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FUZZY NUMBERS AND FUZZY LINEAR EQUATION

M. Sc. THESIS

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JURY ACCEPTANCE AND APPROVAL

DECLARATION

I hereby attest that I have adhered to the prescribed standard thesis format, academic regulations, and ethical principles. Furthermore, I confirm that all materials utilized in this thesis are duly documented in the reference list in accordance with the stipulated regulations and ethical guidelines. I certify that the findings and outcomes presented in this thesis are original and have not been previously obtained elsewhere. Additionally, I affirm that there is no falsification of data within this thesis, and no portion of this work has been submitted as a thesis at this institution or any other university.

25/03/2025

Maher KHAZAAL

ABSTRACT

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In this thesis, we will first give detailed information about fuzzy sets and their properties. Then, after introducing fuzzy numbers, we will focus on the triangular fuzzy numbers, trapezoidal fuzzy numbers and Gaussian-type bell shape fuzzy numbers. We also used the alpha-cut method for each fuzzy number type. The alpha-cuts method is a technique used in fuzzy set theory to handle and analyze fuzzy sets. The alpha-cuts method allows for the conversion of a fuzzy set into a series of crisp sets, making it easier to perform various mathematical operations and analyses. Then, we will give the solution of first order fuzzy linear equations on each type of fuzzy number. A fuzzy set is an extension of the classical concept of a set, introduced by Zadeh in 1965 as a way to deal with uncertainty and ambiguity. Unlike classical sets, where an element either belongs to the set or does not (binary membership), fuzzy sets allow for degrees of membership. This means that an element can partially belong to a fuzzy set with a membership value between 0 and 1. A fuzzy number is a special type of fuzzy set that represents a quantity with uncertain boundaries. Unlike crisp numbers, fuzzy numbers allow for a range of possible values, each with a degree of membership. This concept is useful in situations where data is imprecise, uncertain, or ambiguous. Fuzzy linear equations are an extension of classical linear equations that incorporate the concept of fuzziness to deal with uncertainty and imprecision. In a fuzzy linear equation, the coefficients, variables, and constants can all be represented as fuzzy numbers or fuzzy sets. This allows solving linear equations where some or all of the data is not known with certainty but is instead represented by fuzzy values.

Keywords: Fuzzy Sets, Fuzzy Numbers, Trapezoidal Fuzzy Numbers, Gaussian-Type Bell Shape Fuzzy Numbers, Alpha-Cuts Method, Fuzzy Linear Equations.

ÖZET

BULANIK SAYILAR VE BULANIK DOĞRUSAL DENKLEMLER

Maher KHAZAAL

Yüksek Lisans, Matematik Anabilim Dalı

Tez Danışmanı: Prof. Dr. Naim ÇAĞMAN

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Bu tezde, önce bulanık kümeler ve özellikleri hakkında detaylı bilgi vereceğiz. Daha sonra, bulanık sayıları tanıttıktan sonra, üçgen bulanık sayılara, yamuk bulanık sayılara ve Gauss tipi çan şeklindeki bulanık sayılara odaklanacağız. Ayrıca her bulanık sayı türü için alfa-kesim yöntemini kullandık. Alfa-kesim yöntemi, bulanık küme teorisinde bulanık kümeleri ele almak ve analiz etmek için kullanılan bir tekniktir. Alfa-kesim yöntemi, bulanık bir kümenin bir dizi net kümeye dönüştürülmesini sağlayarak çeşitli matematiksel işlemleri ve analizleri gerçekleştirmeyi kolaylaştırır. Daha sonra, her bulanık sayı türü için birinci dereceden bulanık doğrusal denklemlerin çözümünü vereceğiz. Bulanık küme, 1965 yılında Zadeh tarafından belirsizlik ve muğlaklıkla başa çıkmanın bir yolu olarak ortaya atılan klasik küme kavramının bir uzantısıdır. Bir elemanın kümeye ait olduğu veya olmadığı (ikili üyelik) klasik kümelerin aksine, bulanık kümeler üyelik derecelerine izin verir. Bu, bir elemanın 0 ile 1 arasında bir üyelik değerine sahip bulanık kümeye kısmen ait olabileceği anlamına gelir. Bulanık sayı, belirsiz sınırları olan bir niceliği temsil eden özel bir bulanık küme türüdür. Kesin sayılardan farklı olarak, bulanık sayılar her biri bir üyelik derecesine sahip bir dizi olası değere izin verir. Bu kavram, verilerin kesin olmayan, belirsiz veya muğlak olduğu durumlarda yararlıdır. Bulanık doğrusal denklemler, belirsizlik ve kararsızlıkla başa çıkmak için bulanıklık kavramını içeren klasik doğrusal denklemlerin bir uzantısıdır. Bulanık doğrusal bir denklemde, katsayılar, değişkenler ve sabitlerin hepsi bulanık sayılar veya bulanık kümeler olarak temsil edilebilir. Bu, verilerin bir kısmının veya tamamının kesin olarak bilinmediği, bunun yerine bulanık değerlerle temsil edildiği doğrusal denklemlerin çözülmesine olanak tanır.

Anahtar Kelimeler: Bulanık Kümeler, Bulanık Sayılar, Yamuk Bulanık Sayılar, Gauss Tipi Çan Şeklindeki Bulanık Sayılar, Alpha-Kesim Yöntemi, Bulanık Doğrusal Denklemler.

CONTENTS

PREFACE

First and foremost, I would like to express my gratitude to my esteemed supervisor, Prof. Dr. Naim ÇAĞMAN, for his assistance in everything. Furthermore, I want to thank my family for their financial and moral support. I extend my appreciation to all my dear friends whom I have met since coming to Turkey and who have stood by my side.

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Maher KHAZAAL

1. INTRODUCTION

When it comes to mathematics, the first thing that comes to mind is certainty. However, in our daily conversations, many words that contain uncertainty, such as middle-aged person, long time, expensive car, tall building, are used, the meanings of which vary from person to person and situation to situation. Although such uncertainties that classical logic cannot define are generally accepted as unscientific, many philosophers thought about such uncertainties in the early 19th century. Einstein expressed this situation as follows: “As long as the concepts of mathematics are certain, they do not reflect the truth, and as long as they reflect the truth, they are not certain.” In the 1920s, Heisenberg first introduced the concept of uncertainty, forcing science to be multi-valued. In the early 1930s, Lukasiewicz defined the first three-valued logic system, and in the same period, quantum philosopher Black defined logic with continuous values. Although few Western philosophers adopted multi-valuedness, Lukasiewicz, Gödel and Black continued their theoretical work on the first multi-valued logic and sets, but could not find an application area for themselves. An important turning point in the modern mathematical modeling of uncertainty began in 1965 when Azeri-American Mathematician Lutfi Askerzade Zadeh (1965) from the University of California Berkeley defined fuzzy logic and therefore fuzzy set theory (Zadeh, 1975), (Zadeh, 1965).

Although Zadeh showed in this theory that mathematics, language and human intelligence could be related and that fuzzy logic created a better model of real life, it did not receive much attention from the scientific community and was met with criticism and was even shown as an example of waste of resources by the US National Science Foundation. In 1972, when Ebrahim Mamdani of Iranian origin designed a controller for a steam engine using fuzzy logic theory in England, the world's attention was focused on this subject. After the first commercial application of fuzzy logic was used in the control of a cement factory in Denmark in 1980, most countries in the world, especially Japan, made great progress in this field with research and engineering applications. Today, fuzzy logic is primarily preferred in most decision-making mechanisms. For more detailed information on fuzzy logic, fuzzy sets and their applications, it can be referred to references (Dubois and Prade, 1980), (Klir and Folger, 1988) and (Zimmermann, 1991) from which we obtain this information.

A fuzzy number is a special type of fuzzy set that represents a quantity with uncertain boundaries. Unlike crisp numbers, fuzzy numbers allow for a range of possible values, each with a degree of membership. This concept is useful in situations where the data is imprecise, ambiguous, or vague.

Fuzzy numbers are used in various fields to deal with uncertainty and imprecision. In decision making; in multi-criteria decision analysis, fuzzy numbers represent uncertain criterion weights. In control systems, fuzzy logic controllers use fuzzy numbers to handle uncertain input data. In risk assessment, fuzzy numbers quantify uncertain risks in finance and project management. Th e references we use for fuzzy numbers are; (Shang, G., at al, 2015), (Rezvani and Molani, 2014), (Ocampo and Sarmiento, 2015), (Ban, 2009).

Fuzzy linear equations extend traditional linear equations by allowing coefficients and variables to take on fuzzy values, reflecting real-world situations where parameters are not precisely known. The concept of fuzzy linear equations was further developed by authors such as Dubois and Prade, who introduced various methods to solve these equations by utilizing fuzzy arithmetic and extension principles (Dubois & Prade, 1988), (Dubois & Prade ,1980)

Fuzzy linear equations are used in various fields to model and solve problems with imprecise or uncertain data. In engineering, they are used to design systems with uncertain parameters. In economics, they are used to model market behavior with imprecise data. In decision making, they are used to solve problems with uncertain criteria and constraints. The references we use for fuzzy linear equations are; (Zhao and Li, 2017), (Wang, and Zhang, 2007), (Tanaka and Guo, 2000), (Li and Wang, 2016).

In this thesis, we will first give detailed information about fuzzy sets and their properties. Then, after introducing fuzzy numbers, we will focus on triangular fuzzy numbers, trapezoidal fuzzy numbers and Gaussian bell-shaped fuzzy numbers. We also used the alpha-cut method for each fuzzy number type. The alpha-cuts method is a technique used in fuzzy set theory to handle and analyze fuzzy sets. The alpha-cuts method allows for the conversion of a fuzzy set into a series of crisp sets, making it easier to perform various mathematical operations and analyses. Then, we will give the solution of first-order fuzzy linear equations for each type of fuzzy numbers.

2. BASIC DEFINITIONS AND THEOREMS

2.1 Classical Sets

Information about set theory was compiled from the classic books, (Halmos, 1974), (Jech, 2003) and (Enderton, 1977).

Definition 2.1.1. A set is a collection of distinct objects. The objects can be real, physical things, or abstract, mathematical things. The number of such objects can be finite or infinite. The only important thing is that the objects be distinct, i.e., uniquely identifiable.

From now on the universel set is denoted by U, and the sets are denoted by A, B, C, …

Each of these elements is called a member, or an element, of A.

x ∈ A means x is an element of A. x ∉ A means x is not an element of A.

Definition 2.1.2. An **object** U is an element or member of a set A, written $x \in A$, if x satisfies the rule defining membership is A. We can write $x \notin A$ if x is not an element of A.

Definition 2.1.3. Let U be a nonempty set, called the **universe set**, consisting of all the possible elements of concern in a particular context.

Definition 2.1.4. The **empty set** or null set, denoted $\emptyset$, is the set containing no elements. An important property of a set is the number of elements it contains.

Definition 2.1.5. Two sets ***A*** and B are **equal**, written $A = B$, if A and B contain exactly the same elements.

Theorem 2.2.1. Properties If C, A, and B are any sets, the following properties hold:

i) A = A (reflexive)
ii) If A = B, then B = A (symmetric).
iii) If A = B and B = C then A = C (transitive).

Definition 2.1.6. A set ***A*** is a **subset** of another set *B*, written $A \subset B$ if every element of A is also an element of B.

Note that, by this definition, $A \subset B$ does not exclude the possibility that every element of B is also an element of A.

Definition 2.1.7. A set A is a **proper subset** of B if $A \subset B$ and $A \neq B$. That is, if every element of A is also in B, but some element of B is not in A.

There are two cases of subset:

1. Proper subset ($A \subset U$) means $\exists x \in U$ but $x \notin A$.
2. Subset ($A \subseteq U$) means $A \subset U$ or $U \subset A$.

Theorem 2.1.2. Properties If C, A, and B are any sets, the following properties hold:

i) $\emptyset \subset A$
ii) $A \subset A$ (reflexive)
iii) $If\ A \subseteq B, A \neq B, then\ B \nsubseteq A$ (antisymetrics)
iv) If $C \subset A$ and $A \subset B$, then $C \subset B$ (transitive).
v) $A \subset B\ and\ B \subset A \Leftrightarrow A = B$.

Theorem 2.1.3. Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$. Thus, Subset ($A \subseteq U$) means $A \subset U$ or $A = U$

Definition2.1.8. The **intersection** of two sets A and B, written $A \cap B$ is the set of elements common to both A and B, i.e.,

$$A \cap B = \{x : x \in A\ and\ x \in B\}.$$

Example 2.1.1. If $A = \{x : 1 < x < 8\}$ and $B = \{x : 4 < x < 9\}$, then $A \cap B = \{x : .4 < x < 8\}$.

Definition 2.1.9. Two subsets **A** and **B** are said to be **disjoint** if $A \cap B = \emptyset$.

Theorem 2.1.4.

i) $A \cap A = A$.
ii) $A \cap \emptyset = \emptyset$.
iii) $If\ A \cap B \neq \emptyset, then\ A \neq \emptyset\ and\ B \neq \emptyset$.
iv) $C \cap (A \cap B) = (C \cap A) \cap B\ (associative)$
v) $A \cap B = B \cap A\ (commutative)$

Definition 2.1.10. The **union** of two sets A and B, written $A \cup B$, is the set of elements that are in either A or B or both, i.e.,

$$A \cup B = \{x : x \in A \, or \, x \in B\}$$

Example 2.1.2. if $A = \{x : 1 < x < 8\}$ and $B = \{x : 4 < x < 9\}$, then $A \cup B = \{x : 1 < x < 9\}$.

Definition 2.1.11. The Union of two subsets:

$$A \cup B := \{x \mid x \in A \, or \, x \in B\} = A \cup B.$$

Theorem 2.1.5. (Wei & Zhang, 2011)

i) $A \cup U = A$.

ii) $A \cup \emptyset = A$.

iii) $A \cup B = B \cup A \, (commutative)$

iv) $C \cup (A \cup B) = (C \cup A) \cup B \, (associative)$

v) $A \cup B \neq \emptyset \Leftrightarrow A \neq \emptyset \, or \, B \neq \emptyset$.

Theorem 2.1.6. Properties If C, A, and B are any sets, the following properties hold:

i) $A \cap A = A \cup A = A$

ii) $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)(distributive—\cup \; over \; \cap)$

iii) $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)(distributive—\cap \; over \; \cup)$

iv) $A \cup U = U \, and \, A \cap U = A$

v) $A = B \Leftrightarrow A \cup B \subset A \cap B$

vi) $A \cup (A \cap B) = A \cap (A \cup B) = A$

vii) $If \; C \subset B \, and \, A \subset B, then \, C \cup A \subset B$

viii) $If \; C \subset A \, and \, C \subset B, then \, C \subset A \cap B$

ix) $If \; A \subset B \; then \, C \cup A \subset C \cup B \, and \, C \cap A \subset C \cap B$.

x) $A \subset B \Leftrightarrow A \cup B = B$

xi) $A \cap B \subset A$

xii) $A \subset A \cup B$

Definition 2.1.12. Relative to a universe U, the **complement** of A, written A', is the set of all elements of the universe not contained in S, i.e.,

$$A' = \{x : x \in U \, and \, x \notin A\}$$

Theorem 2.1.7. $(A')' = A$

i) $U' = \emptyset$

ii) $\emptyset' = U$

Example 2.1.3. $If\ U = \{x : x\ is\ a\ real\ number\}$ and $A = \{x : 1 \leq x \leq 2\}$, then $A' = \{x : x < 1\ or\ x > 2\}$.

Theorem 2.1.8. The following are important properties of complements and differences.

i) $A \cup A' = U$

ii) $A \cap A' = \emptyset$

iii) $A = B' \Leftrightarrow B = A'$

iv) $A \subset A' \Leftrightarrow A = \emptyset$

v) $A \subset B \Leftrightarrow B' \subset A'$

vi) $(A \cup B)' = A' \cap B'$

vii) $(A \cap B)' = A' \cup B'$

Definition 2.1.13. Let U be the universe. Membership in a crisp subset A of U is often viewed as a **characteristic** function, χ_A: U $\rightarrow$ $\{0, 1\}$.

defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$$

Theorem 2.1.9. The union intersection of two sets A and B will be defined using the membership function as follows.. For all x $\in$ U

i) $\chi_{\mathrm{A \cup B}}(\mathrm{x}) = \max\{\chi_{\mathrm{A}}(\mathrm{x}), \chi_{\mathrm{B}}(\mathrm{x})\}$

ii) $\chi_{\mathrm{A \cap B}}(\mathrm{x}) = \min\{\chi_A(\mathrm{x}), \chi_{\mathrm{B}}(\mathrm{x})\}$

iii) $\chi_{A'}(\mathrm{x}) = 1 - \chi_A(\mathrm{x})$

Definition 2.1.14. The **cardinality** of a set A, written |A|, is the number of elements in the set. By using the membership function, the cardinality of a set A is definede as.

$$|A| = \sum_{x \in U} \mu_A(x)$$

The cardinality of the empty set is zero ($|\emptyset| = 0$).

Example 2.1.4 Let $U = \mathbb{Z}$ and $A = \{x: |x| \leq 2\}$, Then

$$\mu_A(-3) = 0, \mu_A(-2) = 1, \mu_A(-1) = 1, \mu_A(0) = 1, \mu_A(1) = 1, \mu_A(2) = 1, \mu_A(3) = 0$$

Hence

$$|A| = \sum_{x \in \mathbb{Z}} \mu_A(x) = \mu_A(-3) + \mu_A(-2) + \mu_A(-1) + \mu_A(0) + \mu_A(1) + \mu_A(2) + \mu_A(3)$$

$$|A| = \sum_{x \in \mathbb{Z}} \mu_A(x) = 0 + 1 + 1 + 1 + 1 + 1 + 0 = 5$$

Theorem 2.1.10. Properties If C, A, and B are any sets, the following properties hold:

If A ⊂ B, then |A|≤|B|, and if $A \neq B$, then |A| < |B|.

Theorem 2.1.11. (Wu, 2023) Properties If S and T are any sets, the following properties hold:

i) $|A \cup B| = |A| + |B| - |A \cap B|$.
ii) $|A'| = |U| - |A|$

Example 2.1.5. If $A = \{a_0, a_3, a_4, a_6, a_7\}$ and $B = \{a_1, a_2, a_4, a_5, a_8, a_9\}$ then $A \cup B = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ and $A \cap B = \{a_4\}$. So $|A| = 5$ and $|B| = 6$. But $|A \cap B| = 1$, since there are three elements on both lists. When we form the union, we combine the two lists of elements, but each duplicate element is only listed once. Thus, $|A \cup B| = 5 + 6 - 1 = 10$.

Definition 2.1.15. Two sets A and B are **mutually exclusive** if $A \cap B = \emptyset$. If A and B are mutually exclusive, then $|A \cap B| = 0, so\ |A \cup B| = |A| + |B|$.

Example 2.1.6. If $A = \{a_1, a_3, a_5\}$ and $B = \{a_0, a_2, a_4, a_6, a_8\}$ then $A \cup B = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_8\}$ and $A \cap B = \emptyset$. So $|A| = 3$ and $|B| = 5$. But $|A \cap B| = 0$, since no element appears on both lists. When we form the union, we combine the two lists of elements, and there are no duplicates. Thus, $|A \cup B| = 3 + 5 = 8$.

Definition 2.1.16. Let I={1,2,3,…}, **A** be a set, a partition of **A** which is denoted by π(A) is:

$$\pi(A) := \{A_i \mid i \in I;\ A_i \subseteq A\}$$

satisfied by:

a. $A_i \neq \emptyset$ for all $i \in I$.
b. $A_i \cap A_j = \emptyset$ for all $i, j \in I$ with $i \neq j$.
c. $\bigcup_{i \in I} A_i = A$.

If condition (b) does not satisfy, then π(A) becomes a cover or covering of the set A.

Example 2.1.7. Let $A = \mathbb{Z}$ (the set of all integers). A parition of $\mathbb{Z}$ can be:

$$\pi(A) = \{even\ integers, odd\ integers\}.$$

The subsets are:

$A_1 = \{\dots, -4, -2, 0, 2, 4, \dots.\}$ $(even\ integers)$.

$A_2 = \{\dots, -3, -1, 1, 3, 5, \dots\}$ $(odd\ integers)$.

These subsets are **non-empty**, disjoint, and their union equals $\mathbb{Z}$.

Definition 2.1.17. The Multiplication of a real number r and a subset A of $\mathbb{R}$:

$$rA := \{ra \mid a \in A\}$$

$$r + A := \{r + a \mid a \in A\}.$$

$$rA + s := \{ra + s \mid a \in A\}.$$

Example 2.1.8. Let A={1,2,3,4}, r=2 and s=3. Then:

1) $rA = \{2a | a \in A\} = \{2 . 1 , 2 . 2 , 2 . 3 , 2 . 4\} = \{2 , 4 , 6 , 8\}$
2) $r + A = \{r + a | a \in A\} = \{2 + 1, 2 + 2, 2 + 3, 2 + 4\} = \{3 , 4 , 5 , 6\}$
3) $rA + s = \{ra + s | a \in A\} = \{(2 \times 1) + 3, (2 \times 2) + 3, (2 \times 3) + 3, (2 \times 4) + 3\} = \{5, 7, 9, 11\}$

Definition 2.1.18. The difference between A and B, written $A \setminus B$, is the set of elements in A but not also in B:

$$A \setminus B = \{x : x \in A\ and\ x \notin B\}$$

Example 2.1.9. If $A = \{x : 1 \leq x \leq 2\}$ and $B = \{x : 8 < x < 3\}$, then $A \setminus B = \{x : 1 \leq x \leq 8\}$ and $B \setminus A = \{x : 2 < x \leq 3\}$.

We can see that, in general, $A \setminus B \neq A \setminus B$ (set difference is not commutative).

Theorem 2.1.12. (Song & Liu, 2010) Properties If A and B are any sets, the following properties hold:

i) $A \setminus B = A \setminus (A \cap B)$

ii) $|A \setminus B| = |A| - |A \cap B|$

iii) $A \setminus B = A \cap B'$

iv) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Definition 2.1.19. The symmetric difference between A and B is $(A \cup B) \setminus (A \cap B)$, i.e., the set of elements in A or B, but not both. The cardinality of a symmetric difference satisfies:

$$|(A \cup B) \setminus (A \cap B)| = |A| + |B| - 2|A \cap B|$$

Example 2.1.10. If $A = \{a_1, a_3, a_4, a_5, a_6, a_7\}$ and $B = \{a_0, a_2, a_3, a_4, a_6\}$ then $(A \cup B) \setminus (A \cap B) = \{a_0, a_1, a_2, a_5, a_7\}$, and $|(A \cup B) \setminus (A \cap B)| = 6 + 5 - 3 \times 2 = |5|$.

