THE MASSES AND MIXINGS OF NEUTRAL FERMIONS IN A GRAND UNIFIED E_6 MODEL

by

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ABSTRACT

The modern theory of elementary particle physics is based on the Standard Model and the symmetry groups in question are Lie groups as far as continous symmetries are considered. The strong, weak and electromagnetic interactions among the elementary particles are associated with the Lie algebras of $SU(3)_C$, $SU(2)$ and $U(1)$ in the Standard Model. There are many reasons not to believe that the Standard Model is a final theory. New physics on which we will concentrate in this thesis is based on the idea of unification where three types of known gauge interactions are unified at the same very high energy scale by linking the three of the four forces in nature by combining strong and electroweak forces, called the Grand Unified Theory (GUT). We here ground on the exceptional E_6 group for Grand Unification to give the masses and mixing of neutral leptons. We start by giving some algebraic foundations of exceptional groups including division algebras, octonions and quaternions. Then, we focus on E_6 model where the fundamental representation **27** include the fermionic content of the model. Then a brief review of the assignment of elementary fermions and bosons to irreducible multiplets in E_6 models is followed by a discussion of different, hierarchical symmetry breaking chains from E_6 down to $SU(3)_C \times U(1)_{EM}$. We first discuss the symmetry breaking pattern for the $E_6 \to SO(10) \to SU(5) \to U(2)_{EW} \times SU(3)_C \to U(1)_{QED} \times SU(3)_C$ chain that involves Georgi and Glashow's $SU(5)$ model. Next we concentrate on the symmetry breaking chain $E_6 \to SO(10) \to SU(2)_L \times SU(2)_R \times SU(4)_C \to U(2)_{EW} \times SU(3)_C \to U(1)_{EM} \times SU(3)_C$ with an intermediate Pati-Salam symmetry for which $(B-L)$ is conserved. In particular, the mass/mixing matrix of electrically neutral fermions (i.e.neutrinos) that would be derived from Yukawa couplings is constructed. The pattern of neutrino masses and some bounds on mixing parameters are discussed.

ÖZETÇE

Parçacık fiziğinin modern teorisi Standart Model'i temel almaktadır ve sürekli simetriler düşünüldüğünde söz konusu gruplar Lie gruplarıdır. Standart Model'de temel parçacıklar arasındaki kuvvetli, zayıf ve elektromanyetik etkileşimler $SU(3)_C$, $SU(2)$ ve $U(1)$ Lie grupları ile ilişkilendirilmektedir. Standard Model'in ulaşabileceğimiz son nokta olmadığıa inanmak için bir çok sebep bulunmaktadır. Bizim burada yoğunlaşacagımız Standart Model ötesi yeni fizik, 3 temel ayar etkileşiminin yüksek bir enerji seviyesinde doğadaki 4 kuvvetin 3'ünün birbirine bağlanarak kuvvetli ve elektrozayıf kuvvetlerin birleştiği teori olan Büyük Birleştirme Teorisine dayanmaktadır. Bu tez çalışmasında, istisnai E_6 grubunu temel alarak nötr leptonlar için kütle ve karışımları vermeği amaçlıyoruz. Başlangıç olarak cebirsel temelleri vermek amacıyla bölüm cebirleri, kuatarniyonlar ve oktoniyonları tanımlıyoruz. Daha sonra, fermiyonik içeriği görmek için E_6 modelindeki temel temsillere yoğunlaşıyoruz. Ardından, fermion ve bozonların indirgenemez multipletlere nasıl atandığının kısa bir özetiyle hiyerarşik simetri bozulma zincirlerinin E_6'dan $SU(3)_C \times U(1)_{EM}$ ' e nasıl indirgendiğini tartışıyoruz. Simetri bozulma zincirlerini tartışmak için önce Georgi ve Glashow'un $SU(5)$ modelini içeren $E_6 \to SO(10) \to SU(5) \to U(2)_{EW} \times SU(3)_C \to U(1)_{QED} \times SU(3)_C$ zincirini vereceğiz. Daha sonra, (Baryon-Lepton) kuantum sayısının korunduğu Pati-Salam simetrisini içeren $E_6 \to SO(10) \to SU(2)_L \times SU(2)_R \times SU(4)_C \to U(2)_{EW} \times SU(3)_C \to U(1)_{EM} \times SU(3)_C$ zincirine bakarak nötr leptonların kütle ve karışım matrislerinin çıkarımını vereceğiz. Son olarak, nötrino kütlelerini ve karışım parametre değerlerinde bazı sınırları tartışacağız.

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Chapter 1

INTRODUCTION

Search for harmony has always been essential for science. In that sense symmetries play a critical role in physics especially for particle physics. Symmetry transformations form groups and the mathematics underlying the concept of symmetry is the group theory. The gauge symmetries underlying the interactions via mediating the gauge bosons are described by the gauge theories. The modern theory of high energy physics is based on the Standard Model which is a local quantum gauge field theory supporting the Lie symmetry algebras $SU(3)_C$, $SU(2)$ and $U(1)$ that are associated with strong, weak and electromagnetic interactions among the elementary particles. It is augmented with a fine-tuned Higgs mechanism that generates the observed mass spectrum of all constituent fermions and the intermediate vector bosons. The existence of an elusive heavy scalar Higgs boson in this framework was eventually confirmed at the LHC experiments in 2012.

There are many reasons for not regarding the Standard Model as a final theory. First of all, even though it agrees very well with the experimental data available so far, it has too many independent free parameters to be fixed phenomenologically. This asks for an enlargement of the symmetry schemes in order to cut down the number of free parameters. Secondly, there is more in the Nature than that is evidenced by the Standard Model, because (i) current inflationary cosmological evolution scenarios require the existence of large amounts of Dark Matter and Dark Energy and (ii) the observed neutrino oscillations imply massive neutral fermions with complicated patterns of interactions. Moreover, (iii) the fact that only three families of elementary particles exist still awaits for a theoretical explanation. Finally, (iv) the puzzle of the matter-anti-matter asymmetry of the Universe also requires an explanation. These are the main reasons why we need new physics which holds out more symmetry beyond the Standard Model.

The new physics on which we will concentrate is based on the idea of Grand Unification where three types of known gauge interactions are unified at the same very high energy scale by combining strong and electroweak forces but excluding gravity. In Grand Unified Theories the Standard Model gauge group is embedded into a larger group G. This larger group may be just one simple Lie group or a direct product of a number of simple Lie groups. Since the theory should be based on the Standard Model gauge group, it should be at least a rank 4 group. We will study further gauge symmetries which are members of the Lie family. Most of the GUT's naturally explain the quantization of electric charge, predict the $\sin^2\theta_w$ and massive neutrinos. The most promising candidates of larger Lie groups for unification so far have been $SU(5)$, $SO(10)$, E_6 and the semi-simple $SU(3)_L \times SU(3)_R \times SU(3)^c$. In this thesis we aim to ground on E_6 group for grand unification to give the masses and mixing of neutral fermions. [1,2]

We start by giving some algebraic foundations to understand the exceptional groups in the preliminaries part. To start with we give basics of Division algebras, octonions and quaternions since they form the very basics of the exceptional group algebras. [3,4] Then, the Cayley algebra with the elements as octonions is explained. Later, the non-associative and 27 dimensional Jordan algebra which is used for the exceptional group E_6 is introduced. Next, the Magic square is explained which is a way of constructing several Lie groups out of automorphism algebras. In the last section of preliminaries we give the basics to understand how to use the Mathematica application Lie Art. This application helps us to do tensor product decompositions and the subalgebra branchings of irreducible representations for higher rank groups which is harder to do manually. [5–7]

In the first chapter following the preliminaries, we introduce the Lie groups used for Grand unification such as $SU(5)$, $SO(10)$ and E_6. The $SU(5)$ is important since it is the first and simplest Lie group used for Grand Unification, as proposed by Howard Georgi and Sheldon Glashow in 1974 [8]. The very basic idea was to unify The Standard Model gauge groups $SU(3)\times SU(2)\times U(1)$ into a one single gauge group $SU(5)$. This theory combines leptons and quarks into single irreducible representations and there exists interactions where baryon number is not conserved but the difference between baryon and lepton numbers $(B-L)$ still conserved. This yields a mechanism for proton decay and the rate of proton decay can be predicted from the dynamics of the model. In the first section of this chapter, we give the basics of the algebra of the $SU(5)$ Model therefore we start by defining the basis for the generators, then write the roots and weights and complete with giving the Dynkin diagram of $SU(5)$ which is drawn according to Cartan classification. In the next section, we give the Young Tableaux method to give the irreducible representations of $SU(5)$ which is a useful way to decompose $SU(N)$ algebras into its irreducible representations. Later, we give the particle assignments in $SU(5)$ Grand Unified Model which is important to see the particle content appearing in a larger multiplet rather than the Standard Model, there we give both bosonic and fermionic parts. In the next section, to gather everything in an elegant way we give the Lagrangian of the $SU(5)$ Model where we can see the interactions and mixing of new particles. We also represent the new interactions via Feynman diagrams. We complete the $SU(5)$ Model by describing the Higgs sector, fermion masses and Yukawa interactions. In the last section of this chapter, we give the $SO(10)$ Model in simple terms since the $SU(5)$ part is a detailed part to understand the idea of the unification. Being a higher rank group $SO(10)$ contains a further content by unifying quarks and leptons of the each generation into the single 16 dimensional representation and predicting a right handed neutrino [9].

In chapter 4, we discuss the exceptional group E_6 as a Grand Unification Model. We start by writing the Cartan matrix for E_6 and continue from there to represent the algebra and basics of the whole model. We apply the application the program Lie Art for the tensor decompositions, branching rules and some basics formulations of the E_6 Model since being a high rank groups it has cumbersome calculations. We first aim to draw the weight diagram for the **27** dimensional representation of E_6 since it is the fundamental representation where the fermionic content appears in. For this, we write the levels of the diagram, dimensionality formula for E_6 and follow some specific symmetry breaking branch to get the content. As a most common branching we first choose $E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)$ which is ending with The Standard Model gauge groups. The reason why we first give this branch is that we start by $SU(5)$ and $SO(10)$ so in this way the model seems complete in every way. Next, we form the projection matrices in the chosen symmetry breaking chain to get the particle content explicitly. We start by the highest weight of the fundamental representation **27** of E_6 and continue by lowering the level in the weight diagram and obtain all the fermionic content. Lastly, we write the axis formulas from symmetry axis in the model and see the quantum numbers easily in that way [10].

In chapter 5, we review some of the symmetry breaking patterns in E_6. We begin with analyzing the symmetry breaking mechanisms. For this, we focus on the physical constraints that comes from the patterns instead of putting a Higgs field and minimizing it. These constraints appear from the fact that symmetry breaking directions are related to mass matrices, thus, we start by defining the covariant derivatives and show its relation mass matrix in the associated representation. We also use Cartan subalgebra for this. Initially, we show the relation for $SU(5)$'s adjoint representation **24** and continue with E_6. We derive the axes obtained from the constraints in the symmetry breaking chain $E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)$. This chain is also used as an example to show the effect of the charge conjugation on the weights. We also write the mass matrix for quarks and neutral leptons for a 27plet in E_6. In the next section we concentrate on the symmetry breaking chain $E_6 \supset SO(10) \supset SU(4) \times SU(2)_L \times SU(2)_R$ down to the Standard Model gauge groups where this time $SU(5)$ is not present. This pattern can also be called as symmetry breaking in E_6 via the intermediate Pati-Salam symmetry. We review the basic concepts in this symmetry breaking channel where we use use it for our neutral lepton mass matrix calculations in the next chapter. [11, 12]

In the last chapter before the conclusion, we concentrate on the electrically neutral fermions predicted by the E_6 GUT Model with an intermadiate Pati-Salam symmetry. [13, 14] There appear new particles with this Grand Unified Model and those electrically neutral fermions in question are the conventional neutrinos, right handed neutrino and a sterile neutrino in a 27plet of E_6. [15–17]. We start by defining the projection matrices in $E_6 \supset SO(10) \supset SU(4) \times SU(2)_L \times SU(2)_R$ chain first and complete with $SU(4) \supset SU(3)^c$ projection. We get $U(1)$ charges from every step of projection, the first projection $E_6 \supset SO(10)$ gives Q^t charge and with the hypercharge we get the $(B-L)$ quantum number from $Q = I_3^L + I_3^R + (B-L)/2$. These $U(1)$ charges are used to write the neutral lepton mass matrix. Later, we used Jacobi rotation and some perturbation theories to obtain the eigenvalues of this matrix and mixings and comment on them. Finally, we end this chapter up with quark mass matrix and charged lepton mass matrices. [18, 19].

Chapter 2

PRELIMINARIES

In the interest of studying E_6 we start by introducing octonions since many of the exceptional groups are closely connected to them.

2.1 Division Algebras and Octonions

The Lie algebra of five exceptional groups can be constructed in a unified way in terms of antihermitian matrices. The elements of these antihermitian matrices are members of the Hurtwitz algebra which is a division algebra with the norm obeying the composition law [20].

$$N(x)N(y) = N(xy) \tag{2.1}$$

There are 4 algebras which are both division and composition algebras, called Hurtwitz algebras: Real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{Q}$ and octonions $\mathbb{O}$. Quaternions and octonions are extensions of real and complex numbers which we are familiar with [4]. A quaternion is a linear combination of the real identity element 1 and three different square roots -1, called e_1, e_2, e_3 satisfying the multiplication table seen in Figure 2.1

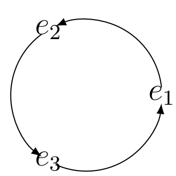


Figure 2.1: Quaternion Units Multiplication Table

Octonions are formed from seven of the squre roots of -1 called $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ which obeys the below multiplication table called as Fano plane seen in Figure 2.2.

Here e_0=1 is called as unit element. Octonion bases, except e_0, obey ${e_i}^2 = -1$ and they are not associative, as for any three bases:

$$\begin{aligned}(e_7e_2)(e_4) &= e_1\\ (e_7)(e_2e_4) &= -e_1\end{aligned} \tag{2.2}$$

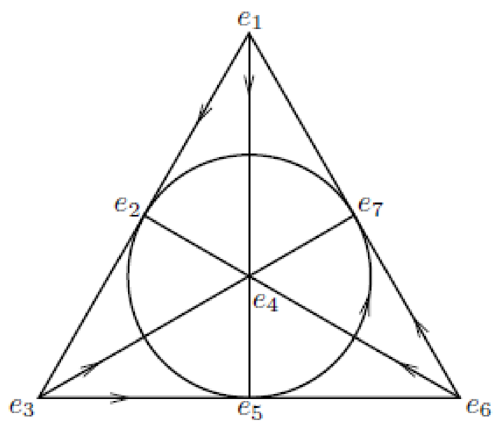


Figure 2.2: Fano Plane

Define an octonion as
$a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$ and the conjugate goes as $\bar{a} = a_0 - a_1e_1 - a_2e_2 - a_3e_3 - a_4e_4 - a_5e_5 - a_6e_6 - a_7e_7$. For any two of them

$$\overline{ab} = \bar{b}\bar{a} \tag{2.3}$$

The norm is defined as

$$|a|^2 = a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \tag{2.4}$$

and the inverse

$$a^{-1} = \frac{\bar{a}}{|a|^2} \tag{2.5}$$

Although octonions are not associative they are alternative and we define the associator of three octonions as

$$[a, b, c] = (ab)c - a(bc) \tag{2.6}$$

The commutator is defined in the usual way

$$[a, b] = ab - ba. \tag{2.7}$$

2.1.1 Cayley Algebra

With the elements as octonions an algebra defined over the field $\mathbb{F}$ is called as Cayley algebra $\mathcal{C}$. This algebra satisfies all the axioms of a field except the associativity. To construct the exceptional Lie algebras we need to define first the automorphism groups of the associated Lie groups. The automorphism group of an algebra is defined as an isomorphism group of that algebra into itself. Automorphism group is formed from transformations which leave the multiplication table invariant. Thus the automorphism group of quaternions is $SU(2)$ which leaves the cyclic multiplication table invariant. To understand the automorphism group of octonions consider the element e_1, there are three pairs of octonionic units forming a quaternionic algebra with e_1: they are $\{e_2, e_3\}$, $\{e_6, e_7\}$ and $\{e_5, e_4\}$, called points to e_1. To preserve the Fano table when we rotate $\{e_2, e_3\}$ plane by α we must rotate $\{e_6, e_7\}$ table by $-\alpha$. This is a 1-parameter family of automorphism and there are three ways to pair up the units in this way but only two of them will be independent, the third one will be fixed by the choice. With 7 different units and 2 different ways to pair up we have a 14 parameter group in hand. This group is the automorphism group of octonions and called as G_2 [6]. In the construction of Lie algebras for the exceptional groups we need different methods rather than the classical ones since lack of associativity results in the Jacobi identity not being satisfied. For this reason the concept of triality is introduced. Octonions are represented by 8 real numbers so the group that leaves the quadratic form invariant is $SO(8)$ which is the norm group for octonions. Taking d_1 as an infinitesimal element of $SO(8)$, the principle of triality states that we can associate d_1 to other two elements in $SO(8)$ say d_2 and d_3 uniquely. They satisfy

$$\overline{d_1(\overline{ab})} = d_2(a)b + ad_2(b) \tag{2.8}$$

for a, b being any two octonions. Thus in $SO(8)$ we can associate any element to other two elements uniquely. Furthermore for the Lie algebra of G_2, we consider the Lie algebra of $SO(8)$ and define the linear mappings

$$\begin{aligned} G_{ij} : C \longrightarrow C \quad i, j = 0, 1, .., 7 \quad \text{where} \quad G_{ij} \in so(8) \\ G_{ij}e_j = e_i \quad G_{ij}e_i = -e_j \quad G_{ij}e_k = 0 \quad k \neq i, j. \end{aligned} \tag{2.9}$$

and

$$F_{ij} \quad : \quad \mathrm{C} \rightarrow \mathrm{C} \tag{2.10}$$

$$F_{ij}a \quad = \quad \frac{1}{2}e_i(\overline{e_j}a) \in \mathrm{C} \tag{2.11}$$

And κ, π, ν as

$$(\kappa d)a = \overline{d\overline{a}} \tag{2.12}$$

$$\pi(G_{ij}) = F_{ij} \tag{2.13}$$

$$\nu = \pi\kappa \tag{2.14}$$

2.1.2 Jordan Algebra

In general, $n \times n$ hermitian matrices over real, complex and quaternions form a Jordan algebra $\mathcal{J}$ which is defined with the product called as Jordan product such that

$$J_1 \circ J_2 = \frac{1}{2}(J_1J_2 + J_2J_1) \tag{2.15}$$

Classical Lie groups which leave the Jordan product invariant are the automorphism groups of the associated division algebra. In particular, $SO(n)$,$SU(n)$ and $Sp(2n)$ are automorphism groups of real numbers, complex numbers and quaternions respectively. For octonions the situation is more specific, only 3×3 Hermitian matrices form a Jordan algebra for octonions. To construct the Lie algebra of exceptional group E_6 we need to introduce the Jordan algebra which is formed from 3×3 Hermitian matrices with entries from Cayley algebra. [21, 22] Any element is defined as

$$A = A(\xi, a) = \begin{pmatrix} \xi_1 & a_3 & \overline{a_2} \\ \overline{a_3} & \xi_2 & a_1 \\ a_2 & \overline{a_1} & \xi_3 \end{pmatrix}$$

where $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and $a_1, a_2, a_3 \in \mathbb{O}$.

Besides the Jordan product, the algebra satisfies the following identities, for the trace

$$\begin{aligned} (A, B) &= tr(A \circ B) \\ tr(A, B, C) &= (A, B \circ C) \end{aligned} \tag{2.16}$$

The multiplication called as the Freudenthal multiplication is defined by

$$A \times B = \frac{1}{2}(2AB - tr(A)B - tr(B)A + (tr(A)tr(B) - (A, B)I)) \tag{2.17}$$

where I is the identity matrix. Trilinear form and determinant are defined respectively as

$$(A, B, C) = (A, B \times C) \tag{2.18}$$

$$det(A) = \frac{1}{3}(A, A, A) \tag{2.19}$$

2.1.3 Magic Square

We have seen that many of the exceptional groups are connected with octonions. Now we will look at the some unified way of constructing the exceptional groups F_4, E_6, E_7 and E_8 out of the automorphism algebras [3]. This method was introduced by Tit and Freudenthal and is called as Tit's construction. When octonions are used as the elements in the matrix representation this construction yields a Lie algebra. The idea starts by obtaining the corresponding Lie algebra of a given Hurtwitz algebra. By this we see many of the groups as a generalization of classical ones. Given a Hurtwitz algebra say c , we get the dimension of the corresponding Lie algebras as

$$dim\mathcal{L} = 3n_c + 2(n_c - 1) + dim(\text{aut}c) \tag{2.20}$$

where n_c corresponds to number of parameters to describe the corresponding Hurtwitz algebra. Thus $n_{\mathbb{R}} = 1, n_{\mathbb{C}} = 2, n_{\mathbb{Q}} = 4$ and $n_{\mathbb{O}} = 8$. To understand the construction consider the traceless antihermitian matrices which contain 36 parameter with octonions having 7 parameters. Together with the automorphism group G_2 which is a 14 parameter group in total we have the 52 parameter group which is same as F_4 [23]. To see this consider the concept of triality which enables us to establish one-to-one correspondence between any three $SO(8)$ elements uniquely. Consider the 3×3 antihermitian matrices with only off diagonal elements

$$A = \begin{pmatrix} 0 & a_3 & \bar{a}_2 \\ \bar{a}_3 & 0 & a_1 \\ a_2 & \bar{a}_1 & 0 \end{pmatrix}$$

there are $3\times 8 = 24$ such matrices and together with the 28 $SO(8)$ transformations we have 52 parameter group F_4. By the principle of triality $SO(8)$ acts like

$$\delta A = \begin{pmatrix} 0 & d_3 a_3 & \overline{d_2 a_2} \\ \overline{d_3 a_3} & 0 & d_1 a_1 \\ d_2 a_2 & \overline{d_1 a_1} & \end{pmatrix}$$

To get a Lie algebra we need to satisfy the algebraic closure and for this to happen commutator of two matrices should result a off-diagonal matrix and triplets of $SO(8)$. Commutator of any two A type matrices results in a matrix of the same kind plus a diagonal matrix

$$X_3(a) = \begin{pmatrix} o & a & a \\ -\bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which should be interpreted as $SO(8)$ transformation. Taking the entries as octonion elements we should observe that Jacobi identity is satisfied where

$$([X_1, X_2], X_3) = \begin{pmatrix} 0 & d_3[X_1, X_2](X_3) & 0 \\ -\bar{d}_3[X_1, X_2](X_3) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Taking the cycling sum over $1, 2$ and 3 we see that Jacobi identity is satisfied.

$$\sum_{cyc} d_3[X_1, X|2](X_3) = 0 \tag{2.21}$$

Thus interpreting the diagonal elements as similarity triplets satisfied the Jacobi identity. F_4 is closed by commutator of two matrices and additional triplets of $SO(8)$. When this method is applied to real numbers, complex numbers and quaternions, we get $SO(3), SU(3)$ and $Sp(6)$ respectively. The embeddings for F_4 is given as

$F_4 \subset SO(3) \times G_2 \quad , \quad F_4 \subset Sp(6) \times SU(2) \quad , \quad F_4 \subset SU(3) \times SU(3)$.
This procedure can be generalized to any two Hurtwitz algebra say c and c' to construct a Lie algebra and the dimension of resulting algebra is given by the formula

$$dim\mathcal{L}(cc') = 3n_c n_{c'} + 2[(n_c - 1)(n_{c'} - 1)] + dim(\text{aut}c) + dim(\text{aut}c') \qquad (2.22)$$

When generalized we see the exceptional groups F_4, E_6, E_7 and E_8 as generalizations of classical groups. These results are summerized in a table called Magic Square given below in Table 2.1.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}$	$\mathbb{O}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$Sp(6)$	F_4
$\mathbb{C}$	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$\mathbb{Q}$	$Sp(6)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Table 2.1: Magic Square

As can be seen also from the table in the first row F_4 is seen as a generalization of the classical group $SU(3)$. When we change the underlying Hurtwitz algebra from complex numbers to octonions we get an additional group which is the automorphism group of octonions G_2 and it reduces to $SU(3)$ in terms of complex numbers.
Taking $c = \mathbb{C}$,$c' = \mathbb{Q}$, Tit's method for construction gives a 35 parameter group from

$$dim\mathcal{L}(cc') = 3 \cdot 2 \cdot 4 + 2[3 \cdot 1] + 2 = 35 \qquad (2.23)$$

where quaternions have 2×2 matrix representation thus we have 6×6 matrices in the resulting Lie algebra which is identified with $SU(6)$.

