

Large-Scale and Nonconvex Eigenvalue Optimization

by

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Large-Scale and Nonconvex Eigenvalue Optimization

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to my family

ABSTRACT

We deal with the optimization of a prescribed eigenvalue of a Hermitian matrix that depends on several parameters analytically. This setting is motivated by classical problems such as a linear semidefinite program under a constant trace assumption, shape optimization and structural design problems, robust stability measures in control theory. Our primary interest is in the problems involving large Hermitian matrices, in particular reducing their sizes by means of orthogonal projections onto appropriate subspaces. But some attention is also paid to the nonconvex nature of these problems.

We start by devising subspace frameworks that yield good subspaces for orthogonal projections. At every step, a projected eigenvalue optimization problem is solved, then the subspaces are expanded so that Hermite interpolation properties are attained between the projected problem and the original large-scale problem at the optimizer of the projected problem. Convergence analyses are conducted in the infinite dimensional setting when the eigenvalues of a self-adjoint compact operator dependent on parameters are optimized, specifically (i) convergence to a global minimizer is proven for the minimization of the J th largest eigenvalue as the subspace dimension goes to infinity, (ii) a superlinear rate-of-convergence result is deduced in the smooth setting with respect to the subspace dimension. We observe the dramatic effects of the latter result in practice; problems on the order of tens of thousands are approximated accurately with projected problems in terms of matrices of sizes on the order of tens.

The second part is devoted to rate-of-convergence analyses in the nonsmooth setting when the eigenvalue is not simple at the optimal parameter value. The analyses here are restricted to the minimization of the largest eigenvalue of a Hermitian matrix dependent on one parameter. We establish a quadratic rate-of-convergence for a gen-

eralized version of the subspace framework from the first part. Additionally, for an algorithm to solve the projected problems that is guaranteed to converge to globally optimal solutions and that is based on piece-wise quadratic model functions, a rapid convergence result is proven in this nonsmooth setting. The rate-of-convergence results are illustrated on several nonsmooth examples arising from the computation of the inner numerical radius, the modulus of the point on the boundary of the field of values closest to the origin.

The third and the final part concerns the minimax problems with the largest eigenvalue in the objective. We tailor a globally convergent algorithm for small problems, and a subspace framework inspired by the ones from the first part for large-scale problems. Once again we prove the rapid convergence of the subspace framework. The theoretical results proven are confirmed in practice on an application, namely the minimization of the numerical radius, which is the modulus of the point in the field of values furthest away from the origin.

Key words: Eigenvalue optimization, large scale, orthogonal projection, eigenvalue perturbation theory, parameter dependent compact operator, matrix-valued function, nonsmooth optimization, global optimization, rate-of-convergence, subspace framework, field of values, numerical radius, definite matrix pairs, inner numerical radius, generalized eigenvalue problem, minimax optimization.

ÖZETÇE

Bu tez çalışmasında bir takım parametrelere analitik olarak bağlı Hermit bir matrisin belirtilen bir özdeğerinin optimizasyonu üzerine yoğunlaşıyoruz. Bu yoğunlaştığımız çerçeve, sabit iz varsayımı altında doğrusal yarı belirgin programlama, şekil optimizasyonu ve yapısal tasarım problemleri, kontrol teorisindeki güçlü dengelik ölçekleri gibi klasik uygulamalar tarafından motive ediliyor. Ana olarak ilgilendiğimiz bu tip büyük Hermit matrisler içeren özdeğer optimizasyonu problemleri, özellikle uygun alt uzaylara dik izdüşümler sayesinde bahsi geçen büyük Hermit matrislerinin boyutlarının indirgenmesi. Ama tez çalışmasında ek olarak bu özdeğer problemlerinin konveks olmayan doğaları ile de ilgileniyoruz.

Çalışmaya iyi alt uzaylar üreten alt uzay çerçeveleri geliştirerek başlıyoruz. Her adımda dik izdüşümle elde edilen indirgenmiş ufak bir özdeğer optimizasyonu problemi çözülüyor, ardından alt uzaylar indirgenmiş problem ile orijinal büyük çaplı problem arasında indirgenmiş problemi optimize eden parametre değerlerinde bir takım Hermit özellikleri sağlanacak şekilde genişletiliyor. Alt uzay çerçeveleri için yakınsama analizini sonsuz boyutta, yani bu çerçevelerin parametrelere bağlı kendine eşlenik kompakt bir operatörün özdeğerlerinin optimizasyonu için uygulandığı durumda, gerçekleştiriyoruz. Bu sonsuz boyuttaki analizde (i) çerçevelerin J inci en büyük özdeğerin minimizasyonu için uygulandığında boyut sonsuza yaklaştıkça global olarak en iyi çözüme yakınsadığını, (ii) yakınsama hızının alt uzayın boyutuna göre ve türevlenebilir durumda en az süper-doğrusal olduğunu kanıtlıyoruz. İkinci hızlı yakınsama sonucunun pratikteki etkisi önemli; büyüklüğü onbinler mertebesinde olan matrisler içeren özdeğer optimizasyonu problemleri, büyüklüğü onlar mertebesinde matrisler içeren indirgenmiş özdeğer optimizasyonu problemleri tarafından yaklaşık olarak ama yüksek doğruluk ile temsil edilebiliyor.

Çalışmanın ikinci kısmını türevlenemeyen durumda, yani optimal parametre değerlerinde özdeğerin çoklu katsayıya sahip olduğu durumda, yakınsama hızı analizlerine adıyoruz. Buradaki analizler özel olarak tek bir parametreye bağlı bir Hermit matris fonksiyonunun en büyük özdeğerinin minimizasyonu problemi üzerinde gerçekleştiriliyor. İlk kısımdaki alt uzay çerçevelerinin genelleştirilmiş bir hali için kuadratik yakınsama sonucunu kanıtlıyoruz. Ayrıca, yine bu türevlenemeyen durumda, indirgenmiş problemlerin global çözümü için kullanılan ve parçalı kuadratik model fonksiyon tabanlı yöntemin hızlı yakınsadığını gösteriyoruz. Elde edilen teorik yakınsama hızı sonuçlarının, pratikte iç sayısal yarıçap hesabından kaynaklanan bir takım türevlenemeyen örnekler üzerinde sağlandığını gözlemliyoruz.

En son olarak en büyük özdeğer fonksiyonunun amaç fonksiyonunda belirttiği minimum-maksimum türü optimizasyon problemlerinin üzerine yoğunlaşıyoruz. Ufak problemler için global olarak en iyi çözüme yakınsayan bir yöntem, ve büyük çaplı problemler için de, ilk kısımdaki çerçevelerden de ilham alarak, alt uzay çerçeveleri tasarlıyoruz. Tasarlanan alt uzay çerçevelerinin yine hızlı yakınsadığını kanıtlıyoruz. Bu teorik sonuçları pratikte sayısal yarıçapın minimizasyonu problemi üzerinde teyit ediyoruz.

Anahtar Kelimeler: Özdeğer optimizasyonu, büyük çaplı, dik izdüşüm, özdeğerlerin saptırılma teorisi, parametrelere bağlı kompakt operatör, matris değerli fonksiyon, türevlenemeyen optimizasyon, global optimizasyon, yakınsama hızı, alt uzay çerçevesi, değerler cismi, sayısal yarıçap, belirgin matris çiftleri, iç sayısal yarıçap, genelleştirilmiş özdeğer problemi, minimum-maksimum optimizasyonu.

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Chapter 1

INTRODUCTION

Eigenvalue optimization has been an active field of research especially since 1970s. The term refers to optimization problems where a prescribed eigenvalue of a matrix dependent on parameters appears either in the objective or in the constraints. Majority of the research so far has focused on the case when the matrix is Hermitian for all permissible parameter values, and the dependence of the Hermitian matrix on the parameters is smooth. Such is the case in the general framework we consider in this thesis work, where we deal with the optimization problems involving the eigenvalues of

$$A(\omega) = f_1(\omega)A_1 + \dots + f_\kappa(\omega)A_\kappa \quad (1.1)$$

over the parameters ω . The matrices $A_1, \dots, A_\kappa \in \mathbb{C}^{n \times n}$ are assumed to be Hermitian, and the functions $f_1, \dots, f_\kappa : \tilde{\Omega} \rightarrow \mathbb{R}$ are real-analytic on their domain $\tilde{\Omega}$, an open subset of $\mathbb{R}^d$. This framework is quite general; for instance any Hermitian matrix-valued function whose entries depend on the variable ω analytically can be written in the form (1.1) by substituting $E_{kj} = 1/2(e_k e_j^* + e_j e_k^*)$, $k = 1, \dots, n$, $j = k, \dots, n$ for $A_1, \dots, A_\kappa$, where $\{e_1, \dots, e_n\}$ denotes the standard basis for $\mathbb{C}^n$.

A well-known special case is a linear semidefinite program (SDP) subject to a constant trace assumption on the primal matrix variable, whose dual can be expressed as a minimization problem in terms of the largest eigenvalue of an affine matrix-valued function $B(\omega) = B_0 + \sum_{j=1}^d \omega_j B_j$ for prescribed Hermitian matrices $B_0, B_1, \dots, B_d$. A linear SDP and its eigenvalue optimization characterization are convex problems. Yet many eigenvalue optimization problems by their nature are not convex, for instance

the minimization of the largest eigenvalue unless $A(\omega)$ is affine, which carries significance in structural design, and various robust stability measures arising in control theory are nonconvex.

This chapter is devoted to a brief background on the subject, as well as a literature survey, motivation and a statement of the scope of this thesis work. We conclude the chapter with an outline of the thesis and its contributions.

1.1 Background

The algorithms that we derive throughout this thesis work for eigenvalue optimization, as well as their convergence analyses, make use of classical results from eigenvalue perturbation theory. Consequently, the next two subsections are devoted to a brief summary of these results that will be referred to in the subsequent sections. We conclude this background section with a presentation of a few applications that give rise to eigenvalue optimization problems in Section 1.1.3.

1.1.1 Variational Characterizations of Eigenvalues of Hermitian Matrices

The eigenvalues $\lambda_1, \dots, \lambda_n$ of a Hermitian matrix $A \in \mathbb{C}^{n \times n}$ are real, whereas the corresponding eigenvectors $v_1, \dots, v_n$ are mutually orthogonal. The following result known as the *Courant-Fischer Theorem* [36] is a consequence of the orthogonality of the eigenvectors; it provides maximin and minimax characterizations for the eigenvalues of A .

Theorem 1.1.1 (Courant-Fischer). *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian with the eigenvalues $\lambda_n \leq \dots \leq \lambda_1$. We have*

$$\lambda_j = \max_{\substack{\mathcal{V} \subseteq \mathbb{C}^n \\ \dim \mathcal{V} = j}} \min_{\substack{x \in \mathcal{V} \\ \|x\|_2 = 1}} x^* A x = \min_{\substack{\mathcal{V} \subseteq \mathbb{C}^n \\ \dim \mathcal{V} = n-j+1}} \max_{\substack{x \in \mathcal{V} \\ \|x\|_2 = 1}} x^* A x$$

for $j = 1, \dots, n$, where the outer maximization and minimization problems are over all subspaces $\mathcal{V}$ of $\mathbb{C}^n$ of dimension j and $n - j + 1$, respectively.

1.1.2 Analyticity of the Eigenvalues of a Hermitian Matrix Dependent on Parameters

Consider an analytic and Hermitian matrix-valued function $A(\omega) : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ dependent on one real parameter. Each of the eigenvalues $\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_n(\omega)$ (note that this listing is not necessarily in a sorted order) of $A(\omega)$ is a root of its characteristic polynomial of the form

$$p(\lambda; \omega) := \alpha_n(\omega)\lambda^n + \dots + \alpha_1(\omega)\lambda + \alpha_0(\omega),$$

where $\alpha_0(\omega), \dots, \alpha_n(\omega)$ have power series representations about 0. As $p(\omega, \tilde{\lambda}_j(\omega)) = 0$, it follows from Puiseux's Theorem [82] that $\tilde{\lambda}_j(\omega)$ has a Puiseux series expansion of the form

$$\tilde{\lambda}_j(\omega) = \sum_{k=0}^{\infty} \beta_k \omega^{k/m}$$

where m is the multiplicity of $\tilde{\lambda}_j(0)$. Suppose $\ell \in \mathbb{N}$ is such that $\beta_0 = \dots = \beta_{\ell-1} = 0$ and $\beta_\ell \neq 0$. Then we have

$$\lim_{\omega \rightarrow 0^+} \frac{\tilde{\lambda}_j(\omega) - \tilde{\lambda}_j(0)}{\omega^{\ell/m}} = \beta_\ell,$$

which implies β_ℓ is real, since $\tilde{\lambda}_j(\omega)$ and $\omega^{\ell/m}$ are real. Furthermore,

$$\lim_{\omega \rightarrow 0^-} \frac{\tilde{\lambda}_j(\omega) - \tilde{\lambda}_j(0)}{(-1)^{\ell/m} \omega^{\ell/m}} = (-1)^{\ell/m} \beta_\ell$$

implying that $(-1)^{\ell/m} \beta_\ell$, as well as $(-1)^{\ell/m}$, must be real. Consequently, ℓ/m must be an integer. This observation leads us to the conclusion that $\tilde{\lambda}_j(\omega)$ can in fact be written as a power series of ω about 0. More precisely, there exist real coefficients c_k such that

$$\tilde{\lambda}_j(\omega) = \sum_{k=0}^{\infty} c_k \omega^k$$

which establishes the real analyticity of $\tilde{\lambda}_j(\omega)$ at 0. The next results states the real analyticity of the eigenvalues of $A(\omega)$ formally; we refer to the books [72, 41] for more details.

Theorem 1.1.2 (Rellich). *Let $A(\omega) : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ be a Hermitian matrix-valued function depending on ω analytically. There exist real analytic functions $\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_n(\omega)$ such that $\{\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_n(\omega)\}$ is precisely the set of eigenvalues of $A(\omega)$ at all ω .*

In the multivariate case, that is when $A(\omega) : \mathbb{R}^d \rightarrow \mathbb{C}^{n \times n}$, there exist continuous functions $\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_n(\omega)$ such that $\{\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_n(\omega)\}$ corresponds to the spectrum of $A(\omega)$ at all ω (as the roots of the characteristic polynomial of $A(\omega)$ varies continuously with respect to ω). Additionally, if $\tilde{\lambda}_j(\tilde{\omega})$ is simple at a given $\tilde{\omega} \in \mathbb{R}^d$, then an application of the implicit function theorem shows that $\tilde{\lambda}_j(\omega)$ is real analytic at $\tilde{\omega}$. On the other hand, in contrast to the univariate case, this real analyticity property may no longer hold in the multivariate case if $\tilde{\lambda}_j(\tilde{\omega})$ is not simple.

Throughout this thesis work, we will often be referring to the eigenvalues of $A(\omega)$ (in the univariate or in the multivariate case) in the sorted order $\lambda_1(\omega) \geq \dots \geq \lambda_n(\omega)$, which are continuous with respect to ω . If the j th largest eigenvalue $\lambda_j(\tilde{\omega})$ is simple at a prescribed $\tilde{\omega} \in \mathbb{R}^d$, then $\lambda_j(\omega) = \tilde{\lambda}_\ell(\omega)$ for all ω in a neighborhood of $\tilde{\omega}$ for some $\ell \in \{1, \dots, n\}$, and the function $\tilde{\lambda}_\ell(\omega)$ that is real analytic at $\tilde{\omega}$ as in the previous paragraphs. Hence, $\lambda_j(\omega)$ is real analytic at $\tilde{\omega}$. The real analyticity, even the differentiability, of $\lambda_j(\omega)$ at $\tilde{\omega}$ may be lost if $\lambda_j(\tilde{\omega})$ is not simple; consider for instance the univariate setting in which case $\lambda_j(\omega)$ is equal to a real analytic function in a left neighborhood of $\tilde{\omega}$, yet typically equal to a different real analytic function in a right neighborhood of $\tilde{\omega}$.

Some of the theoretical results in this thesis work employ the analytical formulas for the derivatives of the eigenvalue functions. Assuming that the j th largest eigenvalue $\lambda_j(\omega)$ of $A(\omega)$ is simple at $\tilde{\omega}$, which ensures the real analyticity of $\lambda_j(\omega)$ at $\tilde{\omega}$, we have

$$\begin{aligned} \frac{d\lambda_j(\tilde{\omega})}{d\omega} &= v_j^* \frac{dA(\tilde{\omega})}{d\omega} v_j, \\ \frac{d^2\lambda_j(\tilde{\omega})}{d\omega^2} &= v_j^* \frac{d^2A(\tilde{\omega})}{d\omega^2} v_j + 2 \left\{ \sum_{m=1, m \neq j}^n \frac{1}{\lambda_j(\tilde{\omega}) - \lambda_m(\tilde{\omega})} \left| v_j^* \frac{dA(\tilde{\omega})}{d\omega^2} v_m \right|^2 \right\} \end{aligned} \quad (1.2)$$

in the univariate case, and

$$\begin{aligned} \frac{\partial \lambda_j(\tilde{\omega})}{\partial \omega_k} &= v_j^* \frac{\partial A(\tilde{\omega})}{\partial \omega_k} v_j \\ \frac{\partial^2 \lambda_j(\tilde{\omega})}{\partial \omega_k \partial \omega_\ell} &= v_j^* \frac{\partial^2 A(\tilde{\omega})}{\partial \omega_k \partial \omega_\ell} v_j \\ &\quad + 2\Re \left\{ \sum_{m=1, m \neq j}^n \frac{1}{\lambda_j(\tilde{\omega}) - \lambda_m(\tilde{\omega})} \left(v_j^* \frac{\partial A(\tilde{\omega})}{\partial \omega_k} v_m \right) \left(v_m^* \frac{\partial A(\tilde{\omega})}{\partial \omega_\ell} v_j \right) \right\} \end{aligned} \quad (1.3)$$

in the multivariate case, where v_j, v_m are unit eigenvectors corresponding to $\lambda_j(\tilde{\omega}), \lambda_m(\tilde{\omega})$.

1.1.3 Applications of Eigenvalue Optimization

We now list some applications of eigenvalue optimization problems with which we perform numerical experiments and illustrate the theoretical results deduced throughout this thesis work.

Quantities Related to Field of Values: For a given $A \in \mathbb{C}^{n \times n}$, Hermitian or non-Hermitian, the field of values of A is defined by

$$F(A) := \{z^* A z \mid z \in \mathbb{C}^n, \ z^* z = 1\}.$$

It is a compact convex set containing the eigenvalues of A . Geometrically, it may be viewed as the image of the unit ball in $\mathbb{C}^n$ under the transformation $z \mapsto z^* A z$. If the matrix A is Hermitian, then the field of values lies on the real axis, indeed $F(A) = [\lambda_{\min}(A), \lambda_{\max}(A)]$, where $\lambda_{\min}(A), \lambda_{\max}(A)$ denote the smallest, largest eigenvalues of A , respectively. Two important quantities related to the field of values are the numerical radius $r(A)$ of A , the modulus of the outermost point in $F(A)$, and the inner numerical radius $\zeta(A)$ of A , the modulus of the point on the boundary of $F(A)$ closest to origin. Formally, the former is defined by [37, 31]

$$r(A) := \max\{|\omega| \mid \omega \in F(A)\},$$

whereas the latter is defined by

$$\zeta(A) := \min\{|\omega| \mid \omega \text{ is on the boundary of } F(A)\}.$$

The numerical radius is used to analyze the convergence of the classical iterative schemes for linear systems [2, 23]. It has the following eigenvalue optimization characterization:

$$r(A) = \max_{\omega \in [0, 2\pi]} \lambda_{\max}(A(\omega)), \quad \text{where } A(\omega) = \frac{Ae^{i\omega} + A^*e^{-i\omega}}{2}.$$

The inner numerical radius carries significance for the numerical solution of the generalized eigenvalue problem [14, 19], checking hyperbolicity of a Hermitian quadratic eigenvalue problem [35] and iterative solutions of a saddle point linear system [52]. Its eigenvalue optimization characterization [14] is given by

$$\zeta(A) = \left| \min_{\omega \in [0, 2\pi]} \lambda_{\max}(A(\omega)) \right|, \quad \text{where } A(\omega) = \frac{Ae^{-i\omega} + A^*e^{i\omega}}{2}.$$

Distance to Instability: The autonomous system

$$x'(t) = Ax(t), \tag{1.4}$$

for a given $A \in \mathbb{C}^{n \times n}$, is called asymptotically stable if and only if $\lim_{t \rightarrow \infty} x(t) = 0$ for all initial conditions, or equivalently all eigenvalues of A lie in the open left-half of the complex plane. Asymptotic stability is an essential property. However, even when the system is asymptotically stable, small perturbations of A may cause some of the eigenvalues of A move onto the imaginary axis resulting in a nearby system that is not asymptotically stable. This is especially a concern in the presence of uncertainty in the entries of A . In such cases, the robust stability of the system, that is the asymptotic stability of the system as well as all nearby systems, becomes an important issue. A measure of the robust stability is given by the distance to instability

$$\mathcal{D}(A) := \min\{\|\Delta\|_2 \mid \exists \omega \in \mathbb{R} \text{ such that } \det(A + \Delta - \omega i I) = 0\},$$

where $\|\cdot\|_2$ denotes the matrix 2-norm. An application of the Eckart-Young theorem [27, Theorem 2.5.3] yields the characterization

$$\mathcal{D}(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(A - \omega i I),$$

with $\sigma_{\min}(\cdot)$ denoting the smallest singular value of its matrix argument.

$\mathcal{H}_\infty$ -Norm: Now let us focus on the linear time-invariant system (LTI)

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.5}$$

with the initial condition $x(0) = 0$, where $u(t)$ and $y(t)$ are the input and output functions, respectively, $x(t)$ is the state function, whereas $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{p \times n}$ with $m, p \leq n$ are the system matrices. Taking the Laplace transformation of (1.5) yields the algebraic equations

$$\begin{aligned} X(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s), \end{aligned} \tag{1.6}$$

expressed in terms of $X(s)$, $U(s)$ and $Y(s)$, the Laplace transforms of the functions $x(t)$, $u(t)$ and $y(t)$, respectively. Finally, at s such that $sI - A$ is invertible, by eliminating $X(s)$ in (1.6), we obtain the relation

$$Y(s) = \{C(sI - A)^{-1}B\} U(s)$$

between the input and the output in the Laplace domain. Here the matrix valued function $H(s) := C(sI - A)^{-1}B$ is referred as the transfer function of the system in (1.5). The $\mathcal{H}_\infty$ -norm of this transfer function is then defined by

$$\|H\|_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(H(i\omega)),$$

and is a measure of the robust stability of (1.5) when A, B, C are subject to perturbations. Here and throughout this thesis work, $\sigma_{\max}(\cdot)$ stands for the largest singular value of its matrix argument.

1.2 Literature, Scope and Motivation

Eigenvalue optimization has a history dating back centuries ago. In the 18th century Lagrange considered designing the strongest column with a prescribed volume

with respect to the cross-sectional areas of the column [68], [50]. He ended up with an optimization problem involving the maximization of the smallest eigenvalue of a fourth order differential operator over a functional space. Analytical solutions for the problem with various boundary conditions caused controversial research for decades [78], [65], [62], [3]. Modern numerical treatments of this structural design problem involved maximization of the smallest eigenvalue of symmetric matrix functions [65], [16] arising from finite-element discretizations.

On a different direction graph partitioning problems in graph theory were turned into minimization of the sum of prescribed number of largest eigenvalues [21], [17]. These graph-partitioning problems are indeed NP-hard; the eigenvalue minimization characterizations are relaxations whose solutions are approximate solutions with a known upper bound on their error. These relaxations as well as applications from control theory motivated a focused research in 1980s for minimizing the sum of prescribed number of largest eigenvalues of matrix-valued functions. Earlier works [26], [67] restricted themselves to the case when the minimization involves a matrix-valued function whose dependence on the parameters is affine, in which case the problem is convex. After the emergence of the interior point methods, the attention has turned into the general nonconvex case [69], [74]. But only local analyses have been performed, and algorithms converging to local minimizers have been suggested.

A shift of paradigm occurred in the late 1980s. In 1984, the interior point methods were introduced by Karmarkar for solutions of linear programs [40], and later extended by Nesterov and Nemirovski [64] to convex programs including linear semidefinite programs (SDPs). A linear SDP is a convex optimization problem that concerns the minimization of a linear objective over a space of symmetric matrices subject to positive semidefiniteness constraints on symmetric matrices. The minimization of the largest eigenvalue, or the maximization of the smallest eigenvalue when the matrix-valued function is an affine function can be posed as an SDP. Therefore, the interior point methods became the standard practice for the solution of these eigenvalue optimization problems when the involved matrix-valued function is affine. Indeed, it became

popular to pose an eigenvalue optimization problem as a linear SDP whenever this is possible.

Yet many eigenvalue optimization problems by their nature are not convex, and cannot be posed as or approximated with a linear SDP. For instance, the maximization of the smallest eigenvalue when the matrix-valued function involved is not affine, and various robust stability measures arising in control theory are nonconvex eigenvalue optimization problems.

1.2.1 Scope

Nearly all of the eigenvalue optimization problems discussed so far (with the exception of $\mathcal{H}_\infty$ -norm computation) can be formulated into

$$\max_{\omega \in \Omega} \lambda_J(\omega) \quad \text{or} \quad \min_{\omega \in \Omega} \lambda_J(\omega), \quad (1.7)$$

where $\lambda_J(\omega)$ denotes the J th largest eigenvalue of a given matrix-valued function $A(\omega)$ of the form (1.1), and the set Ω is a compact subset of $\tilde{\Omega} \subseteq \mathbb{R}^d$ over which $A(\omega)$ in (1.1) is defined. It is well-known that the latter minimization problem in (1.7) is convex in the special case when $J = 1$ and $A(\omega)$ is affine, i.e., it is of the form $A(\omega) := B_0 + \sum_{j=1}^d \omega_j B_j$ for prescribed Hermitian matrices $B_0, B_1, \dots, B_d$. But beyond this special case, the optimization problems in (1.7) are typically nonconvex. Furthermore, as discussed in Section 1.1.2, the J th largest eigenvalue $\lambda_J(\omega)$ is not smooth at an ω where its multiplicity is greater than one. Thus, generally speaking, the problems of the form (1.7) which we deal with throughout this thesis work are nonsmooth and nonconvex optimization problems.

1.2.2 Algorithms for Eigenvalue Optimization

For some of the nonconvex eigenvalue optimization problems, specialized globally convergent algorithms have been derived, but it does not seem possible to generalize them. This is in particular the case for the three applications listed in Section 1.1.3, all of which lead to nonconvex eigenvalue optimization problems. Level-set

algorithms guaranteed to converge to globally optimal solutions of these nonconvex optimization problems have been developed. More specifically, Byers has developed a bisection method for the computation of the distance to instability [13]. Byers' bisection algorithm keeps a lower bound ℓ , and an upper bound u for the distance to instability of a given matrix $A \in \mathbb{C}^{n \times n}$. It checks whether the α level-set of the singular value function $f(\omega) = \sigma_{\min}(A - \omega iI)$ for $\alpha = (\ell + u)/2$ is empty or not by solving a Hamiltonian eigenvalue problem of size double the size of A . Then it updates either the lower or the upper bound depending on whether the level-set is empty or not. The $\mathcal{H}_\infty$ -norm of an LTI system can be viewed as a generalized distance to instability, i.e., the distance to instability of A corresponds to the $\mathcal{H}_\infty$ -norm of the transfer function of the system in (1.5) with $E = I_n$ and $B = C = 0$. Boyd and Balakrishnan [7], as well as Bruinsma and Steinbuch [9], have designed a quadratically convergent algorithm for $\mathcal{H}_\infty$ -norm computation by exploiting Byers' level-set idea. This algorithm at every iteration determines the α level-set of the singular function $g(\omega) = \sigma_{\max}(C(i\omega I - A)^{-1}B)$ for an α which is a lower bound and an estimate for the $\mathcal{H}_\infty$ -norm of the system under consideration; this level-set determination again requires the extraction of the imaginary eigenvalues of a Hamiltonian matrix double the size of A . It then finds the intervals in which $g(\omega) > \alpha$ using the α level-set, and refines α to the largest value attained by $g(\omega)$ over the midpoints of these intervals. For the computation of the numerical radius, He and Watson [31] have incorporated a level-set idea into an algorithm that converges to a locally maximal value of the associated nonconvex eigenvalue function. Inspired by the Boyd-Balakrishnan algorithm for $\mathcal{H}_\infty$ -norm computation, He-Watson algorithm is later refined into a quadratically convergent algorithm by Mengi and Overton [58]. Since all of these level-set methods are based on the computation of the eigenvalues of a Hamiltonian matrix or a matrix pencil of size twice the size of the original problem, they are not suitable for problems beyond medium scale, even for problems involving matrices of sizes on the order of thousands.

There are also a few general purpose algorithms that are applicable to eigenvalue

optimization problems, including those that can be expressed of the form (1.7). Bundle methods are widely used for nonsmooth optimization, in particular for eigenvalue optimization. They are based on piece-wise linear model functions for the nonsmooth objective function [43, 54, 49], but they are most effective for convex problems. Their effectiveness in the nonsmooth setting is limited even to find a locally optimal solution. More recently, it is observed that BFGS with line searches ensuring the satisfaction of the weak Wolfe conditions typically converges to a local optimizer of a nonsmooth objective function, for instance an eigenvalue function [51]. For the determination of a globally optimal solution, the algorithm due to Breiman and Cutler [8] has been adapted to locate a global optimizer of a nonconvex eigenvalue function in [60]. This algorithm is especially meant to locate a global optimizer of a nonconvex eigenvalue optimization problem of the form (1.7) when there are few optimization parameters (e.g., $d = 1$ or $d = 2$). It approximates the objective eigenvalue function with a piece-wise quadratic model function that lies globally above or below the eigenvalue function. Then it optimizes the piece-wise quadratic function globally instead of the eigenvalue function, and refines the piece-wise quadratic function based on the global optimizer. Its global convergence is guaranteed in theory in some cases, especially for the minimization problem in (1.7). However, little is proven regarding its rate-of-convergence.

1.2.3 Large-Scale Problems

The main concern of this thesis is eigenvalue optimization problems of the form (1.7) in the large-scale setting, when the matrix-valued function $A(\omega)$ is of large size, that is when n is large. These problems arise from various applications. For instance, the distance to instability of a large stable matrix with respect to the matrix 2-norm yields large singular value optimization problems [80], which in turn can be expressed as large-scale eigenvalue optimization problems of the form (1.7). The state-of-the-art algorithms for the computation of the distance to instability of a matrix $A \in \mathbb{C}^{n \times n}$ [7, 9] are computationally expensive even for modest values of n . As discussed above,

in various applications, the largest eigenvalue of a matrix-valued function is sought to be minimized. A particular application is the numerical scheme for the design of the strongest column subject to volume constraints [16], where the sizes of the matrices depend on the fineness of a discretization imposed on a differential operator. These matrices can be very large if a fine grid is employed. The standard linear SDP formulations received a lot of attention by the convex optimization community since the 1990s [81]. The dual problem of an SDP under mild assumptions can be cast as an eigenvalue optimization problem [34]. If the size of the matrix variable of an SDP is large, the associated eigenvalue optimization problem involves large matrices. The current SDP solvers are typically not meant to deal with such large-scale problems.