Definition 2.1.20. (Zadeh, 1975) A power set P(A) is a family set containing the subsets of set A. Therefore, the number of elements in the power set P(A) is represented by:

$$|P(A)| = 2^{|A|}.$$

Example 2.1.11. If $\mathrm{A} = \{\mathrm{m}, \mathrm{a}, \mathrm{h}\}$, then:

$|\mathrm{A}| = 3$

$\mathrm{P(A)} = \{\emptyset, \{\mathrm{m}\}, \{\mathrm{a}\}, \{\mathrm{h}\}, \{\mathrm{m}, \mathrm{a}\}, \{\mathrm{m}, \mathrm{h}\}, \{\mathrm{a}, \mathrm{h}\}\{m, a, h\}\}$

$|\mathrm{P(A)}| = 2^3 = 8.$

2.2 Fuzzy Sets

We have seen that belonging or membership of an object to a set is a precise concept; the object is either a member to a set or it is not, hence the membership function can take only two values, 1 or 0. The set tall men in Example 1.5 illustrates the need to increase the describing capabilities of classical sets while dealing with words. To describe gradual transitions Zadeh (1965), the founder of fuzzy sets, introduced grades between 0 and 1 and the concept of graded membership. (Zadeh, 1965)

Let us refer to Example 1.4. Each of the six elements of the universal set $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ either belongs to or does not belong to the set $S = \{x_2, x_3, x_5\}$. According to this, the characteristic function $\mu_S(x)$ takes only the values 1 or 0. Assume now that a characteristic function may take values in the interval [0, 1]. In this way the concept of membership is not any more crisp (either 1 or 0), but becomes fuzzy in the sense of representing partial belonging or degree of membership.

Consider a classical set S of the universe U. S fuzzy set S is defined by a set or ordered pairs, a binary relation

Definition 2.2.1. (Zadeh, 1965) Associates with each element U in S a real number $\mu_S(x)$ in the interval [0,1] which is assigned to U. Larger values of $\mu_S(x)$ indicate higher degrees of membership.

$$S = \{(x, \mu_S(x)) | x \in S, \mu_S(x) \epsilon [0,1]|\} \qquad (2.2.1)$$

where $\mu_S(x)$ is a function called membership function $\mu_S(x)$ specifies the grade or degree to which any element x in S belongs to the fuzzy set S

Let us express the meaning of (2.2.1) in a slightly modified way. The first elements x in the pair $(x, \mu_S(x))$ are given numbers or objects of the classical set S; they satisfy some property (P) under consideration partly (to various degrees). The second elements $\mu_S(x)$ belong to the interval (classical set) $[0, 1]$; they indicate to what extent (degree) the elements x satisfy the property P.

It is assumed here that the membership function $\mu_S(x)$is either piecewise continuous or discrete. The fuzzy set S according to definition (2.2.1) is formally equal to its membership function $\mu_S(x)$. We will identify any fuzzy set with its membership function and use these two concepts as interchangeable. Also we may look at a fuzzy set over a domain S as a function mapping S into [0, 1].

Fuzzy sets are denoted by italic letters S, T, R, …. and the corresponding membership functions by $\mu_S(x), \mu_T(x), \mu_R(x)$, ...

Elements with zero degree of membership in a fuzzy set are usually not listed.

Classical sets can be considered as a special case of fuzzy sets with all membership grades equal to 1. A fuzzy set is called normalized when at least one $x \in S$ attains the maximum membership grade 1; otherwise the set is called nonnormalized. Assume the set S is nonnormalized; then max $\mu_S(x) < 1$ To normalize the set S means to normalize its membership function $\mu_S(x)$, i.e. to divide it by max $\mu_S(x)$, which gives $\frac{\mu_S(x)}{max\ \mu_S(x)}$.

S is called empty set labeled φ if $\mu_S(x) = 0$ for each x ∈ S.

The fuzzy set $S = \{(x_1, \mu_S(x_1))\}$, where x_1 is the only value in $S \subset U$ and $\mu_A(x_1) \in$ [0,1], is called fuzzy singleton.

While the set S is a subset of the universal set U which is crisp, the fuzzy set S is not.

Instead of (1.6), some authors use the notation

$$S = \left\{\frac{\mu_S(x)}{x} \mid x \in S, \mu_S(x) \in [0,1]\right\},$$

where the symbol / is not a division sign but indicates that the top number $\mu_S(x)$is the membership value of the element x in the bottom.

Example 2.2.1. Consider the fuzzy set

$$S = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.5), (x_4, 0.1), (x_5, 0.4), (x_6, 0.9)(x_7, 0.3), (x_8, 0.8)\}$$

which also can be represented as

$$S = 0.2/x_1 + 0.7/x_2 + 0.5/x_3 + 0.1/x_4 + 0.4/x_5 + 0.9/x_6 + 0.3/x_7 + 0.8/x_8$$

It is a discrete fuzzy set consisting of six ordered pairs. The elements U_i , i = 1, . . . , 8, are not necessary numbers; they belong to the classical set

$S = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ which is a subset of a certain universal set U. The membership function $\mu_S(x)$ of S takes the following values on [0, 1]:

$$\mu_S(x_1) = 0.2, \quad \mu_S(x_2) = 0.7, \quad \mu_S(x_3) = 0.5, \quad \mu_S(x_4) = 0.1$$

$$\mu_S(x_5) = 0.4, \quad \mu_S(x_6) = 0.9, \quad \mu_S(x_7) = 0.3, \quad \mu_S(x_8) = 0.8$$

The following interpretation could be given to $\mu_S(x_i)$, i = 1, · · · , 8. The element x_5 is a full member of the fuzzy set S, while the element x_1 is a member of S a little ($\mu_S(x_1) = 0.1$ *is near* 0); x_6 and x_3 are a little more members of S; the element x_4 is almost a full member of S, while x_2 is more or less a member of S.

The fuzzy set S can be given also by the table

$$S \triangleq \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.5), (x_4, 0.1), (x_5, 0.4), (x_6, 0.9), (x_7, 0.3), (x_8, 0.8)\}$$

where the symbol ≜ means "is defined by."

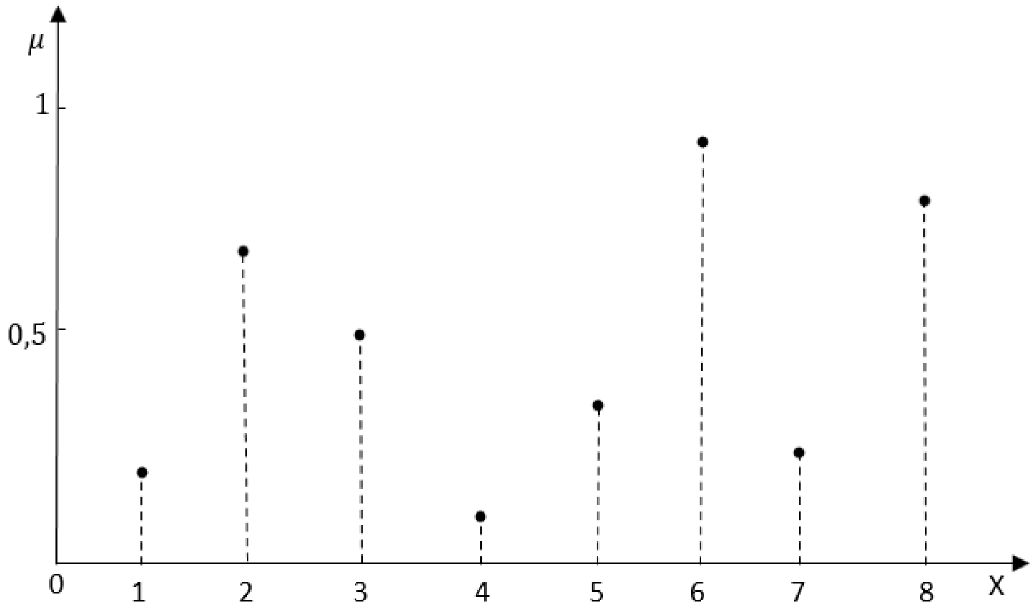


Fig. 2.2.1. Fuzzy set

Now we specify in two different ways the elements xi in S:

Assume that x_i , i = 1, · · · , 8, are integers, namely, $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6, x_7 = 7, x_8 = 8$; they belong to the set S = {1, 2, 3, 4, 5, 6,7,8}, a subset of the universe U = N, the set of all integers. The fuzzy set S becomes

$$S = \{(1, 0.2), (2, 0.7), (3, 0.5), (4, 0.1), (5, 0.4), (6, 0.9), (7,0.3), (8,0.8)\};$$

its membership function $\mu_S(x)$ shown in Fig. 2.2.1 by dots is a discrete one.

Example 2.2.2. Let us describe numbers close to 15.

First consider the fuzzy set

$$S_1 = \left\{\left(x, \mu_{S_1}(x)\right) | x \in [10,20], \mu_{S_1}(x) = \frac{1}{1 + (x - 15)^2}\right\},$$

where $\mu_{S_1}(x)$ (x) shown in Fig. 2.2.2 is a continuous function.

The fuzzy set S_1 represents real numbers close to 10.

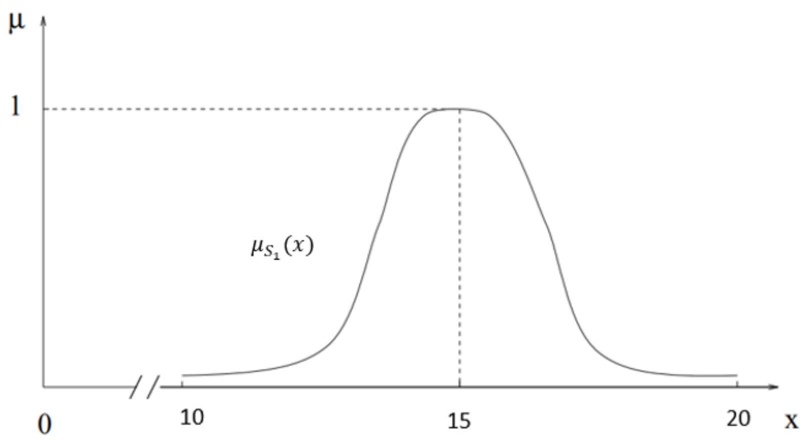


Fig. 2.2.2. Real numbers close to 10.

2.3 Normal Fuzzy Sets

Definition 2.3.1. (Zadeh, 1965) A fuzzy set is **normal** if its core is non-empty. In other words, there exists at least one point x in S in a universe U such that $\mu_S(x) = 1, \forall x \in U$, and

$\exists x \in U \text{ such that } \mu_S(x) = 1$

Normal fuzzy sets are a specific type of fuzzy set in which the membership function takes values in the interval [0, 1]. A fuzzy set is defined by its membership function, which assigns to each element in the universe of discourse a degree of membership between 0 (not a member) and 1 (full membership).

For a fuzzy set to be classified as normal, it must satisfy the condition of normality, meaning that there exists at least one element in the set that has a membership degree of 1. In other words, normal fuzzy sets must have at least one full member. This characteristic distinguishes them from non-normal fuzzy sets, where all membership degrees can be less than 1

Normal fuzzy sets are widely used in various fields, such as control systems, decision-making processes, and pattern recognition, as they allow for a more flexible representation of uncertainty and vagueness compared to traditional binary sets.

Example 2.3.1.

Of a normal fuzzy set (solid) and a sub-normal fuzzy set (dashed). If the two fuzzy sets represent qualities, then the sub-normal fuzzy sets expresses a quality that is never completely met by any element of the domain

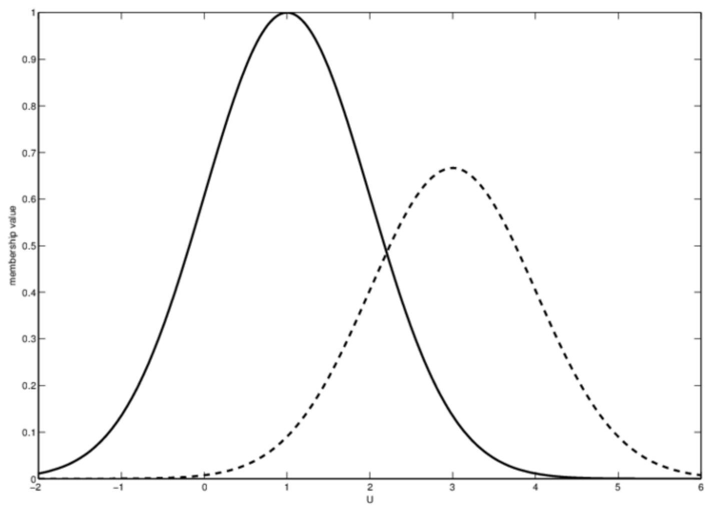


Fig. 2.2.3 Normal fuzzy set and non-normal fuzzy set.

2.4 α-level Sets

Definition 2.4.1. (Buckley & Eslami, 2002) Further we define α-level interval or α-cut, denoted by S_α, as the crisp set of elements x which belong to S at least to the degree α:

$$S_\alpha = \{x | x \in R, \mu_S(x) \geq \alpha\}, \alpha \in [0,1] \qquad (2.4.1)$$

It gives a threshold which provides a level of confidence α in a decision or concept modeled by a fuzzy set. We may use the threshold to discard from consideration those element x in S with grades of membership $\mu_S(x) < \alpha$.

Example 2.4.1. Now we employ for the same purpose the fuzzy set $S = \{(x, \mu_S(x))\}$, where x measured in cm belongs to the interval [160, 200] and $\mu_S(x)$ is defined by (see Fig 2.4.1)

$$\mu_S(x) = \begin{cases} \frac{1}{2(30)^2}(x-140)^2 & for\ \ 160 < x < 170, \\ -\frac{1}{2(30)^2}(x-200)^2 + 1 & for\ \ 170 < x < 200. \end{cases}$$

The membership function $\mu_S(x)$ is a continuous piecewise-quadratic function. The numbers on the horizontal axis x give height in cm and the vertical axis μ shows the degree to which a man can be labeled tall. According to the graph in Fig. 2.4.1, if a person's height is 160 cm, the person is a little tall (degree 0.22), 180 cm stands for almost tall (degree 0.78), 200 cm for tall (degree 1). The segment [0.22, 1] of the vertical axis μ expresses the quantification of the degree of vagueness of the word tall.

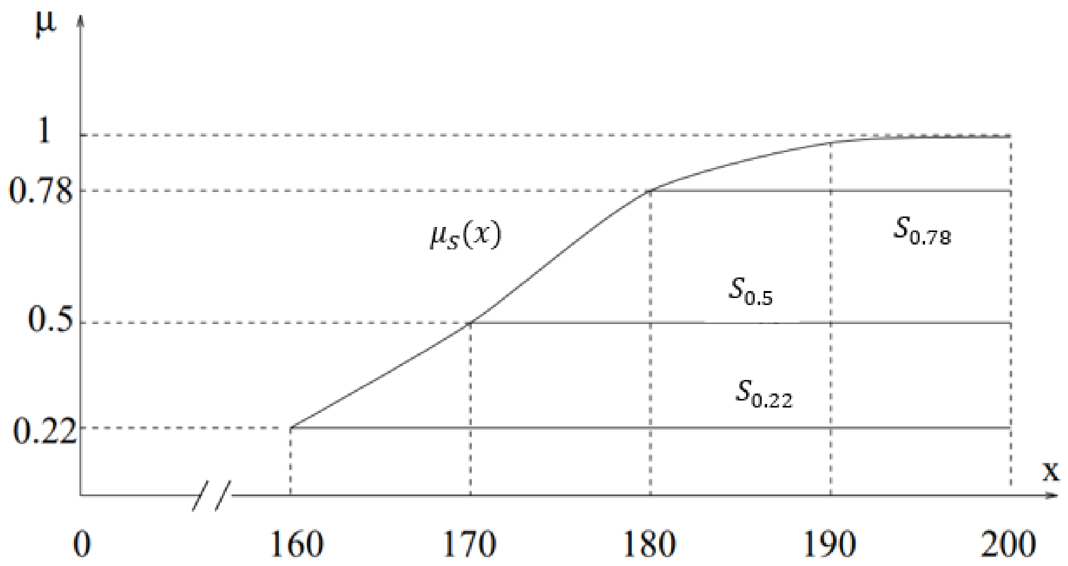


Fig. 2.4.1. Description of tall men by fuzzy set.

Consider Example 2.2.3, the set S , tall men. It has an infinite number of α-level intervals (α-cuts) denoted by Sα where α varies between 0.25 and 1. Some α-cuts shown in Fig. 2.4.1 are given below:

$$S_{0.25} = \{x|x \in R, 165 \leq x \leq 200\}, \mu_S(x) \geq 0.25,$$

$$S_{0.53} = \{x|x \in R, 175 \leq x \leq 200\}, \mu_S(x) \geq 0.53,$$

$$S_{0.8} = \{x|x \in R, 185 \leq x \leq 200\}, \mu_S(x) \geq 0.8$$

For instance we may choose as a threshold the α-cut $S_{0.53}$ thus discarding from consideration men whose height is below 175 cm

A fuzzy set S, where the universe $U = \mathbb{R}$, is convex if and only if the α-level intervals S_α (see (2.2.2)) are convex for all α in the interval (0, 1]. In such a case all α-level intervals S_α consist of one segment (see Fig. 2.3.2(a)). Otherwise the set is nonconvex (see Fig. 2.3.2(b)).

2.5. Convex and Nonconvex Fuzzy Sets

Definition 2.5.1. (Klir & Yuan, 1995) A fuzzy set in a vector space $\mathbb{R}^n$ is **convex** if, for any two elements $x, y \in \mathbb{R}^n$ and for any $\lambda \in [0,1]$, the membership function μ_S satisfies:

$$\mu_S(\lambda x + (1-\lambda)y \geq \min\{\mu_S(x), \mu_S(y)\}$$

Explanation:

- The convexity of a fuzzy set depends on the behavior of its membership function μ_S.
- For a convex fuzzy set, any point along the line segment connecting x and y has a membership value that is at least the minimum membership value of x and y.

Example 2.5.1. A convex fuzzy set satisfies the condition that any point on the line segment connecting two points in the set has a membership value greater than or equal to the minimum membership values of the two endpoints.

Description:

Let be a fuzzy set representing "numbers close to 5."

- The membership function reaches its maximum value $\mu_S(x) = 1$ at $x = 5$.
- The membership decreases linearly as you move away from $x = 5$ in both directions.

• The membership function is **symmetric** and does not exhibit any dips or valleys, ensuring convexity.

The triangular membership function can be defined as:

$$\mu_S(x) = \begin{cases} \frac{x-3}{2} & if \;\; 3 \le x \le 5, \\ \frac{7-x}{2} & if \;\; 5 \le x \le 7, \\ 0 & otherwise \end{cases}$$

Explanation:

• At $x = 5$, the membership value is $\mu_S(5) = 1$ (the peak).

• Between $x = 3$ and $x = 5$, the membership value increases linearly from 0 to 1.

• Between $x = 5$ and $x = 7$, the membership value decreases linearly from 1 to 0.

• Outside the range [3,7], the membership value is 0.

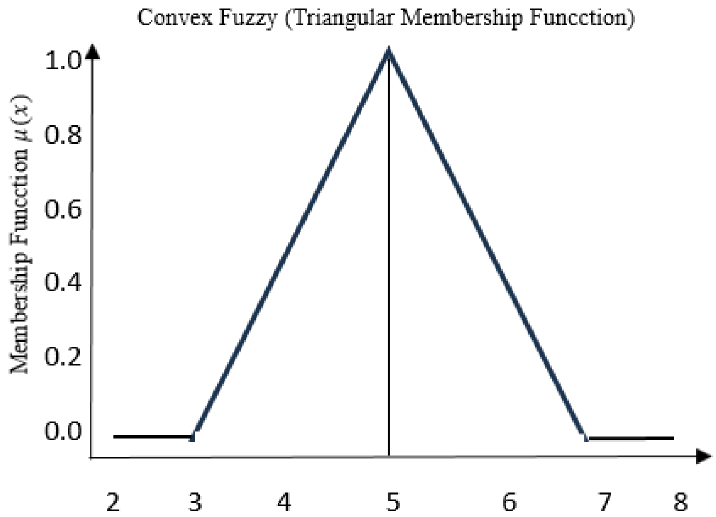


Fig. 2.3.2 (a) Convex fuzzy sets

Graph Representation:

The graph of this function is a symmetric triangle:

• Base: From $x = 3$ to $x = 7$.

• Peak: At $x = 5$, where the membership value is 1.

• The membership values decrease symmetrically as you move away from the peak.

This ensures the fuzzy set is convex because the membership function has no valleys or dips

Definition 2.5.2. (Klir & Yuan, 1995) A fuzzy set is **nonconvex** if it does not satisfy the convexity condition, i.e., there exists at least one pair $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$ such that:

$$\mu_s(\lambda x + (1 - \lambda)y) < \min\{\mu_S(x), \mu_S(y)\}.$$

Explanation: A nonconvex fuzzy set has membership functions that exhibit "dips" or "irregularities," causing points along a line segment to have lower membership values than expected for convexity.

Example 2.5.2. A fuzzy set representing "numbers that are approximately either 2 or 10" might be nonconvex. The membership function peaks around 2 and 10 but drops significantly in between, violating the convexity condition.

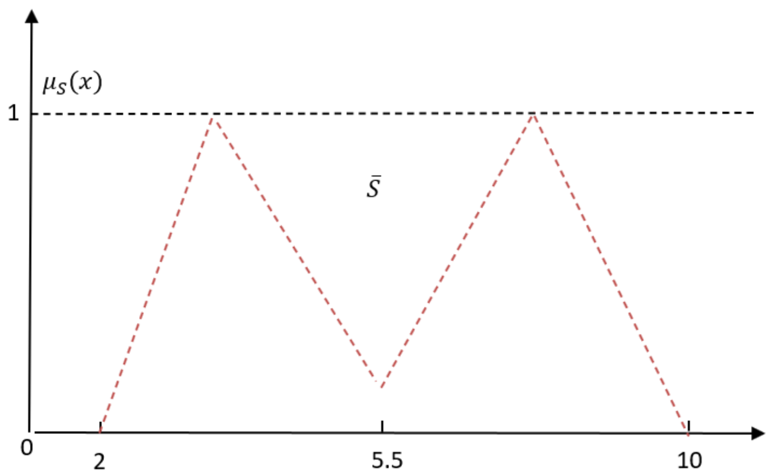


Fig 2.3.2 (b) Nonconvex fuzzy sets.

Summary: Convex Fuzzy Set: Membership values along any line segment between two points are at least the minimum membership value of the endpoints.

Nonconvex Fuzzy Set: Membership values along some line segment fall below the minimum membership value of the endpoints.

This distinction is crucial in optimization, fuzzy logic, and decision-making processes.

2.6 Basic Operations on Fuzzy Sets

In this section we consider the fuzzy sets S and T in the universe U,

$$S = \{(x, \mu_S(x))\}, \mu_S(x) \in [0,1],$$

$$T = \{(x, \mu_T(x))\}, \mu_T(x) \in [0,1].$$

The operations with S and T are introduced via operations on their membership functions $\mu_S(x)\ and\ \mu_T(x)$.

Definition 2.6.1. (Hsu & Huang, 2012) The fuzzy sets S and T are **equal** denoted by $S = T$ if and only if for every $x \in U$,

$$\mu_S(x) = \mu_T(x)$$

Definition 2.6.2. (Yang & Wu, 2019) The fuzzy set S is included in the fuzzy set T denoted by $S \subseteq T$ if for every $x \in U$,

$$\mu_S(x) \leq \mu_T(x).$$

Then S is called a subset of T.

Definition 2.6.3. (Zadeh, 1965) The fuzzy set S is called a **proper subset** of the fuzzy set T denoted $S \subset T$ when S is a subset of B and $S \neq T$, that is for every x ∈ U,

$$\mu_S(x) < \mu_T(x)$$

For instance the nonnormalized sets in Fig. 2.4.1 (a) and (b) are proper.