Taking $c = \mathbb{C}$, $c' = \mathbb{O}$ we have

$$dim\mathcal{L}(cc') = 3 \cdot 2 \cdot 8 + 2[2 \cdot 6] + 14 = 78 \qquad (2.24)$$

This is the exceptional group E_6. One of the Hurtwitz algebra in constructing this exceptional group E_6 is complex and thus resulting group has complex representations. When we look at the magic square there appears a pattern when going between columns and rows. When we look at the E_6 in the last column we observe the subgroup chain. When going from column 4 to column 3 which is expressing octonions in terms of quaternions there apears an extra $SU(2)$ term in expressing the $SU(6)$ as a subgroup of E_6 like $E_6 \supset SU(6) \times SU(2)$. Passing from column 4 to 2 which is expressing octonions in terms of complex numbers this time we have an additional $SU(3)$ resulting to $E_6 \supset SU(3) \times SU(3) \times SU(3)$. Finally representing column4 in terms of column 1 we have the automorphism group of octonions G_2 as an extra group appearing in the subgroup chain. Following from here E_6 has subgroup chains as seen from the magic square go as

$$\begin{aligned} E_6 &\supset SU(3) \times G_2 \\ &\supset SU(3) \times SU(3) \times SU(3) \\ &\supset SU(6) \times SU(2) \end{aligned} \tag{2.25}$$

Taking $c = \mathbb{Q}$ and $c' = \mathbb{Q}$ we get a 66 dimensional algebra again with quaternions represented as 2×2 matrices we have 12×12 matrices identified with $SO(12)$. Taking $c = \mathbb{Q}$ and $c' = \mathbb{O}$ we have this time

$$dim\mathcal{L}(cc') = 3 \cdot 4 \cdot 8 + 2 \cdot 3 \cdot 7 + 14 + 3 = 133 \tag{2.26}$$

parameter group E_7. In the same manner with passing between the columns in magic square E_7 has following subgroup chain.

$$E_7 \supset Sp(6) \times G_2 \tag{2.27}$$
$$\supset SU(6) \times SU(3) \tag{2.28}$$
$$\supset SO(12) \times SU(2) \tag{2.29}$$

Finally taking the two Hurtwitz algebras as octonions we have

$$dim\mathcal{L}(cc') = 3 \cdot 8 \cdot 8 + 2 \cdot 7 \cdot 7 + 14 + 14 = 248 \tag{2.30}$$

with this 248 real parameters from the table it is seen that

$$\begin{aligned} E_8 &\supset F_4 \times G_2 \\ &\supset E_6 \times SU_3 \\ &\supset E_7 \times SU(2). \end{aligned} \tag{2.31}$$

As observed above all exceptional groups apart from G_2 appears as the generalizations of the classical groups in the magic square. It is also shown that the exceptional group appears in the presence of octonions.

We have given the Lie algebra of the exceptional group E_6 which is made possible with Jordan algebra by using octonions. Next the magic square has been introduced which represents the subgroup chains in a specific manner. In the following chapter we will eventually give the complete model based on E_6 [1, 24]. So far we have gathered many group theoretical information to analyze the exceptional groups, with this information we use Dynkin labels obtained from diagrams and analyze the irreducible representations of the exceptional algebra and give the unification model based on the exceptional group E_6 since the Standard Model fits naturally inside it [25–27].

Chapter 3

LIE ALGEBRAS USED FOR GRAND UNIFICATION

3.1 $SU(5)$

First of all we focus on $SU(5)$ group since it is the simplest one to understand the unification process and can be thought as a prototype for GUT. The $SU(5)$ Grand Unified Model which combines the Standard Model gauge groups $SU(3) \times SU(2) \times U(1)$ into a single gauge group is first given by Georgi and Glasow in 1974 thus it is called as Georgi Glashow $SU(5)$ grand unified model [8]. The generators of the $SU(5)$ group are the 5×5 matrices which are the generalization of the Gell Mann matrices

$$T^a = \frac{\lambda^a}{2} \tag{3.1}$$

where $a = 1, 2, ..., 24$ for $SU(5)$. These are Hermitian and traceless matrices with the condition

$$tr(\lambda^a \lambda^b) = \delta^{ab} \tag{3.2}$$

In order to get $SU(3) \times SU(2)$ structure out of $SU(5)$ matrices, we need to keep $SU(3)$ acting on the first 3 rows and columns and $SU(2)$ acting on last two rows and columns. The first eight matrices are the $SU(3)$ Gell-Mann Matrices acting on the first three rows and columns

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 & & \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & -1 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i & & \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \\ & & & & \\ & & & & \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & -i & & \\ 0 & i & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & -2 & & \\ & & & & \\ & & & & \end{pmatrix}.$$

The $21., 22.$ and $23.$ matrices are the Pauli Spin matrices of SU(2) acting on the last two rows and columns of the SU(5) matrices;

$$\lambda^{21} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}, \quad \lambda^{22} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 0 & -i \\ & & & i & 0 \end{pmatrix}, \quad \lambda^{23} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 1 & 0 \\ & & & 0 & -1 \end{pmatrix}.$$

The remaining 12 matrices are obtained from those above by permuting rows and columns:

$$\lambda^{9} = \begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \lambda^{10} = \begin{pmatrix} & & & -i & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \lambda^{11} = \begin{pmatrix} & & & 0 & 0 \\ & & & 1 & 0 \\ & & & 0 & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix},$$

$$\lambda^{12} = \begin{pmatrix} & & & 0 & 0 \\ & & & -i & 0 \\ & & & 0 & 0 \\ 0 & i & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \lambda^{13} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 1 & 0 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \quad \lambda^{14} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & -i & 0 \\ 0 & 0 & i & & \\ 0 & 0 & 0 & & \end{pmatrix},$$

$$\lambda^{15} = \begin{pmatrix} & & & 0 & 1 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \end{pmatrix}, \quad \lambda^{16} = \begin{pmatrix} & & & 0 & -i \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ i & 0 & 0 & & \end{pmatrix}, \quad \lambda^{17} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 1 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \end{pmatrix},$$

$$\lambda^{18} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & -i \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & i & 0 & & \end{pmatrix}, \quad \lambda^{19} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & 1 \\ 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \end{pmatrix}, \quad \lambda^{20} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & -i \\ 0 & 0 & 0 & & \\ 0 & 0 & i & & \end{pmatrix}.$$

The last matrix will be diagonal:

$$\lambda^{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} .$$

We now introduce the Cartan subalgebra which is the maximal Abelian subalgebra of the group and contain l diagonazible, Hermitian and unique matrices for a rank l group, defined as

$$H_i = H_i^\dagger \tag{3.3}$$

$$[H_i, H_i^\dagger] = 0 \quad . \tag{3.4}$$

Normalization condition is $tr(H_i H_j) = \lambda \delta_{ij}$ where $i, j = 1, 2, .., N-1$ for $SU(N)$. $SU(5)$ is a rank 4 group, the four Cartan generator are given expilictly as

$$H^1 = \frac{1}{2} \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} , \quad H^2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} ,$$

$$H^3 = \frac{1}{\sqrt{24}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3 & \\ & & & & 0 \end{pmatrix} , \quad H^4 = \frac{1}{\sqrt{40}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} .$$

The weights are given as the eigenvalues of the Cartan generators in the defining representation via

$$H_i | \lambda >= \lambda_i | \lambda > . \tag{3.5}$$

For $SU(5)$ the weights are given as

$$\begin{aligned}
\lambda_1 &= (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}), \\
\lambda_1 &= (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}), \\
\lambda_1 &= (0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}), \\
\lambda_1 &= (0, 0, -\frac{3}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}), \\
\lambda_1 &= (0, 0, 0, -\frac{4}{2\sqrt{10}}).
\end{aligned} \tag{3.6}$$

The remaining generators are expected to satisfy the eigenvalue equation of the form

$$H_i|E_\alpha >= \alpha_i E_\alpha \quad i = 1, .., l. \tag{3.7}$$

Here E_α's are the non-Hermitian raising and lowering operators with $E_\alpha^\dagger = E_{-\alpha}$ and α_i's are the roots of the algebra. There are l numbers α_i which represents a point in l dimensional Eucledian space called root space. Among all the roots we are interested in the simple ones which are the positive roots that cannot be constructed by other roots. For $SU(5)$ we have the simple roots

$$\begin{aligned}
\alpha_1 &= (1, 0, 0, 0) \\
\alpha_2 &= (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0) \\
\alpha_3 &= (0, -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}, 0) \\
\alpha_4 &= (0, 0, -\frac{3}{2\sqrt{6}}, \frac{5}{2\sqrt{10}}).
\end{aligned} \tag{3.8}$$

The Killing metric for $SU(5)$ is

$$\begin{aligned}
g_{ij} &= tr H_i H_j \\
&= \frac{1}{2}\delta_{ij}.
\end{aligned} \tag{3.9}$$

The Cartan matrix is constructed by the simple roots as,

$$A^{ij} = 2\frac{\alpha_i \alpha_j}{|\alpha|^2}. \tag{3.10}$$

Then the Cartan matrix for $SU(5)$ obtained from the simple roots is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

and the inverse of the Cartan matrix for $SU(5)$ is

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Furthermore for the irreducible representations of $SU(5)$ we need the highest weight. For every irreducible representation of a semi-simple Lie algebra the highest weight can be writen as

$$\Lambda = \sum_{j=1}^{l} a_j \Lambda_j \tag{3.11}$$

where l is the rank of the algebra. For every set of these nonegative integers (a_1, a_2, a_3, a_4) there exist an irreducible representation of the semi-simple Lie algebra with the highest weight Λ. In our case we have 4 fundemental weights $\Lambda_1, \Lambda_2, \Lambda_3$ and Λ_4. Fundemental weights can be constructed from the inverse of The Cartan Matrix with the relation

$$\Lambda_j = \sum_{l=1}^{4} (A^{-1})_{lj} \alpha_l \tag{3.12}$$

Then the fundemental weights of $SU(5)$ can be writen as

$$\begin{aligned} \Lambda_1 &= 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 4\alpha_4 \\ \Lambda_2 &= 3\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \Lambda_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 \\ \Lambda_4 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 \end{aligned} \tag{3.13}$$

resulting

$$\begin{aligned} \Lambda_1 &= \left(\frac{5}{2}, \frac{5}{2\sqrt{3}}, \frac{5}{2\sqrt{6}}, \frac{5}{2\sqrt{10}}\right) \\ \Lambda_2 &= \left(0, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{6}}, \frac{5}{\sqrt{10}}\right) \\ \Lambda_3 &= \left(0, 0, \frac{15}{2\sqrt{6}}, \frac{15}{2\sqrt{10}}\right) \\ \Lambda_4 &= \left(0, 0, 0, \sqrt{10}\right). \end{aligned}$$

Before giving the irreducible representations, let us first constract the Dynkin diagram for $SU(5)$. Cartan matrix represent the properties of the algebra and it is constracted by multiplying the simple roots among themselves as given in 3.10

$$A^{ij} = 2\frac{\alpha^i \alpha^j}{|\alpha^i|^2} \tag{3.14}$$

For $SU(N)$ algebras, lengths of the simple roots are normalized to 2 with a symmetric Cartan matrix and rows of the Cartan matrix gives the Dynkin indicies as we stated before. The whole algebra can be summed up with a Dynkin diagram in an elegant way. To draw the Dynin diagrams we need to look at the relations between the simple roots. We can obtain the angles between simple roots from the Cartan matrix by multiplying it with itself and using the Schwarz inequality defined as

$$\cos\theta_{12} = \frac{\alpha_1 \alpha_2}{|\alpha_1|^2 |\alpha_2|^2} = \frac{1}{4}\sqrt{m_1 m_2}. \tag{3.15}$$

with the limitted possibility $m_1 m_2 < 4$ where m_1 and m_2 are integers. The relations between angles and roots are summurized as follows:

- If $m_1 = m_2 = 0$, $\theta = \frac{\pi}{2}$ roots are orthogonal and there is no restriction for lengths of the roots with $A_{ij} = 0$ roots are not connected

- If $m_1 = m_2 = 1$, $\theta = \frac{\pi}{3}$ and $|\alpha_1| = |\alpha_2|$ with $A_{ij} = -1$ where roots have the same length

- If $m_1 = 2$, $m_2 = 1$, $\theta = \frac{\pi}{4}$ and $|\alpha_2| = \sqrt{2}\,|\alpha_1|$ with $A_{ij} = -2$ where roots have value 2 and 1

- If $m_1 = 3$, $m_2 = 1$, $\theta = \frac{\pi}{6}$ and $|\alpha_2| = \sqrt{3}\,|\alpha_1|$ with $A_{ij} = -3$ where roots have value 3.

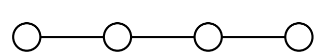
Now we can construct the Dynkin diagrams, first we need to draw a circle for every simple roots and connect these circles by the number of lines given by A_{ij}. Two circles are joined with

- one line if $\theta = \frac{\pi}{3}$

- two lines if $\theta = \frac{\pi}{4}$

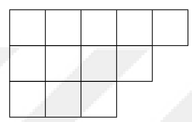
- three lines if $\theta = \frac{\pi}{6}$.

Than, when lengths are inequal draw an arrow pointing the root with smaller length, or with a black dot. The Dynkin diagram for $SU(5)$ is then given as,

$SU(5)$: 

3.1.1 Irreducible Representations by Young Tableaux Method

To give the irreducible representations of $SU(5)$ we can employ a method called Young Tableaux which is used to decompose $SU(N)$ algebras into its irreps. This is a diagram to reduce the higher rank tensors into direct product of irreducible tensors which is the basis for irreducible representations. A Young diagram is a union of boxes arranged as left-justified boxes with each row being at least as long as the row beneath drawn as,



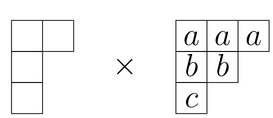
Each irreducible representation of $SU(N)$ is specified by its rank as $(l = n - 1)$ of integers $(a_1, a_2, ..., a_l)$. We can relate the number of boxes with the set of integers $(a_1, a_2, ..., a_l)$ as follows

$$\begin{aligned} p_1 &= \sigma_1 - \sigma_2 \\ p_2 &= \sigma_2 - \sigma_3 \\ &\vdots \\ p_k &= \sigma_l - \sigma_n \end{aligned} \tag{3.16}$$

where $\sigma_1, ..., \sigma_n$ corresponds to number of boxes in the rows labeled by the subscripts. While drawing the tableaux we erase the boxes with complete n box columns for $SU(N)$. Dimension of the representations can be seen from the tableaux by using the following formula

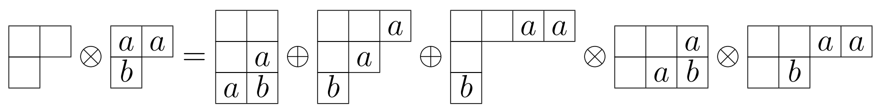
$$\begin{aligned} N = \frac{1}{1!2!\cdots(n-1)!}&(a_1+1)(a_2+a_2+2)\cdots(a_1+a_2+\cdots+a_{n-1}+n-1) \\ &(a_2+1)(a_2+a_3+2)\cdots(a_2+\cdots a_{n-1}+n-2) \\ &\vdots \\ &(a_{n-2}+1)+(a_{n-2}+a_{n-1}+2) \\ &(a_{n-1}+1). \end{aligned} \tag{3.17}$$

The recipe to get higher dimensional representations into the irreducible ones by the Young tableaux method is such that; first, draw the two tableaux and label the rows of the second one by the indicies a,b,c,...,etc.



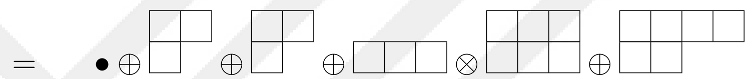
(3.18)

Than, attach the boxes from the second tableaux to the first in all possible ways and one box at a each time. We have two restrictions for the resulting diagram, one of them is no two a's or b's should be in the same diagram and the other restriction is that at any position of given box there should be no more b's than a's to the right and above of it, and it goes the same for b's, c's, etc. Now let's decompose the direct product $\mathbf{8} \otimes \mathbf{8}$ into its irreps. Here $\mathbf{8}$ is the 8 dimensional irreducible representation of $SU(3)$ drawn with the Young tableaux as,



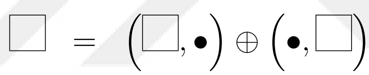
(3.19)

Removing 3 boxes columns since we are in the $SU(3)$ representation



$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27} \tag{3.20}$$

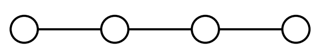
For $SU(N)$ subgroups of type $su(l) \otimes su(m)$ where $l + m = n$ the decomposition in terms of Young tableaux is given such that



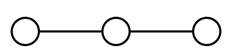
$$n = (l, 1) \oplus (1, m).$$

In the decomposition process we should perform all the allowed splittings in sub-Young tableaux where each tableaux correspond to irreducible tensors. For $SU(5)$ we will get the decomposition as $SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$. Before giving the detailed decomposition for $SU(5)$, let us look at what happens to the Dynkin diagram of $SU(5)$ in the decomposition process. To decompese higher rank representations into smaller ones we start with the Dynkin diagram of the group G of rank l and remove a dot yielding a smaller algebra H of rank $l - 1$. Regular non-semisimple embedding with $U(1)$ is such that $G \supset H \times U(1)$

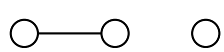
In $SU(5)$ to get the maximal embedding we remove a dot from the Dynkin diagram. Let's show the 4 simple roots circles,

$SU(5)$: 

By removing the last root(circle) from the Dynkin diagram, we get the first embedding which is $SU(4) \times U(1)$;

$SU(4) \times U(1)$: 

Removing the third root (circle) from The Dynkin diagram, we get $SU(3) \times SU(2)$;

$SU(3) \times SU(2)$: 

where first part corresponds to $SU(3)$ and second part to $SU(2)$.

3.1.2 Particle Assignments in $SU(5)$

As we mentioned in the beginning we are interested in the $SU(5)$ Grand Unified Model which combines the Standard model gauge groups $SU(3)\times SU(2)\times U(1)$. Here $SU(3)$ represents the strong interactions of quarks with 8 colored gluons and $SU(2)\times U(1)$ represents the electroweak interactions of quarks and leptons. The Standard model theory treats the creation and annihilation operators as tensor operators under $SU(2)\times U(1)$ Lie algebras. In the defined basis for $SU(5)$ it can be seen easily that $SU(2)$ generators are acting only on rows 4 and 5 and first three rows are unaffected by SU(2) generators thus we can write them as singlets under $SU(2)$ transformations [8]. Same goes for $SU(3)$ generators as acting on first three rows and not effecting the last two rows. Thus the fundamental representation of $SU(5)$ decomposes as $\mathbf{5} = (\mathbf{3{,}1}) \oplus (\mathbf{1{,}2})$ under $SU(3)\times SU(2)$ [24]. Quarks can be represented by $\mathbf{3}$ the fundamental representation in $SU(3)$ and when they are singlets under $SU(2)$ we can furthermore say that $(\mathbf{3},\mathbf{1})$ is the representation for right handed quarks. We are trying to represent fermions in the same $\mathbf{5}$ multiplet, $(\mathbf{1},\mathbf{2})$ can be identified with left handed lepton doublet [8, 28]. We know that we can represent left handed states in terms of their right handed conjugated states as

$$\bar{\psi}_L \equiv \sigma_2\psi_R^*, \quad \text{and} \quad \psi_R \equiv \sigma_2\psi_L^* \tag{3.21}$$

where

$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi, \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma_5)\psi. \tag{3.22}$$

Define $\mathbf{5}$ as the right handed fermion multiplet as,

$$\psi_{\mathbf{5}} = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \\ e^c \\ -\nu^c \end{pmatrix}$$

Furthermore, we have to embed $SU(2)\times U(1)$ into $SU(5)$. In bosonic sector we have photons in $SU(5)$ which are combinations of gauge bosons of $SU(2)\times U(1)$, thus we must identify them with traceless charge operator Q of $SU(5)$. For the ψ multiplet we have traceless condition saying,

$$Tr(Q) = 3Q_q + Q_{e^c} + Q_{\nu^c} = 0. \tag{3.23}$$

Three quarks in this multiplet forms a color triplet. Since the charge operator commutes with $SU(3)$, quarks in this multiplet have the same charges, resulting $Q_q = \frac{1}{3}e$ where e is the electric charge of the electron. Here we see this assignment of $SU(5)$ naturally giving the electric charges of the quarks as in the Standard Model. Finally we can define the charge 1/3 quark as the right handed down quark d^i_R and represent the multiplet as,

$$\psi_{\mathbf{5}} = \begin{pmatrix} d^1_R \\ d^2_R \\ d^3_R \\ e^c \\ \nu^c \end{pmatrix} = \mathbf{(3,1)} \oplus \mathbf{(1,2)}, \quad \text{where} \quad i = 1,2,3 \quad \text{standing for family.}$$

We can uniquely identify the charge operator Q with the choosen basis for $SU(5)$ as

$$Q = Diag(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1, 0) = \frac{1}{2}(\lambda^{23} + \sqrt{\frac{5}{3}}\lambda^{24}). \tag{3.24}$$

Once this fundemental assignment is done, we can name the others. For example $\bar{\mathbf{5}}$ is defined with the left handed fermion multiplet as

$$\bar{\psi}_{\mathbf{5}} = \begin{pmatrix} d^{1c}_R \\ d^{2c}_R \\ d^{3c}_R \\ e^- \\ \upsilon^e \end{pmatrix} = (\bar{\mathbf{3}}, \mathbf{1}) \oplus \mathbf{(1,2)}, \quad \text{where} \quad \mathbf{2} = \bar{\mathbf{2}}.$$

Now we can also assign hypercharge to the Standard Model particles we have by using the famous Gell-Mann-Nishijima formula,

$$\frac{Y}{2} = Q - I_3 \tag{3.25}$$

where Q is the electric charge and I_3 is the isospin. Now we can write the full representation in $SU(3) \times SU(2) \times U(1)$ with the charges as

$$\Psi_5 : \quad \mathbf{5} \longrightarrow (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{2})_{\frac{1}{2}} : \mathbf{u^c}, \bar{\mathbf{l}}(\mathbf{e^c}, \nu^{\mathbf{c}}_{\mathbf{e}}) \tag{3.26}$$

$$\bar{\Psi}_5 : \quad \overline{\mathbf{5}} \longrightarrow (\overline{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} : \mathbf{d^c}, \mathbf{l}(\mathbf{e}, \nu_{\mathbf{e}}) \tag{3.27}$$

where this time the third number hypercharge appears as a subscript. The next representation of $SU(5)$ is the remaining 10 dimensional part which is obtained by taking the product of the fundamental representation $\mathbf{5}$ such that

$$\mathbf{5} \otimes \mathbf{5} = \square \otimes \square = \square\square \oplus \begin{array}{c}\square \\ \square\end{array} = \mathbf{15} \oplus \mathbf{10} \tag{3.28}$$

where the $\mathbf{15}$ is the symmetric and $\mathbf{10}$ is the antisymmetric part. We can continue to assign fermions to get all the Standard Model content in this manner. The representation $\mathbf{10}$ can be represented by antisymmetric 5×5 matrix and organizing