Subspace projections and restrictions have been applied to particular large-scale eigenvalue optimization problems in the past to reduce the sizes of the matrices involved. But general procedures have not been proposed and studied thoroughly. A subspace restriction idea has been employed specifically for the computation of the pseudospectral abscissa of a matrix in [45]. This computational problem involves the optimization of a linear objective function subject to a constraint on the smallest singular value of an affine matrix-valued function. Fast convergence is observed in that work, and confirmed with a superlinear rate-of-convergence result. In the context of standard semidefinite programs, subspace methods have been used for quite a while, in particular the spectral bundle method [34] is based on subspace ideas. The small-scale optimization problems resulting from subspace restrictions and projections are solved by standard SDP solvers [81]. A thorough convergence analysis for them has not been performed, also their efficiency is not fully realized in practice. Extensions for convex quadratic SDPs [53] and linear matrix inequalities [61] have been considered. All these large-scale problems connected to SDPs are convex.

1.2.4 Minimax Optimization Problems

Part of this thesis work focuses on the minimax optimization problems whose objective is the largest eigenvalue of a Hermitian matrix-valued function. Formally, the problem

at our hand is

$$\min_{x \in \Omega} \max_{\omega \in \mathcal{B}} \lambda_1(x, \omega) \quad (1.8)$$

with $\lambda_1(x, \omega)$ denoting the largest eigenvalue,

$$M(x, \omega) := f_1(x, \omega)A_1 + \cdots + f_\kappa(x, \omega)A_\kappa$$

for given Hermitian matrices $A_1, \dots, A_\kappa \in \mathbb{C}^{n \times n}$ and functions $f_1, \dots, f_\kappa : \tilde{\Omega} \times \tilde{\mathcal{B}} \rightarrow \mathbb{R}$ that are real analytic on their domain $\tilde{\Omega} \times \tilde{\mathcal{B}}$. Above, $\Omega, \tilde{\Omega}$ are compact, open subsets of $\mathbb{R}^d$ such that $\Omega \subset \tilde{\Omega}$, whereas $\mathcal{B}, \tilde{\mathcal{B}}$ are closed, open intervals in $\mathbb{R}$ such that $\mathcal{B} \subset \tilde{\mathcal{B}}$.

The minimization of the numerical radius is an example of such a minimax problem and motivated by stabilization considerations for a parameter dependent discrete dynamical systems. More precisely, let us consider the discrete linear system

$$z_{k+1} = Az_k + Bu_k, \quad y_k = Cz_k$$

with the input $u_k \in \mathbb{C}^m$ and the output $y_k \in \mathbb{C}^p$ for given $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$ and $C \in \mathbb{C}^{p \times n}$. By setting $u_k = Ky_k$ for $K \in \mathbb{C}^{m \times p}$, so called the controller, we obtain

$$z_{k+1} = (A + BKC)z_k.$$

The stabilization by output feedback control is one classical problem in control theory, which involves finding a K such that $A + BKC$ is asymptotically stable, i.e., its spectrum is contained strictly inside the unit circle in the complex plane. The problem of finding such a K is an open problem in control theory. Indeed, with the box constraints on the entries of K , it is well known that the problem is NP-hard [63]. Minimizing the spectral radius $\rho(A + BKC)$ of $A + BKC$ (i.e., the maximum of the moduli of the eigenvalues of $A + BKC$) over K maximizes the asymptotic decay rate of the system. However, the non-Lipschitz nature of this problem is a major challenge. On the other hand, minimizing the numerical radius of $A + BKC$ with respect to K turns out to be an optimization problem with a Lipschitz continuous objective. If the minimized numerical radius $r(A + BK_*C)$ is less than one for the optimal K_* , the resulting system $z_{k+1} = (A + BK_*C)z_k$ is asymptotically stable; indeed, as the field

of values contains the eigenvalues, we have $\rho(A + BK_*C) \leq r(A + BK_*C)$. Moreover, $r(A + BK_*C)$ provides upper bounds on the 2-norm of the powers of $A + BK_*C$, hence on the transient behavior of the system $z_{k+1} = (A + BK_*C)z_k$.

Minimax optimization problems with an eigenvalue function in the objective have been considered in the literature by quite a few works. These works are usually motivated by stability optimization for control systems; more specifically they focus on problems such as the minimization of the pseudospectral abscissa, the minimization of the $\mathcal{H}_\infty$ -norm, as well as the maximization of the distance to instability. In recent years, numerical algorithms have been proposed for optimizing matrix stability. A gradient sampling algorithm has been tailored in [10, 11] to deal with potential nonsmoothness of these problems at optimal points. This is later extended to a hybrid algorithm, called **HANSO**, a combination of this gradient sampling algorithm with BFGS [66]. **HIFOO** [1, 12, 29], a special adaptation of **HANSO**, has been developed for fixed-order $\mathcal{H}_\infty$ controller design, but can also be used for optimizing the pseudospectral abscissa and distance to instability, which are special kinds of minimax and maximin optimization problems involving eigenvalue functions in the objective. A common major drawback of these algorithms is that they converge to a locally optimal solution, that is not necessarily a globally optimal solution. Furthermore, they are not designed to cope with large-scale problems. A more recent work [57] concerns the maximization of the distance to instability for a continuous time linear system that depends on parameters. It attempts to solve this problem globally in the small-scale setting, and, in the large-scale setting, it describes the construction of small dimensional subspaces such that the restriction of the problem into these subspaces yield small-scale problems with optimal values same as the original large-scale problem up to machine precision.

1.3 Outline and Contributions

The main body of this thesis work starts in Chapter 2 with a thorough consideration of large-scale eigenvalue optimization problems of the form (1.7). We introduce

procedures that build subspaces such that the restriction of the large-scale problem into these subspaces capture the original large-scale problem accurately. Our subspace construction procedures are supplemented with remarkable theoretical results. In particular, for the minimization problem in (1.7), we prove that the global minimizers of the restricted problems formed by the procedures converge to a global minimizer of the original problem as the subspaces grow in the infinite dimensional setting, that is when the Hermitian matrix-valued function is replaced by a self-adjoint compact operator dependent on parameters. Additionally, both for the minimization problem and for the maximization problem, we prove that this convergence occurs at a superlinear rate under a smoothness assumption, namely that the eigenvalue $\lambda_J(\omega)$ at the converged global optimizer is simple. We illustrate these theoretical findings on several numerical examples. The content of Chapter 2 has recently been published [39] as a research article.

Chapter 3 is devoted to the rate-of-convergence analyses of two algorithms for eigenvalue optimization in the presence of nonsmoothness. Specifically, we focus on the minimization problem in (1.7) when $J = 1$ (i.e., minimization of the largest eigenvalue), when the matrix-valued function depends on one parameter (i.e., $d = 1$) and when $\lambda_1(A(\omega))$ is not simple at the converged global minimizer. We show that the algorithm in [60] based on piece-wise quadratic model functions and intended for small-scale problems converge to the global minimizer of $\lambda_1(A(\omega))$ rapidly. Then we extend the subspace procedures in Chapter 2 for large-scale problems to this nonsmooth setting, and prove rigorously that the sequence of solutions of the restricted problems converges to the global minimizer of the original problem at a quadratic rate as the subspace dimension goes to infinity. As a showcase, the algorithms are employed for the computation of the inner numerical radius on several examples where nonsmoothness is present. The examples arise from applications such as the determination of a preconditioner for a saddle point linear system and checking the hyperbolicity of a quadratic eigenvalue problem. The numerical results on these examples confirm our theoretical findings.

We focus on the minimax optimization problems of the form (1.8) in Chapter 4. First, an adaptation of the algorithm in [60] for these minimax problems is presented that is especially suitable for small-scale problems. Then, inspired from the ideas and observations in Chapter 2, we propose a subspace procedure meant for large-scale problems for the setting of (1.8) when the matrix-valued function $M(x, \omega)$ is of large size. We again prove the global convergence of the subspace procedure, and deduce a superlinear rate-of-convergence result with respect to the subspace dimension in the smooth case when $\lambda_1(M(x, \omega))$ is simple at the optimal x and ω . The algorithms are applied to the minimization of the numerical radius, where we again observe convergence behaviors in practice in harmony with our theoretical results.

We conclude in Chapter 5 with a summary of our findings along with a discussion of a few open problems that are relevant to this thesis work.

Chapter 2

LARGE-SCALE EIGENVALUE OPTIMIZATION

In this chapter, we consider the minimization or maximization of the J th largest eigenvalue of an analytic and Hermitian matrix-valued function, and build on Mengi *et al.* (2014, *SIAM J. Matrix Anal. Appl.*, 35, 699-724). This chapter addresses the setting when the matrix-valued function involved is very large. We describe subspace procedures that convert the original problem into a small-scale one by means of orthogonal projections and restrictions to certain subspaces, and that gradually expand these subspaces based on the optimal solutions of small-scale problems. Global convergence and superlinear rate-of-convergence results with respect to the dimensions of the subspaces are presented in the infinite dimensional setting, where the matrix-valued function is replaced by a compact operator depending on parameters. In practice, it suffices to solve eigenvalue optimization problems involving matrices with sizes on the scale of tens, instead of the original problem involving matrices with sizes on the scale of thousands.

2.1 Introduction

We are concerned with the global optimization problems

$$(\mathbf{MN}) \quad \text{minimize } \{\lambda_J(\omega) \mid \omega \in \Omega\} \quad \text{and} \quad (\mathbf{MX}) \quad \text{maximize } \{\lambda_J(\omega) \mid \omega \in \Omega\}.$$

The feasible region Ω of these optimization problems is a compact subset of $\mathbb{R}^d$. Furthermore, letting $\ell^2(\mathbb{N})$ denote the sequence space consisting of square summable infinite sequences of complex numbers equipped with the inner product $\langle w, v \rangle = \sum_{k=0}^{\infty} \overline{w_k} v_k$ as well as the norm $\|v\|_2 = \sqrt{\sum_{k=0}^{\infty} |v_k|^2}$, the objective function $\lambda_J(\omega)$ is

the J th largest eigenvalue of a compact self-adjoint operator

$$\mathbf{A}(\omega) : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}), \quad \mathbf{A}(\omega) := \sum_{\ell=1}^{\kappa} f_{\ell}(\omega) \mathbf{A}_{\ell} \quad (2.1)$$

for every $\omega \in \tilde{\Omega}$, an open subset of $\mathbb{R}^d$ containing the feasible region Ω . Above $\mathbf{A}_{\ell} : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ and $f_{\ell} : \tilde{\Omega} \rightarrow \mathbb{R}$ for $\ell = 1, \dots, \kappa$ represent given compact self-adjoint operators and real-analytic functions, respectively. Throughout the text, $\mathbf{A}(\omega)$ for each ω and $\mathbf{A}_1, \dots, \mathbf{A}_{\kappa}$ as in (2.1) could intuitively be considered as infinite dimensional Hermitian matrices.

Our interest in the infinite dimensional eigenvalue optimization problems **(MN)** and **(MX)** rise from their finite dimensional counterparts, which for given Hermitian matrices $A_{\ell} \in \mathbb{C}^{n \times n}$ for $\ell = 1, \dots, \kappa$ involve the matrix-valued function

$$A(\omega) := \sum_{\ell=1}^{\kappa} f_{\ell}(\omega) A_{\ell} \quad (2.2)$$

instead of the parameter dependent operator $\mathbf{A}(\omega)$. These problems come with standard challenges due to their nonsmoothness and nonconvexity. But we would like to tackle a different challenge, when the matrices in them are very large, that is n is very large. Thus, the primary purpose of this chapter is to deal with large dimensionality, it does not address the inherent difficulties due to nonconvexity and nonsmoothness. We introduce the ideas in the idealized infinite dimensional setting only because this makes a rigorous convergence analysis possible.

To deal with large dimensionality, we propose restricting the domain and projecting the range of the map $v \mapsto \mathbf{A}(\omega)v$ to small subspaces. This gives rise to eigenvalue optimization problems involving small matrices, which we call reduced problems. Two greedy procedures are presented here to construct small subspaces so that the optimal solution of the reduced problem is close to the optimal solution of the original problem. For both procedures, we observe a superlinear rate of decay in the error with respect to the subspace dimension. The first procedure is more straightforward and constructs smaller subspaces, shown to converge at a superlinear rate when $d = 1$, but lacks a complete formal argument justifying its quick convergence when $d \geq 2$.

The second constructs larger subspaces, but comes with a formal proof of superlinear convergence for all d .

While the proposed procedures operate on **(MN)** and **(MX)** similarly, there are remarkable differences in their convergence behaviors in these two contexts. The proposed subspace restrictions and projections on the map $v \mapsto \mathbf{A}(\omega)v$ lead to global lower envelopes for $\lambda_J(\omega)$. These lower envelopes in turn make the convergence to the globally smallest value of $\lambda_J(\omega)$ possible as the dimensions of the subspaces by the procedures grow to infinity, which we prove formally. Such a global convergence behavior does not hold for the maximization problem: if the subspace dimension is allowed to grow to infinity, the globally maximal values of the reduced problems converge to a locally maximal value of $\lambda_J(\omega)$ that is not necessarily globally maximal. But the maximization problem possesses a remarkable low-rank property: there exists a J dimensional subspace such that when the map $v \mapsto \mathbf{A}(\omega)v$ is restricted and projected to this subspace, the resulting reduced problem has the same globally largest value as $\lambda_J(\omega)$. The minimization problem does not enjoy an analogous low-rank property.

2.1.1 Spectral Properties and Operator Norm of $\mathbf{A}(\omega)$

The spectrum of $\mathbf{A}(\omega)$ is defined by

$$\sigma(\mathbf{A}(\omega)) := \{z \in \mathbb{R} \mid zI - \mathbf{A}(\omega) \text{ is not invertible}\}.$$

This set contains countably many real eigenvalues each with a finite multiplicity; also the only accumulation point of these eigenvalues is 0 (see [41, page 185, Theorem 6.26]). We assume, throughout the text, that $\mathbf{A}(\omega)$ has at least J positive eigenvalues for all $\omega \in \Omega$. This ensures that $\lambda_J(\omega)$ is well-defined over Ω .

The eigenspace associated with each eigenvalue of $\mathbf{A}(\omega)$ is also finite dimensional [46, Corollary 6.44]. Furthermore, the set of eigenvectors of $\mathbf{A}(\omega)$ can be chosen in a way so that they are orthonormal and complete in $\ell^2(\mathbb{N})$ [18, Theorem 4.2.23], i.e., there exist an orthonormal sequence $\{v^{(j)}\}$ in $\ell^2(\mathbb{N})$ and a sequence $\{\mu^{(j)}\}$ of real

numbers such that $\mathbf{A}(\omega)v^{(j)} = \mu^{(j)}v^{(j)}$ for all $j = 1, 2, \dots$, $\mu^{(j)} \rightarrow 0$ as $j \rightarrow \infty$, and any vector $v \in \ell^2(\mathbb{N})$ can be expressed as $v = \sum_{j=1}^{\infty} \alpha_j v^{(j)}$ for some scalars α_j .

Since $\mathbf{A}_1, \dots, \mathbf{A}_\kappa$ and $\mathbf{A}(\omega)$ are compact, they are bounded. Hence, their operator norms

$$\begin{aligned} \|\mathbf{A}_\ell\|_2 &:= \sup \{ \|\mathbf{A}_\ell v\|_2 \mid v \in \ell^2(\mathbb{N}) \text{ such that } \|v\|_2 = 1 \} \quad \text{for } \ell = 1, \dots, \kappa \quad \text{and} \\ \|\mathbf{A}(\omega)\|_2 &:= \sup \{ \|\mathbf{A}(\omega)v\|_2 \mid v \in \ell^2(\mathbb{N}) \text{ such that } \|v\|_2 = 1 \} \end{aligned}$$

are also well-defined.

2.2 The Subspace Procedures

Let $\mathcal{V}$ be a finite dimensional subspace of $\ell^2(\mathbb{N})$, and $V := \{v_1, \dots, v_m\}$ be an orthonormal basis for $\mathcal{V}$. The linear operator

$$\mathbf{V} : \mathbb{C}^m \rightarrow \ell^2(\mathbb{N}), \quad \mathbf{V}\alpha := \sum_{j=1}^m \alpha_j v_j, \quad (2.3)$$

maps the coordinates of a vector $v \in \mathcal{V}$ relative to V to itself. On the other hand, its adjoint $\mathbf{V}^* : \ell^2(\mathbb{N}) \rightarrow \mathbb{C}^m$ projects a vector in $\ell^2(\mathbb{N})$ orthogonally onto $\mathcal{V}$ and represents the orthogonal projection in coordinates relative to V . The procedures that will be introduced in this section are based on operators of the form

$$\mathbf{A}^\mathcal{V}(\omega) := \mathbf{V}^* \mathbf{A}(\omega) \mathbf{V}, \quad (2.4)$$

which is the restriction of $\mathbf{A}(\omega)$ that acts only on $\mathcal{V}$, with its input and output represented in coordinates relative to V . This operator can be expressed in the form

$$\mathbf{A}^\mathcal{V}(\omega) = \sum_{\ell=1}^{\kappa} f_\ell(\omega) \mathbf{A}_\ell^\mathcal{V}, \quad \text{where } \mathbf{A}_\ell^\mathcal{V} := \mathbf{V}^* \mathbf{A}_\ell \mathbf{V},$$

which is beneficial from a computational point of view, because it is possible to form the $m \times m$ matrix representations of the operators $\mathbf{A}_\ell^\mathcal{V}$ in advance. The reduced eigenvalue optimization problems are defined in terms of the J th largest eigenvalue $\lambda_J^\mathcal{V}(\omega)$ of $\mathbf{A}^\mathcal{V}(\omega)$ as

$$\text{minimize } \{ \lambda_J^\mathcal{V}(\omega) \mid \omega \in \Omega \} \quad \text{and} \quad \text{maximize } \{ \lambda_J^\mathcal{V}(\omega) \mid \omega \in \Omega \}. \quad (2.5)$$

The discussions above generalize when $\mathcal{V}$ is infinite dimensional. In the infinite dimensional setting, the operators $\mathbf{A}^{\mathcal{V}}(\omega)$ and $\mathbf{A}_{\ell}^{\mathcal{V}}$ are defined similarly in terms of an infinite countable orthonormal basis $V = \{v_j \mid j \in \mathbb{Z}^+\}$ for $\mathcal{V}$ and the associated operator

$$\mathbf{V}\alpha := \sum_{j=1}^{\infty} \alpha_j v_j. \quad (2.6)$$

The eigenvalue function $\lambda_J^{\mathcal{V}}(\omega)$ is Lipschitz continuous, indeed there exists a uniform Lipschitz constant γ for all subspaces $\mathcal{V}$ of $\ell^2(\mathbb{N})$ as formally stated and proven in the following lemma.

Lemma 2.2.1 (Lipschitz Continuity). *There exists a real scalar $\gamma > 0$ such that for all subspaces $\mathcal{V}$ of $\ell^2(\mathbb{N})$, we have*

$$|\lambda_J^{\mathcal{V}}(\tilde{\omega}) - \lambda_J^{\mathcal{V}}(\omega)| \leq \gamma \cdot \|\tilde{\omega} - \omega\|_2 \quad \forall \tilde{\omega}, \omega \in \tilde{\Omega}.$$

Proof. By Weyl's theorem (see [36, Theorem 4.3.1] for the finite dimensional case; its extension to the infinite dimensional setting is straightforward by exploiting the maximin characterization (2.7) of $\lambda_J^{\mathcal{V}}(\omega)$ given below)

$$\begin{aligned} |\lambda_J^{\mathcal{V}}(\tilde{\omega}) - \lambda_J^{\mathcal{V}}(\omega)| &\leq \|\mathbf{A}^{\mathcal{V}}(\tilde{\omega}) - \mathbf{A}^{\mathcal{V}}(\omega)\|_2 \\ &\leq \sum_{\ell=1}^{\kappa} |f_{\ell}(\tilde{\omega}) - f_{\ell}(\omega)| \|\mathbf{A}_{\ell}^{\mathcal{V}}\|_2 \quad \forall \tilde{\omega}, \omega \in \tilde{\Omega}. \end{aligned}$$

In the last summation $\|\mathbf{A}_{\ell}^{\mathcal{V}}\|_2 \leq \|\mathbf{A}_{\ell}\|_2$ and the real analyticity of $f_{\ell}(\omega)$ implies its Lipschitz continuity, hence, the existence of a constant γ_{ℓ} satisfying

$$|f_{\ell}(\tilde{\omega}) - f_{\ell}(\omega)| \leq \gamma_{\ell} \|\tilde{\omega} - \omega\|_2 \quad \forall \tilde{\omega}, \omega \in \tilde{\Omega}.$$

Combining these observations we obtain

$$|\lambda_J^{\mathcal{V}}(\tilde{\omega}) - \lambda_J^{\mathcal{V}}(\omega)| \leq \left(\sum_{\ell=1}^{\kappa} \gamma_{\ell} \|\mathbf{A}_{\ell}\|_2 \right) \cdot \|\tilde{\omega} - \omega\|_2 \quad \forall \tilde{\omega}, \omega \in \tilde{\Omega}$$

as desired. $\square$

Throughout the rest of this section, we first investigate the relation between $\lambda_J(\omega)$ and $\lambda_J^\mathcal{V}(\omega)$ as well as their globally maximal values. Some of these theoretical results will be frequently used in the subsequent sections. The second and third parts of this section introduce two subspace procedures for the generation of small dimensional subspaces $\mathcal{V}$, leading to reduced eigenvalue optimization problems that approximate the original eigenvalue optimization problems accurately.

2.2.1 Relations between $\lambda_J^\mathcal{V}(\omega)$ and $\lambda_J(\omega)$

We start with a result about the monotonicity of $\lambda_J^\mathcal{V}(\omega)$ with respect to $\mathcal{V}$. This result is an immediate consequence of the following maximin characterization [22, pages 1543-1544] (see also [36, Theorem 4.2.11] for the finite dimensional case) of $\lambda_J^\mathcal{V}(\omega)$ for all subspaces $\mathcal{V}$ of $\ell^2(\mathbb{N})$:

$$\lambda_J^\mathcal{V}(\omega) = \sup_{\mathcal{S} \subseteq \mathcal{V}, \dim \mathcal{S} = J} \min_{v \in \mathcal{S}, \|v\|_2 = 1} \langle \mathbf{A}(\omega)v, v \rangle, \quad (2.7)$$

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product $\langle w, v \rangle := \sum_{k=0}^{\infty} \overline{w_k} v_k$ and the outer supremum is over all J dimensional subspaces of $\mathcal{V}$. Furthermore, if $\mathbf{A}^\mathcal{V}(\omega)$ has at least J positive eigenvalues, that is, if $\lambda_J^\mathcal{V}(\omega) > 0$, or if $\mathcal{V}$ is finite dimensional, then the outer supremum in (2.7) is attained, hence, can be replaced by maximum. The monotonicity result presented next will play a central role later when we analyze the convergence of the subspace procedures.

Lemma 2.2.2 (Monotonicity). *Let $\mathcal{V}_1, \mathcal{V}_2$ be subspaces of $\ell^2(\mathbb{N})$ of dimension larger than or equal to J such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. The following holds:*

$$\lambda_J^{\mathcal{V}_1}(\omega) \leq \lambda_J^{\mathcal{V}_2}(\omega) \leq \lambda_J(\omega). \quad (2.8)$$

A consequence of this monotonicity result is the following interpolatory property.

Lemma 2.2.3 (Interpolatory Property). *Let $\mathcal{V}$ be a subspace of $\ell^2(\mathbb{N})$ of dimension larger than or equal to J , and $\mathbf{V}$ be the operator defined as in (2.3) or (2.6) in terms of an orthonormal basis V for $\mathcal{V}$. If $S := \text{span}\{s_1, \dots, s_J\} \subseteq \mathcal{V}$, where $s_1, \dots, s_J$ are eigenvectors corresponding to the J largest eigenvalues of $\mathbf{A}(\omega)$, then the following hold:*

(i) $\lambda_J(\omega) = \lambda_J^\mathcal{V}(\omega);$

(ii) $\mathbf{V}^* s_J$ is an eigenvector of $\mathbf{V}^* \mathbf{A}(\omega) \mathbf{V}$ corresponding to its eigenvalue $\lambda_J^\mathcal{V}(\omega)$.

Proof. (i) We assume each $s_\ell \in \mathcal{V}$ for $\ell = 1, \dots, J$ is of unit length without loss of generality. It follows that there exist $\alpha_1, \dots, \alpha_J$ of unit length such that $s_\ell = \mathbf{V} \alpha_\ell$ for $\ell = 1, \dots, J$. Now define $\mathcal{S}_\alpha := \text{span}\{\alpha_1, \dots, \alpha_J\} \subseteq \mathbb{C}^m$, and observe

$$\begin{aligned} \lambda_J(\omega) &= \langle \mathbf{A}(\omega) s_J, s_J \rangle = \langle \mathbf{A}(\omega) \mathbf{V} \alpha_J, \mathbf{V} \alpha_J \rangle \\ &= \langle \mathbf{V}^* \mathbf{A}(\omega) \mathbf{V} \alpha_J, \alpha_J \rangle \\ &= \min_{\alpha \in \mathcal{S}_\alpha, \|\alpha\|_2=1} \langle \mathbf{V}^* \mathbf{A}(\omega) \mathbf{V} \alpha, \alpha \rangle \leq \lambda_J^\mathcal{V}(\omega). \end{aligned}$$

The opposite inequality is immediate from Lemma 2.2.2, so $\lambda_J(\omega) = \lambda_J^\mathcal{V}(\omega)$ as claimed.

(ii) The equalities

$$\langle \mathbf{V}^* \mathbf{A}(\omega) \mathbf{V} \alpha_J, \alpha_J \rangle = \min_{\alpha \in \mathcal{S}_\alpha, \|\alpha\|_2=1} \langle \mathbf{V}^* \mathbf{A}(\omega) \mathbf{V} \alpha, \alpha \rangle = \lambda_J^\mathcal{V}(\omega)$$

imply that $\alpha_J = \mathbf{V}^* s_J$ is an eigenvector of $\mathbf{V}^* \mathbf{A}(\omega) \mathbf{V}$ corresponding to the eigenvalue $\lambda_J^\mathcal{V}(\omega)$. $\square$

Maximization of the J th largest eigenvalue over a low dimensional subspace is motivated by the next result. According to the result, it suffices to perform the optimization on a proper J dimensional subspace, which is hard to determine in advance. Here and elsewhere, $\arg \max_{\omega \in \Omega} \lambda_J(\omega)$ and $\arg \min_{\omega \in \Omega} \lambda_J(\omega)$ denote the set of global maximizers and global minimizers of $\lambda_J(\omega)$ over $\omega \in \Omega$, respectively.

Lemma 2.2.4 (Low-Rank Property of Maximization Problems). *For a given subspace $\mathcal{V} \subset \ell^2(\mathbb{N})$ with dimension larger than or equal to J , consider the following assertions:*

(i) $\max_{\omega \in \Omega} \lambda_J^\mathcal{V}(\omega) = \max_{\omega \in \Omega} \lambda_J(\omega);$

(ii) $\mathcal{S}_* := \text{span}\{s_1, \dots, s_J\} \subseteq \mathcal{V}$, where $s_1, \dots, s_J$ are eigenvectors corresponding to the J largest eigenvalues of $\mathbf{A}(\omega_*)$ at some $\omega_* \in \arg \max_{\omega \in \Omega} \lambda_J(\omega)$.

Assertion (ii) implies assertion (i). Furthermore, when $J = 1$, assertions (i) and (ii) are equivalent.

Proof. Suppose $s_1, \dots, s_J \in \mathcal{V}$ satisfies assertion (ii). By Lemma 2.2.3, we have

$$\max_{\omega \in \Omega} \lambda_J(\omega) = \lambda_J(\omega_*) = \lambda_J^{\mathcal{V}}(\omega_*) \leq \max_{\omega \in \Omega} \lambda_J^{\mathcal{V}}(\omega).$$

Lemma 2.2.2 implies the opposite inequality, proving assertion (i).

To prove that the assertions are equivalent when $J = 1$, assume that assertion (i) holds. Letting ω_* be any point in $\arg \max_{\omega \in \Omega} \lambda_1^{\mathcal{V}}(\omega)$, denoting by $\mathbf{V}$ an operator defined as in (2.3) or (2.6) in terms of an orthonormal basis V for $\mathcal{V}$, and denoting by α a unit eigenvector corresponding to the largest eigenvalue of $\mathbf{V}^* \mathbf{A}(\omega_*) \mathbf{V}$ (note that $\lambda_1^{\mathcal{V}}(\omega_*) = \lambda_1(\omega_*) > 0$, so the unit eigenvector α is well-defined), we have

$$\max_{\omega \in \mathbb{R}^d} \lambda_1(\omega) = \lambda_1^{\mathcal{V}}(\omega_*) = \langle \mathbf{V}^* \mathbf{A}(\omega_*) \mathbf{V} \alpha, \alpha \rangle = \langle \mathbf{A}(\omega_*) \mathbf{V} \alpha, \mathbf{V} \alpha \rangle \leq \lambda_1(\omega_*). \quad (2.9)$$

Thus we deduce

$$\lambda_1(\omega_*) = \max_{\omega \in \Omega} \lambda_1(\omega) = \langle \mathbf{A}(\omega_*) \mathbf{V} \alpha, \mathbf{V} \alpha \rangle.$$

Consequently, $\mathbf{V} \alpha$ is a unit eigenvector of $\mathbf{A}(\omega_*)$ corresponding to its largest eigenvalue, where ω_* belongs to $\arg \max_{\omega \in \Omega} \lambda_1(\omega)$. Furthermore, $\text{span}\{\mathbf{V} \alpha\} \subseteq \mathcal{V}$. This proves assertion (ii). $\square$

When $\mathcal{V}$ does not contain the optimal subspace $\mathcal{S}_*$ (as in part (ii) of Lemma 2.2.4), the next result quantifies the gap between the eigenvalues of the original and the reduced operators in terms of the distance from $\mathcal{S}_*$ to the J dimensional subspaces of $\mathcal{V}$. For this result, we define the distance between two finite dimensional subspaces $\tilde{\mathcal{S}}, \mathcal{S}$ of $\ell^2(\mathbb{N})$ of the same dimension by

$$d(\tilde{\mathcal{S}}, \mathcal{S}) := \max_{\tilde{v} \in \tilde{\mathcal{S}}, \|\tilde{v}\|_2=1} \min_{v \in \mathcal{S}} \|\tilde{v} - v\|_2.$$

This distance corresponds to the sine of the largest angle between the subspaces $\tilde{\mathcal{S}}$ and $\mathcal{S}$. Results of a similar nature can be found in the literature; see, for instance, [70, Theorem 11.7.1] and [73, Proposition 4.5] where the bounds are in terms of distances between one dimensional subspaces.