Example 2.6.1. Consider the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the fuzzy sets S and T defined by

$$S = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.5), (x_4, 0.7), (x_5, 0.8), (x_6, 1)\}$$

$$T = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.7)(x_4, 0.9), (x_5, 0.3), (x_6, 0.6)\}$$

$$S \subseteq T = \{(x_1, 0.1), (x_3, 0.5), (x_4, 0.7)\}$$

$$T \subseteq S = \{(x_2, 0.1), (x_5, 0.3), (x_6, 0.6)\}$$

Definition 2.6.4. (Dubois & Prade, 1988) The fuzzy sets S and $\bar{S}$ are **complementary** if

$$\mu_{\bar{S}}(x) = 1 - \mu_S(x) \; or \; \mu_S(x) + \mu_{\bar{S}}(x) = 1. \tag{2.6.4}$$

The membership function $\mu_{\bar{S}}(x)$ is symmetrical to $\mu_S(x)$ with respect to the line $\mu = 0.5$.

Then the **complementary** of S and T are written by

Example 2.6.2. Consider the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the fuzzy sets S and T defined by the table

$$S = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.5), (x_4, 0.7), (x_5, 0.8), (x_6, 1)\}$$

$$T = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.7)(x_4, 0.9), (x_5, 0.3), (x_6, 0.6)\}$$

Then $\mu_{S'}(x) = 1 - \mu_S(x)$

$$S' = \{(x_1, 0.9), (x_2, 0.7), (x_3, 0.5), (x_4, 0.3), (x_5, 0.2)\}$$

$$T' = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.3)(x_4, 0.1), (x_5, 0.7), (x_6, 0.4)\}$$

Theorem 2.6.1. (Wang & Zhang, 2007)

i) $(S')' = S$
ii) $U' = \emptyset$
iii) $\emptyset' = U$

Theorem 2.6.2. (Wang & Zhang, 2007) The following are important properties of complements and differences.

i) $S = T' \Leftrightarrow T = S'$
ii) $S \subset S' \Leftrightarrow S = \emptyset$
iii) $S \subset T \Leftrightarrow T' \subset S'$
iv) $(S \cup T)' = S' \cap T'$
v) $(S \cap T)' = S' \cup T'$

Note 2.6.1. (Zadeh, 1965) $S \cup S' = U$ and $S \cap S' = \emptyset$.

Example 2.6.3. Consider the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the fuzzy sets S and T defined by the table

$$S = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.5), (x_4, 0.7), (x_5, 0.8), (x_6, 1)\}$$

$$S' = \{(x_1, 0.9), (x_2, 0.7), (x_3, 0.5), (x_4, 0.3), (x_5, 0.2)\}$$

and

$$S \cup S' = max\{(x_1, 0.9), (x_2, 0.7), (x_3, 0.5), (x_4, 0.7), (x_5, 0.8), (x_6, 1)\} \neq U$$

$$S \cap S' = \min\{(x_1, 0.1), (x_2, 0.3), (x_3, 0.5), (x_4, 0.3), (x_5, 0.2)\} \neq \emptyset$$

Definition 2.6.5. (Wang & Zhang, 2007) The operation **intersection** of S and T denoted as $S \cap T$ is defined by

$$\mu_{S \cap T}(x) = min\big(\mu_S(x), \mu_T(x)\big), \qquad x \in X. \tag{2.6.5}$$

If $a_1 < a_2, min(a_1, a_2) = a_1$ For instance $min(0.2, 0.9) = 0.2$.

Theorem 2.6.3. (Wang & Zhang, 2007)

i) $A \cap A = A$.
ii) $A \cap \emptyset = \emptyset$.
iii) $If\ A \cap B \neq \emptyset, then\ A \neq \emptyset\ and\ B \neq \emptyset$.
iv) $C \cap (A \cap B) = (C \cap A) \cap B\ (associative)$
v) $A \cap B = B \cap A\ (commutative)$

Example 2.6.4. Consider the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the fuzzy sets S and T defined by the table

$$S = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.5), (x_4, 0.7), (x_5, 0.8), (x_6, 1)\}$$

$$T = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.7)(x_4, 0.9), (x_5, 0.3), (x_6, 0.6)\}$$

Then

$$S \cap T = \{(x_1, 0.1), (x_2, 0.1), (x_3, 0.5), (x_4, 0.7), (x_5, 0.3), (x_6, 0.6)\}$$

Definition 2.6.4. (Zhuang & Chen, 2014) The operation **union** of S and T denoted as $S \cup T$ is defined by

$$\mu_{S\cup T}(x) = max(\mu_S(x), \mu_T(x)), \qquad x \in U. \qquad (2.6.4)$$

If $a_1 < a_2, max(a_1, a_2) = a_2$. For instance max $(0.2, 0.9) = 0.9$

Theorem 2.6.5. (Zhuang & Chen, 2014)

i) $S \cup U = S$.

ii) $S \cup \emptyset = S$.

iii) $S \cup T = T \cup S$ $(commutative)$

iv) $R \cup (S \cup T) = (R \cup S) \cup T$ $(associative)$

v) $S \cup T \neq \emptyset \Leftrightarrow S \neq \emptyset$ or $T \neq \emptyset$.

Example 2.6.5. Consider the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the fuzzy sets S and T defined by the table

$$S = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.5), (x_4, 0.7), (x_5, 1), (x_6, 0.2)\}$$

$$T = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.7)(x_4, 0.9), (x_5, 1), (x_6, 0.1)\}$$

Using (2.3.5) and (2.3.6) gives

$$S \cup T = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 0.9), (x_5, 1), (x_6, 0.2)\}$$

An illustration of how to perform operations on fuzzy sets

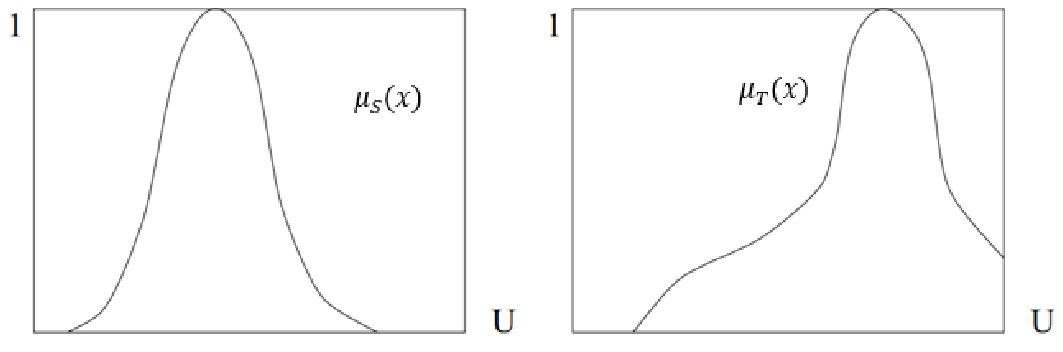


Fig. 2.4.2 Membership function $\mu_S(x), \mu_T(x)$.

Rectangles are used to symbolically depict fuzzy sets and their membership functions, which are considered to be continuous. The complements $\mu_{S'}(x)$ and $\mu_{T'}(x)$ are shown in Fig. 2.4.3, the union $\mu_{S\cup T}(x)$and the intersection $\mu_{S\cap T}(x)$.are shown in Fig. 2.4.2, and $\mu_S(x)$ and $\mu_T(x)$ are presented in Fig. 2.4.2.

Figure 2.4.5 shows that $S \cap T \in S \cup T$.

Law of Excluded Middle and Fuzzy Sets

The classical sets exhibit a significant characteristic known as the law of excluded middle, articulated as $S \cap S' = \emptyset$ and $S \cup S' = U$. It is depicted in Fig. 2.4.6 using Venn diagrams.

The law of excluded middle is inapplicable to fuzzy sets, as $S \cap S' \neq \emptyset$ and $S \cup S' \neq U$. This is seen in Figure 2.4.7.

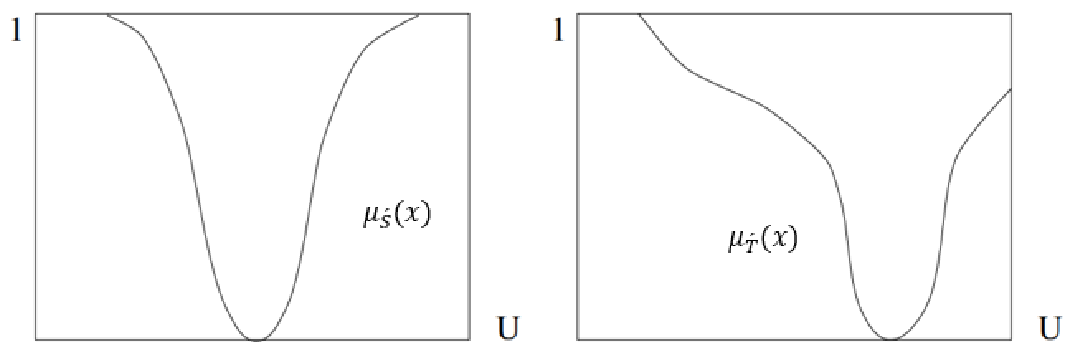


Fig. 2.4.3. Membership function $\mu_{S'}(x), \mu_{T'}(x)$.

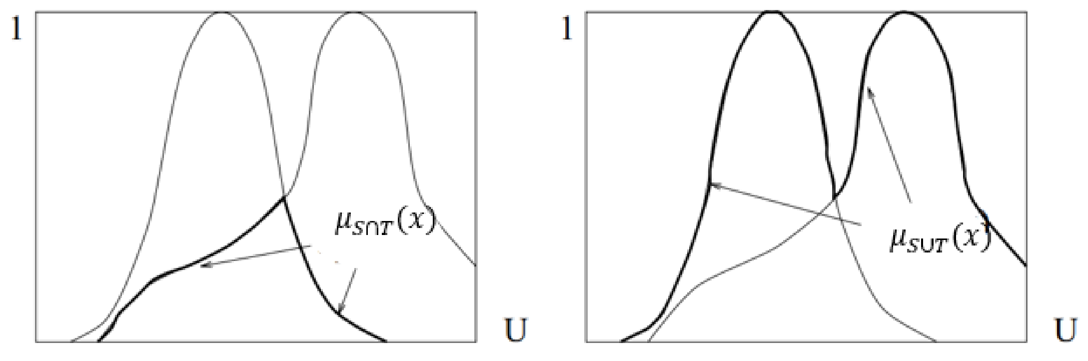
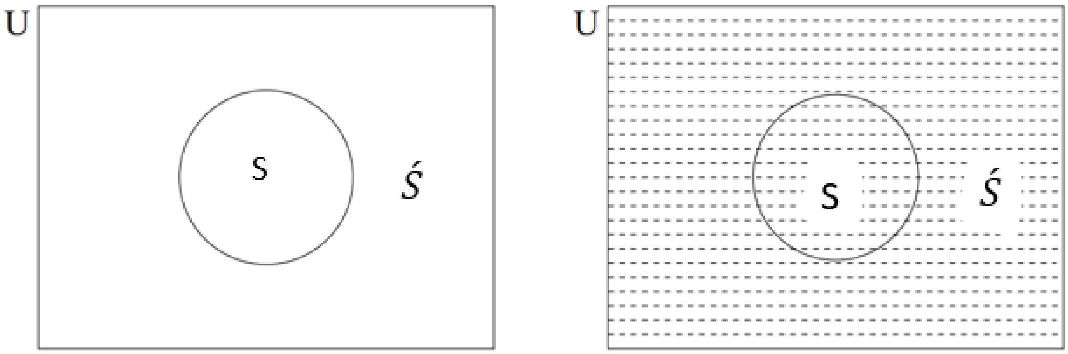


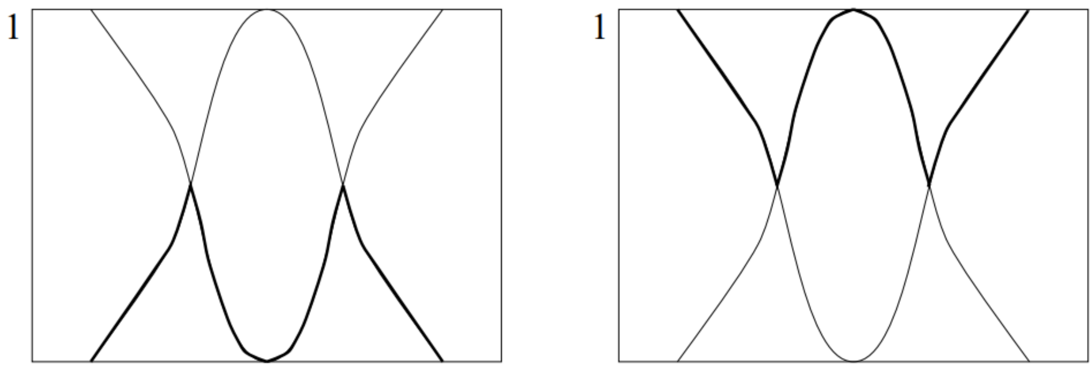
Fig. 2.4.4. Membership function of intersection and union.



$S \cap S' = \emptyset$ $\quad\quad$ $S \cup S' = U$

Fig. 2.4.6 The principle of excluded middle for classical sets

The law of the excluded middle is inherently invalid for fuzzy collections. In classical sets, each item either possesses or lacks a specific attribute, represented by 1 or 0. Fuzzy sets were established to represent the degree to which objects possess a property, ranging from 0 to 1. Numerous hues of gray exist between black and white.



$S \cap \bar{S} \neq \emptyset$ $\quad\quad$ $S \cup \bar{S} \neq U$

Fig. 2.4.7 The law of excluded middle is not valid for fuzzy sets.

The lack of the law of excluded middle in fuzzy set theory makes it less specific than that of classical set theory. However, at the same time, this lack makes fuzzy sets more general and flexible than classical sets and very suitable for describing vagueness and processes with incomplete and imprecise3 information.

3. FUZZY NUMBERS AND FUZZY LINEAR EQUATIONS

In this section we define fuzzy numbers and therir properties.

Definition 3.1. (Ban & Coroianu & Grzegorzewski, 2015) A fuzzy number S on the real line $\mathbb{R}$ is a fuzzy subset of $\mathbb{R}$ that satisfies:

a) Normality: There exists at least one $x \in \mathbb{R}$ such that $\mu_S(x) = 1$.
b) Convexity: For all $x, y \in \mathbb{R}$, and $\lambda \in [0,1]$ $\mu_S(\lambda x + (1-\lambda)y) \geq \min\big(\mu_S(x). \mu_S(y)\big)$.
c) Upper semicontinuity: The membership function μ_S is upper semicontinuous.
d) Bounded support: The support of S, defined as $\{x \in \mathbb{R} \mid \mu_S(x) > 0\}$, is bounded.

Fuzzy numbers will be denoted by bold capital letters S, T, R, . . . , and their membership functions by $\mu_S(x), \mu_T(x), \mu_R(x), \ldots$.

Here we will only focus on the three types of fuzzy numbers that have the widest usage area, which we will give below.

1. Triangular Fuzzy Numbers
2. Trapezoidal Fuzzy Numbers
3. Gaussian-type Bell Shape Fuzzy Number

Definition 3.2. (Dehghan & Hashemi, 2006) A fuzzy linear equation is a type of linear equation in which some or all of the coefficients and/or constants are represented by fuzzy numbers. These equations are used to model situations where data is imprecise, uncertain, or vague.

In a traditional linear equation, you might have something like:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

In a fuzzy linear equation, the coefficients a_i and the constant b could be fuzzy numbers, denoted by $\tilde{a}_i$ and $\tilde{b}$, respectively. This can be written as:

$$\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n = \tilde{b}$$

Fuzzy linear equations are particularly useful in fields such as decision-making, control systems, and operations research, where data may not be precise and there is a need to account for uncertainty. However, here we will focus only on the three types of first degree fuzzy linear equations with one unknown, which we will give below. Let R, S and T be fuzzy numbers.

1. $X + S = R$
2. $SX = R$
3. $SX + R = T$

After that, we will give examples of solutions of all types of fuzzy linear equations on triangular fuzzy numbers, trapezoidal fuzzy numbers and Gaussian fuzzy numbers.

3.1. Triangular Fuzzy Numbers and Fuzzy Linear Equations

Definition 3.1.1. (Tanaka & Guo, 2000) A triangular fuzzy number S is a fuzzy subset of real line ℝ, whose membership function μ_S satisfies the following conditions:

a) $\mu_S(x)$ is a continuous mapping from ℝ to the closed interval [0,1].

b) $\mu_S(x)$ =0, where $-\infty \leq x \leq a_1$.

c) $\mu_S(x)$isstrictlyi ncreasing with constant rateon $a_1 \leq x \leq a_M$.

d) $\mu_S(x) = 1$,where $x = a_M$

e) $\mu_S(x)$isstrictly decreasing with costant rate on $a_M \leq x \leq a_2$

f) $\mu_S(x) = 0$,where . $a_2 \leq x \leq \infty$

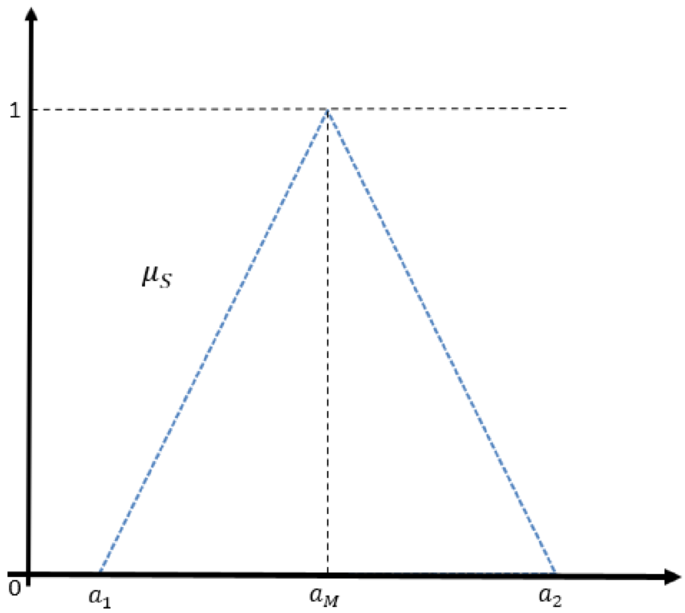


Fig. 3.1.1. Triangular fuzzy number.

Triangular fuzzy number is defined as $= (a_1, a_M, a_2)$, where all a_1, a_M, a_2 are real numbers and its membership function is given below

$$\mu_S(x) = \begin{cases} \dfrac{x - a_1}{a_M - a_1} & for \;\; a_1 \le x \le a_M \\ \dfrac{x - a_2}{a_M - a_2} & for \;\; a_M \le x \le a_2 \\ 0 & otherwise, \end{cases} \tag{3.1.1}$$

Example 3.1.1. Estimating project completion time

Imagine a construction project manager wants to estimate the completion time of a project. Based on previous experience and current circumstances, they provide the following estimates:

1. Optimistic estimate (a_1): 10 weeks (the earliest it could be done).
2. Most likely estimate (a_M): 15 weeks (the most probable duration).
3. Pessimistic estimate (a_2): 20 weeks (the latest it might take).

We can represent these estimates as a TFN with parameters (10, 15, 20). Here's what the membership function looks like:

$$\mu_S(x) = \begin{cases} \dfrac{x - 10}{15 - 10} & for \;\; 10 \le x \le 15 \\ \dfrac{x - 20}{15 - 20} & for \;\; 15 \le x \le 20 \\ 0 & otherwise, \end{cases}$$

Here’s a simple table that shows the membership values for different completion times:

Completion Time (weeks)	Membership Function Value
10	0
12	0.4
15	1
18	0.4
20	0

For time estimates less than 10 weeks or more than 20 weeks, the membership value is 0, indicating they're outside the acceptable range.

For estimates within 10 to 15 weeks, the membership increases linearly, reflecting higher confidence levels (up to 1 at 15 weeks).

For estimates from 15 to 20 weeks, the membership decreases linearly, showing a diminishing level of certainty.

This way, the project manager can better handle the uncertainty and variability in project completion times. This example is just one of many ways to apply TFNs in real-world scenarios!

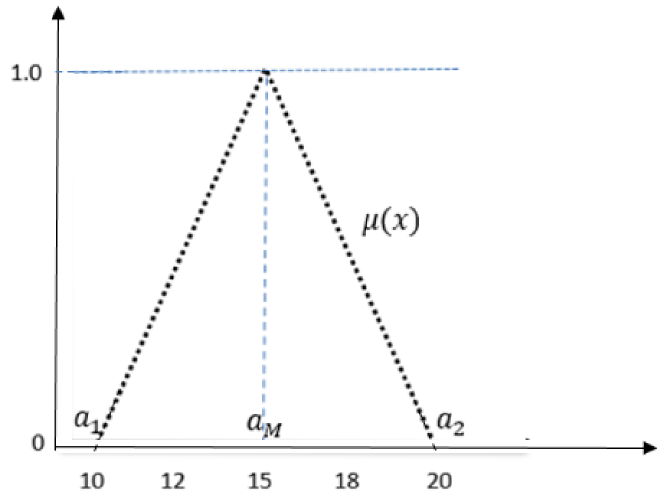


Fig. 3.2.Completion Time (weeks)

3.1.1 Addition of Triangular Fuzzy Numbers

Definition 3.1.2. (Klir & Yuan, 1995) The addition of triangular fuzzy numbers (TFNs) involves performing arithmetic operations on fuzzy numbers. A triangular fuzzy number S can be represented as $S = (a_1, a_M, a_2)$, where a_1 is the lower limit, a_M is the peak or the mode, and a_2 is the upper limit.

Given two triangular fuzzy numbers $S = (a_1, a_M, a_2)$ and $T = (b_1, b_M, b_2)$, their sum $R = S + T$ can be calculated as follows:

$$R = (a_1 + b_1, a_M + b_M, a_2 + b_2)$$

This means that:

The lower limit of R is the sum of the lower limits of S and T.

The peak or mode of R is the sum of the peaks or modes of S and T.

The upper limit of R is the sum of the upper limits of S and T.

Example 3.1.2. Suppose we have two triangular fuzzy numbers:

$$S = (3,5,7), \qquad T = (1,2,3)$$

Their sum $R = S + T$ would be:

$$R = (3 + 1,5 + 2,7 + 3)$$

$$R = (4,7,10)$$

So, the resulting triangular fuzzy number R is (4, 7, 10).

The example demonstrate how the addition of triangular fuzzy numbers works. Each component (lower limit, peak, upper limit) of one fuzzy number is added to the corresponding component of the other fuzzy number to get the resulting triangular fuzzy number.

3.1.2. Subtraction of Triangular Fuzzy Numbers

Definition 3.1.3. (Klir & Yuan, 1995) The subtraction of triangular fuzzy numbers (TFNs) involves performing arithmetic operations on fuzzy numbers. A triangular fuzzy number S can be represented as $S = (a_1, a_M, a_2)$, where a_1 is the lower limit, a_M is the peak or the mode, and a_2 is the upper limit.