$SU(3) \times SU(2)$ decomposition in $SU(5)$ multiplet, we get

$$\mathbf{10} = \left(\begin{array}{c|c} (\bar{\mathbf{3}}, \mathbf{1}) & (\mathbf{3{,}2}) \\ \cdots & \cdots \\ (\mathbf{3{,}2}) & (\mathbf{1{,}1}) \end{array}\right).$$

Arranging color states u_i^c in an antisymmetric matrix, fermion content is written as

$$\mathbf{10} = \psi^{ij} = \psi_{\mathbf{10}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_2 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0. \end{pmatrix}$$

Right handed fermionic part of the Standard Model particle content is given in $\mathbf{5} \oplus \bar{\mathbf{10}}$ representation and left handed in conjugated representation in $\bar{\mathbf{5}} \oplus \mathbf{10}$. Now let us continue with the bosonic part which is assigned to the adjoint representation of $SU(5)$ where the adjoint representation decomposes as

$$\begin{aligned} \mathbf{5} \times \bar{\mathbf{5}} &= A_j^i \oplus 1 \\ &= [(\mathbf{3{,}1})_{-\frac{2}{3}} \oplus (\mathbf{1{,}2})_1] \otimes [(\bar{\mathbf{3}}, \mathbf{1})_{\frac{2}{2}} \oplus (\mathbf{1{,}2})_{-1}]. \end{aligned} \tag{3.29}$$

It can also be represented by Young Tableaux method as

$$\begin{aligned} &= [(\ydiagram{1}, \bullet)_{-\frac{2}{3}} \oplus (\bullet, \ydiagram{1}_1)] \otimes [(\ydiagram{1,1}, \bullet)_{\frac{2}{3}} \oplus (\bullet, \ydiagram{1})_{-1}] \\ &= \left(\ydiagram{2,1}, \bullet\right)_0 \oplus (1,1)_0 \oplus \left(\ydiagram{1}, \ydiagram{1}\right)_{-\frac{5}{3}} \oplus \left(\overline{\ydiagram{1}}, \ydiagram{1}\right)_{\frac{5}{3}} \oplus \left(\bullet, \ydiagram{2}\right)_0 \oplus (1,1)_0 . \end{aligned} \tag{3.30}$$

or

$$\bar{\mathbf{5}} \otimes \mathbf{5} = \ydiagram{1,1,1,1} \otimes \ydiagram{1} = \ydiagram{2,1,1,1} \oplus \ydiagram{1,1,1,1,1} = \mathbf{24} \oplus \mathbf{1} \tag{3.31}$$

where 24 corresponds to adjoint representation which will correspond to gauge bosons. Then the adjoint representation of $SU(5)$ decomposes as

$$\mathbf{24} \rightarrow (\mathbf{8{,}1})_0 + (\mathbf{3{,}1})_0 + (\mathbf{1{,}1})_0 + (\mathbf{3{,}2})_{-\frac{5}{3}} + (\bar{\mathbf{3}}, \mathbf{2})_{\frac{5}{3}}. \tag{3.32}$$

Here $\mathbf{24}$ repsesents gauge bosons appearing in $SU(5)$. Here $(\mathbf{8{,}1})_0$ represents octet of gluons G_μ^a where $a = 1, \cdots, 8$. $(\mathbf{3{,}1})_0$ gives isovector of intermediate bosons W_μ^i where $i = 1, 2, 3$. $(\mathbf{1{,}1})_0$ is isoscalar boson B_μ. The remaining bosons are the new ones predicted by the $SU(5)$ GUT Model. They form a colored isospin doublet denoted by X and Y as

$$(\bar{\mathbf{3}}, \mathbf{2})_{\frac{5}{3}} = \begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix}.$$

Hypercharges of the bosons given in sub-paranthesis here are obtained from tensor products where hypercharges are just added algebraically. Using Gell-Mann-Nishijima formula $Q = I_3 + \frac{Y}{2}$ we can get the electric charges of these new bosons as

$$\begin{aligned} Q(X) &= \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{3} = \frac{4}{3} \\ Q(Y) &= -\frac{1}{2} + \frac{1}{2} \cdot \frac{5}{3} = \frac{1}{3} \end{aligned} \tag{3.33}$$

The adjoint representation **24** can be represented by traceless 5×5 matrix having 24 independent components as

$$\mathbf{24} = \left(\begin{array}{c|c} (\mathbf{8}, \mathbf{1})_0 & (\mathbf{3},\mathbf{2})_{\frac{5}{3}} \\ \hline (\bar{\mathbf{3}}, \mathbf{2})_{-\frac{5}{3}} & (\mathbf{1},\mathbf{3})_0 \end{array}\right) + (\mathbf{1},\mathbf{1})_0.$$

More generally,

$$\hat{A} \equiv \frac{1}{2} \sum_{a=1}^{24} A^a \lambda_a = \frac{1}{2} [\sum_{a=1}^{8} G^a \lambda_a + \sum_{a=9}^{20} A^a \lambda_a + \sum_{21}^{23} W^a \lambda_a + B \lambda_{24}]$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} & \frac{1}{\sqrt{2}} \sum_{a=1}^{8} G^a \lambda_a & & X_1^c & Y_1^c \\ & & & X_2^c & Y_2^c \\ & & & X_3^c & Y_3^c \\ X_1 & X_2 & X_3 & \frac{W^3}{\sqrt{2}} & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & -\frac{W^3}{\sqrt{2}} \end{pmatrix} + \frac{B}{2\sqrt{15}} \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}.$$

There are 12 new bosons. Charges of these new vector bosons point out that these vector bosons are responsible for transitions between leptons and anti-down quarks when acting on $\bar{\mathbf{5}}$ and quark anti-quark transitions when acting on **10**. They are called as leptoquark gauge bosons. These new gauge bosons cause proton decay which is limited experimentally by $\geq 10^{13}$ years. This fact requires these gauge bosons to be very massive. These process of vector bosons to gain mass is analogical to spontaneos symmetry breaking mechanism in the Standard Model. When $SU(5)$ breaks into $SU(3) \times SU(2) \times U(1)$ where the Higgs boson is required and it transforms as $(\mathbf{1}, \mathbf{2})_{\frac{1}{2}}$ and $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$ under $SU(3) \times SU(2) \times U(1)$. This symmetry breaking via Higgs Mechanism requires the Higgs scale ina way that below the scale the gauge bosons are massless. Actually symmetry breaking in $SU(5)$ appears in two stages

$$SU(5) \longrightarrow SU(3) \times SU(2) \times U(1) \longrightarrow SU(3) \times U(1). \tag{3.34}$$

First stage of breaking occurs by introducing the scalar fields in adjoint representation where $SU(5)$ breaks into $SU(3) \times SU(2) \times U(1)$ and the second stage is done by introducing the scalar field in the fundamental representation and choosing the vacuum expectation value in any direction. One of the biggest problems of this two stage symmetry breaking mechanism is the hierarchy problem which is the result of breaking the symmetry at two very different energy scales. These two scales are about 12 order of magnitude apart from each other. This is an important problem because it requires precise fine tunning parameters and with these huge differences we cannot get a reasonable Higgs mass.

3.1.3 Lagrangian of SU(5)

To give a complete $SU(5)$ discussion next thing we should do is to write down the Lagrangian. Let us first define the covariant derivatives. Initially we start by the fermionic part, in fermionic sector we have the representations $(\bar{\mathbf{5}} \oplus \mathbf{10})$ where $\bar{\mathbf{5}} = \psi_i$ and $\mathbf{10} = \psi^{ij}$ [29]. The covariant derivative of ψ_i goes as

$$D_\mu \psi_i = \partial_\mu \psi_i - i\frac{g^5}{2}\sum_{a=1}^{24} A^a_\mu \lambda_a \psi_i = \partial_\mu \psi_i - ig^5 \hat{A}_\mu \psi_i \tag{3.35}$$

where $\hat{A}_\mu$ are the generators for the fundamental representation $\mathbf{5}$. For the antisymmetric part $\mathbf{10} = \psi^{ij}$ where $i, j = 1, .., 5$ and $\psi^{ij} = \frac{1}{\sqrt{2}}(\psi^i\psi^j - \psi^j\psi i)$. We need to define the generators. Consider the infinitesimal transformation

$$\psi^i \to U^i_j \psi^j \quad \text{where} \quad U^i_j = \delta^i_j - i\frac{g^5}{2}\eta^a(\lambda_a)^i_j, \tag{3.36}$$

then,

$$\begin{aligned} \psi^{ij} &\to U^i_{i'}\psi^{i'}U^j_{j'}\psi^{j'} - U^{j'}_{i'}\psi^{i'}U^i_{j'}\psi^{j'} \\ &= U^i_{i'}U^j_{j'}(\psi^{i'}\psi^{j'} - \psi^{j'}\psi^{i'}) \\ &= U^i_{i'}U^{j'}_{j}\psi^{i'j'}. \end{aligned} \tag{3.37}$$

On the other hand we know that $\psi^{ij} \to U^{ij}_{i'j'}\psi^{i'j'}$ where

$$U^{ij}_{i'j'} = \delta^i_{i'}\delta^j_{j'} - i\frac{g^5}{2}\eta^a(T_a)^{ij}_{i'j'} \tag{3.38}$$

Here T_a's are defined to be as the generators of $\mathbf{10}$. Thus,

$$(T_a)^{ij}_{i'j'} = \delta^i_{i'}(\lambda_a)^j_{j'} + \delta^j_{j'}(\lambda_a)^i_{i'}. \tag{3.39}$$

Then the covariant derivative for $\mathbf{10}$ is

$$D_\mu \psi^{ij} = \partial_\mu \psi^{ij} - i\frac{g^5}{2}A^a_\mu T_a \psi^{ij}. \tag{3.40}$$

With the covariant derivatives and fields defined we can write the kinetic part of the Lagrangian as,

$$\begin{aligned}\mathcal{L}_{kin} &= \mathcal{L}_{gauge} + \mathcal{L}_{fermion} \\ &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - i\frac{g}{\sqrt{2}}(A^a_\mu)(A^a_\nu) - i\frac{g}{\sqrt{2}}(A^a_\nu)(A^a_\mu) + \bar{\psi}^i \not{D}\psi^i + Tr(\bar{\psi}^{ij}\not{D}\psi^{ij}).\end{aligned}$$

where $\not{D} = \gamma^\mu D_\mu$,

$$\bar{\psi}^i \not{D}\psi^i = (\bar{\psi}^i \not{\partial}\psi^i - ig^5\bar{\psi}^i \hat{A}\psi^i) \tag{3.41}$$

and

$$\begin{aligned}Tr(\bar{\psi}^{ij}\not{D}\psi^{ij}) &= \bar{\psi}^{ij}(\partial_\mu\psi^{ij} - i\frac{g^5}{2}(T_a)^{ij}_{i'j'}\psi^{i'j'}) \\ &= -Tr(\bar{\psi}^{ij}\partial_\mu\psi^{ij} - 2ig^5\bar{\psi}^{ij}\hat{A}_\mu\psi^{ij}).\end{aligned} \tag{3.42}$$

Coupling of neutral fields W^3_μ and B_μ to matter are determined by covariant derivatives of the form,

$$D_\mu = \partial_\mu - i\frac{g}{2}(W^3_\mu \hat{A}^{11} + B_\mu \hat{A}^{12}) \tag{3.43}$$

where $\hat{A}^{11}$ and $\hat{A}^{12}$ are representations of the λ matrices defined earlier. Than using

$$\begin{aligned}A_\mu &= B_\mu cos\theta_w + W^3_\mu sin\theta_w \\ Z_\mu &= -B_\mu sin\theta_w + W^3_\mu cos\theta_w\end{aligned} \tag{3.44}$$

we can rewrite equation 3.43 in terms of A_μ and Z_μ as

$$\begin{aligned}D_\mu &= \partial_\mu - i\frac{g}{2}[(sin\theta_w\hat{A}^{11} + cos\theta_w\hat{A}^{12})A_\mu] + [(cos\theta_w\hat{A}^{11} - sin\theta_w\hat{A}^{12})Z_\mu] \\ &\equiv \partial_\mu - i[aQA_\mu + g_2Q_zZ_\mu]\end{aligned} \tag{3.45}$$

where charge operator is defined by

$$Q = Diag(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1, 0) = \frac{1}{2}(\hat{A}^{11} + \frac{5}{3}\hat{A}^{12}). \tag{3.46}$$

Using the charge operator in equation 3.45 we get

$$tan\theta_W = \sqrt{\frac{3}{5}}, \quad sin\theta_W = \sqrt{\frac{3}{8}}, \quad \frac{g_5}{2} = \sqrt{\frac{2}{3}}e. \tag{3.47}$$

This prediction is inconsistent with the experiment at first sight but when we go through low energies as Georgi, Quinn and Weinberg did [30] this prediction becomes applicable at the M_X scale and consistent with the experimental $\sin^2\theta$ value. One of the important consequence of $SU(5)$ Model is that quarks and leptons are represented within the same multiplet. This implies that they can transform into each other. We

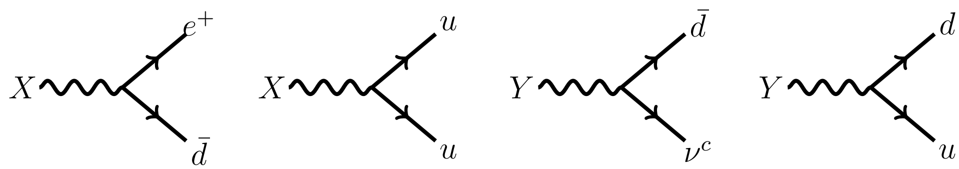


Figure 3.1: X and Y boson couplings to fermions

can give the X and Y boson interactions as coupling to fermions in the Lagrangian [31, 32]. These interactions can be shown by the Feynman diagrams in Figure 3.1.

For the interactions of the new leptoquark gauge bosons X and Y we use A_μ matrix in the kinetic part of the Lagrangian and get

$$\begin{aligned} &\frac{g_5}{\sqrt{2}} \quad \bar{X}^i_\mu(\bar{d}_{iR}\gamma^\mu e^+_R + \epsilon_{ijk}\bar{u}^{ck}_L\gamma^\mu u^j_L + \bar{d}_{iL}\gamma^\mu e^+_L) + h.c. \\ + \quad &\frac{g_5}{\sqrt{2}} \quad \bar{Y}^i_\mu(-\bar{d}_{iR}\gamma^\mu\nu^c_R + \epsilon_{ijk}u^{cj}_L\gamma^\mu d^k_L - u_{iL}\gamma^\mu e^+_L) + h.c. \end{aligned} \tag{3.48}$$

In the Standard Model, baryon and lepton number conservation appears as an accidental symmetry. Baryon (or lepton) number is not conserved in $SU(5)$ since they are represented in the same multiplet [33]. But we see that $(B-L)$ number is conserved in X and Y interactions. We can also see tree level baryon number violation X and Y boson changing interactions in Figure 3.2 as

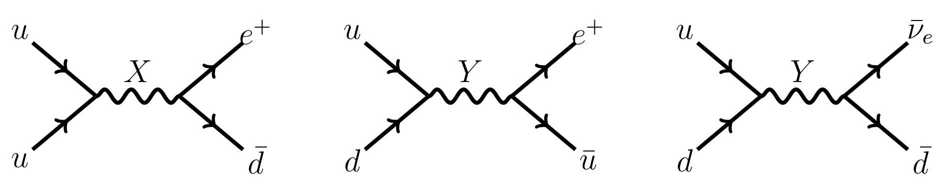


Figure 3.2: X and Y boson changing interactions

As a consequence of a violation of baryon number conservation, proton decay becomes possible. The limit for the proton lifetime is around $\gtrsim 10^{30}yr$ require that baryon number violating interactions to be extremely weak. When we consider proton decay as a result of X and Y boson changing interactions, this will lead to $M_X > 10^{14}GeV$ which is 12 order of magnitude greater than $W^\pm$ and Z masses. This will bring us to an important point that $SU(5)$ theory proposes two stage symmetry breaking mechanism to overcome this problem [34–38].

3.1.4 Spontaneous Symmetry Breaking of SU(5)

When the symmetry breaking occurs in two stages the inconsistencies with the experiments are eliminated in many ways [39].

$$SU(5) \underset{M_X}{\longrightarrow} SU(3)\times SU(2)\times U(1) \underset{M_W}{\longrightarrow} SU(3)\times U(1)_{em} \tag{3.49}$$

In the first stage of breaking X and Y bosons acquire mass of oreder $M_x \sim 10^{15} GeV/c^2$ and leaving other 12 bosons massless. This first step is broken by adjoint of real scalars Σ_a where $a = 1, \cdots, 24$. Coupling of gauge field to Σ is given by kinetic Lagrangian with the field

$$\mathcal{L}_{kin} = \frac{1}{2}\sum_{a=1}^{24}(D_\mu \Sigma_a)^\dagger (D^\mu \Sigma_a) \tag{3.50}$$

where Σ's are representations of the λ matrices defined earlier as $\hat{\Sigma} = \sum_{a=1}^{24}\Sigma^a \hat{\lambda}_a$. To write down the covariant derivates for this adjoint representation, we need to first define the generators. We can follow the same procedure as we did in **10**, recalling the adjoint representation $\mathbf{5}\times\bar{\mathbf{5}} = \mathbf{24}\oplus\mathbf{1}$. A state ψ_i^j in adjoint representation transforms as

$$\psi_i^j \to (U^*)_i^{i'} U_{j'}^j \psi_{i'}^{j'} \tag{3.51}$$

where $U_j^i = \delta_j^i - i\frac{g_5}{2}\xi^a(\hat{\lambda})_j^i$. Then generators of adjoint representation is defined as

$$(T_a)_{ij'}^{i'j} = (\hat{\lambda}_a)_i^{i'}\delta_j^{j'} - \delta_{i'}^i(\hat{\lambda}_a)_{j'}^j. \tag{3.52}$$

Thus covariant derivative in the adjoint representation of $SU(5)$ can be read as

$$\begin{aligned}
D_\mu \Sigma_i^j &= \partial_\mu \Sigma_i^j - i\frac{g_5}{2}A_\mu^a (T_a)_{ij'}^{i'j}\Sigma_{i'}^{j'} \\
&= \partial_\mu \Sigma_i^j - i\frac{g_5}{2}A_\mu^a((\hat{\lambda})_i^{i'}\Sigma_{i'}^j - \Sigma_i^{j'}(\hat{\lambda}_a)_{j'}^j) \\
&= \partial_\mu \Sigma_i^j - i\frac{g_5}{2}A_\mu^a[\hat{\lambda}_a, \Sigma]_i^j \\
D_\mu \hat{\Sigma} &= \partial_\mu \hat{\Sigma} - ig_5[\hat{A}, \hat{\Sigma}].
\end{aligned} \tag{3.53}$$

We expect to obtain the mass matrix when $\hat{\Sigma}$ develops vacuum expectation value (vev) $< \hat{\Sigma}_0 >$ through the kinetic Lagrangian term with the covariant derivative defined above in the adjoint representation. The mass matrix is expected to be in the form

$$\frac{1}{2}g_5^2 Tr[A_\mu, < \Sigma_0 >]^2 \equiv m^2 A_\mu^a A^{\mu,b}. \tag{3.54}$$

With $< \Sigma_0 >$ the gauge bosons X and Y get mass while others remain massless. Since gauge bosons of $SU(3)$ and $SU(2)$ remain massless we can use the block form of matrices we defined for $SU(5)$ as Σ_0 commuting with the corresponding generators

$\hat{\lambda}_1, \cdots, \hat{\lambda}_8$ and $\hat{\lambda}_{20}, \cdots, \hat{\lambda}_{24}$. In the adjoint representation Σ_0 with vanishing trace is restricted to be proportioanl to $\hat{Y}$ as

$$\hat{\Sigma}_0 = \begin{pmatrix} -v & & & & \\ & -v & & & \\ & & -v & & \\ & & & 3v/2 & \\ & & & & 3v/2 \end{pmatrix}$$

Explicit computation shows

$$\begin{aligned} \frac{1}{2}Tr[(D_\mu\hat{\Sigma}_0)^\dagger(D^\mu\hat{\Sigma}_0)] &= \frac{1}{2}Tr(\partial_\mu\hat{\Sigma}_0 + ig_5[\hat{A},\hat{\Sigma}])(\partial^\mu\hat{\Sigma} - ig_5[\hat{A},\hat{\Sigma}]) \\ &= \frac{g_5^2}{2}\sum_{a=1}^{24}|A_\mu^a|^2\sum_{i,k}|[\hat{\lambda},\hat{\Sigma}_0]_{ik}|^2. \end{aligned} \quad (3.55)$$

We obtain the mass terms as $m_X^2 = m_Y^2 = \frac{25}{8}g_5^2v^2$ while other bosons are remained massless. We need to construct potential giving the expected vev which includes general form of the scalar couplings of the adjoint representation. Since we seek for gauge invariance we can form the potential with traces with powers of Σ and we can eliminate the cubic term by imposing the discrete symmetry $\Sigma \to -\Sigma$. Hence imposing the potential

$$V(\Sigma) = -\frac{1}{2}\mu^2Tr\Sigma^2 + \frac{a}{4}(Tr\Sigma^2)^2 + \frac{b}{2}Tr\Sigma^4 \quad (3.56)$$

results

$$V(\hat{\Sigma}_0) = -\frac{1}{2}\frac{15}{4}\mu^2v^2 + \frac{15}{16}v^4(15a+7b) \quad (3.57)$$

which has a minimum when $b > 0, \mu^2 > 0$ as $\mu^2 = \frac{15}{2}av^2 + \frac{7}{2}bv^2$.

For the second stage of symmetry breaking, we know from the Standard Model that we need to introduce a Higgs filed that is transforming as a doublet under $SU(2)$. In the content of $SU(5)$ the simplest possibility is introducing Higgs in **5**, recalling the simplest assignment for the fundemental representation we have

$$H = \begin{pmatrix} h^1 \\ h^2 \\ h^3 \\ h^+ \\ -h^0 \end{pmatrix} = \mathbf{(3,1)} \oplus \mathbf{(1,2)}.$$

Again from the Standard Model we can introduce a similar potential as

$$V(H) = -\frac{1}{2}v^2|H|^2 + \frac{1}{4}\lambda(|H|^2)^2, \quad \text{where} \quad v^2, \lambda > 0. \quad (3.58)$$

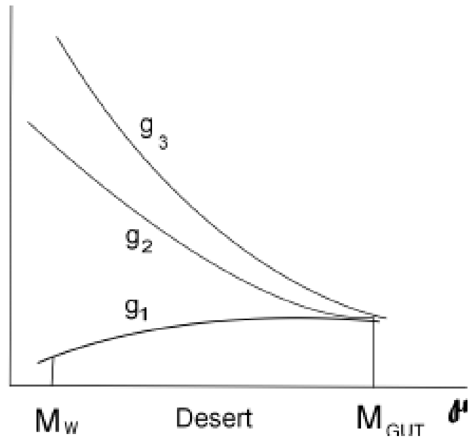


Figure 3.3: Energy dependence of running coupling constatnts in GUT

The vev appears as

$$< (H^5) >=< -h^0 >= v_0, \quad \text{where} \quad v^2 = \lambda v_0^2. \tag{3.59}$$

The thing is that vev here can be along any direction. This will give the same symmetry breaking patterns of $SU(2) \times U(1)$ with

$$M_W^2 = M_Z^2 cos^2\theta_W = \frac{1}{4} g_2^2 v_0^2. \tag{3.60}$$

Problems with two stage symmetry breaking brings out some discussions [40]. First of all, the direction of the vacuum expectation value $v^2 = \lambda v_0^2$ can be in any direction but the desired breaking for $SU(5)$ is to break it into $SU(3) \times SU(2) \times U(1)$. If the vev is taken in the direction 4 or 5 it breaks $SU(2)$ as expected but otherwise if taken in the 1,2 or 3 direction will break $SU(3)$ which is clearly not desired since we want $SU(3)$ to be left unbroken. Also the interaction for the two representation imposed for symmetry breaking Σ and H are not considered here. These cross couplings will bring out the divergent terms. The two stage symmetry breaking can be thought in terms of the running coupling constants since the overall aim is to unify all of them under one unified coupling constant. At the energies greater than M_X, the spontaneous symmetry breaking affects will not matter and we can write the theory under one gauge coupling constant. At energies below M_X the propagators for X and Y bosons can be ignored and we are again safe by the Standard Model with three gauge coupling constants g_1, g_2 and g_3. At energies around M_X these two pictures meet. This can be also illustrated as in Figure 3.3.