Theorem 2.2.5 (Accuracy of Reduced Problems). *Let $\mathcal{V}$ be a subspace of $\ell^2(\mathbb{N})$ with dimension J or larger.*

(i) *For each ω in $\tilde{\Omega}$, we have*

$$\lambda_J(\omega) - \lambda_J^{\mathcal{V}}(\omega) = O(\varepsilon^2), \quad (2.10)$$

where

$$\varepsilon := \min \left\{ d(\tilde{\mathcal{S}}, \mathcal{S}) \mid \tilde{\mathcal{S}} \text{ is a } J \text{ dimensional subspace of } \mathcal{V} \right\}$$

and $\mathcal{S}$ is the subspace spanned by the eigenvectors corresponding to the J largest eigenvalues of $\mathbf{A}(\omega)$.

(ii) *The equality*

$$\max_{\omega \in \Omega} \lambda_J(\omega) - \max_{\omega \in \Omega} \lambda_J^{\mathcal{V}}(\omega) = O(\varepsilon_*^2) \quad (2.11)$$

holds. Here, ε_* is given by

$$\varepsilon_* := \min \left\{ d(\tilde{\mathcal{S}}, \mathcal{S}_*) \mid \tilde{\mathcal{S}} \text{ is a } J \text{ dimensional subspace of } \mathcal{V} \right\}$$

for some subspace $\mathcal{S}_*$ spanned by the eigenvectors corresponding to the J largest eigenvalues of $\mathbf{A}(\omega_*)$ at some $\omega_* \in \arg \max_{\omega \in \Omega} \lambda_J(\omega)$.

Proof. (i) Let $\hat{\mathcal{S}}$ be a J dimensional subspace of $\mathcal{V}$ such that $\varepsilon = d(\hat{\mathcal{S}}, \mathcal{S})$. Furthermore, for a given unit vector $\hat{v} \in \hat{\mathcal{S}}$, let us use the notations

$$v(\hat{v}) := \arg \min \{ \|\hat{v} - v\|_2 \mid v \in \mathcal{S} \} \quad \text{and} \quad \delta(\hat{v}) := \hat{v} - v(\hat{v}),$$

where $\|\delta(\hat{v})\|_2 = O(\varepsilon)$. Observe that the inner minimization problem in the definition of $d(\hat{\mathcal{S}}, \mathcal{S})$ is a least-squares problem, so the minimizer $v(\hat{v})$ of $\|\hat{v} - v\|_2$ over $v \in \mathcal{S}$ defined above is unique and $\delta(\hat{v}) \perp \mathcal{S}$. Additionally, $\|v(\hat{v})\|_2^2 = 1 - O(\varepsilon^2)$ due to the properties $\|\hat{v}\|_2 = \|v(\hat{v}) + \delta(\hat{v})\|_2 = 1$ and $v(\hat{v}) \perp \delta(\hat{v})$.

Now the maximin characterization (2.7) for $\lambda_J^\mathcal{V}(\omega)$ implies

$$\begin{aligned}
\lambda_J^\mathcal{V}(\omega) &\geq \min_{\widehat{v} \in \widehat{\mathcal{S}}, \|\widehat{v}\|_2=1} \langle \mathbf{A}(\omega) \widehat{v}, \widehat{v} \rangle = \min_{\widehat{v} \in \widehat{\mathcal{S}}, \|\widehat{v}\|_2=1} \langle \mathbf{A}(\omega) [v(\widehat{v}) + \delta(\widehat{v})], [v(\widehat{v}) + \delta(\widehat{v})] \rangle \\
&\geq \min_{\widehat{v} \in \widehat{\mathcal{S}}, \|\widehat{v}\|_2=1} \langle \mathbf{A}(\omega) v(\widehat{v}), v(\widehat{v}) \rangle \\
&\quad + \min_{\widehat{v} \in \widehat{\mathcal{S}}, \|\widehat{v}\|_2=1} 2\Re(\langle \mathbf{A}(\omega) v(\widehat{v}), \delta(\widehat{v}) \rangle) \\
&\quad + \min_{\widehat{v} \in \widehat{\mathcal{S}}, \|\widehat{v}\|_2=1} \langle \mathbf{A}(\omega) \delta(\widehat{v}), \delta(\widehat{v}) \rangle \\
&\geq \lambda_J(\omega) \min_{\widehat{v} \in \widehat{\mathcal{S}}, \|\widehat{v}\|_2=1} \|v(\widehat{v})\|_2^2 - O(\varepsilon^2) \\
&= \lambda_J(\omega) - O(\varepsilon^2).
\end{aligned}$$

Above, on the third line, we note that $\langle \mathbf{A}(\omega) v(\widehat{v}), \delta(\widehat{v}) \rangle = 0$ due to the fact $\mathbf{A}(\omega) v(\widehat{v}) \in \mathcal{S}$ and $\delta(\widehat{v}) \perp \mathcal{S}$, and on the second to the last line, we employ $\|v(\widehat{v})\|_2^2 = 1 - O(\varepsilon^2)$.

The desired equality (2.10) follows from $\lambda_J^\mathcal{V}(\omega) \leq \lambda_J(\omega)$ due to Lemma 2.2.2.

(ii) This is an immediate corollary of (i). In particular,

$$O(\varepsilon_*^2) = \lambda_J(\omega_*) - \lambda_J^\mathcal{V}(\omega_*) \geq \max_{\omega \in \Omega} \lambda_J(\omega) - \max_{\omega \in \Omega} \lambda_J^\mathcal{V}(\omega)$$

combined with $\max_{\omega \in \Omega} \lambda_J^\mathcal{V}(\omega) \leq \max_{\omega \in \Omega} \lambda_J(\omega)$ (due to Lemma 2.2.2) yields (2.11). $\square$

2.2.2 The Greedy Procedure

The basic greedy procedure solves the reduced eigenvalue optimization problem (2.5) for a given subspace $\mathcal{V}$. Denoting a global optimizer of the reduced problem with ω_* , the subspace is expanded with the addition of the eigenvectors corresponding to $\lambda_1(\omega_*), \dots, \lambda_J(\omega_*)$, then this is repeated with the expanded subspace. A formal description is given in Algorithm 1, where $\mathcal{S}_k$ denotes the subspace at the k th step of the procedure, and $\lambda_J^{(k)}(\omega) := \lambda_J^{\mathcal{S}_k}(\omega)$. The reduced eigenvalue optimization problems on line 5 are nonsmooth and nonconvex. The description assumes that these problems can be solved globally. The algorithm in [60] works well in practice for this purpose when the number of parameters, d , is small. These reduced problems are computationally cheap to solve, the main computational burden comes from line

6 which requires the computation of eigenvectors of the full problem. In the finite dimensional case, these large eigenvalue problems are typically solved by means of an iterative method, for instance by Lanczos' method.

Algorithm 1 The Greedy Subspace Procedure

Input: A parameter dependent compact self-adjoint operator $\mathbf{A}(\omega)$ of the form (2.1),

and a compact subset $\Omega \subset \tilde{\Omega}$ of $\mathbb{R}^d$.

- 1: $\omega^{(1)} \leftarrow$ a random point in Ω .
 - 2: $s_1^{(1)}, \dots, s_J^{(1)} \leftarrow$ eigenvectors corresponding to $\lambda_1(\omega^{(1)}), \dots, \lambda_J(\omega^{(1)})$.
 - 3: $\mathcal{S}_1 \leftarrow \text{span} \{s_1^{(1)}, \dots, s_J^{(1)}\}$.
 - 4: **for** $k = 2, 3, \dots$ **do**
 - 5: $\omega^{(k)} \leftarrow$ any $\omega_* \in \arg \min_{\omega \in \Omega} \lambda_J^{(k-1)}(\omega)$ for the minimization problem **(MN)**,
 or $\omega^{(k)} \leftarrow$ any $\omega_* \in \arg \max_{\omega \in \Omega} \lambda_J^{(k-1)}(\omega)$ for the maximization problem **(MX)**.
 - 6: $s_1^{(k)}, \dots, s_J^{(k)} \leftarrow$ eigenvectors corresponding to $\lambda_1(\omega^{(k)}), \dots, \lambda_J(\omega^{(k)})$.
 - 7: $\mathcal{S}_k \leftarrow \mathcal{S}_{k-1} \oplus \text{span} \{s_1^{(k)}, \dots, s_J^{(k)}\}$.
 - 8: **end for**
-

As numerical experiments demonstrate (see Section 5), the power of this greedy subspace procedure is that high accuracy is often reached after a small number of steps for reduced problems of small size. This can mostly be attributed to the following interpolatory properties between $\lambda_J^{(k)}(\omega)$ and $\lambda_J(\omega)$.

Lemma 2.2.6 (Hermite Interpolation). *The following hold regarding Algorithm 1:*

- (i) $\lambda_J(\omega^{(\ell)}) = \lambda_J^{(k)}(\omega^{(\ell)})$ for $\ell = 1, \dots, k$;
- (ii) if $J > 1$, then $\lambda_{J-1}(\omega^{(\ell)}) = \lambda_{J-1}^{(k)}(\omega^{(\ell)})$ for $\ell = 1, \dots, k$;
- (iii) if $\lambda_J(\omega^{(\ell)})$ is simple, then $\lambda_J^{(k)}(\omega^{(\ell)})$ is also simple for $\ell = 1, 2, \dots, k$;
- (iv) if $\lambda_J(\omega^{(\ell)})$ is simple, then $\nabla \lambda_J(\omega^{(\ell)}) = \nabla \lambda_J^{(k)}(\omega^{(\ell)})$ for $\ell = 1, 2, \dots, k$.

Proof. (i)-(ii) Lines 2, 3, 6, and 7 of Algorithm 1 imply that $s_1^{(\ell)}, \dots, s_J^{(\ell)} \in \mathcal{S}_k$ for $\ell = 1, \dots, k$. An application of part (i) of Lemma 2.2.3 with $\mathcal{V} = \mathcal{S}_k$ yields $\lambda_j(\omega^{(\ell)}) = \lambda_j^{S_k}(\omega^{(\ell)}) = \lambda_j^{(k)}(\omega^{(\ell)})$ for $\ell = 1, \dots, k$ and $j = J$, as well as $j = J - 1$ if $J > 1$, as desired.

(iii) Suppose $\lambda_J^{(k)}(\omega^{(\ell)})$ is not simple for some $\ell \in \{1, 2, \dots, k\}$. In this case there must exist two mutually orthogonal unit eigenvectors $\hat{\alpha}, \tilde{\alpha} \in \mathbb{C}^m$ corresponding to it. Let us denote by $\mathbf{S}_k$ the operator as in (2.3) in terms of a basis S_k for $\mathcal{S}_k$, and $\mathbf{A}^{S_k}(\omega^{(\ell)}) = \mathbf{S}_k^* \mathbf{A}(\omega^{(\ell)}) \mathbf{S}_k$. It follows from part (i) that

$$\begin{aligned} \lambda_J(\omega^{(\ell)}) &= \lambda_J^{(k)}(\omega^{(\ell)}) = \langle \mathbf{A}(\omega^{(\ell)}) \mathbf{S}_k \hat{\alpha}, \mathbf{S}_k \hat{\alpha} \rangle = \min_{\alpha \in \hat{\mathcal{S}}, \|\alpha\|_2=1} \langle \mathbf{A}(\omega^{(\ell)}) \mathbf{S}_k \alpha, \mathbf{S}_k \alpha \rangle \\ &= \langle \mathbf{A}(\omega^{(\ell)}) \mathbf{S}_k \tilde{\alpha}, \mathbf{S}_k \tilde{\alpha} \rangle = \min_{\alpha \in \tilde{\mathcal{S}}, \|\alpha\|_2=1} \langle \mathbf{A}(\omega^{(\ell)}) \mathbf{S}_k \alpha, \mathbf{S}_k \alpha \rangle \end{aligned}$$

for some J dimensional subspaces $\hat{\mathcal{S}}, \tilde{\mathcal{S}}$ of $\ell^2(\mathbb{N})$ such that $\hat{\alpha} \in \hat{\mathcal{S}}, \tilde{\alpha} \in \tilde{\mathcal{S}}$. This shows that $\mathbf{S}_k \hat{\alpha}, \mathbf{S}_k \tilde{\alpha} \in \ell^2(\mathbb{N})$ are mutually orthogonal eigenvectors corresponding to $\lambda_J(\omega^{(\ell)})$, so $\lambda_J(\omega^{(\ell)})$ is not simple either.

(iv) It follows from part (iii) that $\lambda_J^{(k)}(\omega^{(\ell)})$ is also simple, so both $\lambda_J(\omega)$ and $\lambda_J^{(k)}(\omega)$ are differentiable at $\omega^{(\ell)}$; furthermore the associated unit eigenvectors can be chosen in a way so that they are also differentiable at $\omega^{(\ell)}$ (see [72, pages 57-58, Theorem 1] for the differentiability of $\lambda_J(\omega)$ and the associated unit eigenvector, and [72, pages 33-34, Theorem 1] for the differentiability of $\lambda_J^{(k)}(\omega)$ and the associated unit eigenvector). By Lemma 2.2.3(ii), the eigenvector α_J of $\mathbf{A}^{S_k}(\omega^{(\ell)}) = \mathbf{S}_k^* \mathbf{A}(\omega^{(\ell)}) \mathbf{S}_k$ corresponding to the eigenvalue $\lambda_J^{(k)}(\omega^{(\ell)}) = \lambda_J^{S_k}(\omega^{(\ell)})$ satisfies $\alpha_J = \mathbf{S}_k^* s_J^{(\ell)}$. Equivalently, we have $s_J^{(\ell)} = \mathbf{S}_k \alpha_J$ (since $s_J^{(\ell)} \in \mathcal{S}_k$). By employing the analytical formulas for the derivatives of eigenvalue functions (see [47] for the finite dimensional matrix-valued case whose derivation exploits the Hermiticity of the matrix-valued function; the generalization to the infinite dimensional case, leading to the formula $\partial \lambda_J(\omega) / \partial \omega_q = \langle \partial \mathbf{A}(\omega) / \partial \omega_q s_J, s_J \rangle$, where s_J is a unit eigenvector corresponding to $\lambda_J(\omega)$, is straightforward by making use of the self-adjointness of $\mathbf{A}(\omega)$), for

$q = 1, \dots, d$ we obtain

$$\begin{aligned}
 \frac{\partial \lambda_J^{(k)}(\omega^{(\ell)})}{\partial \omega_q} &= \left\langle \frac{\partial \mathbf{A}^{S_k}(\omega^{(\ell)})}{\partial \omega_q} \alpha_J, \alpha_J \right\rangle \\
 &= \left\langle \frac{\partial \mathbf{A}(\omega^{(\ell)})}{\partial \omega_q} \mathbf{S}_k \alpha_J, \mathbf{S}_k \alpha_J \right\rangle \\
 &= \left\langle \frac{\partial \mathbf{A}(\omega^{(\ell)})}{\partial \omega_q} s_J^{(\ell)}, s_J^{(\ell)} \right\rangle = \frac{\partial \lambda_J(\omega^{(\ell)})}{\partial \omega_q}.
 \end{aligned}$$

This completes the proof. $\square$

2.2.3 The Extended Greedy Procedure

To better exploit the Hermite interpolation properties of Lemma 2.2.6, we extend the basic greedy procedure of the previous subsection with the inclusion of additional eigenvectors in the subspaces at points close to the optimizers of the reduced problems. The purpose here is to achieve

$$\lim_{k \rightarrow \infty} \left\| \nabla^2 \lambda_J(\omega^{(k)}) - \nabla^2 \lambda_J^{(k)}(\omega^{(k)}) \right\|_2 = 0$$

in addition to $\lambda_J(\omega^{(k)}) = \lambda_J^{(k)}(\omega^{(k)})$ and $\nabla \lambda_J(\omega^{(k)}) = \nabla \lambda_J^{(k)}(\omega^{(k)})$. These properties enable us to make an analogy with a quasi-Newton method for unconstrained smooth optimization, and come up with a theoretical superlinear rate-of-convergence result in the next section.

In the extended procedure also the reduced eigenvalue optimization problem (2.5) is solved for a subspace $\mathcal{V}$ already constructed. Denoting the optimizer of the reduced problem with ω_* , in addition to the eigenvectors corresponding to $\lambda_1(\omega_*), \dots, \lambda_J(\omega_*)$, the eigenvectors corresponding to $\lambda_1(\omega_* + h e_{pq}), \dots, \lambda_J(\omega_* + h e_{pq})$ are added into the subspace $\mathcal{V}$ for h decaying to zero if convergence occurs, for $p = 1, \dots, d$ and $q = p, \dots, d$. Here $e_{pq} := (1/\sqrt{2})(e_p + e_q)$ for $p \neq q$ as well as $e_{pp} := e_p$. We provide a formal description of this extended subspace procedure in Algorithm 2 below.

The reduced eigenvalue functions of the extended procedure possesses additional interpolatory properties stated in the lemmas below. Their proofs are similar to the proofs for Lemma 2.2.6, so we omit them.

Algorithm 2 The Extended Greedy Subspace Procedure**Input:** A parameter dependent compact self-adjoint operator $\mathbf{A}(\omega)$ of the form (2.1),and a compact subset $\Omega \subset \tilde{\Omega}$ of $\mathbb{R}^d$.

- 1: $\omega^{(1)} \leftarrow$ a random point in Ω .
- 2: $s_1^{(1)}, \dots, s_J^{(1)} \leftarrow$ eigenvectors corresponding to $\lambda_1(\omega^{(1)}), \dots, \lambda_J(\omega^{(1)})$.
- 3: $\mathcal{S}_1 \leftarrow \text{span} \{s_1^{(1)}, \dots, s_J^{(1)}\}$.
- 4: **for** $k = 2, 3, \dots$ **do**
- 5: $\omega^{(k)} \leftarrow$ any $\omega_* \in \arg \min_{\omega \in \Omega} \lambda_J^{(k-1)}(\omega)$ for the minimization problem (MN),
or $\omega^{(k)} \leftarrow$ any $\omega_* \in \arg \max_{\omega \in \Omega} \lambda_J^{(k-1)}(\omega)$ for the maximization problem (MX).
- 6: $s_1^{(k)}, \dots, s_J^{(k)} \leftarrow$ eigenvectors corresponding to $\lambda_1(\omega^{(k)}), \dots, \lambda_J(\omega^{(k)})$.
- 7: $h^{(k)} \leftarrow \|\omega^{(k)} - \omega^{(k-1)}\|_2$
- 8: **for** $p = 1, \dots, d$, $q = p, \dots, d$ **do**
- 9: $s_{1,pq}^{(k)}, \dots, s_{J,pq}^{(k)} \leftarrow$ eigenvectors corresponding to
- 10: $\lambda_1(\omega^{(k)} + h^{(k)}e_{pq}), \dots, \lambda_J(\omega^{(k)} + h^{(k)}e_{pq})$.
- 11: **end for**
- 12: $\mathcal{S}_k \leftarrow \mathcal{S}_{k-1} \oplus \text{span} \{s_1^{(k)}, \dots, s_J^{(k)}\} \oplus \left\{ \bigoplus_{p=1, q=p}^d \text{span} \{s_{1,pq}^{(k)}, \dots, s_{J,pq}^{(k)}\} \right\}$.
- 13: **end for**

Lemma 2.2.7 (Extended Hermite Interpolation). *The assertions of Lemma 2.2.6 hold for Algorithm 2. Additionally, we have*

$$(i) \quad \lambda_J(\omega^{(\ell)} + h^{(\ell)}e_{pq}) = \lambda_J^{(k)}(\omega^{(\ell)} + h^{(\ell)}e_{pq}) \quad ;$$

$$(ii) \quad \text{if } \lambda_J(\omega^{(\ell)} + h^{(\ell)}e_{pq}) \text{ is simple, then } \lambda_J^{(k)}(\omega^{(\ell)} + h^{(\ell)}e_{pq}) \text{ is also simple;}$$

$$(iii) \quad \text{if } \lambda_J(\omega^{(\ell)} + h^{(\ell)}e_{pq}) \text{ is simple, } \nabla \lambda_J(\omega^{(\ell)} + h^{(\ell)}e_{pq}) = \nabla \lambda_J^{(k)}(\omega^{(\ell)} + h^{(\ell)}e_{pq})$$

for every $\ell = 1, \dots, k$, $p = 1, \dots, d$, and $q = p, \dots, d$.

The main motivation for the inclusion of additional eigenvectors in the subspaces is the deduction of theoretical bounds on the proximity of the second derivatives, which we present next. This result is initially established under the assumption that

the third derivatives of the reduced eigenvalue functions $\omega \mapsto \lambda_J^{(k)}(\omega)$ are bounded uniformly with respect to k provided k is large enough. Subsequently, we show in Proposition 2.2.9 that this assumption is always satisfied, hence can be dropped.

Lemma 2.2.8. *Suppose that the sequence $\{\omega^{(k)}\}$ by Algorithm 2 (or Algorithm 1 when $d = 1$ by defining $h^{(k)} := |\omega^{(k)} - \omega^{(k-1)}|$) is convergent, that its limit $\omega_* := \lim_{k \rightarrow \infty} \omega^{(k)}$ lies strictly in the interior of Ω , and that $\lambda_J(\omega_*)$ is simple. Assume, furthermore, that in an open ball containing ω_* , all third derivatives of functions $\omega \mapsto \lambda_J^{(k)}(\omega)$ are bounded uniformly with respect to $k \geq k_0$ with k_0 sufficiently large. Then the following assertions hold:*

(i) *There exists a constant $C > 0$ such that*

$$\left| \frac{\partial^2 \lambda_J(\omega^{(k)})}{\partial \omega_p \partial \omega_q} - \frac{\partial^2 \lambda_J^{(k)}(\omega^{(k)})}{\partial \omega_p \partial \omega_q} \right|_2 \leq Ch^{(k)} \quad \text{for } p, q = 1, \dots, d, \quad (2.12)$$

in particular,

$$\left\| \nabla^2 \lambda_J(\omega^{(k)}) - \nabla^2 \lambda_J^{(k)}(\omega^{(k)}) \right\|_2 \leq dCh^{(k)}$$

for all k large enough.

(ii) *Additionally, if $\nabla^2 \lambda_J(\omega_*)$ is invertible, then*

$$\left\| [\nabla^2 \lambda_J(\omega^{(k)})]^{-1} - [\nabla^2 \lambda_J^{(k)}(\omega^{(k)})]^{-1} \right\|_2 = O(h^{(k)}) \quad (2.13)$$

for all k large enough.

Proof. (i) First we specify a ball centered at ω_* in which $\lambda_J(\omega)$ and $\lambda_J^{(k)}(\omega)$ for all large k are simple. The argument initially assumes $J > 1$. Letting $\varepsilon := \min\{\lambda_{J-1}(\omega_*) - \lambda_J(\omega_*), \lambda_J(\omega_*) - \lambda_{J+1}(\omega_*)\}$, consider the ball $\mathcal{B}(\omega_*, \varepsilon/(8\gamma))$, where γ is the Lipschitz constant as in Lemma 2.2.1. Without loss of generality, let us assume $\mathcal{B}(\omega_*, \varepsilon/(8\gamma)) \subseteq \Omega$ (i.e., otherwise choose ε even smaller so that $\mathcal{B}(\omega_*, \varepsilon/(8\gamma)) \subseteq \Omega$). Now, due to $\lambda_{J-1}(\omega_*) - \lambda_J(\omega_*) \geq \varepsilon$ as well as $\lambda_J(\omega_*) - \lambda_{J+1}(\omega_*) \geq \varepsilon$, and by Lemma 2.2.1, we have

$$\lambda_{J-1}(\omega) - \lambda_J(\omega) \geq 3\varepsilon/4 \quad \text{and} \quad \lambda_J(\omega) - \lambda_{J+1}(\omega) \geq 3\varepsilon/4 \quad \forall \omega \in \mathcal{B}(\omega_*, \varepsilon/(8\gamma)). \quad (2.14)$$

Next choose k large enough so that $\mathcal{B}(\omega^{(k)}, h^{(k)}) \subset \mathcal{B}(\omega_*, \varepsilon/(8\gamma))$. We will show that $\lambda_j^{(k)}(\omega)$ is also simple in $\mathcal{B}(\omega_*, \varepsilon/(8\gamma))$. In this respect we note that $\lambda_j(\omega^{(k)}) = \lambda_j^{(k)}(\omega^{(k)})$ for $j = J-1, J$ due to parts (i) and (ii) of Lemma 2.2.6, so from (2.14) we have

$$\begin{aligned} \lambda_{J-1}^{(k)}(\omega^{(k)}) - \lambda_J^{(k)}(\omega^{(k)}) &\geq 3\varepsilon/4 \quad \text{and} \\ \lambda_J^{(k)}(\omega^{(k)}) - \lambda_{J+1}^{(k)}(\omega^{(k)}) &\geq \lambda_J^{(k)}(\omega^{(k)}) - \lambda_{J+1}(\omega^{(k)}) \geq 3\varepsilon/4, \end{aligned}$$

where in the last line we also used $\lambda_{J+1}^{(k)}(\omega^{(k)}) \leq \lambda_{J+1}(\omega^{(k)})$ which holds due to monotonicity (Lemma 2.2.2). Another application of Lemma 2.2.1 yields

$$\lambda_{J-1}^{(k)}(\omega) - \lambda_J^{(k)}(\omega) \geq \varepsilon/4 \quad \text{and} \quad \lambda_J^{(k)}(\omega) - \lambda_{J+1}^{(k)}(\omega) \geq \varepsilon/4 \quad \forall \omega \in \mathcal{B}(\omega_*, \varepsilon/(8\gamma)).$$

We remark that the argument above applies to the case $J = 1$ trivially by considering only the gaps $\lambda_J(\omega) - \lambda_{J+1}(\omega)$ as well as $\lambda_J^{(k)}(\omega) - \lambda_{J+1}^{(k)}(\omega)$ and showing they remain bounded away from zero for all ω inside $\mathcal{B}(\omega_*, \varepsilon/(8\gamma))$.

We prove the desired bounds (2.12) first for the sequence $\{\omega^{(k)}\}$ generated by Algorithm 2. The simplicity of the eigenvalue functions $\lambda_J(\omega)$ and $\lambda_J^{(k)}(\omega)$ on $\mathcal{B}(\omega_*, \varepsilon/(8\gamma))$ imply that, for each p, q , the functions

$$\ell_{pq}(t) := \lambda_J(\omega^{(k)} + th^{(k)}e_{pq}) \quad \text{and} \quad \ell_{pq,k}(t) := \lambda_J^{(k)}(\omega^{(k)} + th^{(k)}e_{pq}) \quad (2.15)$$

are analytic on $(0, 1)$. By applying Taylor's theorem to $\ell_{pq}(t), \ell_{pq,k}(t)$ on the interval $(0, 1)$, we obtain

$$\begin{aligned} \ell_{pq}(1) &= \ell_{pq}(0) + \ell'_{pq}(0) + \ell''_{pq}(0)/2 + \ell'''_{pq}(\eta)/6, \\ \ell_{pq,k}(1) &= \ell_{pq,k}(0) + \ell'_{pq,k}(0) + \ell''_{pq,k}(0)/2 + \ell'''_{pq,k}(\eta^{(k)})/6 \end{aligned}$$

for some $\eta, \eta^{(k)} \in (0, 1)$. Now by employing $\ell_{pq}(0) = \ell_{pq,k}(0)$, $\ell'_{pq}(0) = \ell'_{pq,k}(0)$ due to parts (i), (iv) of Lemma 2.2.6, and $\ell_{pq}(1) = \ell_{pq}^{(k)}(1)$ due to part (i) of Lemma 2.2.7, we deduce

$$\frac{\ell''_{pq}(0) - \ell''_{pq,k}(0)}{2} = \frac{\ell'''_{pq,k}(\eta^{(k)}) - \ell'''_{pq}(\eta)}{6}.$$

In the last expression,

$$\ell''_{pq}(0) = [h^{(k)}]^2 e_{pq}^T \nabla^2 \lambda_J(\omega^{(k)}) e_{pq} \quad \text{and} \quad \ell''_{pq,k}(0) = [h^{(k)}]^2 e_{pq}^T \nabla^2 \lambda_J^{(k)}(\omega^{(k)}) e_{pq}$$

as well as $\ell'''_{pq,k}(\eta^{(k)}) = O([h^{(k)}]^3)$ and $\ell'''_{pq}(\eta) = O([h^{(k)}]^3)$, so we have

$$\frac{[h^{(k)}]^2 \left| e_{pq}^T \left[\nabla^2 \lambda_J(\omega^{(k)}) - \nabla^2 \lambda_J^{(k)}(\omega^{(k)}) \right] e_{pq} \right|}{2} \leq \frac{D}{6} [h^{(k)}]^3 \quad (2.16)$$

for some constant D independent of k .

Choosing $q = p$ in (2.16) yields

$$\left| \frac{\partial^2 \lambda_J(\omega^{(k)})}{\partial \omega_p^2} - \frac{\partial^2 \lambda_J^{(k)}(\omega^{(k)})}{\partial \omega_p^2} \right| \leq \frac{D}{3} h^{(k)}.$$

Additionally, for $q \neq p$, rewriting (2.16) as

$$\left| \frac{1}{2} \left(\sum_{\ell=p,q} \frac{\partial^2 \lambda_J(\omega^{(k)})}{\partial \omega_\ell^2} - \frac{\partial^2 \lambda_J^{(k)}(\omega^{(k)})}{\partial \omega_\ell^2} \right) + \left(\frac{\partial^2 \lambda_J(\omega^{(k)})}{\partial \omega_p \partial \omega_q} - \frac{\partial^2 \lambda_J^{(k)}(\omega^{(k)})}{\partial \omega_p \partial \omega_q} \right) \right| \leq \frac{D}{3} h^{(k)},$$

we obtain

$$\left| \frac{\partial^2 \lambda_J(\omega^{(k)})}{\partial \omega_p \partial \omega_q} - \frac{\partial^2 \lambda_J^{(k)}(\omega^{(k)})}{\partial \omega_p \partial \omega_q} \right| \leq \frac{2D}{3} h^{(k)}$$

leading us to (2.12). The proof above establishing the bounds (2.12) also applies to the sequence $\{\omega^{(k)}\}$ generated by Algorithm 1 when $d = 1$ by defining $\tilde{h}^{(k)} := \omega^{(k-1)} - \omega^{(k)}$ (so that $h^{(k)} = |\tilde{h}^{(k)}|$) and letting

$$\ell_{pq}(t) := \lambda_J(\omega^{(k)} + t\tilde{h}^{(k)}e_{pq}) \quad \text{and} \quad \ell_{pq,k}(t) := \lambda_J^{(k)}(\omega^{(k)} + t\tilde{h}^{(k)}e_{pq})$$

instead of (2.15).