Given two triangular fuzzy numbers $S = (a_1, a_M, a_2)$ and $T = (b_1, b_M, b_2)$, their difference $R = S - T$ can be calculated as follows:

$$R = (a_1 - b_2, a_M - b_M, a_2 - b_1)$$

This means that:

The lower limit of R is the difference between the lower limit of S and the upper limit of T.

The peak or mode of R is the difference between the peaks or modes of S and T.

The upper limit of R is the difference between the upper limit of S and the lower limit of T.

Example 3.1.3. Suppose we have two triangular fuzzy numbers:

$$S = (3,5,7), \qquad T = (1,2,3)$$

Their difference $R = S - T$ would be:

$$R = (3 - 3,5 - 2,7 - 1)$$

$$R = (0,3,6)$$

So, the resulting triangular fuzzy number R is (0, 3, 6).

The example demonstrate how the subtraction of triangular fuzzy numbers works. Each component (lower limit, peak, upper limit) of one fuzzy number is subtracted from the corresponding component of the other fuzzy number to get the resulting triangular fuzzy number.

Example 3.1.4. $S + X = T$

$$S = (1,3,5), \qquad T = (4,6,8)$$

The membership functions for these fuzzy numbers can be described as follows:

Now we need to find X such that S + X = T .

Given that X is also a triangular fuzzy number (c_1, c_2, c_3), the solution will be:

$$X = T - S$$

Let's subtract the fuzzy numbers:

$$X = (b_1 - a_3, b_2 - a_2, b_3 - a_1)$$

$$X = (4 - 5,6 - 3,8 - 1)$$

$$X = (-1,3,7)$$

So, X can be represented as:

$$X = (-1,3,7)$$

3.1.3 Multiplication of Triangular Fuzzy Numbers

Definition 3.1.4. (Hunang & Chen, 2015) The multiplication of triangular fuzzy numbers (TFNs) involves the arithmetic operations of fuzzy numbers. A triangular fuzzy number

S can be represented as $S = (a_1, a_M, a_2)$, where a_1 is the lower limit, a_M is the peak or the mode, and a_2 is the upper limit.

Given two triangular fuzzy numbers $S = (a_1, a_M, a_2)$ and $T = (b_1, b_M, b_2)$, their product $R = S \times T$ can be calculated as follows:

$$R = (a_1 \cdot b_1, a_M \cdot b_M, a_2 \cdot b_2)$$

This means that:

The lower limit of R is the product of the lower limits of S and T.

The peak or mode of R is the product of the peaks or modes of S and T.

The upper limit of R is the product of the upper limits of S and T.

Example 3.1.5. Suppose we have two triangular fuzzy numbers:

$$S = (1,2,3), \qquad T = (4,5,6)$$

Their product $R = S \times T$ would be:

$$R = (1 \times 4, 2 \times 5, 3 \times 6)$$

$$R = (4,10,18)$$

So, the resulting triangular fuzzy number R is (4, 10, 18) .

3.1.4 Division of Triangular Fuzzy Numbers

Definition 3.1.5. (Nguyen & Walker, 2006) The division of triangular fuzzy numbers (TFNs) involves performing arithmetic operations on fuzzy numbers. A triangular fuzzy number S can be represented as $S = (a_1, a_M, a_2)$, where a_1 is the lower limit, a_M is the peak or the mode, and a_2 is the upper limit.

Given two triangular fuzzy numbers $S = (a_1, a_M, a_2)$ and $T = (b_1, b_M, b_2)$, their quotient $R = S \div T$ can be calculated as follows:

$$R = \left(\frac{a_1}{b_2}, \frac{a_M}{b_M}, \frac{a_2}{b_1}\right)$$

If $T \neq 0$, This means that:

The lower limit of R is the result of dividing the lower limit of S by the upper limit of T.

The peak or mode of R is the result of dividing the peaks or modes of S and T.

The upper limit of R is the result of dividing the upper limit of S by the lower limit of T.

Example 3.1.6. Suppose we have two triangular fuzzy numbers:

$$S = (6,9,12), \qquad T = (2,3,4)$$

Their quotient $R = S \div T$ would be:

$$R = \left(\frac{6}{4}, \frac{9}{3}, \frac{12}{2}\right)$$

$$R = (1.5,3,6)$$

So, the resulting triangular fuzzy number R is (1.5, 3, 6).

The example demonstrate how the division of triangular fuzzy numbers works. Each component (lower limit, peak, upper limit) of one fuzzy number is divided by the corresponding component of the other fuzzy number to get the resulting triangular fuzzy number.

Example 3.1.7. $S \cdot X = T$, If $S \neq 0$ A fuzzy linear equation on the set of fuzzy numbers can be represented as $S \cdot X = T$ where S and T are fuzzy numbers, and X is the variable fuzzy number we need to find. Triangular fuzzy numbers are often used because they are simple yet effective.

Let's construct an example:

Consider the fuzzy numbers S and T with their triangular membership functions:

S is represented by the triangular fuzzy number $S = (a_1, a_M, a_2)$ with $a_1 \leq x \leq a_M \leq a_2$.

T is represented by the triangular fuzzy number $T = (b_1, b_M, b_2)$ with $b_1 \leq x \leq b_M \leq b_2$.

For simplicity, let's use specific values:

$$S = (1.2.3), \qquad T = (4.5.6)$$

Now, the fuzzy linear equation $S \cdot X = T$ can be written as:

$$(1.2.3) \cdot X = (4.5.6)$$

To solve for X, we use the property of triangular fuzzy numbers. The solution for X is also a triangular fuzzy number, $X = (x_1 . x_2 . x_3)$, where $x_1 . x_2$ and x_3 can be found by dividing the corresponding elements of T by the corresponding elements of S:

$$x_1 = \frac{d}{a} = \frac{4}{1} = 4$$

$$x_2 = \frac{e}{b} = \frac{5}{2} = 2{,}5$$

$$x_3 = \frac{f}{c} = \frac{6}{3} = 2$$

Therefore, the solution is:

$$X = (4.2{,}5.2)$$

Example 3.1.8. $S \cdot X + T = R$, If $S \neq 0$ Let's consider a fuzzy linear equation of the form $S \cdot X + T = R$ with triangular fuzzy numbers. Triangular fuzzy numbers are defined by three parameters,(a_1, a_M, a_2), representing the left endpoint, the peak, and the right endpoint of the triangle.

Given:

$$S = (a_1, a_M, a_2), \qquad T = (b_1, b_M, b_2), \qquad R = (c_1, c_M, c_2)$$

We want to solve for $X = (x_1, x_M, x_2)$ such that:

$$S \cdot X + T = R$$

For simplicity, let's use specific values:

$$S\ =\ (1, 2, 3), \qquad T\ =\ (2, 3, 4), \qquad R\ =\ (5, 6, 7)$$

Now, the fuzzy linear equation can be written as:

$$(1{,}2{,}3) \cdot X + (2{,}3{,}4) = (5{,}6{,}7)$$

To solve for X, we need to isolate X:

$$(1{,}2{,}3) \cdot X = (5{,}6{,}7) - (2{,}3{,}4)$$

The subtraction of triangular fuzzy numbers is performed by subtracting the corresponding parameters:

$$(5{,}6{,}7) - (2{,}3{,}4) = (5 - 2{,}6 - 3{,}7 - 4) = (3{,}3{,}3)$$

Now, we have:

$$(1{,}2{,}3) \cdot X = (3{,}3{,}3)$$

The solution for X can be found by dividing the corresponding parameters of $(3, 3, 3)$ by the parameters of $(1, 2, 3)$:

$$X_1 = \frac{3}{1} = 3,\ X_M = \frac{3}{2} = 1.5,\ X_2 = \frac{3}{3} = 1$$

Therefore, the solution is:

$$X = (3, 1.5, 1)$$

3.2. Trapezoidal Fuzzy Numbers and Fuzzy Linear Equations

Definition 3.2.1. (Hunang & Chen, 2015) A trapezoidal fuzzy number S is a fuzzy subset of real line R, whose membership function μ_S satisfies the following conditions:

a) $\mu_S(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval[0,1].

b) $\mu_S(x)$ =0, where $-\infty \le x \le a_1$.

c) $\mu_S(x)$isstrictlyi ncreasing with constant rateon $a_1 \le x \le b_1$.

d) $\mu_S(x) = 1$,where $b_1 \le x \le b_2$

e) $\mu_S(x)$isstrictly decreasing with costant rate on $b_2 \le x \le a_2$

f) $\mu_S(x) = 0$,where . $a_2 \le x \le \infty$

A fuzzy set $S = (a_1, b_1, b_2, a_2)$ is said to trapezoidal fuzzy number if its membership function is given by where $a_1 \le b_1 \le b_2 \le a_2$

$$\mu_S(x) = \begin{cases} \dfrac{(x - a_1)}{(b_1 - a_1)} & for \quad a_1 \le x \le b_1 \\ 1 & for\ b_1 \le x \le b_2 \\ \dfrac{(a_2 - x)}{(a_2 - b_2)} & for \quad b_2 \le x \le a_2 \\ 0 & otherwise, \end{cases} \tag{3.2.1}$$

It is a particular case of a fuzzy number with a flat.

The supporting interval is $S = [a_1, a_2]$ and the flat segment on level $\alpha = 1$ has projection $[b_1, b_2]$ on the x-axis. With the four values $a_1, a_2, b_1\ and\ b_2$, we can construct the trapezoidal number (3.2.1). It can be denoted by

$$S = (a_1, b_1, b_2, a_2).$$

If $b_1 = b_2 = a_M$ the trapezoidal number reduces to a triangular fuzzy number and is denoted by (a_1, a_M, a_M, a_2). Hence a triangular number (a_1, a_M, a_2) can be written in the form of a trapezoidal number, i.e. $(a_1, a_M, a_2) = (a_1, a_M, a_M, a_2)$.

If $[a_1, b_1] = [b_2, a_2]$, the trapezoidal number is symmetrical with respect to the line . It is in central form and represents the interval $[b_1, b_2]$ and real number close to this interval.

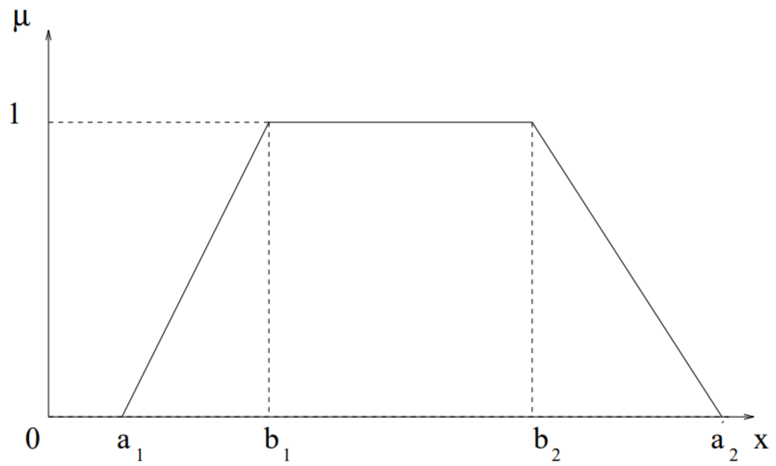


Fig.3.2.1. Trapezoidal fuzzy number.

Example 3.2.1. Estimating delivery time. Imagine you're managing a supply chain and need to estimate the delivery time for a shipment. Based on your experience and current circumstances, your estimates are as follows:

Earliest possible delivery time (a): 5 days

Earliest likely delivery time (b): 7 days

Latest likely delivery time (c): 12 days

Latest possible delivery time (d): 15 days

You can represent these estimates as a TrFN with parameters (5, 7, 12, 15). The membership function would look like this:

$$\mu_S(x) = \begin{cases} \dfrac{(x-5)}{(7-5)} & for \quad 5 \le x \le 7 \\ 1 & for\ 7 \le x \le 12 \\ \dfrac{(15-x)}{(15-12)} & for \quad 12 \le x \le 15 \\ 0, & x \le 5\ or\ x \ge 15 \end{cases}$$

Here’s a table to show the membership values for different delivery times:

Delivery Time (days)	Membership Function Value
5	0
6	0.5
7	1
9	1
12	1
14	0.33
15	0

To visualize it:

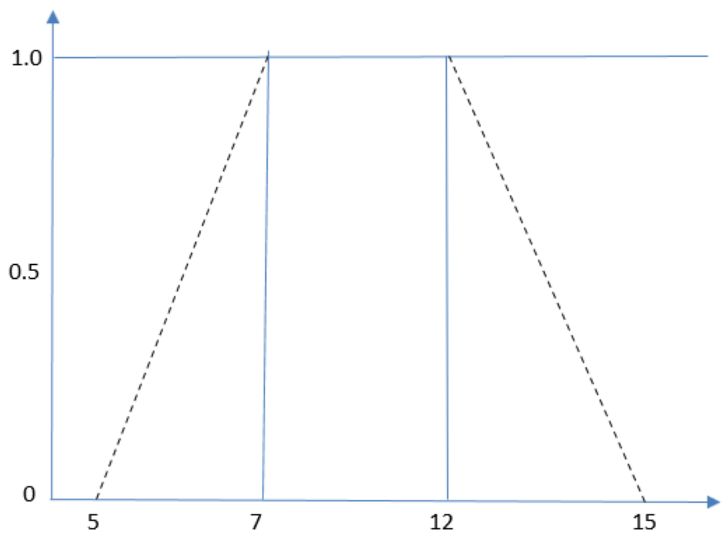


Fig. 3.2.2. The membership value remains at 1, indicating the highest level of certainty.

Delivery Time (days)

For delivery times less than 5 days or more than 15 days, the membership value is 0, indicating they're outside the expected range.

For delivery times between 5 to 7 days, the membership value increases linearly from 0 to 1.

Between 7 to 12 days, the membership value remains at 1, indicating the highest level of certainty.

From 12 to 15 days, the membership value decreases linearly from 1 to 0.

3.2.1. Addition of Trapezoidal Fuzzy Numbers

Definition 3.2.2. (Dehghan & Hashemi, 2006) The addition of trapezoidal fuzzy numbers involves combining two trapezoidal fuzzy numbers to produce a new trapezoidal fuzzy number. A trapezoidal fuzzy number is defined by four real numbers $[a_1, b_1, b_1, a_1]$, where $a_1 \leq b_1 \leq b_1 \leq a_2$. The membership function of this trapezoidal fuzzy number is:

$S = [a_1, b_1, c_1, d_1]$, $T = [a_2, b_2, c_2, d_2]$.

$$R = S + T = [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2] = [a_3, b_3, c_3, d_3]$$

$$R = [a_3, b_3, c_3, d_3]$$

Example 3.2.2. Let's consider two trapezoidal fuzzy numbers:

$$S = [1,3,5,7], \qquad T = [2,4,6,8]$$

To add these two trapezoidal fuzzy numbers, we add the corresponding endpoints:

$$S + T = [1 + 2,3 + 4,5 + 6,7 + 8] = [3,7,11,15]$$

So, the result of adding S and T is the trapezoidal fuzzy number [3, 7, 11, 15].

3.2.2. Subtraction of Trapezoidal Fuzzy Numbers

Definition 3.2.3. (Fodor & Bede, 2006, January) To subtract two trapezoidal fuzzy numbers S and T, each represented by their parameters $S = [a_1, a_2, a_3, a_4]$, and $T = [b_1, b_2, b_3, b_4]$. you simply subtract their corresponding parameters:

$$S - T = [a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1]$$

Example 3.2.3. Let's consider two trapezoidal fuzzy numbers:

$$S = [2,3,5,6], \qquad T = [1,2,4,5]$$

To subtract T from S, we subtract the corresponding elements:

Lower bound: $2 - 5 = -3$

Upper bound: $6 - 1 = 5$

Left peak: $3 - 4 = -1$

Right peak: $5 - 2 = 3$

So, the result of the subtraction $S - T$ is $[-3,5,-1,3]$.

Example 3.2.4 $S + X = T$ Let's define S and T as trapezoidal fuzzy numbers. A trapezoidal fuzzy number S can be represented by a quadruple (a_1, a_2, a_3, a_4), where $a_1 \leq a_2 \leq a_3 \leq a_4$.

Suppose:

$S = (s_1, s_2, s_3, s_4)$, $T = (t_1, t_2, t_3, t_4)$, $X = (x_1, x_2, x_3, x_4)$

Then the fuzzy linear equation $S + X = T$ translates to:

$$(s_1 + x_4, s_2 + x_3, s_3 + x_2, s_4 + x_1) = (t_1, t_2, t_3, t_4)$$

To solve for X, we can find the interval values:

$x_1 = t_1 - s_4$, $x_2 = t_2 - s_3$, $x_3 = t_3 - s_2$, $x_4 = t_4 - s_1$

Let $S = (1,2,3,4)$ and $T = (5,6,7,8)$. Then X can be calculated as:

$x_1 = 5 - 4 = 1$, $x_2 = 6 - 3 = 3$, $x_3 = 7 - 2 = 5$, $x_4 = 8 - 1 = 7$

Thus, $X = (1,3,5,7)$.

3.2.3. Multiplication of Trapezoidal Fuzzy Numbers

Definition 3.2.4. (Rezvani & Molani, 2014) Multiplication of two trapezoidal fuzzy numbers involves the interaction of their respective boundaries. A trapezoidal fuzzy number is represented as (a, b, c, d), where (a, b, c, d) are real numbers that define the shape of the trapezoid. The membership function of a trapezoidal fuzzy number is given by, Here, $\mu_S(x)$ represents the degree of membership of (x) in the fuzzy set S. The values (a) and (d) are the "feet" of the trapezoid, while (b) and (c) are the "shoulders."

When multiplying two trapezoidal fuzzy numbers $S = (a_1, b_1, c_1, d_1)$ and $T = (a_2, b_2, c_2, d_2)$, the resultant fuzzy number is obtained by considering all possible products of the boundaries:

$$S \cdot T = (\min(a_1 \cdot a_2, a_1 \cdot d_2, d_1 \cdot a_2, d_1 \cdot d_2), \min(a_1 \cdot b_2, b_1 \cdot a_2)\,, \max(c_1 \cdot c_2, d_1 \cdot b_2, b_1 \cdot d_2, a_1 \cdot c_2), \max(c_1 \cdot d_2, d_1 \cdot c_2))$$

Example 3.2.5 Let's consider two trapezoidal fuzzy numbers:

$$S = (1,2,3,4), \qquad T = (2,3,4,5)$$

We calculate the products of all boundaries:

From these products, we determine the minimum and maximum values:

$$min(2,5,8,20) = 2, \qquad min(3,4) = 3, \qquad max(12,12,10,4) = 12, \qquad max(15, 16) = 16$$

Thus, the product of the trapezoidal fuzzy numbers S and T is:

$$S \cdot T = (2,3,12,16)$$

3.2.4 Division of Trapezoidal Fuzzy Numbers

Definition 3.14. (Kaur & Kumar, 2012) Division of trapezoidal fuzzy numbers involves dividing one trapezoidal fuzzy number by another. A trapezoidal fuzzy number is typically represented as $[a_1, a_2, a_3, a_4]$, where a_1 and a_3 are the lower and upper bounds, and a_2 and a_4 are the left and right peak points.

$$S = [a_1, a_2, a_3, a_4], \qquad T = (b_1, b_2, b_3, b_4)$$

$$R = \frac{S}{T} = \left[\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right]$$

$$R = \left[\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right]$$

Example 3.16. Let's consider two trapezoidal fuzzy numbers:

$$S = [2,4,6,8], \qquad T = [1,2,3,4]$$

To divide S by T, we divide the corresponding elements, but we need to account for the fuzzy nature by considering all possible combinations. However, division is more complex than addition or multiplication due to the potential for division by zero. For this example, assume all values are non-zero.

Lower bound: $\frac{2}{4}$

Left peak: $\frac{4}{3}$

Right peak: $\frac{6}{2}$

Upper bound: $\frac{8}{1}$

So, the result of the division S/T can be calculated as:

Lower bound: $\frac{2}{4} = 0.5$

Left peak: $\frac{4}{3} \approx 1.33$

Right peak: $\frac{6}{2} = 3$

Upper bound: $\frac{8}{1} = 8$

Thus, the resulting trapezoidal fuzzy number is S/T = [0.5, 1.33, 3, 8].

Example 3.2.6. $S \cdot X = T$. Suppose we have the following trapezoidal fuzzy numbers:

$S = (1,2,3,4)$

$T = (4,,5,6,7)$

We aim to solve for *X* in the equation $S \cdot X = T$.

The multiplication of a trapezoidal fuzzy number S by a scalar X is given by:

$X \cdot S = (X \cdot a_1, X \cdot b_1, X \cdot c_1, X \cdot d_1)$

Therefore, we want to find *X* such that:

$$(X \cdot 1, X \cdot 2, X \cdot 3, X \cdot 4) = (4,5,6,7)$$

To find the interval for *X*, we solve the equations for each component:

$X \cdot 1 = 4 \Longrightarrow X = 4$

$X \cdot 2 = 5 \Longrightarrow X = \dfrac{5}{2} = 2.5$

$X \cdot 3 = 6 \Longrightarrow X = \dfrac{6}{3} = 2$

$X \cdot 4 = 7 \Longrightarrow X = \dfrac{7}{4} = 1.75$

Combining these, the solution for *X* is the interval:

$X = [1.75,2,2.5,4]$

Therefore, the fuzzy linear equation $S \cdot X = T$ with trapezoidal fuzzy numbers results in:

X being the trapezoidal fuzzy number [1.75,2,2.5,4]

Example 3.2.7. A fuzzy linear equation $SX + T = R$ where S, T, and R are trapezoidal fuzzy numbers represented by intervals.

Suppose we have the following trapezoidal fuzzy numbers:

$$S = (1,2,3,4), \qquad T = (2,3,4,5), \qquad R = (10,11,12,13)$$

Here, S, T, and R are written as intervals:

$S = (1,2,3,4)$: Lower bound = 1, First peak = 2, Second peak = 3, Upper bound = 4

$T = (2,3,4,5)$: Lower bound = 2, First peak = 3, Second peak = 4, Upper bound = 5

$R = (10, 11, 12, 13)$: Lower bound = 10, First peak = 11, Second peak = 12, Upper bound = 13

We aim to solve for X in the equation $S \cdot X + T = R$.

For trapezoidal fuzzy numbers, the operations can be approximated as follows:

1. Multiplication of a trapezoidal fuzzy number S by a scalar X:

$$X \cdot S = (X \cdot a, X \cdot b, X \cdot c, X \cdot d)$$

2. Addition of two trapezoidal fuzzy numbers:

$$S + T = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

The equation $S \cdot X + T = R$ becomes:

$$(X \cdot 1, X \cdot 2, X \cdot 3, X \cdot 4) + (2,,3,4,5) = (10,11,12,13)$$

We need to find X such that:

$$(X + 2, 2X + 3, 3X + 4, 4X + 5) = (10,11,12,13)$$

To find X, we solve each component:

1) $X + 2 = 10 \Longrightarrow X = 10 - 2 = 8$

2) $2X + 3 = 11 \Longrightarrow 2X = 11 - 3 = 8 \Longrightarrow x = \frac{8}{2} = 4$

3) $3X + 4 = 12 \Longrightarrow 3X = 12 - 4 = 8 \Longrightarrow X = \frac{8}{3} \approx 2.67$

4) $4X + 5 = 13 \Longrightarrow 4X = 13 - 5 = 8 \Longrightarrow X = \frac{8}{4} = 2$

Combining these, the solution for X is the interval:

$$X = [8,4,2.67,2]$$

Therefore, the fuzzy linear equation $SX + T = R$ with trapezoidal fuzzy numbers results in:

X being the trapezoidal fuzzy number [8,4,2.67,2].