3.1.5 Fermion Masses

We have assigned left handed fermions into the representation $\bar{\mathbf{5}} \oplus \mathbf{10}$ and for the mass term for fermions we look through the product $(\bar{\mathbf{5}} \oplus \mathbf{10}) \times (\bar{\mathbf{5}} \oplus \mathbf{10})$ and see the resulting representations as

$$\begin{aligned}
\bar{\mathbf{5}} \otimes \mathbf{10} &= \mathbf{5} \oplus \bar{\mathbf{45}} \\
\mathbf{10} \otimes \mathbf{10} &= \bar{\mathbf{5}} \oplus \mathbf{45} \oplus \mathbf{50} \\
\bar{\mathbf{5}} \otimes \bar{\mathbf{5}} &= \bar{\mathbf{10}} \oplus \bar{\mathbf{45}}.
\end{aligned} \tag{3.61}$$

We look for singlet components for the gauge invariant mass term where in above products we do not have any. Thus we expect masses to be arising via the spontaneous symmetry breaking mechanism through the gauge invariant couplings of these fermions to Higgs scalars. The form of the masses will depend on which scalars are present. In the minimal model only **5** and **24** of Higgs scalars are present. We do not see **24** in above products thus the adjoint Σ does not couple to fermions. As a result of this fermion mass scale comes around $O(M_W)$ coupling with the scalar Higgs, not on the order of $O(M_X)$ [41]. For the fermion mass terms we look for the terms like

$$\psi_5^T C \psi_{10} H + h.c \tag{3.62}$$

as generating down and electron masses and terms like

$$Tr(\psi_{10}^T C \psi_{10} H + h.c.) \tag{3.63}$$

as generating for up masses appearing in the Lagrangian. From the product coming from the fermion representations above, we see that we can keep **5** as the minimal Higgs representation and add also the **45**. Here **50** is useless since we can not get a colour singlet after electroweak symmetry breaking with **50**. If we want to write down gauge invariant mass terms, they will appear as

$$\mathcal{L}_{FM}^{\psi_5\psi_{10}} = (\psi_{\bar{5}}^{\alpha})_i^T C \psi_{10}^{kl\beta}[(y_{5(de)}^{\alpha\beta})(\delta_k^i (H_{\bar{5}})l - \delta_l^i (H_{\bar{5}})_k) \oplus (y_{45(de)}^{\alpha\beta})(H_{45}^i)_{kl}] + h.c. \tag{3.64}$$

$$\mathcal{L}_{FM}^{\psi_{10}\psi_{10}} = (\psi_{10}^{kl\alpha})^T C (\psi_{10}^{pq\beta})[y_{5(u)}^{\alpha\beta} \epsilon_{klpqr} (H_5^r) \oplus (y_{45(u)}^{\alpha\beta}) \epsilon_{klpst} (H_{\bar{45}})_q^{st}] + h.c. \tag{3.65}$$

where α and β are generation indicies and ϵ is the 5 dimensional Levi-Civita tensor. Here unlike the Standard Model Yukawa terms, we have two Yukawa matrix instead of three. Here y_{de} couples ψ_5 to ψ_{10} giving down and electron masses and y_u and couplings ψ_{10} to ψ_{10} giving up mass terms. For the $SU(5)$ Model we have given the down quarks and charged leptons in the same multiplets thus mass differences can be achieved by discriminating the Higgs field vevs. Giving the extension to the Higgs field as

$$< H >= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_5 \end{pmatrix}.$$

The mass terms for down quarks and electrons obtained from equation 3.64 and the mass term for up quarks obtained from equation 3.65. Again we have this restriction; the masses arising from assigning the down quarks and leptons into the same multiplet come as $m_e = m_d$. At this point **45** can be added to the theory developing a vev for Higgs as

$$< H^a_{b5} >= v_{45}(\delta^a_b - 4\delta^a_4\delta^4_b) \tag{3.66}$$

and giving the ratio $m_e = 3m_d$ which is same for other generations also. Any ratio can be achived by adding both **5** and **45** to the theory. The CKM matrix can be still obtained safely by this two fermion fields. We see that in the minimal model where only **5** and **24** dimensional representations are used we have the prediction $m_b/m_\tau > 3$ for three families but it would fail for more families. Thus we can not still expalin the horizantal symmetries but within the **45** included model we have enough freedom to adjust the fermion masses.

3.2 $SO(10)$

We have given a detailed model of $SU(5)$ to understand the grand unification process. The next candidate should be a larger group which is expected to involve a richer content [23]. It is the rank 5 $SO(10)$ group which is to incorparate with $SU(5)$ also. $SO(10)$ has spinorial representation **16** which decomposes as $\mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1}$ under $SU(5)$ thus $SO(10)$ contains $SU(5)$ and predicts more. Here we won't give a detailed $SO(10)$ discussion like we did for $SU(5)$ but let us give some basics using the group theoretical approach rather than the spinorial representations. $SO(10)$ is a rank 5 group with the simple roots,

$$\begin{aligned}
\alpha_1 &= (2-1000) \\
\alpha_2 &= (-12-100) \\
\alpha_3 &= (0-12-1-1) \\
\alpha_4 &= (00-120) \\
\alpha_5 &= (00-102)
\end{aligned} \tag{3.67}$$

and the spinorial representation **16** has the highest weight (00001). One can start from the highest weight and use the simple roots to get the fermion content easily. Once the fermion content is obtained, it is seen that all the Standard Model fermions appear with the correct quantum numbers as is $SU(5)$ Model and this time there is an extra one which transforms as a singlet under $SU(5)$. This $SU(5)$ singlet field has the proper quantum numbers to be called as left handed anti-neutrino filed or putting in another way conjugate of a right handed neutrino. This is one of the achievements of $SO(10)$ model that right handed neutrino appears naturally. So for the moment let us just show the expected particle content for $SO(10)$ Model as

$$\mathbf{16} = (Q, u^c, d^c, L, \nu^c, e^c) \tag{3.68}$$

where e^c stand for the right handed neutrino. One of the important steps of grand unification is to looking through the breaking schemes, for $SO(10)$ we have subgroup structures as

$$SO(10) \longrightarrow SU(5) \times U(1) \tag{3.69}$$

and

$$SO(10) \longrightarrow SO(6) \times SO(4). \tag{3.70}$$

Here $SO(6)$ is isomorphic to $SU(4)$, and $SO(4)$ to $SU(2) \times SU(2)$. This is the second achievement of $SO(10)$ theory that it contains the Pati-Salam theory also. For the gauge bosons adjoint representation of the $SO(10)$ has 45 generators thus we have 45 gauge bosons in the model. Branching for the **45** under the Standard Model group goes as

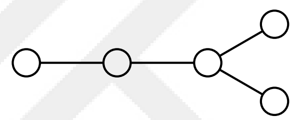
$$\begin{aligned} SO(10) &\longrightarrow SU(3)\times SU(2)\times U(1) \\ \mathbf{45} &\longrightarrow (\mathbf{3},\mathbf{2},\mathbf{2})\oplus(\bar{\mathbf{3}},\mathbf{2},\mathbf{2})\oplus(\mathbf{8},\mathbf{1},\mathbf{1})\oplus(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{1},\mathbf{1}) \end{aligned} \tag{3.71}$$

The Higgs field which gives the masses to new gauge fields and leaving the Standard Model gauge bosons massless is expected to be in

$$\mathbf{16}\otimes\mathbf{16}\longrightarrow\mathbf{10}\oplus\mathbf{120}\oplus\mathbf{126}. \tag{3.72}$$

We can conclude this section by simply giving the Dynkin diagram for $SO(10)$ as

$SO(10)$:



which summarizes the root structure of the group.

3.3 E_6

In Yang Mills types of models we can use group theoretical language to relate particle states to representation vectors. Our motivation here is to introduce the applications of use of Dynkin diagram in model building, summarizing representations, quantum number structure and particle assignments for the exceptional group E_6. In section 3.1.1 we have introduced tensor techniques to embed simple groups into a larger one where we have used the Young Tableaux method. For a higher rank group such as E_6 tensor techniques become impractical, thus we move to Dynkin's methods. In this method we use Dynkin's labels for the representations which will unroll the whole group theoretical structure. For model building procedure we have to embed QED, $SU(2)^w$ and QCD in the unifying group. Embedding can be done through the fundamental representation such that if there is an embedding of the form

$$G \supset G_F \times SU(3)_C \tag{3.73}$$

then the fundamental representation denoted f, has the branching rule

$$f = \sum_i (x_i, y_i^c) \tag{3.74}$$

where x_i is the irreducible representation of G_F and y_i^c is the irreducible representation of $SU(3)_C$. Furthermore, the adjoint irrep is always given by the tensor product $f \times \bar{f}$. There is a theorem for such an embedding which states that if any irrep of G has such a branching rule with y_i^c restricted to $\mathbf{1^c}, \mathbf{3^c}, \mathbf{\bar{3}^c}$ then the fundamental representation must also satisfy the same restriction [24]. Once the color content of the fundamental representation is constructed, color content of the other irreps found from their branching rules. Branching rules describe how the representation of group G is restricted to its subgroups. Now in this chapter we give some specific branching rules and the structure of embedding in E_6. We need subgroup chains and corresponding particle contents in the model. For this, we use projection matrix method where set of projection matrices from E_6 through $U(1)^{em} \times SU(3)_C$ are used. We use this method to see the subgroup chains and how the particle contents and quantum numbers arise from it. E_6 is the most promising exceptional group for unified model building since it is the only exceptional group with non-self conjugate representations which makes flavor-chiral theory possible. E_6 is a rank 6 group which has 78 generators in the adjoint representation. It is a generalization of $SU(5)$ which we have given as the prototype for unification. Generalization is given by subgroup chains of E_6 that end up with $U(1)^{em} \times SU(3)_C$, given as

$$\begin{aligned} E_6 \;&\supset\; SO(10) \times U(1) \quad \mathbf{27} = \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16} \\ &\supset\; SU(2) \times SU(6) \quad \mathbf{27} = (\mathbf{2}, \bar{\mathbf{6}}) \oplus (\mathbf{1}, \mathbf{15}) \\ &\supset\; SU_3 \times SU_3 \times SU_3 \quad \mathbf{27} = (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}, \bar{\mathbf{3}}) \end{aligned} \tag{3.75}$$

where the fundamental representation **27** decompositions are also given. Here only the last one includes QCD as an explicit factor. [42] In the previous chapters we have given the root structure and some group theoretical tools to understand the simple groups SU_3 and SU_5. We have defined the Cartan matrix obtained from the simple roots as

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_i)}. \tag{3.76}$$

As E_6 being a higher rank group we can use LieArt application of Mathematica to see all the algebraic structure easily. To obtain the simple roots we can list all the positive roots in the first place by the command

$$PositiveRoots[E_6] \tag{3.77}$$

Than we can just write the simple roots by following the definition. Finally we can write the Cartan matrix and its inverse by simple roots as

$$A(E_6) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \quad , \quad A^{-1}(E_6) = \begin{pmatrix} \frac{4}{3} & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} & 1 \\ \frac{5}{3} & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} & 2 \\ 2 & 4 & 6 & 4 & 2 & 3 \\ \frac{4}{3} & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} & 2 \\ \frac{2}{3} & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} & 1 \\ 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix}.$$

We know that each weight vector can be written as a linear combination of simple roots α_i as

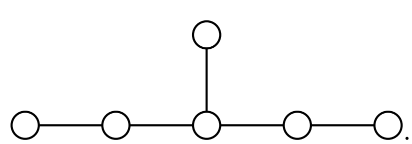
$$\Lambda = \sum_i \bar{\Lambda}_i \frac{2}{(\alpha_i, \alpha_i)} \alpha_i \tag{3.78}$$

where $[\bar{\Lambda}_1, .., \bar{\Lambda}_l]$ gives the weight in the dual basis. Dynkin components or indicies of a weight Λ are defined by

$$a_i = 2\frac{\Lambda, \alpha_i}{\alpha_i, \alpha_i} = \sum_j \bar{\Lambda}_j \frac{2}{(\alpha_j, \alpha_j)} A_{ji} \tag{3.79}$$

in the Dynkin basis. Thus, simple roots have Dynkin components A_{ij} which are just the ith rows in Cartan matrix. With the simple roots we can derive the entire root system. At this point let us give the Dynkin diagram for E_6 which explains the simple roots in an elegant way as

E_6 :



To see the representations we furthermore give the definition of the highest root which is non-degenerate and uniquely defines the irreducible representation. All other weights in the representation can be determined from it. Dynkin's theory states that the highest weight of an irreducible representation can be selected so that the Dynkin labels are non-negative integers. Morever each and every irreducible representation is uniquely identified by a set of integers $(a_1, ..., a_l)$ and each such set is a highest weight of one and only one irreducible representation [43]. Once we have the weight system, we can convert Dynkin labels into eigenvalues of a convenient set of diagonal generators. When we write the highest root in the Dynkin basis $\Lambda = (a_1, .., a_l)$, a specific ith root can be subtracted from the highest root a_i times. A unique number is obtained and it is defined as the level of a weight of an irreducible representation as

$$\bar{R}_i = 2\sum_j (A^{-1})_{ij} \tag{3.80}$$

in dual basis. The highest level is defined as

$$T(\Lambda) = \sum_i \bar{R}_i a_i. \tag{3.81}$$

The components of the level vector is calculated from the Cartan matrix for E_6 such as

$$\begin{aligned}
\bar{R}_1 = 2((A^{-1}_{11}) + (A^{-1}_{12}) + (A^{-1}_{13}) + (A^{-1}_{14}) + (A^{-1}_{15}) + (A^{-1}_{16})) &= 16,\\
\bar{R}_2 = 2((A^{-1}_{21}) + (A^{-1}_{22}) + (A^{-1}_{23}) + (A^{-1}_{24}) + (A^{-1}_{25}) + (A^{-1}_{26})) &= 30,\\
\bar{R}_3 = 2((A^{-1}_{31}) + (A^{-1}_{32}) + (A^{-1}_{33}) + (A^{-1}_{34}) + (A^{-1}_{35}) + (A^{-1}_{36})) &= 42,\\
\bar{R}_4 = 2((A^{-1}_{41}) + (A^{-1}_{42}) + (A^{-1}_{43}) + (A^{-1}_{44}) + (A^{-1}_{45}) + (A^{-1}_{46})) &= 30,\\
\bar{R}_5 = 2((A^{-1}_{51}) + (A^{-1}_{52}) + (A^{-1}_{53}) + (A^{-1}_{54}) + (A^{-1}_{55}) + (A^{-1}_{56})) &= 16,\\
\bar{R}_6 = 2((A^{-1}_{61}) + (A^{-1}_{62}) + (A^{-1}_{63}) + (A^{-1}_{64}) + (A^{-1}_{65}) + (A^{-1}_{66})) &= 22.
\end{aligned} \tag{3.82}$$

Thus the level vector of E_6 is found to be

$$\bar{R} = [16, 30, 42, 30, 16, 22]. \tag{3.83}$$

Now let us look at the weight diagram for the **27** representation of E_6. First of all the level is calculated from the highest root which is (100000) since **27** is the fundamental representation. Thus the level is

$$T(\Lambda) = \sum_i \bar{R}_i a_i \tag{3.84}$$

$$= 16 \cdot 1 + 30 \cdot 0 + 42 \cdot 0 + 30 \cdot 0 + 16 \cdot 0 + 22 \cdot 0 = 16 \tag{3.85}$$

which means we need to subtract simple roots 16 times for this irreducible representation. Weight diagram of **27** of E_6 is given in Table 3.4.

Once we get the weight system of an irreducible representation, it falls into one of the following three categories: non-self conjugate(complex) irreps, self-conjugate irreps or pseudoreal irreps. Self-conjugate irreps are the ones with the property, that the weight at some specific level, say k, are the negatives of those at level $T(\Lambda)-k$. Complex irreps are the ones which do not satisfy this property. Furthermore if $T(\Lambda)$ is odd and irreps are self-conjugate, then such irreps are called pseudoreal irreps. The exceptional group E_6 has complex irreps. Weights of a complex irrep at level k are not the negatives of those at level $T(\Lambda)-k$. Thus the corresponding Dynkin diagram has a non trivial symmetry axis. For E_6 the Dynkin indices has the property $(a_1, a_2, a_3, a_4, a_5, a_6) \neq (a_5, a_4, a_3, a_2, a_1, a_6)$. Irreps of simple groups are also categorized as basic, simple and composite irreps. Basic irreps are those with highest weight satisfying $\sum a_i = 1$. It has only one non-zero Dynkin label. Simple irrep is a basic one where the nonzero a_i corresponding to an endpoint of the Dynkin diagram. Composite irreps are those that satisfy $\sum a_i \geq 2$. Fundemental representation is a simple irrrep. For E_6, Dynkin designation of the fundamental irrep **27** is (100000) and the conjugate $\bar{\mathbf{27}}$ is (000001). The dimension of irreps are found from the formula

$$N(\Lambda) = \prod_{\text{positive roots}} \frac{(\Lambda+\delta, \alpha)}{(\delta, \alpha)} \tag{3.86}$$

where $\delta = (1, 1, .., 1)$ in Dynkin basis, Λ is the highest weight of an irrep and α is a positive root. We can also write the dimensionality formula by LieArt again by the command

$$DIm[Irrep[E_6][a, b, c, d, e, f]] \tag{3.87}$$

where a, b, c, d, e and f are corresponding Dynkin indicies. At this point we want to define the second Casimir invariant and the index of an irrep. Second Casimir invariant is defined as

$$C = f^i_{jk} f^j_{il} x^k x^l = H_i G_{ij} H_j + \sum_{allroots} E_\alpha E_{-\alpha} \tag{3.88}$$

where $f^i_{jk} f^j_{il}$ are the structure constants, x^k, x^l are generators and C as being a Casimir operator commutes with all of the generators:

$$C(\Lambda) = (\Lambda, \Lambda + 2\delta) \tag{3.89}$$

This invariant operators always take integer values, so that we define the index of irrep that is related to lengths of weights according to

$$l(\Lambda) = \frac{N(\Lambda)}{N(adj)} C(\Lambda). \tag{3.90}$$

Furthermore

$$l(\Lambda) \cdot rank(G) = \sum (\Lambda, \Lambda), \tag{3.91}$$

giving the length squared of a weight in an irrep and the sum is over all weights.

For unified model building, we use branching rules to obtain subgroup chains. To derive branching rules the weights of the irrep r of group G are projected onto the weight space of the subgroup G' by the matrix $P(G \supset G')$, then the irreps. are picked out. For large groups computer programs are used [42] and we preferred to work with LieArt. For hand calculations tensor methods are used. The tensor product of irreps are reducible into a direct sum of irreps

$$R_1 \times R_2 = \sum_i R_i. \tag{3.92}$$

Tensor product of the irreps are computed by finding each weight vector a_α and b_β of the reps separetely where $\alpha = 1, .., N(R_1)$ and $\beta = 1, , , N(R_2)$ and forming the $N(R_1)N(R_2)$ weight $a_\alpha + b_\beta$. Then find the highest weight of $a_\alpha + b_\beta$ as the largest value given with $\bar{R} \cdot (a_\alpha + b_\beta)$, and drawing the weight diagram of the irrep with highest weight and remove those weights at level $a_\alpha + b_\beta$, finding the highest weight in the removing set, subtract the weight system of that irrep and so on. The highest weight in $(a_1 .., a_l) \times (b_1 ... b_l)$ is $(a_1 + b_1, ..., a_l + b_l)$. For the second highest weight we have the rule; subtract from the highest weight in the product the minimal sum of simple roots that connect together a nonzero component a_i in the highest weight of R_1 with a nonzero b_j in the highest weight of R_2. For E_6 if a_1 and b_1 are nonzero, subtract the first simple root from the highest weight $(100000) \times (000001)$ has highest weight (100001), 2nd heighest weight is obtained by subtracting the combination $\alpha_6 + \alpha_3 + \alpha_2 + \alpha_1$ from (100001) and gives (000100). Dimension and index sum rules for this irrep go as

$$N(R_1 \times R_2) = N(R_1) \cdot N(R_2) = \sum_i N(R_i) \tag{3.93}$$

giving the dimensionality of irrep R_i. Index of the this irrep is

$$l(R_1 \times R_2) = l(R_1)N(R_2) + N(R_1)l(R_2) = \sum_i l(R_i). \tag{3.94}$$

There are congruency constraints in the irrep. which are grouped as if R_1 and R_2 are both real and not singlets, that is the irreps in $R_1 \times R_2$ are either real or occur in $R + \bar{R}$ conjugate pairs. If R_1 and R_2 are pseudoreal, then the irreps in $R_1 \times R_2$ are real and occur in $R + \bar{R}$ conjugate pairs. If nontrivial R_1 is real and R_2 is pseudoreal then the irreps in $R_1 \times R_2$ are pseudoreal or occur in conjugate pairs. If nontrivial R_1 is self conjugate and R_2 is complex, then all the irreps in $R_1 \times R_2$ are complex. These rules are generated to triality for SU_3, quadrality for SU_4, etc. For E_6 they are given by triality [23]. All irreps in $R_1 \times R_2$ have the same triality which is equal to sum of that for R_1 and R_2. Congruency class for E_6 gives the constraint $c = a_1 - a_2 + a_4 - a_5$ [24]. Our goal is to find all maximal subalgebras of the group G. This is done by subgroup chains which can possibly reduce to $U(1)^{em} \times SU(3)_C$. Maximal subalgebras of simple algebras fall into one of the 2 categories: One of them is the category of the regular subalgebras which is obtained by looking directly at the root diagram. Other category is the special one which is not trivial and discovered by comparing irreps of the group with the ones in the subgroup chain. To obtain the subalgebras we restate the embedding in terms of the projection matrix method. Projection matrix takes the roots and weights of the group G onto the roots and weights of the subgroup G'. Firstly let us look at the simplest example of embedding $SU_3 \supset SU(2) \times U(1)$. In this chain both sides have rank 2 and we can project any root or weight onto a weight of $SU(2) \times U(1)$ by a 2×2 matrix acting as

$$P(SU_3 \supset SU(2) \times U(1)) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (b) \\ (u) \end{pmatrix} \tag{3.95}$$

Here $(a_1 a_2)$ is a SU_3 weight, b is $2I_3$ which is the weight of $SU(2)$, and u is an eigenvalue of $U(1)$ generator. P is given in the Eightfoldway by

$$P(SU_3 \supset SU(2) \times U(1)) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}. \tag{3.96}$$

When we apply this projection matrix to the weight system $(1 \quad 0)$,$(-1 \quad 1)$ and $(0 \quad -1)$ of $\mathbf{3}$ representation of SU_3