(ii) From part (i), as well as by the existence of $\nabla^2 \lambda_J(\omega^{(k)})$, $\nabla^2 \lambda_J^{(k)}(\omega^{(k)})$ for all k large enough so that $\mathcal{B}(\omega^{(k)}, h^{(k)}) \subset \mathcal{B}(\omega_*, \varepsilon/(8\gamma))$ and by the continuity of $\nabla^2 \lambda_J(\omega)$ at $\omega = \omega_*$, we have

$$\lim_{k \rightarrow \infty} \nabla^2 \lambda_J(\omega^{(k)}) = \lim_{k \rightarrow \infty} \nabla^2 \lambda_J^{(k)}(\omega^{(k)}) = \nabla^2 \lambda_J(\omega_*).$$

Exploiting this and the invertibility of $\nabla^2 \lambda_J(\omega_*)$, we deduce that $\nabla^2 \lambda_J(\omega^{(k)})$ and $\nabla^2 \lambda_J^{(k)}(\omega^{(k)})$ are invertible for all k large enough.

For such a k , letting $A = \nabla^2 \lambda_J(\omega^{(k)})$, $\tilde{A} = \nabla^2 \lambda_J^{(k)}(\omega^{(k)})$ and $X = [\nabla^2 \lambda_J(\omega^{(k)})]^{-1}$, $\tilde{X} = [\nabla^2 \lambda_J^{(k)}(\omega^{(k)})]^{-1}$, the classical adjugate formula (i.e., $[B^{-1}]_{ij} = \frac{(-1)^{i+j} \det(B_{ji})}{\det(B)}$ for an invertible B , where B_{ji} denotes the matrix obtained from B by removing its j th row and the i th column) yields

$$\tilde{x}_{ij} = \frac{(-1)^{i+j} \det(\tilde{A}_{ji})}{\det(\tilde{A})} \quad \text{for } i, j = 1, \dots, d.$$

By part (i), in particular (2.12), we have $\det(\tilde{A}_{ji}) = \det(A_{ji}) + O(h^{(k)})$ and $\det(\tilde{A}) = \det(A) + O(h^{(k)})$, so we deduce

$$\tilde{x}_{ij} = \frac{(-1)^{i+j} \det(A_{ji}) + O(h^{(k)})}{\det(A) + O(h^{(k)})} = \frac{(-1)^{i+j} \det(A_{ji})}{\det(A)} + O(h^{(k)}) = x_{ij} + O(h^{(k)})$$

leading to (2.13). $\square$

Proposition 2.2.9. *Suppose that the sequence $\{\omega^{(k)}\}$ by Algorithm 1 Algorithm 2 is convergent, its limit $\omega_* := \lim_{k \rightarrow \infty} \omega^{(k)}$ lies strictly in the interior of Ω , and that $\lambda_J(\omega_*)$ is simple. Then for k sufficiently large all third derivatives of functions $\omega \mapsto \lambda_J^{(k)}(\omega)$ exist in an open ball containing ω_* , and they can be bounded uniformly with respect to k .*

Proof. For the sake of clarity we first consider the case when $d = 1$. Following from the assumptions of the proposition and from the elements spelled out in the proof of Lemma 2.2.8, there exists an open interval $\mathcal{I} := (\omega_* - r_1, \omega_* + r_1)$, and numbers $k_1 \in \mathbb{N}$ and $\ell \in \mathbb{R}, \ell > 0$, such that for every $k \geq k_1$ and all $\omega \in \mathcal{I}$, the eigenvalue $\lambda_J^{(k)}(\omega)$ is simple, as well as

$$\min \left\{ \left| \lambda_J^{(k)}(\omega) - \lambda_{J-1}^{(k)}(\omega) \right|, \left| \lambda_J^{(k)}(\omega) - \lambda_{J+1}^{(k)}(\omega) \right| \right\} \geq \ell. \quad (2.17)$$

Since functions f_ℓ are assumed real analytic, there exist analytic extensions on the complex plane at ω_* , which we refer to by functions $\hat{f}_\ell$, $\ell = 1, \dots, \kappa$. We denote by $\hat{\mathbf{A}}^{S_k}(\omega)$ the corresponding extension of $\mathbf{A}^{S_k}(\omega)$. Following the ideas spelled out in the proof of Lemma 3 of [44], we can express

$$\begin{aligned} \left\| \hat{\mathbf{A}}^{S_k}(\omega) - \hat{\mathbf{A}}^{S_k}(\omega_*) \right\|_2 &\leq \sum_{\ell=1}^{\kappa} \|\mathbf{S}_k^* \mathbf{A}_\ell \mathbf{S}_k\|_2 \left| \hat{f}_\ell(\omega) - \hat{f}_\ell(\omega_*) \right| \\ &\leq \sum_{\ell=1}^{\kappa} \|\mathbf{A}_\ell\|_2 \left| \hat{f}_\ell(\omega) - \hat{f}_\ell(\omega_*) \right|. \end{aligned} \quad (2.18)$$

Note that if ω is nonreal, the difference $\hat{\mathbf{A}}^{\mathcal{S}_k}(\omega) - \hat{\mathbf{A}}^{\mathcal{S}_k}(\omega_*)$ might be non-Hermitian. However, an important conclusion from the above inequalities is that $\|\hat{\mathbf{A}}^{\mathcal{S}_k}(\omega) - \hat{\mathbf{A}}^{\mathcal{S}_k}(\omega_*)\|_2$ can be bounded from above uniformly independent of k , that is independent of the dimension of the subspace $\mathcal{S}_k$, for all ω inside a disk centered at ω_* . Furthermore, this upper bound can be chosen arbitrarily close to 0 by reducing the radius of the disk.

From (2.17), (2.18), and Theorem 5.1 of [77], we conclude that there exists a number $r_2 \in (0, r_1)$ independent of $k \geq k_1$ (i.e., the dimension of the subspace), such that the eigenvalue function $\hat{\lambda}_J^{(k)}$, obtained by replacing $\mathbf{A}^{\mathcal{S}_k}$ with its analytic extension $\hat{\mathbf{A}}^{\mathcal{S}_k}$, remains simple on and inside the disk $\{\omega \in \mathbb{C} : |\omega - \omega_*| \leq r_2\}$ for every $k \geq k_1$. Hence $\hat{\lambda}_J^{(k)}$ is well-defined and analytic inside this disk for every $k \geq k_1$. For $\tilde{\omega} \in \mathbb{R}$ satisfying $|\tilde{\omega} - \omega_*| < r_2/2$, we have

$$\frac{d^3 \hat{\lambda}_J^{(k)}}{d\omega^3}(\tilde{\omega}) = \frac{3!}{2\pi i} \oint_{|\omega - \tilde{\omega}| = r_2/2} \frac{\hat{\lambda}_J^{(k)}(\omega)}{(\omega - \tilde{\omega})^4} d\omega$$

with i the imaginary unit. We can bound

$$\begin{aligned} \left| \frac{d^3 \hat{\lambda}_J^{(k)}}{d\omega^3}(\tilde{\omega}) \right| &\leq \frac{96}{r_2^3} \max_{\omega \in \mathbb{C}, |\omega - \tilde{\omega}| = r_2/2} \left| \hat{\lambda}_J^{(k)}(\omega) \right| \\ &\leq \frac{96}{r_2^3} \sum_{\ell=1}^{\kappa} \left\| \mathbf{A}_\ell^{\mathcal{S}_k} \right\|_2 \left(\max_{\omega \in \mathbb{C}, |\omega - \tilde{\omega}| = r_2/2} \left| \hat{f}_\ell(\omega) \right| \right) \\ &\leq \frac{96}{r_2^3} \sum_{\ell=1}^{\kappa} \left\| \mathbf{A}_\ell \right\|_2 \left(\max_{\omega \in \mathbb{C}, |\omega - \tilde{\omega}| = r_2/2} \left| \hat{f}_\ell(\omega) \right| \right), \end{aligned} \quad (2.19)$$

hence, an upper bound is established that does not depend on $k \geq k_1$.

For the case $d > 1$ only a slight adaptation is needed for the mixed derivatives. We sketch the main principles by means of the case when $d = 2$. From the ideas behind (2.19) a bound on $\hat{\lambda}_J^{(k)}$ induces a bound, uniform in k , on its partial derivative $\frac{\partial^2 \hat{\lambda}_J^{(k)}}{\partial \omega_1^2}$ in an open set containing ω_* . Likewise the bound on the latter induces a bound on $\frac{\partial^3 \hat{\lambda}_J^{(k)}}{\partial \omega_1^2 \partial \omega_2}$. $\square$

2.3 Convergence Analysis

In this section we analyze the convergence properties of the two subspace procedures introduced. The first part concerns global convergence: it elaborates on whether

the iterates of Algorithm 1 and Algorithm 2 necessarily converge as the subspace dimensions grow to infinity, and if they do converge, where they converge to. The second part establishes a superlinear rate-of-convergence result for the iterates of Algorithm 2 (and Algorithm 1 when $d = 1$).

2.3.1 Global Convergence

The subspace procedures, when applied to the minimization problem (MN), converge globally. A formal statement of this global convergence property together with its proof are given in what follows. Note that the sequence $\{\omega^{(k)}\}$ generated by Algorithm 1 or Algorithm 2 belongs to the bounded set Ω , so it must have convergent subsequences.

Theorem 2.3.1. *The following hold for both Algorithm 1 and Algorithm 2.*

(i) *The limit of every convergent subsequence of the sequence $\{\omega^{(k)}\}$ for the minimization problem (MN) is a global minimizer of $\lambda_J(\omega)$ over Ω .*

(ii) $\lim_{k \rightarrow \infty} \lambda_J^{(k)}(\omega^{(k+1)}) = \lim_{k \rightarrow \infty} \min_{\omega \in \Omega} \lambda_J^{(k)}(\omega) = \min_{\omega \in \Omega} \lambda_J(\omega).$

Proof. (i) Let $\{\omega^{(\ell_k)}\}$ be a convergent subsequence of $\{\omega^{(k)}\}$. By the monotonicity property, that is by Lemma 2.2.2, we have

$$\begin{aligned} \min_{\omega \in \Omega} \lambda_J(\omega) &\geq \min_{\omega \in \Omega} \lambda_J^{(\ell_{k+1}-1)}(\omega) \\ &= \lambda_J^{(\ell_{k+1}-1)}(\omega^{(\ell_{k+1})}) \geq \lambda_J^{(\ell_k)}(\omega^{(\ell_{k+1})}), \end{aligned} \tag{2.20}$$

and, by the interpolatory property, that is part (i) of Lemma 2.2.6,

$$\min_{\omega \in \Omega} \lambda_J(\omega) \leq \lambda_J(\omega^{(\ell_k)}) = \lambda_J^{(\ell_k)}(\omega^{(\ell_k)}). \tag{2.21}$$

But observe that the Lipschitz continuity of $\lambda_J^{(\ell_k)}(\omega)$ (see Lemma 2.2.1) implies

$$\lim_{k \rightarrow \infty} \left| \lambda_J^{(\ell_k)}(\omega^{(\ell_{k+1})}) - \lambda_J^{(\ell_k)}(\omega^{(\ell_k)}) \right| = \gamma \lim_{k \rightarrow \infty} \|\omega^{(\ell_{k+1})} - \omega^{(\ell_k)}\|_2 = 0.$$

Consequently, taking the limits in (2.20) and (2.21) as $k \rightarrow \infty$ and employing the interpolatory property (part (i) of Lemma 2.2.6), we deduce

$$\lim_{k \rightarrow \infty} \lambda_J(\omega^{(\ell_k)}) = \lim_{k \rightarrow \infty} \lambda_J^{(\ell_k)}(\omega^{(\ell_k)}) = \lim_{k \rightarrow \infty} \lambda_J^{(\ell_k)}(\omega^{(\ell_{k+1})}) = \min_{\omega \in \Omega} \lambda_J(\omega). \quad (2.22)$$

Finally, by the continuity of $\lambda_J(\omega)$, the sequence $\{\omega^{(\ell_k)}\}$ must converge to a global minimizer of $\lambda_J(\omega)$.

(ii) Let $\lambda_* := \min_{\omega \in \Omega} \lambda_J(\omega)$. We first show that the sequence $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ is convergent. In this respect observe that for every pair of positive integers p, k such that $p > k$, due to monotonicity (Lemma 2.2.2) we have

$$\begin{aligned} \lambda_* &\geq \min_{\omega \in \Omega} \lambda_J^{(p)}(\omega) = \lambda_J^{(p)}(\omega^{(p+1)}) \\ &\geq \lambda_J^{(k)}(\omega^{(p+1)}) \geq \min_{\omega \in \Omega} \lambda_J^{(k)}(\omega) = \lambda_J^{(k)}(\omega^{(k+1)}). \end{aligned}$$

Hence the sequence $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ is monotonically increasing bounded above by λ_* proving its convergence.

Now let $\{\omega^{(\ell_k)}\}$ be a convergent subsequence of $\{\omega^{(k)}\}$ as in part (i), and let us consider the sequence $\{\lambda_J^{(\ell_{k+1}-1)}(\omega^{(\ell_{k+1})})\}$, which is a subsequence of $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$. We complete the proof by establishing the convergence of this subsequence to λ_* . The monotonicity and the boundedness of $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ from above by λ_* imply

$$\lambda_J^{(\ell_k)}(\omega^{(\ell_{k+1})}) \leq \lambda_J^{(\ell_{k+1}-1)}(\omega^{(\ell_{k+1})}) \leq \lambda_*. \quad (2.23)$$

Above, as shown in part (i) (in particular, see (2.22)), $\lim_{k \rightarrow \infty} \lambda_J^{(\ell_k)}(\omega^{(\ell_{k+1})}) = \lambda_*$, so we must also have $\lim_{k \rightarrow \infty} \lambda_J^{(\ell_{k+1}-1)}(\omega^{(\ell_{k+1})}) = \lambda_*$ as desired. $\square$

As for the maximization problem **(MX)**, it does not seem possible to conclude with such a convergence result to a global maximizer. This is because only a lower bound (but not an upper bound) is available in terms of the reduced eigenvalue functions for the maximum value of $\lambda_J(\omega)$ over $\omega \in \Omega$. However, the sequence $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ is still convergent as shown next.

Theorem 2.3.2. *The sequence $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ generated by Algorithms 1 and 2 for the maximization problem **(MX)** converges.*

Proof. Letting $\lambda^* := \max_{\omega \in \Omega} \lambda_J(\omega)$, the sequence by Algorithms 1 and 2 satisfies

$$\begin{aligned} \lambda_J^{(k-1)}(\omega^{(k)}) &\leq \lambda_J(\omega^{(k)}) = \lambda_J^{(k)}(\omega^{(k)}) \\ &\leq \max_{\omega \in \Omega} \lambda_J^{(k)}(\omega) = \lambda_J^{(k)}(\omega^{(k+1)}) \leq \lambda^*, \end{aligned} \quad (2.24)$$

where the first and the last inequality follow from the monotonicity, and the equality in the first line is a consequence of the interpolatory property. These inequalities lead to the conclusion that the sequence $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ is monotone increasing and bounded above by λ^* , hence convergent. $\square$

We observe in practice that the sequence $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ converges to a $\tilde{\lambda}$ such that $\lambda_J(\tilde{\omega}) = \tilde{\lambda}$ for some $\tilde{\omega}$ that is a local maximizer of $\lambda_J(\omega)$, that is not necessarily a global maximizer.

We illustrate these convergence results for the minimization as well as for the maximization of the largest eigenvalue $\lambda_1(\omega)$ of

$$A(\omega) = \frac{Ae^{i\omega} + A^*e^{-i\omega}}{2} = \cos(\omega) \left(\frac{A + A^*}{2} \right) + \sin(\omega) \left(\frac{iA - iA^*}{2} \right) \quad (2.25)$$

over $\omega \in [0, 2\pi]$ and for a particular matrix A . The maximum of the largest eigenvalue of $A(\omega)$ over $\omega \in [0, 2\pi]$ corresponds to the numerical radius of A [37, 31]; see Section 1.1.3 for more on the numerical radius. Here we particularly choose A as the 400×400 matrix whose real part comes from a five-point finite difference discretization of a Poisson equation, and whose complex part has random entries selected independently from a normal distribution with zero mean and standard deviation equal to 20. An application of the basic subspace procedure (Algorithm 1) for the maximization of $\lambda_1(\omega)$ starting with $\omega^{(1)} = 2$ results in convergence to a local maximizer $\tilde{\omega} \approx 2.353$, whereas initiating the procedure with $\omega^{(1)} = 5.5$ leads to convergence to $\omega^* \approx 6.145$, the unique global maximizer of $\lambda_1(\omega)$ over $[0, 2\pi]$. This is depicted on the top row in Figure 2.1 on the left and on the right, respectively. The situation is quite different when the subspace procedure is applied to minimize $\lambda_1(\omega)$. It converges to the global minimizer $\omega_* \approx 3.959$ of $\lambda_1(\omega)$ regardless of whether the procedure is initiated with $\omega^{(1)} = 2$ or $\omega^{(1)} = 5.5$ as depicted on the bottom row of Figure 2.1. Notice that

the subspace procedure for the maximization problem constructs reduced eigenvalue functions that capture $\lambda_1(\omega)$ well only locally around the maximizers. In contrast, for the minimization problem the reduced eigenvalue functions capture $\lambda_1(\omega)$ globally, but their accuracy is higher around the minimizers.

2.3.2 Rate-of-Convergence

Next we are concerned with how quickly the iterates of the subspace procedures converge when they do converge to a smooth stationary point of $\lambda_J(\omega)$. Note that by Theorem 2.3.1, for the minimization problem, if the sequence $\{\omega^{(k)}\}$ by Algorithm 2 (or Algorithm 1) converges to a point ω_* where $\lambda_J(\omega)$ is simple, then ω_* must be a smooth stationary point of $\lambda_J(\omega)$. The analysis below applies to Algorithm 2 (and Algorithm 1 when $d = 1$) in a unified way, both for the minimization problem and for the maximization problem.

Theorem 2.3.3 (Superlinear Convergence). *Suppose that the sequence $\{\omega^{(k)}\}$ by Algorithm 2 (or Algorithm 1 when $d = 1$) converges to a point ω_* in the interior of Ω such that*

(i) $\lambda_J(\omega_*)$ is simple, (ii) $\nabla \lambda_J(\omega_*) = 0$, and (iii) $\nabla^2 \lambda_J(\omega_*)$ is invertible.

Then there exists a constant $\alpha > 0$ such that

$$\frac{\|\omega^{(k+1)} - \omega_*\|_2}{\|\omega^{(k)} - \omega_*\|_2 \max \{\|\omega^{(k)} - \omega_*\|_2, \|\omega^{(k-1)} - \omega_*\|_2\}} \leq \alpha \quad \forall k. \quad (2.26)$$

Proof. The arguments in the first two paragraphs of the proof of Lemma 2.12 establish the existence of a ball $\mathcal{B}(\omega_*, \eta)$ containing $\mathcal{B}(\omega^{(k)}, h^{(k)})$ for all k large enough, say for $k \geq k'$, such that $\lambda_J(\omega)$ as well as $\lambda_J^{(k)}(\omega)$ for $k \geq k'$ are simple for all $\omega \in \mathcal{B}(\omega_*, \eta)$. The Hessians $\nabla^2 \lambda_J(\omega)$ and $\nabla^2 \lambda_J^{(k)}(\omega)$ for $k \geq k'$ are Lipschitz continuous inside this ball.

Additionally, following the arguments in the proof of part (ii) of Lemma 2.12, the invertibility of $\nabla^2 \lambda_J(\omega_*)$ ensures that $\nabla^2 \lambda_J(\omega^{(k)})$ and $\nabla^2 \lambda_J^{(k)}(\omega^{(k)})$ are invertible (indeed $\|[\nabla^2 \lambda_J(\omega^{(k)})]^{-1}\|_2$ and $\|[\nabla^2 \lambda_J^{(k)}(\omega^{(k)})]^{-1}\|_2$ are bounded from above by some constant) for all k large enough, say for $k \geq k'' \geq k'$.

Now for $k \geq k''$ by Taylor's theorem with integral remainder,

$$0 = \nabla \lambda_J(\omega_*) = \nabla \lambda_J(\omega^{(k)}) + \int_0^1 \nabla^2 \lambda_J(\omega^{(k)} + \alpha(\omega_* - \omega^{(k)})) (\omega_* - \omega^{(k)}) d\alpha.$$

Employing $\nabla \lambda_J(\omega^{(k)}) = \nabla \lambda_J^{(k)}(\omega^{(k)})$ (Lemma 2.2.6 (iii)) in this equation and left multiplying both sides of the equation by the inverse of $\nabla^2 \lambda_J(\omega^{(k)})$ yield

$$\begin{aligned} 0 &= [\nabla^2 \lambda_J(\omega^{(k)})]^{-1} \cdot \nabla \lambda_J^{(k)}(\omega^{(k)}) + (\omega_* - \omega^{(k)}) + \\ &[\nabla^2 \lambda_J(\omega^{(k)})]^{-1} \int_0^1 [\nabla^2 \lambda_J(\omega^{(k)} + \alpha(\omega_* - \omega^{(k)})) - \nabla^2 \lambda_J(\omega^{(k)})] (\omega_* - \omega^{(k)}) d\alpha. \end{aligned} \quad (2.27)$$

A second order Taylor expansion of $\nabla \lambda_J^{(k)}(\omega)$ about $\omega^{(k)}$, noting $\nabla \lambda_J^{(k)}(\omega^{(k+1)}) = 0$, implies

$$[\nabla^2 \lambda_J^{(k)}(\omega^{(k)})]^{-1} \nabla \lambda_J^{(k)}(\omega^{(k)}) = -(\omega^{(k+1)} - \omega^{(k)}) + O([h^{(k+1)}]^2).$$

Using this equality in (2.27) leads us to

$$\begin{aligned} 0 &= \omega_* - \omega^{(k+1)} + \left\{ [\nabla^2 \lambda_J(\omega^{(k)})]^{-1} - [\nabla^2 \lambda_J^{(k)}(\omega^{(k)})]^{-1} \right\} \nabla \lambda_J(\omega^{(k)}) \\ &\quad + O([h^{(k+1)}]^2) + \\ &[\nabla^2 \lambda_J(\omega^{(k)})]^{-1} \int_0^1 [\nabla^2 \lambda_J(\omega^{(k)} + \alpha(\omega_* - \omega^{(k)})) - \nabla^2 \lambda_J(\omega^{(k)})] (\omega_* - \omega^{(k)}) d\alpha, \end{aligned} \quad (2.28)$$

which, by taking the 2-norms and employing the triangle inequality, yields

$$\begin{aligned} \|\omega^{(k+1)} - \omega_*\|_2 &\leq \left\| [\nabla^2 \lambda_J(\omega^{(k)})]^{-1} - [\nabla^2 \lambda_J^{(k)}(\omega^{(k)})]^{-1} \right\|_2 \|\nabla \lambda_J(\omega^{(k)})\|_2 \\ &\quad + O([h^{(k+1)}]^2) + \\ &\left\| [\nabla^2 \lambda_J(\omega^{(k)})]^{-1} \right\|_2 \int_0^1 \|\nabla^2 \lambda_J(\omega^{(k)} + \alpha(\omega_* - \omega^{(k)})) - \nabla^2 \lambda_J(\omega^{(k)})\|_2 \|\omega^{(k)} - \omega_*\|_2 d\alpha. \end{aligned} \quad (2.29)$$

To conclude with the desired superlinear convergence result, we bound the terms on the right-hand side of the inequality in (2.29) from above in terms of $\|\omega^{(k+1)} - \omega_*\|_2$, $\|\omega^{(k)} - \omega_*\|_2$, and $\|\omega^{(k-1)} - \omega_*\|_2$. To this end, first note that the terms in the third line of (2.29) is $O(\|\omega^{(k)} - \omega_*\|_2^2)$; this is because of the boundedness of

$\|[\nabla^2 \lambda_J(\omega^{(k)})]^{-1}\|_2$ and the Lipschitz continuity of $\nabla^2 \lambda_J(\omega)$ inside $\mathcal{B}(\omega_*, \eta)$. Second, $\|\nabla \lambda_J(\omega^{(k)})\|_2 = O(\|\omega^{(k)} - \omega_*\|)$, as can be seen from a Taylor expansion of $\nabla \lambda_J(\omega)$ about $\omega^{(k)}$ and by exploiting $\nabla \lambda_J(\omega_*) = 0$. Finally, due to part (ii) of Lemma 2.2.8, we have $\|[\nabla^2 \lambda_J(\omega^{(k)})]^{-1} - [\nabla^2 \lambda_J(\omega^{(k)})]^{-1}\|_2 = O(h^{(k)})$. Applying all these bounds to (2.29) gives rise to

$$\|\omega^{(k+1)} - \omega_*\|_2 \leq O(h^{(k)}\|\omega^{(k)} - \omega_*\|_2) + O([h^{(k+1)}]^2) + O(\|\omega^{(k)} - \omega_*\|_2^2).$$

The desired result (2.26) follows noting $h^{(k)} \leq 2 \max\{\|\omega^{(k)} - \omega_*\|_2, \|\omega^{(k-1)} - \omega_*\|_2\}$ and $[h^{(k+1)}]^2 \leq 2(\|\omega^{(k)} - \omega_*\|_2^2 + \|\omega^{(k+1)} - \omega_*\|_2^2)$. $\square$

Regarding, specifically, the minimization of the largest eigenvalue when $d = 1$, the order of the superlinear rate of convergence for Algorithm 1 is shown to be at least $1 + \sqrt{2}$ in [44, Theorem 1].

It does not seem straightforward to extend the rate of convergence result above to Algorithm 1 when $d \geq 2$, because relations such as the ones given by Lemma 2.2.8 between the second derivatives of $\lambda_J(\omega)$ and $\lambda_J^{(k)}(\omega)$ are not evident. We observe a superlinear rate-of-convergence for Algorithm 1 in practice as well. Algorithm 2 requires a slightly fewer iterations to reach a prescribed accuracy, but Algorithm 1 attains the prescribed accuracy with subspaces of smaller dimension. These observations are illustrated in Table 2.1 on the problem of minimizing the largest eigenvalue $\lambda_1(\omega)$ of

$$A(\omega) = A_0 + \omega_1 A_1 + \cdots + \omega_d A_d \quad (2.30)$$

for given Hermitian matrices $A_0, A_1, \dots, A_d \in \mathbb{C}^{n \times n}$ for $d = 2, 3$, where we restrict $\omega_1, \dots, \omega_d$ to the interval $[-60, 60]$. Such eigenvalue optimization problems are convex [67], and arise from a classical structural design problem [16, 50]. Additionally, as discussed in Section 2.5.4, they are closely related to SDPs. In Table 2.1 the minimal values $\lambda_1^{(k)}(\omega^{(k+1)})$ of the reduced eigenvalue functions $\lambda_1^{(k)}(\omega)$, as well as the error $\|\omega^{(k+1)} - \omega_*\|_2$, converge superlinearly with respect to k both for $d = 2$ and for $d = 3$. In both cases the extended subspace procedure on the right column achieves

10 decimal digits accuracy (in the sense that $\lambda_1(\omega_*) - \lambda_1^{(k)}(\omega^{(k+1)}) \leq 10^{-10}$) after 8 iterations but with subspaces of dimension 32 and 56 for $d = 2$ and $d = 3$, respectively. On the other hand, the basic subspace procedure on the left column requires 11 and 15 iterations, which are also the dimensions of the subspaces constructed, for $d = 2$ and $d = 3$, respectively, to achieve the same accuracy. Observe also that the minimal values $\lambda_1^{(k)}(\omega^{(k+1)})$ seem to converge at a rate even faster than the rate of decay of the errors $\|\omega^{(k+1)} - \omega_*\|_2$.

In the one parameter case (i.e., $d = 1$) Theorem 2.3.3 applies to Algorithm 1 as well to establish its superlinear convergence. The convergence of Algorithm 1 on the example of Figure 2.1 (concerning the minimization or maximization of the largest eigenvalue of the matrix-valued function in (2.25)) starting with $\omega^{(1)} = 5.5$ is depicted in Table 2.2. For the maximization and minimization the iterates $\{\omega^{(k)}\}$ converge to the global maximizer (≈ 6.145) and global minimizer (≈ 3.959) of $\lambda_1(\omega)$ at a superlinear rate, which is realized earlier for the maximization problem. The optimal values of the reduced eigenvalue function $\lambda_1^{(k)}(\omega^{(k+1)})$ converge to the globally maximal value and minimal value of $\lambda_1(\omega)$ at an even faster rate.

2.4 Variations and Extensions

2.4.1 A Greedy Subspace Procedure without Past

The rate-of-convergence analysis of the previous section and, in particular, the proof of Theorem 2.3.3, for Algorithm 2 (for Algorithm 1 when $d = 1$) makes use of the eigenvectors added into the subspace in the last iteration (in the last two iterations) only. Hence Theorem 2.3.3 and its superlinear convergence assertion still hold for Algorithm 2 even if its line 11 is changed as

$$\mathcal{S}_k \leftarrow \text{span} \left\{ s_1^{(k)}, \dots, s_J^{(k)} \right\} \oplus \left\{ \bigoplus_{p=1, q=p}^d \text{span} \left\{ s_{1,pq}^{(k)}, \dots, s_{J,pq}^{(k)} \right\} \right\},$$

that is even if the previous subspace is completely discarded. We refer to this variant of Algorithm 2 as *Algorithm 2 without past*. Similarly for Algorithm 1, when $d = 1$, the superlinear convergence assertion of Theorem 2.3.3 still holds if only the eigenvectors

$d = 2$							
ALGORITHM 1				ALGORITHM 2			
k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\ \omega^{(k+1)} - \omega_*\ _2$	k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\ \omega^{(k+1)} - \omega_*\ _2$
6	6	27.8402784689	0.3245035160	3	12	7.0566934870	0.3112458616
7	7	27.9933586806	0.1134918678	4	16	24.9336629638	0.5250223403
8	8	28.4522934270	0.0247284535	5	20	27.6531948787	0.1830729388
9	9	28.5008552358	0.0007732659	6	24	28.4440816653	0.0142076774
10	10	28.5010522075	0.0000243843	7	28	28.5010327361	0.0001556439
11	11	28.5010523924	0.0000000694	8	32	28.5010523924	0.0000000344
$d = 3$							
ALGORITHM 1				ALGORITHM 2			
k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\ \omega^{(k+1)} - \omega_*\ _2$	k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\ \omega^{(k+1)} - \omega_*\ _2$
10	10	28.1244577720	0.1137180915	3	21	22.9802867532	0.8996337159
11	11	28.1897386962	0.0482696055	4	28	26.1225009081	0.6602340340
12	12	28.2359716412	0.0051064698	5	35	27.0151485822	0.1737449902
13	13	28.2388852363	0.0005705897	6	42	28.0850898434	0.0374522435
14	14	28.2389043164	0.0000124871	7	49	28.2387293186	0.0002142688
15	15	28.2389043663	0.0000001102	8	56	28.2389043663	0.0000001112

Table 2.1: The minimization of the largest eigenvalues of two particular matrix-valued functions of the form (2.30) with two and three parameters using Algorithms 1 and 2. The minimal values $\lambda_1^{(k)}(\omega^{(k+1)})$ of the reduced eigenvalue functions, as well as the errors $\|\omega^{(k+1)} - \omega_*\|$ of the minimizers are listed with respect to k and $p := \dim \mathcal{S}_k$. (Top row) $d = 2$, A_2, A_1, A_0 are 400×400 such that A_2 is random symmetric, A_1 is random Hermitian, whereas A_0 is obtained by adding $2A_2 - A_1$ to another random symmetric matrix; (bottom row) $d = 3$, A_3, A_2, A_1, A_0 are 400×400 such that A_3, A_2, A_1 are random symmetric, while A_0 is obtained from a random symmetric matrix by adding $A_3 + 2A_2 - 3A_1$.

