimensional algebra A and an ideal I of A, we have the identity, $(J(A) + I)/I = J(A/I)$.*

Proof. $(J(A) + I)/I = \{x + I : x \in J(A)\}$ is a nil ideal of A/I and hence it is contained in $J(A/I)$, by Lemma 2.13.

Conversely, we use the third isomorphism theorem for rings,

$$(A/I)/((JA + I)/I) \cong A/(JA + I) \cong (A/JA)/((JA + I)/JA)$$

where the last term, $(A/JA)/((JA + I)/JA)$ is a semisimple ring by Corollary 2.9. By the ring isomorphism, the first term, namely $(A/JA)/((JA + I)/JA)$ is a semisimple ring as well. Again, using Corollary 2.9, we infer that $(A/JA)/((JA + I)/JA)$ is a semisimple (A/JA)-module. Therefore, by Lemma 2.15, we obtain that $J(A/I) \subseteq (J(A) + I)/I$. □

Corollary 2.21. *Given a finite dimensional algebra A, the Jacobson radical of A is equal to the intersection of maximal ideals.*

Proof. Assume that A is finite dimensional algebra and define $I := \bigcap_{\mathfrak{m} \text{ max ideal}} \mathfrak{m}$,

which contains the Jacobson radical by Lemma 2.17. By the third isomorphism theorem for rings, $A/I \cong (A/JA)/(I/JA)$ is a quotient of a semisimple algebra. Hence, by Corollary 2.9, A/I is itself a semisimple algebra. Then, by Corollary 2.16 and by the previous lemma, we get $J(A)/I = J(A) + I/I = J(A/I) = 0$, i.e. $J(A) = I$. □

A ring with a unique maximal left ideal is called a *local* ring. By definition, the unique maximal left ideal of a local ring is equal to its Jacobson radical.

Lemma 2.22. *Given a unital ring R, the following are equivalent:*
1. R has a unique maximal left ideal.
2. R has a unique maximal right ideal.
3. $R/J(R)$ is a division ring.
4. The set of non-units of R form an ideal.
If (1)-(4) hold, then $J(R)$ is the unique maximal ideal of R. The converse is true for commutative rings.

Proof. (4)⇒(1): Let X denote the set of non-units of R, which is not the whole ring since $1_R \in R - X$. By the hypothesis, X is an ideal. For any maximal left ideal $\mathcal{L}$ of R, we have that $\mathcal{L} \subseteq X$, because a proper left ideal can not contain any units. By the maximality of $\mathcal{L}$, we have $\mathcal{L} = X$. Therefore, X is the unique maximal left ideal of R.

(1)⇒(4): By hypothesis, the ideal $J(R)$ is the unique maximal left ideal of R. To show that $X \subseteq J(R)$, we shall use a contrapositive argument: For $x \notin J(R)$, the left ideal Rx can not be proper, because then it would have to be contained in a maximal left ideal, but this maximal left ideal can not be the *unique* maximal left ideal $J(R)$ of R. Therefore, we know that $Rx = R$, in particular there is some y such that $yx = 1$. Then, y is also not contained in the *proper ideal* $J(R)$, which means, by the same argument, that $zy = 1$ for some z. Therefore, $z = z1 = zyx = 1x = x$, and y is invertible and not an element of X. Hence, we obtain $X \subseteq J(R)$. Conversely, as $J(R)$ can not contain any units, it is included in X, therefore, $X = J(R)$ is an ideal.

(1)⇒(3): The unique maximal left ideal of R equals $J(R)$, and so, the non-zero

elements of $R/J(R)$ are invertible by the equivalence of (1) and (4).

(3)⇒(1): A maximal left ideal $\mathcal{L}$ of R contains the Jacobson radical $J(R)$. Then, $\mathcal{L}/J(R)$ is a proper left ideal in a division ring, and so it must be zero, in other words, $\mathcal{L} = J(R)$.

By the left-right symmetric nature of (3), the equivalence (1)⇔(3) implies the equivalence (2)⇔(3).

If (1)-(4) hold, then we have a unique maximal left ideal $\mathcal{L}$. By the definition of the Jacobson radical, $\mathcal{L} = J(R)$ is an ideal. Then $\mathcal{L}$ is also the unique maximal ideal of R because in any chain of ideals, $\mathcal{L} \subseteq \mathfrak{m} \subsetneq R$, the ideal $\mathfrak{m}$ is in particular a left ideal, so, by the maximality property of $\mathcal{L}$, equality follows. □

For commutative rings, having a unique maximal ideal is also used for defining local rings. In our case, a local ring is any ring that satisfies one of the equivalent properties (1)-(4).

Lemma 2.23. *If R is a local ring, then R has a unique simple module S up to isomorphism.*

Proof. Let R be a local ring with two simple modules S_1 and S_2. Let S_i be generated by nonzero elements $s_i \in S_i$, meaning that $S_i = Rs_i$. Then, we have R-module epimorphisms, $f_i : R \to S_i$ given by $r \mapsto rs_i$. Note that $\ker(f_i)$ is a left ideal in R for each i and the isomorphism $R/\ker(f_i) \cong S_i$ shows that $\ker(f_i)$ is a maximal left ideal. As R has a unique maximal left ideal, we have, $S_1 \cong R/\ker(f_1) = R/\ker(f_2) \cong S_2$. □

Simple rings are local, so we have the following application:

Lemma 2.24. *For a field $\mathbb{F}$, the ring $M_n(\mathbb{F})$ is simple and $\mathbb{F}^n$ is the unique simple $M_n(\mathbb{F})$-module up to isomorphism.*

Proof. Let E_{ij} be the the elementary row operation matrix with 1 at the ith row and jth column and zero elsewhere. Note that the set of E_{ij}'s constitute a

basis for $M_n(F)$. Now, $\mathbb{F}^n$ is a simple $M_n(\mathbb{F})$-module as follows: Given a nonzero submodule $W \subseteq \mathbb{F}^n$, we can choose a nonzero element $w \in W$ with a non-zero entry w_j on the jth row. Multiplying w with the matrix $w_j^{-1}E_{ij}$, we see that the ith standard basis vector e_i is inside W for each i, and therefore, we obtain $W = \mathbb{F}^n$.

We similarly show that $M_n(\mathbb{F})$ is a simple ring: Let W be an ideal of $M_n(\mathbb{F})$ with a non-zero element A with a non-zero entry a_{ij} on the ith row and jth column. By the definition of an ideal, $a_{ij}^{-1}E_{ki}AE_{jl} = E_{kl}$ lies in W for all k and l, so that W contains all the basis elements of $M_n(\mathbb{F})$ and must be equal to $M_n(\mathbb{F})$ itself.

By the previous lemma, the local ring $M_n(\mathbb{F})$ has a unique simple module and $\mathbb{F}^n$ must be that module. □

An element $e \in R$ is called an *idempotent* provided $e^2 = e$ holds. An idempotent $e \in R$ always induces a direct sum decomposition of R into R-modules/left ideals, ${}_RR = Re \bigoplus R(1-e)$ and a direct sum decomposition of R into ideals, $R = ReR \bigoplus R(1-e)R$ and also a direct sum decomposition of R into subrings, $R = eRe \bigoplus (1-e)R(1-e)$.

Two idempotents f and g are said to be *orthogonal* if $fg = gf = 0$. An *orthogonal decomposition* of e is an equality $e = \sum_{i=1}^{n} e_i$ for $n > 1$ such that e_i and e_j are orthogonal idempotents for all $i \neq j$. Note that an orthogonal decomposition of an idempotent e in a finite dimensional algebra A implies a direct sum of subalgebras $eAe = e_1Ae_1 \bigoplus \ldots \bigoplus e_nAe_n$ of smaller dimensions, so that there cannot be infinitely many orthogonal idempotents in A. A non-zero idempotent e is called *primitive* if it does not have any nontrivial orthogonal decomposition. An orthogonal decomposition of e where all the summands are primitive is said to be a *primitive decomposition* of e. Note that if an idempotent does not have a primitive decomposition, this implies the existence of infinitely many orthogonal idempotents $\{e_i\}$, which shows that in a finite dimensional algebra, every idempotent has a primitive decomposition. Given idempotents e and f, we say that f *belongs to* e and write $f \leq e$, provided $f = efe$, or equivalently, $f = ef = fe$. If f is a primitive idempotent, then clearly, f belongs to an idempotent e if and only if f appears in a primitive decomposition of e.

The identity 1_R has a unique decomposition $1_R = b_1 + \ldots + b_n$ into primitive idempotents in a commutative ring R, because if $1_R = b_1 + \ldots + b_n = c_1 + \ldots + c_m$ are two primitive decompositions, then multiplying the equality $b_1 + \ldots + b_n = c_1 + \ldots + c_m$ by c_1, we get $b_1c_1 + \ldots + b_nc_1 = c_1^2 = c_1$, which implies, by the primitivity of c_1, that $b_ic_1 = c_1$ for some i. Then, similarly, we multiply the original equation $b_1 + \ldots + b_n = c_1 + \ldots + c_m$ by b_i to get, $b_i = b_ic_1 + \ldots + b_ic_m$, which implies, by the primitivity of b_i, that $b_i = b_ic_1$. Then, $b_i = b_ic_1 = c_1$. Doing this for all c_j, we see that $n = m$ and the decompositions are the same up to a permutation.

For a ring R, the primitive idempotents of the center $Z(R)$ are called *blocks* of R. Additionally, the principal ideal Rb generated by a block b is also called a *block*, which causes no ambiguities, as they are of different types. We shall denote the set of blocks of R by $\mathcal{B}(R)$ from now on. With this notation, for every ring R, the identity has a unique primitive decomposition $1_R = \sum_{b \in \mathcal{B}(R)} b$ into blocks b and R has a unique direct sum decomposition $R = \bigoplus_{b \in \mathcal{B}(R)} Rb$ into blocks Rb.

Lemma 2.25. *For a primitive idempotent f in a ring R with identity, there is a unique block $b \in R$ such that $f \leq b$.*

Proof. Let $1_R = b_1 + \ldots + b_n$ be the unique decomposition of 1_R into blocks. Multiplying this equation by f, we get $0 \neq f = fb_1 + \ldots + fb_n$, so that there is at least one b_i such that $fb_i \neq 0$. Now, $f = b_if + (1 - b_i)f$ is a decomposition of f into orthogonal idempotents, and by the definition of primitivity, both summands can not be non-zero at the same time, and so, $(1 - b_i)f = 0$, i.e., $f = b_if$. Now, suppose, for a contradiction, that f belongs to another b_j for $j \neq i$. Then, multiplying $f = fb_i$ by b_j and using the orthogonality of b_i and b_j, we get $0 \neq f = fb_j = fb_ib_j = 0$, a contradiction. □

In a ring R, two idempotents e and f are called *conjugate* if there is a unit $u \in R^\times$ such that $e = ufu^{-1}$. This is clearly an equivalence relation on the idempotents of R. Another important equivalence relation is the following: Two idempotents $e, f \in R$ are defined to be *associate* provided there are elements $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$. By an easy checking, this is also an equivalence relation on the idempotents of R. We aim to show that these two equivalence relations agree on a finite dimensional algebra.

Lemma 2.26. *Two idempotents e and f in R are associate if and only if $Re \cong Rf$ as R-modules.*

Proof. If $xy = e$ and $yx = f$ for some $x \in eRf$ and $y \in fRe$, then $\varphi : Re \to Rf$ given by $a \mapsto ax$ is an R-module isomorphism: For injectivity, note that if $a \in \ker(\varphi)$, then $ax = 0$, from which, $a = ae = axy = 0y = 0$. For surjectivity, given an element $b \in Rf$, then $b = bf = byx$ is the image of by under φ.

Conversely, if $\varphi : Re \to Rf$ is an R-module isomorphism, then by surjectivity, there is some $y \in Re$ such that $\varphi(y) = f$. The elements $x := \varphi(e)$ and y are sufficient for our purposes: $x = \varphi(e) = \varphi(e^2)f = e\varphi(e)f \in eRf$ and since $\varphi(fy) = f\varphi(y) = f^2 = f = \varphi(y)$, we get $fy = y$ by the injectivity of φ, therefore, $y = fye \in fRe$. Finally, $yx = y\varphi(e) = \varphi(ye) = \varphi(y) = f$ and since $\varphi(xy) = x\varphi(y) = xf = x = \varphi(e)$, we get $xy = e$, by the injectivity of φ again. □

For a general ring R, as opposed to a finite dimensional algebra, the relationship between being conjugate and being associate for two idempotents e and f is a little more complicated [Coh03, Lemma 4.3.4]:

Lemma 2.27. *Two idempotents e and f are conjugate if and only if both e and f are associate and $1 - e$ and $1 - f$ are associate.*

Proof. If two idempotents are conjugate, they are also associate: Given $u \in R^\times$ such that $e = ufu^{-1}$, then the definitions $x := uf = eu$ and $y := fu^{-1} = u^{-1}e$ yield that e and f are associate. Also, note that using the equation $e = ufu^{-1}$, one can show that $1 - e = u(1 - f)u^{-1}$, i.e. $1 - e$ and $1 - f$ are conjugate, hence they are also associate, by the argument above.

Conversely, assume that e and f are associate and also that $1 - e$ and $1 - f$ are associate. This means that there are some $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$ and also there are some $z \in (1 - e)R(1 - f)$ and $w \in (1 - f)R(1 - e)$ such that $zw = 1 - e$ and $wz = 1 - f$. Then, $xw = xf(1 - f)w = 0$ and similarly, $zy = 0$. As a result we have $(x + z)(y + w) = e + 0 + 0 + (1 - e) = 1$ and similarly, $(y + w)(x + z) = 1$, showing that $u := x + z$ is invertible with inverse $u^{-1} = y + w$.

Finally, $ufu^{-1} = (x+z)yx(y+w) = (xyx)(y+w) = (xy)(xy) = e^2 = e$, showing that e and f are conjugate. □

For an Artinian module M over a ring R, the Krull-Schmidt Theorem [Thé95, Theorem 4.4] says that the direct sum decomposition $M = M_1 \oplus \ldots \oplus M_n$ is unique up to permutations and isomorphisms. More precisely, if $M = M_1 \oplus \ldots \oplus M_n$ and $M = M'_1 \oplus \ldots \oplus M'_m$ are two direct sum decompositions of an Artinian R-module M into indecomposables, then $n = m$ and we can reorder the submodules such that $M_i \cong M'_i$ as R-modules. If A is a finite dimensional algebra, then the Krull-Schmidt Theorem is valid for the Artinian module ${}_AA$.

Corollary 2.28. *Given a finite dimensional algebra A and two primitive idempotents $e, f \in A$, then, e and f are conjugate if and only if they are associate.*

Proof. By the previous lemma, being conjugate implies being associate. Conversely, if e and f are associate, then $Ae \cong Af$ by Lemma 2.26. Then, by the Krull-Schmidt Theorem, the equalities of A-modules $Ae \oplus A(1-e) = {}_AA = Af \oplus A(1-f)$ show that $A(1-e) \cong A(1-f)$ as well, from which, $1-e$ and $1-f$ are associate, by Lemma 2.26. Applying the previous lemma, we see that e and f are conjugate. □

The bijective correspondence between primitive idempotents and indecomposable modules shall be important later on:

Lemma 2.29. *Given a ring R and an idempotent $e \in R$, then the primitive decompositions $e = e_1 + \ldots + e_n$ of e are in bijective correspondence with the direct sum decompositions $Re = M_1 \oplus \ldots \oplus M_n$ of Re into indecomposable R-modules M_i. In particular, $e \in R$ is primitive if and only if Re is indecomposable as an R-module.*

Proof. If $Re = M_1 \oplus \ldots \oplus M_n$, then $e = m_1 + \ldots + m_n$ for some $m_i \in M_i$. But then, $m_1 = m_1e = m_1(m_1 + \ldots + m_n) = m_1^2 + m_1m_2 + \ldots + m_1m_n$, so that $m_1 - m_1^2 = m_1m_2 + \ldots + m_1m_n \in M_1 \cap (M_2 \oplus \ldots \oplus M_n) = 0$. In particular, $m_1 = m_1^2$ and $m_1m_2 + \ldots + m_1m_n = 0$. Similarly, the element $-m_1m_2 = m_1m_3 + \ldots + m_1m_n$ is in the intersection $M_2 \cap (M_3 \oplus \ldots \oplus M_n) = 0$. Continuing in this fashion, we obtain $m_1m_j = 0$ for each $j \geq 2$. By the same argument applied to m_i for $i \geq 2$ in place of

m_1, we get that $m_i^2 = m_i$ and $m_i m_j = m_j m_i = 0$, in other words, $e = m_1 + \ldots + m_n$ is a primitive decomposition.

Conversely, if $e = e_1 + \ldots + e_n$ is a primitive decomposition, then $Re = Re_1 \bigoplus \ldots \bigoplus Re_n$ is a direct sum decomposition, and each Re_i are indecomposable, for otherwise we would have a decomposition $Re_i = M \bigoplus N$ for some submodules M and N, which contradicts with the primitivity of e_i, by an argument as above. These processes are easily checked to be inverses to each other, hence they provide a one-to-one correspondence. □

Lemma 2.30. *In a finite dimensional algebra A, if $e = f_1 + \ldots + f_n = d_1 + \ldots + d_m$ are two primitive decompositions of the same idempotent $e \in A$, then $n = m$ and we can reorder the primitive idempotents such that $f_i = {}^{u_i}d_i$ for some units u_i, in other words, primitive compositions are unique up to permutations and conjugations.*

Proof. By the Krull-Schmidt Theorem, Ae has a unique direct sum decomposition into indecomposable modules. Then, the equations
$Ae = Af_1 \bigoplus \ldots \bigoplus Af_n = Ad_1 \bigoplus \ldots \bigoplus Ad_m$ show that $n = m$ and after reordering, we obtain $Ad_i \cong Af_i$. Therefore, d_i and f_i are conjugate by Lemma 2.26 and Corollary 2.29. □

For a given idempotent $\overline{e} = e + I$ of A/I, the process of finding an idempotent f of R such that $\overline{f} = \overline{e}$ is called *lifting* the idempotent $\overline{e}$. This is actually possible for all nilpotent ideals.

Lemma 2.31. *Given a nilpotent ideal I of a ring R and an idempotent $e + I$ of R/I, then there is an idempotent $f \in R$ such that $f + I = e + I$.*

Proof. Let $I^n = 0$ and consider the following chain of quotient rings and canonical ring epimorphisms:

$$R \cong R/I^n \xrightarrow{\pi_n} R/I^{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_{i+2}} R/I^{i+1} \xrightarrow{\pi_{i+1}} R/I^i \xrightarrow{\pi_i} \ldots \xrightarrow{\pi_2} R/I \xrightarrow{\pi_1} R/R \cong 0$$

We shall define a succession of idempotents $e_i + I^i \in A/I^i$ such that $\pi_i(e_i + I^i) = e_{i-1} + I^{i-1}$: Given the idempotent $e_{i-1} + I^{i-1}$ in A/I^{i-1}, then we can find an element $a + I^i$ such that $\pi_i(a + I^i) = e_{i-1} + I^{i-1}$, by the surjectivity of π_i. Then, note that

$a^2 + I^{i-1} = e_{i-1}^2 + I^{i-1} = e_{i-1} + I^{i-1} = a + I^{i-1}$, in other words, $a^2 - a \in I^{i-1}$. But then, since $(I^{i-1})^2 \subseteq I^i$, we have that $(a^2 - a)^2 \in I^i$. Now, define $e_i := 3a^2 - 2a^3$ and check that $\pi_i(e_i + I^i)$ is indeed $e_{i-1} + I^{i-1}$. Also, $e_i + I^i$ is an idempotent, since $e_i^2 - e_i = (3a^2 - 2a^3)(3a^2 - 2a^3 - 1) = (2a-3)(2a+1)(a^2 - a)^2 \in I^i$, by a direct computation. □

This lemma shall be used to lift idempotents from $A/J(A)$, for a finite dimensional algebra A, the Jacobson radical of which is proven to be nilpotent.

Lemma 2.32. *Given a finite dimensional algebra A, and an idempotent $e \in A$, then e is primitive in A if and only if e is primitive in eAe. Also, e is primitive in A if and only if $\overline{e} = e + J(A)$ is primitive in $A/J(A)$.*

Proof. For the first part, we move contrapositively: If $e = f + g$ is an orthogonal decomposition in eAe, then it is also an orthogonal decomposition in A. Conversely, if $e = f + g$ is an orthogonal decomposition in A, then $efe = f$ and $ege = g$, so that $e = efe + ege = f + g$ is an orthogonal decomposition in eAe.

For the second equivalence, suppose that e is not primitive, i.e., $e = f + g$ for some nonzero orthogonal idempotents f and g. Then, the decomposition $\overline{e} = \overline{f} + \overline{g}$ shows that $\overline{e}$ is not primitive, provided $\overline{f}$ and $\overline{g}$ are non-zero, which is indeed the case, as a nonzero idempotent can not be nilpotent, hence can not be in the radical. Conversely, if $\overline{e}$ is not primitive, then $\overline{e} = e + J(A) = (f + J(A)) + (g + J(A))$ for nonzero orthogonal idempotents $f + J(A)$ and $g + J(A)$. Then, by the previous lemma, there are nonzero idempotents $\tilde{f}$ and $\tilde{g}$ in A such that $\tilde{f} + J(A) = f + J(A)$ and $\tilde{g} + J(A) = g + J(A)$. Then, $e + J(A) = (\tilde{f} + J(A)) + (\tilde{g} + J(A))$ implies that $e - \tilde{f} - \tilde{g} \in J(A)$, and again, since $J(A)$ can not contain a nonzero idempotent, $e - \tilde{f} - \tilde{g} = 0$. By a similar argument, $\tilde{f}$ and $\tilde{g}$ are orthogonal, meaning that $e = \tilde{f} + \tilde{g}$ is not primitive. □

We shall be using the Schur's Lemma:

Lemma 2.33 (Schur's Lemma)**.** *Given a field k, a finite dimensional algebra A over k and a simple A-module V, then the ring $End_A(V)$ is a division ring. If k is algebraically closed, then there is a ring isomorphism $End_A(V) \cong k$.*

Proof. For a non-zero A-module endomorphism $f : V \to V$, where V is a simple A-module, note that $\mathrm{im}(f)$ is a non-zero submodule of V, which implies that $\mathrm{im}(f) = V$ by the simplicity of V. Similarly, $\ker(f)$ is a proper submodule of V, which implies that $\ker(f)$ is zero. Hence, f is an isomorphism and has an inverse. If, furthermore, k is algebraically closed, we proceed as follows: For any $f \in \mathrm{End}_A(V)$, note that V is a finite dimensional k-vector space, so that it has a finite k-basis. Consider the matrix representation M of f in this basis. As k is algebraically closed, $\det(M - \lambda I)$ has roots in the polynomial ring $k[\lambda]$, i.e. M has some eigenvalues λ. Having determinant 0, the morphism $M - \lambda I$ is a non-invertible element of $\mathrm{End}_A(V)$, which is a proven to be a division ring, therefore $M - \lambda I = 0$, i.e. $M = \lambda I$. This argument shows that λ is the unique eigenvalue of f. Then, the mapping $\mathrm{End}_A(V) \to k$ given by $f \mapsto \lambda$, sending f to its unique eigenvalue λ is a k-algebra isomorphism. □

Given a direct sum of modules $U_1 \oplus \ldots \oplus U_n$, it is possible to define a matrix ring $M_n(\mathrm{Hom}_A(U_j, U_i))$ in which, the elements ϕ have entries $\phi_{ij} \in \mathrm{Hom}_A(U_j, U_i)$ (note the reversal of indices.)