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}. \tag{3.97}$$

This result can be identified as $SU(2)$ doublet with hypercharge $Y = \frac{1}{3}$ and a singlet with $Y = -\frac{2}{3}$. It can be summarized in a branching rule as

$$\mathbf{3} = \mathbf{2}(1) + \mathbf{1}(-2) \tag{3.98}$$

We know there is a diagonal generator Q characterized by an axis in root space. A state $|\Lambda >$ with weight Λ is an eigenstate of Q;

$$Q|\Lambda > \;=\; Q(\Lambda)|\Lambda > \tag{3.99}$$

$$Q(\Lambda) \;=\; (Q, \Lambda) \tag{3.100}$$

To compute Q, we put the Q axis in the dual basis with components $[\bar{q}_1, ...\bar{q}_l]$ and weight Λ in Dynkin basis so that

$$Q(\Lambda) = \sum_i \bar{q}_i a_i. \tag{3.101}$$

To get $\bar{q}_i$, we find the complete explicit embedding of Q by working backwards for one irrep to get $\bar{q}_i$ from $Q(\Lambda) = \sum_i \bar{q}_i a_i$. Then compute $Q(\Lambda)$ for other weights from $Q(\Lambda) = (Q, \Lambda)$. Let's continue with the SU_3 example: Select the root $(2-1)$ in Dynkin basis, I_3 value of the roots $(2-1)$ is $+1$, (11) is $\frac{1}{2}$ and $(1-2)$ is $\frac{1}{2}$. Thus any root or weight Λ with labels $(a_1 a_2)$ has I_3 value;

$$I_3(a_1a_2) = \bar{I}_3 \cdot \Lambda = \frac{a_1}{2} \tag{3.102}$$

$$\bar{I}_3 = \frac{1}{2}[1 \quad 0] \tag{3.103}$$

with normalized hypercharge axis

$$\bar{Y} = \frac{1}{3}[1 \quad 2] \quad \text{with} \quad Y(a_1a_2) = \bar{Y} \cdot \Lambda. \tag{3.104}$$

Lastly Q^{em} is computed from Gell-Mann-Nishijima formula $Q^{em} = I_3 + \frac{Y}{2}$ by adding the axis

$$\bar{Q}^{em} = \frac{1}{2}[1 \quad 0] + \frac{1}{6}[1 \quad 2] = \frac{1}{3}[2 \quad 1] \tag{3.105}$$

with $Q^{em} = \bar{Q}^{em} \cdot \Lambda$. Maximal subalgebras and branching rules of E_6 for the fundamental representation **27** is given in [24]: The regular subalgebras with corresponding branching rules are

$$\begin{aligned}
E_6 \quad &\supset \quad SO(10) \times U(1) \quad \mathbf{27} = \mathbf{1} + \mathbf{10} + \mathbf{16} \\
&\supset \quad SU(2) \times SU(6) \quad \mathbf{27} = (\mathbf{2}, \bar{\mathbf{6}}) + (\mathbf{1}, \mathbf{15}) \\
&\supset \quad SU_3 \times SU_3 \times SU_3 \quad \mathbf{27} = (\bar{\mathbf{3}}, \bar{\mathbf{3}}, \bar{\mathbf{1}}) + (\mathbf{1}, \bar{\mathbf{3}}, \bar{\mathbf{3}}).
\end{aligned}$$

Special subalgebras go as

$$\begin{aligned}
E_6 \quad &\supset \quad SU_3 \quad \mathbf{27} = \mathbf{27} \\
&\supset \quad SU_3 \quad \mathbf{27} = \mathbf{27} \\
&\supset \quad Sp_8 \quad \mathbf{27} = \mathbf{27} \\
&\supset \quad F_4 \quad \mathbf{27} = \mathbf{1} + \mathbf{26} \\
&\supset \quad SU_3 \times G_2 \quad \mathbf{27} = (\bar{\mathbf{6}}, \mathbf{1}) + (\mathbf{3} \oplus \mathbf{7}).
\end{aligned}$$

When we consider the charge conjugation operator that carries the particles into their antiparticles, the possibility of defining it comes with the constraint that C must commute with electric charge and with the first, third, forth, sixth and eighth generators of $SU(3)_C$ while commuting with the second, fifth, and seventh generators of $SU(3)_C$ in the Gell-Mann basis. For the simple groups G, a candidate for C may be an element of the group or not. It should anticommute with some set A of Hermitian generators of G, and commute with the remaining set S generating the symmetric subgroup. For the exceptional group E_6 we have a possible outer automorphism for C, that carries each complex irrep to its complex conjugated one. Candidate charge conjugation operators are given in Table 3.5. There we see that a given simple group G, the symmetric subgroup G_s left invariant by C, the number $rank(A)$ of the generators of the Cartan subalgebra of G that anticommute with C. When we consider the unitary operator CP, it is unique for each G in Table 3.5 that changes the sign of all Cartan subalgebra. In the assignment of fermion families to the complex **27** of E_6, we have the usual color-flavor decomposition $SU_3 \times SU_3 \times SU(3)_C$ where $SU(2)^w$ is in the first SU_3, the **27** contains $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}, \mathbf{3}^c) + (\mathbf{1}, \bar{\mathbf{3}}, \bar{\mathbf{3}}^c)$ [1]. From table 3.5 we see that the only candidate for C leaves the symmetric subgroup $SU(2) \times SU(6)$ invariant and that $SU(2) \times SU(6)$ contains two generators from the Cartan subalgebra of E_6. We are interested in E_6 as a candidate to contain $SU(3)_C$. Then satisfactory maximal subgroups of E_6 are given as $F_4, SO(10) \times U(1), SU(2) \times SU(6), SU_3 \times SU_3 \times SU_3$. Tensor products of irreps of E_6, and branching rules are given in detail in [24]. In the irreps of E_6, the congruency is reduced to triality in terms of Dynkin indices as $a_1 - a_2 + a_4 - a_5 (mod 3)$. There is only one embedding of $U(1)^{em} \times SU(3)_C$in E_6 that is suitable for model building which is the one where the multiplet **27** has nine $\mathbf{1^c}$'s, three $\mathbf{3^c}$'s, and three $\bar{\mathbf{3}}^{\mathbf{c}}$'s with electric charge spectrum of three $\mathbf{3^c}$'s being $2/3, -1/3, -1/3$. The projection matrices that project a weight of E_6 to $U(1)^{em} \times SU(3)_C$ can be chosen such that $SU(3)_C$ and Q^{em} axes of the subgroup chains coincide. The idea is to project the highest weight of an irrep onto the highest weight of the subgroup chain:

$$E_6 \supset SO(10) \times U(1)^t \times \supset SU(5) \times U(1)^r \times U(1)^t \supset SU(2)^w \times SU(3)_C \times U(1)^w \times U(1)^r \times U(1)^t. \tag{3.106}$$

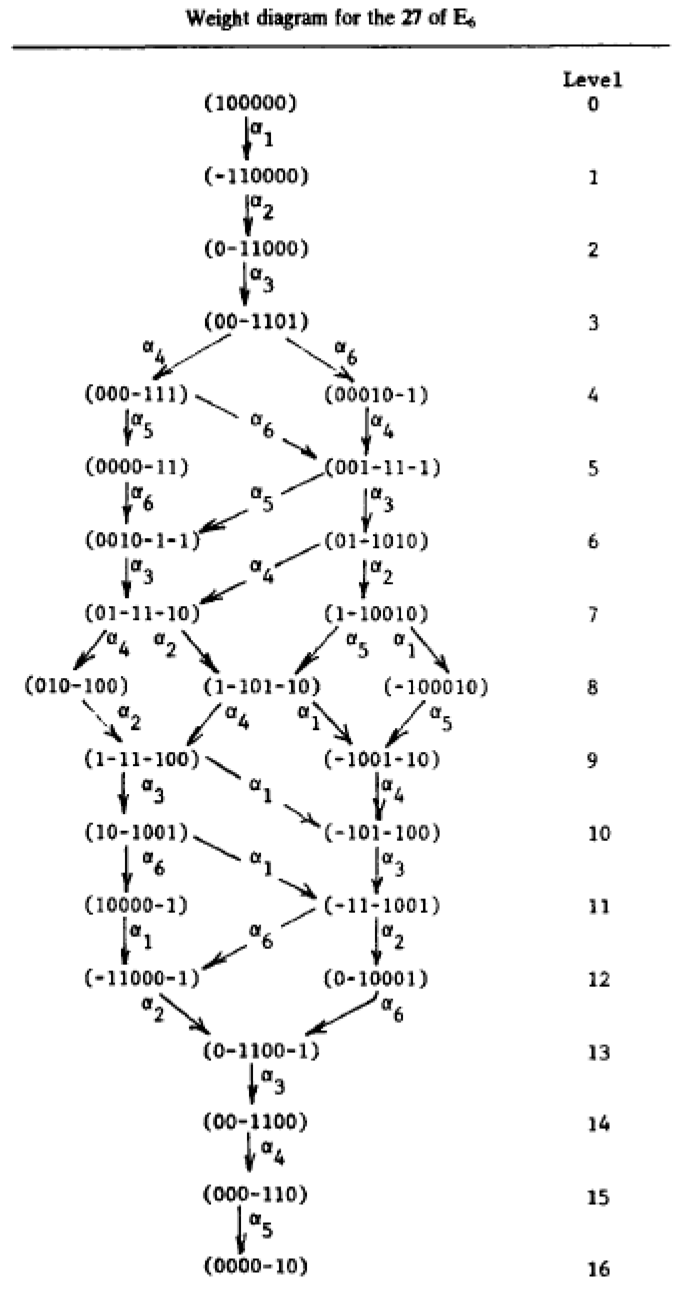


Figure 3.4: Table 1 [24]

Symmetric subgroups of simple groups

G	G_S	rank(A)	Action of C on irrep R
SU_n	SO_n	$n-1$	$\bar{R}$
SU_{p+q}	$SU_p \times SU_q \times U_1$	$\min(p,q)$[a]	R
SU_{2n}	Sp_{2n}	$n-1$	$\bar{R}$
SO_{p+q}	$SO_p \times SO_q$	$\min(p,q)$[a]	R (p or q even)
			$\bar{R}$ or R′ (p and q odd)[b]
SO_{2n}	$SU_n \times U_1$	$[n/2]$	R
Sp_{2n}	$SU_n \times U_1$	n	R
Sp_{2p+2q}	$Sp_{2p} \times Sp_{2q}$	$\min(p,q)$[a]	R
G_2	$SU_2 \times SU_2$	2	R
F_4	$SU_2 \times Sp_6$	4	R
	SO_9	1	R
E_6	Sp_8	6	$\bar{R}$
	$SU_2 \times SU_6$	4	R
	$SO_{10} \times U_1$	2	R
	F_4	2	$\bar{R}$
E_7	SU_8	7	R
	$SU_2 \times SO_{12}$	4	R
	$E_6 \times U_1$	3	R
E_8	SO_{16}	8	R
	$SU_2 \times E_7$	4	R

[a] The case p or q equal unity defines a symmetric subgroup with SU_1 or SO_1 empty; the Lie algebra of Sp_2 is isomorphic to that of SU_2.

[b] If $p+q=4n+2$, C reflects complex spinor irreps into their conjugates; if $p+q=4n$, C reflects the real or pseudoreal spinor irreps into the nonequivalent spinor of the same dimension.

Figure 3.5: Table 2 [24]

Chapter 4

PARTICLE ASSIGNMENTS IN E_6

In this chapter our aim is to give an outline for the particle assignments in the fundamental representation of E_6 model. The fundamental representation of E_6 has dimension 27 thus it can also be represented as **27** which gives the fermion content in E_6 GUT Model. To see how the fermionic content in **27** of E_6 arise we can start by constructing the weight diagram for **27** of E_6. The weight diagram can be constructed by starting from the highest weight of that representation which is (100000) for **27**. The ith simple root can be subtracted from the highest weight a_i times. Then, this procedure should be repeated at each level. Finally, we get the weight system of the representation. For **27** the level is 16 as given in Equation 3.84, thus we obtain the corresponding weight diagram as given in Table 3.4. Here we are looking for subgroup decompositions of E_6 containing QCD group $SU(3)_C$ as a subgroup, so the physical subgroup chains are given as;

$$\begin{aligned}
E_6 &\supset SO(10) \supset SU(5) \supset SU(2)^w \times SU(3)_C \\
&\supset SO(10) \supset SU(2)^w \times SU(2) \times SU_4 \\
&\supset SU_3 \times SU_3 \times SU(3)_C.
\end{aligned}$$

Here $U(1)$'s are omitted in all the chains above. To observe the particle content explicitly, we can start with the physical subgroup chains and weight diagram that we obtained in Table 3.4. We can use projection of highest weight method to obtain particle content. Concentrating on the subgroup chain $E_6 \supset SO(10) \supset SU(5) \supset SU(2)^w \times SU(3)_C$. We have to go from E_6 down to $SU(2)^w \times SU(3)_C$ step by step. In every step we obtain projection matrix by projecting the highest weight of the representation to highest weight of targeting subgroup. Thus we obtained P_1, by projecting the highest of E_6 to highest weight of $SO(10)$ as,

$$P_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, projecting $SO(10)$ to $SU(5)$ gives,

$$P_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

and finally $SU(5)$ to $SU(2)^w \times SU(3)_C$ gives

$$P_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now let's apply these projection matrices to weight diagram of **27** in Table 3.4. Starting from the first level which has the weight (100000) and apply the projection matrices as

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Here P_1 is applied to first level weight and the result is a weight in **16** of $SO(10)$ which can be checked by putting the resulting weight into the dimensionality formula given before in Equation 3.86. Continue with P_2 applied to this weight

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

giving a $SU(5)$ weight in **10** representation. Finally applying P_3 to this weight and we get

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Here we obtained 3×1 matrix. This final 3×1 matrix identifies the (100000) weight in **27** as weights in $SU(2)$ and SU_3 respectively. First entry a_1 gives I_3 value in $SU(2)$ and second and third entries $a_2 a_3$ together are identified as color roots. In this example, since the first entry is $SU(2)$ weight it basically gives the isospin as

$I_3 = a_1/2 = 1/2$ and the SU_3 color root is (10). This procedure is given for SU_3 implicitly in chapter 4. This isospin value and color root together give the information about the electric charge Q which is +2/3. Then by Gell-Mann-Nijhijima formula $Q = I_3 + \frac{Y^w}{2}$ we can obtain the hypercharge as $Y^w = 1/3$. This procedure is applied to all weights in the weight diagram **27** of E_6. Now let's also observe the weight in final level 16 as the second example which is $(0000-10)$;

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

this is a $SU(5)$ weight in **16** representation. Continue by applying P_2

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

getting a weight in **10** representation of $SO(10)$. Finally with P_3 applied to this weight

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix},$$

and we obtained an $I_3 = -\frac{1}{2}$ quark with color root $(0{-}1)$. This method sets out all the particle content in **27** . We categorized all of them in Table 5.1. In this table which is constructed by projection matrix method we identified 9 color singlets (leptons), 3 quarks and 3 antiquarks in **27** irrep of E_6 model. Now we can obtain a more general formula to get the quantum number structure valid for any weight. In doing this the content we get helps us which we have done for SU_3 before. This time we need to pay attention to subgroup structure also. Consider **10**(**16**) quarks in Table 5.1 which have weights $(100000), (0000-11), (1-10010), (1000-1), (0-10001), (0000-10)$. Here **10**(**16**) represents that in projection of E_6 down to $SO(10)$ and $SU(5)$ we first get the weight in $SO(10)$ in irrep **16** and than get the weight in $SU(5)$ by projection matrix P_2 which is in representation **10**. The final weight we obtained by this procedure is represented as **10**(**16**). We know their I_3 values from projections to $SU(2)\times SU_3$, since they are quarks and isospins are known we can easily detect their electric charges.

Once we have these known quantum numbers we can derive general axis formulas as we have done for SU_3 before, this time for E_6 axis formulas are given as,

$$\begin{aligned} I_3(a_1a_2a_3a_4a_5a_6) &= \bar{I_3}\cdot\lambda \\ Y^w(a_1a_2a_3a_4a_5a_6) &= \bar{Y_w}\cdot\lambda \\ Q^{em}(a_1a_2a_3a_4a_5a_6) &= \bar{Q^{em}}\cdot\lambda. \end{aligned} \tag{4.1}$$

and apply for 6 weights in **10**(**16**) mentioned above. Thus we get the axis formulas for E_6 as,

$$\bar{Q^{em}} = \frac{1}{3}[212010], \tag{4.2}$$

$$I_3^w = \frac{1}{2}[111110], \tag{4.3}$$

$$Y^w = \frac{1}{3}[1-11-3-10]. \tag{4.4}$$

This is a very handy result since now we have the general axis formula and we can obtain the quantum numbers for any weight if we know the Dynkin indices. Consider the $(01-1010)$ weight in the 6th level of **27** of E_6. Applying I_3 axis formulas above to get isospin, hypercharge and electric charge as

$$\begin{aligned} I_3 &= \frac{1}{2}(111110)(01-1010) = \frac{1}{2}, \\ Q^{em} &= \frac{1}{3}(212010)(01-1010) = 0, \\ Y^w &= \frac{1}{3}(1-11-3-10)(01-1010) = -1. \end{aligned} \tag{4.5}$$

These formulas can be applied to any weight in E_6 to get quantum numbers. We can also identify and list the nonzero roots of E_6. Same method is used to get content of these roots also and we can list the set of generators for flavor interactions. $U(1)$ factors are in Cartan subalgebra of E_6 and they correspond to axes in root space. Recall the subgroup chain $E_6 \supset SO(10) \supset SU(5) \supset SU(2)^w \times SU(3)_C$ we have two more $U(1)$ axis rather than weak isospin charge. Let's call the one coming from $E_6 \supset SO(10)$ as $U(1)^t$ and the other coming from $SO(10) \supset SU(5)$ as $U(1)^r$. Other two of the Cartan subalgebra members (axises) are members of $SU(3)_C$. All of these physical roots and axes in E_6 weight space is summarized in Table 5.2. We see the whole fermionic content predicted by E_6 GUT. The first important outcome is we have all the Standard Model leptons and quarks. We also have extra particles. Here we have followed the breaking chain including $SO(10)$ as a subgroup, thus we have right handed neutrino appearing in **16**plet of $SO(10)$ representation. We know that all the matter around us is composed of spin 1/2 particles which are described by Weyl spinors. All of the known particles except neutrinos are described as left handed or right handed according to their chirality, and, the Standard Model neutrinos are all left handed particles, they are neutral particles that do not have electricl or color charge thus they participate only in weak interactions [44]. More detailed information about the right handed neutrino predicted by E_6 GUT Model is given the next chapters. Looking through the particle content in the 27-plet of E_6 we see new particles predicted which we call as exotic particles. These particles are called exotic due to their quantum numbers which we represent with capital letters such as exotic quarks with Q and exotic leptons with L in Table 5.1. We cannot assign a unique baryon number for these particles and in the next chapter it is shown that $(B - L)$ (baryon minus lepton number) is a local symmetry in the left-right extended theory for the exotic fermions in E_6 [45]. For the gauge couplings to be unified, the exotic fermions should be at the TeV scale. They do not interact directly with quarks and leptons, so they are produced only through the electroweak gauge bosons. [46]

Chapter 5

SYMMETRY BREAKING PATTERNS IN E_6

5.1 *Symmetry Breaking via Intermediate* $SU(5)$ *and Relation to The Mass Matrix*

So far we have obtained the physical representations and fermionic particle content for E_6 GUT Model. We can move further to analyze symmetry breaking mechanisms. The usual way for observing the symmetry breaking is to set up a Higgs potential and minimize it. Here we will instead try to apply some physical constraints which will be applicable for any symmetry breaking mechanism. Our aim is to break the group G into the subgroup H, this is achieved by representations of spinless fields $\phi(\mathbf{r})$. These fields take vacuum expectation values (vev) $< \phi(\mathbf{r}) >$. We will look at the subgroups H of G which leaves $\phi(\mathbf{r})$ invariant, H is also called as little group of $\phi(\mathbf{r})$. For each little group the branching rule for the representation $\mathbf{r}$ into irreps of H tells that there exist at least one singlet representation. These little groups are further called as maximal little group if there exists no $\phi(\mathbf{r})$ in dim($\mathbf{r}$) space with little group H* such that G $\supset$ H* $\supset$ H. We can recall that in the single adjoint breaking each maximal little group of the adjoint irrep of G is found by removing one dot from Dynkin diagram. In the adjoint representation of of E_6 which is **78** breaking occurs like $SO(10)\times U(1), SU(2)\times SU(5)\times U(1), SU(2)\times SU_3\times SU_3\times U(1), SU(6)\times U(1)$. At every step there appears a $U(1)$ factor and a removed dot from Dynkin diagram as we have given in Chapter 3. Maximal little groups become important after symmetry breaking. They are restricted by color and flavor embedding. In analyzing the symmetry breaking mechanism we will concentrate on symmetry breaking direction in weight space and how it is related to mass matrices of the particles. We have introduced spin zero fields $\phi(\mathbf{r})$ with vevs $\langle\phi(\mathbf{r})\rangle$. If we know these vacuum expectation values, we can obtain vector boson mass matrix of a Yang Mills theory with local symmetry G in the lowest order approximation. The covariant derivative in the Lagrangian gives theoretical structure

$$(D_\mu\phi)^\dagger D_\mu\phi \quad \text{where} \quad D_\mu = \partial_\mu - igA^a_\mu T_a. \tag{5.1}$$

Here T_a is the representation matrix of the scalar fields, A is the gauge potential. The mass matrix is seen from

$$\begin{aligned} (D_\mu\phi_\nu)^\dagger(D_\mu\phi_\nu) &= \partial_\mu\phi^\nu\partial^\mu\phi^{nu} + ig\partial^\mu(\phi^\nu A^a_\mu T_a\phi_\nu) \\ &= g^2 T_a T_b A^{b\mu} A^a_\mu \phi_\nu\phi^\nu \end{aligned} \tag{5.2}$$

where $\phi_\nu = \langle \phi(\mathbf{r}) \rangle$. In matrix notation following [47],

$$(D_\mu \phi_\nu)^\dagger (D_\mu \phi_\nu) = M^{ab} A_{\mu a} A_b^\mu \tag{5.3}$$

and

$$(M^2)_{ab} = g^2 (\phi_\nu^\dagger)_i (T_{-a})_{ij} (T_b)_{jk} (\phi_\nu)_k \tag{5.4}$$

where $(T_a)_{ij}$ represents the matrix element of the ath generator in representation $\mathbf{r}$ which the spin zero fields are assigned. $a = 1, \cdots N(adj)$, $i = 1, \cdots N(\mathbf{r})$. T_a is the ladder operator with root a, thus $(T_a)^\dagger = T_{-a}$. In another way

$$(T_a)_{ij} = \langle \mathbf{r}, \lambda | X_a | \mathbf{r}, \lambda' \rangle \tag{5.5}$$

where X_a is the generator with root a. CP conjugation reverses the signs of weight of a field and takes the representation to its conjugate one $\bar{\mathbf{r}}$. Thus the mass matrix can be written in the form,

$$(M^2)_{ab} = g^2 \sum_{\lambda\lambda'\lambda''} \phi_\nu^\dagger(\bar{\mathbf{r}}, -\lambda'') \langle \bar{\mathbf{r}}, -\lambda'' | X_{-a} | \bar{\mathbf{r}}, -\lambda \rangle \langle \mathbf{r}, \lambda | X_b | \mathbf{r}, \lambda' \rangle \phi_\nu(\mathbf{r}, \lambda'). \tag{5.6}$$

Here $\phi_\nu(\mathbf{r}, \lambda)$ is the tensor operator and $\phi_\nu^\dagger$ is the adjoint one. To obtain the vector boson mass matrix classically Higgs potential is minimized by computing the vev's $\phi_v(\mathbf{r}, \lambda)$, then diagonalizing M^*. Little group is obtained by the transformations leaving $\phi_v(\mathbf{r})$ invariant which are the eigenstates of equation 5.6 with zero eigenvalues. At this step we need to employ the tensor operator. Irreducible tensor operator of order k as a set of $2k+1$ operators T_q^k is defined for $q = -k, \cdots, +k$ as

$$U T_q^k U^\dagger = \sum_{q'} T_{q'}^k D_{q'q}^k(U) \quad , \quad \text{for all operators U.} \tag{5.7}$$

Tensor operator $T(\mathbf{r}, \lambda)$ satisfies the commutation relations,

$$[X_a, T(\mathbf{r}, \lambda)] = \sum_{\lambda'} \langle \mathbf{r}, \lambda' | X_a | \mathbf{r}, \lambda \rangle T(\mathbf{r}, \lambda'), \tag{5.8}$$

the adjoint tensor operator changes weights when acting on kets by $-\lambda$. In computation of mass matrix we need the commutation relations of Cartan subalgebra, recall;

$$[X_a, H_i] = 0, \tag{5.9}$$

if X_a is in Cartan subalgebra and this results to conclude that vector bosons are massless. We can obtain a diagonal mass matrix if the remaining generators E_a can be written in Cartan-Weyl basis. Recalling the relations,

$$[E_i, E_\alpha] = \alpha_i E_\alpha \tag{5.10}$$

and

$$H_i E_\alpha|\lambda\rangle = E_\alpha H_i|\lambda\rangle + \alpha_i E_\alpha|\lambda\rangle \tag{5.11}$$
$$= (\lambda_i + \alpha_i)(E_\alpha|\lambda\rangle). \tag{5.12}$$

Now we can write equation 5.6 as

$$(M^2)_{ab} = -g^2 \sum_\lambda Tr[X_{-a}, \phi^\dagger_\nu(\mathbf{r}, -\lambda)][X_b, \phi_\nu, (\mathbf{r}\lambda)]. \tag{5.13}$$

When $\phi_\nu(\mathbf{r}, \lambda)$ has zero weight and $\mathbf{r}$ is the adjoint representation of group G, we can expand ϕ_ν as

$$\phi_\nu(\mathbf{r}, \lambda = 0) = \sum_{i=1}^{l} \bar{c}_i H_i \tag{5.14}$$

here $\bar{c}_i$ is the symmetry breaking axis in weight space with components $[\bar{c}_1, \cdots, \bar{c}_l]$ where l is the rank of the group G. Substituting 5.14 into 5.13 we get

$$(M^2)_{ab} = g^2 Tr\{(\sum_i a_i\bar{c}_i)X_{-a}(\sum_j b_j\bar{c}_j)X_b\} \tag{5.15}$$
$$= g^2\delta_{ab}(\sum_i \bar{c}_i a_i)^2 \tag{5.16}$$

where in the last step normalization condition $Tr(X_{-a}X_b) = \delta_{ab}$ is used. Here $\sum a_i\bar{c}_i = (c, a)$ is a scalar product in weight space. We can read from here that mass is proportional to the projection of the weight of the vector boson onto the symmetry breaking axis c. This result can be used to see whether the **24** adjoint breaking in $SU(5)$ take $SU(5)$ to $SU(2) \times SU_3 \times U(1)$. Rank of two sides are the same, thus breaking is in Cartan subalgebra. Remembering the dot removing procedure in SU_N, breaking we must have a vector boson here. We know that gluons must be left massless after the breaking, this requires c to be perpendicular to color roots. Color roots are obtained by the projection matrix P_3 applied to roots in Table 5.1, $(1001), (11-10), (0-111)$. With this requirement $\bar{c}$ must have the form,

$$\bar{c} = [a, b, a+b, -a]. \tag{5.17}$$

Other requirement comes from the breaking $I^w = 0$ and c must be perpendicular to I^w_+ root which is $(-111-1)$ giving $a = -2b$. Then $\bar{c}$ is proportional to $[-21-12]$ and c is proportional to $(1-11-1)$. Magnitudes of the scalar products (c, a) for lepto-diquark roots are equal and nonzero. This gives an explicit representation of the breaking of $SU(5)$ to $SU(2) \times SU_3 \times U(1)$ by **24**.