MAXIMIZATION			
k	$\lambda_1^{(k)}(\omega^{(k+1)})$	$ \omega^{(k+1)} - \omega_* $	$\omega^{(k+1)}$
1	0.9213417656	0.6098999764	5.5354507094
2	0.9272678333	0.4809562556	5.6643944303
3	0.9452307484	0.2569647709	5.8883859150
4	0.9596256053	0.0659648186	6.0793858672
5	0.9617115893	0.0019013105	6.1434493753
6	0.9617265293	0.0000002797	6.1453504061
MINIMIZATION			
k	$\lambda_1^{(k)}(\omega^{(k+1)})$	$ \omega^{(k+1)} - \omega_* $	$\omega^{(k+1)}$
10	0.8905833092	1.2006093051	5.1592214939
11	0.8953986993	2.7480373651	1.2105748237
12	0.9039755162	0.2734960246	3.6851161642
13	0.9112584133	0.0063764350	3.9522357538
14	0.9112669429	0.0000006079	3.9586115809
15	0.9112669442	0.0000000000	3.9586121888

Table 2.2: Iterates of Algorithm 1 on the example depicted on the right-hand column of Figure 2.1 involving the maximization and minimization of the largest eigenvalue of a matrix-valued function of the form (2.25) starting with $\omega^{(1)} = 5.5$. The optimal values $\lambda_1^{(k)}(\omega^{(k+1)})$ of the reduced eigenvalue functions, the errors $|\omega^{(k+1)} - \omega_*|$ of the optimizers, and the optimizers $\omega^{(k+1)}$ are listed with respect to k .

from the last two iterations are kept inside the subspace, that is, if line 7 of Algorithm 1 is replaced by

$$\mathcal{S}_k \leftarrow \text{span} \left\{ s_1^{(k-1)}, \dots, s_J^{(k-1)} \right\} \oplus \text{span} \left\{ s_1^{(k)}, \dots, s_J^{(k)} \right\}.$$

We refer to this variant of Algorithm 1 for the case $d = 1$ as *Algorithm 1 without past*.

The main issue with these subspace procedures without past is the convergence. In Section 2.3.1 the convergence of the sequence $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ is established both for the minimization problem and for the maximization problem (by Theorems 2.3.1 and 2.3.2, respectively). When the eigenvectors from the past iterations are discarded, the convergence of $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ is not guaranteed anymore for the minimization problem, because the monotonicity of $\lambda_J^{(k)}(\omega)$ with respect to k is no longer true. On the other hand, all the equalities and inequalities in (2.24) concerning the monotonicity and boundedness of $\{\lambda_J^{(k)}(\omega^{(k+1)})\}$ for the maximization problem can be verified to hold, so this sequence is still guaranteed to converge.

The remarks of the previous paragraph are illustrated in Table 2.3 which concerns the application of Algorithm 1 without past to the example of Table 2.2. For the maximization problem, the maximizers $\lambda_1^{(k)}(\omega^{(k+1)})$ of the reduced eigenvalue functions $\lambda_1^{(k)}(\omega)$ converge to the globally largest value of $\lambda_1(\omega)$ at a superlinear rate with respect to k , even though for every k the subspace dimension is two. For the minimization problem, the values in the table depict that the sequence $\{\lambda_1^{(k)}(\omega^{(k+1)})\}$ does not converge.

The subspace procedures without past work effectively in practice for the maximization problem. On the other hand, for the minimization problem, it may fail to converge. In [44] a convergent subspace framework making use of three dimensional subspaces is devised for the Crawford number computation, which involves the maximization of the smallest eigenvalue (equivalently, the minimization of the largest eigenvalue) of a matrix-valued function depending on one variable. The three dimensional subspaces are formed of eigenvectors from the past iterations, but not necessarily the eigenvectors from the last three iterations. The approach is built on the concavity of the smallest eigenvalue function involved and the knowledge of an

MAXIMIZATION		MINIMIZATION	
k	$\lambda_1^{(k)}(\omega^{(k+1)})$	k	$\lambda_1^{(k)}(\omega^{(k+1)})$
1	0.9213417656	45	-0.3266668608
2	0.9322945953	46	-0.5147292664
3	0.9595070069	47	-0.3348522694
4	0.9615567360	48	-0.4859510415
5	0.9617263748	49	-0.3259949226
6	0.9617265294	50	-0.5140279537

Table 2.3: The optimal values $\lambda_1^{(k)}(\omega^{(k+1)})$ of the reduced eigenvalue functions by the variant of Algorithm 1 that keeps only the subspaces from the last two iterations are listed with respect to k for the example of the right column of Figure 2.1 (which involves the maximization and minimization of the largest eigenvalue of a matrix-valued function of the form (2.25) starting with $\omega^{(1)} = 5.5$).

interval containing the global maximizer. It does not seem easy to extend the ideas in [44] to the general nonconvex setting that involves the minimization of $\lambda_J(\omega)$.

2.4.2 Subspace Procedures for Singular Value Optimization

The ideas of the previous sections can be extended to maximize or minimize the J th largest singular value $\sigma_J(\omega)$ of a compact operator

$$\mathbf{B}(\omega) := \sum_{\ell=1}^{\kappa} f_{\ell}(\omega) \mathbf{B}_{\ell} \quad (2.31)$$

over a compact subset Ω of $\mathbf{R}^d$. As before $\mathbf{B}(\omega)$ is defined over ω that belongs to an open subset $\tilde{\Omega}$ containing the feasible region Ω . In representation (2.31) of $\mathbf{B}(\omega)$ above, $f_{\ell} : \tilde{\Omega} \rightarrow \mathbb{R}$ is a real-analytic function and $\mathbf{B}_{\ell} : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ is a compact operator for $\ell = 1, \dots, \kappa$. Once again, the infinite dimensional problem is motivated by the finite dimensional case where the matrix-valued function

$$B(\omega) := \sum_{\ell=1}^{\kappa} f_{\ell}(\omega) B_{\ell} \quad (2.32)$$

for given $B_\ell \in \mathbb{C}^{p \times q}$ $\ell = 1, \dots, \kappa$ with large dimensions, takes the role of $\mathbf{B}(\omega)$.

Let $\mathcal{U}, \mathcal{V}$ be two subspaces of $\ell^2(\mathbb{N})$ with equal dimension, and let U, V be orthonormal bases for these subspaces. The subspace procedures to optimize $\sigma_J(\omega)$ are based on the optimization of the J th largest singular value $\sigma_J^{\mathcal{U}, \mathcal{V}}(\omega)$ of an operator of the form

$$\mathbf{B}^{\mathcal{U}, \mathcal{V}}(\omega) := \mathbf{U}^* \mathbf{B}(\omega) \mathbf{V}, \quad (2.33)$$

where $\mathbf{V}$ ($\mathbf{U}$) represents the operator as in (2.3) or (2.6) that maps the coordinates of a vector in $\mathcal{V}$ ($\mathcal{U}$) relative to the basis V (U) to itself. This operator can be written as

$$\mathbf{B}^{\mathcal{U}, \mathcal{V}}(\omega) = \sum_{\ell=1}^{\kappa} f_\ell(\omega) \mathbf{B}_\ell^{\mathcal{U}, \mathcal{V}}, \quad \text{where} \quad \mathbf{B}_\ell^{\mathcal{U}, \mathcal{V}} := \mathbf{U}^* \mathbf{B}_\ell \mathbf{V},$$

which helps to reduce computational costs.

The subspace projections and restrictions here for singular value optimization are closely related to the ones employed for eigenvalue optimization. Indeed $\sigma_J(\omega)$ and $\sigma_J^{\mathcal{U}, \mathcal{V}}(\omega)$ correspond to the J th largest eigenvalues of

$$\begin{aligned} \begin{bmatrix} 0 & \mathbf{B}(\omega) \\ \mathbf{B}^*(\omega) & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & \mathbf{U}^* \mathbf{B}(\omega) \mathbf{V} \\ \mathbf{V}^* \mathbf{B}^*(\omega) \mathbf{U} & 0 \end{bmatrix} \\ = \begin{bmatrix} \mathbf{U}^* & \mathbf{V}^* \end{bmatrix} \begin{bmatrix} 0 & \mathbf{B}(\omega) \\ \mathbf{B}^*(\omega) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}, \end{aligned}$$

respectively. Hence the results in Section 2.2, specifically Lemma 2.2.1 - 2.2.4 and Theorem 2.2.5, extend to relate the singular values $\sigma_J(\omega)$ and $\sigma_J^{\mathcal{U}, \mathcal{V}}(\omega)$. For instance, monotonicity amounts to the following: for four subspaces $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2$ of $\ell^2(\mathbb{N})$ such that $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and $\mathcal{V}_1 \subseteq \mathcal{V}_2$, we have

$$\sigma_J^{\mathcal{U}_1, \mathcal{V}_1}(\omega) \leq \sigma_J^{\mathcal{U}_2, \mathcal{V}_2}(\omega) \leq \sigma_J(\omega).$$

The extended greedy subspace procedure for singular value optimization forms $\mathcal{U}$ and $\mathcal{V}$ from the left singular vectors and right singular vectors of $\mathbf{B}(\omega)$ at the optimizers of the reduced problems and at nearby points. A precise description is

given in Algorithm 3 below, where $\mathcal{U}_k$ and $\mathcal{V}_k$ denote the left and the right subspace at step k and $\sigma_J^{(k)}(\omega) := \sigma_J^{\mathcal{U}_k, \mathcal{V}_k}(\omega)$. The basic greedy procedure is defined similarly by modifying lines 11 and 12 as

$$\mathcal{U}_k \leftarrow \mathcal{U}_{k-1} \oplus \text{span} \left\{ u_1^{(k)}, \dots, u_J^{(k)} \right\} \quad \text{and} \quad \mathcal{V}_k \leftarrow \mathcal{V}_{k-1} \oplus \text{span} \left\{ v_1^{(k)}, \dots, v_J^{(k)} \right\}. \quad (2.34)$$

In the description, by a consistent pair of left and right singular vectors u and v corresponding to a singular value σ of $\mathbf{B}(\omega)$, we mean the vectors satisfying $\mathbf{B}(\omega)v = \sigma u$ and $\mathbf{B}^*(\omega)u = \sigma v$ simultaneously.

Algorithm 3 The Extended Greedy Subspace Procedure for Singular Value Optimization

Input: A parameter dependent compact operator $\mathbf{B}(\omega)$ of the form (2.32), and a

compact subset $\Omega \subset \tilde{\Omega}$ of $\mathbb{R}^d$.

- 1: $\omega^{(1)} \leftarrow$ a random point in Ω .
 - 2: $u_1^{(1)}, \dots, u_J^{(1)}$ and $v_1^{(1)}, \dots, v_J^{(1)} \leftarrow$ consistent left and right singular vectors corresponding to $\sigma_1(\omega^{(1)}), \dots, \sigma_J(\omega^{(1)})$.
 - 3: $\mathcal{U}_1 \leftarrow \text{span} \left\{ u_1^{(1)}, \dots, u_J^{(1)} \right\}$ and $\mathcal{V}_1 \leftarrow \text{span} \left\{ v_1^{(1)}, \dots, v_J^{(1)} \right\}$.
 - 4: **for** $k = 2, 3, \dots$ **do**
 - 5: $\omega^{(k)} \leftarrow$ any $\omega_* \in \arg \min_{\omega \in \Omega} \sigma_J^{(k-1)}(\omega)$ for the minimization problem, or
 $\omega^{(k)} \leftarrow$ any $\omega_* \in \arg \max_{\omega \in \Omega} \sigma_J^{(k-1)}(\omega)$ for the maximization problem.
 - 6: $u_1^{(k)}, \dots, u_J^{(k)}$ and $v_1^{(k)}, \dots, v_J^{(k)} \leftarrow$ consistent left and right singular vectors corresponding to $\sigma_1(\omega^{(k)}), \dots, \sigma_J(\omega^{(k)})$.
 - 7: $h^{(k)} \leftarrow \|\omega^{(k)} - \omega^{(k-1)}\|_2$
 - 8: **for** $p = 1, \dots, d,$ $q = p, \dots, d$ **do**
 - 9: $u_{1,pq}^{(k)}, \dots, u_{J,pq}^{(k)}$ and $v_{1,pq}^{(k)}, \dots, v_{J,pq}^{(k)} \leftarrow$ consistent left and right singular vectors corresponding to $\sigma_1(\omega^{(k)} + h^{(k)}e_{pq}), \dots, \sigma_J(\omega^{(k)} + h^{(k)}e_{pq})$.
 - 10: **end for**
 - 11: $\mathcal{U}_k \leftarrow \mathcal{U}_{k-1} \oplus \text{span} \left\{ u_1^{(k)}, \dots, u_J^{(k)} \right\} \oplus \left\{ \bigoplus_{p=1, q=p}^d \text{span} \left\{ u_{1,pq}^{(k)}, \dots, u_{J,pq}^{(k)} \right\} \right\}$.
 - 12: $\mathcal{V}_k \leftarrow \mathcal{V}_{k-1} \oplus \text{span} \left\{ v_1^{(k)}, \dots, v_J^{(k)} \right\} \oplus \left\{ \bigoplus_{p=1, q=p}^d \text{span} \left\{ v_{1,pq}^{(k)}, \dots, v_{J,pq}^{(k)} \right\} \right\}$.
 - 13: **end for**
-

Observe that the reduced singular value function $\sigma_J^{(k)}(\omega)$ is the same as $\lambda_J^{(k)}(\omega)$

formed by Algorithm 2 when it is applied to

$$\mathbf{A}(\omega) := \begin{bmatrix} 0 & \mathbf{B}(\omega) \\ \mathbf{B}^*(\omega) & 0 \end{bmatrix}, \quad (2.35)$$

so Algorithm 3 applied to $\mathbf{B}(\omega)$ and Algorithm 2 applied to $\mathbf{A}(\omega)$ lead the same sequence $\{\omega^{(k)}\}$. Similarly, the basic greedy subspace procedure for singular value optimization is equivalent to Algorithm 1 operating on $\mathbf{A}(\omega)$. All of the convergence results deduced in Section 4.3.2 for eigenvalue optimization, in particular,

1. global convergence for the minimization problem (Theorem 2.3.1),
2. convergence of the sequence of maximal values of the reduced problems for the maximization problem (Theorem 2.3.2),
3. superlinear rate of convergence for smooth optimizers (Theorem 2.3.3),

carry over to this singular value optimization setting. Some of these results in Section 4.3.2 are proven assuming the simplicity of $\lambda_J(\omega_*)$ at a particular ω_* , which translates into a simplicity and a positivity assumption on $\sigma_J(\omega_*)$ in the singular value setting. Regarding issue 2 above, we observe in practice the convergence of $\{\sigma_J^{(k)}(\omega^{(k+1)})\}$ for the maximization problem to σ_* such that $\sigma_J(\tilde{\omega}) = \sigma_*$ for some local maximizer $\tilde{\omega}$, that is not necessarily a global maximizer, similar to what we observe in the eigenvalue optimization setting.

2.4.3 Optimization of the J th Smallest Singular Value

The minimum (maximum) of the J th smallest eigenvalue $\lambda_{-J}(\omega)$ of $\mathbf{A}(\omega)$ is equal to the negative of the maximum (minimum) of the J th largest eigenvalue of $-\mathbf{A}(\omega)$. Hence Algorithms 1 and 2 can be adapted to optimize $\lambda_{-J}(\omega)$.

The optimization of the J th smallest singular value has a different nature. In particular it cannot be converted into an optimization problem involving the J th largest singular value. This is partly seen by the observation that the J th smallest singular

value $\sigma_{-J}(\omega)$ of an operator $\mathbf{B}(\omega)$ of the form (2.31) corresponds to an eigenvalue of $\mathbf{A}(\omega)$ as in (2.35), right in the middle of its spectrum. For the restricted operator $\mathbf{B}^{\mathcal{U},\mathcal{V}}(\omega)$ defined as in (2.33) and its J th smallest singular value $\sigma_{-J}^{\mathcal{U},\mathcal{V}}(\omega)$, monotonicity is lost as the subspaces $\mathcal{U}, \mathcal{V}$ expand. In particular $\sigma_{-J}^{\mathcal{U},\mathcal{V}}(\omega) \geq \sigma_{-J}(\omega)$ does not necessarily hold; the restriction of the domain of $\mathbf{B}(\omega)$ to $\mathcal{V}$ causes an increase in the J th smallest singular value, whereas the projection of the range onto $\mathcal{U}$ causes a decrease in the singular value. As a consequence of the loss of monotonicity, the interpolation properties do not hold anymore either.

A neater and theoretically more sound approach is to employ only the restrictions from the right-hand side, that is a subspace procedure that operates on $\mathbf{B}^{\mathcal{V}}(\omega) := \mathbf{B}(\omega)\mathbf{V}$ and its J th smallest singular value. The resulting subspace procedures are equivalent to those for eigenvalue optimization applied to $\mathbf{B}^*(\omega)\mathbf{B}(\omega)$, so monotonicity and interpolation properties as well as all the theoretical convergence properties are regained.

2.4.4 A Comparison with the Cutting-Plane Methods

The cutting-plane method was introduced by Kelley to solve convex minimization problems [42]. For unconstrained minimization problems, the approach underestimates convex functions with piecewise linear functions globally, solves the resulting linear program, and refines the underestimator with the addition of the new linear approximation around the optimizer of the linear program [6, Section 9.3.2].

Let us consider again the minimization of the largest eigenvalue $\lambda_1(\omega)$ of an affine and Hermitian matrix-valued function

$$A(\omega) = A_0 + \omega_1 A_1 + \cdots + \omega_d A_d$$

for given Hermitian matrices $A_0, \dots, A_d \in \mathbb{C}^{n \times n}$. Recall that the largest eigenvalue $\lambda_1(\omega)$ is convex [67], so the cutting-plane method is applicable for its minimization as outlined in Algorithm 4. In this description $\widehat{\lambda}_1^{(k)}(\omega)$ represents the piecewise linear

underestimator for $\lambda_1(\omega)$ and is defined by

$$\begin{aligned}\widehat{\lambda}_1^{(k)}(\omega) &:= \max_{\ell=1,\dots,k} \lambda_1(\widehat{\omega}^{(\ell)}) + \nabla \lambda_1(\widehat{\omega}^{(\ell)})^T (\omega - \widehat{\omega}^{(\ell)}) \\ &= \max_{\ell=1,\dots,k} (\widehat{v}^{(\ell)})^* A(\omega) \widehat{v}^{(\ell)},\end{aligned}$$

where $\widehat{v}^{(\ell)}$ denotes a unit eigenvector corresponding to the largest eigenvalue of $A(\widehat{\omega}^{(\ell)})$. Furthermore, in line 4 of the outline, $\widehat{\omega}^{(k)}$ denotes the unique minimizer of the convex function $\widehat{\lambda}_1^{(k-1)}(\omega)$.

Algorithm 4 The Cutting-Plane Method to Minimize the Largest Eigenvalue

Input: A matrix-valued function $A : \mathbb{R}^d \rightarrow \mathbb{C}^{n \times n}$ that is affine and Hermitian

- 1: $\widehat{\omega}^{(1)} \leftarrow$ a random vector in $\mathbb{R}^d$.
 - 2: $\widehat{v}^{(1)} \leftarrow$ a unit eigenvector corresponding to largest eigenvalue of $A(\widehat{\omega}^{(1)})$.
 - 3: **for** $k = 2, 3, \dots$ **do**
 - 4: $\widehat{\omega}^{(k)} \leftarrow \arg \min_{\omega \in \Omega} \widehat{\lambda}_1^{(k-1)}(\omega)$.
 - 5: $\widehat{v}^{(k)} \leftarrow$ a unit eigenvector corresponding to largest eigenvalue of $A(\widehat{\omega}^{(k)})$.
 - 6: **end for**
-

The basic greedy subspace procedure Algorithm 1 for the minimization of the largest eigenvalue $\lambda_1(\omega)$ in this finite dimensional matrix-valued setting minimizes $\lambda_1^{(k)}(\omega)$, which, by the monotonicity property, underestimates $\lambda_1(\omega)$. The sequences $\{\omega^{(k)}\}$ and $\{\widehat{\omega}^{(k)}\}$ generated by Algorithm 1 and Algorithm 4 are not the same in general, but, for simplicity, let us suppose $\omega^{(\ell)} = \widehat{\omega}^{(\ell)}$ for $\ell = 1, \dots, k$ and for some k . In this case, for each $\omega \in \Omega$, we have

$$\begin{aligned}\lambda_1(\omega) &\geq \lambda_1^{(k)}(\omega) = \max_{\alpha \in \mathbb{C}^m, \|\alpha\|_2=1} \alpha^* V_k^* A(\omega) V_k \alpha \\ &\geq \max_{\ell=1,\dots,k} (\widehat{v}^{(\ell)})^* A(\omega) \widehat{v}^{(\ell)} = \widehat{\lambda}_1^{(k)}(\omega),\end{aligned}$$

where the columns of the matrix V_k form an orthonormal basis for the subspace $\text{span}\{\widehat{v}^{(1)}, \dots, \widehat{v}^{(k)}\}$. This illustrates that, for this special case concerning the minimization of the largest eigenvalue of an affine matrix-valued function, the underestimators used by the basic greedy subspace procedure are more accurate than those used by the cutting-plane method.

The better accuracy of the subspace procedure is apparent in Figure 2.2 for the matrix-valued function $A(\omega) = A_0 + \omega A_1$, where A_0, A_1 are 10×10 random symmetric matrices. In this figure, the function $\widehat{\lambda}_1^{(2)}(\omega)$ used by the cutting-plane method is the maximum of two linear approximations about $\widehat{\omega}^{(1)} = 0$ and $\widehat{\omega}^{(2)} = 0.2$, while a two dimensional subspace is used by the subspace procedure with $\omega^{(1)} = 0$ and $\omega^{(2)} = 0.2$.

2.5 Numerical Experiments

The next section specifies the Matlab software accompanying this work, and some important implementation details of this software. The rest of the section is devoted to three applications of large-scale eigenvalue and singular value optimization, namely, the numerical radius, the distance to instability from a matrix or a pencil, and the minimization of the largest eigenvalue of an affine matrix-valued function, which is closely related to semi-definite programming. On these examples we illustrate the power of the subspace procedures, introduced and analyzed in the previous sections, in practice.

2.5.1 Software and Implementation Details

MATLAB implementations accompanying this work are made publicly available on the internet [38]. The current version includes generic routines for the following purposes (for a prescribed positive integer J):

- minimization of the J th largest eigenvalue (based on Algorithm 1);
- maximization of the J th largest eigenvalue (based on Algorithm 1 without past if $d = 1$, or Algorithm 2 without past if $d > 1$);
- minimization of the J th smallest singular value (based on an adaptation of Algorithm 1 without past if $d = 1$, or Algorithm 2 without past if $d > 1$, for singular value optimization).

The Matlab routines above terminate if one of the conditions

- (1) $\left| \lambda_J^{(k)}(\omega^{(k+1)}) - \lambda_J^{(k-1)}(\omega^{(k)}) \right| \leq \text{tol}$ (or $\left| \sigma_J^{(k)}(\omega^{(k+1)}) - \sigma_J^{(k-1)}(\omega^{(k)}) \right| \leq \text{tol}$),
- (2) $\# \text{ subspace iterations} > \sqrt{n}$,

hold for a prescribed tolerance `tol`, where n is the size of the matrices A_ℓ in (2.2) for eigenvalue optimization, or the maximum of the dimensions p and q of the matrices B_ℓ in (2.32) for singular value optimization. For all of the experiments in this section `tol` = 10^{-12} is used unless otherwise specified. The second condition is never used in practice for the examples in this section, because the other condition is fulfilled after a few subspace iterations.

The reduced eigenvalue optimization and singular value optimization problems are solved by means of the the MATLAB package `eigopt` [60, Section 10]. These routines keep a lower bound and an upper bound for the optimal value of the reduced problem, and terminate when they differ by less than a prescribed tolerance. We set this tolerance equal to 0.1 tol . Additionally, `eigopt` requires a global lower (upper) bound γ for the minimization (maximization) problem on the second derivatives of the eigenvalue functions or the singular value functions; particular choices for the three applications in this section are specified below. Finally, `eigopt` performs the optimization on a box, which must be supplied by the user. Once again, particular box choices for the applications in this section are specified below.

Large-scale eigenvalue and singular value problems are solved iteratively by means of `eigs` and `svds` in Matlab. If the optimal values of the reduced eigenvalue or singular value functions at two consecutive iterations are close enough (i.e., if they differ by an amount less than 10^{-2}), then we provide the shift $\lambda_{\text{cur}} + 5|\lambda_{\text{cur}} - \lambda_{\text{pre}}|$ (or $\sigma_{\text{cur}} + 5|\sigma_{\text{cur}} - \sigma_{\text{pre}}|$, in the case of singular values) - with $\lambda_{\text{cur}}, \lambda_{\text{pre}}$ ($\sigma_{\text{cur}}, \sigma_{\text{pre}}$) denoting the optimal values of the current, previous reduced J th largest eigenvalue functions (J th smallest singular value functions) - to `eigs` (`svds`) and require it to compute the J eigenvalues (singular values) closest to this shift. Otherwise, `eigs` and `svds` are called without shifts.

2.5.2 Numerical Radius

The numerical radius $r(A)$ of a matrix $A \in \mathbb{C}^{n \times n}$, as indicated in Section 1.1.3, has an eigenvalue optimization characterization

$$r(A) = \max_{\omega \in [0, 2\pi]} \lambda_1(\omega) \quad \text{where } A(\omega) = \frac{Ae^{i\omega} + A^*e^{-i\omega}}{2}.$$

We apply Algorithm 1 without past as discussed in Section 2.4.1 for the computation of the numerical radius of two families of matrices. The box and γ supplied to `eigopt` are $[0, 2\pi]$ and $2\|A\|_2$, respectively. The latter is not guaranteed to be an upper bound on the second derivatives of the eigenvalue function, but it works well in practice in our experience.

Example 1: The Grcar matrix is a Toeplitz matrix with 1s on the main, first, second, third superdiagonals, -1s on the first subdiagonal, which exhibits ill-conditioned eigenvalues. It is used as a test matrix in the previous works [31, 79] concerning the estimation of the numerical radius. Table 2.4 lists the computed values of the numerical radius by the subspace procedure and runtimes in seconds for the Grcar matrices of sizes varying between 320 – 20480. As the matrix sizes increase, the solution of large-scale eigenvalue problems take nearly all of the computation time. In contrast to this, the time spent to solve reduced eigenvalue optimization problems is very small and it is more or less constant as the sizes of the matrices increase. The reported values of the numerical radius for the Grcar matrices of sizes 320 and 640 match with the results reported in [79] up to 12 decimal digits.

Example 2: This example concerns the gear matrix which is another Toeplitz matrix with 1s on the superdiagonal, subdiagonal, 1 at position $(1, n)$, and -1 at $(n, 1)$. This matrix also has ill-conditioned, real eigenvalues lying in the interval $(-2, 2)$. The computed values of the numerical radius by the subspace procedure for the gear matrix of size n with n varying in 320 – 20480, as well as the runtimes, are given in Table 2.5. Once again, the computation time for increasing n is dominated by large-scale eigenvalue computations. The results reported for the gear matrices of sizes 320 and 640 match again with those reported in [79] up to prescribed accuracy.

n	# iter	$r(A)$	total time	eigval comp	reduced prob
320	11	3.240793870067	8.3	0.8	7.4
640	12	3.241243679341	9.1	1.4	7.7
1280	13	3.241357030535	9.7	2.5	7.2
2560	15	3.241385481170	10.9	4.6	6.3
5120	16	3.241392607964	14.4	8.8	5.4
10240	18	3.241394391431	30.8	26.6	3.9
20480	19	3.241394837519	87.4	82.8	4.0

Table 2.4: The computed value of the numerical radius by Algorithm 1 without past (i.e, the subspaces $\mathcal{S}_k$ are spanned by the eigenvectors at the last two iterations, hence, two dimensional), the number of subspace iterations, and the runtimes in seconds are listed for the Grcar matrix of size n for various n . Specifically, the number of subspace iterations (2nd column) to reach the prescribed accuracy, the numerical radius (3rd column), the total runtime (4th column), the time spent for large-scale eigenvalue computations (5th column), and the time spent for the solution of the reduced eigenvalue optimization problems (6th column) are reported w.r.t. the sizes of the matrices.

2.5.3 Distance to Instability

In Section 1.1.3, we define the distance to instability from a square matrix $A \in \mathbb{C}^{n \times n}$ with all eigenvalues on the open left half of the complex plane as

$$\mathcal{D}(A) := \min\{\|\Delta\|_2 \mid \exists \omega \in \mathbb{R}, \det(A + \Delta - \omega i I) = 0\}$$

which is suggested in [80] as a measure of robust stability for the autonomous system $x'(t) = Ax(t)$ and it has the characterization

$$\mathcal{D}(A) = \min_{\omega \in \mathbb{R}} \sigma_{-1}(\omega),$$

where $\sigma_{-1}(\omega)$ denotes the smallest singular value of $B(\omega) = A - \omega i I$.

n	# iter	$r(A)$	total time	eigval comp	reduced prob
320	5	1.999904217490	5.2	0.4	4.8
640	5	1.999975979457	5.5	0.8	4.6
1280	6	1.999993985476	6.2	1.2	5.0
2560	5	1.999998495194	5.8	1.9	3.9
5120	5	1.999999623651	6.3	3.0	3.3
10240	5	1.999999905895	7.7	5.6	2.1
20480	5	1.999999976471	16.4	14.6	1.6

Table 2.5: This table lists the computed value of the numerical radius by Algorithm 1 without past (i.e, the subspaces $\mathcal{S}_k$ are spanned by the eigenvectors at the last two iterations, hence two dimensional) for the gear matrix of size n for various n , as well as the number of subspace iterations, and runtimes in seconds. For the meaning of each column see the caption of Table 2.4.