Lemma 2.34. *Given an algebra A over a field and given finitely many A-modules $\{U_i\}$, then there is an algebra isomorphism $\Omega : \mathrm{End}_A(U_1 \oplus \ldots \oplus U_n) \to M_n(\mathrm{Hom}_A(U_j, U_i))$ defined by $\phi \mapsto \Omega(\phi)$ such that $\Omega(\phi)_{ij} = \mathcal{P}_i \circ \phi \circ \mathcal{I}_j$, where $\mathcal{P}$ and $\mathcal{I}$ are projection and inclusion functions, respectively.*

Proof. Noting that $\mathcal{P}_j \circ \mathcal{I}_j = \mathrm{id}_{U_j}$ and $\sum_j \mathcal{I}_j \circ \mathcal{P}_j = \mathrm{id}_{(\oplus_j U_j)}$, we have,

$$(\Omega(\psi)\Omega(\phi))_{ik} = \sum_j \Omega(\psi)_{ij}\Omega(\phi)_{jk} = \sum_j \mathcal{P}_i\psi\mathcal{I}_j\mathcal{P}_j\phi\mathcal{I}_k$$

$$= \mathcal{P}_i\psi(\sum_j \mathcal{I}_j\mathcal{P}_j)\phi\mathcal{I}_k = \mathcal{P}_i\psi\phi\mathcal{I}_k = (\Omega(\psi\phi))_{ik}$$

therefore, Ω is a homomorphism. For surjectivity, note that we can see the matrices $M_n(\mathrm{Hom}_A(U_j, U_i))$ as a subset of $\mathrm{End}_A(U_1 \oplus \ldots \oplus U_n)$ in the obvious way and Ω acts as identity on the former set. For injectivity, suppose that $\Omega(\psi) = \Omega(\phi)$, i.e. for each i, j, we have $\mathcal{P}_j\phi\mathcal{I}_i = \mathcal{P}_j\psi\mathcal{I}_i$. Multiplying on the left by $\mathcal{I}_j$ and on the right by $\mathcal{P}_i$ and summing over i, we have that $\phi = \mathcal{I}_j\mathcal{P}_j\phi(\sum_i \mathcal{I}_i\mathcal{P}_i) = \mathcal{I}_j\mathcal{P}_j\psi(\sum_i \mathcal{I}_i\mathcal{P}_i) = \psi$. □

Lemma 2.35. *If A is finite dimensional simple algebra over an algebraically closed field k, then $A \cong End_k(V)$ for a unique simple A-module V. Choosing a k-basis of V, we have a k-algebra isomorphism $A \cong M_n(k)$, where $dim_k(V) = n$.*

Proof. Note that the uniqueness of V follows by Lemma 2.24 once we show that A is isomorphic to $M_n(k)$. By the finite dimension hypothesis, there exists a minimal left ideal V of A, which is in particular a simple A-module. Consider the map $A \to \mathrm{End}_k(V)$ given by $a \mapsto f_a$, where $f_a(v) = av$. The kernel of this map is an ideal, and it is proper, as it does not include the identity of A. By the simplicity of A, this kernel is zero, and we have an embedding of A into $\mathrm{End}_k(V)$. By recourse to dimensions, we shall show that this is an isomorphism. Note that by choosing a k-basis $\{v_1, \ldots, v_n\}$ of V, we have that $\mathrm{End}_k(V) \cong M_n(k)$, therefore, $\dim_k(\mathrm{End}_k(V)) = n^2$. We now show that $\dim_k(A) = n^2$ as well: Denoting $\bigoplus_{n \text{ times}} V$ by V^n, define an A-module homomorphism $\Gamma : A \to V^n$ by $a \mapsto (av_1, \ldots, av_n)$. The kernel of Γ, $\ker(\Gamma) = \bigcap_i \mathrm{ann}_A(v_i) = \mathrm{ann}_A(V)$ is an ideal of A and it is proper, as it does not include the identity of A. Again, by the simplicity of A, this kernel is zero, which means that A is a submodule of V^n up to isomorphism. As V is simple, it must be that ${}_AA \cong V^r$ for some $r \leq n$. If $r < n$, we get a contradiction as follows: By an easy generalization of the previous lemma, $\Gamma \in \mathrm{Hom}_A(V^r, V^n)$ corresponds to a matrix $M \in \mathrm{M}_{n\times r}(\mathrm{End}_A(V)) \cong M_{n\times r}(k)$ such that the action of M on V^r is the same as that of Γ. Note that we have $\Gamma(1_A) = (1_Av_1, \ldots, 1_Av_n) = (v_1, \ldots, v_n)$, which means that, via the isomorphism ${}_AA \cong V^r$, there are some u_i for $1 \leq i \leq r$ satisfying

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = \begin{bmatrix} v_1 \\ \\ \vdots \\ \\ v_n \end{bmatrix}$$

Performing elementary matrix operations, we get a non-trivial linear relation $0 = \sum_{i=1}^{n} \lambda_i v_i$ on the last row, contradicting the linear independence of $\{v_i\}$. Hence, $r = n$ and $A \cong V^n$ as A-modules, in particular, as k-vector spaces. Therefore, $\dim_k(A) = \dim_k(V^n) = n^2$. □

Lemma 2.36. *For a finite dimensional algebra A, an idempotent $e \in A$ is primitive if and only if eAe is a local subalgebra of A.*

Proof. Let us write $\overline{e}$ for $e + J(A) \in A/J(A)$. We first show that $eJ(A)e = J(eAe)$ as follows: $eJ(A)e = J(A) \cap eAe$ is a nilpotent ideal of eAe, so it is included in $J(eAe)$. Conversely, $eAe/eJ(A)e = \overline{e}A/J(A)\overline{e}$ is a product of simple algebras, so that $eJ(A)e$ is included in $J(eAe)$. Now, by Lemma 2.32, $\overline{e}$ is primitive in $A/J(A)$, in particular, it is primitive in $\overline{e}(A/J(A))\overline{e}$, which is a product of simple algebras by Lemma 2.19. However, the identity of a product $B_1 \times \ldots \times B_n$ of algebras is never primitive, as it has an obvious decomposition into non-trivial orthogonal idempotents. Hence, $\overline{e}A/J(A)\overline{e}$ is isomorphic to a single simple k-algebra S. By Lemma 2.35, it is actually isomorphic to $M_n(k)$ for some $n \geq 1$. As the identity element of $M_n(k)$ for $n > 1$ is not primitive $\left(\text{for example, } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$, n can not be bigger than 1. Hence, $eAe/J(eAe) = eAe/eJ(A)e \cong M_1(k) \cong k$ is a division algebra, and so, eAe is a local algebra, by Lemma 2.22.

The converse implication is easier: A division ring D can not contain any non-trivial idempotents: If e is a nonzero idempotent in D, it has an inverse e^{-1}, so that $1_D = ee^{-1} = eee^{-1} = e1_D = e$. In particular, the identity of a division ring is primitive. Now, if eAe is local, then $eAe/J(eAe) = \overline{e}A/J(A)\overline{e}$ is a division algebra, which implies that its identity, $\overline{e}$, is primitive in $\overline{e}A/J(A)\overline{e}$. Then, $\overline{e}$ is also primitive in $A/J(A)$, from which, e is primitive in A by Lemma 2.32. □

In a unital local ring R, one can not have a nontrivial idempotent e, for this would imply the direct sum decomposition of ideals $R = ReR \oplus R(1-e)R$, in particular it would imply distinct maximal ideals. For the converse, we have the following corollary:

Corollary 2.37. *A finite dimensional algebra A over an algebraically closed field k is local if and only if it has no nontrivial idempotents.*

Proof. If A has no non-trivial idempotents, 1_A is primitive, so by the previous lemma, $1_A A 1_A = A$ is local. □

Lemma 2.38. *Given primitive idempotents e and f of a finite dimensional algebra A, then e and f are conjugate if and only if eAf is not contained in $J(A)$.*

Proof. Suppose, first that $e = ufu^{-1}$ for some unit $u \in A$. If $eAf \subseteq J(A)$, then $euf \in J(A)$, but then, by hypothesis, $euf = uff = uf \in J(A)$ and we get $e = ufu^{-1} \in J(A)$, a contradiction.

Conversely, if $eAf \notin J(A)$, we shall aim to show that the subalgebra $eAfAe$ is not contained in the Jacobson radical. Denote the ideal $AeAfA$ by I, which contains eAf. By hypothesis, I is not contained in the Jacobson radical, in particular, it is not nilpotent. Then, *a fortiori*, I^2 is not nilpotent, which means it is also not contained in the Jacobson radical. Now, if $eAfAe$ were contained the Jacobson radical, then $I^2 = AeAfAeAfA \subseteq AeAfAeA \subseteq J(A)$, a contradiction. Hence, we have obtained $eAfAe \notin J(A)$. Then there must be some $a, b \in A$ such that $eafbe \notin J(A)$. Let $x = eaf$ and $y = fbe$, and note that $xy \notin J(A)$, in particular $xy \notin J(A) \cap eAe = J(eAe)$, the Jacobson radical of a local ring. By Lemma 2.22, xy is actually a unit in eAe, in other words $(xy)u = u(xy) = e$, for some $u \in eAe$. Then, as $(yux)(yux) = yu(xyu)x = yuex = yux$ is a non-zero idempotent in the local algebra fAf, it must be equal to its unique idempotent f, in other words, $yux = f$. At this point, we have $(ux)y = e$ and $y(ux) = f$, i.e., e and f are associate, in particular, conjugate. □

Altough the following proof seems short, it depends on almost all the previous lemmas:

Lemma 2.39 (Rosenberg's Lemma)**.** *Let A be a finite dimensional algebra over an algebraically closed field k. If e is a primitive idempotent in a sum $I_1 + \ldots + I_n$ of ideals of A, then e lies in one of the summands I_i.*

Proof. As $e^3 \in eI_1e + \ldots + eI_ne$ is non-zero, there is some i such that $eI_ie \neq 0$. The non-zero ideal eI_ie of the local ring eAe is not contained in its Jacobson radical $J(eAe)$, so it must equal to the whole ring: $eAe = eI_ie$. Then, we have $e = e.1_A.e \in eAe = eI_ie \subseteq I_i$. □

For a commutative ring R with unity and a group G, the map $\epsilon : RG \to R$, defined by the mapping $g \mapsto 1_R$, extended R-linearly to RG, is called *the augmentation map*. The kernel of ϵ is denoted as IG and called the *augmentation ideal*. The augmentation ideal IG has an R-basis $\{g - 1_G : g \in G - \{1_G\}\}$ as follows: The elements of this set is linearly independent because if $0 = \sum_{g \neq 1_G} \lambda_g (g - 1_G) = (-\sum_{g \neq 1} \lambda_g) 1_G + \sum_{g \neq 1_G} \lambda_g g$ then each λ_g is zero as G is an R-basis of RG. On the other hand, given an element $x = \sum_{g \in G} \lambda_g g$ in the kernel, we have that $\sum_{g \in G} \lambda_g = \epsilon(x) = 0$ by the definition of the kernel, so that $x = x - 0 = \sum_{g \in G} \lambda_g g - \sum_{g \in G} \lambda_g = \sum_{g \neq 1_G} \lambda_g (g - 1_G)$ is in the span of the set $\{g - 1_G : g \in G - \{1_G\}\}$.

Lemma 2.40. *Given a field k of characteristic p and a finite p-group G, the augmentation ideal and the Jacobson ideal of kG coincide.*

Proof. Let $|G| = p^n$ for $n \geq 0$. We first claim that IG is a nilpotent ideal with $(IG)^{p^n} = 0$, which we prove by induction on n. For $n = 0$, the group G is trivial, so that $IG = 0$ and the induction starts. Let $|G| = p^n$ with p-groups of smaller orders satisfying the statement. As a p-group, G has non-trivial center, so let H be a subgroup of the center of G, with $|H| = p$. Consider the k-algebra epimorphism, $\pi : kG \to k\overline{G}$, where, $\overline{G} = G/H$. We claim that the ideal $\ker(\pi)$ is generated by IH as a left ideal, or formally, $\ker(\pi) = (kG)(IH)$. To show this, pick an element a in the kernel and write,

$$a = \sum_{g \in G} \lambda_g g = \sum_{g \in [G/H]} g \sum_{x \in H} \lambda_{g,x} x$$

By the definition of π, we have $0 = \pi(a) = \sum_{g \in [G/H]} (\sum_{x \in H} \lambda_{g,x}) \overline{g}$, which implies that $\sum_{x \in H} \lambda_{g,x} = 0$ for each $g \in [G/H]$, since $k\overline{G}$ is a k-vector space over $\overline{G}$. Therefore, $\sum_{x \in H} \lambda_{g,x} x \in IH$ for each g and so, $a = \sum_{g \in [G/H]} g(\sum_{x \in H} \lambda_{g,x} x)$ lies in the left ideal $(kG)(IH)$. Conversely, as the basis elements $\{h - 1 : h \in H - \{1\}\}$ are mapped to zero, the whole k-space, that is, IH lies in the kernel, from which, the left ideal in kG generated by IH also lies in the kernel, as π is a kG-algebra homomorphism.

Now, by the induction hypothesis, we know that $(I\overline{G})^{p^{n-1}} = 0$ and we also know that $(IH)^p = 0$. Then, $(IG)^{p^{n-1}} \subseteq \ker(\pi)$ because for $a \in (IG)^{p^{n-1}}$, we have $\pi(a^{p^{n-1}}) = \pi(a)^{p^{n-1}} \in (I\overline{G})^{p^{n-1}} = 0$. On the other hand, $IH \subseteq kH \subseteq kZ(G) \subseteq Z(kG)$, i.e. IH commutes with every other element in kG, so that we have

$[\ker(\pi)]^p = [(kG)(IH)]^p = (kG)^p(IH)^p = 0$, therefore, $(IG)^{p^n} \subseteq \ker(\pi)^p = 0$ and the proof is complete.

As a nilpotent ideal, IG is contained in the Jacobson radical.

On the other hand, kG/IG has dimension 1 as a vector space over k, hence we have an algebra isomorphism $kG/IG \cong k$. Then, as k is a simple kG-module, by Lemma 2.15, $J(kG)$ is contained in IG and we have equality. □

Corollary 2.41. *Given a field k of characteristic p and a p-group G, k is the unique simple kG-module up to isomorphism.*

Proof. $kG/J(kG) \cong k$ is a division ring, therefore kG is a local algebra, and it has a unique simple module. The trivial kG-module k is a simple kG-module so it must be that module. □

Lemma 2.42 (Maschke's Theorem)**.** *If k is a field, G is a finite group and $char(k) \nmid |G|$, then any kG-module V is semisimple.*

Proof. Let W be a submodule of V and let $\pi : V \to W$ be a projection onto W, i.e. π is a k-linear transformation such that $\pi(w) = w$ for all $w \in W$. Such a transformation can be defined using a k-basis of V. Then we get $V = W \bigoplus \ker(\pi)$ as k-vector spaces, but $\ker(\pi)$ might not be G-invariant. Consider the map $\psi : V \to W$ be defined by $v \mapsto \frac{1}{|G|}\sum_{g\in G} g\pi(g^{-1}v)$, where we use the fact that $\frac{1}{|G|}$ is a well-defined element of k as $\mathrm{char}(k)$ does not divide $|G|$, by hypothesis. Now, ψ is also a projection onto W, as follows: For any $w \in W$, we have $\psi(w) = \frac{1}{|G|}\sum_{g\in G} g\pi(g^{-1}w) = \frac{1}{|G|}\sum_{g\in G} gg^{-1}w = \frac{1}{|G|}|G|w = w$. Therefore, we have $V = W \bigoplus \ker(\psi)$ as k-vector spaces. But this time, $\ker(\psi)$ is G-invariant: Given $x \in G$ and $v \in \ker(\psi)$, we calculate $\psi(xv) = \frac{1}{|G|}\sum_{g\in G} g\pi(g^{-1}xv) = \frac{1}{|G|}\sum_{g\in G} x(x^{-1}g)\pi(g^{-1}xv) = x\frac{1}{|G|}\sum_{y\in G} y\pi(y^{-1}v) = x\psi(v) = 0$ by a change of variables $y = g^{-1}x$. Hence, every submodule W of is a direct summand of V as a kG-submodule. □

Theorem 2.43 (Clifford's Theorem)**.** *If k is any field, G any finite group, V a simple kG-module and N is a normal subgroup of G, then $Res^G_N(V)$ is semisimple as a kN-module.*

Proof. Let U be any simple kN-submodule of $\mathrm{Res}^G_N(V)$. For $g \in G$, gU is also a kN-submodule because for $n \in N$, we have $ngU = g(g^{-1}ng)U \subseteq gU$ as N is normal in G. Also, gU is a simple kN-submodule because if it had a nonzero proper submodule W, then $g^{-1}W$ would similarly be a nonzero proper submodule of U, in contradiction with the simplicity of U. Note that $\sum_{g\in G} gU$ is a nonzero kG-submodule of V, hence $V = \sum_{g\in G} gU$ by the simplicity of V as a kG-module. Then, $\mathrm{Res}^G_N(V) = \sum_{g\in G} gU$ is a semisimple kN-module. □

Given two normal p-subgroups P and Q of G, their multiple PQ is also a normal p-subgroup, so that G has a largest normal p-subgroup, denoted $O_p(G)$. Note that given a kG-module V and a subgroup N with trivial action on V, meaning that $nv = v$ for all $v \in V$ and $n \in N$, then $k(G/N)$ has a well-defined action on V, in fact, it acts the same as kG. Hence, quotienting out subgroups of G with trivial action on a module V does not bring any loss of information.

For the next lemma, we shall use the fact that if M is an S module, R is a subring of S and $\mathrm{Res}^S_R(M)$ is simple, then M must also be simple as an S-module because otherwise M would have a nonzero proper S-submodule N, and $\mathrm{Res}^S_R(N)$ would in turn be a nonzero proper R-submodule of $\mathrm{Res}^S_R(M)$, giving a contradiction.

Lemma 2.44. *For a field k of characteristic p and a finite group G, we have $O_p(G) = \{g \in G : gs = s$ for all simple kG-modules S and $s \in S\}$, in other words, $O_p(G)$ is the smallest subgroup of G acting trivially on all simple kG-modules. In particular, simple kG-modules are the same as simple $k(G/O_p(G))$-modules.*

Proof. Given a simple kG-module S, by Clifford's Theorem, $\mathrm{Res}^G_{O_p(G)}(S)$ is semisimple as a $kO_p(G)$-module. Then, by Corollary 2.40, $O_p(G)$ acts trivially on S. Conversely, $H := \{g \in G : gs = s$ for all simple kG-modules S and $s \in S\}$ is a subgroup of G because given $h, h' \in H$, for all simple S and $s \in S$, we have $hh's = s$ and $h^{-1}s = s$. Also, H is normal in G because for $h \in H$ and $g \in G$,

we have $gs \in S$ by the G-invariance of S, so that $hgs = gs$, which is equivalent to $(g^{-1}hg)s = s$. Assume, without loss of generality, that H is non-trivial. We shall show that H is a p-group: Suppose, for a contradiction, that there is some non-identity $h \in H$ with order relatively prime to p. Then, $p \nmid |\langle h \rangle|$, which implies, by Maschke's Theorem, that $\mathrm{Res}^G_{\langle h \rangle}(kG)$ is a semisimple $k\langle h \rangle$-module. As $k\langle h \rangle$ is a subring of kG, by the remark above, kG is a semisimple kG-module as well. But by Corollary 2.40, the only simple kG-module is the trivial kG-module, so that $h \in kG$ acts trivially on kG. In particular, $h^2 = h$, i.e. $h = 1$, a contradiction. Hence, H is a normal p-subgroup of G and it is contained in the biggest normal p-subgroup, $O_p(G)$. □

The structure of a ring is reflected on its modules: The concept of Morita equivalence for rings we now define depends on this observation. Given an S-module N and an (R, S)-bimodule M, we can construct an R-module $M \otimes_S N$, which bears interesting connections to N. If particular, if $M = {}_S S_S$, we have an isomorphism, $\sigma_N : S \otimes_S N \to N$ given by $s \otimes n \mapsto s.n$ Symmetrically, we have an isomorphism $\sigma_M : M \otimes_R R \to M$ for any right R-module M (The subscripts of these isomorphisms shall be omitted from now on.) These observations are precursors of the following definition of Morita equivalence [Thé95, p. 65]:

Definition 2.45. Rings R and S are called Morita equivalent if there exists an (R, S)-bimodule M, an (S, R)-bimodule N, an isomorphism of (R, R)-bimodules $\varepsilon : M \otimes N \to R$ and an isomorphism of (S, S)-bimodules $\eta : N \otimes M \to S$ such that the following diagrams commute:

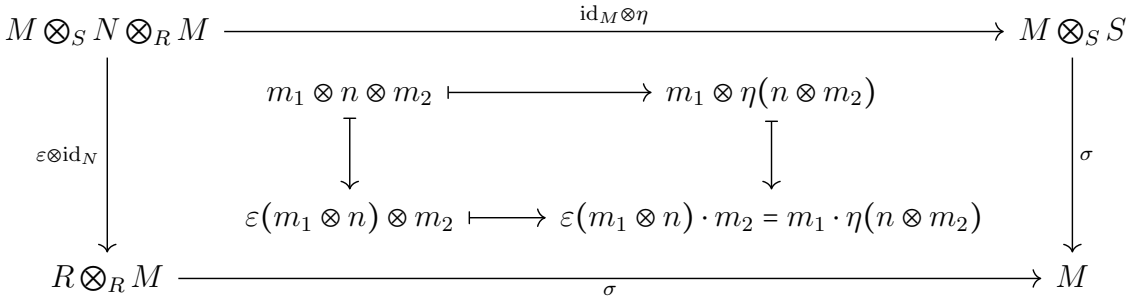


Diagram 2.1

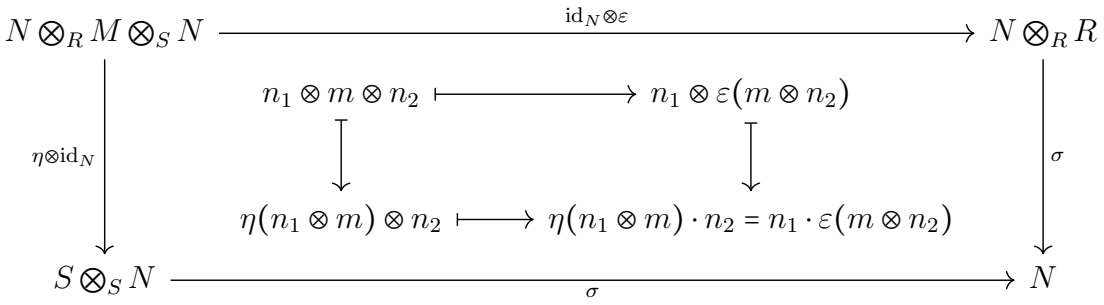


Diagram 2.2

Lemma 2.46. *If R and S rings with identity and there exists an (R,S)-bimodule M, an (S,R)-bimodule N, and some epimorphisms $\eta : N \otimes M \to S$ and $\varepsilon : M \otimes N \to R$ of bimodules such that the diagrams 2.1 and 2.2 commute, then ε and η are, in fact, isomorphisms, i.e. R and S are Morita equivalent.*

Proof. We show the kernel of these maps to be zero: Let $x = \sum_i m_i \otimes n_i$ be any element of the kernel of ε. As ε is an epimorphism and R is a ring with identity, let $\varepsilon(\sum_j m'_j \underset{S}{\otimes} n'_j) = 1_R$ for some m'_j and n'_j. As $M \otimes_S N$ is a left R-module,

$$\begin{aligned}
x = 1_R.x &= \varepsilon(\textstyle\sum_j m'_j \underset{S}{\otimes} n'_j).(\sum_i m_i \underset{S}{\otimes} n_i) \\
&= \textstyle\sum_{i,j} \varepsilon(m'_j \underset{S}{\otimes} n'_j).m_i \underset{S}{\otimes} n_i \\
&= \textstyle\sum_{i,j} m'_j.\eta(n'_j \underset{R}{\otimes} .m_i) \underset{S}{\otimes} n_i && \text{by the commutativity of Diagram 2.1} \\
&= \textstyle\sum_{i,j} m'_j \underset{S}{\otimes} \eta(n'_j \underset{R}{\otimes} .m_i).n_i && \text{by the property of tensor products} \\
&= \textstyle\sum_{i,j} m'_j \underset{S}{\otimes} n'_j.\varepsilon(m_i \underset{S}{\otimes} n_i) && \text{by the commutativity of Diagram 2.2} \\
&= (\textstyle\sum_j m'_j \underset{S}{\otimes} n'_j).\varepsilon(\sum_i m_i \underset{S}{\otimes} n_i) \\
&= (\textstyle\sum_j m'_j \underset{S}{\otimes} n'_j).0 = 0
\end{aligned}$$

The kernel of η is zero by a similar argument, so they are both isomorphisms. □

It is clear that Morita equivalence is an equivalence relation, which we shall denote by $\equiv_{\mathrm{Mor}}$. A standard example of Morita equivalence is between R and $M_n(R)$ for $n > 1$.