We can carry out the same analysis for adjoint breaking of E_6. Direction of big breaking we are considering here in 6 dimensional weight space is called as B. I is the weak breaking which gives the masses of weak bosons. Here the direction of B must be perpendicular to color roots since otherwise gluons gain masses. Furthermore B must also be perpendicular to root of weak isospin raising operator since we also do not want the weak bosons to get a superlarge masses. We have already defined the color roots as $(000001), (0100-10), (0-10011)$. The plane formed by these roots and perpendicularity condition implies B is in the form $\bar{B} = [-cdabd0]$. The weak isospin root is $I_+^w = (10001-1)$, so we can eliminate one more parameter in B. The perpendicularity between weak isospin and B implies $\bar{B} = [-ccabc0]$. Now we can look at every weight's scalar product with $\bar{B}$ and see the parametrization of vector boson mass eigenvalues for each root up to a factor.

The little breaking must have the components $|\Delta I^w| = 0|$, or $|\Delta I^w| = \frac{1}{2}$. We have $\bar{B} = [-cdabd0]$ if the big breaking has zero weight it is constrained by $a = b + c$. Let us only consider $|\Delta I^w| = \frac{1}{2}$ term and breaking of E_6 occurs as a consequence of single, nonzero, extreme weight L. Extreme weight is defined as for any root a if $\lambda + a$ is a weight, $\lambda - a$ is not. Then, $(Q^{em}, L) = 0$ and $(L, I_3^w) = \frac{1}{2}$ and L is also perpendicular to color roots. These constraints require L to be

$$\bar{L} = [-d, d \pm 1, d + e, e, d \pm 1, 0] \tag{5.18}$$

or using

$$a_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \sum_j \bar{\lambda_j} \frac{2}{(\alpha_j, \alpha_j)} A_{ji} \tag{5.19}$$

and write L as

$$\bar{L} = (-3d \mp 1, 2d - e \pm 2, d + e \mp 1, e - 2d \mp 1). \tag{5.20}$$

All the $Q^{em} = 0$ and $|\Delta I_3^w| = \frac{1}{2}$ and weights of **27** are of this type. Three candidates are $(00010-1), (-10-100)$ and $(01-1010)$. A boson with a nonzero weight gets mass squared contribution proportional to (L, a). B is not restricted only by **78** because there are other representations of E_6 with nonzero weights such as **78,650, 2430, 2925** or larger ones. If B is due to adjoint representation of Higgs scalar alone, the only independent Casimir invariants of E_6 are of order 2,5,6,8,9 and 12 so Higgs potential can depend on the length of **78** only. Because of this there is no restriction on a, b and c in tree approximation. This results to no restriction an a,b and c in tree approximation. On the one loop corrections of effective potential selecting $a = 0, b = -c$ the $SO(10) \times U(1)$ will be left unbroken. We can hope B to be singled out by breaking. If the adjoint breaking is along the only one root with $|\Delta I^w| = 0$ which is $(0-111-1-1)$ then the $SU(6)$ is left unbroken. If the little breaking is in **27**, then there exists three candidates with $|\Delta I^w| = \frac{1}{2}$ weight. Each breaks $SU(2) \times U(1) \times U(1) \times U(1)$ to $U(1)^{em}$. In simple Higgs models with a reducible representation of scalars vacuum values of different irreps are perpendicular to weight space for a range of parameters. Weak breaking in $SU(5)$ transforms as $\mathbf{5}+\bar{\mathbf{5}}$ or $\mathbf{10}$ of $SO(10)$. Weight of L can be seen from Table 21 as $(0-110-10)$ and $(10-1100)$ which are presented also in $\mathbf{27}$ of E_6. Furthermore L is perpendicular to B if $a = 2c$ and $b = 3c$. With this further restriction if B is in this specified direction, then B breaks E_6 down to $SU(2)^w \times SU(2) \times U(1) \times U(1)^t \times SU_3$ and L breaks it down to $U(1)^{em} \times SU(3)_C$. About the charge conjugation, $\mathbf{27}$ has two charge $-\frac{1}{3}$ quarks and their antiparticles so there exist a field to study the effect of charge conjugation operator C on the quark masses. The symmetry subgroups of E_6 are $Sp_8, SU(2) \times SU(6), SO(10) \times U(1)$ and F_4. Among these the reflection associated with Sp_8 and F_4 reflect $\mathbf{27}$ to $\bar{\mathbf{27}}$. CP is associated with Sp_8 [47]. C must be associated with $SU(2)\times SU(6)$, since it is inner and flips the signs of enough diagonal generators. C leaves invariant 2 of the 6 diagonal quantum numbers in E_6 [24]. We have identified the embedding of color and flavor in E_6 by the subgroup chain,

$$E_6 \supset SO(10)\times U(1)^t \supset SU(5)\times U(1)^r\times U(1)^t \supset SU(2)^w\times U(1)^w\times SU(3)_C\times U(1)^r\times U(1)^t. \tag{5.21}$$

We have also given the projections E_6 down to $SU(2) \times U(1) \times SU(3)$ by projection matrices $P_1(E_6 \supset SO(10)), P_2(SO(10) \supset SU(5)), P_3(SU(5) \supset SU(2) \times SU(3))$. The weights of $\mathbf{27}$ after these projection matrices applied are representations in $SO(10), SU(5)$ and $SU(2) \times SU(3)$. Now we can impose the effect of charge conjugation on these weights and predict the charge conjugation matrix C matrix for E_6 as,

$$C(E_6) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix inverts the color roots and reverses the signs of electric charge and $2Q^r + Y^w$ while leaving $3Y^w + 4Q^r - 10I_3^w$ and Q^t invariant. Three of the neutral lepton weights in $\mathbf{27}$ are eigenvectors of $C(E_6)$ with eigenvalues 1, these are $(-101-100), (01-1010)$ and $(1-101-10)$. Other 2 neutral weights transformed into each other by C. Remaining charged weights transform as required by C. Let us look at charge $-2/3$ up quark as an example. Weights are transformed by C as,

$$\begin{aligned} (100000) &\longrightarrow (00-1100), \\ (1-10010) &\longrightarrow (01-11-10), \\ (10000-1) &\longrightarrow (00-1101). \end{aligned} \tag{5.22}$$

Charge $-1/3$ quarks and charged leptons have the same weight structure, so we can move on with another quark example. The charge $-1/3$ quark is in $\mathbf{10}$ of $SU(5)$ and

16 of $SO(10)$, denoted as **10(16)**. These weights are $(0000-11), (0-10001)$ and $(0000-10)$ and reflected by C as

$$\begin{aligned} (0000-11) &\longrightarrow (0-1100-1), \\ (0-10001) &\longrightarrow (0010-1-1), \\ (0000-10) &\longrightarrow (0-11000). \end{aligned} \tag{5.23}$$

These reflected weights are in $\bar{\mathbf{5}}(\mathbf{16})$. Other charge $\frac{-1}{3}$ quark is in **5(10)** has weights $(-110000), (-100010), (-11000-1)$ and reflected by C as,

$$\begin{aligned} (-110000) &\longrightarrow (000-110), \\ (-100010) &\longrightarrow (010-100), \\ (-11000-1) &\longrightarrow (000-111). \end{aligned} \tag{5.24}$$

C partners here are in $\bar{\mathbf{5}}(\mathbf{10})$. Concentrating on a single quarks the mass operator $\psi\bar{\psi}$ breaks into two parts $\bar{psi}_L\psi_R + \psi_L\bar{\psi}_R$ since $1 \pm \gamma^5$ commutes with local symmetry transformations. For a single quark mass matrix is in the form ,

$$\begin{array}{c} \\ q_L \\ \bar{q_L} \\ \bar{q_R} \\ q_R \end{array} \begin{array}{c} \begin{array}{cccc} \bar{q_R} & q_R & q_L & \bar{q_L} \end{array} \\ \begin{pmatrix} 0 & 0 & M_{11} & M_{12} \\ 0 & 0 & M_{21} & M_{22} \\ M_{11}^\dagger & M_{21}^\dagger & 0 & 0 \\ M_{12}^\dagger & M_{22}^\dagger & 0 & 0 \end{pmatrix} \end{array}.$$

The mass matrix M for the color state **10** for quarks and **-10** for antiquarks according to general mass matrix form above we can write the mass matrix as,

$$\begin{array}{c} \\ D\mathbf{5}(\mathbf{10}) \\ d\mathbf{10}(\mathbf{16}) \\ \bar{d}\bar{\mathbf{5}}(\mathbf{10}) \\ \bar{D}\bar{\mathbf{5}}(\mathbf{16}) \end{array} \begin{array}{c} \begin{array}{cccc} D\mathbf{5}(\mathbf{10}) & d\mathbf{10}(\mathbf{16}) & \bar{d}\bar{\mathbf{5}}(\mathbf{10}) & \bar{D}\bar{\mathbf{5}}(\mathbf{16}) \end{array} \\ \begin{pmatrix} 0 & 0 & [(-110-110)] & (-10100-1) \\ 0 & 0 & (000-101) & [(0-110-10)] \\ [(-110-110)] & (000-101) & 0 & 0 \\ (-10100-1) & [(0-110-10)] & 0 & 0 \end{pmatrix} \end{array}.$$

There are two candidate assignments with extreme C behavior for the weak isospin conserving mass. Either the $(-10100-1)$ mass is nonzero d states is left massless and C is maximally violated or $(-110-110)$ mass is nonzero, D is massless. Here three of the neutral weights in **27** are eigenvectors of $C(E_6)$ with eigenvalues $+1$; $(-101-100), (01-1010)$ and $(1-101-10)$. The same can be done for the neutral lepton and charged lepton mass matrices.

5.2 *Breaking via Intermediate Pati-Salam Chain*

In the previous section we have given the breaking chain from E_6 down to the Standard Model gauge groups with intermediate $SU(5)$ presence. Now we can look at another chain including intermediate Pati-Salam Model instead of $SU(5)$. Pati-Salam Model is also called as Partial Unification Theory and it is proposed at the same year with Georgi-Glashow's $SU(5)$ Model in 1974 by Abdus Salam and Jagesh Pati [8,13]. The model is based on the gauge groups $SU(4) \times SU(2)_L \times SU(2)_R$ where $SU(4)$ part can be expressed as an extension of QCD with the lepton number as a forth color and $SU(2)_R$ is right symmetric part of the left symmetry $SU(2)_L$. Again we have three generations of fermions but this time with the representations $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ and $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$. The Pati Salam and $SU(5)$ Models both can be embedded in $SO(10)$ for further unification, however, Pati-Salam Model can be thought as an alternative to $SU(5)$ Model in many ways. First of all, in $SU(5)$ Model there is no left-right symmetry which is is important to explain the differences between left handed and right handed particles in the theory. In $SU(5)$ weak hypercharge and color charges are uncoupled and Pati-Salam theory explains $U(1)$ charge as unified with $SU(4)$'s color charge which in turn explains the B-L number symmetry and conserves $(B - L)$ number and introducing the $U(1)$ as $U(1)_{B-L}$. The other portion of the weak hypercharge is in $SU(2)_R$ part. The final point in the symmetry breaking chain we aim to arrive is the Standard Model gauge group, thus we eventually expect those separate parts to unify as a usual weak hypercharge $U(1)_Y$. The left-right symmetry provided by $SU(2)_L$ and $SU(2)_R$ restores the parity and this is one of the strong parts of the model. Pati-Salam Model is also consistent with the stability of proton. Another motivation for us to use Pati-Salam intermediate symmetry is the in recent discoveries about neutrino masses and mixings. The new symmetries breaking patterns finally ending with the Standard Model can be responsive in neutrino physics. In this way, Pati-Salam breaking chain provides neutrino Majorana masses and mixings in leptonic sector where we will be focusing in the next chapter. Yukawa terms and mixings in quark sector is also provided by the model. For the Grand Unification, we centered upon the exceptional group E_6, and Pati-Salam 's left right symmetric model fits well by being embedded in $SO(10)$ breaking chain.

5.3 Trinification

Another important GUT is called Trinification and it is based on the subgroups $SU(3) \times SU(3) \times SU(3)^c$, here the first two $SU(3)$ are taken to be $SU(3)_L$ and $SU(3)_R$ acting respectively on left handed and right handed quark components, the last one is the color group [1]. The symmetry breaking chain with the trinification in E_6 is important in a way that it is the only chain containing QCD as an explicit factor and the breaking occurs just in one step as

$$E_6 \longrightarrow SU(3)_L \times SU(3)_R \times SU(3)^c. \tag{5.25}$$

Fermions in this model again appears in three families and has the representations

$$(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}). \tag{5.26}$$

Quarks and their charge conjugates have the representations

$$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{\mathbf{2}}{\mathbf{3}}) \oplus \mathbf{2}(\bar{\mathbf{3}}, \mathbf{1}, \frac{\mathbf{1}}{\mathbf{3}}), \tag{5.27}$$

and the extra fields which are left handed have

$$(\mathbf{1}, \mathbf{2}, \frac{\mathbf{1}}{\mathbf{2}}) \oplus \mathbf{2}(\mathbf{1}, \mathbf{2}, -\frac{\mathbf{1}}{\mathbf{2}}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}) \oplus \mathbf{2}(\mathbf{1}, \mathbf{1}, \mathbf{0}). \tag{5.28}$$

This model also predicts right handed neutrinos and 12 additional new fermions in addition to the Standard Model. The Higgs field is expected to appear in $((\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}))$ or $(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3})$ representation and resulting the symmetry breaking of $SU(3)_L \times SU(3)_R \longrightarrow SU(2) \times U(1)$ thus getting the Standard Model gauge group again.

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Table 5.1: Weights and content of the **27** of E_6

Weight	Level	Color	Q^{em}	I_3^w	$SU(5)(SO(10))$	$SO(10)$ weight
(00010−1)	4	(00)	0	1/2	$\bar{\mathbf{5}}(\mathbf{16})$	(1−1010)
(−1001−10)	9	(00)	−1	−1/2	$\bar{\mathbf{5}}(\mathbf{16})$	(1000−1)
(1−11−100)	9	(00)	1	0	$\mathbf{10}(\mathbf{16})$	(−101−10)
(10−1001)	10	(00)	0	0	$\mathbf{1}(\mathbf{16})$	(−11−101)
(001−11−1)	5	(00)	1	1/2	$\mathbf{5}(\mathbf{10})$	(0−1100)
(−101−100)	10	(00)	0	−1/2	$\mathbf{5}(\mathbf{10})$	(001−1−1)
(01−1010)	6	(00)	0	1/2	$\bar{\mathbf{5}}(\mathbf{10})$	(00−111)
(−11−1001)	11	(00)	−1	−1/2	$\bar{\mathbf{5}}(\mathbf{10})$	(01−100)
(1−101−10)	8	(00)	0	0	$\mathbf{1}(\mathbf{1})$	(00000)
(100000)	0	(10)	2/3	1/2	$\mathbf{10}(\mathbf{16})$	(00001)
(1−10010)	7	(−11)	2/3	1/2	$\mathbf{10}(\mathbf{16})$	(−10010)
(10000−1)	11	(0−1)	2/3	1/2	$\mathbf{10}(\mathbf{16})$	(0−1001)
(0000−11)	5	(10)	−1/3	−1/2	$\mathbf{10}(\mathbf{16})$	(010−10)
(0−10001)	12	(−11)	−1/3	−1/2	$\mathbf{10}(\mathbf{16})$	(−1100−1)
(0000−10)	16	(0−1)	−1/3	−1/2	$\mathbf{10}(\mathbf{16})$	(000−10)
(−110000)	1	(10)	−1/3	0	$\mathbf{5}(\mathbf{10})$	(10000)
(−100010)	8	(−11)	−1/3	0	$\mathbf{5}(\mathbf{10})$	(0001−1)
(−11000−1)	12	(0−1)	−1/3	0	$\mathbf{5}(\mathbf{10})$	(1−1000)
(000−111)	4	(01)	1/3	0	$\bar{\mathbf{5}}(\mathbf{10})$	(−11000)
(010−100)	8	(1−1)	1/3	0	$\bar{\mathbf{5}}(\mathbf{10})$	(000−11)
(000−110)	15	(−10)	1/3	0	$\bar{\mathbf{5}}(\mathbf{10})$	(−10000)
(0−11000)	2	(01)	1/3	0	$\bar{\mathbf{5}}(\mathbf{16})$	(0010−1)
(0010−1−1)	6	(1−1)	1/3	0	$\bar{\mathbf{5}}(\mathbf{16})$	(1−11−10)
(0−1100−1)	13	(−10)	1/3	0	$\bar{\mathbf{5}}(\mathbf{16})$	(0−110−1)
(00−1101)	3	(01)	−2/3	0	$\mathbf{10}(\mathbf{16})$	(01−110)
(01−11−10)	7	(1−1)	−2/3	0	$\mathbf{10}(\mathbf{16})$	(10−101)
(00−1100)	14	(−10)	−2/3	0	$\mathbf{10}(\mathbf{16})$	(00−110)

Table 5.2: Physical roots and Axes in E_6 Weight Space

	Dynkin Basis	Dual Basis
	(000001)	[123212]
Color Roots	(0100−10)	[122101]
	(0−10011)	[001111]
Weak Isospin Root	(100001−1)	[111110]
Q^{em} axis	1/3(3−23−32−2)	1/3[212010]
I_3^w axis	1/2(10001−1)	1/2[111110]
Y^w axis	1/3(3−46−61−1)	1/3[1−11−3−10]
Q^t axis	3(1−101−10)	[1−101−10]
Q^r axis	(−3−141−1−4)	[−114110]
B axis	(−3c, 3c−a, 2a−b−c, 2b−a−c, 2c−b, −a)	[−c c a b c 0]
L axis	(−3d∓1, 2d−e±2, d+e∓1,e−2d∓1, 2d-e±2,d+e)	[−d, d±1, d+e, e, d±1, (

Chapter 6

NEUTRAL FERMIONS IN E_6 WITH INTERMEDIATE PATI- SALAM SYMMETRY

Arguably the best known GUT (Grand Unification Theory) is due to Georgi and Glashow [8] that is based on the block-diagonal embedding

$$U(2)_{EW} \times SU(3)_C \hookrightarrow SU(5).$$

$SU(5)$ is a rank-4 Lie group that supports a single (running) coupling constant. The grand unification may occur at extremely high energy scales at $\sim 10^{15} GeV$ [30]. The broken symmetry pattern at lower energy scales is achieved through a two step hierarchy

$$SU(5) \to U(2)_{EW} \times SU(3)_C \to U(1)_{QED} \times SU(3)_C. \tag{6.1}$$

The rank drops by one at the second step. This gives rise to a $U(1)_Y$ charge associated with the hypercharge quantum number Y. The ultimate price to pay for grand unification is the necessity of proton decay. Such lepto-quark processes would probe physics almost at the Planck scales. The present-day observational lower limit on the proton life-time independent on the decay mode stands at $\gtrsim 10^{34}$ years and poses a great challenge for future experiments [28, 36, 37, 39].

There are two other popular GUT's that historically followed $SU(5)$ soon after; one based on the rank-5 Lie group $SO(10)$ [9,38] and the other on rank-6 exceptional Lie group E_6 [1, 10, 11, 48]. The symmetry breaking hierarchy for $SO(10)$ GUT of Fritzsch and Minkowski [9] reads

$$SO(10) \to SU(5) \to U(2)_{EW} \times SU(3)_C \to U(1)_{QED} \times SU(3)_C. \tag{6.2}$$

The drop by one in rank at the final step is again associated with $U(1)_Y$ while the drop by one in rank at the first step is associated with a new $U(1)^t$ charge.
The symmetry breaking pattern for the Model I E_6 GUT of Gürsey, Ramond and Sikivie [1, 2, 12] on the other hand goes as

$$E_6 \to SO(10) \to SU(5) \to U(2)_{EW} \times SU(3)_C \to U(1)_{QED} \times SU(3)_C. \tag{6.3}$$

Here the first two steps involve two separate $U(1)$ charges. We note that both of these GUTs are anomaly-free and left-right symmetric.