This shows how to solve fuzzy linear equations with trapezoidal fuzzy numbers and intervals.

3.3. Gaussian-type Bell Shape Fuzzy Number

Definition 3.3.1. (Buckley & Eslami, 2002) A Bell-Shaped Fuzzy Number is a type of fuzzy number whose membership function is characterized by a symmetric bell curve. It is commonly expressed using the Gaussian function, though other forms like generalized bell functions are also used. Bell-shaped fuzzy numbers are particularly useful in modeling uncertainty with smooth and gradual transitions.

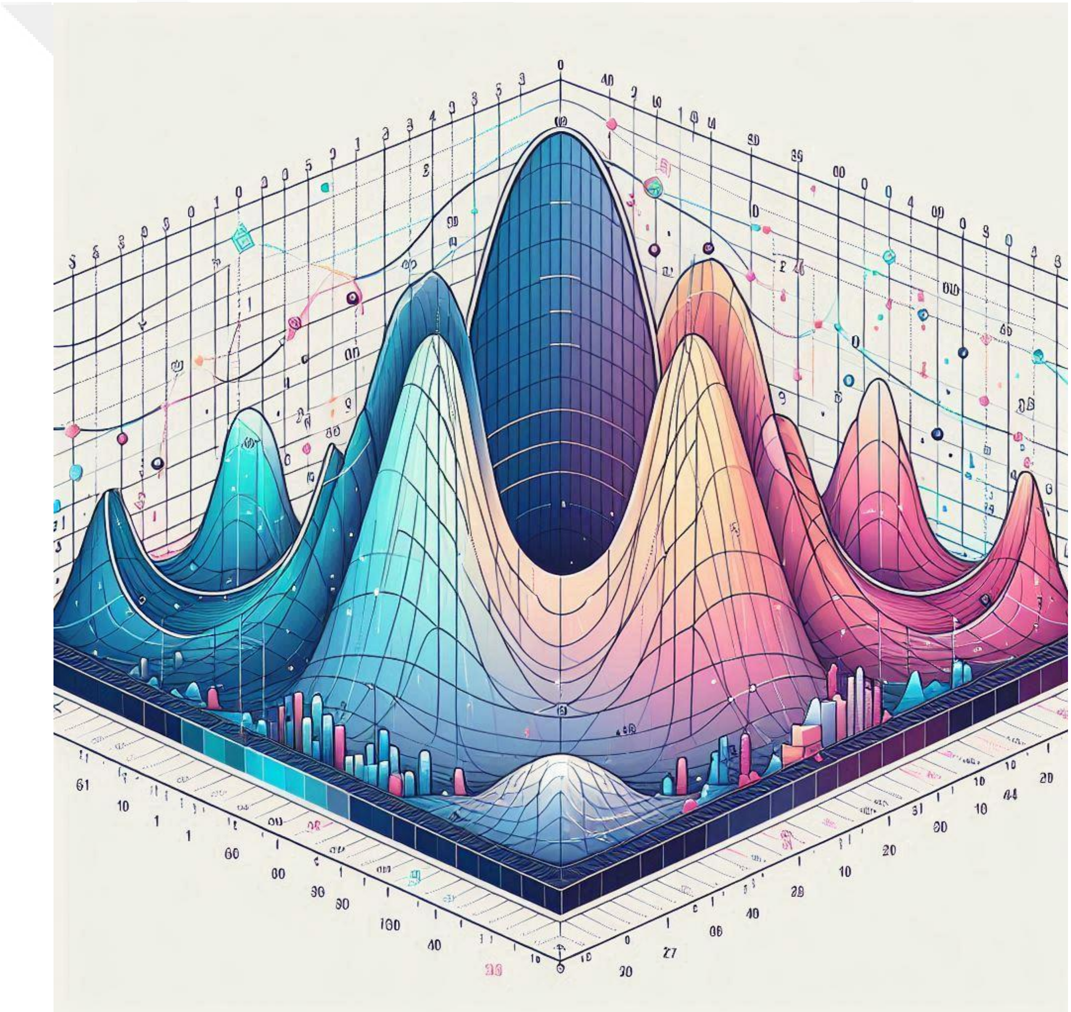


Fig 3.3.1. Gaussian Fuzzy Number

The general form of a Gaussian-type bell-shaped fuzzy number S can be written as:

$$\mu_S(x) = \exp\left(-\frac{(x-c)^2}{2\cdot\sigma^2}\right)$$

where:

c: the center or peak point of the bell (the most certain value).

σ: the spread or width of the bell shape, controlling the degree of fuzziness.

x: the membership value at point (ranges from 0 to 1)

Examples 3.3.1. Consider a Gaussian fuzzy number S with the following parameters:

Center $c = 5$

Standard deviation $\sigma = 1$

The membership function for this fuzzy number is:

$$\mu_S(x) = \exp\left(-\frac{(x-5)^2}{2\cdot 1^2}\right) = \exp\left(-\frac{(x-5)^2}{2}\right)$$

This function will yield the following degrees of membership for some values of x:

x	$\mu_S(x)$
3	0.1353
4	0.6065
5	1.0000
6	0.6065
7	0.1353

As you can see, the membership degree is highest at the center (x = 5) and decreases symmetrically as (x) moves away from the center.

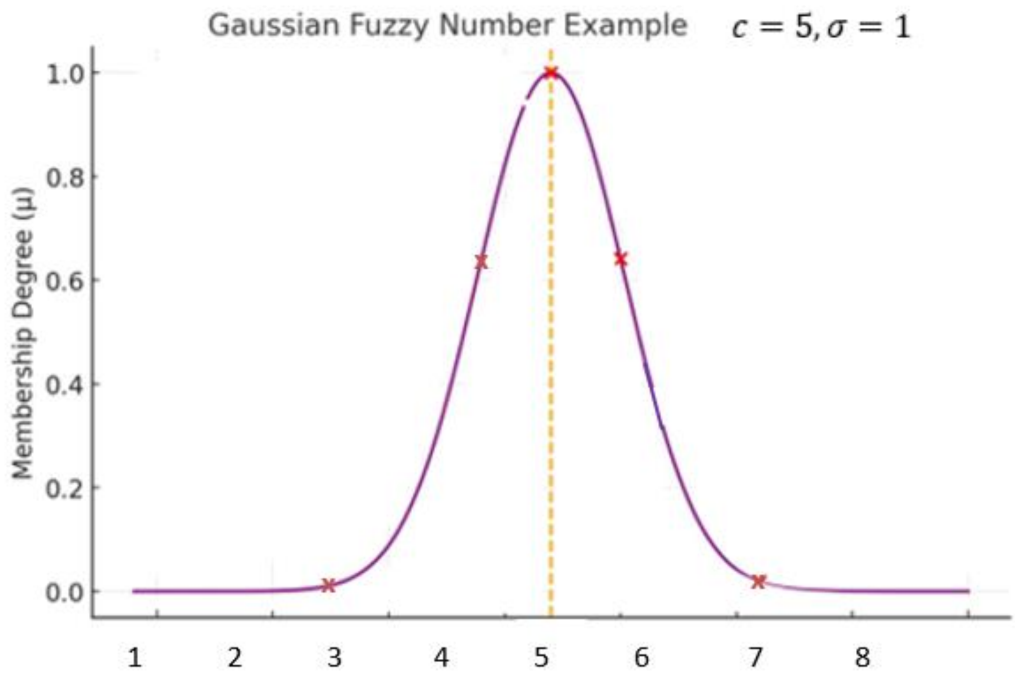


Fig 3.3.2. Gaussian fuzzy number ($c = 5, \sigma = 1$)

The Gaussian fuzzy number with these parameters and example points is shown below:

3.3.1. Addition of Gaussian-Type Bell Shape Fuzzy Numbers

Definition 3.3.2. (Zhuang & Chen, 2014) Let's denote two Gaussian fuzzy numbers as S and T with the following membership functions:

$$\mu_S(x) = \exp\left(-\frac{(x - c_1)^2}{2\sigma_1^2}\right)$$

$$\mu_T(x) = \exp\left(-\frac{(x - c_1)^2}{2\sigma_2^2}\right)$$

Where c_1 and c_2 are the centers, and σ_1 and σ_2 are the standard deviations of the fuzzy numbers S and T, respectively.

The result of adding two fuzzy numbers $R = S + T$ is a new Gaussian fuzzy number with a combined membership function. The parameters of the resultant fuzzy number R can be obtained as follows:

Combined Center

The center c_3 of the resulting fuzzy number R is the sum of the centers of S and T:

$$c_3 = c_1 + c_2$$

Combined Standard Deviation

The standard deviation σ_3 of the resulting fuzzy number R can be found using the following formula:

$$\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2}$$

Example 3.3.2. Let's consider two Gaussian fuzzy numbers S and T with the following parameters:

$$S: c_1 = 3, \quad \sigma_1 = 2, \qquad T: c_2 = 5, \quad \sigma_2 = 3$$

The parameters for the resulting fuzzy number $R = S + T$:

Center: $c_3 = 3 + 5 = 8$

Standard Deviation: $\sigma_3 = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} \approx 3.61$

The membership function for the resulting fuzzy number R is:

$$\mu_R(x) = \exp\left(-\frac{(x-8)^2}{2 \cdot 3.61^2}\right)$$

X	$\mu_R(x)$
5	0.2912
6	0.4868
7	0.7575
8	1.000
9	0.7575
10	0.4868
11	0.2912

As you can see, the membership degree is highest at the center $x = 8$ and symmetrically decreases as we move away from the center.

3.3.2. Subtraction of Gaussian-Type Bell Shape Fuzzy Numbers

Definition 3.3.3. (Klir & Yuan, 1995) Given two Gaussian fuzzy numbers S and T, with their respective membership functions defined as:

$$\mu_S(x) = \exp\left(-\frac{(x - c_1)^2}{2 \cdot \sigma_1^2}\right)$$

$$\mu_T(x) = \exp\left(-\frac{(x - c_2)^2}{2 \cdot \sigma_2^2}\right)$$

Where c_1 and c_2 are the centers, and σ_1 and σ_1 are the standard deviations.

The result of subtracting these two fuzzy numbers $R = S - T$ is another Gaussian fuzzy number with the following parameters:

Combined Center

The center c_3 of the resulting fuzzy number R is the difference between the centers of S and T:

$$c_3 = c_1 - c_2$$

Combined Standard Deviation

The standard deviation σ_3 of the resulting fuzzy number R is calculated similarly to the addition case:

$$\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2}$$

Example 3.3.3. Let's consider two Gaussian fuzzy numbers S and T with the following parameters:

$$S\colon\ c_1 = 8, \quad \sigma_1 = 3, \qquad T\colon\ c_2 = 5, \quad \sigma_2 = 2$$

The parameters for the resulting fuzzy number $R = S - T$:

Center: $c_3 = 8 - 5 = 3$

Standard Deviation: $\sigma_1 = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13} \approx 3.61$

The membership function for the resulting fuzzy number R is:

$$\mu_R(x) = \exp\left(-\frac{(x-3)^2}{2 \cdot 3.61^2}\right)$$

To illustrate the results, you can calculate the membership degrees for some values of (x) using the new membership function. For example:

X	$\mu_R(x)$
0	0.2912
1	0.4868
2	0.7575
3	1.0000
4	0.7575
5	0.4868
6	0.2912

As you can see, the membership degree is highest at the center (x = 3) and symmetrically decreases as we move away from the center.

Example 3.3.4. $S + X = T$, of a fuzzy linear equation where S and T are bell-shaped fuzzy numbers.

Bell-shaped fuzzy numbers are often represented by Gaussian membership functions, which are commonly used to model uncertainty in fuzzy systems.

A Gaussian fuzzy number S can be characterized by its mean μ and standard deviation σ. The membership function for a Gaussian fuzzy number is given by:

$$\mu_S(x) = exp\left(-\frac{(x-c_S)^2}{2 \cdot \sigma^2}\right)$$

Let's define S and T as bell-shaped fuzzy numbers with their respective means and standard deviations.

Let's say:

$$S = (c_S, \sigma_S) = (3,1)$$

$$T = (c_T, \sigma_T) = (6,1.5)$$

The membership functions for these fuzzy numbers are:

For S:

$$\mu_S(x) = exp\left(-\frac{(x-3)^2}{2 \cdot 1^2}\right)$$

For T:

$$\mu_T(x) = exp\left(-\frac{(x-6)^2}{2 \cdot 1.5^2}\right)$$

Now we need to find X such that $S + X = T$.

Given that X is also a bell-shaped fuzzy number characterized by (c_X, σ_X), the solution will be:

$$X = T - S$$

Let's subtract the fuzzy numbers:

$$c_X = c_T - c_S = 6 - 3 = 3$$

$$\sigma_S = \sqrt{\sigma_T^2 - \sigma_S^2} = \sqrt{1.5^2 - 1^2} = \sqrt{2.25 - 1} = \sqrt{1.25} \approx 1.12$$

So,X can be represented as:

$$X = (c_X, \sigma_X) = (3, 1.12)$$

The membership function for X would be:

$$\mu_X(x) = exp\left(-\frac{(x-3)^2}{2 \cdot 1.12^2}\right)$$

3.3.3. Multiplication of Gaussian-Type Bell Shape Fuzzy Numbers

Definition 3.3.4. (Jain & Biesinger & Linford, 2018) Given two Gaussian fuzzy numbers S and T, with their respective membership functions defined as:

$$\mu_S(x) = exp\left(-\frac{(x-c_S)^2}{2 \cdot \sigma_S^2}\right)$$

$$\mu_T(x) = \exp\left(-\frac{(x-c_T)^2}{2 \cdot \sigma_T^2}\right)$$

Where c_S and c_T are the centers, and σ_S and σ_T are the standard deviations.

The result of multiplying these two fuzzy numbers $R = S \cdot T$ is another Gaussian fuzzy number with the following parameters:

Combined Center

The center c_R of the resulting fuzzy number R is the product of the centers of S and T:

$$c_R = c_S \cdot c_T$$

Combined standard deviation

The standard deviation σ_R of the resulting fuzzy number R is found using the following formula:

$$\sigma_R = c_R \sqrt{\left(\frac{\sigma_S}{c_S}\right)^2 + \left(\frac{\sigma_T}{c_T}\right)^2}$$

Example 3.3.5. Let's consider two Gaussian fuzzy numbers S and T with the following parameters:

S: $c_S = 3, \quad \sigma_S = 2, \quad T\colon\ c_T = 4, \quad \sigma_T = 3$

The parameters for the resulting fuzzy number $R = S \cdot T$ are:

Center: $c_R = 3 \cdot 4 = 12$

Standard deviation:

$$\sigma_R = 12 \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{3}{4}\right)^2} = 12 \sqrt{\frac{4}{9} + \frac{9}{16}} = 12 \sqrt{\frac{145}{144}} \approx 12 \cdot 1.0035 = 12.042$$

So, the Gaussian fuzzy number R has:

Center $c_R = 12$

Standard deviation $\sigma_R = 12.042$

The membership function for the resulting fuzzy number R will be:

$$\mu_R(x) = \exp\left(-\frac{(x-12)^2}{2 \cdot 12.042^2}\right)$$

x	$\mu_R(x)$
8	0.9385
10	0.9824
12	1.0000
14	0.9824
16	0.9385

As you can see, the membership degree is highest at the center $x = 12$ and symmetrically decreases as we move away from the center.

3.3.4 Division of Gaussain-Type Bell Shape Fuzzy Numbers

Definition 3.3.5. (Mohammadi & Fadavi, 2018) Given two Gaussian fuzzy numbers S and T, with their respective membership functions defined as:

$$\mu_S(x) = \exp\left(-\frac{(x - c_S)^2}{2 \cdot \sigma_S^2}\right)$$

$$\mu_T(x) = \exp\left(-\frac{(x - c_T)^2}{2 \cdot \sigma_T^2}\right)$$

Where c_S and c_T are the centers, and σ_1 and σ_1 are the standard deviations.

The result of dividing these two fuzzy numbers $R = S/T$ is another Gaussian fuzzy number with the following parameters:

Combined Center

The center c_R of the resulting fuzzy number R is the quotient of the centers of S and T:

$$c_R = \frac{c_S}{c_T}$$

Combined standard deviation

The standard deviation σ_R of the resulting fuzzy number R is found using the following formula:

$$\sigma_R = c_R \sqrt{\left(\frac{\sigma_S}{c_S}\right)^2 + \left(\frac{\sigma_T}{c_T}\right)^2}$$

Example 3.3.5. Let's consider two Gaussian fuzzy numbers S and T with the following parameters:

$$S: c_S = 8, \qquad \sigma_S = 2, \qquad T: c_T = 4, \qquad \sigma_T = 1$$

The parameters for the resulting fuzzy number $R = S/T$:

Center: $c_R = \frac{8}{4} = 2$

Standard Deviation:

$$\sigma_R = 2\sqrt{\left(\frac{2}{8}\right)^2 + \left(\frac{1}{4}\right)^2} = 2\sqrt{\frac{1}{16} + \frac{1}{16}} = 2\sqrt{\frac{2}{16}} = 2\sqrt{\frac{1}{8}} \approx 2 \cdot 0.3536 = 0.7072$$

So, the Gaussian fuzzy number R has:

Center $c_R = 2$

Standard deviation $\sigma_R \approx 0.7072$

The membership function for the resulting fuzzy number R will be:

$$\mu_R(x) = \exp\left(-\frac{(x-2)^2}{2 \cdot 0.7072^2}\right)$$

To calculate the membership degrees for some values of (x) using the new membership function, you can use the following example values:

x	$\mu_R(x)$
1	0.7788
1.5	0.9608
2	1.0000
2.5	0.9608
3	0.7788

As you can see, the membership degree is highest at the center (x = 2) and symmetrically decreases as we move away from the center.

Example 3.3.6. Let's consider the following values:

S: center $c_S = 6$, standard deviation $\sigma_S = 2$

T: center $c_T = 12$, standard deviation $\sigma_T = 3$

Using the theorem:

Center of X:

$$c_X = \frac{12}{6} = 2$$

Standard Deviation of X:

$$\sigma_X = |2|\sqrt{\left(\frac{3}{12}\right)^2 + \left(\frac{2}{6}\right)^2} = 2\sqrt{\frac{1}{16} + \frac{1}{9}} = 2\sqrt{\frac{16+9}{144}} = 2\sqrt{\frac{25}{144}} = 2 \cdot \frac{5}{12} = \frac{10}{12} = \frac{5}{6}$$

$$\approx 0.833$$

Thus, the Gaussian fuzzy number X has:

Center $c_X = 2$

- Standard deviation $\sigma_X \approx 0.833$

The membership function for the resulting fuzzy number X will be:

$$\mu_X(x) = \exp\left(-\frac{(x-2)^2}{2 \cdot 0.833^2}\right)$$

To calculate the membership degrees for some values of (x) using the new membership function, you can use the following example values:

x	$\mu_X(x)$
1.5	0.8255
1.8	0.9636
2	1.0000
2.2	0.9636
2.5	0.8255

As you can see, the membership degree is highest at the center ($x = 2$) and symmetrically decreases as we move away from the center.

Example 3.3.7. Let's consider an example with specific Gaussian-type fuzzy numbers:

Given:

$$S = (c_S = 4, \quad \sigma_S = 1), \quad T = (c_T = 1, \quad \sigma_T = 0.5),$$
$$R = (c_R = 6, \quad \sigma_R = 1.5)$$

Find X:

1) Isolate XS:

$$XS = R - T = \left(c_R - c_T, \sqrt{\sigma_R^2 + \sigma_T^2}\right) = \left(6 - 1, \sqrt{1.5^2 + 0.5^2}\right) = \left(5, \sqrt{2.25 + 0.25}\right)$$
$$= \left(5, \sqrt{2.5}\right) \approx (5{,}1.58)$$

2) Inverse of S:

$$S^{-1} = \left(\frac{1}{4}, \frac{1}{1}\right) = (0.25{,}1)$$

3) Solve for X:

$$X = (5{,}1.58) \cdot (0.25{,}1) = (5 \cdot 0.25{,}1.58 \cdot 1) = (1.25{,}1.58)$$

Result

The Gaussian-type bell-shaped fuzzy number X that satisfies the equation $XS + T = R$ is:

$$X = (1.25, 1.58)$$

Parameter	Fuzzy Number S	Fuzzy Number T	Fuzzy Number R	Resulting Fuzzy Number X
Mean (c)	4	1	6	1.25
Standard Deviation σ	1	0.5	1.5	1.58

Here is the example in tabular format for better clarity:

4. FUZZY NUMBERS AND FUZZY LINEAR EQUATIONS USING α-cuts METHOD

In this section, we used the alpha-cut method for each fuzzy number type. The alpha-cuts method is a technique used in fuzzy set theory to handle and analyze fuzzy sets. The alpha-cuts method allows for the conversion of a fuzzy set into a series of crisp sets, making it easier to perform various mathematical operations and analyses. Then, we will give the solution of first order fuzzy linear equations on each type of fuzzy number.

Definition 4.1. (Buckley & Eslami, 2002) Further we define **α-level interval or α-cut**, denoted by S_α, as the crisp set of elements x which belong to S at least to the degree α:

$$S_\alpha = \{x \mid x \in R, \mu_S(x) \geq \alpha\}, \alpha \in [0, 1]$$

It establishes a threshold that offers a confidence level α in a choice or notion represented by a fuzzy set. We may utilize the threshold to exclude from consideration those elements x in S for which the membership grades $\mu_S(x) < \alpha$. are less than α.

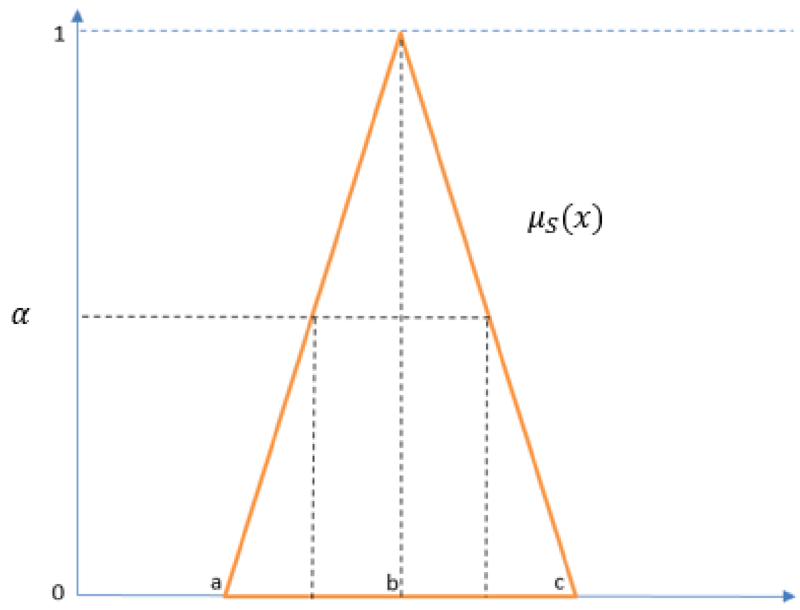


Fig 4.1. fuzzy numbers whit α-level interval or α-cut

Example 4.1. Triangular fuzzy number. Consider a triangular fuzzy number S with a membership function $\mu_S(x)$ defined by three points: (a, 0), (b, 1), and (c, 0). These points define the shape of the triangle:

a is the left endpoint where $\mu_S(x) = 0$.

b is the peak where $\mu_S(x) = 1$.

c is the right endpoint where $\mu_S(x) = 0$ again.