Lemma 2.47. *Given a ring R with identity, we have a Morita equivalence, $R \equiv_{\mathrm{Mor}} M_n(R)$.*

Proof. Let $M = N = R^n$ and define,

$$\begin{array}{rcl} \varepsilon : R^n \otimes_{M_n(R)} R^n & \to & R \\ \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} & \mapsto & x_1y_1 + \ldots + x_ny_n \end{array}$$

$$\begin{array}{rcl} \eta : R^n \otimes_R R^n & \to & M_n(R) \\ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \otimes \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix} & \mapsto & \begin{bmatrix} x_1y_1 & \dots & x_1y_n \\ \vdots & & \vdots \\ x_ny_1 & \dots & x_ny_n \end{bmatrix} \end{array}$$

The map ε is well-defined by a direct computation:

$$\varepsilon\left(\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}\begin{bmatrix} z_{11} & \dots & z_{1n} \\ \vdots & & \vdots \\ z_{n1} & \dots & z_{nn} \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = \textstyle\sum_{i,j} x_i z_{ij} y_j$$

$$= \varepsilon\left(\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \otimes \begin{bmatrix} z_{11} & \dots & z_{1n} \\ \vdots & & \vdots \\ z_{n1} & \dots & z_{nn} \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right)$$

Similarly, η is well-defined. Commutativity of Diagram 2.1 holds as follows:

$$\varepsilon\left(\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) . \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1y_1z_1 + \ldots + x_ny_nz_1, & \dots & , x_1y_1z_n + \ldots + x_ny_nz_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} . \eta\left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \otimes \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}\right)$$

Commutativity of Diagram 2.2 is also satisfied by a similar calculation:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} . \varepsilon \left(\begin{bmatrix} y_1 & \dots & y_n \end{bmatrix} \otimes \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \right)$$
$$= \begin{bmatrix} x_1 y_1 z_1 + \ldots + x_1 y_n z_n \\ \vdots \\ x_n y_1 z_1 + \ldots + x_n y_n z_n \end{bmatrix}$$
$$= \eta \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \otimes \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix} \right) . \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

□

A property preserved under Morita equivalence is called a *Morita-invariant* property. One example is the following: Given Morita equivalent rings $R_i \equiv_{\text{Mor}} S_i$ for $i = 1, 2$, there is a canonical Morita equivalence $R_1 \times R_2 \equiv_{\text{Mor}} S_1 \times S_2$. By induction, we conclude that direct products of Morita equivalent rings are also Morita equivalent, i.e. direct products are Morita-invariant.

Lemma 2.48. *If R and S are Morita equivalent, then there is a category equivalence between ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$.*

Proof. The functor $F = M \otimes_S \_ : {}_S\mathfrak{M} \to {}_R\mathfrak{M}$ is defined on objects by $F(V) = M \otimes_S V$ and on morphisms by $F(f) = \mathrm{id}_M \otimes_S f : \ m \otimes v \mapsto m \otimes f(v)$. The functor $G = N \otimes_R \_$ is defined symmetrically. To show that GF is naturally isomorphic to $\mathrm{id}_{{}_S\mathfrak{M}}$, pick an object V and define a map, $\alpha_V = \sigma(\eta \otimes \mathrm{id}_V) : GF(V) = N \otimes M \otimes V \to \mathrm{id}_{{}_S\mathfrak{M}}(V) = V$ by $n \otimes m \otimes v \mapsto \eta(n \otimes m) \cdot v$. The diagrams

$$\begin{array}{ccc} N \otimes M \otimes V & \xrightarrow{\mathrm{id}_N \otimes \mathrm{id}_M \otimes f} & N \otimes M \otimes V' \\ \downarrow{\scriptstyle \alpha_V} & & \downarrow{\scriptstyle \alpha_{V'}} \\ V & \xrightarrow[f]{} & V' \end{array}$$

commute as f is an S-module homomorphism, so that $\{\alpha_V\}_V$ is a natural transformation. It remains to show that each $\alpha_V : N \otimes M \otimes V \to V$ is an isomorphism.

Using the surjectivity of η, we fix some n_i and m_i such that $\eta(\sum_i n_i \otimes m_i) = 1_S$. Define a map $\beta_V : V \to N \bigotimes M \bigotimes V$ by $v \mapsto \sum_i n_i \otimes m_i \otimes v$. We claim that α_V and β_V are inverse to each other: We have, $\alpha_V\beta_V(v) = \alpha_V(\sum_i n_i \otimes m_i \otimes v) = \sum_i \eta(n_i \otimes m_i) \otimes \cdot v = \eta(\sum_i n_i \otimes m_i) \cdot v = 1_S \cdot v = v$. On the other hand,

$$\begin{aligned}
\beta_V\alpha_V(n \otimes m \otimes v) &= \beta_V(\eta(n \otimes m) \cdot v) \\
&= \textstyle\sum_i n_i \otimes m_i \otimes \eta(n \otimes m) \cdot v \\
&= \textstyle\sum_i n_i \otimes m_i \cdot \eta(n \otimes m) \otimes v \\
&= \textstyle\sum_i n_i \otimes \varepsilon(m_i \otimes n) \cdot m \otimes v \quad \text{by Diagram 2.1} \\
&= \textstyle\sum_i n_i \cdot \varepsilon(m_i \otimes n) \otimes m \otimes v \\
&= \textstyle\sum_i \eta(n_i \otimes m_i) \cdot n \otimes m \otimes v \quad \text{by Diagram 2.2} \\
&= \eta(\textstyle\sum_i n_i \otimes m_i) \cdot n \otimes m \otimes v \\
&= 1_S \cdot n \otimes m \otimes v \\
&= n \otimes m \otimes v
\end{aligned}$$

Therefore, $\{\alpha_V\}$ is a natural isomorphism between GF and $\mathrm{id}_{{}_S\mathfrak{M}}$. By a similar argument, FG is naturally isomorphic to the functor $\mathrm{id}_{{}_R\mathfrak{M}}$ and we acquire a category equivalence between ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$. □

The converse of this lemma is also true: If there is a category equivalence between ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$, then R and S are Morita equivalent [Lam99]. In fact, in some books, Morita equivalence between R and S is *defined* to be a category equivalence of ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$.

Lemma 2.49. *If R and S are Morita equivalent via an (R,S)-bimodule M and an (S,R)-bimodule N and isomorphisms $\eta : N \otimes M \to S$ and $\varepsilon : M \otimes N \to R$ of bimodules, then the functors $M \otimes_S \_$ and $N \otimes_R \_$*

1. preserve injections and surjections

2. preserve exact sequences (they are exact functors)

3. preserve split exact sequences

4. preserve simple modules, in the sense that if V is a simple S-module, then $M \otimes_S V$ is a simple R-module.

Proof. In the following, whatever is proven for $M \otimes_S \_$ holds for $N \otimes_R \_$ by symmetry.

1. $M \otimes_S \_$ preserves injections: For an injective S-module homomorphism $f : V \to V'$, we shall show that $\mathrm{id}_M \otimes f : M \otimes_S V \to M \otimes V'$ is also injective. For this, it enough to show that $W \coloneqq \ker(\mathrm{id}_M \otimes f)$ is zero. Defining the inclusion map $i : W \to M \otimes_S V$, we note that the composition $(\mathrm{id}_M \otimes f)i$ is zero:

$$W \xrightarrow{\quad i \quad} M \otimes_S V \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_S V'$$

Applying $N \otimes_R \_$, we get a diagram, which commutes as f is a homomorphism:

$$\begin{array}{ccccc} N \otimes_R W & \xrightarrow{\mathrm{id}_N \otimes i} & N \otimes_R M \otimes_S V & \xrightarrow{\mathrm{id}_N \otimes \mathrm{id}_M \otimes f} & N \otimes_R M \otimes_S V' \\ & & \downarrow{\scriptstyle \sigma(\eta \otimes \mathrm{id}_V)} & & \downarrow{\scriptstyle \sigma(\eta \otimes \mathrm{id}_{V'})} \\ & & V & \xrightarrow{\qquad f \qquad} & V' \end{array}$$

Note that the composite map $f\sigma(\eta \otimes \mathrm{id}_V)(\mathrm{id}_N \otimes i)$ is zero. f is an injection by hypothesis, and $\sigma(\eta \otimes \mathrm{id}_V)$ is an isomorphism by the definition of Morita equivalence, therefore, $(\mathrm{id}_N \otimes i)$ is zero. We apply $M \otimes_S \_$ to get another diagram, which commutes, as i is a homomorphism:

$$\begin{array}{ccc} M \otimes_S N \otimes_R W & \xrightarrow{\mathrm{id}_M \otimes \mathrm{id}_N \otimes i} & (M \otimes_S N) \otimes_R (M \otimes_S V) \\ \downarrow{\scriptstyle \sigma(\varepsilon \otimes \mathrm{id}_W)} & & \downarrow{\scriptstyle \sigma(\varepsilon \otimes \mathrm{id}_{M \otimes V})} \\ W & \xrightarrow{\qquad i \qquad} & M \otimes_S V \end{array}$$

From this diagram, we obtain that the composite map $i\sigma(\varepsilon \otimes \mathrm{id}_W)$ is zero, and $\sigma(\varepsilon \otimes \mathrm{id}_W)$ is an isomorphism by the proof of the previous lemma, so that i is zero and so W is zero. By a similar argument, we show that $M \otimes_S \_$ preserves surjections: For a surjective S-module homomorphism $f : V \to V'$, we need show that, $\mathrm{id}_M \otimes f : M \otimes_S V \to M \otimes V'$ is also surjective. For this, it enough to show that $W \coloneqq \mathrm{coker}(\mathrm{id}_M \otimes f)$ is zero. Defining the canonical projection $q : M \otimes_S V' \to W$, we note that the composition $q(\mathrm{id}_M \otimes f)$ is zero:

$$M \otimes_S V \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_S V' \xrightarrow{\quad q \quad} W$$

Applying $N \otimes_R \_$, we get a diagram, which commutes, as f is a homomorphism:

$$\begin{array}{ccccc} V & & \xrightarrow{\quad f \quad} & & V' \\ \Big\downarrow{\scriptstyle (\varepsilon^{-1}\otimes \mathrm{id}_V)\sigma^{-1}} & & & & \Big\downarrow{\scriptstyle (\varepsilon^{-1}\otimes \mathrm{id}_{V'})\sigma^{-1}} \\ N \otimes_R M \otimes_S V & \xrightarrow{\mathrm{id}_N\otimes \mathrm{id}_M \otimes f} & N \otimes_R M \otimes_S V' & \xrightarrow{\mathrm{id}_N \otimes q} & N \otimes_R W \end{array}$$

Note that the composite map $(\mathrm{id}_N \otimes q)(\varepsilon^{-1} \otimes \mathrm{id}_{V'})\sigma^{-1} f$ is zero. As f is surjective and $(\varepsilon^{-1} \otimes \mathrm{id}_{V'})\sigma^{-1}$ is an isomorphism, $\mathrm{id}_N \otimes q$ is zero. We apply $M \otimes_S \_$ to get another diagram, which commutes as q is a homomorphism:

$$\begin{array}{ccc} (M \otimes_S N) \otimes_R M \otimes_S V' & \xrightarrow{\mathrm{id}_M \otimes \mathrm{id}_N \otimes q} & M \otimes_S N \otimes_R W \\ \Big\downarrow{\scriptstyle \sigma(\varepsilon \otimes \mathrm{id}_{M\otimes V'})} & & \Big\downarrow{\scriptstyle \sigma(\varepsilon \otimes \mathrm{id}_W)} \\ M \otimes_S V' & \xrightarrow{\quad q \quad} & W \end{array}$$

From this diagram, we obtain that the composite map $q\sigma(\varepsilon \otimes \mathrm{id}_{M\otimes V'})$ is zero, and $\sigma(\varepsilon \otimes \mathrm{id}_{M\otimes V'})$ is an isomorphism by the proof of the previous lemma, so that q is zero and so W is zero.

2. Let $\mathcal{S} : 0 \to V \xrightarrow{f} V' \xrightarrow{g} V'' \to 0$ be a short exact sequence. Firstly, note that as $N \otimes_R M \otimes_S \_$ is naturally isomorphic to the exact functor $\mathrm{id}_{S\mathfrak{M}}$ and as natural transformations preserve exactness, $N \otimes_R M \otimes_S \_$ is itself an exact functor, which implies that the sequence, $N \otimes_R M \otimes_S \mathcal{S}$:

$$0 \to N \otimes_R M \otimes_S V \xrightarrow{\mathrm{id}_N \otimes \mathrm{id}_M \otimes f} N \otimes_R M \otimes_S V' \xrightarrow{\mathrm{id}_N \otimes \mathrm{id}_M \otimes g} N \otimes_R M \otimes_S V'' \to 0$$

is exact. Our aim is to show the sequence,

$$M \otimes \mathcal{S} : \ 0 \to M \otimes_S V \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_S V' \xrightarrow{\mathrm{id}_M \otimes g} M \otimes_S V'' \to 0$$

to be exact. As $M \otimes \_$ preserves injections and surjections, $\mathrm{id}_M \otimes f$ is injective and $\mathrm{id}_M \otimes g$ is surjective, which ensures the exactness at the first and last terms. For the middle term, note that the composite $(\mathrm{id}_M \otimes g)(\mathrm{id}_M \otimes f) = \mathrm{id}_M \otimes gf$ is zero

by hypothesis, so that $\mathrm{im}(\mathrm{id}_M \otimes f) \subseteq \ker(\mathrm{id}_M \otimes g) =: W$. This means that $\mathrm{id}_M \otimes f$ restricts to an injection $\overline{\mathrm{id}_M \otimes f} : M \otimes_S V \to W$. To complete the chain, we define an inclusion, $j : W \to M \otimes_S V'$. It remains to show that $W \subseteq \mathrm{im}(\mathrm{id}_M \otimes f)$ in the following diagram:

$$M \otimes_S V \xrightarrow{\overline{\mathrm{id}_M \otimes f}} W \xrightarrow{j} M \otimes_S V' \xrightarrow{\mathrm{id}_M \otimes g} M \otimes_S V''$$
$$\mathrm{id}_M \otimes f \text{ (curving back from } M \otimes_S V' \text{ to } W\text{)}$$

Applying $N \otimes_R \_$, we get the following sequence:

$$N \otimes_R M \otimes_S V \xrightarrow{\mathrm{id}_N \otimes \overline{\mathrm{id}_M \otimes f}} N \otimes_R W \xrightarrow{\mathrm{id}_N \otimes j} N \otimes_R M \otimes_S V'$$
$$\mathrm{id}_N \otimes \mathrm{id}_M \otimes f \text{ (curving back from } N \otimes_R M \otimes_S V' \text{ to } N \otimes_R W\text{)}$$
$$\downarrow \mathrm{id}_N \otimes \mathrm{id}_M \otimes g$$
$$N \otimes_R M \otimes_S V''$$

Denoting $\mathrm{im}(\varphi)$ as $\varphi[D]$ for a function $\varphi : D \to E$, we calculate,

$$\begin{aligned}(\mathrm{id}_N \otimes j)[N \otimes_R W] &\subseteq \ker(\mathrm{id}_N \otimes \mathrm{id}_M \otimes g) \\ &= (\mathrm{id}_N \otimes \mathrm{id}_M \otimes f)[N \otimes_R M \otimes_S V] \\ &= (\mathrm{id}_N \otimes j)(\mathrm{id}_N \otimes \overline{\mathrm{id}_M \otimes f})[N \otimes_R M \otimes_S V]\end{aligned}$$

since $(\mathrm{id}_M \otimes g)j = 0$ and $N \otimes_R M \otimes_S \mathcal{S}$ is exact.

As $N \otimes_R \_$ preserves injections, $\mathrm{id}_N \otimes j$ is injective and has a left inverse, which we apply on the set inclusion, $(\mathrm{id}_N \otimes j)[N \otimes_R W] \subseteq (\mathrm{id}_N \otimes j)(\mathrm{id}_N \otimes \overline{\mathrm{id}_M \otimes f})[N \otimes_R M \otimes_S V]$ to get that $N \otimes_R W \subseteq (\mathrm{id}_N \otimes \overline{\mathrm{id}_M \otimes f})[N \otimes_R M \otimes_S V]$, so that $\mathrm{id}_N \otimes \overline{\mathrm{id}_M \otimes f}$ is a surjection. As $N \otimes_R \_$ preserves injections, $\mathrm{id}_N \otimes \overline{\mathrm{id}_M \otimes f}$ is actually an isomorphism. Then, as $M \otimes \_$ preserves bijections, $\mathrm{id}_M \otimes \mathrm{id}_N \otimes \overline{\mathrm{id}_M \otimes f}$ is also an isomorphism, which implies, by the proof of the previous lemma, that $\overline{\mathrm{id}_M \otimes f}$ is an isomorphism. Therefore, $W \subseteq \mathrm{im}(\overline{\mathrm{id}_M \otimes f}) = \mathrm{im}(\mathrm{id}_M \otimes f)$ and the proof is complete.

3. If

$$\mathcal{S} : \; 0 \to V \xrightarrow{f} V' \underset{g}{\overset{h}{\leftrightarrows}} V'' \to 0$$

is split exact sequence, where a right inverse h of g is a homomorphism, then the (homomorphism) $\mathrm{id}_M \otimes h$ is a right inverse of $\mathrm{id}_M \otimes g$, and so $M \otimes \mathcal{S}$ also splits. Conversely, if

$$M \otimes \mathcal{S} : \ 0 \to M \otimes_S V \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_S V' \xrightarrow{\mathrm{id}_M \otimes g} M \otimes_S V'' \to 0$$

is split, then $N \otimes M \otimes \mathcal{S}$ is also split. By the proof of the previous lemma, $\mathcal{S}$ is isomorphic to $N \otimes M \otimes \mathcal{S}$, and so, $\mathcal{S}$ also splits.

4. Firstly, if W is nonzero, then $M \otimes W$ is nonzero, for otherwise $W \cong R \otimes W \cong N \otimes M \otimes W = 0$, a contradiction. Now, given a module V, V is not simple if and only if there is an exact sequence $\mathcal{S} : \ 0 \to V \xrightarrow{f} V' \xrightarrow{g} V'' \to 0$ with the first and last terms nonzero. But then, the sequence

$$M \otimes \mathcal{S} : \ 0 \to M \otimes_S V \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_S V' \xrightarrow{\mathrm{id}_M \otimes g} M \otimes_S V'' \to 0$$

is an exact sequence with the first and last terms nonzero, so that $M \otimes V$ is not simple. The proof of the converse implication is similar. □

Note: By category theory, the functor $M \otimes \_$ is left adjoint to $\mathrm{Hom}_R(\_, M)$. Since a left adjoint functor is right exact and a right adjoint functor is left exact, we know that $M \otimes \_$ is right exact. Similarly, $\_ \otimes M$ is left adjoint to $\mathrm{Hom}_R(M, \_)$, so that $\_ \otimes M$ is also right exact. So, we already knew from category theory that $M \otimes \_$ was right exact, but this lemma states that it is fully exact, which is a stronger property.

The last result of this chapter shall be that semisimplicity is a Morita-invariant property. Given two unital rings R and S such that $R \equiv_{\mathrm{Mor}} S$ via an (R, S)-bimodule M and an (S, R)-bimodule N, if S is semisimple, we can show R to be semisimple as follows: Lemma 2.10 says S being semisimple is equivalent to the fact that every finitely generated S-module is semisimple. Let W be a finitely generated R-module. Then, $N \otimes_R W$ is a semisimple S-module, i.e. there are simple S-modules V_i such that $N \otimes_R W \cong V_1 \times \ldots \times V_n$. Applying $M \otimes_S \_$, we get $W \cong M \otimes_S N \otimes_R W \cong M \otimes_S (V_1 \times \ldots \times V_n) \cong (M \otimes_S V_1) \times \ldots \times (M \otimes_S V_n)$. Since

$M \otimes_S \_$ preserves simple modules, W must be semisimple. As W was chosen arbitrarily, Lemma 2.10 yields that R is semisimple.

Chapter 3

Brauer Pairs

Brauer pairs defined when a finite group G acts on a k-algebra A, where k is an algebraically closed field of characteristic p. In this chapter, we shall start with kG-modules and generalize their structure until we obtain interior p-permutation G-algebras. Then, we shall define Brauer pairs and prove two theorems concerning them. Let us first develop some tools to analyse the structure of G-algebras:

For finite groups $K \leq H \leq G$, and a kG-module M, we define the following maps, called *transfer* and *restriction* maps, respectively:

$$t_K^H : M^K \to M^H \qquad\qquad r_K^H : M^H \to M^K$$
$$t_K^H(m) := \sum_{x \in [H/K]} xm \qquad\qquad r_K^H(m) := m$$

where $[H/K]$ is a set of representatives of the left cosets of K in H, and M^K denotes the set $\{m \in M : km = m \text{ for all } k \in K\}$ of elements of M fixed by all elements of K, which is a submodule of M. Note that a different choice of representatives of cosets gives the same function t_K^H, because if $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are two sets of representatives, we can reorder the second set so that we have $x_i K = y_i K$, i.e. there are some $k_i \in K$ such that $x_i k_i = y_i$. Then, $y_1 m + \ldots + y_n m = x_1 k_1 m + \ldots + x_n k_n m = x_1 m + \ldots + x_n m$ as m is already fixed by K.

A useful notation is to write $M^H_{<H}$ for the sum of submodules $\sum_{K<H} t^H_K(M^K)$. The quotient map, called the *Brauer map*, $\mathrm{br}_H : M^H \to M(H)$ makes sense for $|H| > 1$, where $M(H)$ denotes the quotient module, $M^H/M^H_{<H}$, called the *Brauer quotient*. The map br_1 is defined to be the identity map $M \to M$.

Lemma 3.1. *Given a direct sum of kG-modules $M \oplus N$ and a subgroup H of G, the fixed points operation $()^H$ is compatible with direct sum: $(M \oplus N)^H = M^H \oplus N^H$*

Proof. Given a in the left-hand side, $a = m + n$ for $m \in M$ and $n \in N$. Then, $h \cdot m + h \cdot n = h \cdot a = a = m + n$ for $h \in H$, so, $h \cdot m - m = n - h \cdot n$ lies inside the intersection $M \cap N = 0$. The converse implication is clear. □

Let us move to G-algebras: Given a field k, a finite group G and a k-algebra A, then we call A a *G-algebra* provided G acts on A compatibly with the addition and multiplication of A. More formally, a pair (A, ψ) is called a *G-algebra* provided ψ is a group homomorphism $G \to \mathrm{Aut}_k(A)$. For ease of notation, we shall prefer to write ${}^g a$ instead of $\psi(g)(a)$ for $g \in G$ and $a \in A$. Given a G-algebra A, the set $A^K := \{m \in A : {}^k a = a \text{ for all } k \in K\}$ of fixed elements constitutes a unitary subalgebra of A.

Conjugation by a unit (let us use the notation $c_u : a \mapsto uau^{-1}$) is an important example of an automorphism of A, called an *inner automorphism*. The special case that G acts as the inner automorphisms of A is treated separately. Formally, a pair (A, ψ) is called an *interior G-algebra* if ψ is a homomorphism $G \to A^\times$, where $A^\times$ is the set of units of A. Instead of $\psi(g)a$, we write $g \cdot a$. There is a canonical group homomorphism $A^\times \to \mathrm{Aut}(A)$ sending a unit u of A to the conjugation map c_u, and so an interior G-algebra is indeed a special case of G-algebra. Given an interior G-algebra, we shall denote the induced action $g \cdot a \cdot g^{-1}$ as ${}^g a$. Interior G-algebras have properties that regular G-algebras do not have. For example, consider the relationship between the center $Z(A)$ of A and the H-fixed points A^H of A, for a subgroup $H \leq G$: By elementary calculations we can see that $Z(A) \cap A^H = [Z(A)]^H \subseteq Z(A^H)$ holds for any G-algebra A. If we further assume A to be an interior G-algebra, the center is always G-fixed, i.e. $Z(A) \subseteq A^G$, because

for $a \in Z(A)$ and $g \in G$, we have $\psi(g)a = a\psi(g)$, which is equivalent to say that $g \cdot a \cdot g^{-1} = a$.

The conjugation action of G on the k-algebra kG is the primary example of an interior G-algebra. More generally, a group representation $\rho : G \to \mathrm{Aut}_k(V)$ is seen to be an interior G-algebra with $A := \mathrm{End}_k(V)$ and $A^\times = \mathrm{Aut}_k(V)$. Hence, the concept of G-algebra is a natural generalization of the action of a group G on some vector space. The following lemma collects some properties of the restriction, transfer and conjugation maps on a G-algebra A.