Here also, we adopt Algorithm 1 without past (see Section 2.4.1), noting that at step $k + 1$ the two dimensional right subspace is the span of the right singular vectors corresponding to $\sigma_{-1}(\omega^{(k)})$ and $\sigma_{-1}(\omega^{(k-1)})$. We illustrate numerical results on two families of sparse matrices. We set γ , the global lower bound for the second derivatives of the singular value function for `eigopt`, equal to $-2\|A\|_2$, which is a heuristic that works well in practice. The boxes supplied to `eigopt` are $[-60, 60]$ and $[150, 160]$ for the first and second families, respectively. These boxes indeed contain the global minimizers.

Example 3: Tolosa matrices arise from the stability analysis of an airplane. They are used as test examples in previous works [32, 25] concerning the computation of the distance to instability. These are stable matrices with all eigenvalues lying in the left half of the complex plane, but they are nearly unstable (see Figure 4 of [25] for the spectrum of the 340×340 Tolosa matrix), indeed their perturbations at a distance 0.002 have eigenvalues on the right half-plane. We run the subspace procedure on the Tolosa matrices of size 340, 1090, 2000, 4000, which are all available through

n	# iter	$\mathcal{D}(A)$	total time	singval comp	reduced prob
340	4	0.001999796888	9.1	1.0	8.0
1090	4	0.001999796888	9.7	1.4	8.2
2000	4	0.001999796888	9.9	1.8	8.1
4000	4	0.001999796888	10.3	2.7	7.5

Table 2.6: The runtimes in seconds, number of iterations (2nd column) and computed values of $\mathcal{D}(A)$ (3rd column) are listed for Algorithm 1 without past applied to compute the distance to instability from the Tolosa matrices of sizes 340, 1090, 2000, 4000. The subspaces $\mathcal{S}_k$ are two dimensional and spanned by the right singular vectors computed at the last two iterations. The total runtimes, the time for large-scale singular value computations, and the time for the reduced optimization problems in seconds are provided in the 4th, 5th, and 6th columns, respectively.

the matrix market [5]. According to Table 2.6 only four iterations suffice to reach prescribed accuracy. The time required for large-scale singular value computations increases with respect to the size of the matrices. However the majority of the time is consumed for the solution of the reduced problems, but this is only because the matrices are relatively small. The computed value of $\mathcal{D}(A)$ is the same in each case and match with the results reported in [25] up to prescribed accuracy.

Example 4: This example is taken from [32]. A finite difference discretization of an Orr-Sommerfeld operator with step size h for planar Poiseuille flow leads to an $n \times n$ generalized eigenvalue problem $B_n v = \lambda L_n v$ or an $n \times n$ standard eigenvalue problem $L_n^{-1} B_n v = \lambda v$, where $n = 2/h - 1$,

$$L_n = \frac{1}{h^2} \text{tridiag}(1, -(2 + h^2), 1),$$

$$B_n = \frac{1}{100} L_n^2 - i(U_n L_n + 2I),$$

and $U_n = \text{diag}(1 - x_1^2, \dots, 1 - x_n^2)$ with $x_k = 1 + k \cdot h$ for $k = 1, \dots, n$. Matrix $A_n = L_n^{-1} B_n$ is stable with eigenvalues on the left half-plane, yet nearly unstable.

We apply the subspace procedure for the computation of the distance to instabil-

ity for the Orr-Sommerfeld matrix A_n of sizes $n = 400, 1000, 2000, 4000, 8000, 16000$. Note that A_n is not sparse, yet applications of Arnoldi's method for large-scale smallest singular value computations on $A_n - \omega i I$ require the solutions of the linear systems of the form $(A_n - \omega i I)v_{k+1} = v_k$ for v_{k+1} for a given v_k . We equivalently solve the sparse linear system

$$(B_n - \omega i L_n)v_{k+1} = L_n v_k \quad (2.36)$$

in practice. In Table 2.7 we again observe that the total computation time is dominated by large-scale singular value computations as n increases, whereas the contribution of the time for the solution of the reduced problems to the total running time is very little for large n . The computed values of $\mathcal{D}(A)$ in the table are listed only to six decimal digits, because for large n `eigs` could not compute singular values beyond 7-8 decimal digits in a reliable fashion. We attribute this to the fact that the norm of B_n and L_n increase considerably as n increases, so it is not possible to solve the linear system (2.36) with high accuracy, for instance, $\|B_{16000}\|_2 \approx 6.467 \cdot 10^{13}$. For smaller n the computed solutions are accurate up to 12 decimal digits, for instance the computed value of the distance to instability for A_{400} by the subspace procedure is 0.001978172281 which differs from the result reported in [25] by an amount less than 10^{-12} .

2.5.4 Minimization of the Largest Eigenvalue

A problem that drew substantial interest in the late 1980s and early 1990s [67, 24] concerns the minimization of the largest eigenvalue of $\lambda_1(\omega)$ of

$$A(\omega) := A_0 + \omega_1 A_1 + \cdots + \omega_d A_d \quad (2.37)$$

for given symmetric matrices $A_0, \dots, A_d \in \mathbb{R}^{n \times n}$. This problem is already discussed in Section 2.3.2 in the more general case, when $A_0, \dots, A_d$ are complex and Hermitian. Numerical results over there on random matrices indicate that the sequences generated by both the basic subspace procedure and the extended one converge at least

n	# iter	$\mathcal{D}(A)$	total time	singval comp	reduced prob
400	9	0.001978	5.0	0.9	4.1
1000	8	0.001978	5.3	1.7	3.4
2000	7	0.001978	5.9	3.0	2.8
4000	8	0.001978	7.9	5.5	2.3
8000	8	0.001979	15.2	11.7	3.4
16000	7	0.001938	30.8	27.7	2.9

Table 2.7: The number of iterations, computed values of the distance to instability, and runtimes are given for Algorithm 1 without past applied to estimate the distance to instability from the Orr-Sommerfeld matrices. Once again the subspaces are always two dimensional and spanned by the right singular vectors computed at the last two iterations. For the meaning of each column we refer to the caption of Table 2.6.

superlinearly. An important application is in the context of semidefinite programming: under mild assumptions the dual of an SDP can be expressed as an unconstrained minimization problem with the objective function $\lambda_1(\omega) + b^T \omega$ for some $b \in \mathbb{R}^d$ and with $\lambda_1(\omega)$ denoting the largest eigenvalue of a matrix-valued function $A(\omega)$ of the form (2.37).

We apply the subspace procedures Algorithms 1 and 2 for a particular notoriously difficult family of matrix-valued functions depending on two parameters. In this example the subspaces from the previous iterations are kept fully. Note that the Matlab software is based on Algorithm 1; additionally we apply Algorithm 2 for comparison purposes. As for the parameters for `eigopt`, since the largest eigenvalue function is convex, γ (the global lower bound on the second derivatives of the eigenvalue function) in theory can be chosen zero, instead we set $\gamma = -10^{-6}$ for numerical reliability. We have specified the box containing the minimizer as $[-10, 10] \times [-10, 10]$.

Example 5: In [67], for $A_0 \in \mathbb{R}^{n \times n}$ with its (k, j) entry equal to

$$\begin{cases} \min\{k, j\} & \text{if } |k - j| > 1, \\ \min\{k, j\} + 0.1 & \text{if } |k - j| = 1, \\ 0 & k = j, \end{cases}$$

the spectral radius (i.e., the absolute value of the eigenvalue furthest away from the origin) of $C_n(\omega) = A_0 - \sum_{j=1}^d \omega_j e_j e_j^T$ is minimized. It is observed in that paper on the $n = 10$ case that at the optimal ω , the eigenvalue with the largest modulus has multiplicity three. This problem would correspond to an SDP relaxation of a max-cut problem, if A_0 had been a Laplace matrix of a graph [34, Section 7].

Here we minimize the spectral radius of

$$\tilde{C}_n(\omega) = \frac{1}{100n} A_0 - \omega_1 I_u - \omega_2 I_l \quad (2.38)$$

for $n = 250, 500, 1000, 2000$, where $I_u = \text{diag}(I_{n/2}, 0_{n/2})$ and $I_l = \text{diag}(0_{n/2}, I_{n/2})$. The scaling in front of A_0 is to make sure that the unique minimizer of the problem is inside $[-10, 10] \times [-10, 10]$. This problem can equivalently be posed as the minimization of the largest eigenvalue of $A(\omega) := \text{diag}(\tilde{C}_n(\omega), -\tilde{C}_n(\omega))$, so fits within the problem class described by (2.37). Table 2.8 lists the minimal spectral radius values computed by the basic and extended subspace procedures along with number of subspace iterations and computation time. The total computation times are again dominated by the solutions of large eigenvalue problems. Furthermore, even though the basic subspace procedure usually requires more iterations to reach the prescribed accuracy, overall it solves fewer large eigenvalue problems and takes less computation time as compared to the extended subspace procedure.

In all cases the computed minimizer ω_* is such that the largest eigenvalue of $A(\omega_*)$ has algebraic multiplicity three, so $\lambda_1(\omega)$ is not differentiable at ω_* . For instance, for $n = 500$, that is, when the matrix-valued function $A(\omega)$ is of size 1000, the five largest eigenvalues of $A(\omega_*)$ are listed in Table 2.9 on the left. Even the gaps between the remaining 497 positive eigenvalues are very small, as indeed among 500 positive

n	# iter	p	ρ_*	total time	eigval comp	reduced prob
250	7	7	0.509646245274	3.8	1.5	2.2
500	7	7	1.016261471669	4.8	2.8	1.6
1000	8	8	3.584040976076	13.9	11.8	1.2
2000	7	7	4.055903987776	68.7	65.8	0.7
n	# iter	p	ρ_*	total time	eigval comp	reduced prob
250	6	24	0.509646245274	4.7	2.8	1.6
500	6	24	1.016261471669	7.6	5.4	1.1
1000	7	28	3.584040976076	22.1	17.8	1.0
2000	7	28	4.055903987776	115.9	106.7	0.8

Table 2.8: The table lists the number of subspace iterations, the dimension of the subspace $\mathcal{S}_k$ at termination (p), computational times in seconds, and the computed minimal value (ρ_*) of the spectral radius when Algorithm 1 (on the top) and Algorithm 2 (at the bottom) are applied to minimize the largest eigenvalue of $A(\omega) = \text{diag}(\tilde{C}_n(\omega), -\tilde{C}_n(\omega))$ for the matrix-valued function $\tilde{C}_n(\omega)$ defined as in (2.38). Note that $A(\omega)$, whose largest eigenvalue is minimized is of size $2n$. Also note that the subspaces from all of the previous iterations are kept in these examples.

eigenvalues 495 of them lie in an interval of length 0.017. The fact that most of the eigenvalues belong to a small interval causes poor convergence properties for **eigs**. On the other hand, it appears that the nonsmoothness does not affect the superlinear convergence of the iterates $\{\lambda_1^{(k)}(\omega^{(k+1)})\}$, as depicted in Table 2.9 on the right. This quick convergence is also apparent from the number of subspace iterations in Table 2.8. Theorem 2.3.3 does not apply to this nonsmooth case. It is an open problem to come up with a formal argument explaining the quick convergence in this nonsmooth setting.

Eigenvalues of $A(\omega_*)$	
1.016261471669	
1.016261471669	
1.016261471669	
1.016234975093	
1.016234803766	

k	$\lambda_1^{(k)}(\omega^{(k+1)})$
3	1.016260414001
4	1.016261417096
5	1.016261471669
6	1.016261471669

Table 2.9: (Left) The five largest eigenvalues of $A(\omega) = \text{diag}(\tilde{C}_{500}(\omega), -\tilde{C}_{500}(\omega))$ at ω_* , where $\tilde{C}_n(\omega)$ is defined as in (2.38) and ω_* is the minimizer of the largest eigenvalue of $A(\omega)$ for $n = 500$. (Right) The last four iterates $\{\lambda_1^{(k)}(\omega^{(k+1)})\}$ of Algorithm 1 (which keeps all of the subspaces from the previous iterations) when the large eigenvalue computations are performed directly by calling `eig` in MATLAB.

2.6 Concluding Remarks

We have proposed subspace procedures to cope with large-scale eigenvalue and singular value optimization problems. To optimize the J th largest eigenvalue of a Hermitian and analytic matrix-valued function $A(\omega)$ over ω for a prescribed integer J , the subspace procedures operate on a small matrix-valued function that acts like $A(\omega)$ in a small subspace. The subspace is expanded with the addition of the eigenvectors of $A(\omega)$ at the optimizer of the eigenvalue function of the small matrix-valued function and, possibly, at nearby points. A similar strategy is adopted to optimize the J th largest singular value of an analytic matrix-valued function $B(\omega)$. In that context, it is advantageous to use two different subspace restrictions on the input and the output to $B(\omega)$ so that the resulting small matrix-valued function acts like the original one only in these small input and output spaces. The subspaces are expanded with the inclusion of the left and right singular vectors of $B(\omega)$ at the optimizers of the small problems and at nearby points. The optimization of the J th smallest singular value involves some subtlety; here, it seems suitable to apply restrictions only on the input to $B(\omega)$, which is equivalent to the frameworks for eigenvalue optimization applied to $B(\omega)^*B(\omega)$. The preferred subspace procedures for particular cases are summarized

in the table below.

Problem	Preferred Subspace Procedure
(1) $\min_{\omega \in \Omega} \lambda_J(\omega)$	Algorithm 1
(2) $\max_{\omega \in \Omega} \lambda_J(\omega)$	Algorithm 2 (Algorithm 1) without past if $d > 1$ (if $d = 1$)
(3) $\min_{\omega \in \Omega} \sigma_J(\omega)$	Algorithm 3(b)
(4) $\max_{\omega \in \Omega} \sigma_J(\omega)$	Algorithm 3 (Algorithm 3(b)) without past if $d > 1$ (if $d = 1$)
(5) $\min_{\omega \in \Omega} \sigma_{-J}(\omega)$	Algorithm 2(s) (Algorithm 1(s)) without past if $d > 1$ (if $d = 1$)
(6) $\max_{\omega \in \Omega} \sigma_{-J}(\omega)$	Algorithm 1(s)

In the table, Algorithm 3(b) refers to the basic greedy procedure for singular value optimization that uses two-sided projections, i.e., Algorithm 3 but with lines 11 and 12 replaced by (2.34). Additionally, Algorithms 1(s) and 2(s) refer to the adaptations of Algorithm 1 and Algorithm 2 for singular value optimization, which form the subspace from the right singular vectors rather than the eigenvectors. Note that the minimization and maximization of a J th smallest eigenvalue are not listed in the table, since they can be posed as the maximization and minimization of a J th largest eigenvalue, respectively.

We have performed convergence and rate-of-convergence analyses for these subspace procedures by extending $A(\omega)$ and $B(\omega)$ to infinite dimension, by replacing them with compact operators $\mathbf{A}(\omega) : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ and $\mathbf{B}(\omega) : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$, the former of which is also self-adjoint. Most remarkably, Theorem 2.3.1 establishes global convergence for Algorithms 1-2 when the J th largest eigenvalue is minimized,

and Theorem 2.3.3 establishes a superlinear rate-of-convergence for Algorithm 1 when $d = 1$ and Algorithm 2 for minimizing and maximizing the J th largest eigenvalue. The superlinear convergence result is established under the simplicity assumption on the J th largest eigenvalue at the optimizer, even though we observe superlinear convergence in numerical experiments (e.g., see Example 5 in Section 2.5.4) where this simplicity assumption is violated. The convergence results also extend to Algorithm 3 for the optimization of the J th largest singular value. The convergence properties of the proposed subspace frameworks for the six problems in the table above are as follows.

(1)&(3)&(6): Proven global convergence at a proven (observed) superlinear rate if $d = 1$ ($d > 1$);

(2)&(4)&(5): Proven convergence but not necessarily to a global solution, observed local convergence at a proven superlinear rate.

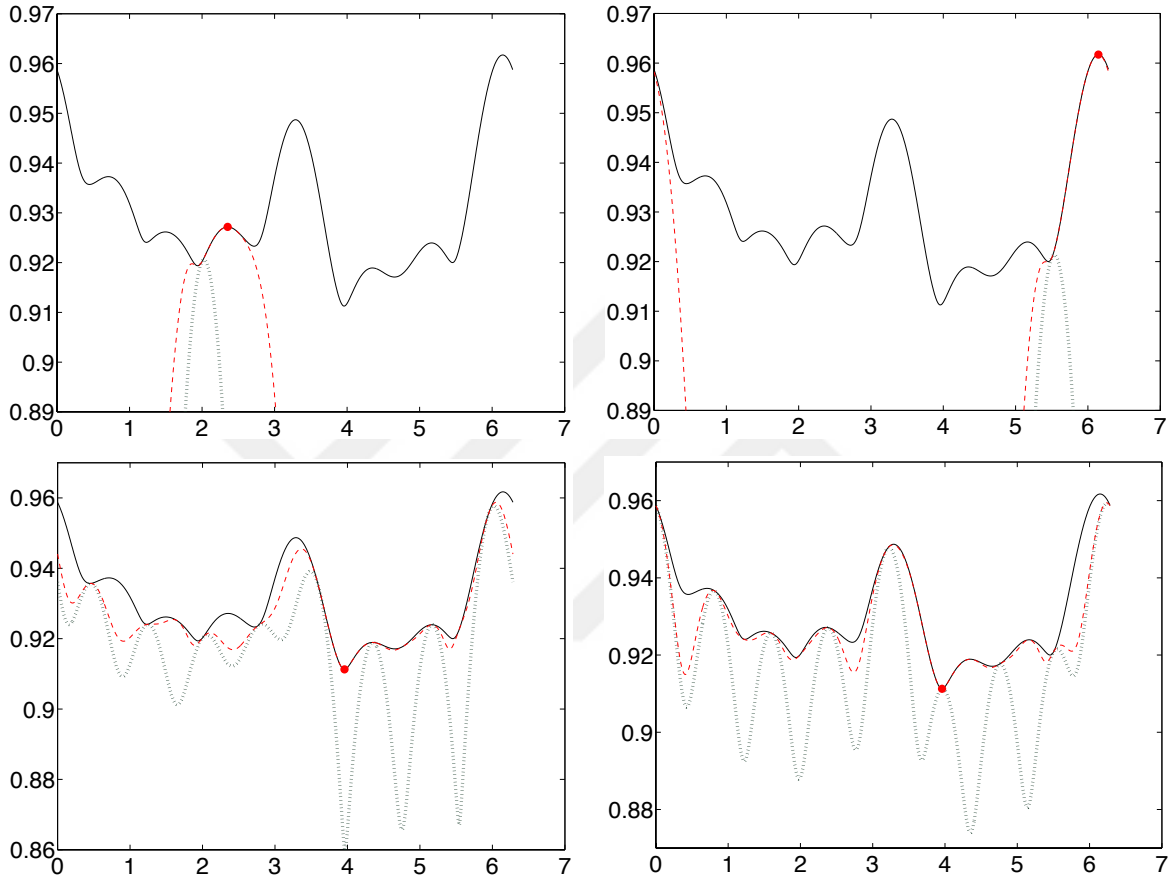


Figure 2.1: Top and bottom rows illustrate Algorithm 1 to maximize and minimize, respectively, the largest eigenvalue $\lambda_1(\omega)$ (solid curves) of $A(\omega)$ as in (2.25) over $\omega \in [0, 2\pi]$ (the horizontal axis). For both the maximization and the minimization, Algorithm 1 is initiated with two different initial points $\omega^{(1)} = 2$ on the left column and $\omega^{(1)} = 5.5$ on the right column. The dotted curves on the top and at the bottom represent the reduced eigenvalue functions with one and eight dimensional subspaces, respectively. The dashed curves correspond to the reduced eigenvalue functions right before convergence, with five dimensional subspaces on the top and fifteen dimensional subspaces at the bottom. The dots represent the converged points, that is denoting the converged ω value with $\hat{\omega}$ and letting $\hat{\lambda} := \lambda_1(\hat{\omega})$ it marks $(\hat{\omega}, \hat{\lambda})$.

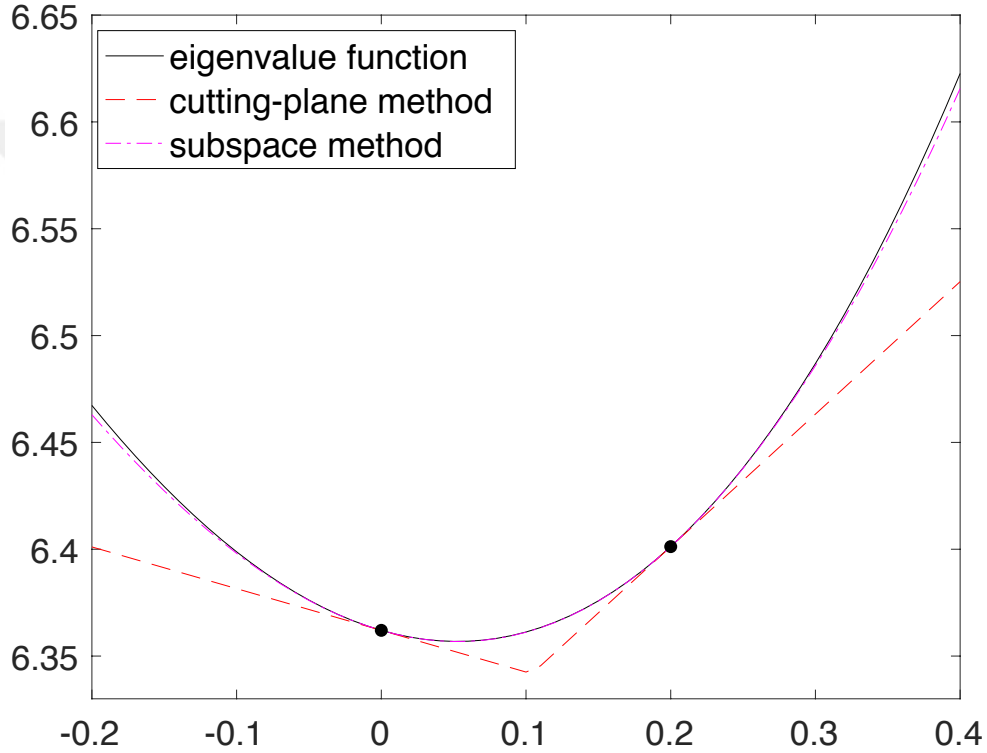


Figure 2.2: A comparison of the approximations by the cutting-plane method and the subspace procedure: (solid curve) the largest eigenvalue of $A(\omega) = A_0 + \omega A_1$, for randomly selected 10×10 symmetric A_0, A_1 with respect to $\omega \in [-0.2, 0.4]$; (dashed curve) the function $\hat{\lambda}_1^{(2)}(\omega)$ used by the cutting-plane method with $\hat{\omega}^{(1)} = 0$ and $\hat{\omega}^{(2)} = 0.2$; (dashed-dotted curve) the function $\lambda_1^{(2)}(\omega)$ used by the subspace procedure with $\omega^{(1)} = 0$ and $\omega^{(2)} = 0.2$; additionally, the black dots represent $(0, \lambda_1(0))$ and $(0.2, \lambda_1(0.2))$.

Chapter 3

NONSMOOTH RATE-OF-CONVERGENCE ANALYSES OF ALGORITHMS FOR EIGENVALUE OPTIMIZATION

This chapter focuses on two recent algorithms to minimize the largest eigenvalue of a Hermitian matrix dependent on one parameter. The first algorithm uses piece-wise quadratic support functions underestimating the eigenvalue function and is meant for small- to medium-scale problems. The other is the basic greedy subspace framework, introduced in the previous chapter, and effective in dealing with the problems when the matrix-valued function is large. In the smooth setting when the largest eigenvalue is simple at its global minimizer, we observe in practice that the former converges at a linear rate, while the latter converges at a superlinear rate both in theory and practice. In this chapter, we analyze the rate of convergence of both algorithms in the presence of the nonsmoothness, when the largest eigenvalue is not simple and zero is strictly inside the generalized Clark derivative. We generalize the subspace framework to cope with the nonsmooth setting. We prove rigorously that the subsequences formed by the odd terms as well as the even terms of the sequence generated by the first algorithm converge to a global minimizer at a quadratic rate, and also establish formally the quadratic rate-of-convergence of the generalized subspace framework. The algorithms are applied to, and the deduced results are illustrated on the computation of the inner numerical radius, the modulus of the point on the boundary of the field of values closest to the origin, which carries significance for instance for the numerical solution of a definite generalized symmetric eigenvalue problem.

3.1 Introduction

A pair of Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$ is said to be definite if

$$\begin{aligned} \gamma(A, B) &:= \min_{z \in \mathbb{C}^n, \|z\|_2=1} \sqrt{(z^*Az)^2 + (z^*Bz)^2} \\ &= \min_{z \in \mathbb{C}^n, \|z\|_2=1} |z^*(A + iB)z| \\ &= \min \{|w| \mid w \in F(A + iB)\} > 0, \end{aligned} \tag{3.1}$$

where $F(C)$ denotes the field of values of $C \in \mathbb{C}^{n \times n}$, the subset of $\mathbb{C}$ defined by

$$F(C) := \{z^*Cz \in \mathbb{C} \mid z \in \mathbb{C}^n, \|z\|_2 = 1\}.$$

The definiteness of (A, B) carries significance for the numerical solution of the generalized eigenvalue problem $Ax = \lambda Bx$ [14, 19]. If (A, B) is known to be definite, then the generalized eigenvalue problem can be transformed into another related problem $\tilde{A}x = \lambda \tilde{B}x$ with $\lambda_{\min}(\tilde{B}) = \gamma(A, B)$, where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of its matrix argument. Furthermore, the transformed problem can be solved by calculating a Cholesky factorization $\tilde{B} = R^*R$, and computing the eigenvalues of the Hermitian matrix $R^{-*}\tilde{A}R^{-1}$. This is a plausible procedure with a small backward error provided $\gamma(A, B)$ is not small. Two other application areas concern Hermitian quadratic eigenvalue problems, in particular checking the hyperbolicity of such problems [35], and saddle point linear systems with symmetric indefinite coefficient matrices, in particular setting up a conjugate gradient iteration for such systems [52].

Motivated by such applications, Cheng and Higham have focused on procedures for efficient determination of whether a given pair (A, B) of Hermitian matrices is definite or not [14]. If the Hermitian pair is not definite, in the same paper, the authors have also considered the computation of a pair $(A + \Delta A_*, B + \Delta B_*)$ where $(\Delta A_*, \Delta B_*)$ solves the following minimization problem for a prescribed positive real number δ :

$$d_\delta(A, B) := \min \left\{ \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\|_2 \mid \gamma(A + \Delta A, B + \Delta B) \geq \delta \right\}. \tag{3.2}$$

The nonconvex eigenvalue optimization problem

$$\min_{\theta \in [0, 2\pi)} \lambda_1(A \cos \theta + B \sin \theta) \quad (3.3)$$

has shown to be closely related to both the determination of whether the pair (A, B) is definite or not, as well as the computation of an optimal $(\Delta A_*, \Delta B_*)$ such that $\gamma(A + \Delta A_*, B + \Delta B_*) \geq \delta$ and $\left\| \begin{bmatrix} \Delta A_* & \Delta B_* \end{bmatrix} \right\|_2 = d_\delta(A, B)$. Note that above $\lambda_1(\cdot)$ represents the largest eigenvalue of its matrix argument. The minimal value in (3.3) has a geometric meaning; in absolute value it corresponds to the inner numerical radius of $A + iB$, i.e., the modulus of the point on the boundary of $F(A + iB)$ closest to the origin. It is argued in [14] that the major challenge for the estimation of $d_\delta(A, B)$ is the global solution of the optimization problem in (3.3).

The optimization problem in (3.3) is only a special instance of a family of eigenvalue optimization problems

$$\min_{\Omega} \lambda_1(\omega), \quad \text{where} \quad A(\omega) := \sum_{j=1}^{\kappa} f_j(\omega) A_j, \quad (3.4)$$

where Ω is an interval in $\mathbb{R}$, the matrices $A_1, \dots, A_\kappa \in \mathbb{C}^{n \times n}$ are Hermitian and the functions $f_1, \dots, f_\kappa : \tilde{\Omega} \rightarrow \mathbb{R}$ are real analytic on their domain $\tilde{\Omega}$, which is an open interval in $\mathbb{R}$ containing Ω . Recall that an eigenvalue optimization problem of the form (3.4) is typically nonconvex excluding the very special affine case $A(\omega) = A_1 + \omega A_2$.

The proposed algorithms in [60] and Chapter 2 are in many cases effective in solving nonconvex eigenvalue optimization problems of the form (3.4) globally. The former of these [60] employs piece-wise quadratic functions to model the objective eigenvalue function, and is meant for small- to medium-scale problems. The latter introduces a subspace framework to deal with problems when the size of $A(\omega)$ is large. It repeatedly projects $A(\omega)$ to small subspaces, and minimizes the largest eigenvalue of the resulting projected matrix-valued function.

Here we present the adaptations of the algorithms in [60] and Chapter 2 for the solution of (3.3). This paves the way for efficient determination of whether a Hermitian pair (A, B) is definite or not, as well as efficient computation of the distance $d_\delta(A, B)$

in (3.2) and a nearest definite pair $(\tilde{A}, \tilde{B})$ such that $\gamma(\tilde{A}, \tilde{B}) \geq \delta$. The adaptation of the algorithm in [60] is guaranteed to converge to the global minimizer of (3.3), and performs well in practice on small- to medium-scale problems. The subspace framework in Chapter 2 extends the range of applicability to quite large Hermitian matrix pairs.