Let's use specific values: a = 2, b = 5, and c = 8. The membership function $\mu_S(x)$is defined as:

$$\mu_S(x) = \begin{cases} 0 & if \quad x \leq a \\ \dfrac{x-a}{b-a} & if \; a < x \leq b \\ \dfrac{c-x}{c-b} & if \;\; b < x < c \\ 0 & if \quad x \geq c \end{cases}$$

Finding α-cuts

To find the α-cut S_α for a given α $(0 < \alpha \leq 1)$, we need to determine the intervals on the x-axis where $\mu_S(x)$ is at least α.

1. Identify the α level: Choose a value for α. For this example, let's use α = 0.5.

2. Determine the intervals: Calculate the x-values where $\mu_S(x)$ = α (in this case, 0.5).

For the left side of the peak (a to b):

$$\frac{x-a}{b-a} = 0.5 \rightarrow x - a = 0.5(b-a) \rightarrow x = a + 0.5(b-a)$$

$$x = 2 + 0.5(5-2) = 2 + 1.5 = 3.5$$

For the right side of the peak (b to c):

$$\frac{c-x}{c-b} = 0.5 \rightarrow c - x = 0.5(c-b) \rightarrow x = c - 0.5(c-b)$$

$$x = 8 - 0.5(8-5) = 8 - 1.5 = 6.5$$

3. Form the α-cut interval: The α-cut $S_{0.5}$ is the interval [3.5, 6.5].

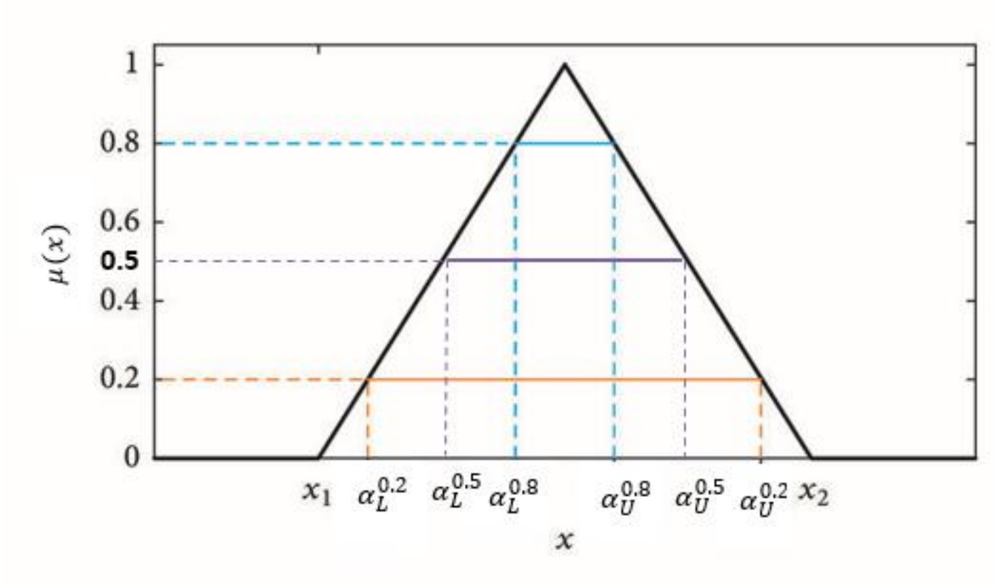


Fig 4.2. α-cuts for different α Levels

For α = 0.2: Calculate the x-values as before. The interval might be [2.6, 7.4].

For α = 0.8: Calculate the x-values. The interval might be [4.4, 5.6].

These intervals represent the α-cuts of the fuzzy number for different α levels. The intervals become narrower as α increases, focusing on elements with higher membership degrees.

4.1 Triangular Fuzzy Numbers with α-cuts

Definition 4.1.1. (Shang, Zaiyue & Cungen, 2015) An α-cut of a triangular fuzzy number $S = (a_1, a_2, a_3)$ is a precise interval that delineates the range of values for which the membership function of the fuzzy number meets or exceeds a specified threshold α ($0 \leq \alpha \leq 1$).

L_S^α denotes the lower limit of the α-cut, which may be articulated as:

$$L_S^\alpha = a_1 + (a_2 - a_1) \cdot \alpha$$

U_S^α is the upper bound of the α-cut can be defined as:

$$U_S^\alpha = a_3 - (a_3 - a_2) \cdot \alpha$$

For a triangular fuzzy number $S = (a_1, a_2, a_3)$, the functions L_S^α and U_S^α can be defined as:

$$L_S^\alpha = a_1 + (a_2 - a_1) \cdot \alpha$$

$$U_S^\alpha = a_3 - (a_3, a_2) \cdot \alpha$$

Mathematically, the α-cut of S is defined by

$$S_\alpha = [a_1 + (a_2 - a_1) \cdot \alpha, a_3 - (a_3, a_2) \cdot \alpha]$$

Example 4.1.1. Let's consider a triangular fuzzy number $S = (3,5,7)$. To find the α-cut for $\alpha = 0.5$, we substitute α into the formula:

$$S_{0.5} = [3 + (5 - 3) \cdot 0.5, 7 - (7 - 5) \cdot 0.5]$$

$$S_{0.5} = [3 + 1, 7 - 1]$$

$$S_{0.5} = [4,6]$$

So, the α-cut of S at $\alpha = 0.5$ is the interval [4, 6].

4.1.1. Addition Triangular Fuzzy Numbers Using α-cuts Method

Definition 4.1.2. (Ocampo & Sarmiento, 2015) Let $S = (a, b, c)$ and $T = (p, q, r)$ be two imprecise quantities characterized by their membership functions

$$\mu_S(x) = \begin{cases} \dfrac{x - a}{b - a} & a \le x \le b \\ \dfrac{c - x}{c - b} & b \le x \le c \end{cases}$$

$$\mu_T(x) = \begin{cases} \dfrac{x - p}{p - q} & p \le x \le q \\ \dfrac{r - x}{r - q} & q \le x \le r \end{cases}$$

Then $\alpha_S = \{(b - a)\alpha + a, c - (c - b)\alpha\}$ and $\alpha_T = \{(q - p)\alpha + p, r - (r - q)\alpha\}$. The α-cuts of fuzzy numbers S and T, respectively. To compute the sum of fuzzy numbers S and T, we initially add the α-cuts of S and T utilizing interval arithmetic.

$$\begin{aligned} \alpha_S + \alpha_T &= [(b - a)\alpha + a, c - (c - b)\alpha] + [(q - p)\alpha + p, r - (r - q)\alpha] \\ &= [(b - a + q - p)\alpha + a + p, c + r - (c - b + r - q)\alpha] \qquad (4.1.2) \end{aligned}$$

To get the membership function $\mu_{S+T}(x)$, we equal x to both the first and second components in (4.1.2), resulting in

To find the membership function $\mu_{S+T}(x)$ we equate to x both the first and second component in (4.1.2) which gives

$$x = [(b - a + q - p)\alpha + a + p] \; and \; x = [c + r - (c - b + r - q)\alpha]$$

Now, expressing α in terms of *x* and setting $\alpha = 0$ and $\alpha = 1$ in (4.3) we get α together with the domain of *x*,

$$\alpha = \frac{x - (a + p)}{(b + q) - (a + p)} \qquad (b + q) \leq x \leq (a + p)$$

$$\alpha = \frac{(c + r) - x}{(c + r) - (b + q)} \qquad (b + q) \leq x \leq (c + r)$$

which gives

$$\mu_{S+T}(x) = \begin{cases} \dfrac{x - (a + p)}{(b + q) - (a + p)} & (b + q) \leq x \leq (a + p) \\ \dfrac{(c + r) - x}{(c + r) - (b + q)} & (b + q) \leq x \leq (c + r) \end{cases}$$

Example 4.1.2 Consider two triangular fuzzy numbers: S = (1, 3, 5), T = (2, 4, 6), Let's calculate the sum $R = S + T$ using the α-cuts method for $\alpha = 0.5$:

1) Calculate α-cuts for S:

$$S_L^{0.5} = 1 + 0.5(3 - 1) = 1 + 1 = 2$$

$$S_U^{0.5} = 5 - 0.5(5 - 3) = 5 - 1 = 4$$

2) Calculate α-cuts for T:

$$T_L^{0.5} = 2 + 0.5(4 - 2) = 2 + 1 = 3$$

$$T_U^{0.5} = 6 - 0.5(6 - 4) = 6 - 1 = 5$$

3) Calculate α-cuts for R:

$$R_L^{0.5} = S_L^{0.5} + T_L^{0.5} = 2 + 3 = 5$$

$$R_U^{0.5} = S_U^{0.5} + T_U^{0.5} = 4 + 5 = 9$$

Therefore, the α-cut at $\alpha = 0.5$ for the sum $R = S + T$ is the interval [5, 9].

4.1.2 Subtraction of Triangular Fuzzy Numbers Using α-cuts Method

Definition 4.1.3. (Li & Wang, 2016) Let $S = \{a, b, c\}$ and $T = \{p, q, r\}$ be two fuzzy numbers.Then $\alpha_S = \{(b-a)\alpha + a, c-(c-b)\alpha\}$ and $\alpha_T = \{(q-p)\alpha + p, r-(r-q)\alpha\}$, The α-cuts of fuzzy numbers S and T are as follows. To compute the subtraction of fuzzy numbers S and T, we initially subtract the α-cuts of S and T utilizing interval arithmetic.

$$\begin{aligned}\alpha_S + \alpha_T &= [(b-a)\alpha + a, c-(c-b)\alpha] - [(q-p)\alpha + p, r-(r-q)\alpha] \\ &= [(b-a)\alpha + a - (r-(r-q)\alpha), c-(c-b)\alpha + (q-p)\alpha + p] \\ &= [(a-r) + (b-a+r-q)\alpha, (c-p) + (c-b+q-p)\alpha] \qquad (4.1.3)\end{aligned}$$

To get the membership function $\mu_{S+T}(x)$, we equal x to both the first and second components in (4.1.3), resulting in

$$x = [(a-r) + (b-a+r-q)\alpha, \qquad and\ x = (c-p) + (c-b+q-p)\alpha]$$

By expressing α in relation to x and substituting α = 0 and α = 1 into (4.1.3), we obtain α along with the domain of x.

$$\alpha = \frac{x-(a-p)}{(b-q)-(a-p)} \qquad (b-q) \le x \le (a-p)$$

$$\alpha = \frac{(c-r)-x}{(c-r)-(b-q)} \qquad (b-q) \le x \le (c-r)$$

which gives

$$\mu_{S-T}(x) = \begin{cases} \dfrac{x-(a-p)}{(b-q)-(a-p)} & (b-q) \le x \le (a-p) \\ \dfrac{(c-r)-x}{(c-r)-(b-q)} & (b-q) \le x \le (c-r) \end{cases}$$

Example 4.1.3. Let S = (2, 5, 8) and T = (1, 4, 7).

1) α-cuts calculation:

For S:

$$S_{0.5} = [2 + 0.5(5 - 2), 8 - 0.5(8 - 5)] = [3.5,6.5]$$

For T:

$$T_{0.5} = [1 + 0.5(4 - 1), 7 - 0.5(7 - 4)] = [2.5,5.5]$$

2) Subtract α-cuts: If R=S-T, then

$$R_{0.5} = [3.5 - 5.5,6.5 - 2.5] = [-2,4]$$

3) Resulting triangular fuzzy number:

Lower bound: $a_3 = 2 - 7 = -5$

Peak: $b_3 = 5 - 4 = 1$

Upper bound: $c_3 = 8 - 1 = 7$

Thus, the resulting triangular fuzzy number R is (-5, 1, 7).

Example 4.1.4. Let's put some actual numbers to it for clarity:

Assume:

$$S = (1,4,7), \qquad T = (3.6.9)$$

The α-cuts are:

$$S_\alpha = [1 + \alpha(4 - 1), 7 + \alpha(4 - 7)] = [1 + 3\alpha, 7 - 3\alpha]$$
$$T_\alpha = [3 + \alpha(6 - 3), 9 + \alpha(6 - 9)] = [3 + 3\alpha, 9 - 3\alpha]$$

So, solving for *X*:

$$X_\alpha = [3 + 3\alpha, 9 - 3\alpha] - [1 + 3\alpha, 7 - 3\alpha]$$
$$= [(3 - 1) + 3\alpha - 3\alpha, (9 - 7) - 3\alpha + 3\alpha] = [2,2]$$

Hence, *X* is:

$$X = (2,2,2)$$

There you go! We used triangular fuzzy numbers and their α-cuts to solve the fuzzy linear equation $S + X = T$.

4.1.3. Multiplication of Triangular Fuzzy Numbers Using α-cuts Method

Definition 4.1.4. (Li & Wang, 2016) Let $S = \{a, b, c\}$ and $T = \{p, q, r\}$ be two positive fuzzy numbers. Then $\alpha_S = \{(b-a)\alpha + a, c-(c-b)\alpha\}$ and $\alpha_T = \{(q-p)\alpha + p, r-(r-q)\alpha\}$ and are the α cuts of fuzzy numbers S and T respectively. To calculate multiplication of fuzzy We initially multiply the α-cuts of S and T utilizing interval arithmetic.

$$\alpha_S * \alpha_T = [(b-a)\alpha + a, c-(c-b)\alpha] * [(q-p)\alpha + p, r-(r-q)\alpha]$$

$$= [\big((b-a)\alpha + a\big) * \big((q-p)\alpha + p\big), (c-(c-b)\alpha) * (r-(r-q)\alpha) \quad (4.1.4)$$

To get the membership function $\mu_{S*T}(x)$, we equal x to both the first and second components in (4.1.4), resulting in

$$x = (b-a)(q-p)\alpha^2 + \big((b-a)\alpha\big)p + \big((q-p)\alpha\big)a + ap \text{ and}$$

$$x = (c-b)(r-q)\alpha^2 - \big((r-q)\alpha\big)c + \big((c-b)\alpha\big)r + cr$$

Now, expressing α in terms of x and setting α = 0 and α =1 in (4.1.4) we get α together with the domain of x,

$$\alpha = \frac{-((b-a)p+(q-p)\alpha+\sqrt{\big((b-a)p+(q-p)\alpha\big)^2-4(b-a)(q-p)(ap-x)}}{2(b-a)(q-p)}, \qquad ap \le x \le bq$$

Which gives,

$$\alpha = \frac{-\big((r-q)c+(c-b)r\big)-\sqrt{\big((r-q)c+(c-b)r\big)^2-4(c-b)(r-q)(cr-x)}}{2(c-b)(r-q)}, \qquad bq \le x \le cr$$

$$\mu_{S*T}(x) = \begin{cases} \dfrac{-((b-a)p+(q-p)\alpha+\sqrt{\big((b-a)p+(q-p)\alpha\big)^2-4(b-a)(q-p)(ap-x)}}{2(b-a)(q-p)}, & ap \le x \le bq \\ \dfrac{-\big((r-q)c+(c-b)r\big)-\sqrt{\big((r-q)c+(c-b)r\big)^2-4(c-b)(r-q)(cr-x)}}{2(c-b)(r-q)}, & bq \le x \le cr \end{cases}$$

Example 4.1.5

Let's take two triangular fuzzy numbers $S = (1,3,5)$ and $Y = (2,4,6)$.

For α = 0.5:

Calculate the α-cuts:

$$S_{0.5} = [1 + 0.5(3 - 1), 5 - 0.5(5 - 3)] = [2,4]$$

$$T_{0.5} = [2 + 0.5(4 - 2), 6 - 0.5(6 - 4)] = [3,5]$$

Multiply the intervals:

$$R_{0.5} = [\min(2 \times 3, 2 \times 5, 4 \times 3, 4 \times 5), \max(2 \times 3, 2 \times 5, 4 \times 3, 4 \times 5)]$$

$$R_{0.5} = [6,20]$$

By repeating this process for other α values, you can form the final triangular fuzzy number.

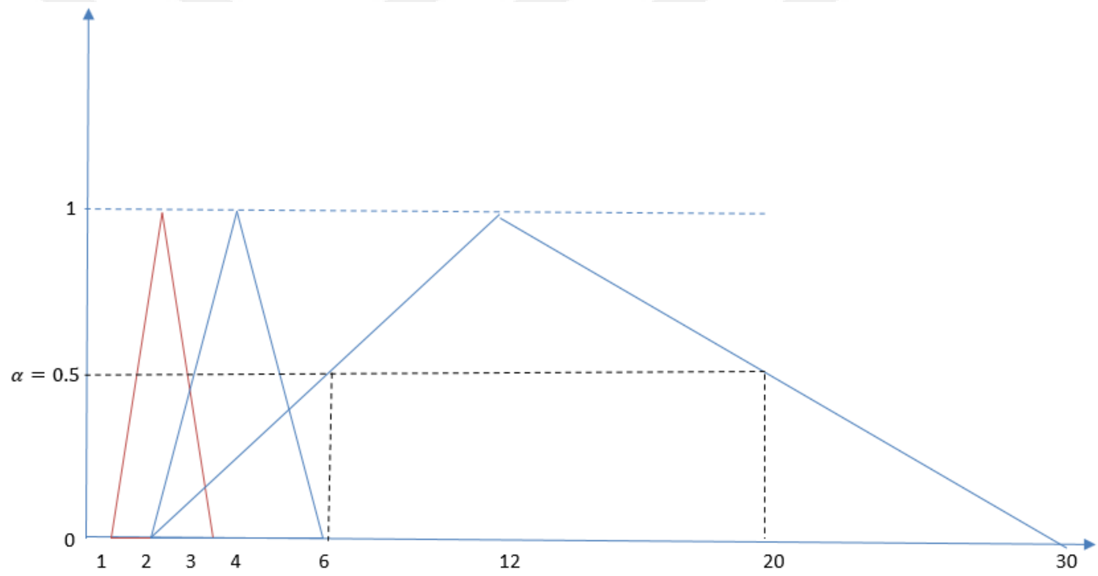


Fig 4.3. Example 4.6. α = 0.5

4.1.4. Division of Triangular Fuzzy Numbers Using α-cuts Method

Definition 4.1.5. (Ocampo & Sarmiento, 2015) Let $S = \{a, b, c\}$ and $T = \{p, q, r\}$ and be two positive fuzzy numbers.

Then $\alpha_S = \{(b - a)\alpha + a, c - (c - b)\alpha\}$ and $\alpha_T = \{(q - p)\alpha + p, r - (r - q)\alpha\}$, The α-cuts of fuzzy numbers S and T, respectively. To compute the division of fuzzy numbers S and T, we initially split the α-cuts of S and T utilizing interval arithmetic.

$$\frac{\alpha_S}{\alpha_T} = \frac{[(b-a)\alpha + a, c-(c-b)\alpha]}{[(q-p)\alpha + p, r-(r-q)\alpha]} = \left[\frac{(b-a)\alpha + a}{r-(r-q)\alpha}, \frac{c-(c-b)\alpha}{(q-p)\alpha + p}\right] \qquad (4.1.5)$$

To get the membership function $\mu_{S/T}(x)$, we equal x to both the first and second components in (4.1.5), resulting in

$$\alpha = \frac{(b-a)\alpha + a}{r-(r-q)\alpha}$$

and

$$\alpha = \frac{c-(c-b)\alpha}{(q-p)\alpha + p}$$

By expressing α in terms of x and substituting α = 0 and α = 1 into (4.1.5), we obtain α along with the domain of x.

$$\alpha = \frac{xr-a}{(b-a)+(q-r)x}, \qquad \frac{a}{r} \leq x \leq \frac{b}{q}$$

and

$$\alpha = \frac{c-px}{(c-b)+(q-p)x}, \qquad \frac{b}{q} \leq x \leq \frac{c}{r}$$

which gives

$$\mu_{S/T}(x) = \begin{cases} \dfrac{xr-a}{(b-a)+(q-r)x}, & \dfrac{a}{r} \leq x \leq \dfrac{b}{q} \\ \dfrac{c-px}{(c-b)+(q-p)x}, & \dfrac{b}{q} \leq x \leq \dfrac{c}{r} \end{cases}$$

Example 4.1.6. Let's use the same triangular fuzzy numbers S and T as in the previous examples:

1. Fuzzy Number S: Lower limit $a = 1$, Peak $b = 3$, Upper limit $c = 5$
2. Fuzzy Number T: Lower limit $a = 2$, Peak $b = 5$, Upper limit $c = 7$

Calculating the α-cuts: For $\alpha = 0.5$: S at $\alpha = 0.5$

$$S_{0.5} = [1 + 0.5(3-1), 5 - 0.5(5-3)]$$

$$S_{0.5} = [2,4]$$

T at $\alpha = 0.5$

$$T_{0.5} = [2 + 0.5(5 - 2), 7 - 0.5(7 - 5)]$$

$$T_{0.5} = [3.5,6]$$

Division of α-cuts:

R at $\alpha = 0.5$

$$R_{0.5} = \left[min\left(\frac{2}{3.5}, \frac{2}{6}, \frac{4}{3.5}, \frac{4}{6}\right), max\left(\frac{2}{3.5}, \frac{2}{6}, \frac{4}{3.5}, \frac{4}{6}\right)\right]$$

$$R_{0.5} = [min(0.571,0.333,1.143,0.667), max(0.571,0.333,1.143,0.667)]$$

$$R_{0.5} = [0.333,1.143]$$

So, the 0.5-cut for the quotient of these two fuzzy numbers S and T is the interval [0.333, 1.143].

Using α-cuts for division of fuzzy numbers allows us to handle the intervals representing different levels of certainty effectively. This approach ensures the arithmetic operations remain consistent with the inherent imprecision of fuzzy numbers.

Example 4.1.7. A fuzzy linear equation in the form of $SX = T$ using triangular fuzzy numbers (TFNs) with α-cuts.

Given the fuzzy linear equation:

$$SX = T$$

We'll use some specific values for our triangular fuzzy numbers:

$$S = (1,3,5), T = (2,6,10)$$

Now let's find the α-cuts for S and T:

$$S_\alpha = [1 + \alpha(3 - 1), 5 + \alpha(3 - 5)] = [1 + 2\alpha, 5 - 2\alpha]$$

$$T_\alpha = [2 + \alpha(6 - 2), 10 + \alpha(6 - 10)] = [2 + 4\alpha, 10 - 4\alpha]$$

Substituting the α-cuts:

$$X_\alpha = \frac{[2 + 4\alpha, 10 - 4\alpha]}{[1 + 2\alpha, 5 - 2\alpha]}$$

This simplifies to:

For the **lower bound**:

$$X_\alpha^L = \frac{2 + 4\alpha}{1 + 2\alpha}$$

For the **upper bound**:

$$X_\alpha^U = \frac{10+4\alpha}{5+2\alpha}$$

So, the α-cut for *X* is:

$$X_\alpha = \left[\frac{2+4\alpha}{1+2\alpha}, \frac{10+4\alpha}{5+2\alpha}\right]$$

Therefore, the solution for *X* is represented by the α-cuts:

$\alpha = 0.5$

$$X_{0.5} = \left[\frac{2+4(0.5)}{1+2(0.5)}, \frac{10+4(0.5)}{5+2(0.5)}\right] = \left[\frac{2+2}{1+1}, \frac{10+2}{5+1}\right] = [2,2]$$

This gives us an example of solving the fuzzy linear equation $SX = T$ with triangular fuzzy numbers and their α-cuts.