Lemma 3.2. *Denoting the automorphism $a \mapsto {}^g a$ by c_g, we have the following properties for a G-algebra A:*
(i) $r_L^K r_K^H = r_L^H$ and $t_K^H t_L^K = t_L^H$ if $L \le K \le H$
(ii) $r_H^H = t_H^H = id_{A^H}$
(iii) $c_{gh} = c_g c_h$
(iv) $c_h : A^H \to A^H$ is the identity if $h \in H$
(v) $c_g r_K^H = r_{{}^g K}^{{}^g H} c_g$ and $c_g t_K^H = t_{{}^g K}^{{}^g H} c_g$ if $K \le H$
(vi) (Mackey axiom) if $L, K \le H$,

$$r_L^H t_K^H = \sum_{h \in [L\backslash H/K]} t^L_{L \cap {}^h K} r^{{}^h K}_{L \cap {}^h K} c_h$$

If we omit restriction maps, $t_K^H(a) = \sum_{h \in [L\backslash H/K]} t^L_{L \cap {}^h K}({}^h a)$ for $a \in A^K$
(vii) (Frobenius axiom) For $a \in A^K$ and for $b \in A^H$, we have $t_K^H(ab) = t_K^H(a)b$ and symmetrically, $t_K^H(ba) = b t_K^H(a)$

Proof. Proof of (v): For $g \in G$, we can see that $\{x_1 K, \ldots, x_n K\}$ is a set of representatives of the left cosets H/K if and only if $\{{}^g x_1 \, {}^g K, \ldots, {}^g x_n \, {}^g K\}$ is a set of representatives of ${}^g H / {}^g K$. By symmetry, $\{K x_1, \ldots, K x_n\}$ is a set of representatives of the right cosets $K\backslash H$ if and only if $\{K^g x_1^g, \ldots, K^g x_n^g\}$ is a set of representatives of $K^g \backslash H^g$. These observations can be denoted briefly as ${}^g[H/K] = [{}^g H / {}^g K]$ and $[K\backslash H]^g = [K^g \backslash H^g]$. Similarly, for $h \in H$, we can see that $\{x_1 K, \ldots, x_n K\}$ is a set of representatives of the left cosets H/K if and only if $\{x_1 h K^h, \ldots, x_n h K^h\}$ is a set of representatives of H/K^h. By symmetry, $\{K x_1, \ldots, K x_n\}$ is a set of representatives of the right cosets $K\backslash H$ if and only if $\{{}^h K h x_1, \ldots, {}^h K h x_n\}$ is a set of representatives of ${}^h K \backslash H$. We write $[H/K].h =$

$[H/K^h]$ and $h.[K\backslash H] = [{}^hK\backslash H]$ to denote these observations briefly. (Another useful bit of knowledge is that $\{x_1, \ldots, x_n\}$ is a set of representatives of the left cosets H/K if and only if $\{x_1^{-1}, \ldots, x_n^{-1}\}$ is a set of representatives of the right cosets $K\backslash H$, i.e. $[H/K]^{-1} = [K\backslash H]$, however, we shall not use this here.) Now let us calculate, for any $a \in A^K$,

$$c_g t_K^H(a) = {}^g\Big(\sum_{x\in[H/K]} {}^x a\Big) = \sum_{x\in[H/K]} {}^{gx}a = \sum_{x\in[H/K]} {}^{(gxg^{-1})g}a$$
$$= \sum_{y\in{}^g[H/K]} {}^{yg}a = \sum_{y\in[{}^gH/{}^gK]} {}^{yg}a = t_{{}^gK}^{{}^gH}({}^g a) = t_{{}^gK}^{{}^gH} c_g(a)$$

We refer the reader to [Kül91, Chapter 7] for the proofs of the rest of the items. □

Notes: 1- Mackey Axiom (vi) has the following generalization: For each $a \in A^L$ and $b \in A^K$,

$$t_L^H(a)t_K^H(b) = \sum_{h\in[L\backslash H/K]} t^H_{{}^hL\cap K}({}^h ab)$$

2- The inclusion r_K^H is a unitary k-algebra homomorphism. The map t_K^H is not multiplicative, but the Frobenius axiom (vii) makes $A_K^H := t_K^H(A^K)$ into an ideal of A^H. Then, $A_{<H}^H := \sum_{K<H} A_K^H$ is an ideal in A^H and the Brauer map $\mathrm{br}_H : A^H \to A(H) := A^H/A_{<H}^H$ is now a k-algebra homomorphism.

Lemma 3.3. *For $g \in G$ and for subgroups $K \le H \le G$, we have,*
1. ${}^g(A^H) = A^{{}^gH}$
2. ${}^g(A_K^H) = A_{{}^gK}^{{}^gH}$
3. $A_K^H = A_{{}^hK}^H$ *for all* $h \in H$.
4. $A^H = A_P^H$ *for any Sylow p-subgroup P of H. In particular, if H is not a p-group, then $A(H) = 0$ and $br_H : A^H \to A(H)$ is zero map.*

Proof. 1. For any $a \in A^H$ and for any $h \in H$, we see that ${}^{ghg^{-1}g}a = {}^{gh}a = {}^g a$, so that ${}^g a$ is fixed by ${}^gH$. The converse implication is similar.
2. When we apply $c_g t_K^H$ on A^K, using axiom (v), we get the second identity.

3. We note,

$$\begin{aligned}
A_K^H &= t_K^H(A^K) \\
&= \{t_K^H(a) : a \in A^K\} \\
&= \{\sum_{x \in [H/K]} {}^x a : a \in A^K\} \\
&= \{\sum_{x \in [H/K]} {}^{xh^{-1}h} a : a \in A^K\} \\
&= \{\sum_{y \in [H/K]h^{-1}} {}^{yh} a : a \in A^K\} \\
&= \{\sum_{y \in [H/K^{h^{-1}}]} {}^{yh} a : a \in A^K\} \\
&= \{\sum_{y \in [H/{}^hK]} {}^{yh} a : a \in A^K\} \\
&= \{t_{{}^hK}^H({}^h a) : a \in A^K\} \\
&= \{t_{{}^hK}^H(b) : b \in A^{{}^hK}\} \\
&= t_{{}^hK}^H(A^{{}^hK}) = A_{{}^hK}^H
\end{aligned}$$

4. For the last item, note that the index $|H : P|$ is invertible in k. Then, given an element $a \in A^H$, as a is already fixed by H, we see that $t_P^H(\frac{1}{|H:P|}a) = a$ by the Frobenius axiom. □

We can define an action of G on the Brauer quotients $A(H)$ such that ${}^g(A(H)) = A({}^gH)$, as we now show: Using the previous lemma, we conjugate by g, to get, ${}^g(A_{<H}^H) = \sum_{K<H} {}^g(A_K^H) = \sum_{K<H} A_{{}^gK}^{{}^gH} = \sum_{R<{}^gH} A_R^{{}^gH} = A_{<{}^gH}^{{}^gH}$ by a change of variables $R = {}^gK$. Hence, we make the following definition on the individual elements: $c_g \mathrm{br}_H(a) = c_g(a + A_{<H}^H) := {}^g a + A_{<{}^gH}^{{}^gH} = \mathrm{br}_{{}^gH}(c_g(a))$. We can denote this relation as $c_g \mathrm{br}_H = \mathrm{br}_{{}^gH} c_g$. One implication of this is that $A(H)$ is an $N_G(H)$-algebra: Given $a \in A_{<H}^H$ and $g \in N_G(H)$, we have ${}^g a \in {}^g(A_{<H}^H) = A_{<{}^gH}^{{}^gH} = A_{<H}^H$. In particular, br_H is now an $N_G(H)$-algebra homomorphism.

The following relationship between the Brauer homomorphism and the transfer map shall become useful:

Lemma 3.4. *Given a a subgroup H, some p-subgroups P and Q of G such that*

$P \leq H$ and given an element $a \in A^P$, the following equality holds:

$$br_Q(t_P^H(a)) = \sum_{\substack{g \in [Q\backslash H/P] \\ Q \leq {}^g P}} br_Q({}^g a)$$

Proof. By the Mackey axiom, $t_P^G(a) = \sum_{g\in[Q\backslash G/P]} t^Q_{Q\cap {}^g P}({}^g a)$. Applying br_Q, we get $\mathrm{br}_Q t_P^G(a) = \sum_{g\in[Q\backslash G/P]} \mathrm{br}_Q t^Q_{Q\cap {}^g P}({}^g a)$. Note that if $Q \cap {}^g P < Q$, then $t^Q_{Q\cap {}^g P}({}^g a)$ lies in the ideal $A^Q_{Q\cap {}^g P}$, which is in the kernel $A^Q_{<Q}$ of br_Q. Hence, only the summands with index g such that $Q \cap {}^g P = Q$ survive under the action of br_Q. But the equality $Q \cap {}^g P = Q$ is equivalent to saying $Q \leq {}^g P$, so the result follows. □

Corollary 3.5. *Given a p-subgroup P and $a \in A^P$, the following holds: $br_P(t_P^G(a)) = br_P(t_P^{N_G(P)}(a)) = t_P^{N_G(P)} br_P((a))$. In particular, $br_P : A_P^G \to [A(P)]_P^{N_G(P)}$ is an $N_G(P)$-algebra epimorphism.*

Proof. Replacing P for Q and G for H in the previous lemma, we get,

$$\mathrm{br}_P(t_P^G(a)) = \sum_{g\in[P\backslash G/P] P \leq {}^g P} \mathrm{br}_P({}^g a)$$

But $P \leq {}^g P$ implies $P = {}^g P$ as we are working with finite groups. In particular, the indices on the right-hand side satisfy $g \in N_G(P)$. Hence, $\mathrm{br}_P(t_P^G(a)) = \sum_{g\in[P\backslash N_G(P)/P]} \mathrm{br}_P({}^g a)$. On the other hand, replacing P for Q and $N_G(P)$ for H in the previous lemma, we obtain, by a similar argument, that $\mathrm{br}_P(t_P^{N_G(P)}(a)) = \sum_{g\in[P\backslash N_G(P)/P]} \mathrm{br}_P({}^g a)$, and so the first equality holds. As br_P is already shown to be an $N_G(P)$-algebra homomorphism, the second equality is also true. Now, consider the restricted map $\mathrm{br}_P : A_P^G \to \mathrm{br}_P(A_P^G) \subseteq A(P)$. By the identity we have just proven, the image $\mathrm{br}_P(A_P^G)$ is exactly $[A(P)]_P^{N_G(P)}$ and so, we have an $N_G(P)$-algebra epimorphism $A_P^G \to [A(P)]_P^{N_G(P)}$ □

Lemma 3.6. *For a G-set X, $x \in X$, $g \in G$, $H \leq G$ we have*
i) ${}^g(C_H(x)) = C_H({}^g x)$
ii) ${}^g(C_H(X)) = C_H({}^g X)$
iii) ${}^g(N_H(X)) = N_H({}^g X)$

Proof. The if direction and only-if direction in parts (i) and (iii) are similar, so we show only one direction:

i) Given ${}^{g}h$ in the left-hand side,
${}^{({}^{g}h)g}x = {}^{ghg^{-1}g}x = {}^{gh}x = {}^{g}x$, as $h \in C_H(x)$ by hypothesis.
ii) ${}^{g}(C_H(X)) = \bigcap_{x\in X} {}^{g}(C_H(x)) = \bigcap_{x\in X} C_H({}^{g}x) = \bigcap_{y\in({}^{g}X)} C_H(y) = C_H({}^{g}X)$
iii) Given ${}^{g}h$ in the left-hand side, ${}^{({}^{g}h)g}X = {}^{ghg^{-1}g}X = {}^{gh}X = {}^{g}X$, as $h \in N_H(X)$ by hypothesis. □

If we are given a G-set X, we can always form the free k-vector space kX on the set X, and kX is naturally a kG-module, an example of a *permutation* kG-module. In general, a k-vector space M with some G-invariant k-basis X is defined to be a *permutation* kG*-module*. The operations of induction, restriction and conjugation as they are commonly defined on kG-modules preserve permutation modules, as we now show.

If $H \leq G$ are finite groups, M is a kH-module and N is a kG-module, then we have the following definitions [Web16]:
i) The induced module $\mathrm{Ind}_H^G(M)$ is defined to be the kG-module, $kG \underset{kH}{\otimes} M$.
ii) The restricted module $\mathrm{Res}_H^G(N)$ is defined to be the kH-module N with the same kH-action.
iii) The conjugated module $\mathrm{Con}_H^g(M)$ is defined to be the $k({}^{g}H)$-module M with the following action: ${}^{g}h \cdot m := hm$.

Lemma 3.7. *Given a finite group G, a subgroup $H \leq G$, a permutation kH-module M with basis X and a permutation kG-module N with basis Y, then*
i) $\mathrm{Ind}_H^G(M)$ *is a permutation kG-module with basis $\{g \otimes x : g \in [G/H], x \in X\}$.*
ii) $\mathrm{Res}_H^G(N)$ *is a permutation kH-module with basis Y.*
iii) For $g \in G$, the conjugate module $\mathrm{Con}_H^g(M) = M$ *is a permutation $k({}^{g}H)$-module with basis $g \cdot X$.*

Proof. The second and third parts are clear. For the first part, M has a k-basis $X = \bigsqcup_i X_i$, where $\{X_i\}_i$ denote H-orbits on X. Using the definition of the induced

kG-module,

$$\begin{aligned}
\mathrm{Ind}_H^G(kX) &= kG\bigotimes_{kH} kX \\
&= (\bigoplus_{g\in[G/H]} kgH)\bigotimes_{kH}(\bigoplus_i kX_i) \\
&\cong \bigoplus_{g\in[G/H]}\bigoplus_i (kgH\bigotimes_{kH} kX_i) \\
&= \bigoplus_{g\in[G/H]}\bigoplus_i g(kH\bigotimes_{kH} kX_i) \\
&\cong \bigoplus_{g\in[G/H]}\bigoplus_i g(kX_i)
\end{aligned}$$

where, in the latest item, each component has a k-basis $\{g\cdot x : x\in X_i\}$, so, using the fact that $kX\bigoplus kY = k(X\bigsqcup Y)$, the direct sum has basis $\bigsqcup_{g,i} = \{g\cdot x : x\in X_i\} = \{g\cdot x : g\in[G/H],\ x\in X\}$. Chasing back through the isomorphisms, $\mathrm{Ind}_H^G(kX)$ has basis $\{g(1\otimes x) : g\in[G/H],\ x\in X\} = \{g\otimes x : g\in[G/H],\ x\in X\}$. □

Given a kG-module M, if for all p-subgroups P of G, the restricted module $\mathrm{Res}_P^G(M)$ is a permutation module, then M is called a *p-permutation kG-module*. To determine if a given kG-module M is a p-permutation kG-module, it is enough to take a Sylow p-subgroup S of G and show that $\mathrm{Res}_S^G(M)$ is a permutation module, by the Sylow Theorems and the lemma just proven.

Let us try to find the Brauer quotient of a permutation kP-module kX for a p-group P.

Lemma 3.8. *Given a permutation kG-module kX for a finite group G, then,*
1. $(kX)^G$ has a basis $\{[y]_G^+ : y\in[G\backslash X]\}$ and for a p-subgroup P of G.
2. $(kX)_P^G$ has a basis $\{[x]_G^+ : x\in[G\backslash X]$ and a Sylow subgroup S of $C_G(x)$ is contained in P, up to G-conjugacy$\}$.

Proof. Let us denote a set of representatives of the G-orbits in X by $[G\backslash X]$. Also, $C_G(y)$ shall denote the stabilizer $\mathrm{Stab}_G(y)$ in G of y and $[y]_G^+$ shall denote the

orbit sum $\sum_{g\in[G/C_G(y)]} g\cdot y$. Now, given a permutation kG-module kX,

$$\begin{aligned}
(kX)^G &= \{\sum_{x\in X}\lambda_x x : \sum_{x\in X}\lambda_x g\cdot x = \sum_{x\in X}\lambda_x x \text{ for all } g\in G\} \\
&= \{\sum_{y\in X}\lambda_y y : \sum_{y\in X}\lambda_{g^{-1}\cdot y} y = \sum_{y\in X}\lambda_y y \text{ for all } g\in G\} \qquad \text{by putting } y := g\cdot x \\
&= \{\sum_{y\in X}\lambda_y y : \lambda_{g^{-1}y} = \lambda_y \text{ for all } g\in G\} \qquad \text{as } X \text{ is free over } k \\
&= \{\sum_{y\in[G\backslash X]}\lambda_y [y]_G^+\}
\end{aligned}$$

Hence, the orbit sums of G-orbits on X form a k-basis for $(kX)^G$. For the other statement, note that P also acts on X by dividing the G-orbits into smaller parts: $[x]_G = \bigsqcup_{j=1}^{n(x)}[x_j]_P$ for some $x_j\in[P\backslash X]$. Then, $[x]_G^+ = \sum_{j=1}^{n(x)}[x_j]_P^+ = [x_j]_G^+$ denotes the relation between their orbit sums. Note that by the Orbit-Stabilizer Theorem, we have $|C_G(x)| = \dfrac{|G|}{|[x]_G|} = \dfrac{|G|}{|[x_j]_G|} = |C_G(x_j)|$ for each j. Now, we shall apply t_P^G on the set $(kX)^P = k\{[x_j]_P^+\} = k\{t^P_{C_P(x_j)}(x_j)\}$, to obtain,

$$\begin{aligned}
t_P^G((kX)^P) &= k\{t_P^G t^P_{C_P(x_j)}(x_j)\} \\
&= k\{t^G_{C_P(x_j)}(x_j)\} \\
&= k\{t^G_{C_G(x_j)} t^{C_G(x_j)}_{C_P(x_j)}(x_j)\} \\
&= k\{|C_G(x_j) : C_P(x_j)|.t^G_{C_G(x_j)}(x_j)\} \\
&= k\{|C_G(x_j) : C_P(x_j)|.[x_j]_G^+\} \\
&= k\{[x_j]_G^+ : p\nmid \frac{|C_G(x_j)|}{|C_P(x_j)|},\ 1\le j\le n(x)\} \\
&= k\{[x]_G^+ : p\nmid \frac{|C_G(x)|}{|C_P(x_j)|},\ 1\le j\le n(x)\} \\
&= k\{[x]_G^+ : C_P(x_j)\in \mathrm{Syl}_p(C_G(x)), 1\le j\le n(x)\} \\
&\subseteq k\{[x]_G^+ : \text{a Sylow subgroup of } C_G(x) \text{ lies in } P\} \\
&\subseteq k\{[x]_G^+ : \text{a Sylow subgroup of } C_G(x) \text{ lies in } P, \text{ up to } G\text{-conjugacy}\}
\end{aligned}$$

Therefore, $t_P^G((kX)^P)$ is spanned by the last set. As the special orbits sums $[x]_G^+$ in the last set are linearly independent over k, they indeed form a k-basis for $t_P^G((kX)^P) = (kX)_P^G$. □

Lemma 3.9. *Given a field k of characteristic p, a p-permutation kG-module*

$M = kX$ with some G-invariant basis X and a p-subgroup P of G, then the equality $(kX)^P = k(X)^P \bigoplus (kX)^P_{<P}$ holds, i.e., the Brauer quotient $M(P) = (kX)(P)$ is isomorphic to the span kX^P of the P-fixed points of X, as a p-permutation $kN_G(P)$-module.

Proof. First assume that G is any finite group. By the previous lemma, $(kX)^G$ has basis $\{[y]^+_G : y \in [G\backslash X]\}$. Among these basis elements $[y]^+_G$, the ones having size 1 clearly belong to $k(X^G)$ and the ones having size > 1 have $C_G(y) < G$, so that $[y]^+_G = \sum_{g\in[G/C_G(y)]} g \cdot y \in \sum_{K<G} t^G_K((kX)^K) = kX^G_{<G}$. Hence, $(kX)^G = k(X^G) + kX^G_{<G}$. If we assume $G = P$ is a p-group and k has characteristic p, then the intersection is zero as follows: Suppose not, and say that a basis element y with P-orbit size 1 belongs to the intersection. Then, $y = \sum_i \lambda_i t^P_{K_i}(a_i)$ for some $K_i < P$ and $a_i \in (kX)^{K_i}$. In this equation, the coefficient of y should be equal on the left-hand side and right-hand side. However, since P is a p-group, $t^P_{K_i}(y) = |P : K_i|y = 0$ in kX, so that, the coefficient of y on the right-hand side can only be zero. □

We have seen that the action of br_P on a permutation kP-module just truncates the support of an element to kX^P.

Corollary 3.10. *If Q and P are p-subgroups of G such that $Q \le P$, we have $br_P br_Q = br_P$.*

Proof. X^P is a subset of X^Q, and so, truncating the support first to X^Q and then to X^P has the same effect as truncating the support diretly to X^P. □

When ${}^gH \le K$ holds for some $g \in G$, we shall briefly denote this situation as $H \le_G K$.

Lemma 3.11. *If $H \le G$ and P is a p-subgroup of H, then* $\ker(\mathrm{br}_P) \cap (kX)^H = \sum_{Q\in\mathcal{R}}(kX)^H_Q$, *where $\mathcal{R} = \{Q : P \not\leq_H Q \le H\}$.*

Proof. ($\supseteq$) Let $Q \in \mathcal{R}$ be chosen. By Lemma 3.8, $(kX)^H_Q$ has k-basis, $\{[x]^+_H : x \in [H\backslash X] :$ there is S such that $Q \ge_H S \in \mathrm{Syl}_p(C_H(x))\}$. Now, suppose, for a

contradiction, that $[x]_H^+ = \sum_{h\in[H/C_H(x)]} {}^h x$ does not lie in the kernel $\ker(\mathrm{br}_P)$ for some $x \in [H\backslash X]$. Then, for some h, we have ${}^h x \in X^P$, which is equivalent to saying that $P \leq C_H({}^h x)$. Then, by Lemma 3.6, ${}^{h^{-1}}P \leq C_H(x)$, i.e. $P \leq_H C_H(x)$. Then, by the original Sylow Theorem, $P \leq_H S$ for some Sylow p-subgroup S of $C_H(x)$ and for this Sylow p-subgroup S, we have that $P \leq_H S \leq_H Q$, contradicting with the hypothesis.

($\subseteq$) Choose an element $a \in \ker(\mathrm{br}_P) \cap (kX)^H$ and write $a = \sum_{x\in X} \lambda_x x$. We shall proceed by induction on the size of the support of a, i.e. on the cardinality of the set $\{x : \lambda_x \neq 0\}$. If this cardinality is 0, then $a = 0$ and the basis step is complete. Now, for any basis element x in the support of a, we must have $x \notin X^P$, because otherwise a part of x would survive under the action of br_P, contradicting that $a \in \ker(\mathrm{br}_P)$. The statement $x \notin X^P$ is equivalent to $P \not\leq C_H(x)$. Also, for any $h \in H$, as a is fixed by H, the equality $\lambda_x x + \ldots = a = {}^h a = \lambda_x {}^h x + \ldots$ shows that ${}^h x$ is in the support of a as well (with the same coefficient as that of x), so that $P \not\leq C_H({}^h x)$, or equivalently, ${}^{h^{-1}}P \not\leq C_H(x)$. Since this holds for all $h \in H$, we have that $P \not\leq_H C_H(x)$. Now, if we put $Q := C_H(x)$, note that $Q \in \mathcal{R}$ and $[x]_H^+ = t^H_{C_H(x)}(x) = t^H_Q(x) \in (kX)^H_Q$. On the other hand, the support of $a - \lambda_x [x]_H^+$ is strictly less than that of a, so that by the inductive hypothesis, the rest of a lies on the right-hand side as well. □

A G-algebra A is called a *p-permutation G-algebra* if it has a p-permutation kG-module structure at the same time as it has a G-algebra structure given by the same G-action. From now on, we always assume that A is an interior p-permutation G-algebra over an algebraically closed field k of characteristic p.

At the present, we are ready to define the main protagonist of the story, that is, the Brauer pair. Given an interior p-permutation G-algebra A over an algebraically closed field k of characteristic p, a *Brauer pair* (P, c) on A is a pair of a p-subgroup P of G and a nonzero block c of $A(P)$. The set of Brauer pairs on A shall be denoted $\mathfrak{Br}(A)$. We define an action of G on $\mathfrak{Br}(A)$ by ${}^g(P, c) := ({}^g P, {}^g c)$. To see that this action is well-defined, note that the conjugate group ${}^g P$ is also a p-subgroup of G and ${}^g c$ is a block of ${}^g(A(H)) = A({}^g H)$ as follows: As c_g is an automorhism, a is central in B if and only if ${}^g a$ is central in ${}^g B$ for a subalgebra B of A. Also, an idempotent e is primitive in B if and only if ${}^g e$ is primitive

in ${}^gB$. The same facts hold for the induced action of G on quotient algebras of subalgebras of A, in particular, on $A(H)$.

We define a poset structure on Brauer pairs as follows: Write $(Q,d) \lhd (P,c)$ when $Q \trianglelefteq P$ and $d \in A^P$ and $c \leq \mathrm{br}_P(d)$. This relation is clearly reflexive and anti-symmetric. So, the transitive closure of $\lhd$ gives a poset structure $\leq$ on Brauer pairs. We write $(Q,d) < (P,c)$ if $(Q,d) \lhd (P,c)$ and $Q \neq P$. To show that $\mathfrak{Br}(A)$ is a G-poset, let $(Q,d) \lhd (P,c)$ be given. The first sentence in the definition above, $Q \lhd P$ means that $P \leq N_G(Q)$, which implies that ${}^gP \leq {}^gN_G(Q) = N_G({}^gQ)$, so that ${}^gQ \lhd {}^gP$ holds. The second sentence, $d \in A^P$ implies that ${}^gd \in {}^g(A^P) = A^{{}^gP}$. Finally, the third sentence $c \leq \mathrm{br}_P(d)$ implies that $c_g(c) \leq c_g\mathrm{br}_P(d) = \mathrm{br}_{{}^gP}({}^gd)$. Therefore, we obtain ${}^g(Q,d) \lhd {}^g(P,c)$ and $\mathfrak{Br}(A)$ becomes a G-poset. We now show that it is actually a refinement of the p-subgroup poset of G.