A slightly different approach is adopted in one of the earliest grand unified models proposed by Pati and Salam [13,14,49,50]. It involves the following embedding of the colour symmetry

$$SU(3)_C \hookrightarrow SU(4)_C$$

and treats lepton number as a fourth colour. Accordingly a left-right symmetric extension of the SM is determined through the embedding

$$SU(3)_C \times SU(2)_L \times U(1)_Y \hookrightarrow SU(4)_C \times SU(2)_L \times SU(2)_R \cong SO(6) \times SO(4).$$

This alternative route will be referred to as Pati-Salam symmetry in what follows. If we now further embed $SO(10) \hookrightarrow E_6$, we consider the hierarchy

$$\begin{aligned} E_6 \to SO(10) \to SU(2)_L\times \quad SU(2)_R \quad & \times SU(4)_C \to U(2)_{EW} \times SU(3)_C \\ \to \qquad & U(1)_{QED} \times SU(3)_C. \end{aligned} \tag{6.4}$$

Here a basic fermion multiplet in each generation becomes $\{27\}$ of E_6 rather than $\{16\}$ of $SO(10)$, thus implying eleven extra new types of fermions [51–53]. Such a Grand Unified E_6 model with intermediate Pati-Salam symmetry thus combines the leptons and quarks in each generation into a single fundamental representation of E_6 and allows for interactions among them in such a way that the baryon number on its own is not conserved but the difference between the baryon and lepton numbers $(B-L)$ would be conserved [16, 33, 45, 54, 55]. The most recent work on grand unification based on non-commutative geometries provides further motivation that supports the Pati-Salam approach [56, 57].

In this chapter, we consider a Grand Unified E_6 model with intermediate Pati-Salam symmetry. In particular, we look at electrically neutral leptons (neutrinos) and study their masses and mixing within each family. In section 6.1, the assignment of fermion multiplets is discussed in general [49, 50, 56–58]. We make use of the computer program LieArt in Mathematica to work out the Lie algebraic weights and the roots of E_6, given the subgroup decomposition. The relevant projection operators for our symmetry breaking hierarchy are explicitly constructed. We also comment briefly on the spectrum of coloured quarks and charged leptons. In section:2, a mass matrix for neutral leptons is written down and its eigenvalues and eigenvectors are studied in second order perturbation theory. Furthermore certain bounds involving masses and mixing parameters are derived.

6.1 Fermion Assignments

In the E_6 model, bosons are assigned to the adjoint representation $\{78\}$ and fermions are assigned to the fundamental representation $\{27\}$. Therefore, in accordance with the irreducible decomposition

$$\{27\} \times \{27\} = \{\bar{27}\} + \{351\} + \{3\bar{5}1\}, \tag{6.5}$$

it is possible to assign the Higgs scalar bosons to the representation $\{\bar{27}\}$ of E_6. Our aim here is to determine the generic mass matrix of the fermions in $\{27\}$ by examining the invariant Yukawa couplings via the symmetry breaking chain

$$E_6 \longrightarrow SO(10) \times U(1)^t \longrightarrow SU(2)_L \times SU(2)_R \times SU(4) \longrightarrow$$
$$SU(2)_L \times SU(2)_R \times SU(3)_C \times U(1)_{B-L} \longrightarrow U(1)_Y \times SU(2)_I \times SU(3)_C.$$

To highlight the fermionic content predicted by our model, a good starting point would be the Cartan matrix of the GUT group E_6 for which we write down all the simple roots and can get the information we need for any representation we are interested in. We make use of the computer program *LieArt* [7] for the tensor decompositions, branching rules and some basic formulations of the E_6. The Cartan matrix of E_6 is explicitly written as

$$E_6 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}. \tag{6.6}$$

Each weight vector can be given as a linear combination of simple roots α_i as

$$\Lambda = \sum_i \bar{\Lambda}_i \frac{2}{(\alpha_i, \alpha_i)} \alpha_i$$

where $[\bar{\Lambda}_1, .., \bar{\Lambda}_l]$ gives the weight in the dual basis. Dynkin indices of a weight Λ are defined as

$$a_i = 2\frac{(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \sum_j \bar{\Lambda}_j \frac{2}{(\alpha_j, \alpha_j)} A_{ji}$$

in the Dynkin basis. Then every irreducible representation is uniquely identified by an ordered set of integers $(a_1, ..., a_l)$, and each such set is a highest weight of one and only one irreducible representation $[5, 23, 24]$. We are interested in the fundamental representation of E_6 where all the fermions in the SM and more appears in. We simply start by the highest weight and substract the simple roots and obtain the weight diagram for the representation. Highest weight for the $\{27\}$ of E_6 is (100000). We can see below the weight diagram generated by LieArt [7]:

WeightSystem[Irrep[[E6][1,0,0,0,0,0]]

$1\;0\;0\;0\;0\;0,\quad -1\;1\;0\;0\;0\;0,\quad 0\;-1\;1\;0\;0\;0,$
$0\;0\;-1\;1\;0\;1,\quad 0\;0\;0\;-1\;1\;1,\quad 0\;0\;0\;1\;0\;-1,\quad 0\;0\;0\;0\;-1\;1,$
$0\;0\;1\;-1\;1\;-1,\quad 0\;0\;1\;0\;-1\;-1,\; 0\;1\;-1\;0\;1\;0,\; 0\;1\;-1\;1\;-1\;0,$
$1\;-1\;0\;0\;1\;0, -1\;0\;0\;0\;1\;0, 0\;1\;0\;-1\;0\;0,\; 1\;-1\;0\;1\;-1\;0,$
$-1\;0\;0\;1\;-1\;0,\; 1\;-1\;1\;-1\;0\;0,\; -1\;0\;1\;-1\;0\;0,\; 1\;0\;-1\;0\;0\;1,$
$-1\;1\;-1\;0\;0\;1,\; 1\;0\;0\;0\;0\;-1,\; -1\;1\;0\;0\;0\;-1,\; 0\;-1\;0\;0\;0\;1,$
$0\;-1\;1\;0\;0\;-1,\quad 0\;0\;-1\;1\;0\;0,\quad 0\;0\;0\;-1\;1\;0,\quad 0\;0\;0\;0\;-1\;0$
$0\;-1\;1\;0\;0\;-1,\quad 0\;0\;-1\;1\;0\;0,\quad 0\;0\;0\;-1\;1\;0,\quad 0\;0\;0\;0\;-1\;0$

Since we are interested in physical content, we should look for color and flavor embeddings in the model. The Dynkin indices can be converted into eigenvalues of a set of diagonal generators with

$$Q(\Lambda) = \sum_i \bar{Q}_i a_i.$$

The electric charge operator measured in this way for E_6 is

$$Q^{EM} = \frac{1}{3}[212010].$$

We should now concentrate on the subgroup structure and the related symmetry breaking chain to gather a detailed information about the particle content. E_6 GUT has physical interest when $SU(3)_C$ is contained as a subgroup. In this work we concentrate on the chain where $SU(4)$ contains $SU(3)_C$. We follow the projection of highest weight of the representation to the highest weight it branches to. Thus we work with different projection matrices at each step of the symmetry breaking chain. For the chain $E_6 \supset SO(10) \supset SU(2)_L \times SU(2)_R \times SU(4)$ where $SU(4) \supset SU(3)_C$, we have the following projection matrices:

$$P_1(E_6 \supset SO(10)) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where P_1 projects the weights to the subgroup $SO(10)$, and

$$P_{13}(SO(10) \supset SU(2)_L \times SU(2)_R \times SU(4)) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix},$$

where P_{13} projects the weights in $SO(10)$ to the weights in the $SU(2)_L \times SU(2)_R \times SU(4)$ representation, and

$$P_5(SU(4) \supset SU(3)_C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where P_5 projects the weights in $SU(4)$ to the weights in $SU(3)_C$. At every step of symmetry breaking chain we get different $U(1)$ charges. We project $SO(10)$ weight to the subgroup $SU(2)_L \times SU(2)_R \times SU(4)$ and the first Dynkin index in the projected weight $(a_1 a_2 a_3 a_4 a_5)$ gives the weak isospin $I_3^L = a_1/2$, the second Dynkin index gives $I_3^R = a_2/2$ and the remaining indices $(a_3 a_4 a_5)$ fix the weight in $SU(4)$. Then we take this $SU(4)$ weight and project it once more relative to the subgroup $SU(3)_C$ and get the color weight in $SU(3)_C$. Therefore, the modified Gell-Mann-Nishijima formula that gives the electric charge reads

$$Q^{EM} = I_3^L + I_3^R + \frac{1}{2}(B - L). \tag{6.7}$$

(A) The labelling and the physical interpretation of the corresponding (first generation) neutral fermions are as follows:

- ν_e is the left-handed electron neutrino with quantum numbers $I_3 = 1/2, Y = -1, B - L = -1$,

- N_e^C is the charge-conjugate of right-handed electron neutrino with quantum numbers $I_3 = 0, Y = 0, B - L = 1$,

- ν_E is the left-handed exotic neutrino with quantum numbers $I_3 = 1/2, Y = -1, B - L = 0$,

- N_E^C is the charge conjugate of right-handed exotic neutrino with quantum numbers $I_3 = -1/2, Y = 1, B - L = 0$,

- ν_S is the left-handed sterile neutrino with quantum numbers $I_3 = 0, Y = 0, B - L = 0$.

Quantum Numbers of Neutral Leptons						
Particle Assignment	Q^t	I_3^L	I_3^R	$(B-L)$	Y	Q^{EM}
ν_e	1	1/2	0	-1	-1	0
N_e^C	1	0	-1/2	1	0	0
N_E^C	-2	-1/2	1/2	0	1	0
ν_E	-2	1/2	-1/2	0	-1	0
ν_S	4	0	0	0	0	0

Table 6.1: $U(1)$-charges for first generation neutral leptons in left-right symmetric symmetry breaking chain in E_6.

Assuming that $\{\bar{27}\}$ dominates the Higgs boson sector, we now look for particles with charges in $\{\bar{27}\}$ representations that come with minus signs, by looking at Table 6.1. We write the charges in $\{\bar{27}\}$ in $(\nu_e, N_e^C, \nu_E, N_E^C, \nu_S)$ basis as [41, 59]:

	ν_e (1,−1,−1)	N_e^c (1,1,0)	ν_E (−2,0,−1)	N_E^c (−2,0,1)	ν_S
ν_e (1,−1,−1)		(2,0,−1)	(−1,−1,−2)	(−1,−1,0)	
N_e^c (1,1,0)	(2,0,−1)			(−1,1,1)	
ν_E (−2,0,−1)	(−1,−1,−2)			(−4,0,0)	(2,0,−1)
N_E^c (−2,0,1)	(−1,−1,0)	(−1,1,1)	(−4,0,0)		(2,0,1)
ν_S			(2,0,−1)	(2,0,1)	

Empty entries above mean that the corresponding charges do not occur in the $\{\bar{27}\}$ representation*.

(B) The labelling and the physical interpretation of the corresponding (first generation) colour-triplet quarks are as follows:

- $\mathbf{u}_L$ is the left-handed part of up-quarks, with quantum numbers $I_3 = 1/2, Y = 1/3, B-L = 1/3$,

- $\mathbf{d}_L$ is the left-handed part of down-quarks, with quantum numbers $I_3 = -1/2, Y = 1/3, B-L = 1/3$,

*The alternative symmetry breaking chain with intermediate $SU(5)$ symmetry instead of the Pati-Salam symmetry leads to a mass matrix with the same non-vanishing entries as above except the entry $m_{13} \sim (-1,-1,-2)$ that vanishes. This is the case considered in Rosner [59].

- $\mathbf{D}_R^C$ is the charge conjugate of right-handed part of exotic quarks, with quantum numbers $I_3 = 0, Y = 2/3, B - L = 2/3$,

- $\mathbf{u}_R^C$ is the charge conjugate of right-handed part of up-quarks, with quantum numbers $I_3 = 0, Y = -1/3, B - L = -1/3$,

- $\mathbf{d}_R^C$ is the charge conjugate of right-handed part of down-quarks, with quantum numbers $I_3 = 0, Y = 2/3, B - L = 1/3$,

- $\mathbf{D}_L$ is the left-handed part of exotic quarks, with quantum numbers $I_3 = 0, Y = -2/3, B - L = 2/3$.

Quantum Numbers of Color Triplet Quarks						
Particle Assignment	Q^t	I_3^L	I_3^R	$(B-L)$	Y	Q^{EM}
$\mathbf{u}_L$	1	$1/2$	0	$1/3$	$1/3$	$2/3$
$\mathbf{d}_L$	1	$-1/2$	0	$1/3$	$1/3$	$-1/3$
$\mathbf{D}_R^C$	-2	0	0	$2/3$	$2/3$	$1/3$
$\mathbf{d}_R^C$	1	0	$1/2$	$-1/3$	$2/3$	$1/3$
$\mathbf{u}_R^C$	1	0	$-1/2$	$-1/3$	$-4/3$	$-2/3$
$\mathbf{D}_L$	-2	0	0	$-2/3$	$-2/3$	$-1/3$

Table 6.2: $U(1)$-charges for first generation, color triplet quarks in left-right symmetric symmetry breaking chain in E_6.

We also write down the allowed charges in $\{\bar{27}\}$, relative to the above basis of quarks as

	$\mathbf{u}_L$ (1,1,1/3)	$\mathbf{d}_L$ (1,−1/3,−4/3)	$\mathbf{D}_R^C$ (1,−1/3,1/3)	$\mathbf{d}_L^C$ (−1,1.1/3)	$\mathbf{u}_L^C$ (−2,2/3,2/3)	$\mathbf{D}_L$ (−2,−2/3,−2/3)
$\mathbf{u}_L$ (1,1,1/3)					(2,0,−1)	
$\mathbf{d}_L$ (1,−1/3,−4/3)			(−1,1,1)	(2,0,,1)		
$\mathbf{D}_R^C$ (1,−1/3,1/3)		(−1,1,1)			(−1,−1,−2)	(−4,0,0)
$\mathbf{d}_R^C$ (−1,1.1/3)		(2,0,1)				(−1,−1,0)
$\mathbf{u}_R^C$ (−2,2/3,2/3)	(2,0,−1)		(−1,−1,−2)			
$\mathbf{D}_L$ (−2,−2/3,−2/3)			(−4,0,0)	(−1,−1,0)		

(C) The labelling and the physical interpretation of the corresponding (first generation) charged leptons are as follows:

- e_L is the left-handed part of electron, with quantum numbers $I_3 = 1/2, Y = -1, B - L = -1$,

- e_R^C is the charge conjugate of right-handed part of electron, with quantum numbers $I_3 = 0, Y = 2, B - L = 1$,

- E_L is the left-handed part of exotic lepton with quantum numbers $I_3 = 1/2, Y = -, B - L = 0$,

- E_R^C is the charge conjugate of right-handed part of exotic lepton with quantum numbers $I_3 = -1/2, Y = -1, B - L = 0$.

Quantum Numbers of Charged Leptons						
Particle Assignment	Q^t	I_3^L	I_3^R	$(B-L)$	Y	Q^{EM}
e_L	1	−1/2	0	−1	−1	−1
e_R^C	1	0	1/2	1	2	1
E_L	−2	−1/2	−1/2	0	−1	−1
E_R	−2	1/2	1/2	0	1	1

Table 6.3: $U(1)$-charges for electrically charged leptons in left-right symmetric symmetry breaking chain in E_6.

	$e(1,-1,-1)$	$e^c(1,1,2)$	$E(-2,0,-1)$	$E^c(-2,0,1)$
$e(1,-1,-1)$	–	$(2,0,1)$	$(-1,-1,-2)$	$(-1,-1,0)$
$e^c(1,1,2)$	$(2,0,1)$	–	$(-1,1,1)$	–
$E(-2,0,-1)$	$(-1,-1,-2)$	$(-1,1,1)$	–	$(-4,0,0)$
$E^c(-2,0,1)$	$(-1,-1,0)$	–	$(-4,0,0)$	–

The allowed charges in $\{\bar{27}\}$, relative to the basis of charged leptons given above will be

6.2 Masses and Mixing of Neutral Leptons

The spectrum of electrically neutral fermion fields in our E_6-model consists of three generations of (i) active left-chiral and right-chiral neutrinos [44], (ii) heavy exotic (Dirac) neutrinos [60–63], and (iii) light sterile (Majorana) neutrinos [59, 64]. We won't be dealing with mixings among different generations in the present paper. We introduce a minimal seesaw mechanism below to separate out the observable left-chiral active neutrinos from the right-chiral active neutrinos which remain unobservable at current energy scales. Such a separation in mass is not predicted by the E_6 models, however, we follow a common practice [65] and implement a minimal seesaw mechanism by hand in order to remain consistent with current neutrino observations. We associate large Dirac masses with exotic neutrinos. The sterile neutrino masses could be small or even may be set to zero at first approximation to remain in favour with recent inflationary big bang cosmology scenarios, [66–74].

In order to set the notation in general, let us consider a left-handed, 2-component complex spinor field ψ_L and a right-handed, 2-component complex spinor field ψ_R that transform as $(1/2,0)$ and $(0,1/2)$ irreducible representations of the Lorentz group, respectively. Their charge conjugate fields $\psi_L^C = i\sigma_2\psi_L^*$ and $\psi_R^C = i\sigma_2\psi_R^*$ carry opposite chiralities. A 4-component Dirac spinor field can be given by

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

thus transforming under $(1/2.0) \oplus (0,1/2)$. A Majorana spinor field is self-charge conjugate and satisfies $\psi_R^C = \psi_L$ (so that $\psi_L^C = \psi_R$ as well).
We consider the following formal expression for the Lagrangian density of a Dirac fermion:

$$\begin{aligned} \mathcal{L}_{Fermion} &= Her(i\bar{\Psi}(\gamma \cdot \nabla)\Psi) + i\bar{\Psi}\mathcal{M}\Psi \\ &= Her(i\bar{\psi}_L(\sigma \cdot \nabla)\psi_L) + Her(i\bar{\Psi}_R^C(\sigma \cdot \nabla)\psi_R^C) \\ &\quad + iM_L\bar{\psi}_L\psi_L + iM_R\bar{\psi}_R^C\psi_R^C + iM\left(\bar{\psi}_L\psi_R^C + \bar{\psi}_R^C\psi_L\right) \end{aligned}$$

where M_L, M_R are two Majorana masses and M is a Dirac mass. By convention, all the 2-spinors that appear in the Lagrangian density are taken as left-handed. If ψ_L describes a left-handed neutrino field, then ψ_R^C describes an (independent) left-handed anti-neutrino field. Then we write down by inspection the following 2×2 mass matrix

$$\mathcal{M} = \begin{pmatrix} M_L & M \\ M & M_R \end{pmatrix}.$$

Keeping with the same conventions as above, we describe all the masses and mixing of neutral fermions in our E_6-model by the following 5×5 real symmetric matrix:

$$\mathcal{M} = \begin{pmatrix} 0 & m & m_{13} & m_{14} & 0 \\ m & M & 0 & m_{24} & 0 \\ m_{13} & 0 & 0 & M' & m_{35} \\ m_{14} & m_{24} & M' & 0 & m_{45} \\ 0 & 0 & m_{35} & m_{45} & -m'' \end{pmatrix}. \tag{6.8}$$

We have introduced three mass parameters m, M', m'' and one minimal seesaw parameter M. Here and in what follows, lower case m's correspond to "small" masses while the upper case M's correspond to "large" masses. All the mass parameters are assumed positive. Furthermore there are five mixing parameters $m_{13}, m_{14}, m_{24}, m_{35}, m_{45}$. Let us remark here that if the symmetry breaking chain involves an intermediate $SU(5)$ symmetry, then the mass/mixing matrix cannot support the entry m_{13} . Then it would be set to zero above which is the case studied by Rosner [59]. In our problem though, the symmetry breaking chain passes through an intermediate stage with Pati-Salam symmetry so that $m_{13} \neq 0$.

A diagonalisation of the above matrix to determine the exact mass eigenvalues doesn't seem feasible. Nevertheless we demand that the signs and magnitudes of the free parameters should be so chosen that there will be two real positive and three real negative eigenvalues, with one positive eigenvalue and one negative eigenvalue being equal to each other in absolute value. In such a case, all the masses assigned to the neutral fermions would be real and positive. On the other hand, since we do not have sufficient clue concerning the magnitude of the exotic and/or sterile neutrino masses and their mixings, no numerical estimates would be possible. Then , we do the best we can and evaluate below some perturbative expressions for the mass eigenvalues. The starting point for our model building process will be the lowest order neutrino mass matrix

$$\mathcal{M}_0 = \begin{pmatrix} 0 & m & 0 & 0 & 0 \\ m & M & 0 & 0 & 0 \\ 0 & 0 & 0 & M' & 0 \\ 0 & 0 & M' & 0 & 0 \\ 0 & 0 & 0 & 0 & -m'' \end{pmatrix}.$$

The upper left 2×2 block refers to active neutrinos where m is a Dirac mass and M is introduced to induce a minimal seesaw mechanism. The middle 2×2 block refers to an exotic fermion whose Dirac mass is M'. The lower right corner refers to a single, self-conjugate sterile neutrino for which we introduce a Majorana mass m''. The eigenvalues of $\mathcal{M}_0$ are found to be

$$M_+ = \frac{M}{2} + \sqrt{(\frac{M}{2})^2 + m^2}, \quad -m_- = \frac{M}{2} - \sqrt{(\frac{M}{2})^2 + m^2}, \quad M', \quad -M', \quad -m'',$$

with the corresponding eigenvectors

$$|1>= \frac{1}{\sqrt{M_+^2 + m^2}} \begin{pmatrix} m \\ M_+ \\ 0 \\ 0 \\ 0 \end{pmatrix}, |2>= \frac{1}{\sqrt{m_-^2 + m^2}} \begin{pmatrix} m \\ -m_- \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$|3>= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, |4>= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, |5>= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In first approximation, for $M >> m$, we have

$$M_+ \cong M, \quad m_- \cong \frac{m^2}{M}$$

with the corresponding eigenvectors

$$\frac{1}{\sqrt{M^2 + m^2}} \begin{pmatrix} m \\ M \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{(\frac{m^2}{M})^2 + m^2}} \begin{pmatrix} m \\ -\frac{m^2}{M} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the mass of the active right-handed neutrino $\sim M$, could be "large" so that it is a heavy neutrino; while the mass of the active left-handed neutrino $\sim \frac{m^2}{M}$, remains "small" so that it is a light neutrino.

We regard the mixings as perturbation on the zeroth order mass matrix given above:

$$\mathcal{M}_1 = \mathcal{M} - \mathcal{M}_0.$$

Then we calculate the following second order perturbative corrections to the mass eigenvalues:

$$\begin{aligned}
\Delta M_+ &= -\frac{|<1|\mathcal{M}_1|3>|^2}{M_+ - M'} - \frac{|<1|\mathcal{M}_1|4>|^2}{M_+ + M'}, \\
\Delta m_- &= -\frac{|<2|\mathcal{M}_1|3>|^2}{-m_- - M'} - \frac{|<2|\mathcal{M}_1|4>|^2}{-m_- + M'}, \\
\Delta M'_+ &= -\frac{|<3|\mathcal{M}_1|1>|^2}{M' - M_+} - \frac{|<3|\mathcal{M}_1|2>|^2}{M' + m_-}, \\
\Delta M'_- &= -\frac{|<4|\mathcal{M}_1|1>|^2}{-M' - M_+} - \frac{|<4|\mathcal{M}_1|2>|^2}{-M' + m_-}, \\
\Delta m'' &= -\frac{|<5|\mathcal{M}_1|3>|^2}{-M'} - \frac{|<5|\mathcal{M}_1|4>|^2}{M'},
\end{aligned}$$

where the relevant matrix elements turn out to be

$$<1|\mathcal{M}_1|3>= \frac{m(m_{13}+m_{14}) + M_+ m_{24}}{\sqrt{2(m^2+M_+^2)}}, \quad <1|\mathcal{M}_1|4>= \frac{m(m_{13}-m_{14}) - M_+ m_{24}}{\sqrt{2(m^2+M_+^2)}};$$

$$<2|\mathcal{M}_1|3>= \frac{m(m_{13}+m_{14}) - m_- m_{24}}{\sqrt{2(m^2+m_-^2)}}, \quad <2|\mathcal{M}_1|4>= \frac{m(m_{13}-m_{14}) + m_- m_{24}}{\sqrt{2(m^2+m_-^2)}};$$

$$<3|\mathcal{M}_1|5>= \frac{m_{35}+m_{45}}{\sqrt{2}}, \quad <4|\mathcal{M}_1|5>= \frac{m_{35}-m_{45}}{\sqrt{2}}.$$

Then, for instance, the sterile neutrino mass, assuming $m''=0$ to zeroth order in perturbation theory, is given by

$$m_S \cong \frac{2|m_{35}m_{45}|}{M'}, \tag{6.9}$$

so that it is inversely proportional to the exotic neutrino mass. Its magnitude is controlled by the mixing of exotic and sterile neutrinos.