The main contribution of this chapter is on the theoretical side. Global convergence of the algorithms has already been established; the subspace framework in Chapter 2 is globally convergent for the family of optimization problems (3.4) provided the projected problems are solved globally, whereas the algorithm in [60] is globally convergent for problems of the form (3.4) provided a global lower bound is known on $\lambda_1''(\omega)$ over all ω where $\lambda_1(\omega)$ is differentiable. We are also quite informed about the rate-of-convergences of these algorithms in the smooth case when $\lambda_1(\omega_*)$ is simple at a converged global minimizer ω_* ; it is proven and illustrated in the previous chapter that the subspace framework converges at a superlinear rate both in theory and in practice (see also [44]), while we observe that the algorithm in [60] converges at a linear rate (even though a formal proof is open, numerical experiments indicate a linear convergence convincingly). However, little is known about the rate-of-convergences in the presence of nonsmoothness when $\lambda_1(\omega_*)$ is not simple. In this chapter, we analyze the rate-of-convergences of the algorithm in Chapter 2 and [60] on the problems of the form (3.4) in the nonsmooth setting. We assume $\lambda_1(\omega_*)$ is multiple, moreover $0 \in \text{Int } \partial\lambda_1(\omega_*)$ (i.e., zero lies strictly in the interior of $\partial\lambda_1(\omega_*)$), where the generalized Clarke derivative $\partial f(\omega)$ at ω_* of a univariate function $f(\omega)$ that is differentiable almost everywhere excluding a set Γ of measure zero is given by [15]

$$\partial f(\omega_*) := \text{Co} \left\{ \lim_{k \rightarrow \infty} f'(\omega^{(k)}) \mid \omega^{(k)} \rightarrow \omega_*, \omega^{(k)} \notin \Gamma \forall k \right\} \quad (3.5)$$

with $\text{Co}(\cdot)$ denoting the convex hull. The condition $0 \in \text{Int } \partial\lambda_1(\omega_*)$ is equivalent to $\lambda_1(\omega)$ having non-zero left-hand and non-zero right-hand derivatives at ω_* , which is generically the case. In our analysis for the subspace framework, we keep an additional assumption, namely we assume $0 \notin \text{bd } \partial\lambda_J(\omega_*)$ (i.e., zero is not on the boundary of $\partial\lambda_J(\omega_*)$) for the J th largest eigenvalue $\lambda_J(\omega)$ of $A(\omega)$ if $\lambda_J(\omega_*) = \lambda_1(\omega_*)$. This

assumption also holds generically in the nonsmooth setting, and amounts to requiring that $\lambda_J(\omega)$ has non-zero left-hand and right-hand derivatives at ω_* . We prove that if the maximum of the errors of the last two iterates of the algorithm in [60] is h , the error of the next iterate is $O(h^2)$. We also generalize the subspace framework in Chapter 2 to cope with the nonsmooth setting, and show rigorously that the iterates of the proposed generalized framework converge at a quadratic rate.

3.2 Background on Definite Pairs and Nearest Definite Pairs

3.2.1 Connections with Fields of Values

Recall that a given Hermitian pair (A, B) is definite if

$$\begin{aligned}\gamma(A, B) &= \min_{z \in \mathbb{C}^n, \|z\|=1} \sqrt{(z^* A z)^2 + (z^* B z)^2} \\ &= \min\{|z| \mid z \in F(A + iB)\} > 0.\end{aligned}$$

The quantity $\gamma(A, B)$ is called the Crawford number. If a given Hermitian pair (A, B) is known to be definite, then the following result [14, 76] is helpful for the numerical solution of the generalized eigenvalue problem $Ax = \lambda Bx$.

Theorem 3.2.1. *Suppose that (A, B) is a definite Hermitian pair. Letting*

$$A_\theta := A \cos \theta + B \sin \theta \quad \text{and} \quad B_\theta := -A \sin \theta + B \cos \theta,$$

for $\theta \in [0, 2\pi)$, and $\phi \in [0, 2\pi)$ be such that $\gamma(A, B)e^{i\phi} = \arg \min_{z \in F(A+iB)} |z|$, the following assertions hold:

- (i) $A_\theta + iB_\theta = e^{-i\theta}(A + iB)$.
- (ii) *The pair (u_θ, v_θ) is an eigenvalue of (A_θ, B_θ) (i.e., $\det(v_\theta A_\theta - u_\theta B_\theta) = 0$) if and only if the pair (u, v) defined by*

$$\begin{bmatrix} v \\ u \end{bmatrix} := \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} v_\theta \\ u_\theta \end{bmatrix}$$

is an eigenvalue of (A, B) (i.e., $\det(vA - uB) = 0$).

- (iii) The matrix B_φ is positive definite, indeed $\lambda_{\min}(B_\varphi) = \gamma(A, B)$, where $\varphi := -\pi/2 + \phi$.
- (iv) The inclusions $F(A_\varphi + iB_\varphi) \subseteq H_\varphi := \{\omega \in \mathbb{C} \mid \text{Im}(\omega) \geq \gamma(A, B)\}$, and $i\gamma(A, B) \in F(A_\varphi + iB_\varphi)$ hold.

Here is some geometric intuition behind Theorem 3.2.1. The matrices A_θ and B_θ correspond to the real and imaginary parts of the matrix whose field of values is obtained by rotating the field of values of $A + iB$ by an angle of θ in the clock-wise direction. Moreover, $\text{Re}(z^*(A_\theta + iB_\theta)z) = z^*A_\theta z$. Hence, the (orthogonal) projection of $F(A_\theta + iB_\theta)$ onto the real axis is given by $[\lambda_{\min}(A_\theta), \lambda_{\max}(A_\theta)]$, where $\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$ stand for the smallest, largest eigenvalue, respectively, of their matrix arguments. By a similar argument the (orthogonal) projection of $F(A_\theta + iB_\theta)$ onto the imaginary axis is given by $i[\lambda_{\min}(B_\theta), \lambda_{\max}(B_\theta)]$. This in turn implies that, for a definite pair (A, B) , we have $\lambda_{\min}(B_\theta) \leq \gamma(A, B)$ for all θ . Part (iii) of the theorem asserts that the equality $\lambda_{\min}(B_\theta) = \gamma(A, B)$ is attained for the choice of $\theta = \varphi = -\pi/2 + \phi$, which is also a global maximizer of $\lambda_{\min}(B_\theta)$. When $F(A + iB)$ is rotated by an angle of φ in the clock-wise direction, the resulting field of values $F(A_\varphi + iB_\varphi)$ becomes tangential to the horizontal line $\{\omega \in \mathbb{C} \mid \text{Im}(\omega) = \gamma(A, B)\}$ at $i\gamma(A, B)$, and lies fully on or above this line.

The definiteness of a pair (A, B) can be inferred from the inner numerical radius of $A + iB$, where, for a matrix $C \in \mathbb{C}^{n \times n}$, its inner numerical radius is defined by

$$\zeta(C) := \min \{|z| \mid z \text{ is on the boundary of } F(C)\}.$$

It follows from (3.1) that the pair (A, B) is definite if and only if $0 \notin F(A + iB)$ if and only if $\gamma(A, B) = \zeta(A + iB)$. The next result describes a concrete approach to determine the definiteness of (A, B) by solving an eigenvalue optimization problem associated with $\zeta(A + iB)$.

Theorem 3.2.2 (Cheng & Higham [14]). *The inner numerical radius satisfies*

$$\zeta(C) = \left| \min_{\theta \in [0, 2\pi)} \lambda_1(\theta) \right| \quad (3.6)$$

where $\lambda_1(\theta)$ is the largest eigenvalue of $H(\theta) = (Ce^{-i\theta} + C^*e^{i\theta})/2$. Letting $\theta_* := \arg \min_{\theta \in [0, 2\pi)} \lambda_1(\theta)$, we furthermore have

- (i) $0 \in F(C)$ if and only if $\lambda_1(\theta_*) \geq 0$, and
- (ii) the point $\zeta(C)e^{i\phi}$ is on the boundary of $F(C)$ with

$$\phi := \begin{cases} \theta_* & \text{if } 0 \in F(C) \\ \theta_* + \pi & \text{if } 0 \notin F(C) \end{cases}$$

The eigenvalue optimization characterization (3.6) is a consequence of the facts that the inner numerical radius of C and $Ce^{-i\theta}$ are the same, since $F(Ce^{-i\theta}) = e^{-i\theta}F(C)$, as well as

$$\lambda_{\min} \left(\frac{Ce^{-i\theta} + C^*e^{i\theta}}{2} \right) \leq \operatorname{Re}(z) \leq \lambda_{\max} \left(\frac{Ce^{-i\theta} + C^*e^{i\theta}}{2} \right) \quad \forall z \in F(Ce^{-i\theta})$$

where the bounds are attained at points on the boundary of $F(Ce^{-i\theta})$.

Next suppose that (A, B) is not definite. For a prescribed real number $\delta > 0$, we consider $d_\delta(A, B)$ defined as in (3.2), which corresponds to a distance from (A, B) to a nearest definite pair that is δ -away from indefiniteness. It is clear from this definition that if (A, B) is definite with $\gamma(A, B) > \delta$, then we have $d_\delta(A, B) = 0$. Moreover, $d_\delta(A, B)$ has a characterization in terms of $\zeta(A + iB)$, which is stated next [14].

Theorem 3.2.3. *For a given Hermitian pair (A, B) , let $C := A + iB$ and $\theta_* := \arg \min_{\theta \in [0, 2\pi)} \lambda_1(\theta)$, where*

$$H(\theta) = A \cos \theta + B \sin \theta = (Ce^{-i\theta} + C^*e^{i\theta})/2.$$

- (i) *We have*

- $d_\delta(A, B) = \delta + \zeta(C)$ if $0 \in F(C)$
(equivalently if $\lambda_1(\theta_*) \geq 0$), or
- $d_\delta(A, B) = \max\{\delta - \zeta(C), 0\}$ if $0 \notin F(C)$
(equivalently if $\lambda_1(\theta_*) < 0$).

- (ii) Furthermore, letting $H(\theta_*) = Q \text{diag}(\lambda_i) Q^*$ with $\lambda_n \leq \dots \leq \lambda_1$ be a spectral decomposition, two sets of optimal perturbations in both cases are given by

$$\Delta A_1 = \cos \theta_* Q \text{diag}(\min\{-\delta - \lambda_i, 0\}) Q^*,$$

$$\Delta B_1 = \sin \theta_* Q \text{diag}(\min\{-\delta - \lambda_i, 0\}) Q^*$$

and

$$\Delta A_2 = -d_\delta(A, B) \cos \theta_* I, \quad \Delta B_2 = -d_\delta(A, B) \sin \theta_* I.$$

- (iii) The inner numerical radius $\zeta(A + \Delta A_j + i(B + \Delta B_j))$ is attained at $\zeta(A + \Delta A_j + i(B + \Delta B_j))e^{i(\theta_* + \pi)}$ for $j = 1, 2$.

3.2.2 Procedure to Locate a Nearest Definite Pair

Algorithm 5 below computes the distance $d_\delta(A, B)$ defined as in (3.2) for a given real number $\delta > 0$ by exploiting Theorem 3.2.3. It also computes optimal perturbations $\Delta A, \Delta B$, and determines an angle ψ such that $(\tilde{A}, \tilde{B})$ is a definite pair with positive definite $\tilde{B}$ and $\lambda_{\min}(\tilde{B}) = \max\{\delta, \gamma(A, B)\}$, where $\tilde{A} + i\tilde{B} = e^{-i\psi}(A + \Delta A + i(B + \Delta B))$.

We deduce that the matrix $\tilde{B}$ is positive definite with $\lambda_{\min}(\tilde{B}) = \delta$ for the particular choice of ψ in line 7 by employing Theorems 3.2.3 and 3.2.1. In particular, by part (iii) of Theorem 3.2.3, the inner numerical radius $\zeta(A + \Delta A, B + \Delta B)$ is attained at $\zeta(A + \Delta A, B + \Delta B)e^{i(\theta_* + \pi)}$. Hence, it follows from part (iii) of Theorem 3.2.1 that, for the angle

$$\psi := \theta_* + \pi - \pi/2 = \theta_* + \pi/2,$$

the matrices $\tilde{A}, \tilde{B}$ defined by $\tilde{A} + i\tilde{B} = e^{-i\psi}(A + \Delta A + i(B + \Delta B))$ satisfy $\lambda_{\min}(\tilde{B}) = \gamma(A + \Delta A, B + \Delta B) = \max\{\delta, \gamma(A, B)\}$.

Note that if (A, B) is definite with $\gamma(A, B) \geq \delta$, then Algorithm 5 returns $d_\delta(A, B) = 0$ and $\Delta A = \Delta B = 0$. But ψ in line 7 in this case satisfies $\tilde{A} + i\tilde{B} = e^{-i\psi}(A + iB)$ with $\lambda_{\min}(\tilde{B}) = \gamma(A, B)$.

It is argued in [14] that the most challenging part of Algorithm 5 is the solution of the optimization problem $\min_{\theta \in [0, 2\pi)} \lambda_1(\theta)$ globally in line 1. In the next section we present two algorithms to solve these nonconvex eigenvalue optimization problems.

Algorithm 5 Distance to a Nearest Definite Pair

Input: Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$, a parameter $\delta > 0$

Output: $d_\delta(A, B)$, matrices $\Delta A, \Delta B$ s.t. $\gamma(A + \Delta A, B + \Delta B) = \max\{\delta, \gamma(A, B)\}$ and $\|[\Delta A \ \Delta B]\|_2 = d_\delta(A, B)$, as well as $\psi \in [0, 2\pi)$ s.t. $(\tilde{A}, \tilde{B}) = e^{-i\psi}(A + \Delta A, B + \Delta B)$ is a definite pair with $\lambda_{\min}(\tilde{B}) = \max\{\delta, \gamma(A, B)\}$.

- 1: $\theta_* \leftarrow \arg \min_{\theta \in [0, 2\pi)} \lambda_1(A \cos \theta + B \sin \theta)$
 - 2: $H(\theta_*) \leftarrow A \cos \theta_* + B \sin \theta_*$
 - 3: $Q \text{diag}(\lambda_i) Q^* \leftarrow \text{spectral decomposition of } H(\theta_*) \text{ with } \lambda_n \leq \dots \leq \lambda_1$
 - 4: $d_\delta(A, B) \leftarrow \max\{\delta + \lambda_1, 0\}$
 - 5: $\Delta A \leftarrow \cos \theta_* Q \text{diag}(\min\{-\delta - \lambda_i, 0\}) Q^*$
 - 6: $\Delta B \leftarrow \sin \theta_* Q \text{diag}(\min\{-\delta - \lambda_i, 0\}) Q^*$
 - 7: $\psi \leftarrow \theta_* + \pi/2$ is such that $(\tilde{A}, \tilde{B})$ is a definite pair with positive definite $\tilde{B}$ and $\lambda_{\min}(\tilde{B}) = \max\{\delta, \gamma(A, B)\}$, where $\tilde{A} + i\tilde{B} = e^{-i\psi}(A + \Delta A + i(B + \Delta B))$
-

3.3 Computation of Inner Numerical Radius

3.3.1 Level-Set Algorithm

The problem at our hand is

$$\zeta(C) = \left| \min_{\theta \in [0, 2\pi)} \lambda_1(\theta) \right|,$$

Our first algorithm to minimize $\lambda_1(\theta)$ globally is analogous to the one to compute the numerical radius described in [58], and is an extension of the Boyd-Balakrishnan algorithm to compute $\mathcal{H}_\infty$ -norm [7].

Following the practice in [58], for a given estimate $\alpha > 0$ for $\lambda_1(\theta)$, the algorithm finds the α -level set of $\lambda_1(\theta)$, that is it determines the set of θ such that $\lambda_1(\theta) = \alpha$. The open intervals $I_1, \dots, I_k$ satisfying

$$\lambda_1(\theta) < \alpha \quad \forall \theta \in I_\ell$$

for $\ell = 1, \dots, k$ are inferred from this α -level set. The refined estimate $\tilde{\alpha}$ is set equal to the minimum value attained by $\lambda_1(\theta)$ over the set of the midpoints of $I_1, \dots, I_k$.

The key and challenging issue for the algorithm is determining the α -level set of $\lambda_1(\theta)$, and the following result [58] is helpful for this purpose.

Theorem 3.3.1. *Let $\alpha > 0$ be a given real number, and*

$$R(\alpha) = \begin{bmatrix} 2\alpha I & -C \\ I & 0 \end{bmatrix}, \quad S = \begin{bmatrix} C^* & 0 \\ 0 & I \end{bmatrix}.$$

The pencil $L(\lambda) = R(\alpha) - \lambda S$ has $e^{i\theta}$ as an eigenvalue for some $\theta \in \mathbb{R}$ or singular if and only if the Hermitian matrix $H(\theta) = (Ce^{-i\theta} + C^*e^{i\theta})/2$ has α as an eigenvalue.

We put Theorem 3.3.1 into use to find the α -level set of $\lambda_1(\theta)$ as follows. First we extract the unit generalized eigenvalues $e^{i\theta'}$ of the pencil $L(\lambda) = R(\alpha) - \lambda S$. The set of such θ' is a superset of the α -level set of $\lambda_1(\theta)$ by Theorem 3.3.1. To determine the exact α -level set, we compute the eigenvalues of $H(\theta')$ for each θ' , and keep the ones for which $H(\theta')$ has α as indeed the largest eigenvalue.

Now we are ready to present the algorithm. At the j th iteration, given an estimate $r^{(j)}$ for $\lambda_1(\theta)$ such that $r^{(j)} > \lambda_1(\theta)$ and a set of midpoints $\phi_1^{(j)}, \dots, \phi_{m_j}^{(j)}$ of the open intervals where $\lambda_1(\theta)$ takes values smaller than $r^{(j)}$, the minimum value that $\lambda_1(\theta)$ attains over the midpoints $\phi_1^{(j)}, \dots, \phi_{m_j}^{(j)}$ is computed. We set $r^{(j+1)}$ equal to this minimum value. Observe that $r^{(j+1)}$ is a refined estimate, and still an upper bound for $\lambda_1(\theta)$. Subsequently, the open intervals $I_1^{(j+1)}, \dots, I_{m_{j+1}}^{(j+1)}$ where $\lambda_1(\theta) < r^{(j+1)}$ is determined by exploiting Theorem 3.3.1 followed by the maximum eigenvalue checks. Here we note that $I_{m_{j+1}}^{(j+1)}$ may be of the form $I_{m_{j+1}}^{(j+1)} = (\ell_{m_{j+1}}^{(j+1)}, 2\pi) \cup [0, u_{m_{j+1}}^{(j+1)})$ with $\ell_{m_{j+1}}^{(j+1)} > u_{m_{j+1}}^{(j+1)}$. Finally, the midpoints $\theta_1^{(j+1)}, \dots, \theta_{m_{j+1}}^{(j+1)}$ of $I_1^{(j+1)}, \dots, I_{m_{j+1}}^{(j+1)}$ are computed. Algorithm 6 formally describes this level-set method.

The sequence $\{r^{(j)}\}$ generated by Algorithm 6 is guaranteed to converge to the minimal value of $\lambda_1(\theta)$ globally, whose absolute value is equal to $\zeta(C)$. This global convergence result can be deduced from the continuity of $\lambda_1(\theta)$ and the fact that the length of the greatest open interval where $\lambda_1(\theta)$ is smaller than the current estimate is at least halved at each iteration [7, Theorem 4.2].

Theorem 3.3.2. *The sequence $\{r^{(j)}\}$ generated by Algorithm 6 converges to $\lambda_* := \min_{\theta \in [0, 2\pi)} \lambda_1(\theta)$.*

Algorithm 6 Level-set based algorithm for inner numerical radius

Input: A matrix $C \in \mathbb{C}^{n \times n}$.

Output: The sequence $\{r^{(j)}\}$.

- 1: $\phi_1^{(0)} \leftarrow \{0\}$ and $m_0 \leftarrow 1$.
- 2: **for** $j = 0, 1, \dots$ **do**
- 3: $r^{(j+1)} \leftarrow \min\{\lambda_1(\phi_k^{(j)}) \mid k = 1, \dots, m_j\}$.
- 4: Determine all of the intervals $I_k^{(j+1)} = \left(\ell_k^{(j+1)}, u_k^{(j+1)}\right)$ or $I_k^{(j+1)} = \left(\ell_k^{(j+1)}, u_k^{(j+1)} + 2\pi\right)$ with $\ell_k^{(j+1)}, u_k^{(j+1)} \in [0, 2\pi)$ such that

$$\lambda_1(\theta) < r^{(j+1)} \quad \forall \theta \in I_k^{(j+1)} \quad \text{and} \quad \lambda_1(\ell_k^{(j+1)}) = \lambda_1(u_k^{(j+1)}) = r^{(j+1)}$$
 for $k = 1, \dots, m_{j+1}$.
- 5: Form the set $\{\phi_1^{(j+1)}, \dots, \phi_{m_{j+1}}^{(j+1)}\}$, where $\phi_k^{(j+1)}$ is the midpoint of the open interval $I_k^{(j+1)}$ defined by

$$\phi_k^{(j+1)} = \begin{cases} \frac{\ell_k^{(j+1)} + u_k^{(j+1)}}{2}, & \text{if } \ell_k^{(j+1)} < u_k^{(j+1)} \\ \frac{\ell_k^{(j+1)} + u_k^{(j+1)} + 2\pi}{2} \bmod 2\pi, & \text{otherwise} \end{cases}.$$

6: **end for**

Algorithm 6 converges locally at a quadratic rate under the assumption that $\lambda_1(\theta)$ is simple at one of its global minimizers. It can be proven for instance by following the ideas in the proof for the quadratic convergence result [7, Theorem 5.1] regarding the level-set algorithm for $\mathcal{H}_\infty$ -norm computation. A formal proof is included in Appendix A.

Theorem 3.3.3. *Let $\theta_* := \arg \min_{\theta \in [0, 2\pi)} \lambda_1(\theta)$, and suppose $\lambda_1(\theta)$ is simple at θ_* . Then the sequence $\{r^{(j)}\}$ generated by Algorithm 6 converges to $\lambda_* := \min_{\theta \in [0, 2\pi)} \lambda_1(\theta)$ at a quadratic rate.*

Remark. In the case that $\lambda_1(\theta)$ is not simple at θ_* , we have no longer quadratic convergence. This is unlike the analogous algorithms to compute the numerical radius and $\mathcal{H}_\infty$ -norm, which converge quadratically regardless of the multiplicity of the

eigenvalue at the maximizer of the largest eigenvalue function involved. The reason is that the largest eigenvalue function at a maximizer is guaranteed to be at least twice continuously differentiable. On the contrary, this differentiability property is not true at a minimizer of the largest eigenvalue function if the eigenvalue is not simple.

3.3.2 Support Based Algorithm

The algorithm that we employ in this section for computing the inner numerical radius is borrowed from [60]. Throughout the section, we consider the general setting of (3.4). The algorithm that we discuss here to solve (3.4) globally is based on the boundedness of the second derivatives of objective eigenvalue function. It replaces the eigenvalue functions with piece-wise quadratic support functions that underestimate $\lambda_1(\omega)$ globally.

We first introduce formally the quadratic support functions, which are the main ingredients of the algorithm and based on the analytical properties of $\lambda_1(\omega)$. The next result and Theorem 1.1.2 state the analytical properties that are relevant to the derivation [72, 47].

Lemma 3.3.4. *Let $A(\omega) : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ be a Hermitian matrix-valued function as in (3.4). The following hold:*

(i) *For given $\tilde{\omega}, p \in \mathbb{R}$, letting $\phi(\alpha) := \lambda_1(\tilde{\omega} + \alpha p)$, the left-hand derivative $\phi'_-(\alpha)$ and the right-hand derivative $\phi'_+(\alpha)$ of $\phi(\alpha)$ exist everywhere, furthermore they satisfy $\phi'_+(\alpha) \geq \phi'_-(\alpha)$ at all $\alpha \in \mathbb{R}$.*

(ii) *If $\lambda_1(\omega)$ is simple, then it is twice continuously differentiable. At all such ω , we have*

$$\frac{d\lambda_1(\omega)}{d\omega} = v^* \frac{dA(\omega)}{d\omega} v$$

and

$$\frac{d^2\lambda_1(\omega)}{d\omega^2} = v^* \frac{d^2A(\omega)}{d\omega^2} v + 2 \sum_{k=2}^n \frac{1}{\lambda_1(\omega) - \lambda_k(\omega)} \left| v_k^* \frac{dA(\omega)}{d\omega} v \right|^2$$

where $\lambda_k(\omega)$ denotes the k th largest eigenvalue of $A(\omega)$, and v, v_k are unit eigenvectors corresponding to $\lambda_1(\omega), \lambda_k(\omega)$, respectively, for $k = 2 \dots, n$.

In the next result we present the quadratic support functions. The proof of the fact that these functions are global under-estimators for the largest eigenvalue functions follows from part (ii) of Lemma 3.3.4. We omit the proof because of its similarity to the proof of [56, Theorem 2.2], the analogous result that constructs upper support functions for smallest eigenvalue functions.

Theorem 3.3.5 (Quadratic Lower Support Functions). *Suppose $\lambda_1(\omega)$ is simple at $\omega^{(k)} \in \Omega$. Additionally, suppose γ satisfies $\lambda_1''(\omega) \geq \gamma$ for all $\omega \in \Omega$ such that $\lambda_1(\omega)$ is simple. Then, we have*

$$\lambda_1(\omega) \geq q_k(\omega) := \lambda_k + \lambda_k'(\omega - \omega^{(k)}) + \frac{\gamma}{2}(\omega - \omega^{(k)})^2 \quad \forall \omega \in \Omega \quad (3.7)$$

where $\lambda_k := \lambda_1(\omega^{(k)})$ $\lambda_k' := \lambda_1'(\omega^{(k)})$.

We call $q_k(\omega)$ as in (3.7) the quadratic support function about $\omega^{(k)}$. Such a quadratic support function is defined in terms of a lower bound γ for the second derivatives of the eigenvalue function. This lower bound can occasionally be obtained from the expression

$$\frac{d^2 \lambda_1(\omega)}{d\omega^2} = v^* \frac{d^2 A(\omega)}{d\omega^2} v + 2 \sum_{k=2}^n \frac{1}{\lambda_1(\omega) - \lambda_k(\omega)} \left| v_k^* \frac{dA(\omega)}{d\omega} v \right|^2.$$

Since the summation term on the right-hand side is non-negative, we must have

$$\frac{d^2 \lambda_1(\omega)}{d\omega^2} \geq v^* \frac{d^2 A(\omega)}{d\omega^2} v \geq - \left\| \frac{d^2 A(\omega)}{d\omega^2} \right\|_2.$$

Hence $\gamma = -\max_{\omega \in \Omega} \|d^2 A(\omega)/d\omega^2\|_2$ is a theoretically sound choice.

In the case of computing the inner numerical radius of $A + iB$ for a given Hermitian pair (A, B) , we minimize $\lambda_1(\theta)$ of $H(\theta)$ where $H(\theta) = A \cos \theta + B \sin \theta$ over $\theta \in [0, 2\pi]$.

Since

$$\begin{aligned} \left| \frac{d^2 \lambda_1(\theta)}{d\theta^2} \right| &= |v^*(A \cos \theta + B \sin \theta)v| \\ &\geq -\|A \cos \theta + B \sin \theta\|_2 \geq -\|A\|_2 - \|B\|_2 \end{aligned}$$

with v representing a unit eigenvector corresponding to the largest eigenvalue of $H(\theta)$, we can set $\gamma = -\|A\|_2 - \|B\|_2$.

Finally we present the algorithm based on these support functions. For a given point $\omega^{(0)}$, initially the quadratic support function $q_0(\omega)$ about $\omega^{(0)}$ is constructed. Then the algorithm generates a sequence $\{\omega^{(k)}\}$ consisting of estimates for a global minimizer of $\lambda_1(\omega)$, a sequence $\{\ell^{(k)}\}$ consisting of lower bounds for the globally smallest value of $\lambda_1(\omega)$ and a sequence $\{\bar{q}_k(\omega)\}$ consisting of piece-wise quadratic model functions for $\lambda_1(\omega)$. At the k th iteration, the point $\omega^{(k+1)}$ is set equal to a global minimizer of the piece-wise quadratic model function $\bar{q}_k(\omega) := \max_{j=0,\dots,k} q_j(\omega)$. This is followed by the construction of the quadratic support function $q_{k+1}(\omega)$ about $\omega^{(k+1)}$, and inclusion of $q_{k+1}(\omega)$ in the piece-wise quadratic model function. A formal description is given in Algorithm 7.

Algorithm 7 Support Based Algorithm

Input: The matrix-valued function $A(\omega)$, a closed interval $\Omega \subset \mathbb{R}$, the lower bound

γ on $\lambda_1''(\omega)$ for all $\omega \in \Omega$ such that $\lambda_1(\omega)$ is simple.

Output: The sequences $\{\omega^{(k)}\}$ and $\{\ell^{(k)}\}$.

- 1: $\omega^{(0)} \leftarrow$ an initial point in Ω .
 - 2: $q_0(\omega) \leftarrow$ the initial quadratic support function about $\omega^{(0)}$
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $\bar{q}_k(\omega) \leftarrow \max \{q_j(\omega) \mid j = 0, \dots, k\}$
 - 5: $\omega^{(k+1)} \leftarrow \arg \min_{\omega \in \Omega} \bar{q}_k(\omega)$ and $\ell^{(k+1)} \leftarrow \bar{q}_k(\omega^{(k+1)})$
 - 6: $q_{k+1}(\omega) \leftarrow$ the quadratic support function about $\omega^{(k+1)}$
 - 7: **end for**
-

The next result draws global convergence conclusions regarding the sequences $\{\omega^{(k)}\}$ and $\{\ell^{(k)}\}$ by Algorithm 7. This has been proven in [60, Theorem 8.1].

Theorem 3.3.6. *Every convergent subsequence of the sequence $\{\omega^{(k)}\}$ by Algorithm 7 converges to a global minimizer of $\lambda_1(\omega)$ over $\omega \in \Omega$. Furthermore,*

$$\lim_{k \rightarrow \infty} \ell^{(k)} = \min_{\omega \in \Omega} \lambda_1(\omega).$$

If the global minimizer ω_* of $\lambda_1(\omega)$ over $\omega \in \Omega$ is unique, then the convergence of the sequence $\{\omega^{(k)}\}$ itself to ω_* can be asserted. This is formally presented next.

Theorem 3.3.7. *Suppose that $\lambda_1(\omega)$ has a unique global minimizer over all $\omega \in \Omega$, say at ω_* . Then the sequence $\{\omega^{(k)}\}$ by Algorithm 7 converges to ω_* .*

Proof. Assume for a contradiction that $\{\omega^{(k)}\}$ does not converge to ω_* . Equivalently there exists $\epsilon_0 > 0$ such that for all $K \in \mathbb{N}$ and for some $k \in \mathbb{N}$ such that $k > K$, we have $|\omega^{(k)} - \omega_*| \geq \epsilon_0$. Hence, we can construct a subsequence $\{\omega^{(\ell_k)}\}$ of $\{\omega^{(k)}\}$ such that for all k the inequality

$$|\omega^{(\ell_k)} - \omega_*| \geq \epsilon_0 \quad (3.8)$$

is satisfied. But this subsequence $\{\omega^{(\ell_k)}\}$ is bounded, so it has a subsequence $\{\omega^{(n_{\ell_k})}\}$ that is convergent. Now the uniqueness of ω_* combined with Theorem 3.3.6 leads us to $\lim_{k \rightarrow \infty} \omega^{(n_{\ell_k})} = \omega_*$, which contradicts (3.8). $\square$

3.3.3 Rate-of-Convergence of the Support Based Algorithm

In numerous numerical experiments, we observe that Algorithm 7 converges at a linear rate in the smooth case if $\lambda_1(\omega)$ is simple at its global minimizers, but we do not have a formal proof of this observation at this point.