Lemma 3.12. *Given $Q \lhd P$ and a block idempotent e of $A(P)$, there is a unique block d of $A(Q)$ such that $(Q,d) \lhd (P,c)$.*

Proof. Firstly, for $x \in P$ we have that $\mathrm{br}_P({}^xa) = {}^x\mathrm{br}_P(a) = \mathrm{br}_P(a)$ where, the first equality follows from the fact that br_P is an $N_G(P)$-algebra homomorphism and the second follows from the fact that $\mathrm{br}_P(a) \in A^P$. As Q is normal in P, we have that ${}^x(A^Q) = A^{({}^xQ)} = A^Q$ meaning that P acts on the blocks $\mathcal{B}(Q)$ of A^Q. Using the (unique) decomposition of 1 into block idempotents $\mathcal{B}(Q)$,

$$1 = \mathrm{br}_P(1) = \sum_{b \in \mathcal{B}(Q)} \mathrm{br}_P(b) = \sum_{b \in [P \backslash \mathcal{B}(Q)]} \sum_{[P/C_P(y)]} \mathrm{br}_P(b)$$

As P is a p-group and k is of char p, only the fixed elements in this sum survive, so that, we have $1 = \mathrm{br}_P(1) = \sum_{b \in \mathcal{B}(Q)^P} \mathrm{br}_P(b)$. Multiplying this equality by c, we obtain $c = c.1 = \sum_{b \in \mathcal{B}(Q)^P} c.\mathrm{br}_P(b)$. In this last equation, there must be at least one b such that $c.\mathrm{br}_P(b) \neq 0$, call that idempotent d. Given a ring epimorphism $f : R \to S$, it is easily seen that $f(Z(R)) \subseteq Z(S)$. In our case, br_P is surjecive, so $\mathrm{br}_P(d)$ is central in $\mathrm{im}(\mathrm{br}_P) = kX^P$. Let $\mathrm{br}_P(d) = d_1 + \ldots + d_n$ be a primitive decomposition of $\mathrm{br}_P(d)$ in $Z(kX^P)$. Then, $0 \neq \mathrm{br}_P(d)c = d_1c + \ldots + d_nc$ implies that there is some i such that $d_ic \neq 0$. But then, the orthogonal decompositions $d_i = d_ic + d_i(1-c)$ and $c = cd_i + c(1-d_i)$ are only possible if $c = d_ic = d_i$ by the primitivity of c and d_i in $Z(kX^P)$. Therefore, $\mathrm{br}_P(d)c = d_ic = c$. For uniqueness, suppose that $c \leq \mathrm{br}_P(d')$

for some block $d' \neq d$. Then, $\mathrm{br}_P(d') = \mathrm{br}_P(d')c = \mathrm{br}_P(d')\mathrm{br}_P(d)c = \mathrm{br}_P(d'd)c = 0$, as different blocks are orthogonal to each other. □

It appears that the G-poset of Brauer pairs on A does refine the poset of p-subgroups of G, but not so much: If Q is normal in P, there can only one Brauer pair (Q, d) under (P, c) for each block c. Actually, it turns out that we do not need the hypothesis $Q \lhd P$ here, this is true for all Brauer pairs: For each Brauer pair (P, c), and for each subgroup $Q \leq P$, there is a unique block d of $A(Q)$ such that $(Q, d) \leq (P, c)$. We shall show this, but we need to be careful that there might be more than one normal chain of subgroups under a group. First, we start with the following lemma:

Lemma 3.13. *$\lhd$ is transitive between subgroups $R \lhd Q \lhd P$ such that $R \lhd P$, i.e. for such subgroups, $(R, e) \lhd (Q, d) \lhd (P, c)$ implies $(R, e) \lhd (P, c)$.*

Proof. First, we show that e is stable under P. Let $x \in P$ be chosen. We claim that $(R, {}^x e) \lhd (P, c)$ holds, which implies, by the uniqueness property of e, that $e = {}^x e$. For this purpose, we check the three properties in the definition of $(R, {}^x e) \lhd (P, c)$:
1. $R \lhd P$ is given by the hypothesis.
2. $d \leq \mathrm{br}_Q({}^x e)$: By the definition of $(Q, d) \lhd (P, c)$, we know that d is fixed under P and by the definition of $(R, e) \lhd (Q, d)$, we have $d \leq \mathrm{br}_Q(e)$. Hence, we check that $d = {}^x d = {}^x(d\mathrm{br}_Q(e)) = {}^x d\, {}^x\mathrm{br}_Q(e) = d\mathrm{br}_Q({}^x e)$, where we used the fact that br_Q is an $N_G(Q)$-algebra homomorphism.
3. ${}^x e$ is fixed under Q: Conjugating the inclusion $Q \subseteq C_G(e)$ by $x \in P$, we get $Q = {}^x Q \subseteq C_G({}^x e)$ by Lemma 3.6. Hence, we obtain ${}^x(R, e) = ({}^x R, {}^x e) = (R, {}^x e) \lhd (Q, d)$. Then, by the uniqueness property of e, we conclude that ${}^x e = e$ and stability under P is established. Lastly, to show that $c \leq \mathrm{br}_P(e)$, we note, $c = c\mathrm{br}_P(d) = c\mathrm{br}_P(d\mathrm{br}_Q(e)) = c\mathrm{br}_P(d)\mathrm{br}_P(\mathrm{br}_Q(e)) = c\mathrm{br}_P(e)$ by Corollary 3.10. □

By a repeated application of Lemma 3.13, we get the equivalence: $(Q, d) \lhd (P, c)$ if and only if $(Q, d) \leq (P, c)$ and $Q \lhd P$.

Theorem 3.14. *For each Brauer pair (P, c), and for each subgroup $Q \leq P$, there exists a unique block d of $A(Q)$ such that $(Q, d) \leq (P, c)$.*

Proof. The case $Q \lhd P$ is covered in Lemma 3.12 and the remark above. So, assume that $N_P(Q) < P$.

For the existence statement, we consider the chain of growing normalizers for a p-group: $Q \lhd N_P(Q) \lhd \ldots \lhd P$. Inductively, we can find blocks at each level such that $(Q,d) \lhd (N_P(Q),e) \lhd \ldots \lhd (P,c)$ holds, which means, by the definition of $\leq$, that $(Q,d) \leq (P,c)$.

For uniqueness, suppose that $(Q,d') = (Q_0,d_0) \lhd (Q_1,d_1) \lhd \ldots \lhd (Q_{n-1},d_{n-1}) \lhd (Q_n,d_n) = (P,c)$ is another chain. We can assume that inclusions here are proper. We apply strong induction on n, where $|Q : P| = p^n$. So, assume that for all $k < n$ and for all Brauer pairs (R,e) and for all p-subgroups T of R such that $|T : R| = p^k$, there is a unique Brauer pair $(T,u) \leq (R,e)$. The basis step, $n = 1$ is covered by Lemma 3.12 already. For $n > 1$, note that by the chain of inclusions $Q \subsetneq Q_1 \leq N_P(Q) \subsetneq P$, we have the strict inequality $|N_P(Q) : Q_1| < |P : Q|$ so that by the inductive hypothesis, d_1 is the unique idempotent such that $(Q_1,d_1) \leq (N_P(Q),e)$. Then, by the transivity of $\leq$, we obtain $(Q,d') \leq (N_P(Q),e)$. Similarly, we have $|N_P(Q) : Q| < |P : Q|$ so that by the inductive hypothesis, d is the unique idempotent such that $(Q,d) \leq (N_P(Q),e)$. Hence, $d = d'$. □

Definition 3.15. Given a Brauer pair (P,c) on A and a block b of A, (P,c) is said to be *associated to* b if $(1,b) \leq (P,c)$.

By Theorem 3.14, the blocks b of A partition the poset of Brauer pairs on A: Each Brauer pair is associated to a unique block b.

It is discovered that a technique of assigning a *defect group* to each block Ab of a G-algebra $A = \bigoplus_b Ab$ helps understanding the structure of that block. Defect groups are defined using different tools in different settings. In the theory of Brauer pairs, defect groups correspond to maximal Brauer pairs associated to a block. We call (D,c) a *maximal Brauer pair associated to* b provided $|D|$ is maximal such that $(1,b) \leq (D,c)$ for some c. In this situation, D is called a *defect group* of b (briefly, a *defect* of b). We now give four additional equivalent ways of defining a defect of a block (Note that, for the following lemma, the observations $\mathrm{br}_1(b) = b \neq 0$, $b = t_G^G(b) \in A_G^G$ and $(1,b) \leq (1,b)$ ensure that the items (1), (2) and

(4) make sense):

Lemma 3.16. *Given a p-permutation G algebra A, a block b of A and a p-subgroup D of G, the following are equivalent:*
1- $|D|$ is maximal such that $(1,b) \leq (D,c)$ for some c (D is a defect of b).
2- $|D|$ is maximal such that $br_D(b) \neq 0$.
3- For all P, $P \leq_G D$ if and only if $br_P(b) \neq 0$.
4- $|D|$ is minimal such that $b \in A_D^G$
5- For all H, $D \leq_G H$ if and only if $b \in A_H^G$.

Proof. (1)⇒(2): By definition, $c \leq \mathrm{br}_D(b)$, so that we know $\mathrm{br}_D(b) \neq 0$. Also, for any p-subgroup P, we know that $b \in Z(A) \subseteq Z(A^P)$. Now, given a ring epimorphism $f : R \to S$, it is easy to see that $f(Z(R)) \subseteq Z(S)$. Therefore, as $\mathrm{br}_P : A^P \to A(P)$ is a surjection, we have $\mathrm{br}_P(Z(A^P)) \subseteq Z(A(P))$, in particular, $\mathrm{br}_P(b)$ falls in the center of $A(P)$. So, assume that $\mathrm{br}_P(b)$ is non-zero and let $\mathrm{br}_P(b) = c_1 + \ldots + c_n$ be a block decomposition of b in $A(P)$. Then, for some c_i, we have $c_i \leq \mathrm{br}_P(b)$, in particular, $(1,b) \leq (P,c_i)$, which implies, by the maximality of $|D|$, that $|P| \leq |D|$.

(2)⇒(1): The proof of this implication is very similar to that of (1) ⇒ (2).

(4)⇒(5): Let H be a subgroup such that $b \in A_H^G$. Then, letting $b = t_D^G(a) = t_H^G(b)$ for some $a \in A^D$ and $b \in A^H$, we have, $b = b^2 = t_D^G(a)t_H^G(b) = \sum_{g \in [D\backslash G/H]} t_{{}^gD\cap H}^G({}^gab) \in \sum_{g \in [D\backslash G/H]} A_{{}^gD\cap H}^G$. Then, by Rosenberg's Lemma, there is some g such that $b \in A_{{}^gD\cap H}^G$. By the minimality of $|D|$, we get $|{}^gD| = |D| \leq |{}^gD \cap H|$, which implies ${}^gD = {}^gD \cap H$, which implies ${}^gD \subseteq H$. Conversely, let H be a subgroup such that $D \leq_G H$, i.e. there is some subgroup E such that $D =_G E \leq H$. Then, by Lemma 3.3.3, we obtain $b \in A_D^G = A_E^G$. But also, $A_E^G = t_E^G(A^E) = t_H^G t_E^H(A^E) = t_H^G(A_E^H) \subseteq t_H^G(A^H) = A_H^G$, therefore, $b \in A_H^G$ as well.

(5)⇒(4): Trivial

(2)⇒(4): Let $|E|$ be minimal such that $b \in A_E^G$. By the implication "(4)→(5)" we have proven already, E satisfies, $E \leq_G H$ if and only if $b \in A_H^G$ for all H. By hypothesis, $\mathrm{br}_D(b) \neq 0$, so by Lemma 3.11 (putting G instead of H), $b \notin \sum_{D \not\leq_G Q \leq G} A_Q^G$. Then, as $b \in A_E^G$, we must have $D \leq_G E$. Now, suppose, for a

contradiction, that $\mathrm{br}_E(b) = 0$. Then, by Lemma 3.11 again, $b \in \sum_{E \nleq_G Q \leq G} A_Q^G$, so that by Rosenberg's Lemma, $b \in A_Q^G$ for some Q such that $E \nleq_G Q$, which contradicts with the property E satisfies. Hence, $\mathrm{br}_E(b) \neq 0$, which implies, by the hypothesis, that $|E| \leq |D|$, so that $E =_G D$ and $b \in A_E^G = A_D^G$ by Lemma 3.3.3. Also, for any D_1 such that $b \in A_{D_1}^G$, we have $|E| \leq |D_1|$ by the property of E, in particular, $|D| = |E| \leq |D_1|$. Therefore, $|D|$ is minimal such that $b \in A_D^G$.

(5)⇒(2): Suppose, for a contradiction, that $\mathrm{br}_D(b) = 0$. Then, by Lemma 3.11, $b \in \sum_{D \nleq_G Q \leq G} A_Q^G$, so that, by Rosenberg's Lemma, $b \in A_Q^G$ for some Q such that $D \nleq_G Q$, in contradiction with the hypothesis. Therefore, $\mathrm{br}_D(b)$ must be non-zero. Now let P be any subgroup such that $\mathrm{br}_P(b) \neq 0$. If $P \nleq_G D$, then, using the hypothesis, we get $b \in \sum_{P \nleq_G Q \leq G} A_Q^G$, so that, by Lemma 3.11, $b \in \ker(\mathrm{br}_P)$, a contradiction. Hence, $P \leq_G D$, in particular, $|P| \leq |D|$.

(2)⇒(3): By the implication (2)→(4), we know that $|D|$ is minimal such that $b \in A_D^G$. Let P be a subgroup such that $P \leq_G D$. Suppose, for a contradiction, that $\mathrm{br}_P(b) = 0$. Then, by Lemma 3.11, $b \in \sum_{P \nleq_G Q \leq G} A_Q^G$, so that, by Rosenberg's Lemma, $b \in A_Q^G$ for some Q that satisfies $P \nleq_G Q$. If it were the case that $D \nleq_G Q$, then $b \in \sum_{D \nleq_G Q \leq G} A_Q^G$, which implies, by Lemma 3.11, that $b \in \ker(\mathrm{br}_D)$, in contradiction with the hypothesis. Hence, $D \leq_G Q$, so that we obtain $P \leq_G D \leq_G Q$, another contradiction. Hence $\mathrm{br}_P(b) \neq 0$. Conversely, assume that $\mathrm{br}_P(b) \neq 0$. Suppose, for a contradiction, that $P \nleq_G D$. Then, $b \in \sum_{P \nleq_G Q \leq G} A_Q^G$, which implies, by Lemma 3.11, that $b \in \ker(\mathrm{br}_P)$, a contradiction. Hence, we have $P \leq_G D$.

(3)⇒(2): Trivial □

Corollary 3.17. *Defect groups of a block constitute a single orbit under the action of G.*

Proof. Suppose that D and E are two defect groups of a block b. By the previous lemma, they satisfy,
$P \leq_G D$ if and only if $b \in A_P^G$ and,
$P \leq_G E$ if and only if $b \in A_P^G$
Substituting D for P in the second sentence, we get that $D \leq_G E$. Substituting E for P in the first sentence, we get $E \leq_G D$. Therefore, we obtain $D =_G E$.

Conversely, if D is a defect of b and E is a G-conjugate of D, then it is easy to see that E satisfies the fifth condition of the previous lemma. □

Our definition of a *maximal b-Brauer pair* does not directly reflect maximality with respect to the relation $\leq$. However, according to the following lemma, our definition actually captures the common sensical property of being maximal with respect to $\leq$, for a Brauer pair (P, c).

Lemma 3.18. *Given a p-subgroup P of G, then $|P|$ is maximal such that $(1, b) \leq (P, c)$ for some c (P is a defect of b) if and only if $(1, b) \leq (P, c)$ holds maximally for all c.*

Proof. ($\Rightarrow$): If $(1, b) \leq (P, c) \leq (Q, d)$ such that $(P, c) \neq (Q, d)$ then $P \leq Q$ and $|Q| \leq |P|$, so that $P = Q$, a contradiction.

($\Leftarrow$): Conversely, let D be a defect of b, and choose some non-zero $c \in \mathcal{B}(A(P))$ such that $(1, b) \leq (P, c)$. Note that $\mathrm{br}_P(b) \geq c \neq 0$, so that by Lemma 3.16.3, we have $P \leq_G D$, in particular, $|P| \leq |D|$. Also, as $|D|$ is maximal such that $(1, b) \leq (D, e)$ for some e, we obtain that $|D| \leq |P|$. Therefore, $|D| = |P|$ and $P =_G D$, i.e. P is a defect of b. □

Hence, it turns out that the chains of b-Brauer pairs are never truncated early with respect to their cardinality.

The subgroup of G stabilizing a Brauer pair (P, c) is $N_G(P) \cap C_G(c)$. We shall denote this subgroup as $N_G(P, c)$.

Theorem 3.19. *Let b be a block of an interior p-permutation G-algebra A, D be a defect of b and P be a normal p-subgroup of G. Then, we have the following facts:*
1. $b \in [A(P)]^G$
2. $P \leq_G D$ and $\mathrm{br}_P(b) \neq 0$.
3. If P is a defect of b, then G acts transitively on the maximal Brauer pairs (P, b_i), associated to b.

Proof. 1. Firstly, we show that $\ker(\mathrm{br}_P)$ is contained in $J(A)$. Let S be any simple A-module. As A is an interior G-algebra, there is a group homomorphism $\psi : G \to A^\times$. We can linearly extend this map to a k-algebra homomorphism $kG \to A$, thereby making S into a simple kG-module. By Lemma 2.44, P acts trivially on S, i.e. for all $s \in S$, we have $\psi(p) \cdot s = s$ by definition of the action of kG on S. Now, for all proper subgroups Q of P, we can show that A_Q^P annihilates S: Let $t_Q^P(a)$ be an element of A_Q^P and let $s \in S$. Then, $t_Q^P(a) \cdot s = \sum_{p \in [P\backslash Q]} {}^p a \cdot s = \sum_{p \in [P\backslash Q]} [\psi(p) a \psi(p^{-1})] \cdot s = \sum_{p \in [P\backslash Q]} a \cdot s = |P : Q| \cdot s = 0$ as p divides the index $|P : Q|$. As S and Q were selected arbitrarily, we obtain $\ker(\mathrm{br}_P) = \sum_{Q<P} A_Q^P \subseteq J(A)$. As A is a p-permutation algebra over k, Lemma 3.9 applies, so that br_P acts as a projection, inducing a direct sum decomposition, $A^P = A(P) \oplus A_{<P}^P = \mathrm{im}(\mathrm{br}_P) \oplus \ker(\mathrm{br}_P)$. As shown before, using the fact that A is an interior G-algebra, we have $Z(A) \subseteq A^G \subseteq A^P$, so let b be a block of A and write $b = x + y$ such that $x \in \mathrm{im}(\mathrm{br}_P)$ and $y \in \ker(\mathrm{br}_P) \subseteq J(A)$, where y is a nilpotent element, say, with order smaller than p^n for a natural number n. As shown before, using the surjectivity of br_P, we get that $\mathrm{br}_P(b) = x$ is in the center of $A(P)$. Using the Binomial Theorem, the idempotency of b and the centrality of x, we obtain, $b = b^{p^n} = (x + y)^{p^n} = \sum_{k=0}^{p^n} \binom{p^n}{k} x^k y^{(p^n - k)} = x^{p^n} + y^{p^n}$ where, the last step follows as we work in characteristic p. Since y is nilpotent, we get that $b = x^{p^n} \in A(P)$. Lastly, x is G-fixed because for any $g \in G$,

$$
\begin{aligned}
x &= \mathrm{br}_P(x) + \mathrm{br}_P(y) && \text{as } \mathrm{br}_P \text{ is a projection and } y \in \ker(\mathrm{br}_P) \\
&= \mathrm{br}_P(b) \\
&= \mathrm{br}_P({}^g b) && \text{as } b \text{ is } G\text{-fixed} \\
&= {}^g\mathrm{br}_P(b) && \text{as } \mathrm{br}_P \text{ is } N_G(P) = G\text{-algebra homomorphism} \\
&= {}^g\mathrm{br}_P(x) + {}^g\mathrm{br}_P(y) \\
&= {}^g x && \text{as } \mathrm{br}_P \text{ is a projection and } y \in \ker(\mathrm{br}_P)
\end{aligned}
$$

2. As D is a defect of b, we have that $b \in A_D^G$, which has a k-basis, $\{[x]_G^+ : x \in [G\backslash X] \text{ and } C_G(x) \text{ has a Sylow subgroup } S \text{ such that } S \leq_G D\}$ by Lemma 3.8. By the first part, $b \in A(P)^G = [k(X^P)]^G$, which has a k-basis, $\{[x]_G^+ : x \in [G\backslash X^P]\}$ by Lemma 3.8 again. As P is normal in G, the subset X^P of X is also G-invariant, so that we can choose representatives that satisfy $[G\backslash X^P] \subseteq [G\backslash X]$. Using this, note that the elements $\{[x]_G^+ : x \in [G\backslash X]\}$ are linearly independent, so that the supports of b in the two bases above must intersect, meaning that there is some

$[x]_G^+$ supporting b such that $x \in X^P$ and $C_G(x)$ has a Sylow subgroup S such that $S \leq_G D$. As x is fixed by P, we get that $P \leq C_G(x)$, from which we obtain $P \leq_G S$ by the original Sylow Theorem. Therefore, $P \leq_G S \leq_G D$, in particular, $P \leq_G D$. Then, $\mathrm{br}_P(b) \neq 0$ follows by Lemma 3.16.3.

3. Assuming P is a defect of b, let $b = b_1 + \ldots + b_n$ be the decomposition of b into blocks b_i of $A(P)$. By the first part, $\mathrm{br}_P(b) = b \geq b_i$ and $b \in A^P$, so that we get $(1, b) \leq (P, b_i)$, i.e. the Brauer pairs (P, b_i) are associated to b. As P is a defect of b, these Brauer pairs are maximal Brauer pairs associated to b. Are they all of the maximal Brauer pairs associated to b? Yes, indeed, if (Q, d) is any maximal b-Brauer pair, then Q is a defect of b and it is G-conjugate to P, by Corollary 3.17. As P is normal in G, we have $P = Q$. Then, $b = \mathrm{br}_Q(b) \geq d$ implies that $b_i = d$ for some i, as the block decomposition of b is unique. Now, to show that G acts transitively on (P, b_i), it shall be enough to show that G acts transitively on b_i. Let $H_i := N_G(P, b_i)$ and choose a set of representatives $[G/H_1] = \{g_1, \ldots, g_n\}$ of H_1 in G. As b is G-fixed, we have ${}^{g_i}b = b$, i.e. ${}^{g_i}b_1 + \ldots + {}^{g_i}b_n = b_1 + \ldots + b_n$. The action of G as automorphisms preserves orthogonality, primitivity and centrality of idempotents, so that the left-hand side of this equation is a block decomposition. By the uniqueness of block decomposition, ${}^{g_i}b_1 = b_i$ for some i. Each g_i sends b_1 to different b_i, because if ${}^{g_i}b_1 = {}^{g_j}b_1$, then we get ${}^{g_j^{-1}g_i}b_1 = b_1$, i.e. $g_j^{-1}g_i \in H_1$, and so $g_i = g_j$ by the definition of $[G/H_i]$. Therefore, we see that G acts transitively on b_i and in particular, $b = b_1 + \ldots + b_n = \sum_{g_i \in [G/H_1]} {}^{g_i}b_1 = t_{H_1}^G(b_1)$. By similar arguments, $b = t_{H_j}^G(b_j)$ for each j. □

We have seen that, given an interior p-permutation G-algebra A and a block b of A, if the defect P of b is normal in G, then the maximal Brauer pairs associated to b are G-conjugate in the poset of Brauer pairs associated to b. In other words, they assume the role of Sylow p-subgroups S of G. In fact, if $A = kG$, then the condition that $P \lhd G$ is not necessary: For each block b of kG, the maximal Brauer pairs associated to b are G-conjugate. However, for an interior p-permutation G-algebra A, this is not necessarily true. [AKO11] contains a treatment of the general case: Theorem 2.20a in [AKO11] states that given a p-permutation G-algebra A, then the maximal Brauer pairs on A associated to a primitive idempotent f in A^G are G-conjugate. For this purpose, we need further machinery.

Definition 3.20. Given a Brauer pair (P, c) on A, a primitive idempotent i of A^P is associated to (P, c) provided $0 \neq \mathrm{br}_P(i) \leq c$, or equivalently, $\mathrm{br}_P(i)c \neq 0$.