If we consider the mass matrix appearing with the $U(1)$ charges from the symmetry breaking chain which does not include the term M on the diagonal and since it has no exact solution, we can follow different methods to get approximate values. One of our main assumptions is that the exotic mass $M' \equiv M_{34}$ is much larger than the others. Then we can diagonalise the mass matrix with respect to it by a Jacobi transformation

$$J_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & -s & 0 \\ 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and obtain the matrix $\mathcal{M}' = J_{34}^T \mathcal{M} J_{34}$ as

$$\mathcal{M}' = \begin{pmatrix} 0 & m_{12} & (m_{13}+m_{14})/\sqrt{2} & (m_{14}-m_{13})/\sqrt{2} & 0 \\ m_{12} & 0 & m_{24}/\sqrt{2} & m_{24}/\sqrt{2} & 0 \\ (m_{13}+m_{14})/\sqrt{2} & m_{24}/\sqrt{2} & M_{34} & 0 & (m_{35}+m_{45})/\sqrt{2} \\ (m_{14}-m_{13})/\sqrt{2} & m_{24}/\sqrt{2} & 0 & -M_{34} & (m_{45}-m_{35})7\sqrt{2} \\ 0 & 0 & (m_{35}+m_{45})/\sqrt{2} & (m_{45}-m_{35})/\sqrt{2} & 0 \end{pmatrix}.$$

To constrain the eigenvalues we can use Gersgorin Discs theorem [75] which states that, if $\mathcal{M}' \in Mat(n, \mathbf{R})$ we can always write $\mathcal{M}' = \mathcal{D}+\mathcal{E}$ where $\mathcal{D} = diag(a_{11}, \cdots, a_{nn})$ is just the main diagonal part of $\mathcal{M}'$ and $\mathcal{E}$ has zero main diagonal, then

$$|a_{ii}| > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}| \quad \text{for at least one value of i=1,..,n,}$$

and every eigenvalue of matrix $\mathcal{M}'$ satisfies,

$$|\lambda - a_{ii}| \geq R_i^{'} = \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}| \quad \text{for all i=1,..,n,}$$

where $\sum R_i^{'} = \sum_{j\neq i} |a_{ij}|$ represents the absolute values of non-diagonal elements in the ith row.

Thus for matrix $\mathcal{M}'$ we have the eigenvalues constarined according to,

$$\begin{aligned} |\lambda_1| &\geq m_{12} + \frac{(m_{14}+m_{13})}{\sqrt{2}} + \frac{|m_{14}-m_{13}|}{\sqrt{2}}, \\ |\lambda_2| &\geq m_{12} + \frac{2m_{24}}{\sqrt{2}}, \\ |\lambda_3 - M_{34}| &\geq \frac{(m_{14}+m_{13})}{\sqrt{2}} + \frac{m_{24}}{\sqrt{2}} + \frac{(m_{35}+m_{45})}{\sqrt{2}}, \\ |\lambda_4 + M_{34}| &\geq \frac{|m_{14}-m_{13}|}{\sqrt{2}} + \frac{m_{24}}{\sqrt{2}} + \frac{|m_{45}-m_{35}|}{\sqrt{2}}, \\ |\lambda_5| &\geq \frac{(m_{35}+m_{45})}{\sqrt{2}} + \frac{|m_{45}-m_{35}|}{\sqrt{2}}. \end{aligned}$$

Another method is to use perturbation theorem for the eigenvalues of the matrix [75]. Letting $\mathcal{D} = diag(a_{11}, \cdots, a_{nn}) \in Mat(n, \mathbf{R})$ and $\mathcal{E} = [e_{ij}] \in Mat(n, \mathbf{R})$, consider the perturbed matrix $\mathcal{D}+\mathcal{E}$. Then the eigenvalues of $\mathcal{D}+\mathcal{E}$ are contained in the discs,

$$\{z \in C : |z - \lambda - e_{ii}| \leq R_i^{'}(\mathcal{E}) = \sum_{\substack{j\neq i\\ j=1}}^{n} |e_{ij}|\} \quad i = 1, \cdots, n,$$

which are contained in the discs,

$$\{z \in C : |z - \lambda_i| \leq R_i^{'}(\mathcal{E}) = \sum_{j=1}^{n} |e_{ij}|\}.$$

Thus if $\hat{\lambda}$ is an eigenvalue of $\mathcal{D}+\mathcal{E}$, there is some eigenvalue λ_i such that,

$$|\hat{\lambda}-\lambda_i| \leq |||\mathcal{E}|||_\infty.$$

Here the matrix norm

$$\begin{aligned}
|||\mathcal{E}|||_\infty &= max\{m_{12}+\sqrt{2}m_{14}, m_{12}+\sqrt{2}m_{24}, \\
&\quad \frac{m_{14}}{\sqrt{2}}+\frac{m_{24}}{\sqrt{2}}+\frac{(m_{45}+m_{35})}{\sqrt{2}}, \\
&\quad \frac{m_{14}}{\sqrt{2}}+\frac{m_{24}}{\sqrt{2}}+\frac{(m_{45}-m_{35})}{\sqrt{2}}, \sqrt{2}m_{45}\}.
\end{aligned}$$

These two methods taken together would constrain the mass eigenvalues to certain intervals determined in terms of the strengths of mixing parameters. In the absence of numerical data, we are not able to carry the analysis further.

Chapter 7

CONCLUSION

We have considered an E_6 grand unified model with intermediate Pati-Salam symmetry where the gauge bosons live in the adjoint representation $\{78\}$ of E_6 while each single generation of fermions is assigned to a copy of the fundamental representation $\{27\}$. We take the Higgs bosons in $\{\bar{27}\}$ and in accord with the assumed symmetry breaking chain work out all the physical, conserved $U(1)$ charges of the fermions. Furthermore the generic Yukawa couplings allowed by our symmetry breaking chain are also pointed out. We concentrate our attention in particular on the electrically neutral fermion sector, neglecting here any horizontal mixing between different generations. First of all we have active neutrinos of both chiralities. In order to differentiate between the masses of the left-handed and right-handed active neutrinos, a minimal seesaw mechanism is implemented by hand. Although such a mechanism wouldn't be allowed by our underlying GUT symmetry, phenomenologically heavy right-handed active neutrinos are not observed, unlike the physical light left-handed active neutrinos. We further have a heavy exotic (Dirac) neutrino and a light (or even massless to the lowest order of approximation) sterile (Majorana) neutrino. Then we write down a 5×5 real symmetric matrix that defines the masses and the mixing of these neutral leptons. We first discuss tree level masses by turning off all the mixing terms. We note that when the mixings are turned on, the active and sterile neutrinos do not mix, while both of these types can mix with the exotic neutrinos on their own. We have shown up to second order perturbation approximation that the sterile neutrino mass would be inversely proportional to the exotic neutrino mass, and directly proportional to the product of the sterile-exotic mixing coefficients. The effects of the mixing of active and exotic neutrinos on the sterile neutrino mass are expected to show up at higher order perturbations [76]. By the plausible assumption that exotic neutrinos would be the heaviest among all neutrino types, we also consider an expansion of the mass matrix around its maximum eigenvalue, taking the maximum to be the exotic neutrino mass M_{34}. Then we apply the Gersgorin discs theorem that allows us to put constraints on actual mass eigenvalues in terms of the mixing parameters. The neutral lepton masses and mixings in a E_6 GUT with intermediate $SU(5)$ symmetry has been analysed before by Rosner [59]. The 5×5 matrix of masses and mixing we derive here differ from Rosner's expression by the non-vanishing mixing parameter $m_{13} \neq 0$. This particular term enters into all our mass eigenvalue expressions and also appears in the constraint conditions we found. Yet we have not been able to pinpoint a physical process that would distinguish between the two different symmetry breaking patterns. We note that the same single non-vanishing mixing parameter that differentiates between the two symmetry breaking patterns also appears in the mass and mixing matrices of the colour triplet quarks and the charged leptons.

The bosonic sector of SM consists of 8 (massless) gluons, 3 heavy weak intermediate bosons and one massless photon. Together with a single real scalar Higgs boson, this adds up to 28 physical bosonic degrees of freedom. The fermionic sector of SM includes 3 generations each of charged leptons, chiral neutrinos and iso-doublets of 3-coloured quarks. This adds up to 15 physical, fermionic degrees of freedom for each generation. In the $SO(10)$ GUTs, fermions are assigned to the fundamental representation $\{16\}$. Therefore to postulate the existence of a single heavy right handed neutrino over the SM fermion spectrum will be sufficient to fill up the whole multiplet [69]. On the other hand in the E_6 GUTs fermions are assigned to $\{27\}$, so that one needs to postulate further the existence of heavy exotic fermions that amounts to 11 extra fermionic degrees of freedom for each generation. In the bosonic sector, on the other hand, the adjoint representation $\{45\}$ of $SO(10)$ asks for 18 extra intermediate vector bosons to be postulated while the adjoint representation $\{78\}$ of E_6 requires 51 new gauge bosons. Thus, both of these models require one to postulate the existence of large numbers of gauge bosons that would mediate lepto-quark interactions. The fact that the fermionic sector of $SO(10)$ GUTs are economical in that respect over the E_6 GUTs does not provide sufficient reason on its own to justify not to go beyond $SO(10)$ in a grand unification theory. Strong support for E_6 GUTs come from string theory motivated grand unification models [77–80]. For instance, effective heterotic string field theory models in 10-dimensions should contain either one of $SO(32)$ or $E_8 \times E_8$ gauge symmetry groups for anomaly cancellation. It is suggestive to consider the well-known chain of Coxeter-Dynkin diagrams for the E-series in the Cartan classification of complex semi-simple Lie algebras:

$$E_8 \to E_7 \to E_6 \to E_5 \cong SO(10) \to E_4 \cong SU(5) \to SU(4) \to SU(3).$$

If we identify the E_8 at the upper end of this tower with one of the E_8's in heterotic string theory and identify the $SU(3)$ at the lower end of the tower with the unbroken colour symmetry of elementary particles,then it is natural not to stop at an $SO(10)$ GUT and move over to E_6 GUTs towards E_8 [71].

The electroweak unification occurs at energy scales of $\sim 10^2 GeV$ at which the local (Abelian) gauge symmetry of QED is diagonally embedded into a larger (non-Abelian) gauge symmetry group:

$$U(1)_{QED} \hookrightarrow U(2)_{EW} \cong U(1)_Y \times SU(2)_I.$$

The strong interactions on the other hand are described in terms of quarks and gluons in QCD based on a symmetry group

$$SU(3)_C \times G_F.$$

The flavour group G_F accommodates a global horizontal symmetry among the quark generations. The local colour symmetry group $SU(3)_C$ generates asymptotically free forces which might explain the observed confinement of colour. This is a hypothesis yet to be theoretically verified. While going over to GUTs, moving up in the hierarchy steps at ever increasing energy scales, the unbroken colour symmetry $SU(3)_C$ is always preserved. Therefore two big issues in grand unification schemes remain: (i) to find room for three and only three generations of fermions and (ii) to verify colour confinement hypothesis that is central to the success of QCD. $SO(10)$ GUTs are more economical in numbers but on both of the essential issues above E_6 GUTs seem more suggestive. An algebraic approach to colour confinement problem had been proposed long ago by Gürsey [12]. The fact that the structure of all exceptional Lie algebras including E_6 depends on the non-associtative properties of octonions is well-known [6]. The 26 dimensional exceptional Jordan algebra of Hermitian 3×3 matrices over complex octonions plays a unique role in the algebraic approaches to grand unification. There are other related early work along these lines [81], [74], yet this is still an open research direction.

BIBLIOGRAPHY

[1] F. Gürsey, P. Ramond, and P. Sikivie. A universal gauge theory model based on e6. *Physics Letters B*, 60(2):177–180, 1976.

[2] F. Gürsey and M. Serdaroĝlu. E6 gauge field theory model revisited. *Il Nuovo Cimento A (1965-1970)*, 65(3):337–354, 1981.

[3] Wen-long Lin. Octonions and exceptional groups. *Chinese Journal of Physics*, 30(4):579–597, 1992.

[4] J. Baez. The octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205, 2002.

[5] H. Georgi. *Lie Algebras In Particle Physics: from Isospin To Unified Theories (Frontiers in Physics)*. Westview Press, 1999.

[6] C. Manogue and T. Dray. Octonions, e6, and particle physics. In *Journal of Physics: Conference Series*, volume 254, page 012005. IOP Publishing, 2010.

[7] R. Feger and T. W. Kephart. Lieart a mathematica application for lie algebras and representation theory. *Computer Physics Communications*, 192:166–195, 2015.

[8] H. Georgi and S.L. Glashow. Unity of all elementary-particle forces. *Physical Review Letters*, 32(8):438–441, 1974.

[9] H. Fritzsch and P. Minkowski. Unified interactions of leptons and hadrons. *Annals of Physics*, 93(1):193–266, 1975.

[10] R. Barbieri and D.V. Nanopoulos. An exceptional model for grand unification. *Physics Letters B*, 91(3-4):369–375, 1980.

[11] Y. Achiman and B. Stech. Quark-lepton symmetry and mass scales in an e6 unified gauge model. *Physics Letters B*, 77(4-5):389–393, 1978.

[12] F. Gürsey and M. Serdaroğlu. Basic fermion masses and mixings in the e_6 model. *Lettere Al Nuovo Cimento (1971–1985)*, 21(1):28–32, 1978.

[13] J.C. Pati, S. Rajpoot, and A. Salam. Neutral-current phenomena within the left-right-symmetric unified theory of quarks and leptons. *Physical Review D*, 17(1):131, 1978.

[14] J.C. Pati and A. Salam. Lepton number as the fourth" color". *Physical Review D*, 10(1):275, 1974.

[15] S. Okubo. Representations and classifications of compact gauge groups for unified weak and electromagnetic interactions. *Physical Review D*, 18(10):3792, 1978.

[16] R.N. Mohapatra and R.E. Marshak. Local b- l symmetry of electroweak interactions, majorana neutrinos, and neutron oscillations. *Physical Review Letters*, 44(20):1316, 1980.

[17] S Cecotti, J-P Derendinger, S Ferrara, L Girardello, and M Roncadelli. Properties of e6 breaking and superstring theory. *Physics Letters B*, 156(5-6):318–326, 1985.

[18] D. London and J.L. Rosner. Extra gauge bosons in e 6. *Physical Review D*, 34(5):1530, 1986.

[19] RN Mohapatra. Neutrino mass physics at lhc. *Nuclear Physics B-Proceedings Supplements*, 229:393–399, 2012.

[20] J.F. Adams. *Lectures on exceptional Lie groups*. University of Chicago Press, 1996.

[21] JF Cornwell. *Group Theory in Physics*, volume 1. Academic Press London, 1984.

[22] Steinkirch M.V. *Introduction to Group Theory for Physicist*. State University of Stony Brook, 2011.

[23] P. Ramond. *Group theory: a physicist's survey*. Cambridge University Press, 2010.

[24] R. Slansky. Group theory for unified model building. *Physics Reports*, 79(1):1–128, 1981.

[25] F. Costa. *Symmetries and Group Theory in Particle Physics*. Springer, 2012.

[26] Ling-Fong Li. Group theory of the spontaneously broken gauge symmetries. *Physical Review D*, 9(6):1723, 1974.

[27] P Ramond. Introduction to exceptional lie groups and algebras. *Cal Tech. CALT-68*, 577:58, 1976.

[28] M. Gell-Mann, P. Ramond, and R. Slansky. Color embeddings, charge assignments, and proton stability in unified gauge theories. *Reviews of Modern Physics*, 50(4):721, 1978.

[29] G. Ross. Grand unified theories, frontiers in physics. 1984.

[30] H Georgi, H.R. Quinn, and S. Weinberg. Hierarchy of interactions in unified gauge theories. *Physical Review Letters*, 33(7):451, 1974.

[31] B. Bajc. *Grand unification and proton decay.* samozal, 2014.

[32] A. Greljo. Towards unification: Su (5) and so (10). 2012.

[33] S. Weinberg. Baryon-and lepton-nonconserving processes. *Physical Review Letters*, 43(21):1566, 1979.

[34] M.C. Brak. The hierarchy problem in the standard model and little higgs theories. *Master's thesis, NIKHEF*, 2004.

[35] I. Nišandzic. Su (5) and so (10) unification. 2011.

[36] J.C. Pati. Discovery of proton decay: a must for theory, a challenge for experiment. In *AIP Conference Proceedings*, volume 533, pages 37–53. AIP, 2000.

[37] G. Senjanović. Proton decay and grand unification. In *AIP Conference Proceedings*, volume 1200, pages 131–141. AIP, 2010.

[38] H. Georgi. The state of the art gauge theories. In *AIP Conference Proceedings*, volume 23, pages 575–582. AIP, 1975.

[39] P. Langacker. Grand unified theories and proton decay. *Physics Reports*, 72(4):185–385, 1981.

[40] S. Coleman and E. Weinberg. Radiative corrections as the origin of spontaneous symmetry breaking. *Physical Review D*, 7(6):1888, 1973.

[41] R.W. Robinett and J.L. Rosner. Mass scales in grand unified theories. *Physical Review D*, 26(9):2396, 1982.

[42] W.G. Mckay, J. Patera, and D. Sannikoff. *The computation of branching rules for representations of semisimple Lie algebras.* Computers in Nonassociative Rings and Algebras, Editors RE Beck and B. Kolman, New York, Academic Press, 1977.

[43] E.B. Dynkin. Semisimple subalgebras of semisimple lie algebras. *Selected Papers*, pages 175–308, 1972.

[44] M. Drewes. The phenomenology of right handed neutrinos. *International Journal of Modern Physics E*, 22(08):1330019, 2013.

[45] E.A. Paschos, U. Sarkar, and H. So. Baryon and lepton number assignment in e 6 models. *Physical Review D*, 52(3):1701, 1995.

[46] S.L. Chen and E. Ma. Exotic fermions and bosons in the quartification model. *Modern Physics Letters A*, 19(17):1267–1271, 2004.

[47] R. Slansky. Fun with e6. In *Recent Developments in High-Energy Physics*, pages 141–164. Springer, 1980.

[48] Q Shafi. E6 as a unifying gauge symmetry. *Physics Letters B*, 79(3):301–303, 1978.

[49] R.N. Mohapatra and J.C. Pati. " natural" left-right symmetry. *Physical Review D*, 11(9):2558, 1975.

[50] Gt Senjanovic and R.N. Mohapatra. Exact left-right symmetry and spontaneous violation of parity. *Physical Review D*, 12(5):1502, 1975.

[51] H. Ruegg and T. Schucker. Masses for fermions in a unified gauge model based on e6. *Nuclear Physics B*, 161(2-3):388–396, 1979.

[52] H. Georgi and D.V. Nanopoulos. Masses and mixing in unified theories. *Nuclear Physics B*, 159(1-2):16–28, 1979.

[53] R. Barbieri and D.V. Nanopoulos. Hierarchical fermion masses from grand unification. *Physics Letters B*, 95(1):43–46, 1980.

[54] J. Harada. Hypercharge and baryon minus lepton number in e6. *Journal of High Energy Physics*, 2003(04):011, 2003.

[55] R.N. Mohapatra. From old symmetries to new symmetries: quarks, leptons and bl. *International Journal of Modern Physics A*, 29(29):1430066, 2014.

[56] A. H. Chamseddine, A. Connes, and W.D. van Suijlekom. Beyond the spectral standard model: emergence of pati-salam unification. *arXiv preprint arXiv:1304.8050*, 2013.

[57] A.H. Chamseddine, A. Connes, and W.D. van Suijlekom. Grand unification in the spectral pati-salam model. *arXiv preprint arXiv:1507.08161*, 2015.

[58] M. Yoshimura. Yet another symmetry breaking to be discovered. *Progress of Theoretical and Experimental Physics*, 2016(7):07B103, 2016.

[59] J.L. Rosner. Three sterile neutrinos in e 6. *Physical Review D*, 90(3):035005, 2014.

[60] Y Tosa and Robert Eugene Marshak. Exotic fermions. *Physical Review D*, 32(3):774, 1985.

[61] T.G. Rizzo. Exotic fermions in e6 models and $\mu \to e + \gamma$. *Physical Review D*, 34(7):2160, 1986.

[62] T.G. Rizzo. Phenomenology of exotic particles in e 6 theories. *Physical Review D*, 34(5):1438, 1986.

[63] R.P. Feger and T.W. Kephart. Grand unification and exotic fermions. *Physical Review D*, 92(3):035005, 2015.

[64] Fornengo N. Gariazzo S. Giunti C. Hannestad S. Archidiacono, M. and M. Laveder. Light sterile neutrinos after bicep-2. *Journal of Cosmology and Astroparticle Physics*, 2014(06):031, 2014.

[65] E. Ma. Neutrino masses in an extended gauge model with e 6 particle content. *Physics Letters B*, 380(3):286–290, 1996.

[66] S.F. King and C. Luhn. Neutrino mass and mixing with discrete symmetry. *Reports on Progress in Physics*, 76(5):056201, 2013.

[67] G. Nemevšek, M.and Senjanović and V. Tello. Connecting dirac and majorana neutrino mass matrices in the minimal left-right symmetric model. *Physical review letters*, 110(15):151802, 2013.

[68] E. Ma. Pathways to naturally small neutrino masses. *Physical Review Letters*, 81(6):1171, 1998.

[69] J.C. Pati. Advantages of unity with su (4)-color: Reflections through neutrino oscillations, baryogenesis and proton decay. *International Journal of Modern Physics A*, 32(09):1741013, 2017.

[70] Z. Kakushadze and S-H. Tye. Classification of three-family grand unification in string theory. i. the so (10) and e 6 models. *Physical Review D*, 55(12):7878, 1997.

[71] S. Davis. The reduction of the exceptional groups of string theory and the standard model. *Quantum Physics Letters*, 4(1):1, 2015.

[72] F. Gürsey. Octonionic structures in particle physics. *Group Theoretical Methods in Physics*, pages 508–521, 1979.

[73] L. C. Biedenharn and P. Truini. Exceptional groups and elementary particle structures. *Physica A: Statistical Mechanics and its Applications*, 114(1-3):257–270, 1982.

[74] R. Kerner. Ternary algebraic structures and their applications in physics. *arXiv preprint math-ph/0011023*, 2000.

[75] Johnson C. Horn. R. *Matrix Analysis*. Cambridge University Press, 1985.

[76] C. Giunti. Light sterile neutrinos: status and perspectives. *Nuclear Physics B*, 908:336–353, 2016.

[77] J.A. Harvey. Patterns of symmetry breaking in the exceptional groups. *Nuclear Physics B*, 163:254–260, 1980.

[78] D. Lüst and S. Theisen. Exceptional groups in string theory. *International Journal of Modern Physics A*, 4(17):4513–4533, 1989.

[79] N. Arkani-Hamed and Y. Grossman. Light active and sterile neutrinos from compositeness. *Physics Letters B*, 459(1):179–182, 1999.

[80] S. Ito, M.and Kuwakino, N. Maekawa, S. Moriyama, K. Takahashi, K. Takei, S. Teraguchi, and T. Yamashita. E 6 grand unified theory with three generations from heterotic string theory. *Physical Review D*, 83(9):091703, 2011.

[81] L. Biedenharn and L. Horwitz. Non-associative algebras and exceptional gauge groups. *Differential Geometric Methods in Mathematical Physics*, pages 152–166, 1981.