Example 4.1.8. We'll use some specific values for our triangular fuzzy numbers:

$$S = (1,3,5), \qquad T = (2,4,6), \qquad R = (3,9,15)$$

Now let's find the α-cuts for *S*, *T*, and *R*:

$$S_\alpha = [1+\alpha(3-1), 5+\alpha(3-5)] = [1+2\alpha, 5-2\alpha]$$

$$T_\alpha = [2+\alpha(4-2), 6+\alpha(4-6)] = [2+2\alpha, 6-2\alpha]$$

$$R_\alpha = [3+\alpha(9-3), 15+\alpha(9-15)] = [3+6\alpha, 15-6\alpha]$$

Substituting the α-cuts:

$$X_\alpha = \frac{[3+6\alpha, 15-6\alpha] - [2+2\alpha, 6-2\alpha]}{[1+2\alpha, 5-2\alpha]}$$

This simplifies to:

$$X_\alpha = \frac{[(3-2)+6\alpha-2\alpha, (15-6)-6\alpha+2\alpha]}{[1+2\alpha, 5-2\alpha]}$$

$$X_\alpha = \frac{[1+4\alpha, 9-4\alpha]}{[1+2\alpha, 5-2\alpha]}$$

Now, let's simplify the fraction:

For α in the interval [0,1]:

$$X_\alpha = \left[\frac{1+4\alpha}{1+2\alpha}, \frac{9-4\alpha}{5-2\alpha}\right]$$

This provides us with the α-cut for *X*.

Therefore, the solution for *X* is represented by the α-cuts:

$$X_\alpha = \left[\frac{1+4\alpha}{1+2\alpha}, \frac{9-4\alpha}{5-2\alpha}\right]$$

This gives us an example of solving the fuzzy linear equation $SX + T = R$ with triangular fuzzy numbers and their α-cuts.

4.2. Trapezoidal Fuzzy Numbers and Fuzzy Linear Equations Using α-cuts Method

Definition 4.2.1. (Gowri & Kavitha, 2021) Trapezoidal fuzzy number (TrFN) A trapezoidal fuzzy number S is defined by four parameters:

a) the lower limit where the membership value starts to increase,
b) the lower boundary of the core where the membership value is 1,
c) the upper boundary of the core where the membership value starts to decrease,
d) the upper limit where the membership value returns to zero.

The membership function $\mu_S(x)$ for a trapezoidal fuzzy number is given by:

$$\mu_S(x) = \begin{cases} 0 & if \quad x < a, \\ \dfrac{x-a}{b-a} & if \quad a \leq x \leq b, \\ 1 & if \quad b < x < c, \\ \dfrac{d-x}{d-c} & if \quad c \leq x \leq d, \\ 0 & if \quad x > d. \end{cases}$$

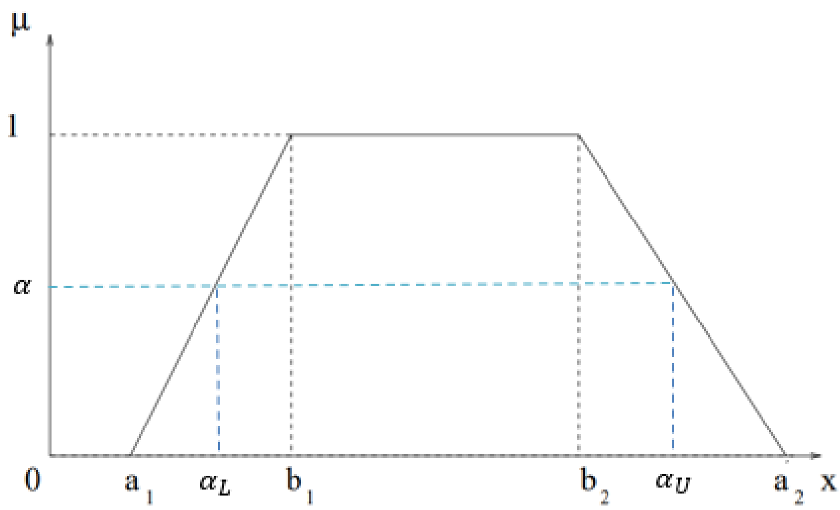


Fig 4.2.1 Trapezoidal fuzzy numbers using α-cuts Method

An α-cut S_α of a trapezoidal fuzzy number S represents the set of elements that have membership values greater than or equal to α. For a trapezoidal fuzzy number, every α-cut S_α is an interval $[S_1^\alpha, S_2^\alpha]$, where:

$$S_1^\alpha = a + \alpha(b - a)$$

$$S_2^\alpha = d - \alpha(d - c)$$

For $b \leq x \leq c$ the membership value is 1, so the interval holds for all α within this range.

Example 4.2.1. Trapezoidal fuzzy number S with $a = 2,\ b = 4, c = 7, and\ d = 9$.

For $\alpha = 0.5$:

$$S_{0.5} = [2 + 0.5(4 - 2), 9 - 0.5(9 - 7)]$$

$$S_{0.5} = [2 + 1, 9 - 1]$$

$$S_{0.5} = [3,8]$$

So, the 0.5-cut for this trapezoidal fuzzy number S is the interval [3, 8].

By using α-cuts, we can effectively represent and simplify operations on trapezoidal fuzzy numbers at different levels of certainty. This approach is beneficial for applications that involve uncertainty and imprecision.

4.2.1. Addition of Trapezoidal Fuzzy Numbers Using α-cuts Method

Definition 4.2.2. (Ocampo & Sarmiento, 2015) Let $S = \{a, b, c, d\}$ and $T = \{p, q, r, s\}$ be two imprecise quantities characterized by their membership functions

Then $\alpha_S = \{(b - a)\alpha + a, d - (d - c)\alpha\}$ and $\alpha_T = \{(q - p)\alpha + p, s - (s - r)\alpha\}$, The α-cuts of fuzzy numbers S and T, respectively. To compute the sum of fuzzy numbers S and T, we initially add the α-cuts of S and T utilizing interval arithmetic $\alpha_S + \alpha_T$

$$= [(b - a)\alpha + a, d - (d - c)\alpha] + [(q - p)\alpha + p, s - (s - r)\alpha]$$

$$= [(b - a + q - p)\alpha + a + p, d + s - (d - c + s - r)\alpha] \tag{4.2.2}$$

To get the membership function $\mu_{S+T}(x)$, we equal x to both the first and second components in (4.2.2), resulting in

$$x = [(b - a + q - p)\alpha + a + p] and\ x = [d + s - (d - c + s - r)\alpha]$$

By expressing α in relation to x and substituting α = 0 and α = 1 in (4.2.2), we obtain α along with the domain of x.

$$\alpha = \frac{x-(a+p)}{(b+q)-(a+p)} \qquad (b+q) \le x \le (a+p)$$

$$\alpha = \frac{(d+s)-x}{(d+s)-(c+r)} \qquad (c+r) \le x \le (d+s)$$

which gives

$$\mu_{S+T}(x) = \begin{cases} \dfrac{x-(a+p)}{(b+q)-(a+p)} & (b+q) \le x \le (a+p) \\ \dfrac{(d+s)-x}{(d+s)-(c+r)} & (c+r) \le x \le (d+s) \end{cases}$$

Example 4.2.2. Let's take two trapezoidal fuzzy numbers $S = (1,2,4,5)$ and $T = (2,3,5,6)$.

For α = 0.5:

Calculate the α-cuts:

$$S_{0.5} = [1 + 0.5(2-1), 5 - 0.5(5-4)] = [1.5,4.5]$$

$$T_{0.5} = [2 + 0.5(3-2), 6 - (6-5)] = [2.5,5.5]$$

Add the intervals:

$$R_{0.5} = [1.5 + 2.5, 4.5 + 5.5] = [4,10]$$

By repeating this process for other α levels, you can form the final trapezoidal fuzzy number. $R = (3,5,9,11)$

4.2.2. Subtraction of Trapezoidal Fuzzy Numbers Using α-cuts Method

Definition 4.2.3. (Ocampo & Sarmiento, 2015) Let $S = \{a,b,c,d\}$ and $T = \{p,q,r,s\}$ be two imprecise integers characterized by their membership functions, Then

$$\alpha_S = \{(b-a)\alpha + a, d-(d-c)\alpha\} \quad \text{and} \quad \alpha_T = \{(q-p)\alpha + p, s-(s-r)\alpha\}$$

The α-cuts of fuzzy numbers S and T, respectively. To compute the subtraction of fuzzy numbers S and T, we initially sum the α-cuts of S and T utilizing interval arithmetic

$\alpha_S - \alpha_T,$

$= [(b-a)\alpha + a, d-(d-c)\alpha] - [(q-p)\alpha + p, s-(s-r)\alpha]$

$= [(b-a)\alpha + a - (s-(s-r)\alpha), d-(d-c)\alpha + (q-p)\alpha + p]$

$$= [(a-s) + (b-a+s-r)\alpha, (d-p) + (d-c+q-p)\alpha] \qquad (4.2.3)$$

To get the membership function $\mu_{S-T}(x)$, we equal x to both the first and second components in (4.2.3), resulting in

$$x = [(a-s) + (b-a+s-r)\alpha] and\ x = [(d-p) + (d-c+q-p)\alpha]$$

By expressing α in relation to x and substituting α = 0 and α = 1 into (4.2.3), we obtain α along with the corresponding domain of x.

$$\alpha = \frac{x-(a-s)}{(b-r)-(a-s)} \qquad (a-s) \le x \le (p-r)$$

$$\alpha = \frac{(d-p)-x}{(d-p)-(c-q)} \qquad (c-q) \le x \le (d-p)$$

which gives

$$\mu_{S-T}(x) = \begin{cases} \dfrac{x-(a-s)}{(b-r)-(a-s)} & (a-s) \le x \le (p-r) \\ \dfrac{(d-p)-x}{(d-p)-(c-q)} & (c-q) \le x \le (d-p) \end{cases}$$

Example 4.2.3. Let's take two trapezoidal fuzzy numbers $S = (1,2,4,5)$ and $T = (2,3,5,6)$.

For α = 0.5:

Calculate the α-cuts:

$$S_{0.5} = [1 + 0.5(2-1), 5 - 0.5(5-4)] = [1.5,4.5]$$

$$T_{0.5} = [2 + 0.5(3-2), 6 - 0.5(6-5)] = [2.5,5.5]$$

Subtract the intervals:

$$R_{0.5} = [1..5 - 5.5,4.5 - 2.5] = [-4,2]$$

$$R = [-5,-3,1,3]$$

Example 4.2.4. Let's put some actual numbers to it for clarity: Assume:

$$S = (1,2,4,5),\ T = (2,3,5,6)$$

The α-cuts are:

$$S_\alpha = [1 + \alpha(2-1), 5 + (4-5)] = [1+\alpha, 5-\alpha]$$

$$T_\alpha = [2 + \alpha(3-2), 6 + (5-6)] = [2+\alpha, 6-\alpha]$$

So, solving for X:

$$X_\alpha = [2+\alpha, 6-\alpha] - [1+\alpha, 5-\alpha] = [(2-1) + \alpha - \alpha, (6-5) - \alpha + \alpha] = [1,1]$$

Hence, X is:

$$X = (1,1,1,1)$$

There you go! We used trapezoidal fuzzy numbers and their α-cuts to solve the fuzzy linear equation $S + X = T$.

4.2.3. Multiplication of Trapezoidal Fuzzy Numbers Using α-cuts Method

Definition 4.2.4. (Li & Wang, 2016) Let $S = \{a, b, c, d\}$ and $T = \{p, q, r, s\}$ Let there be two positive fuzzy integers. Thus, $\alpha_S = \{(b-a)\alpha + a, s - (s-c)\alpha\}$ and $\alpha_T = \{(q-p)\alpha + p, s - (s-r)\alpha\}$, representing the α-cuts of fuzzy numbers S and T, respectively. To compute the multiplication of fuzzy integers S and T, we initially multiply the α-cuts of S and T utilizing interval arithmetic.

$$\alpha_S * \alpha_T = [(b-a)\alpha + a, d - (d-c)\alpha] * [(q-p)\alpha + p, s - (s-r)\alpha]$$

$$= [\big((b-a)\alpha + a\big) * \big((q-p)\alpha + p\big), (d-(d-c)\alpha) * (s-(s-r)\alpha) \quad (4.2.4)$$

To get the membership function $\mu_{S*T}(x)$, we equal x to both the first and second components in (4.2.4), resulting in

$$x = (b-a)(q-p)\alpha^2 + \big((b-a)\alpha\big)p + \big((q-p)\alpha\big)a + ap \ \text{ and}$$

$$x = (d-c)(s-r)\alpha^2 - \big((s-r)\alpha\big)d + \big((d-c)\alpha\big)s + ds$$

By expressing α in terms of x and substituting α = 0 and α = 1 into (4.2.3), we obtain α along with the domain of x.

$$\alpha = \frac{-((b-a)p + (q-p)\alpha + \sqrt{\big((b-a)p + (q-p)\alpha\big)^2 - 4(b-a)(q-p)(ap-x)}}{2(b-a)(q-p)}, \qquad ap \le x \le bq$$

Which gives,

$$\alpha = \frac{-\left((s-r)d+(d-c)s\right)-\sqrt{\left((s-r)d+(d-c)s\right)^2-4(d-c)(s-r)(ds-x)}}{2(d-c)(s-r)}, \qquad cr \le x \le ds$$

$$\mu_{S*T}(x) = \begin{cases} \dfrac{-((b-a)p+(q-p)\alpha+\sqrt{\left((b-a)p+(q-p)\alpha\right)^2-4(b-a)(q-p)(ap-x)}}{2(b-a)(q-p)}, & ap \le x \le bq \\ \dfrac{-\left((s-r)d+(d-c)s\right)-\sqrt{\left((s-r)d+(d-c)s\right)^2-4(d-c)(s-r)(ds-x)}}{2(d-c)(s-r)}, & cr \le x \le ds \end{cases}$$

Example 4.2.5. Let's take two trapezoidal fuzzy numbers $S = (1,2,4,5)$ and $T = (2,3,5,6)$.

For α = 0.5:

Calculate the α-cuts:

$$S_{0.5} = [1 + 0.5(2-1), 5 - 0.5(5-4)] = [1.5,4.5]$$

$$T_{0.5} = \left[2 + 0.5(3-2), 6 - 0.5(6-5) = [2.5,5.5]\right]$$

Multiply the intervals:

$$R_{0.5} = [\min(1.5 \cdot 2.5, 1.5 \cdot 5.5, 4.5 \cdot 2.5, 4.5 \cdot 5.5), \\ \max(1.5 \cdot 2.5, 1.5 \cdot 5.5, 4.5 \cdot 2.5, 4.5 \cdot 5.5)]$$

$$R_{0.5} = [3.75,24.75]$$

$$R = (2,6,20,30)$$

4.2.4 Division of Trapezoidal Fuzzy Numbers Using α-cuts Method

Definition 4.2.5. (Ocampo & Sarmiento, 2015) Let $S = \{a, b, c\}$ and $T = \{p, q, r\}$ Let them be two positive fuzzy integers. Thus, $\alpha_S = \{(b-a)\alpha + a, d - (d-c)\alpha\}$ and $\alpha_T = \{(q-p)\alpha + p, s - (s-r)\alpha\}$ represent the α-cuts of the fuzzy numbers S and T, respectively. To compute the division of fuzzy numbers S and T, we initially split the α-cuts of S and T utilizing interval arithmetic.

$$\frac{\alpha_S}{\alpha_T} = \frac{[(b-a)\alpha + a, d - (d-c)\alpha]}{[(q-p)\alpha + p, s - (s-r)\alpha]} = \left[\frac{(b-a)\alpha + a}{s - (s-r)\alpha}, \frac{d - (d-c)\alpha}{(q-p)\alpha + p}\right] \tag{4.2.5}$$

To determine the membership function $\mu_{S/T}(x)$, we equal x to both the first and second components in (4.2.5), resulting in:

$$\alpha = \frac{(b-a)\alpha + a}{s-(s-r)\alpha}$$

and

$$\alpha = \frac{d-(d-c)\alpha}{(q-p)\alpha + p}$$

By expressing α in relation to x and substituting α = 0 and α = 1 into (4.2.5), we obtain α along with the corresponding domain of x.

$$\alpha = \frac{xs-a}{(b-a)+(r-s)x}, \qquad \frac{a}{s} \le x \le \frac{b}{r}$$

and

$$\alpha = \frac{d-px}{(d-c)+(q-p)x}, \qquad \frac{c}{q} \le x \le \frac{d}{p}$$

which gives

$$\mu_{S/T}(x) = \begin{cases} \dfrac{xs-a}{(b-a)+(r-s)x}, & \dfrac{a}{s} \le x \le \dfrac{b}{r} \\ \dfrac{d-px}{(d-c)+(q-p)x}, & \dfrac{c}{q} \le x \le \dfrac{d}{p} \end{cases}$$

Example 4.2.6. Division of Trapezoidal Fuzzy Numbers Using α-cuts Method

$$S = [2,4,6,8], T = [1,3,5,7]$$

1) Determine the α-cuts for S and T

Let's assume the value of $\alpha = 0.1$. Then, we find the α-cuts for S and T:

$$S_{0.1} = [2+0.1(4-2), 6+0.1(8-6)] = [2.2,6.2]$$

$$T_{0.1} = [1+0.1(3-1), 5+0.1(7-5)] = [1.2,5.2]$$

2) Divide the α-cuts

To divide S by T, We divide the corresponding α-cuts:

$$\frac{S_{0.1}}{T_{0.1}} = \left[\frac{2.2}{5.2}, \frac{6.2}{1.2}\right] = [0.423,5.167]$$

3) Express the Result in Standard Form

The result is another trapezoidal fuzzy number:

$R = [0.423,5.167]$

So, the division of the trapezoidal fuzzy numbers S and T using the α-cuts method with $\alpha = 0.1$ results in another trapezoidal fuzzy number R.

Example 4.2.7. We'll use some specific values for our trapezoidal fuzzy numbers:

$$S = (1,2,4,5), T = (2,3,7,8)$$

Now let's find the α-cuts for *S* and *T*:

$$S_\alpha = [1 + \alpha(2 - 1), 5 + \alpha(4 - 5)] = [1 + \alpha, 5 - \alpha]$$

$$T_\alpha = [2 + \alpha(3 - 2), 8 + \alpha(7 - 8)] = [2 + \alpha, 8 - \alpha]$$

Substituting the α-cuts:

$$X_\alpha = \frac{[2 + \alpha, 8 - \alpha]}{[1 + \alpha, 5 - \alpha]}$$

This simplifies to:

For the **lower bound**:

$$X_\alpha^L = \frac{2 + \alpha}{1 + \alpha}$$

For the **upper bound**:

$$X_\alpha^U = \frac{8 - \alpha}{5 - \alpha}$$

So, the α-cut for *X* is:

$$X_\alpha = \left[\frac{2 + \alpha}{1 + \alpha}, \frac{8 - \alpha}{5 - \alpha}\right]$$

Therefore, the solution for *X* is represented by the α-cuts:

$\alpha = 0.5$

$$X_{0.5} = \left[\frac{2 + 0.5}{1 + 0.5}, \frac{8 - 0.5}{5 - 0.5}\right] = \left[\frac{2.5}{1.5}, \frac{7.5}{4.5}\right] \approx [1.667.1.667]$$

This gives us an example of solving the fuzzy linear equation $SX = T$ with trapezoidal fuzzy numbers and their α-cuts.

Example 4.2.8. We'll use some specific values for our trapezoidal fuzzy numbers:

$$S = (1,2,4,5), T = (2,3,5,6), R = (3,5,10,12)$$

Now let's find the α-cuts for *S*, *T*, and *R*:

$$S_\alpha = [1 + \alpha(2 - 1), 5 + \alpha(4 - 5)] = [1 + \alpha, 5 - \alpha]$$

$$T_\alpha = [2+\alpha(3-2), 6+\alpha(5-6)] = [2+\alpha, 6-\alpha]$$

$$R_\alpha = [3+\alpha(5-3), 12+\alpha(10-12)] = [3+2\alpha, 12-2\alpha]$$

Substituting the α-cuts:

$$S_\alpha X_\alpha = [3+2\alpha, 12-2\alpha] - [2+\alpha, 6-\alpha]$$

This simplifies to:

$$S_\alpha X_\alpha = [(3-2)+2\alpha-\alpha, (12-6)-2\alpha+\alpha] = [1+\alpha, 6-\alpha]$$

Now, solving for X_α:

$$X_\alpha = \frac{[1+\alpha, 6-\alpha]}{[1+\alpha, 5-\alpha]}$$

This can be simplified for both bounds:

For the **lower bound**:

$$X_\alpha^L = \frac{1+\alpha}{1+\alpha} = 1$$

For the upper bound:

$$X_\alpha^U = \frac{6-\alpha}{5-\alpha}$$

Therefore, the α-cut for *X* is:

$$X_\alpha = \left[1, \frac{6-\alpha}{5-\alpha}\right]$$

So, the solution for *X* is represented by the α-cuts:

$$X_\alpha = \left[1, \frac{6-\alpha}{5-\alpha}\right]$$

This gives us an example of solving the fuzzy linear equation $SX+T=R$ with trapezoidal fuzzy numbers and their α-cuts.

4.3. Gaussian Bell Shape Fuzzy Numbers and Fuzzy Linear Equations Using α-cuts Method

Definition 4.3.1. (Dutta & Boruah & Ali, 2011) α-Cuts Method. The α-cuts method is a way to represent fuzzy numbers using crisp (non-fuzzy) intervals at different levels of membership (α). An α-cut of a fuzzy number *S* is the set of all elements in the universe of discourse that have a membership degree greater than or equal to a given threshold α, where $0 \le \alpha \le 1$.