Note: To see this equivalence, we refer to the Lifting Theorem of Idempotents [Kes+19, Theorem 6.18], part of which goes, if $\varphi : A \to B$ is surjective homomorphism of finite dimensional k-algebras, and $e \in A$ is a primitive idempotent, then $\varphi(e)$ is either zero or primitive in B. Now, to see the equivalence, first assume that $\mathrm{br}_P(i)c$ is non-zero. Then, in particular, $\mathrm{br}_P(i)$ non-zero, and so, it must be a primitive idempotent of $A(P)$, by the Lifting Theorem of Idempotents. Then, by Lemma 2.25, there is a unique idempotent that $\mathrm{br}_P(i)$ belongs to, and that idempotent must be c, considering the hypothesis. The converse implication is clear.

For each Brauer pair (P, c), we can find a primitive idempotent $i \in A^P$ associated to (P, c): Let $1_A = \sum_{i \in I} i$ be a primitive decomposition of 1_A in A^P. Applying br_P and multiplying by c, we get $0 \neq c = c\mathrm{br}_P(1_A) = \sum_{i \in I} c\mathrm{br}_P(i)$, which shows that there must be at least one $i \in I$ such that $c\mathrm{br}_P(i) \neq 0$, and that i is associated to c.

Conversely, for each primitive idempotent $i \in A^P$ such that $\mathrm{br}_P(i) \neq 0$, there exists a unique Brauer pair (P, c) such that i is associated to (P, c). Again, this follows by the Lifting Theorem for Idempotents and Lemma 2.25.

We shall need the following additional characterization of the inclusion relation of Brauer pairs:

Lemma 3.21. *Given two Brauer pairs (Q, d) and (P, c) such that $Q \leq P$, the following are equivalent:*
1. $(Q, d) \leq (P, c)$
2. For a primitive idempotent $i \in A^P$ associated to (P, c), i is also associated to (Q, d).
3. There exist primitive idempotents $i \in A^P$ associated to (P, c) and $j \in A^Q$ associated to (Q, d) such that $j \leq i$.

Proof. (1)⇒(2): By induction, it is enough to show the statement for a normal pair of Brauer pairs, $(Q, d) \lhd (P, c)$: Let $i \in A^P$ be an idempotent associated to (P, c), i.e. let $0 \neq \mathrm{br}_P(i) \leq c$. Suppose, for a contradiction, that $\mathrm{br}_Q(i)d = 0$. Then, applying

br_P on the last equation, we get $\mathrm{br}_P(i)\mathrm{br}_P(d) = 0$. But then, using the fact that $\mathrm{br}_P(d).c = c$, we obtain, $\mathrm{br}_P(i) = \mathrm{br}_P(i)c = \mathrm{br}_P(i)(\mathrm{br}_P(d)c) = (\mathrm{br}_P(i)\mathrm{br}_P(d))c = 0$, a contradiction.

(2)⇒(3): Choose an idempotent $i \in A^P$ associated to (P, c). By hypothesis, we know that i is also associated to d, in particular, that $d.\mathrm{br}_Q(i) \neq 0$. Let $i = \sum_{j\in J} j$ be a primitive decomposition of i in A^Q. Applying br_Q and multiplying by d, we get $0 \neq d.\mathrm{br}_Q(i) = \sum_{j\in J} d.\mathrm{br}_Q(j)$, which shows that there is at least one $j \in A^Q$ such that $d.\mathrm{br}_Q(j) \neq 0$, i.e. j is associated to d. Lastly, $j \leq i$ is true by construction.

(3)⇒(1): Let (Q, d_0) be the unique Brauer pair such that $(Q, d_0) \leq (P, c)$. We aim to show that $d = d_0$. By hypothesis, there are some $i \in A^P$ associated to (P, c) and $j \in A^Q$ associated to (Q, d) such that $j = ij$. Apply br_Q to the last equation to get $\mathrm{br}_Q(j) = \mathrm{br}_Q(j).\mathrm{br}_Q(i)$. Since j is associated to d, we have $0 \neq d.\mathrm{br}_Q(j) = d.\mathrm{br}_Q(j).\mathrm{br}_Q(i)$, in particular, $0 \neq d.\mathrm{br}_Q(i)$, i.e. i is associated to d. By the implication "(1)⇒(2)", we know that i is also associated to d_0. If $d \neq d_0$ were true, d and d_0 would be orthogonal. But, i being associated to d in particular means that $\mathrm{br}_Q(i) = d\mathrm{br}_Q(i)$. Multiplying this equality by d_0, we obtain the contradiction, $0 \neq d_0\mathrm{br}_Q(i) = d_0 d\mathrm{br}_Q(i) = 0$. □

Lemma 3.22. *If $i \in A^P$ is associated to (P, c) and $u \in (A^P)^\times$ is a unit of A^P, then ${}^u i$ is also associated to (P, c).*

Proof. By the hypothesis,

$$\begin{aligned} 0 &\neq \mathrm{br}_P(i)c \\ &= \mathrm{br}_P(u^{-1}uiu^{-1}u)c \\ &= \mathrm{br}_P(u^{-1})\mathrm{br}_P({}^u i)c\mathrm{br}_P(u) \end{aligned}$$

In particular, $\mathrm{br}_P({}^u i)c$ must be non-zero as well. □

Actually, *pointed groups* are defined by considering the equivalence classes of the primitive idempotents of A^P, where two idempotents i and j are considered equivalent if they are conjugate, i.e., if there is a unit $u \in (A^P)^\times$ such that $i = {}^u j$. The equivalence classes of idempotents, α, are called *points* and the pairs (P, α), where P is a p-subgroup of G and α is a point, are called *pointed groups*, denoted

P_α. Pointed groups constitute a further refinement of the G-poset of Brauer pairs. Our exposition of Brauer pairs uses a thinned down version of the theory of pointed groups.

We shall now associate each b-Brauer pair (P, c) to some of the primitive idempotents $f \in A^G$. Let $b = f_1 + \ldots + f_n$ be a primitive decomposition of b in A^G. Applying br_P and multiplying by c, we obtain, $0 \neq c = c\mathrm{br}_P(b) = c\mathrm{br}_P(f_1) + \ldots + c\mathrm{br}_P(f_n)$, implying that some of the idempotents f_i satisfy $c\mathrm{br}_P(f_i) \neq 0$.

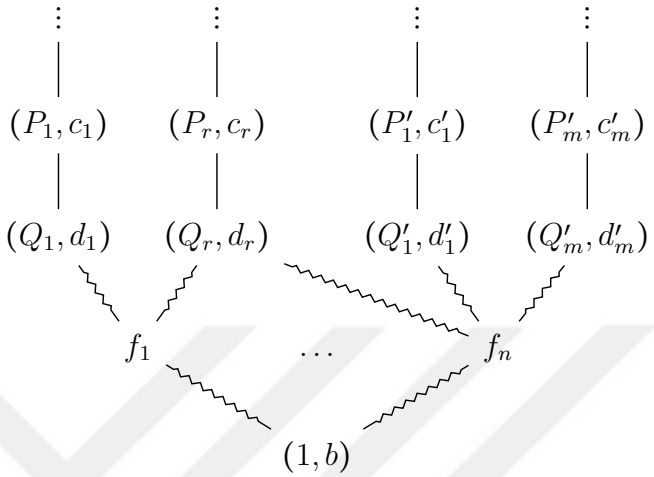
Definition 3.23. Given a primitive idempotent f in A^G and a Brauer pair (P, c) on A, we say that (P, c) is associated to f, or briefly, (P, c) is an f-Brauer pair, provided $0 \neq c\mathrm{br}_P(f)$.

Note that f might not remain primitive in A^P, and neither is $\mathrm{br}_P(f)$ necessarily primitive in $A(P)$, i.e. Lemma 2.25 does not apply here to give an equivalence of conditions as in Definition 3.20.

We now define maximal f-Brauer pairs for primitive idempotents $f \in A^G$, which shall be related to maximal b-Brauer pairs for $b \in \mathcal{B}(A)$ in the future.

Definition 3.24. Given a primitive idempotent f in A^G, a maximal f-Brauer pair is an f-Brauer pair such that it is maximal with respect to the inclusion relation of Brauer pairs.

The association of Brauer pairs to primitive idempotents of A^G does not partition the set of all Brauer pairs: A Brauer pair may be associated to more than one primitive idempotent f_i.



The respective associations of a primitive idempotent i of A^P to (P, c) and (P, c) to a primitive idempotent f of A^G are compatible in the following sense:

Lemma 3.25. *Given a primitive idempotent f in A^G and a Brauer pair (P, c),*
1. If (P, c) is associated to f, then there exists a primitive idempotent $i \in A^P$ associated to (P, c) such that $i \leq f$.
2. If $i \in A^P$ is a primitive idempotent associated to (P, c) such that $i \leq f$, then (P, c) is associated to f.

Proof. 1. By hypothesis, we have $0 \neq \mathrm{br}_P(f)c$. As $b \in Z(A) \subseteq A^G \subseteq A^P$, f has a primitive decomposition $f = \sum_{i \in I} i$ in A^P. Applying br_P and multiplying by c, we get $0 \neq c\mathrm{br}_P(f) = \sum_{i \in I} c\mathrm{br}_P(i)$, so that for at least one i, $c\mathrm{br}_P(i)$ is non-zero, meaning that i is associated to (P, c). Also, $i \leq f$ by construction.

2. Firstly, $i \leq f$ means $i = if$. Then, applying br_P and using the hypotheses, we obtain $0 \neq \mathrm{br}_P(i) = \mathrm{br}_P(i)\mathrm{br}_P(f) = \mathrm{br}_P(i)c\mathrm{br}_P(f)$, so that $c\mathrm{br}_P(f) \neq 0$, in other words, (P, c) is associated to f. □

The set of Brauer pairs associated to a primitive idempotent $f \in A^G$ are closed under conjugation and subpairs. More precisely,

Lemma 3.26. *If (P, c) is associated to a primitive idempotent $f \in A^G$ and $(Q, d) \leq (P, c)$ and $g \in G$, then (Q, d) and ${}^g(P, c)$ are also associated to f.*

Proof. Firstly, let us show that the subpair (Q, d) is associated to f. By the first part of the previous lemma, there is a primitive idempotent $i \in A^P$ associated to (P, c) such that $i \leq f$. By Lemma 3.21.2, i is also associated to (Q, d). Then, by the second part of the previous lemma, (Q, d) is associated to f. Secondly, ${}^g(P, c)$ is associated to f as follows: (P, c) being associated to f means $0 \neq \mathrm{br}_P(f)c$. Applying the automorphism $g \in G$, we get $0 \neq {}^g(\mathrm{br}_P(f)c) = \mathrm{br}_{{}^gP}({}^gf)\,{}^gc = \mathrm{br}_{{}^gP}(f)\,{}^gc$, where we have used the fact that f is fixed by G. Hence, ${}^g(P, c) = ({}^gP, {}^gc)$ is associated to f by definition. □

The next two lemmas shall be essential in the proof of the Sylow Theorem for Brauer pairs, the last result of the present chapter.

Lemma 3.27. *If f is a primitive idempotent in A^G, then there is a p-subgroup P and a primitive idempotent $i \in A^P$ such that, $i \leq f$, $\mathrm{br}_P(i) \neq 0$ and $f \in t_P^G(A^P i A^P)$.*

Proof. Let P be a minimal subgroup of G such that $f \in A_P^G$, i.e., $f = t_P^G(a)$ for some $a \in A^P$. Let $f = \sum_{i \in I} i$ be a primitive decomposition of f in A^P. As f is an idempotent, $f = f^2 = f t_P^G(a) = t_P^G(fa) = t_P^G(\sum_{i \in I} ia) = \sum_{i \in I} t_P^G(ia) \in \sum_{i \in I} t_P^G(A^P i A^P)$, where the third equality follows by the Frobenius axiom (vii) in Lemma 3.2. As $t_P^G(A^P i A^P)$ is an ideal of A^G, and f is a primitive idempotent in A^G, we obtain, by Rosenberg's Lemma (Lemma 2.39), that $f \in t_P^G(A^P i A^P)$ for some i. By construction, $i \leq f$. Now we aim to show that $\mathrm{br}_P(i) \neq 0$. If this fails, this means $i \in A_Q^P$ for some $Q < P$. Now, f being inside the ideal $t_P^G(A^P i A^P)$ means that there is some $a \in A^P i A^P$ such that $f = t_P^G(a)$. Then, $i \in A_Q^P$ implies that $a \in A_Q^P$ as well, as A_Q^P is an ideal of A^P. So, let $a = t_Q^P(b)$ for some $b \in A^Q$. Then, by axiom (i) of Lemma 3.2, we get $f = t_P^G(a) = t_P^G t_Q^P(b) = t_Q^G(b) \in A_Q^G$, contradicting the minimality property of P. Finally, by Lemma 3.3.4, $\mathrm{br}_P(i)$ being non-zero implies that P is a p-subgroup of G. □

Lemma 3.28. *Let (P, c) and (Q, d) be f-Brauer pairs, i a primitive idempotent associated to (P, c), j a primitive idempotent associated to (Q, d) such that $i, j \leq f$. Suppose, also, that $f \in t_P^G(A^P i A^P)$. Then, $(Q, d) \leq_G (P, c)$ holds.*

Proof. By hypothesis, there an element $a \in A^P i A^P$ such that $f = t_P^G(a)$. In Lemma

3.4, it was shown that br_Q acts in the following way on $t_P^G(a)$:

$$\mathrm{br}_Q(t_P^H(a)) = \sum_{g\in[Q\backslash H/P]Q\leq {}^gP} \mathrm{br}_Q({}^ga)$$

As $j \leq f$, i.e., $j = jf$, we can calculate $\mathrm{br}_Q(j)$, which is non-zero as j is associated to (Q,d):

$$0 \neq \mathrm{br}_Q(j) = \mathrm{br}_Q(jf) = \mathrm{br}_Q(j)\mathrm{br}_Q(t_P^G(a)) = \sum_{g\in[Q\backslash G/P]Q\leq {}^gP} \mathrm{br}_Q(j\,{}^ga)$$

Since the non-zero idempotent $\mathrm{br}_Q(j)$ can not lie in the Jacobson radical, there must be an index g such that $Q \leq {}^gP$ and the non-zero summand $\mathrm{br}_Q(j\,{}^ga)$ does not lie in $J(A(Q))$. Now, if $\varphi : A \to B$ is an epimorphism of finite dimensional algebras, then $\varphi(J(A))$ is nilpotent a nilpotent ideal in B, and so, $\varphi(J(A)) \subseteq J(B)$. Applying this fact to the algebra epimorphism br_Q, we know that $\mathrm{br}_Q(J(A^Q)) \subseteq J(A(Q))$. From this, we can infer that $j\,{}^ga$ is not inside $J(A^Q)$, for otherwise we would obtain $\mathrm{br}_Q(j\,{}^ga) \in J(A(Q))$. Now, as ${}^ga \in {}^g(A^PiA^P) = A^{{}^gP}\,{}^giA^{{}^gP}$, there are some elements $x_m, y_m \in A^{{}^gP} \subseteq A^Q$ such that ${}^ga = \sum_m x_m\,{}^giy_m$. Then, since $j\,{}^ga = \sum_m jx_m\,{}^giy_m \notin J(A^Q)$ there must be at least one index m such that $jx_m\,{}^giy_m \notin J(A^Q)$, in particular $jx_m\,{}^gi \notin J(A^Q)$. Now, let ${}^gi = \sum_{k\in K} k$ be a primitive decomposition of ${}^gi$ in A^Q. Then again, since $jx_m\,{}^gi = \sum_{k\in K} jx_mk \notin J(A^Q)$ there must be at least one primitive idempotent $k \in A^Q$ such that $jx_mk \notin J(A^Q)$, in particular, $jA^Qk \nsubseteq J(A^Q)$. Then, by Lemma 2.38, j and k are conjugate, i.e. there is a unit $u \in (A^Q)^\times$ such that ${}^uj = k \leq {}^gj$. Note that as a conjugate of j, the idempotent ${}^uj$ is associated to (Q,d) as well. At this point, we have primitive idempotents ${}^uj$ associated to (Q,d) and ${}^gi$ associated to ${}^g(P,c)$ such that ${}^uj \leq {}^gj$. Therefore, an application of Lemma 3.21.3 yields that $(Q,d) \leq {}^g(P,c)$, in other words, that $(Q,d) \leq_G (P,c)$. □

Theorem 3.29. *(Sylow Theorem for Brauer pairs) For a primitive idempotent f of A^G, the maximal Brauer pairs associated to f are conjugate under G.*

Proof. By Lemma 3.27, there is a p-subgroup P and a primitive idempotent $i \in A^P$ such that $i \leq f$, $\mathrm{br}_P(i) \neq 0$ and $f \in t_P^G(A^PiA^P)$. Also, there exists a unique Brauer pair (P,c) such that i is associated to (P,c), as it was shown after Definition 3.20. By Lemma 3.25.2, (P,c) is an f-Brauer pair. Our claim is that for every

other f-Brauer pair (Q, d), there is some $g \in G$ such that ${}^g(Q, d) \leq_G (P, c)$, which implies both that (P, c) is a maximal f-Brauer pair and that maximal f-Bauer pairs are G-conjugate. So let (Q, d) be any f-Brauer pair. By Lemma 3.25.1, there is a primitive idempotent $j \in A^Q$ associated to (Q, d) such that $j \leq f$. At this point, we note that i and j are associated to (P, c) and (Q, d), respectively, and $i, j \leq f$ and $f \in t_P^G(A^P i A^P)$, in other words, the conditions of the previous lemma are satisfied and we obtain $(Q, d) \leq_G (P, c)$.

□

Notes: 1. Putting $A = kG$, we get the Sylow Theorem for Brauer pairs on kG, which says that maximal b-Brauer pairs are G-conjugate, for a block $b \in Z(kG) = (kG)^G$. Further, putting $A = k$, we get the original Sylow Theorem that the Sylow subgroups of a finite group G are conjugate under the action of G. 2. If (D, e) is a maximal b-Brauer pair, for each $f \in A^G$ such that (D, e) is associated to f, we can see that (D, e) is also a maximal f-Brauer pair: Let (P, c) be a maximal f-Brauer pair. Then, by the proof of the previous theorem, there is some $g \in G$ such that ${}^g(D, e) \leq (P, c)$. Then we have $(1, b) \leq {}^g(Q, d) \leq (P, c)$ and by transitivity, (P, c) is a b-Brauer pair. By the maximality of ${}^g(D, e)$ as a b-Brauer pair, we get ${}^g(D, e) = (P, c)$, i.e., $(D, e) = {}^g(P, c)$ is a maximal f-Brauer pair. On the other hand, if (P, c) is a maximal f-Brauer pair such that $f \leq b$, then (P, c) is a b-Brauer pair, but it does not necessarily follow that (P, c) is a *maximal* b-Brauer pair. The defects of b might have strictly bigger cardinality than P.

We have one last lemma, to be frequently used in the later chapters.

Lemma 3.30. *If $Q \leq P \leq R$ and $(P, c) \leq (R, e) \geq (Q, d)$, then $(Q, d) \leq (P, c)$.*

Proof. Let d' be the unique block of $A(Q)$ such that $(Q, d') \leq (P, c)$. By transitivity, $(Q, d') \leq (R, e)$. By Theorem 3.14, $d = d'$. Hence, $(Q, d) = (Q, d') \leq (P, c)$. □

Chapter 4

Category Algebras

In this chapter, we define category algebras and show that semisimplicity is preserved under category equivalence. As a motivatation for the definition of a category algebra, consider the following: Given a category $\mathcal{C}$ and a commutative ring R with identity, a *representation* of $\mathcal{C}$ over R is defined to be a functor from $\mathcal{C}$ into the category of R-modules: $F : \mathcal{C} \to {}_R\mathfrak{M}$. For any group G, we can form a category $\mathcal{C}$ with one object and exactly one invertible morphism for each element of G. A representation of this category $\mathcal{C}$ over a commutative ring R corresponds to a homomorphism $\rho : G \to \mathrm{End}_R(M)$ for some R-module M, i.e. a representation of G over R, hence, we see that representations of categories are, in some sense, a generalization of group representations. Just as representations $\rho : G \to \mathrm{End}_R(M)$ of a group G over R are in bijective correspondence with RG-modules M, representations of a category $\mathcal{C}$ over R are in bijective correspondence with $R\mathcal{C}$-modules, where $R\mathcal{C}$ is a category algebra, which we now define:

Letting $\{C_i\}_{i\in I}$ denote the objects of $\mathcal{C}$, for some possibly infinite indexing set I, the *category algebra* $R\mathcal{C}$ of $\mathcal{C}$ over R is the free R-module over the set $\bigsqcup_{i,j} \mathcal{C}(C_i, C_j)$ of morphisms of $\mathcal{C}$, with multiplication $f.g = \left\{ \begin{array}{ll} f \circ g & \text{if } \mathrm{dom}(f) = \mathrm{im}(g) \\ 0 & \text{otherwise} \end{array} \right\}$.

Now, we can show the trivial correspondence between $R\mathcal{C}$-modules and functors $F : \mathcal{C} \to {}_R\mathfrak{M}$: Given an $R\mathcal{C}$-module M, we define $F : \mathcal{C} \to {}_R\mathfrak{M}$ by $C \mapsto \mathrm{id}_C \cdot M$ on objects. On morphisms, we put $\mathcal{C}(C_1, C_2) \ni f \mapsto F(f) : \mathrm{id}_{C_1} \cdot M \to \mathrm{id}_{C_2} \cdot M$, where

$F(f)$ is the mapping, $\mathrm{id}_{C_1} \cdot m \mapsto (f \circ \mathrm{id}_{C_1}) \cdot m = (\mathrm{id}_{C_2} \circ f \circ \mathrm{id}_{C_1}) \cdot m$. It is routine to check that F is a functor. Conversely, given a functor $F : \mathcal{C} \to {}_R\mathfrak{M}$, we define an R-module $M := \bigoplus_{C \in \mathcal{C}} F(C)$, after which, we define an $R\mathcal{C}$-action on M as follows: Given $f \in \mathcal{C}(C_1, C_2)$ and $m \in M$, we let $f \cdot m := \left\{ \begin{array}{ll} F(f)(m) & \text{if } m \in F(C_1) \\ 0 & \text{otherwise} \end{array} \right\}$. The necessary confirmations that M is indeed an $R\mathcal{C}$-module are again routine. Sometimes, it is more natural to think about $R\mathcal{C}$-modules. So, some of the terminology from module theory carry over to representations of $\mathcal{C}$. As an example, simple $R\mathcal{C}$-modules correspond to simple representations of $\mathcal{C}$ and classifying simple representations of $\mathcal{C}$ is a step toward better understanding the representation theory of a category $\mathcal{C}$, just as it happens in group representation theory.

The R-algebra $R\mathcal{C}$ has subalgebras $R\mathcal{C}(C_1, C_2)$ for each pair of objects C_1 and C_2. Note that id_C is the identity element of the subalgebra $R\mathcal{C}(C, C)$, but if $C_1 \neq C_2$, then $R\mathcal{C}(C_1, C_2)$ might lack an identity and multiplication of every two elements in $R\mathcal{C}(C_1, C_2)$ is zero. It is clear that $R\mathcal{C}$ has a direct sum decomposition into subalgebras: $R\mathcal{C} = \bigoplus_{C_i, C_j \in \mathcal{C}} R\mathcal{C}(C_i, C_j)$. If $\mathcal{C}$ has finitely many objects, $R\mathcal{C}$ has the identity, $\mathrm{id}_{R\mathcal{C}} = \sum_{C \in \mathcal{C}} \mathrm{id}_C$. We shall utilize a matrix algebra, $M_n(R\mathcal{C}(C_j, C_i))_{1 \leq i,j \leq n}$, elements of which are matrices ϕ with coefficients $\phi_{ij} \in \mathcal{C}(C_j, C_i)$, where C_i for $i = 1, \ldots, n$ are some objects in $\mathcal{C}$.

$$\phi = \begin{bmatrix} \phi_{11} & \ldots & \phi_{1n} \\ \vdots & & \vdots \\ \phi_{n1} & \ldots & \phi_{nn} \end{bmatrix} \in \begin{bmatrix} \mathcal{C}(C_1, C_1) & \ldots & \mathcal{C}(C_n, C_1) \\ \vdots & & \vdots \\ \mathcal{C}(C_1, C_n) & \ldots & \mathcal{C}(C_n, C_n) \end{bmatrix}$$

Recall that two objects C_1 and C_2 of a category is isomorphic if there are maps $f : C_1 \to C_2$ and $g : C_2 \to C_1$ such that $gf = \mathrm{id}_{C_1}$ and $fg = \mathrm{id}_{C_2}$

Lemma 4.1. *If $\{C_i\}_{i=1}^n$ are finitely many objects of $\mathcal{C}$, then there is an R-algebra isomorphism $M_n(R\mathcal{C}(C_j, C_i))_{1 \leq i,j \leq n} \cong \bigoplus_{i,j} R\mathcal{C}(C_i, C_j)$.*

Proof. We show this isomorphism for $n = 2$ only, but the general case is similar in nature.

$$\begin{array}{rccl} \Gamma: & M_n(R\mathcal{C}(C_j,C_i)) & \to & R\mathcal{C}(C_1,C_1)\oplus R\mathcal{C}(C_1,C_2)\oplus R\mathcal{C}(C_2,C_1)\oplus R\mathcal{C}(C_2,C_2) \\ & \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix} & \mapsto & \phi_{11}+\phi_{12}+\phi_{21}+\phi_{22} \end{array}$$

Γ is clearly a bijection. To show that it is a homomorphism, we calculate,

$$\begin{array}{ccc} \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}.\begin{bmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{bmatrix} & \mapsto & \begin{array}{l}(\phi_{11}+\phi_{12}+\phi_{21}+\phi_{22})\\ \cdot(\psi_{11}+\psi_{12}+\psi_{21}+\psi_{22})\end{array} \\ = & & = \\ \begin{bmatrix}\phi_{11}\psi_{11}+\phi_{12}\psi_{21} & \phi_{11}\psi_{12}+\phi_{12}\psi_{22}\\ \phi_{21}\psi_{11}+\phi_{22}\psi_{21} & \phi_{21}\psi_{12}+\phi_{12}\psi_{22}\end{bmatrix} & \mapsto & \begin{array}{l}\phi_{11}\psi_{11}+\phi_{12}\psi_{21}+\phi_{11}\psi_{12}+\phi_{12}\psi_{22}\\ +\phi_{21}\psi_{11}+\phi_{22}\psi_{21}+\phi_{21}\psi_{12}+\phi_{12}\psi_{22}\end{array} \end{array}$$

□

Lemma 4.2. *If $\{C_i\}_{i=1}^n$ are isomorphic objects, then there is an R-algebra isomorphism, $M_n(R\mathcal{C}(C_j,C_i))_{1\le i,j\le n}\cong M_n(R\mathcal{C}(C_1,C_1))$.*

Proof. Again, we show this isomorphism for $n=2$ only, but the general case is similar in nature. By assumption, we have inverse isomorphisms: $C_1 \underset{\beta}{\overset{\alpha}{\leftrightarrows}} C_2$ We define,

$$\begin{array}{rccl} & M_n(R\mathcal{C}(C_j,C_i)) & \to & M_n(R\mathcal{C}(C_1,C_1)) \\ \Gamma: & \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix} & \mapsto & \begin{bmatrix}\phi_{11} & \phi_{12}\,\beta\\ \alpha\,\phi_{21} & \alpha\,\phi_{22}\,\beta\end{bmatrix} \\ & \begin{bmatrix}s & t\,\alpha\\ \beta\,u & \beta\,v\,\alpha\end{bmatrix} & \leftarrow\!\shortmid & \begin{bmatrix}s & t\\ u & v\end{bmatrix} \quad :\Omega \end{array}$$

Γ and Ω are evidently inverse maps. To show that Γ is an algebra homomorphism, we calculate,

$$\begin{array}{ccc} \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}.\begin{bmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{bmatrix} & = & \begin{bmatrix}\phi_{11}\psi_{11}+\phi_{12}\psi_{21} & \phi_{11}\psi_{12}+\phi_{12}\psi_{22}\\ \phi_{21}\psi_{11}+\phi_{22}\psi_{21} & \phi_{21}\psi_{12}+\phi_{12}\psi_{22}\end{bmatrix} \\ \downarrow & & \downarrow \\ \begin{bmatrix}\phi_{11} & \phi_{12}\,\beta\\ \alpha\,\phi_{21} & \alpha\,\phi_{22}\,\beta\end{bmatrix}.\begin{bmatrix}\psi_{11} & \psi_{12}\,\beta\\ \alpha\,\psi_{21} & \alpha\,\psi_{22}\,\beta\end{bmatrix} & = & \begin{bmatrix}\phi_{11}\psi_{11}+\phi_{12}\psi_{21} & (\phi_{11}\psi_{12}+\phi_{12}\psi_{22})\,\beta\\ \alpha\,(\phi_{21}\psi_{11}+\phi_{22}\psi_{21}) & \alpha\,(\phi_{21}\psi_{12}+\phi_{12}\psi_{22})\,\beta\end{bmatrix} \end{array}$$

□

A standard result in category theory is the fact that $\mathcal{A}$ and $\mathcal{B}$ are equivalent categories if and only if there is a full, faithful and essentially surjective functor $F : \mathcal{A} \to \mathcal{B}$.

Theorem 4.3. *If $\mathcal{A}$ and $\mathcal{B}$ are equivalent categories with finitely many objects and $R\mathcal{A}$ is semisimple, then so is $R\mathcal{B}$.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be a fullly faithful and essentially surjective functor, given by the assumption. Let $\mathrm{Ob}(\mathcal{B}) = \{B_i : i \in I\}$ and let $\mathrm{Ob}(\mathcal{A}) = \{A_l : l \in L\}$ for some finite sets I and L. For each $B_i \in \mathrm{Ob}(\mathcal{B})$, we are provided an A_i such that $F(A_i) \cong B_i$, via some inverse morphisms $F(A_i) \underset{\beta_i}{\overset{\alpha_i}{\leftrightarrows}} B_i$, by the essential surjectivity of F. We would like to relate $\mathbb{K}\mathcal{B}(B_i, B_{i'})$ to $\mathbb{K}\mathcal{B}(A_i, A_{i'})$ via these isomorphisms. But first, to be able to treat treat isomorphic objects together, we define an equivalence relation $\sim_I$ on I as follows: Let $i \sim_I i'$ if and only if $B_i \cong B_{i'}$. We define a similar equivalence relation $\sim_L$ on L by identifying isomorphic objects. Then, we choose some set J of representative elements of the equivalence classes of $\sim_I$. For each $j, j' \in J$, if A_j and $A_{j'}$ are isomorphic, then, as a functor preserves isomorphisms, we have $B_j \cong F(A_j) \cong F(A_{j'}) \cong B_{j'}$, which implies that $j = j'$. Hence, $\{A_j\}_{j \in J}$ is a set of non-isomorphic objects. In particular, we can choose a set K of representatives of the equivalence classes of $\sim_L$ such that $J \subseteq K$. Now, we define two maps,

$$\begin{aligned} \bigoplus_{j \in J} R\mathcal{B}(B_j, B_j) &\to \bigoplus_{j \in J} R\mathcal{A}(A_j, A_j) \subseteq \mathbb{K}\mathcal{A} \\ \psi : \quad g &\mapsto f \text{ such that } F(f) = \alpha_j g \beta_j \\ \beta_j F(f) \alpha_j &\mapsfrom f \quad : \phi \end{aligned}$$

extended to their domains R-linearly. The map ψ is well defined as F is full and faithful. One can see that ϕ and ψ are inverses to each other, so they are bijective.

Lastly, ϕ is a homomorphism by the following calculation: For $f_1, f_2 \in \mathcal{A}(A_j, A_j)$, we have $\phi(f_1 f_2) = \beta_j F(f_1 f_2) \alpha_j = \beta_j F(f_1) \alpha_j \beta_j F(f_2) \alpha_j = \phi(f_1)\phi(f_2)$.

Therefore, ϕ is a bijective R-algebra homomorphism, i.e. an R-algebra isomorphism.

Using this isomorphism and the previous lemmas, we construct a sequence of equivalences,

$$\begin{aligned}
R\mathcal{B} &= \bigoplus_{i,i' \in I} R\mathcal{B}(B_i, B_{i'}) && \text{by definition of } R\mathcal{B} \\
&= \bigoplus_{j \in J} \bigoplus_{i,i' \in [j]_\sim} R\mathcal{B}(B_i, B_{i'}) && \text{by definition of } \sim \\
&\cong \bigoplus_{j \in J} M_{n_j}(R\mathcal{B}(B_i, B_{i'}))_{i,i' \in [j]_\sim} && \text{by Lemma 4.1, where } n_j = |[j]_\sim| \\
&\cong \bigoplus_{j \in J} M_{n_j}(R\mathcal{B}(B_j, B_j)) && \text{by Lemma 4.2} \\
&\equiv_{\mathrm{Mor}} \bigoplus_{j \in J} R\mathcal{B}(B_j, B_j) && \text{by Lemma 2.47} \\
&\cong \bigoplus_{j \in J} R\mathcal{B}(A_j, A_j) && \text{by the isomorphism above}
\end{aligned}$$

By symmetry, $R\mathcal{A}$ is Morita equivalent to $\bigoplus_{k \in K} R\mathcal{A}(A_k, A_k)$ as well. Since $J \subseteq K$, we can also relate $\bigoplus_{k \in K} R\mathcal{A}(A_k, A_k)$ and $\bigoplus_{j \in J} R\mathcal{A}(A_j, A_j)$ via an algebra epimorphism γ:

$$\begin{aligned}
\bigoplus_{k \in K} R\mathcal{A}(A_k, A_k) &\to \bigoplus_{j \in J} R\mathcal{A}(A_j, A_j) \\
\gamma : \quad f &\mapsto \begin{cases} f & \text{if } \mathrm{im}(f) = \mathrm{dom}(f) \in \{A_j\}_{j \in J} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

extended to its domain R-linearly (by a routine checking, γ is an R-algebra epimorphism.)

Hence, $\bigoplus_{j \in J} R\mathcal{A}(A_j, A_j)$ is a quotient algebra of $\bigoplus_{k \in K} R\mathcal{A}(A_k, A_k)$.

At this point, the R-algebra $\bigoplus_{k \in K} R\mathcal{A}(A_k, A_k)$ is Morita equivalent to $R\mathcal{A}$, which is semisimple by hypothesis. As semisimplicity is a Morita-invariant property of algebras, $\bigoplus_{k \in K} R\mathcal{A}(A_k, A_k)$ is semisimple. The algebra $\bigoplus_{j \in J} R\mathcal{A}(A_j, A_j)$ is a quotient algebra of $\bigoplus_{k \in K} R\mathcal{A}(A_k, A_k)$, so it is also semisimple by Corollary 2.9. Finally, $R\mathcal{B}$ is Morita equivalent to $\bigoplus_{j \in J} R\mathcal{A}(A_j, A_j)$, therefore, it is also semisimple.

□

Chapter 5

The Biset Category of Finite Groups

In this chapter, we define the biset cateory of finite groups and list some of the properties bisets satisfy. We have recourse to the book [Bou90] and the article [Bar16] for the definitions.

Given finite G-sets X and Y, we can form their disjoint union $X \sqcup Y$ and their Cartesian product $X \times Y$ to obtain new G-sets. Regarding these operations as addition and multiplicaion leads us to the definition of the Burnside ring $B(G)$.

We would like to consider G-sets up to isomorphism and group up isomorphic copies of G-sets. Given a finite group G, its *Burnside group* $B(G)$ is defined to be the free abelian group on the set of isomorphism classes $[X]$ of G-sets, quotiented out by the subgroup generated by elements of the form $[X] + [Y] - [X \sqcup Y]$. To give $B(G)$ a ring structure, its multiplication operation is defined as $\overline{[X]}.\overline{[Y]} := \overline{[X \times Y]}$.

The Burnside ring $B(G)$ is so constructed that it contains all the isomorphism classes of finite G-sets and nothing else. This is because for each G-set X, there exists a unique image $\overline{[X]}$ in $B(G)$. More precisely, X and Y are isomorphic as G-sets if and only if their images $\overline{[X]}$ and $\overline{[Y]}$ in $B(G)$ are equal. This claim shall be a corollary of the following theorem of Burnside [Bou90, Theorem 2.4.5]:

Theorem 5.1 (Burnside's Theorem)**.** *Given a finite group G and (finite) G-sets X and Y, the following are equivalent:*
1) X and Y are isomorphic as G-sets.
2) For all subgroups H of G, the subsets X^H and Y^H have the same cardinality, where X^H denotes the elements of X fixed by H.

Proof. (1)⇒(2): Letting $f : X \to Y$ be a G-set isomorphism, the restricted function $f|_{X^H} : X^H \to Y^H$ is a bijection. (2)⇒(1): We can write $X = \bigsqcup_{x\in[G\backslash X]} Gx \cong \bigsqcup_{x\in[G\backslash X]} G/C_G(x)$, where Gx denotes the orbit of x and $C_G(x)$ is the stabilizer in G of x. We chose a set of representatives $\mathcal{S}(G)$ of G-conjugacy classes of subgroups of G. Then, write $X \cong \bigsqcup_{K\in\mathcal{S}(G)} n_K(G/K)$ for some $n_K \geq 0$, where $n_K(G/K)$ denotes the disjoint union of n_K copies of the G-set G/K (n_K are well-defined, as stabilizers of two points x and y from the same orbit are G-conjugate). Similarly, let $Y = \bigsqcup_{K\in\mathcal{S}(G)} m_K(G/K)$. Note that for $H, K \in \mathcal{S}(G)$, the fixed points $(G/K)^H$ are $\{gK : hgK = gK \text{ for all } h \in H\}$, or in other words, $(G/K)^H = \{gK : H \leq {}^gK\}$. As fixed point operation commutes with disjoint unions, $X^H \cong \bigsqcup_{K\in\mathcal{S}(G)} n_K(G/K)^H$. The hypothesis tells us that $|X^H| = \sum_{K\in\mathcal{S}(G)} n_K|(G/K)^H| = \sum_{K\in\mathcal{S}(G)} m_K|(G/K)^H| = |Y^H|$. To write this equation shorter, we shall use $\mathcal{S}(G)$ as an index set, for which purpose, we define a total ordering $\prec$ on $\mathcal{S}(G)$: Given two $H, K \in \mathcal{S}(G)$, if $|H| < |K|$ then define $H \prec K$. The rest of the pair of elements of $\mathcal{S}(G)$ share the same cardinality, so we put a total order on them (The subgroups of G of equal cardinality constitute finite sets and a set of n elements can be totally ordered in $n!$ many different ways.) Now define a matrix Γ with the entries $\Gamma_{HK} = |(G/K)^H|$. The hypothesis now says that $\sum_{K\in\mathcal{S}(G)} \Gamma_{HK}(n_K - m_K) = 0$. Further, we can define a vector v by $v_H = n_K - m_K$ and write $\sum_{K\in\mathcal{S}(G)} \Gamma_{HK} v_K = 0$, in other words, we have a matrix equality, $\Gamma v = 0$. However, for each pair satisfying $H \succ K$, the set $\{gK : H \leq {}^gK\}$ is empty and $\Gamma_{HK} = |(G/K)^H| = 0$, i.e. Γ is an upper triangular matrix, in particular, it is invertible, hence we obtain $v = 0$. Therefore, $n_K - m_K = 0$ for each K and we obtain the G-set isomorphisms, $X \cong \bigsqcup_{K\in\mathcal{S}(G)} n_K G/K \cong Y$. □

Corollary 5.2. *Given finite G-sets X and Y, we have $X \cong Y$ if and only if $\overline{[X]} = \overline{[Y]}$ in $B(G)$.*

Proof. If $\overline{[X]} = \overline{[Y]}$ in $B(G)$, then there are G-sets U_i, V_i, Z_i and W_i such that

$$[X] = [Y] + \sum_i [U_i] + [V_i] - [U_i \sqcup V_i] + \sum_j [Z_j \sqcup W_j] - [Z_j] - [W_j] \tag{1}$$

Sending the terms with negative coefficient to the opposite side of the equation, we get

$$[X] + \sum_j [Z_j] + \sum_j [W_j] + \sum_i [U_i \sqcup V_i] = [Y] + \sum_i [U_i] + \sum_i [V_i] + \sum_j [Z_j \sqcup W_j] \tag{2}$$

Note that arranging the order of the terms, we can put this equation in such a form that n'th term on the left-hand side is equal to n'th term on the right-hand side. In general, if there are G-set isomorphisms $S_i \cong T_i$ for $i = 0, \ldots, n$, then we can construct the obvious G-set isomorphism, $\bigsqcup_i S_i \cong \bigsqcup_i T_i$. Hence, an equality of isomorphism classes of G-sets $\sum_i [S_i] = \sum_i [T_i]$ implies an equality of isomorphism classes of G-sets, $[\bigsqcup_i S_i] = [\bigsqcup_i T_i]$. By this principle, we obtain the following G-set isomorphism from Equation (2):

$$X \sqcup (\bigsqcup_j Z_j) \sqcup (\bigsqcup_j W_j) \sqcup \bigsqcup_i (U_i \sqcup V_i) \cong Y \sqcup (\bigsqcup_i U_i) \sqcup (\bigsqcup_i V_i) \sqcup \bigsqcup_j (Z_i \sqcup W_i) \tag{3}$$

On the other hand, the fixed point operator commutes with disjoint unions, i.e. for each subgroup H of G, we have $(S \sqcup T)^H = S^H \sqcup T^H$ for G-sets S and T. More generally, $|(\bigsqcup_i S_i)^H| = |\bigsqcup_i S_i^H| = \sum_i |S_i^H|$. Using this fact, we count the fixed points of the Isomorphism (3):

$$|X^H| + \sum_j |Z_j^H| + \sum_j |W_j^H| + \sum_i (|U_i^H| + |V_i^H|) = |Y^H| + \sum_i |U_i^H| + \sum_i |V_i^H| + \sum_j (|Z_j^H| + |W_j^H|) \tag{4}$$

As a result, $|X^H| = |Y^H|$ for each subgroup H, therefore $X \cong Y$ by the theorem of Burnside. □

At this point, we have successfully encoded every G-set X in $B(G)$. Actually, as every G-set X is a disjoint union of transitive G-sets, we express $B(G)$ simply as the $\mathbb{Z}$-algebra, $B(G) = \bigoplus_{H \in \mathcal{S}(G)} \mathbb{Z}(G/H)$.

For finite groups H and G, a set X is called an (H, G)-*biset* provided that H has a left action on X, G has a right action on X and their actions are compatible:

$(h \cdot x) \cdot g = h \cdot (x \cdot g)$. Equivalently, X is a left $H \times G^{\mathrm{op}}$-set. To see this equivalence, suppose, first, that X is an (H, G)-biset. Then, defining $(h, g) \cdot x := h \cdot x \cdot g$ gives an $H \times G^{\mathrm{op}}$-action as follows:

$$\begin{aligned}
[(h_1, g_1).(h_2, g_2)] \cdot x &= (h_1 h_2, g_{1 \cdot_{\mathrm{op}}} g_2) \cdot x \\
&= (h_1 h_2, g_2 g_1) \cdot x \\
&= (h_1 h_2) \cdot x \cdot (g_2 g_1) \\
&= h_1 \cdot (h_2 \cdot x \cdot g_2) \cdot g_1 \\
&= (h_1, g_1) \cdot ((h_2, g_2) \cdot x)
\end{aligned}$$

Conversely, given a left $H \times G^{\mathrm{op}}$-set X, we define a left H-action $h \cdot x := (h, 1) \cdot x$ and a right G-action $x \cdot g := (1, g) \cdot x$ and check that X is a left H-set (clear) and a right G-set: $(x \cdot g_1) \cdot g_2 = (1, g_2) \cdot ((1, g_1) \cdot x) = [(1, g_2).(1, g_1)] \cdot x = (1, g_{2 \cdot_{\mathrm{op}}} g_1) \cdot x = (1, g_1 g_2) \cdot x = x \cdot (g_1 g_2)$ such that these actions are compatible: $(h \cdot x) \cdot g = (1, g) \cdot ((h, 1) \cdot x) = [(1, g).(h, 1)] \cdot x = (h, g) \cdot x = [(h, 1).(1, g)] \cdot x = (h, 1) \cdot ((1, g) \cdot x) = h \cdot (x \cdot g)$.

For example, the quotient group $\dfrac{H \times G}{L}$ for a subgroup L of $H \times G$ is an (H, G)-biset by the action $(h_1, g_1) \cdot (h, g) L := (h_1 h, g_1^{-1} g) L$.

The biset category $\mathcal{C}$ of finite groups is defined as follows: The objects are all finite groups and morphism sets are $\mathcal{C}(G, H) := B(H, G) = B(H \times G^{\mathrm{op}})$, the set of isomorphism classes of (H, G)-bisets.

As the map given by $g \mapsto g^{-1}$ is a group isomorphism $G \to G^{\mathrm{op}}$, the isomorphism classes of left $H \times G^{\mathrm{op}}$-sets and the isomorphism classes of left $H \times G$-sets are in one-to-one correspondence. In other words, we do not need to differentiate between (H, G^{op})-bisets and (H, G)-bisets. As a result, we write $\mathcal{C}(G, H) = B(H, G) = \bigoplus_{L \in \mathcal{S}(H \times G)} \mathbb{Z}\left[\dfrac{H \times G}{L}\right]$.

To define the composition operation of $\mathcal{C}$, take any $X \in \mathcal{C}(G, H)$ and $Y \in \mathcal{C}(H, K)$ and define an H-action on $X \times Y$ by $h \cdot (x, y) := (x \cdot h, h^{-1} \cdot y)$. Representatives of orbits of this action shall be denoted by $(x, {}_H y)$ and the set of orbits shall be denoted by $X \times_H Y$. Now, $[X].[Y] := [X \times_H Y]$ gives the definition of composition in $\mathcal{C}$.

From now on, by an (H,G)-*biset*, let us mean an isomorphism class of an (H,G)-biset for convenience of exposition.

Some important bisets to mention include the elementary bisets: Given a group homomorphism $\alpha : H \leftarrow G$, then H is a transitive (H,G)-biset, where H acts naturally on the left and G acts on the right by means of α: $h_1 \cdot h \cdot g_1 \coloneqq h_1 h\alpha(g)$. Similarly, H has a transitive (G,H)-biset structure defined via α. Using these transitive bisets, we make the following definitions:

1. The *induction biset* ${}_H\mathrm{ind}_G^\alpha$ is defined as the (H,G)-biset, $\left[\frac{H\times G}{\triangle(H,\alpha,G)}\right] = [H]$,

2. The *restriction biset* ${}_G\mathrm{res}_H^\alpha$ is defined as the (G,H)-biset, $\left[\frac{G\times H}{(G,\alpha,H)\triangle}\right] = [H]$,

where, $\triangle(H,\alpha,G) = \{(\alpha(g),g) : g \in G\}$ is the graph of α in $H \times G$ and $(G,\alpha,H)\triangle = \{(g,\alpha(g)) : g \in G\}$ is the graph of α in $G\times H$. These definitions make sense because we have an (H,G)-biset isomorphism $\frac{H\times G}{\triangle(H,\alpha,G)} \cong {}_H H_G$ defined on the cosets by $(h,g)\triangle \mapsto h\alpha(g^{-1})$ and a (G,H)-biset isomorphism $\frac{G\times H}{(G,\alpha,H)\triangle} \cong {}_G H_H$ defined on the cosets by $(g,h)\triangle \mapsto \alpha(g)h^{-1}$, where $\triangle$ is short for $\triangle(H,\alpha,G)$ and $(G,\alpha,H)\triangle$, respectively.

Note: Inductions and restrictions of bisets satisfy, ${}_K\mathrm{ind}_H^\beta \mathrm{ind}_G^\alpha = {}_K\mathrm{ind}_G^{\beta\alpha}$ and ${}_G\mathrm{res}_H^\beta \mathrm{res}_K^\alpha = {}_G\mathrm{res}_K^{\beta\alpha}$. Let us summarize some notational conventions: When α is an inclusion $\alpha : H \hookleftarrow K$, we omit α in the notation and write, ${}_H\mathrm{ind}_K^\alpha = {}_H\mathrm{ind}_K$ and ${}_K\mathrm{res}_H^\alpha = {}_K\mathrm{res}_H$. When α is a surjection $\alpha : H \leftarrow K$ for a normal subgroup N of H, we write ${}_H\mathrm{ind}_K^\alpha = {}_H\mathrm{def}_K^\alpha$ and ${}_K\mathrm{res}_H^\alpha = {}_K\mathrm{inf}_H^\alpha$ and call them *deflation biset* and *inflation biset*, respectively. When α is the canonical surjection $\alpha : H/N \leftarrow H$ for a normal subgroup N of H, we omit the superscripts and write ${}_{H/N}\mathrm{ind}_H^\alpha = {}_{H/N}\mathrm{def}_H$ and ${}_H\mathrm{res}_{H/N}^\alpha = {}_H\mathrm{inf}_{H/N}$. When $\alpha : H \leftarrow K$ is an isomorphism, we write ${}_H\mathrm{ind}_K^\alpha = {}_H\mathrm{iso}_K^\alpha$ call it an *isogation biset*. Finally, when $c_g : {}^gH \leftarrow H$ is a conjugation isomorphism for some $g \in G$, we write ${}_{gH}\mathrm{ind}_H^{c_g} = {}_{gH}\mathrm{con}_H^g$ and call it a *conjugation biset*. These elementary bisets are important, as they generate the biset category in the sense that any transitive (H,G)-biset $\left[\frac{H\times G}{L}\right]$, for any two groups H,G, can be written in the form ${}_H\mathrm{ind}_D\mathrm{inf}_{D/C}\mathrm{iso}_{B/A}^\alpha\mathrm{inf}_B\mathrm{res}_G$ for some suitable subgroups $C \lhd D \leq H$, $A \lhd B \leq G$ and an isomorphism $\alpha : D/C \leftarrow B/A$ [Bou90, Remark 3.1.2].