For a Gaussian fuzzy number *S*, the α-cut S_α is represented as an interval:

$$S_\alpha = [S_L^\alpha, S_U^\alpha]$$

where S_L^α and S_R^α are the left and right endpoints of the interval at level α. These endpoints can be calculated by solving the membership function equation for (x) at the given α level:

$$\mu_S(x) = \alpha$$

This translates to:

$$x = c_S \pm \sqrt{-2\sigma_S^2 \ln(\alpha)}$$

So, the α-cut S_α is:

$$S_\alpha = \left[c_S - \sqrt{-2\sigma_S^2 \ln(\alpha)}, c_S + \sqrt{-2\sigma_S^2 \ln(\alpha)}\right]$$

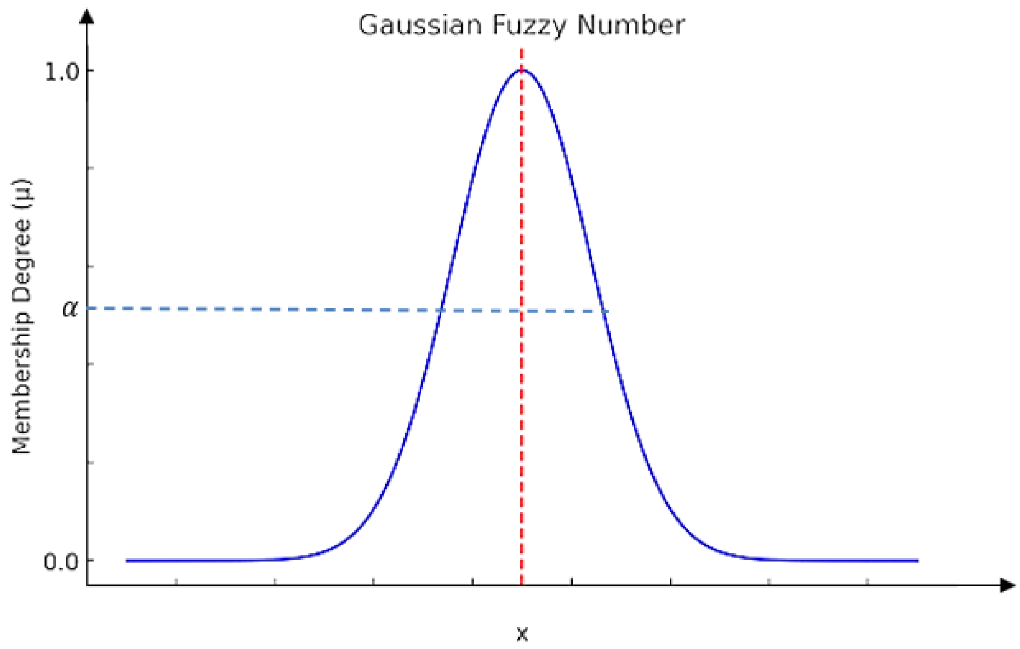


Fig 4.3.1 Fuzzy linaer equaition using α-cuts method

Example 4.3.1 Let's consider a Gaussian fuzzy number with the following parameters:

Center $c_S = 5$, Standard deviation $\sigma_S = 2$

We want to find the α-cut for $\alpha = 0.5$.

Step-by-Step Calculation:

$$\mu_S(x) = exp\left(-\frac{(x - c_S)^2}{2 \cdot \sigma_S^2}\right)$$

$c_S = 5, \qquad \sigma_S = 2$

α-cut Calculation: Find S_L^α and S_U^α for $\alpha = 0.5$:

$$\mu_S(x) = 0.5$$

Taking the natural logarithm on both sides:

$$\exp\left(-\frac{(x-5)^2}{2 \cdot 2^2}\right) = 0.5$$

$$-\frac{(x-5)^2}{8} = \ln(0.5)$$

Solving for $(x-5)^2$:

$$(x-5)^2 = -8ln0.5$$

Calculate $\ln(0.5)$:

$$\ln(0.5) \approx -0.693$$

Substitute the value:

$$(x-5)^2 = 8 \times 0.693$$

Calculate S_L^α and S_U^α:

$$x = 5 \pm \sqrt{5.544}$$

$$\sqrt{5.544} \approx 2.354$$

Therefore:

$$S_L^{0.5} = 5 - 2.354 \approx 2.646$$

$$S_U^{0.5} = 5 + 2.354 \approx 7.354$$

So, the α-cut at $\alpha_{0.5}$ for the Gaussian fuzzy number is:

$$S_{0.5} = [2.646{,}7.354]$$

Using the α-cuts method, we represented the Gaussian fuzzy number as an interval at $\alpha = 0.5$. This method helps in performing various operations more straightforwardly.

4.3.1. Addition of Gassian Bell Shape Fuzzy Numbers Using α-cuts Method

Definition 4.3.2. (Lin, Cao & Liao, 2018) The addition of two Gaussian bell-shaped fuzzy numbers using the α-cuts method involves performing the addition operation at different levels of α to obtain the intervals (α-cuts) for the resulting fuzzy number. The α-cuts method allows for handling uncertainty and fuzziness by converting the fuzzy arithmetic operations into interval arithmetic at specified levels of α.

For Gaussian fuzzy numbers, the α-cuts are determined based on the mean and standard deviation of the Gaussian distribution, and the addition is carried out on these intervals.

Formula for α-cuts

For Gaussian fuzzy numbers S and T with means c_S and c_T, and standard deviations σ_S and σ_T

$$[S_L^\alpha, S_U^\alpha] = [c_S - k_\alpha \sigma_S, c_S + k_\alpha \sigma_S]$$

$$[T_L^\alpha, T_U^\alpha] = [c_T - k_\alpha \sigma_T, c_T + k_\alpha \sigma_T]$$

The α-cuts for the resulting fuzzy number $R = S + T$ are given by:

$$[R_L^\alpha, R_U^\alpha] = [S_L^\alpha + T_L^\alpha, S_U^\alpha + S_U^\alpha]$$

Example 4.3.2. Consider two Gaussian fuzzy numbers S and T:

S with mean $c_S = 4$ and standard deviation $\sigma_S = 1$.

T with mean $c_T = 3$ and standard deviation $\sigma_T = 2$.

For α = 0.4, $k_\alpha \approx 0.253$.

Determine the α-cuts

For S:

$$S_L^{0.4} = 4 - 0.253 \times 1 = 4 - 0.253 = 3.747$$

$$S_U^{0.4} = 4 + 0.253 \times 1 = 4 + 0.253 = 4.253$$

For T:

$$T_L^{0.4} = 3 - 0.253 \times 2 = 3 - 0.506 = 2.494$$

$$T_U^{0.4} = 3 + 0.253 \times 2 = 3 + 0.506 = 3.506$$

Add the Intervals

To add S and T using the α-cuts method:

For $\alpha = 0.4$:

$$[R_L^{0.4}, R_U^{0.4}] = [T_L^{0.4} - S_U^{0.4}, T_U^{0.4} - S_L^{0.4}]$$

$$R_L^{0.4} = 3.747 + 2.494 = 6.241$$

$$R_U^{0.4} = 4.253 + 3.506 = 7.759$$

So, the α-cut for R at $\alpha = 0.4$ is [6.241, 7.759].

4.3.2. Subtraction of Gaussian Bell Shape Fuzzy Numbers Using α-cuts Method

Definition 4.3.3. (Wei & Zhang, 2011) The subtraction of two Gaussian bell-shaped fuzzy numbers using the α-cuts method involves performing the subtraction operation at different levels of α to obtain the intervals (α-cuts) for the resulting fuzzy number. The α-cuts method allows for handling uncertainty and fuzziness by converting the fuzzy arithmetic operations into interval arithmetic at specified levels of α.

For Gaussian fuzzy numbers, the α-cuts are determined based on the mean and standard deviation of the Gaussian distribution, and the subtraction is carried out on these intervals.

Formula for α-cuts:

For Gaussian fuzzy numbers S and T with means c_S and c_T, and standard deviations σ_S and σ_T.

$$[S_L^\alpha, S_U^\alpha] = [c_S - k_\alpha \sigma_S, c_S + k_\alpha \sigma_S]$$

$$[T_L^\alpha, T_U^\alpha] = [c_T - k_\alpha \sigma_T, c_T + k_\alpha \sigma_T]$$

The α-cuts for the resulting fuzzy number $R = S - T$ are given by:

$$[R_L^\alpha, R_U^\alpha] = [S_L^\alpha - T_U^\alpha, S_U^\alpha - S_L^\alpha]$$

Example 4.3.3. Subtraction of Gaussian Bell Shape Fuzzy Numbers Using α-cuts Method (α = 0.5)

Consider two Gaussian fuzzy numbers S and T:

S with mean $c_S = 6$ and standard deviation $\sigma_S = 1.5$.

T with mean $c_S = 4$ and standard deviation $\sigma_S = 1$.

For α = 0.5, $k_\alpha \approx 0.674$.

Determine the α-cuts

For S:

$$S_L^{0.5} = 6 - 0.674 \times 1.5 = 6 - 1.011 = 4.989$$

$$S_U^{0.5} = 6 + 0.674 \times 1.5 = 6 + 1.011 = 7.011$$

For T:

$$T_L^{0.5} = 4 - 0.674 \times 1 = 4 - 674 = 3.326$$

$$T_U^{0.5} = 4 + 0.674 \times 1 = 4 + 0.674 = 4.674$$

Subtract the Intervals

To subtract S and T using the α-cuts method:

For $\alpha = 0.5$:

$$R_L^{0.5} = 4.989 - 4.674 = 0.315$$

$$R_U^{0.5} = 7.011 - 3.326 = 3.685$$

So, the α-cut for R at $\alpha = 0.5$ is [0.315, 3.685].

Example 4.3.4 Consider the fuzzy linear equation:

$$X + S = T$$

Given fuzzy numbers S and T:

For S, Mean $c_S = 4$, Standard deviation $\sigma_S = 1$.

For T, Mean $c_T = 10$, Standard deviation $\sigma_T = 2$.

For α = 0.5, $k_\alpha \approx 0.674$.

Determine the α-cuts

For S:

$$S_L^{0.5} = 4 - 0.674 \times 1 = 4 - 0.674 = 3.326$$

$$S_U^{0.5} = 4 + 0.674 \times 1 = 4 + 0.674 = 4.674$$

For T:

$$T_L^{0.5} = 10 - 0.674 \times 2 = 10 - 1.348 = 8.652$$

$$T_U^{0.5} = 10 + 0.674 \times 2 = 10 + 1.348 = 11.348$$

Formulate the Interval Arithmetic

At α = 0.5, the fuzzy equation can be written as:

$$[X_L^{0.5} + X_L^{0.5}] + [S_U^{0.5} + S_U^{0.5}] = [T_L^{0.5}, T_U^{0.5}]$$

Solve for X

To isolate X, subtract the intervals of S from T:

$$[X_L^{0.5}, X_U^{0.5}] = [T_L^{0.5} - S_U^{0.5}, T_U^{0.5} - S_L^{0.5}]$$

For $\alpha = 0.5$:

$$X_L^{0.5} = 8.652 - 4.674 = 3.978$$

$$X_U^{0.5} = 11.348 - 3.326 = 8.022$$

So, the α-cut for X at $\alpha = 0.5$ is [3.978, 8.022].

By repeating this process for different values of α (from 0 to 1), you can construct the fuzzy number X that satisfies the fuzzy linear equation $X + S = T$.

4.3.3. Multiplication of Gaussian Bell Shape Fuzzy Numbers Using α-cuts Method

Definition 4.3.4. (Zimmermann, 2011) The multiplication of two Gaussian bell-shaped fuzzy numbers using the α-cuts method involves performing the multiplication operation at different levels of α to obtain the intervals (α-cuts) for the resulting fuzzy number. The α-cuts method allows for handling uncertainty and fuzziness by converting the fuzzy arithmetic operations into interval arithmetic at specified levels of α.

For Gaussian fuzzy numbers, the α-cuts are determined based on the mean and standard deviation of the Gaussian distribution, and the multiplication is carried out on these intervals.

Formula for α-cuts:

For Gaussian fuzzy numbers S and T with means c_S and c_T, and standard deviations σ_S and σ_T.

$$[S_L^{\alpha}, S_U^{\alpha}] = [c_S - k_{\alpha}\sigma_S, c_S + k_{\alpha}\sigma_S]$$

$$[T_L^{\alpha}, T_U^{\alpha}] = [c_T - k_{\alpha}\sigma_T, c_T + k_{\alpha}\sigma_T]$$

The α-cuts for the resulting fuzzy number $R = S \cdot T$ are given by:

$$[R_L^{\alpha}, R_U^{\alpha}] = [\min(S_L^{\alpha} \cdot T_L^{\alpha}, S_L^{\alpha} \cdot T_U^{\alpha}, S_U^{\alpha} \cdot T_L^{\alpha}, S_U^{\alpha} \cdot T_U^{\alpha}), max(S_L^{\alpha} \cdot T_L^{\alpha}, S_L^{\alpha} \cdot T_U^{\alpha}, S_U^{\alpha} \cdot T_L^{\alpha}, S_U^{\alpha} \cdot T_U^{\alpha})]$$

Example 4.3.5. Consider two Gaussian fuzzy numbers S and T:

S with mean $c_S = 3$ and standard deviation $\sigma_S = 1.2$.

T with mean $c_T = 2$ and standard deviation $\sigma_T = 0.8$.

For α = 0.5, $k_\alpha = 0.674$.

Determine the α-cuts

For S:

$$S_L^{0.5} = 3 - 0.674 \times 1.2 = 3 - 0.8088 = 2.1912$$

$$S_U^{0.5} = 3 + 0.674 \times 1.2 = 3 + 0.8088 = 3.8088$$

For T:

$$T_L^{0.5} = 2 - 0.674 \times 0.8 = 2 - 0.5392 = 1.4608$$

$$T_U^{0.5} = 2 + 0.674 \times 0.8 = 2 + 0.5392 = 2.5392$$

Multiply the Intervals

To multiply S and T using the α-cuts method:

For$\alpha = 0.5$:

$$R_L^{0.5} = [\min(2.1912 \cdot 1.4608, 2.1912 \cdot 2.5392, 3.8088 \cdot 1.4608, 3.8088 \cdot 2.5392)] \\ = [\min(3.2004, 5.5668, 5.5630, 9.6694)] = 3.2004$$

$$R_U^{0.5} = [\max(2.1912 \cdot 1.4608, 2.1912 \cdot 2.5392, 3.8088 \cdot 1.4608, 3.8088 \cdot 2.5392)] \\ = [\max(3.2004, 5.5668, 5.5630, 9.6694)] = 9.6694$$

So, the α-cut for R at $\alpha = 0.5$ is [3.2004, 9.6694].

By repeating this process for different values of α (from 0 to 1), you can construct the Gaussian bell-shaped fuzzy number representing the multiplication of S and T.

4.3.4 Division of Gaussian Bell Shape Fuzzy Numbers Using α-cuts Method

Definition 4.3.5. (Ban, 2009) The division of two Gaussian bell-shaped fuzzy numbers using the α-cuts method involves performing the division operation at different levels of α to obtain the intervals (α-cuts) for the resulting fuzzy number. The α-cuts method allows for handling uncertainty and fuzziness by converting the fuzzy arithmetic operations into interval arithmetic at specified levels of α.

For Gaussian fuzzy numbers, the α-cuts are determined based on the mean and standard deviation of the Gaussian distribution, and the division is carried out on these intervals.

Formula for α-cuts:

For Gaussian fuzzy numbers S and T with means c_S and c_T, and standard deviations σ_S and σ_T:

$$[S_L^\alpha, S_U^\alpha] = [c_S - k_\alpha \sigma_S, c_S + k_\alpha \sigma_S]$$

$$[T_L^\alpha, T_U^\alpha] = [c_T - k_\alpha \sigma_T, c_T + k_\alpha \sigma_T]$$

The α-cuts for the resulting fuzzy number $R = S/T$ are given by:

$$[R_L^\alpha, R_U^\alpha] = \left[\min\left(\frac{S_L^\alpha}{T_L^\alpha}, \frac{S_L^\alpha}{T_U^\alpha}, \frac{S_U^\alpha}{T_L^\alpha}, \frac{S_U^\alpha}{T_U^\alpha}\right), max(\frac{S_L^\alpha}{T_L^\alpha}, \frac{S_L^\alpha}{T_U^\alpha}, \frac{S_U^\alpha}{T_L^\alpha}, \frac{S_U^\alpha}{T_U^\alpha})\right]$$

Example 4.3.6. Consider two Gaussian fuzzy numbers S and T:

S with mean $c_S = 6$ and standard deviation $\sigma_S = 1.5$.

T with mean $c_T = 0.3$ and standard deviation $\sigma_T = 0.6$.

For α = 0.5, $k_\alpha = 0.674$.

Determine the α-cuts

For S:

$$S_L^{0.5} = 6 - 0.674 \cdot 1.5 = 6 - 11.011 = 4.989$$

$$S_U^{0.5} = 6 + 0.674 \cdot 1.5 = 6 + 1.011 = 7.011$$

For T:

$$T_L^{0.5} = 3 - 0.674 \cdot 0.6 = 3 - 0.4044 = 2..5956$$

$$T_U^{0.5} = 3 + 0.674 \cdot 0.6 = 3 + 0.4044 = 3.4044$$

Divide the Intervals

To divide S by T using the α-cuts method:

For $\alpha = 0.5$:

$$R_L^{0.5} = min\left(\frac{4.989}{2.5956}, \frac{4.989}{3.4044}, \frac{7.011}{2.5956}, \frac{7.011}{3.4044}\right) = min(1.922{,}1.465{,}2.701{,}2.058) = 1.465$$

$$R_L^{0.5} = max\left(\frac{4.989}{2.5956}, \frac{4.989}{3.4044}, \frac{7.011}{2.5956}, \frac{7.011}{3.4044}\right) = max\ (1.922{,}1.465{,}2.701{,}2.058) = 2.701$$

So, the α-cut for R at $\alpha = 0.5$ is [1.465, 2.701].

By repeating this process for different values of α (from 0 to 1), you can construct the Gaussian bell-shaped fuzzy number representing the division of S and T.

Example 4.3.7. Consider the fuzzy linear equation:

$$X \cdot S = T$$

Given fuzzy numbers S and T:

For S, Mean $c_S = 4$, Standard deviation $\sigma_S = 1$.

For T, Mean $c_T = 20$, Standard deviation $\sigma_T = 2$.

For α = 0.5, $k_\alpha \approx 0.674$.

Determine the α-cuts

For S:

$$S_L^{0.5} = 4 - 0.674 \cdot 1 = 4 - 0.674 = 3.326$$

$$S_U^{0.5} = 4 + 0.674 \cdot 1 = 4 + 0.674 = 4.674$$

For T:

$$T_L^{0.5} = 20 - 0.674 \cdot 2 = 20 - 1.348 = 18.652$$

$$T_U^{0.5} = 20 + 0.674 \cdot 2 = 20 + 1.348 = 21.348$$

Formulate the Interval Arithmetic

At α = 0.5, the fuzzy equation can be written as:

$$[X_L^\alpha, X_U^\alpha] \cdot [S_L^\alpha, S_U^\alpha] = [T_L^\alpha, T_U^\alpha]$$

Solve for X

To isolate X, divide the intervals of T by the intervals of S:

For $\alpha = 0.5$:

$$X_L^{0.5} = min\left(\frac{18.652}{3.326}, \frac{18.652}{4.674}, \frac{21.348}{3.326}, \frac{21.348}{4.674}\right)$$
$$= \min(5.607, \quad 3.991, \quad 6.417, \quad 4.568) = 3.991$$

$$X_U^{0.5} = max\left(\frac{18.652}{3.326}, \frac{18.652}{4.674}, \frac{21.348}{3.326}, \frac{21.348}{4.674}\right)$$

$$= \max(5.607, \quad 3.991, \quad 6.417, \quad 4.568) = 6.417$$

So, the α-cut for X at $\alpha = 0.5$ is [3.991, 6.417].

By repeating this process for different values of α (from 0 to 1), you can construct the fuzzy number X that satisfies the fuzzy linear equation $X \cdot S = T$.

Example 4.3.8. Consider the fuzzy linear equation:

$$X \cdot S + T = R$$

Given fuzzy numbers S, T, and R:

For S, Mean $c_S = 3$, Standard deviation $\sigma_S = 0.5$

For T, Mean $c_T = 2$, Standard deviation $\sigma_T = 0.4$

For R, Mean $c_R = 8$, Standard deviation $\sigma_R = 1$

For α = 0.5, $k_\alpha \approx 0.674$.

Determine the α-cuts

For S:

$$S_L^{0.5} = 3 - 0.674 \cdot 0.5 = 3 - 0.337 = 2.663$$

$$S_U^{0.5} = 3 + 0.674 \cdot 0.5 = 3 + 0.337 = 3.337$$

For T:

$$T_L^{0.5} = 2 - 0.675 \cdot 0.4 = 2 - 0.2696 = 1.7304$$

$$T_U^{0.5} = 2 + 0.674 \cdot 0.4 = 2 + 0.2696 = 2.2696$$

For R:

$$R_L^{0.5} = 8 - 0.674 \cdot 1 = 8 - 0.674 = 7.326$$

$$R_U^{0.5} = 8 + 0.674 \cdot 1 = 8 + 0.674 = 8.674$$

Formulate the Interval Arithmetic

At α = 0.5, the fuzzy equation can be written as:

$$[X_L^{0.5}, X_U^{0.5}] \cdot [S_L^{0.5}, S_U^{0.5}] + [T_L^{0.5}, T_U^{0.5}] = [R_L^{0.5}, R_U^{0.5}]$$

Solve for X

To isolate X, first subtract the intervals of T from the intervals of R:

$$[R_L^{0.5} - T_U^{0.5}, R_U^{0.5} - T_L^{0.5}] = [7.326 - 2.2696, 8.674 - 1.7304] = [5.0564, 6.9436]$$

Then divide the resulting intervals by the intervals of S:

$$[X_L^{0.5}, X_U^{0.5}] = \left[min\left(\frac{5.0564}{2.663}, \frac{5.0564}{3.337}, \frac{6.9436}{2.663}, \frac{6.9436}{3.337}\right), \max\left(\frac{5.0564}{2.663}, \frac{5.0564}{3.337}, \frac{6.9436}{2.663}, \frac{6.9436}{3.337}\right)\right]$$

For $\alpha = 0.5$:

$$X_L^{0.5} = min\left(\frac{5.0564}{2.663}, \frac{5.0564}{3.337}, \frac{6.9436}{2.663}, \frac{6.9436}{3.337}\right) = \min(1.515, 1.898, 2.081, 2.607) = 1.515$$

$$X_U^{0.5} = \max\left(\frac{5.0564}{2.663}, \frac{5.0564}{3.337}, \frac{6.9436}{2.663}, \frac{6.9436}{3.337}\right) = \max(1.515, 1.898, 2.081, 2.607) = 2.607$$

So, the α-cut for X at $\alpha = 0.5$ is [1.515, 2.607].

By repeating this process for different values of α (from 0 to 1), you can construct the fuzzy number X that satisfies the fuzzy linear equation $X \cdot S + T = R$.

5. CONCLUSION

This study reveals the true power of fuzzy mathematics, proving that uncertainty is not a weakness but a reality that can be understood and controlled. Traditional mathematics demands precision, but the real world is full of vague, incomplete, and imprecise data. This research shows that fuzzy equations offer a way to work with uncertainty instead of against it, making them essential in modern problem-solving. One of the most striking findings is that fuzzy equations can be solved with accuracy, even when exact values are unknown. By using α-cuts and membership functions, this study demonstrates that uncertainty can be measured and processed logically. Methods like quasi linearization and upper-lower solutions provide structured ways to find answers, proving that even imprecise data can lead to reliable solutions. The study also highlights how fuzzy arithmetic addition, subtraction, multiplication, and division allows calculations to remain meaningful in uncertain conditions. By structuring numbers within fuzzy intervals, this research shows that mathematics can adapt to the complexity of real-world problems. Beyond theory, this work emphasizes the huge impact of fuzzy mathematics on fields like artificial intelligence, engineering, and economics. From decision-making to predictive modeling, fuzzy systems are shaping the future. In the end, this study delivers a powerful message: true intelligence is not about avoiding uncertainty but mastering it. The world is full of unknowns, and fuzzy mathematics is the key to making sense of them.

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