Multiplication of elementary bisets satisfy some properties. For example, given subgroups $K \leq H \geq L$, the biset ${}_K\mathrm{res}_H\mathrm{ind}_L$ can be calculated in a reverse fashion:

$$ {}_K\mathrm{res}_H\mathrm{ind}_L = \sum_{KgL \subseteq H} {}_K\mathrm{ind}\,{}_{gL\cap K}\mathrm{con}^g_{L\cap K^g}\mathrm{res}_L $$

This relation is called the *Mackey relation*. It was discovered that the Mackey relation, along with some other properties often occur in different settings. The definition of Mackey functor is intended to capture this phenomenon. One of the equivalent definitions of a Mackey functor, given in [Thé95, p. 500], is the following: Given a group G and a ring R, let $\mathcal{S}(G)$ be the set of subgroups of G. A *Mackey functor* M is a family $M(H)$ of R-modules, where H runs over the subgroups $\mathcal{S}(G)$ of G, together with R-linear maps,

$$ \begin{aligned} I^H_K &: M(K) \to M(H) \\ R^H_K &: M(H) \to M(K) \\ c_g &: M(H) \to M({}^gH) \end{aligned} $$

where $K \leq H$ and $g \in G$, such that the following axioms are satisfied:

(1) $R^K_L R^H_K = R^H_L$ and $I^H_K I^K_L = I^H_L$ if $L \leq K \leq H$
(2) $R^H_H = R^H_H = \mathrm{id}_{M(H)}$
(3) $c_{gh} = c_g c_h$
(4) $c_h : M(H) \to M(H)$ is the identity if $h \in H$
(5) $c_g R^H_K = R^{{}^gH}_{{}^gK} c_g$ and $c_g I^H_K = I^{{}^gH}_{{}^gK} c_g$
(6) (Mackey axiom) $R^H_L I^H_K = \sum_{h\in[L\backslash H/K]} I^L_{L\cap {}^hK} c_h R^K_{L^h\cap K}$ if $L, K \leq H$

From this definition, we see that the map $H \mapsto A^H$ for a fixed H-algebra A is a Mackey functor, with transfer, restriction and conjugation as maps. Referring to [Bou90, Lemma 2.3.24] for a proof of the Mackey relation (6), we also see that the map $H \mapsto B(H)$, sending H to the Burnside ring $B(H)$ is a Mackey functor, with elementary bisets ${}_K\mathrm{res}_H$, ${}_H\mathrm{ind}_K$ and ${}_{gH}\mathrm{con}^g_H$ as maps.

Chapter 6

The Biset Category of Brauer Pairs

Given a finite group G, we define the biset category $\mathcal{D}$ of Brauer pairs by considering Brauer pairs $\mathfrak{Br}(A)$ on an interior p-permutation G-algebra A as objects, instead of subgroups of G. The morphism sets $\mathcal{D}((Q,d),(P,c))$ are once again the Burnside ring $B(P,Q)$ of (P,Q)-bisets. The biset category $\mathcal{D}$ of Brauer pairs has a subcategory $\mathcal{B}$ of interest, called the ordinary Mackey category of Brauer pairs. In this chapter, we construct $\mathcal{B}$ and prove a structural theorem concerning it. The main references of this chapter are [Bou90] and [Bar16].

Given Brauer pairs (P,c), (Q,d) and an element $g \in G$, we define three elementary morphisms in $\mathcal{D}$:

$$
\begin{aligned}
{}_{(Q,d)}\mathrm{res}_{(P,c)} &= {}_{Q}\mathrm{res}_{P} \\
{}_{(P,c)}\mathrm{ind}_{(Q,d)} &= {}_{P}\mathrm{ind}_{Q} \\
{}^{g}{}_{(Q,d)}\mathrm{con}^{g}{}_{(Q,d)} &= {}^{g}{}_{Q}\mathrm{con}^{g}{}_{Q}
\end{aligned}
$$

Instead of letting H run on the subgroups $\mathcal{S}(G)$, if we let (P,c) run on the set $\mathcal{B}(A)$ of Brauer pairs on A, we observe that the six conditions appearing in the definition of a Mackey functor are satisfied: Given $(Q,d) \leq (P,c) \leq (R,e)$,

(1.i) ${}_{(Q,d)}\mathrm{res}_{(R,e)} = {}_{(Q,d)}\mathrm{res}_{(P,c)}\mathrm{res}_{(R,e)}$

(1.ii) ${}_{(R,e)}\mathrm{ind}_{(Q,d)} = {}_{(R,e)}\mathrm{ind}_{(P,c)}\mathrm{ind}_{(Q,d)}$

(2) ${}_{(Q,d)}\mathrm{res}_{(Q,d)} = {}_{(Q,d)}\mathrm{ind}_{(Q,d)} = \mathrm{id}_{(Q,d)}$
(3) ${}_{{}^{gh}(L,d)}\mathrm{con}^{gh}_{(L,d)} = {}_{{}^{gh}(Q,d)}\mathrm{con}^{g}{}_{{}^{h}(Q,d)}\mathrm{con}^{h}_{(Q,d)}$
(4) ${}_{{}^{h}(R,e)}\mathrm{con}^{h}_{(R,e)}$ is the identity $\mathrm{id}_{(R,e)}$ if and only if $h \in N_G((R,e))$.
(5.i) ${}_{{}^{g}(P,c)}\mathrm{con}^{g}_{(P,c)}\mathrm{res}_{(H,e)} = {}_{{}^{g}(P,c)}\mathrm{res}_{{}^{g}(R,e)}\mathrm{con}^{g}_{(R,e)}$
(5.ii) ${}_{{}^{g}(R,e)}\mathrm{con}^{g}_{(R,e)}\mathrm{ind}_{(P,c)} = {}_{{}^{g}(R,e)}\mathrm{ind}_{{}^{g}(P,c)}\mathrm{con}^{g}_{(P,c)}$
(6) (Mackey axiom) if $(P,c) \leq (R,e) \geq (Q,d)$,
${}_{(P,c)}\mathrm{res}_{(R,e)}\mathrm{ind}_{(Q,d)} = \sum_{P{}^{g}Q \subseteq R} {}_{(P,c)}\mathrm{ind}_{({}^{g}Q\cap P, {}^{g}t)}\mathrm{con}^{g}_{(L\cap P^{g},t)}\mathrm{res}_{(Q,d)}$

Let us prove the sixth item: We already know by [Bou90, Lemma 2.3.24] that the Mackey relation

$$ {}_{P}\mathrm{res}_{R}\mathrm{ind}_{Q} = \sum_{P{}^{g}Q \subseteq R} {}_{P}\mathrm{ind}_{{}^{g}Q\cap P}\mathrm{con}^{g}{}_{Q\cap P^{g}}\mathrm{res}_{Q} $$

is true of bisets. To show that it also holds for Brauer pairs, i.e. to show the truth of the sentence,

$$ {}_{(P,c)}\mathrm{res}_{(R,e)}\mathrm{ind}_{(Q,d)} = \sum_{P{}^{g}Q \subseteq R} {}_{(P,c)}\mathrm{ind}_{({}^{g}Q\cap P, {}^{g}a)}\mathrm{con}^{g}{}_{(Q\cap P^{g},a)}\mathrm{res}_{(Q,d)} $$

we need to show that, $({}^{g}Q \cap P, {}^{g}a) \leq (P,c)$, where $(Q \cap P^{g}, a)$ is the unique Brauer pair less than (Q,d) given by Theorem 3.15. But this is indeed the case, because we have $(P,c) \leq (H,e) \geq (Q \cap P^{g}, a)$ and ${}^{g}Q \cap P \leq P$, so that by Lemma 3.31, we obtain $({}^{g}Q \cap P, {}^{g}a) \leq (P,c)$.

We have replaced the objects of the biset category of finite groups to obtain the biset category of Brauer pairs and showed that this new category continues to satisfy the six relations expected of them. The article [Bar16] is concerned, in part, with the ordinary Mackey category of finite groups, a subcategory of the biset category of finite groups, generated by the left-free transitive morphisms. We shall show that this category also has a counterpart, with Brauer pairs as objects.

Definition 6.1. A transitive (H,G)-biset is left-free if it is free as a left H-set. Right-free bisets are defined similarly.

We show that left-free (H,G)-bisets are precisely the bisets ${}_{H}\mathrm{ind}^{\alpha}_{V}\mathrm{res}_{G} =$

${}_H\mathrm{ind}_{\alpha(V)}\mathrm{def}^{\alpha}_{V}\mathrm{res}_{G} = \left[\frac{H\times G}{\triangle(H,\alpha,V)}\right]$, where $V \leq G$ and $\alpha : H \leftarrow V$ is a group homomorphism. We shall be using the following notation: Given a subgroup L of $H \times G$, the $p_1(L)$ is defined to be the subgroup $\{h \in H : (h, g) \in L \text{ for some } g \in G\}$ of H. The subgroup $p_2(L)$ of G is defined similarly. Given a transitive (H, G)-biset X, firstly, note that $X \cong \dfrac{H \times G}{L}$ where $L = \mathrm{Stab}_{H\times G}(x) = \{(h, g) : h \cdot x \cdot g = x\}$ is the stabilizer in $H \times G$ of some $x \in X$. Define $V := p_2(L)$ and define a group homomorphism $\alpha : H \leftarrow V$ by $h \leftarrow\!\shortmid\, g$ such that $(h, g) \in L$. Note that α is well-defined, because X is H-free. Now, it is easy to see that L is equal to the subgroup $\triangle(H, \alpha, V)$ of $H \times G$. Conversely, by a straightforward calculation, $\left[\frac{H\times G}{\triangle(H,\alpha,V)}\right]$ is seen to be a left-free transitive (H, G)-biset.

By an application of the Mackey relation, we obtain the following decomposition: ${}_H\mathrm{ind}^{\alpha}_{V}\mathrm{res}_{G} = {}_H\mathrm{ind}_{\alpha(V)}\mathrm{def}^{\alpha}_{V}\mathrm{res}_{G}$

By a similar argument, the right-free (H, G)-bisets are precisely the bisets, ${}_H\mathrm{ind}_{V}\mathrm{res}^{\alpha}_{G} = {}_H\mathrm{ind}_{V}\mathrm{inf}^{\alpha}_{\alpha(V)}\mathrm{res}_{G}$

The ordinary Mackey category of *finite groups* is defined by recourse to a category called an ordinary Mackey system [Bar16, p. 8], the definition of which we include here for ease of reference:

Let $\mathfrak{K}$ be a set of finite groups, closed under taking subgroups. A category $\mathcal{F}$ is a *Mackey system* on $\mathfrak{K}$ if the set of objects of $\mathcal{F}$ is $\mathfrak{K}$ and the sets of morphisms of $\mathcal{F}$ are group homomorphisms satisfying the following four axioms:
MS1 For all $V \leq G \in \mathfrak{K}$, the inclusion $G \leftarrow\!\shortmid\, V$ is included in $\mathcal{F}$.
MS2 For all $V \leq G \in \mathfrak{K}$ and $g \in G$, the conjugation map ${}^gV \ni {}^gv \leftarrow\!\shortmid\, v \in V$ is a morphism in $\mathcal{F}$.
MS3 For any morphism $\alpha : F \leftarrow G$ in $\mathcal{F}$, the associated homomorphism $\alpha(G) \leftarrow G$ is in $\mathcal{F}$.
MS4 For any morphism α in $\mathcal{F}$ such that α is a group isomorphism, α^{-1} is also in $\mathcal{F}$.

We call $\mathcal{F}$ an *ordinary Mackey system* provided all the morphisms in $\mathcal{F}$ are injective.

Given an ordinary Mackey system $\mathcal{F}$ on $\mathfrak{K}$, the ordinary Mackey category $\mathcal{M}_{\mathcal{F}}$

on $\mathcal{F}$ is defined as follows: The set of objects of $\mathcal{M}_\mathcal{F}$ is $\mathfrak{K}$ and the sets of morphisms $\mathcal{M}_\mathcal{F}(H,G)$ are $\mathbb{Z}\{{}_H\mathrm{ind}_V^\alpha\mathrm{res}_G : \alpha \in \mathcal{F}(H,V)\}$, the $\mathbb{Z}$-linear combinations of left-free transitive morphisms ${}_H\mathrm{ind}_V^\alpha\mathrm{res}_G$ such that α comes from $\mathcal{F}$.

The morphism sets of the ordinary Mackey category $\mathcal{M}_\mathcal{F}$ of finite groups are closed by an application of Proposition 3.1 of [Bar16], which lays out the Mackey relation for the multiplication of two left-free transitive bisets. For our purpose, a version of this proposition specialized to conjugation bisets shall be sufficient:

Lemma 6.2. *Given $g,h \in G$, we have an identity,*

$$ {}_{(P,c)}\mathrm{ind}_{(V,v)}^{g}\mathrm{res}_{(Q,d)}\mathrm{ind}_{(W,w)}^{h}\mathrm{res}_{(R,e)} = \sum_{VyW\subseteq Q} {}_{(P,c)}\mathrm{ind}_{(V^{yh}\cap W,t)}^{gyh}\mathrm{res}_{(R,e)} $$

Proof. Proposition 3.1 of [Bar16] contains a proof of this equality for the Biset Category of subgroups already, but we need to check that ${}^{gyh}(V^{yh}\cap W, f) = ({}^gV \cap {}^{gyh}W, {}^{gyh}f) \leq (P,c)$. The diagram below might be helpful to visualize the following arguments:

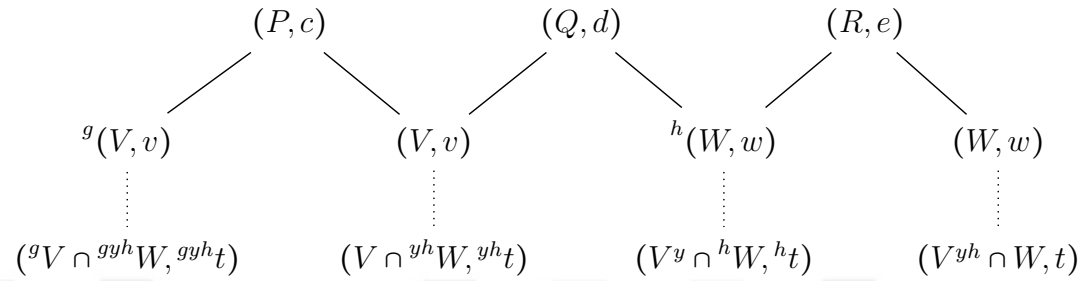
As $V^{yh}\cap W \leq W \leq R$ and $(W,w) \leq (R,e) \geq (V^{yh}\cap W, t)$, an application of Lemma 3.31 yields $(V^{yh}\cap W,t) \leq (W,w)$.

Conjugating this inequality by h, we get ${}^h(V^{yh}\cap W,t) \leq {}^h(W,w)$, i.e. $(V^y \cap {}^hW, {}^ht) \leq {}^h(W,w) \leq (Q,d)$. Since y in an element of Q, it fixes (Q,d), so that we have $(V\cap {}^{yh}W, {}^{yh}t) \leq {}^y(Q,d) = (Q,d)$.

Next, as $V\cap {}^{yh}W \leq V \leq Q$ and $(V,v) \leq (Q,d) \geq (V\cap {}^{yh}W, {}^{yh}t)$ we get $(V\cap {}^{yh}W, {}^{yh}f) \leq (V,v)$ by Lemma 3.31. Conjugating this inequality by g, we get ${}^g(V\cap {}^{yh}W, {}^{yh}t) \leq {}^g(V,v)$, i.e. $({}^gV\cap {}^{gyh}W, {}^{gyh}t) \leq {}^g(V,v) \leq (P,c)$.

Lastly, as ${}^gV\cap {}^{gyh}W \leq {}^gV \leq P$ and $({}^gV\cap {}^{gyh}W, {}^{gyh}t) \leq (P,c) \geq {}^g(V,v)$ we get

$({}^gV \cap {}^{gyh}W, {}^{gyh}t) \leq {}^g(V,v)$ by Lemma 3.31.



□

To define the ordinary Mackey category $\mathcal{B}$ of Brauer pairs on A, it shall be convenient to define a precursor category $\mathcal{G}$ having Brauer pairs as objects and with morphism sets $\mathcal{G}((P,c),(Q,d)) = \{{}_{(P,c)}\mathrm{ind}^g_{(Q,d)} \mid g \in G,\ (P,c) \geq {}^g(Q,d))\}$ for any two objects (P,c) and (Q,d). The morphism sets of this category are easily seen to be closed under composition, using the properties of bisets listed above.

Now we can define the category $\mathcal{B}$, called the *ordinary Mackey category of Brauer pairs on* A, with Brauer pairs as objects and with the morphism sets generated by the left-free transitive morphisms ${}_{(P,c)}\mathrm{ind}^g_{(V,v)}\mathrm{res}_{(Q,d)}$ such that the morphisms ${}_{(P,c)}\mathrm{ind}^g_{(V,v)}$ are coming from $\mathcal{G}$:

$$\begin{aligned}\mathcal{B}((P,c),(Q,d)) &= \mathbb{Z}\{{}_{(P,c)}\mathrm{ind}^g_{(V,v)}\mathrm{res}_{(Q,d)} \mid {}_{(P,c)}\mathrm{ind}^g_{(V,v)} \in \mathcal{G}((P,c),(V,v))\} \\ &= \mathbb{Z}\{{}_{(P,c)}\mathrm{ind}^g_{(V,v)}\mathrm{res}_{(Q,d)} \mid g \in G,\ (P,c) \geq {}^g(V,v),\ (Q,d) \geq (V,v)\}\end{aligned}$$

Note that the morphism sets of $\mathcal{B}$ are closed under composition, by Lemma 5.4.

To be able to apply Theorem 3.14 on $\mathcal{B}$, we define *localized* versions of all the categories above: Let $\mathcal{G}_f$ be the full subcategory of $\mathcal{G}$ with objects restricted to the Brauer pairs associated to a fixed primitive idempotent $f \in A^G$. Now, fix a maximal f-Brauer pair (D, e_D) and further restrict the set of objects to the f-Brauer pairs under (D, e_D). This category shall be denoted $\mathcal{LG}_f$ (To avoid clutter, we prefer $\mathcal{LG}_f$ over a more informative notation like ${}_{(D,e_D)}\mathcal{G}_f$.) We have similar definitions for $\mathcal{B}_f$ and $\mathcal{LB}_f$. The morphism sets of the subcategories $\mathcal{G}_f$, $\mathcal{LG}_f$, $\mathcal{B}_f$ and $\mathcal{LB}_f$ are closed under composition by Lemma 3.26 (which says that

the set of f-Brauer pairs is closed under taking subpairs and under conjugation).

The Sylow Theorem for Brauer pairs (Theorem 3.30) can be used to get an equivalence of categories, $\mathcal{LB}_f \simeq \mathcal{B}_f$ as follows:

Lemma 6.3. *The inclusion functor* $\mathrm{i} : \mathcal{LB}_f \to \mathcal{B}_f$ *is an equivalence of categories.*

Proof. We claim that i is a full, faithful and essentially surjective functor. Given two Brauer pairs (P, c) and (Q, d), the inclusion function $\mathrm{i} : \mathcal{LB}_f((P, c), (Q, d)) \to \mathcal{B}_f((P, c), (Q, d))$ is clearly full and faithful. To show that i is essentially surjective, pick an object $(Q, d) \in \mathrm{Ob}(\mathcal{B}_f)$. Let (Q, d) be contained in a maximal f-Brauer pair (P, c), which, by the Sylow theorem for Brauer pairs, is G-conjugate to (D, e_D), i.e. there is some $g \in G$ such that ${}^g(D, e_D) = (P, c)$. Then, we see that ${}^{g^{-1}}(Q, d) \in \mathrm{Ob}(\mathcal{LB}_f)$ such that its image in $\mathcal{B}_f$ is isomorphic to (Q, d) via the isomorphism ${}_{(Q,d)}\mathrm{con}^{g}{}_{{}^{g^{-1}}(Q,d)}$, the inverse of which is ${}_{{}^{g^{-1}}(Q,d)}\mathrm{con}^{g^{-1}}{}_{(Q,d)}$. □

Here is our last result on the ordinary Mackey category of Brauer pairs on A:

Theorem 6.4. *Given a field* $\mathbb{K}$ *of characteristic zero and a primitive idempotent* $f \in A^G$*, then the category algebra* $\mathbb{K}\mathcal{B}_f$ *over* $\mathbb{K}$ *is semisimple.*

Proof. Let $\mathcal{F}$ be the category with subgroups of D as objects and with morphism sets, $\mathcal{F}(P, Q) = \{c_g : P \leftarrow Q \mid g \in G, (P, e_P) \geq {}^g(Q, e_Q)\}$. It is evident that $\mathcal{F}$ satisfies the definition of an ordinary Mackey system. We have a strict isomorphism of categories $\mathcal{F} \cong \mathcal{LG}_f$ by the functor $P \mapsto (P, e_P)$, where e_P is the unique block of $A(P)$ s.t. $(P, e_P) \leq (D, e_D)$. Then, by construction, we obtain further strict isomorphism of categories $\mathcal{M}_{\mathcal{F}} \cong \mathcal{M}_{(\mathcal{LG}_f)} = \mathcal{LB}_f$. But, the category algebra $\mathbb{K}\mathcal{M}_{\mathcal{F}}$ is semisimple by [Bar16, Theorem 4.6], so that $\mathbb{K}\mathcal{LB}_f$ is also semisimple. By the previous lemma, $\mathcal{LB}_f$ and $\mathcal{B}_f$ are equivalent categories, and by Lemma 4.3, equivalence of categories preserves the semisimplicity of category algebras. Therefore, $\mathbb{K}\mathcal{B}_f$ is also semisimple. □

As a matter of fact, [Bar16, Theorem 4.6] directly applies to $\mathbb{K}\mathcal{B}$, using the following argument:

Theorem 6.5. *Given a field $\mathbb{K}$ of characteristic zero, then the category algebra $\mathbb{K}\mathcal{B}$ is semisimple.*

Proof. For each $(P, c) \in \mathrm{Ob}(\mathcal{B})$, let $G_{(P,c)}$ be a group equipped with a group isomorphism $\theta_{(P,c)} : P \to G_{(P,c)}$. Further, if we are given two Brauer pairs (P, c) and (Q, d) and an element $g \in G$ such that $(P, c) \geq {}^g(Q, d)$, then we define a group homomorphism, $\alpha_g := \theta_{(P,c)} \circ c_g \circ \theta^{-1}_{(Q,d)}$

$$\begin{array}{ccc} P & \xrightarrow{\theta_{(P,c)}} & G_{(P,c)} \\ {\scriptstyle c_g}\uparrow & & \uparrow{\scriptstyle \alpha_g} \\ Q & \xrightarrow[\theta_{(Q,d)}]{} & G_{(Q,d)} \end{array}$$

We shall construct a strict equivalence of categories between $\mathcal{B}$ and an ordinary Mackey category of finite groups, the category algebra of which is proven to be semisimple in [Bar16, Theorem 4.6].

We build an ordinary Mackey system $\mathcal{F}$ as follows: Define $\mathfrak{K} := \{G_{(P,c)} : (P, c) \in \mathrm{Ob}(\mathcal{B})\}$, which is clearly closed under taking subgroups. The set of objects of $\mathcal{F}$ is exactly $\mathfrak{K}$. If we are given two objects $G_{(P,c)}$ and $G_{(Q,d)}$, then we define $\mathcal{F}(G_{(Q,d)}, G_{(P,c)}) := \{\alpha_g : g \in G \text{ and } (P, c) \geq {}^g(Q, d)\}$. Evidently, the category $\mathcal{F}$ is an ordinary Mackey system of finite groups.

Note that we have a strict equivalence of categories $\mathcal{G} \cong \mathcal{F}$ where $\mathcal{G}$ is the precursor category defined on page 80. Hence, by construction, we infer another strict equivalence of categories, $\mathcal{B} \cong \mathcal{M}_{\mathcal{F}}$.

By [Bar16, Theorem 4.6], we know that the category algebra $\mathbb{K}\mathcal{M}_{\mathcal{F}}$ is semisimple. Therefore, $\mathbb{K}\mathcal{B}$ is also semisimple. □

Note that, we obtain the subcategory $\mathcal{B}_f$ of $\mathcal{B}$, by replacing A in the definition of $\mathcal{B}$ with fAf, where $f \in A^G$ is a primitive idempotent. This provides us a second proof of Theorem 6.4.

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