Instead we turn our attention to the nonsmooth case, in particular provide a formal rate-of-convergence analysis in this case. Remarkably the presence of nonsmoothness accelerates Algorithm 7. This is quite a contrast to the level-set method of Section 3.3.1, whose quadratic convergence in the smooth case is hindered and limited to a linear convergence by the existence of nonsmoothness.

Throughout the rest we assume $\lambda_1(\omega)$ has a unique global minimizer, say at ω_* , and that $\lambda_1(\omega_*)$ is multiple with $0 \in \text{Int } \partial\lambda_1(\omega_*)$ where $\partial\lambda_1(\omega_*)$ is the generalized Clarke derivative defined as in (3.5). (Recall that the condition $0 \in \text{Int } \partial\lambda_1(\omega_*)$ holds generically when $\lambda_1(\omega_*)$ is multiple.) The eigenvalue function $\lambda_1(\omega)$ is continuous and piecewise real analytic at ω_* . Indeed there has to be two real analytic eigenvalue functions $\tilde{\lambda}_1(\omega)$, $\tilde{\lambda}_2(\omega)$ such that $\tilde{\lambda}_1(\omega_*) = \tilde{\lambda}_2(\omega_*)$ and $\lambda_1(\omega) = \max\{\tilde{\lambda}_1(\omega), \tilde{\lambda}_2(\omega)\}$ in

a neighborhood of ω_* [72]. Moreover, without loss of generality, we can assume there exists an open interval $\mathcal{I}$ containing ω_* such that

$$\lambda_1(\omega) = \begin{cases} \tilde{\lambda}_1(\omega) & \omega \in \mathcal{I}, \omega \geq \omega_* \\ \tilde{\lambda}_2(\omega) & \omega \in \mathcal{I}, \omega < \omega_* \end{cases}$$

and $\lambda'_{*,+} := \tilde{\lambda}'_1(\omega_*) > 0$, $\lambda'_{*,-} := \tilde{\lambda}'_2(\omega_*) < 0$. Note that $\lambda'_{*,+}$ and $\lambda'_{*,-}$ correspond to the right-hand and left-hand derivatives, respectively, of $\lambda_1(\omega)$ at ω_* . In what follows, we also use the notations

$$\begin{aligned} \lambda''_{*,+} &:= \tilde{\lambda}''_1(\omega_*), \quad \lambda'''_{*,+} := \tilde{\lambda}'''_1(\omega_*), \\ \lambda''_{*,-} &:= \tilde{\lambda}''_2(\omega_*), \quad \lambda'''_{*,-} := \tilde{\lambda}'''_2(\omega_*), \end{aligned}$$

which correspond to higher order one-sided derivatives of $\lambda_1(\omega)$ at ω_* , as well as $\lambda_* := \lambda_1(\omega_*)$. Additionally, the short-hands $\lambda_k, \lambda'_k, \lambda''_k$ represent $\lambda_1(\omega^{(k)})$, $\lambda'_1(\omega^{(k)})$, $\lambda''_1(\omega^{(k)})$ at an iterate $\omega^{(k)}$ of Algorithm 7. Finally, the lower bound γ for the second derivatives of $\lambda_1(\omega)$ is assumed to be negative throughout this subsection without loss of generality.

The following result characterizes the global minimizer $\omega^{(k+1)}$ of $\bar{q}_k(\omega)$ for large k . This point always turns out to be the intersection point of two quadratic support functions about the iterates that are closest to ω_* among the iterates on the left-hand and on the right-hand side of ω_* . Moreover if the distance between the iterates about which these two support functions are constructed is h , then $\omega^{(k+1)}$ is located at a distance of $\Theta(h^2)$ to ω_* .

Lemma 3.3.8 (Minimizers of the Support Functions). *Suppose ω_* is the unique global minimizer of $\lambda_1(\omega)$ over $\omega \in \Omega$, and the eigenvalue $\lambda_1(\omega_*)$ is multiple, $0 \in \text{Int} \partial \lambda_1(\omega_*)$. The sequence $\{\omega^{(k)}\}$ by Algorithm 7 satisfies the following for all large k :*

- (i) *The point $\omega^{(k+1)}$ is the intersection point of $q_{\ell(k)}(\omega)$ and $q_{r(k)}(\omega)$, where $\ell(k), r(k) \in \{1, \dots, k\}$ are given by*

$$\begin{aligned} \ell(k) &:= \arg \min \{ \omega_* - \omega^{(j)} \mid j \in \{0, \dots, k\} \text{ s.t. } \omega_* > \omega^{(j)} \} \quad \text{and} \\ r(k) &:= \arg \min \{ \omega^{(j)} - \omega_* \mid j \in \{0, \dots, k\} \text{ s.t. } \omega_* < \omega^{(j)} \}. \end{aligned}$$

(ii) Letting $h := \max\{\omega_* - \omega^{(\ell(k))}, \omega^{(r(k))} - \omega_*\}$, we have

$$\omega^{(k+1)} = \alpha \cdot \omega_* + \beta \cdot \left(\frac{\omega^{(\ell(k))} + \omega^{(r(k))}}{2} \right) + O(h^3), \quad (3.9)$$

where $\alpha, \beta \in \mathbb{R}^+$ are such that $\alpha + \beta = 1$ and $\beta = \Theta(h)$.

(iii) Furthermore, $|\omega^{(k+1)} - \omega_*| = \Theta(h^2)$.

Proof. (i) The real analyticity of $\tilde{\lambda}_1(\omega)$, $\tilde{\lambda}_2(\omega)$ imply that these eigenvalue functions are continuously differentiable. Observe that $\lambda'_1(\omega) = \tilde{\lambda}'_1(\omega)$ for all $\omega \in \mathcal{I}$ such that $\omega > \omega_*$, and $\lambda'_1(\omega) = \tilde{\lambda}'_2(\omega)$ for all $\omega \in \mathcal{I}$ such that $\omega < \omega_*$, so there exists an interval $\tilde{\mathcal{I}} := (\omega_* - \delta, \omega_* + \delta) \subseteq \mathcal{I}$ for some $\delta > 0$ such that

$$\begin{aligned} \lambda'_1(\omega) &\geq \frac{\lambda'_{*,+}}{2} & \omega \in \tilde{\mathcal{I}}, \omega > \omega_*, \\ \lambda'_1(\omega) &\leq \frac{\lambda'_{*,-}}{2} & \omega \in \tilde{\mathcal{I}}, \omega < \omega_*. \end{aligned}$$

We can choose δ as small as we wish. In particular, in the subsequent arguments, we assume $\delta \leq \min\{(\lambda'_{*,-})/(4\gamma), -(\lambda'_{*,+})/(4\gamma)\}$ without loss of generality.

Theorem 3.3.7 shows that $\omega^{(k)} \rightarrow \omega_*$ as $k \rightarrow \infty$, so for all k large enough $\omega^{(k)}, \omega^{(r(k))}, \omega^{(\ell(k))} \in \tilde{\mathcal{I}}$. For such a large k , we have

$$q'_{\ell(k)}(\omega^{(\ell(k))}) = \lambda'_1(\omega^{(\ell(k))}) \leq \frac{\lambda'_{*,-}}{2}.$$

Additionally, the inequality $\delta \leq (\lambda'_{*,-})/(4\gamma)$ implies

$$q'_{\ell(k)}(\omega) = \lambda'_1(\omega^{(\ell(k))}) + \gamma(\omega - \omega^{(\ell(k))}) \leq \frac{\lambda'_{*,-}}{2} - \gamma\delta \leq \frac{\lambda'_{*,-}}{4}.$$

for all $\omega \in (\omega_* - \delta, \omega^{(\ell(k))})$. This means $q_{\ell(k)}(\omega)$ is decreasing on this interval, so

$$\bar{q}_k(\omega^{(\ell(k))}) = q_{\ell(k)}(\omega^{(\ell(k))}) < q_{\ell(k)}(\omega) \leq \bar{q}_k(\omega) \quad \forall \omega \in (\omega_* - \delta, \omega^{(\ell(k))}).$$

Consequently, $\omega^{(k+1)}$, the global minimizer of $\bar{q}_k(\omega)$, cannot lie in $(\omega_* - \delta, \omega^{(\ell(k))})$.

Analogous arguments apply to $q_{r(k)}(\omega)$ for $\omega \in \tilde{\mathcal{I}}$ such that $\omega > \omega^{(r(k))}$. In particular, $q'_{r(k)}(\omega^{(r(k))}) = \lambda'_1(\omega^{(r(k))}) \geq \frac{\lambda'_{*,+}}{2}$, and, employing $\delta \leq -(\lambda'_{*,+})/(4\gamma)$, we have $q'_{r(k)}(\omega) = \lambda'_1(\omega^{(r(k))}) + \gamma(\omega - \omega^{(r(k))}) \geq \frac{\lambda'_{*,+}}{2} + \gamma\delta \geq \frac{\lambda'_{*,+}}{4}$ for all

$\omega \in (\omega^{(r(k))}, \omega_* + \delta)$. This leads us to the conclusion $\bar{q}_k(\omega^{(r(k))}) = q_{r(k)}(\omega^{(r(k))}) < q_{r(k)}(\omega) \leq \bar{q}_k(\omega) \quad \forall \omega \in (\omega^{(r(k))}, \omega_* + \delta)$. Hence, $\omega^{(k+1)} \notin (\omega^{(r(k))}, \omega_* + \delta)$ as well.

As $\omega^{(k+1)} \in \tilde{\mathcal{I}} = (\omega_* - \delta, \omega_* + \delta)$, the only possibility left out is that $\omega^{(k+1)} \in [\omega^{(\ell(k))}, \omega^{(r(k))}]$. But the global minimizer of $\bar{q}_k(\omega)$ in $[\omega^{(\ell(k))}, \omega^{(r(k))}]$ is the intersection point of $q_{\ell(k)}(\omega)$ and $q_{r(k)}(\omega)$, as the former of these quadratic functions is decreasing, while the latter is increasing on this interval.

(ii) It follows from part (i) that $\omega^{(k+1)}$ satisfies $q_{\ell(k)}(\omega^{(k+1)}) = q_{r(k)}(\omega^{(k+1)})$. Now solving this equation for $\omega^{(k+1)}$ yields

$$\omega^{(k+1)} = \frac{\lambda_{\ell(k)} - \lambda_{r(k)} + \lambda'_{r(k)}\omega^{(r(k))} - \lambda'_{\ell(k)}\omega^{(\ell(k))} - \frac{\gamma}{2} \left([\omega^{(r(k))}]^2 - [\omega^{(\ell(k))}]^2 \right)}{\left(\lambda'_{r(k)} - \lambda'_{\ell(k)} \right) - \gamma (\omega^{(r(k))} - \omega^{(\ell(k))})}.$$

By applications of Taylor's theorem with third order remainders to $\lambda_{r(k)}$, $\lambda_{\ell(k)}$, $\lambda'_{r(k)}$, $\lambda'_{\ell(k)}$, specifically by expressing them in terms of $\lambda_1(\omega)$ and one-sided derivatives of $\lambda_1(\omega)$ at ω_* , we obtain

$$\begin{aligned} \omega^{(k+1)} = & \left\{ \left[\lambda_* + \lambda'_{*,-}(\omega^{(\ell(k))} - \omega_*) + \lambda''_{*,-}(\omega^{(\ell(k))} - \omega_*)^2 + O(h^3) \right] - \right. \\ & \left[\lambda_* + \lambda'_{*,+}(\omega^{(r(k))} - \omega_*) + \lambda''_{*,+}(\omega^{(r(k))} - \omega_*)^2 + O(h^3) \right] + \\ & \left[\lambda'_{*,+} + \lambda''_{*,+}(\omega^{(r(k))} - \omega_*) + \lambda'''_{*,+}(\omega^{(r(k))} - \omega_*)^2 + O(h^3) \right] \times \\ & \left[(\omega^{(r(k))} - \omega_*) + \omega_* \right] - \\ & \left[\lambda'_{*,-} + \lambda''_{*,-}(\omega^{(\ell(k))} - \omega_*) + \lambda'''_{*,-}(\omega^{(\ell(k))} - \omega_*)^2 + O(h^3) \right] \times \\ & \left[(\omega^{(\ell(k))} - \omega_*) + \omega_* \right] - \\ & (\gamma/2) \left[[\omega^{(r(k))}]^2 - [\omega^{(\ell(k))}]^2 \right] \Big\} / \\ & \left\{ \left[\lambda'_{*,+} + \lambda''_{*,+}(\omega^{(r(k))} - \omega_*) + \lambda'''_{*,+}(\omega^{(r(k))} - \omega_*)^2 + O(h^3) \right] - \right. \\ & \left[\lambda'_{*,-} + \lambda''_{*,-}(\omega^{(\ell(k))} - \omega_*) + \lambda'''_{*,-}(\omega^{(\ell(k))} - \omega_*)^2 + O(h^3) \right] - \\ & \left. \gamma \left[\omega^{(r(k))} - \omega^{(\ell(k))} \right] \right\}. \end{aligned}$$

This can be rearranged into

$$\omega^{(k+1)} = \frac{(\alpha_+ - \alpha_-) \cdot \omega_* + \eta \cdot \left(\frac{\omega^{(\ell(k))} + \omega^{(r(k))}}{2} \right)}{\alpha_+ - \alpha_- + \eta} + O(h^3)$$

where

$$\begin{aligned} \alpha_+ &:= [\lambda'_{*,+} + \lambda''_{*,+}(\omega^{(r(k))} - \omega_*) + \lambda'''_{*,+}(\omega^{(r(k))} - \omega_*)^2], \\ \alpha_- &:= [\lambda'_{*,-} + \lambda''_{*,-}(\omega^{(\ell(k))} - \omega_*) + \lambda'''_{*,-}(\omega^{(\ell(k))} - \omega_*)^2], \\ \eta &:= -\gamma (\omega^{(r(k))} - \omega^{(\ell(k))}). \end{aligned}$$

The desired result follows from $\alpha_+ - \alpha_- > 0$, indeed $\alpha_+ - \alpha_- = \Theta(1)$, and by setting $\alpha := (\alpha_+ - \alpha_-)/(\alpha_+ - \alpha_- + \eta)$, $\beta := \eta/(\alpha_+ - \alpha_- + \eta)$.

(iii) This is immediate from equation (3.9) by observing

$$\begin{aligned} |\omega^{(k+1)} - \omega_*| &= \left| \alpha \cdot \omega_* + \beta \cdot \left(\frac{\omega^{(\ell(k))} + \omega^{(r(k))}}{2} \right) - (\alpha + \beta) \cdot \omega_* \right| + O(h^3), \\ &= \left| \beta \cdot \left[\left(\frac{\omega^{(\ell(k))} + \omega^{(r(k))}}{2} \right) - \omega_* \right] \right| + O(h^3) = \Theta(h^2). \end{aligned}$$

□

The main rate-of-convergence result concerning Algorithm 7 in the nonsmooth case is presented next. It asserts that if the maximum of the errors of the last two iterates is h , the error of the next iterate is $O(h^2)$. A similar rapid convergence conclusion is drawn for the sequence $\{\ell^{(k)}\}$ as well.

Theorem 3.3.9 (Rate-of-convergence). *Suppose that the global minimizer of $\lambda_1(\omega)$ over Ω is unique, say at ω_* . Additionally, suppose that the eigenvalue $\lambda_1(\omega_*)$ is multiple, and $0 \in \text{Int } \partial\lambda_1(\omega_*)$. Following assertions hold for the sequence $\{\omega^{(k)}\}$ generated by Algorithm 7 for all k large enough:*

(i) $k \in \{\ell(k), r(k)\}$.

(ii) $|\omega^{(k+1)} - \omega_*| = O(\max\{|\omega^{(k)} - \omega_*|, |\omega^{(k-1)} - \omega_*|\}^2)$.

(iii) If γ is large enough in absolute value, then

$$\lambda_* - \ell^{(k+1)} = O((\lambda_* - \ell^{(k-1)})^2).$$

Proof. (i) If $\omega^{(k)} > \omega_*$, then it is apparent from (3.9) that for large k we must have $\omega^{(k)} \in (\omega_*, (\omega^{(\ell(k-1))} + \omega^{(r(k-1))})/2)$ implying $\omega^{(k)} - \omega_* < \omega^{(r(k-1))} - \omega_*$. Hence, in this case, $r(k) = k$.

Similarly, if $\omega^{(k)} < \omega_*$, then $\omega^{(k)} \in ((\omega^{(\ell(k-1))} + \omega^{(r(k-1))})/2, \omega_*)$ for large k by (3.9). This in turn implies $\omega_* - \omega^{(k)} < \omega_* - \omega^{(\ell(k-1))}$, so $\ell(k) = k$.

(ii) Let us suppose $k = r(k)$ without loss of generality. (Otherwise, $k = \ell(k)$ by part (i) and a similar arguments applies.) This means that $\ell(k) \neq k$, so $\ell(k) = \ell(k-1)$. If $\ell(k) = \ell(k-1) = k-1$, then part (iii) of Lemma 3.3.8 implies

$$\begin{aligned} |\omega^{(k+1)} - \omega_*| &= \Theta(\max\{\omega_* - \omega^{(\ell(k))}, \omega^{(r(k))} - \omega_*\}^2) \\ &= \Theta(\max\{\omega_* - \omega^{(k-1)}, \omega^{(k)} - \omega_*\}^2). \end{aligned}$$

Hence, let us suppose $\ell(k-1) \neq k-1$. But then $r(k-1) = k-1$ by part (i). Furthermore, as $\omega^{(r(k))} = \omega^{(k)} \in (\omega_*, (\omega^{(\ell(k-1))} + \omega^{(r(k-1))})/2)$, we must have

$$\omega^{(k-1)} - \omega_* = \omega^{(r(k-1))} - \omega_* > \omega_* - \omega^{(\ell(k-1))} = \omega_* - \omega^{(\ell(k))}$$

from which we deduce

$$\max\{|\omega^{(k-1)} - \omega_*|, |\omega^{(k)} - \omega_*|\} \geq \max\{\omega_* - \omega^{(\ell(k))}, \omega^{(r(k))} - \omega_*\}.$$

Hence, letting $h := \max\{|\omega^{(k-1)} - \omega_*|, |\omega^{(k)} - \omega_*|\}$, we have $\max\{\omega_* - \omega^{(\ell(k))}, \omega^{(r(k))} - \omega_*\} = O(h)$. It follows from part (iii) of Lemma 3.3.8 that $|\omega^{(k+1)} - \omega_*| = O(h^2)$, completing the proof.

(iii) Part (i) of Lemma 3.3.8 asserts that the point $\omega^{(k-1)}$ is the intersection point of $q_{\ell(k-2)}(\omega)$ and $q_{r(k-2)}(\omega)$. Without loss of generality, let us assume $\omega^{(r(k-2))} - \omega_* > \omega_* - \omega^{(\ell(k-2))}$. We have

$$\begin{aligned} \ell^{(k-1)} &= q_{r(k-2)}(\omega^{(k-1)}) \\ &= \lambda_{r(k-2)} + \lambda'_{r(k-2)}(\omega^{(k-1)} - \omega^{(r(k-2))}) + \frac{\gamma}{2}(\omega^{(k-1)} - \omega^{(r(k-2))})^2. \end{aligned}$$

Now applications of Taylor's theorem about ω_* yields

$$\begin{aligned} \ell^{(k-1)} &= \left[\lambda_* + \lambda'_{*,+}(\omega^{(r(k-2))} - \omega_*) + \frac{\lambda''_{*,+}}{2}(\omega^{(r(k-2))} - \omega_*)^2 \right] + \\ &\quad \left[\lambda'_{*,+} + \lambda''_{*,+}(\omega^{(r(k-2))} - \omega_*) \right](\omega^{(k-1)} - \omega^{(r(k-2))}) + \frac{\gamma}{2}(\omega^{(k-1)} - \omega^{(r(k-2))})^2 \\ &\quad + O((\omega^{(r(k-2))} - \omega_*)^3), \end{aligned}$$

where we use $\omega^{(k-1)} - \omega_* = \Theta((\omega^{(r(k-2))} - \omega_*)^2)$ due to part (iii) of Lemma 3.3.8.

Letting $h := \lambda_* - \ell^{(k-1)}$, the last equation yields

$$\begin{aligned} h &= -\lambda'_{*,+}(\omega^{(k-1)} - \omega_*) - \frac{\lambda''_{*,+}}{2}(\omega^{(r(k-2))} - \omega_*)^2 \\ &\quad - \lambda''_{*,+}(\omega^{(r(k-2))} - \omega_*)(\omega^{(k-1)} - \omega^{(r(k-2))}) \\ &\quad - \frac{\gamma}{2}(\omega^{(k-1)} - \omega^{(r(k-2))})^2 + O((\omega^{(r(k-2))} - \omega_*)^3) \\ &= -\lambda'_{*,+}(\omega^{(k-1)} - \omega_*) + \frac{\lambda''_{*,+}}{2}(\omega^{(r(k-2))} - \omega_*)^2 \\ &\quad - \frac{\gamma}{2}(\omega^{(r(k-2))} - \omega_*)^2 + O((\omega^{(r(k-2))} - \omega_*)^3). \end{aligned}$$

Assuming γ is large enough, the terms on the order of $\Theta(\omega^{(k-1)} - \omega_*) = \Theta((\omega^{(r(k-2))} - \omega_*)^2)$ do not cancel out. This means $\omega^{(k-1)} - \omega_* = \Theta(h)$ and $(\omega^{(r(k-2))} - \omega_*)^2 = \Theta(h)$.

Letting $h_2 := \lambda_* - \ell^{(k)} \leq \lambda_* - \ell^{(k-1)}$ (recall that the sequence $\{\ell^{(k)}\}$ is increasing bounded from above by ℓ_*), and following similar steps, we also deduce $\omega^{(k)} - \omega_* = \Theta(h_2) = O(h)$. Now it follows from part (ii) that $|\omega^{(k+1)} - \omega_*| = O(h^2)$. The point $\omega^{(k+1)}$ is the intersection point of $q_{\ell(k)}(\omega)$ and $q_{r(k)}(\omega)$, where $\ell(k) = k$ or $r(k) = k$, so

$$\begin{aligned} \ell^{(k+1)} &= \bar{q}_k(\omega^{(k+1)}) = q_k(\omega^{(k+1)}) \\ &= \lambda_k + \lambda'_k(\omega^{(k+1)} - \omega^{(k)}) + \frac{\gamma}{2}(\omega^{(k+1)} - \omega^{(k)})^2. \end{aligned}$$

Assume for now $\omega^{(k)} > \omega_*$. Recalling $\omega^{(k)} - \omega_* = \Theta(h_2) = O(h)$, and once again applying Taylor's theorem to λ_k, λ'_k about ω_* give rise to

$$\begin{aligned} \ell^{(k+1)} &= \left[\lambda_* + \lambda'_{*,+}(\omega^{(k)} - \omega_*) + O((\omega^{(k)} - \omega_*)^2) \right] + \\ &\quad \left[\lambda'_{*,+} + O(\omega^{(k)} - \omega_*) \right] (\omega^{(k+1)} - \omega^{(k)}) + \frac{\gamma}{2}(\omega^{(k+1)} - \omega^{(k)})^2 \\ &= \lambda_* + \lambda'_{*,+}(\omega^{(k+1)} - \omega_*) + O(h^2) = \lambda_* + O(h^2), \end{aligned}$$

which in turn implies $\lambda_* - \ell^{(k+1)} = O(h^2)$ as desired. If $\omega^{(k)} < \omega_*$, all of the equalities above still hold but applying Taylor's theorem to the left-hand side of ω_* . This results in the same expressions except that occurrences of $\lambda'_{*,+}$ are replaced by $\lambda'_{*,-}$. $\square$

k	$r^{(k)}$
2	0. <u>8</u> 687683091642120
3	0. <u>81</u> 19559545628993
4	0. <u>811</u> 8872240421637
5	0. <u>8118</u> 872239262381
6	0. <u>81188</u> 72239262371

Table 3.1: The iterates $r^{(k)}$ of Algorithm 6 are listed with respect to k . These listed values indicate a quadratic rate-of-convergence as expected in theory.

3.3.4 Numerical Experiments

Distance to a Nearest Definite Pair. Consider the matrices $A = \text{diag}(-3 : 3) \in \mathbb{R}^{7 \times 7}$ and $B \in \mathbb{R}^{7 \times 7}$ defined by $b_{ij} = 1/(i + j)$ except $b_{11} = b_{77} = -1$. This is an indefinite Hermitian pair example taken from [14]. We run both algorithms to compute the inner numerical radius $\zeta(A + iB)$ and the distance $d_\delta(A, B)$ for $\delta = 10^{-8}$. The computed distance by both algorithms is $d_\delta(A, B) = 0.8118872239262$. Figure 3.1 illustrates the field of values of $A + iB$, $A + \Delta A + i(B + \Delta B)$ and $\tilde{A} + i\tilde{B} = e^{-i\psi}(A + \Delta A + i(B + \Delta B))$, where $\Delta A, \Delta B$ and ψ are as in Algorithm 5. Note that $\tilde{B}$ is positive definite with $\lambda_{\min}(\tilde{B}) = 10^{-8}$. In each plot, the point where the inner numerical radius is attained is marked with a diamond, and the eigenvalues of $A + iB$ are marked with circles. In this example, $\lambda_1(A \cos \theta + B \sin \theta)$ turns out to be simple at the global minimizer θ_* and Table 3.1 illustrates the quadratic convergence of Algorithm 6.

Testing Hyperbolicity of Quadratic Eigenvalue Problems. The quadratic eigenvalue problem (QEP)

$$Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0 \quad (3.10)$$

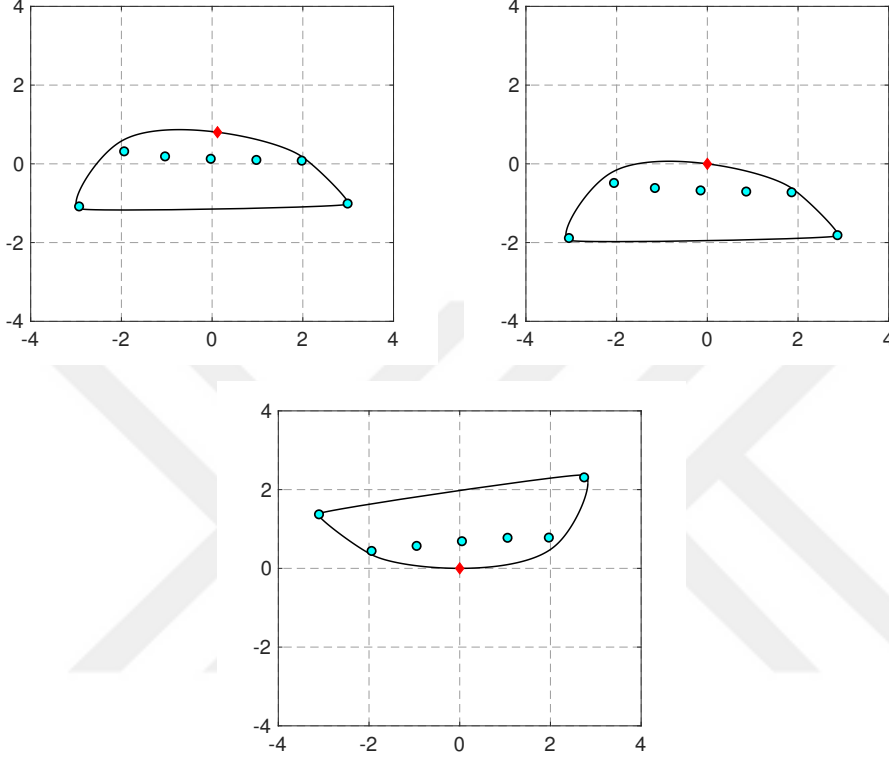


Figure 3.1: The figure depicts the field of values of $A + iB$ (top left), $(A + \Delta A) + i(B + \Delta B)$ (top right), $e^{-i\psi}((A + \Delta A) + i(B + \Delta B))$ (bottom), where $\Delta A, \Delta B, \psi$ are those returned by Algorithm 5 for the Hermitian pair (A, B) taken from [14] and $\delta = 10^{-8}$. In each case, the diamond marks the point where the inner numerical radius is attained, whereas the circles mark the eigenvalues.

is said to be hyperbolic if $A, B, C \in \mathbb{C}^{n \times n}$ are Hermitian, A is positive definite, and

$$(x^* B x)^2 > 4(x^* A x)(x^* C x) \quad \forall x \in \mathbb{C}^n \text{ s.t. } x \neq 0. \quad (3.11)$$

Let x be an eigenvector and λ be the corresponding eigenvalue satisfying (3.10). By multiplying (3.10) by x^* from left, we obtain the quadratic equation $\lambda^2 x^* A x + \lambda x^* B x + x^* C x = 0$ with solutions

$$\frac{-x^* B x \pm \sqrt{(x^* B x)^2 - 4(x^* A x)(x^* C x)}}{2x^* A x}. \quad (3.12)$$

At least one of these two solutions must be the eigenvalue λ corresponding to x ,

which implies, assuming the existence of n linearly independent eigenvectors and due to (3.11), an hyperbolic QEP has at least n real eigenvalues. More is true; it is shown in [48, Section 7.6] that if the QEP is hyperbolic with positive definite B and positive semidefinite C , then counting also the multiplicities, n numbers as in (3.12) involving positive square-roots correspond to n eigenvalues of (3.10), referred as primary eigenvalues. The remaining n numbers as in (3.12) but involving negative square-roots also correspond to n eigenvalues; these are called the secondary eigenvalues. The eigenvectors associated with the primary eigenvalues form a linearly independent set, and the same is true for the secondary eigenvalues.

It is well known [35] that the hyperbolicity of the QEP in (3.10) is equivalent to the definiteness of (A_1, B_1) with

$$A_1 = \begin{bmatrix} -C & 0 \\ 0 & A \end{bmatrix} \quad B_1 = - \begin{bmatrix} B & A \\ A & 0 \end{bmatrix}. \quad (3.13)$$

Consider in particular the QEP with

$$A = I_4, \quad B = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 12 & -4 & 0 \\ 0 & -4 & 12 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

corresponding to a mass, damping, stiffness matrix, respectively. Applications of Algorithms 6 and 7 for the computation of the inner numerical radius of $A_1 + iB_1$ yield the global minimizer $\theta_* = 2.5682098635$ of $\lambda_1(A_1 \cos \theta + B_1 \sin \theta)$ with $\lambda_1(A_1 \cos \theta_* + B_1 \sin \theta_*) = -0.4897656697$. It follows from Theorem 3.2.2 that the field of values of $A_1 + iB_1$ does not contain the origin, implying that the pair (A_1, B_1) is definite and hence the QEP is hyperbolic.

Comparison of Algorithms 6 and 7. Next we compare the performances of the level-set based and support based algorithms to compute the inner numerical radius of $A_n + iB_n$ for various n , where A_n is the Fiedler matrix and B_n is the Moler matrix of size $n \times n$. In Table 3.2, we provide CPU times and the number of iterations required

LEVEL-SET			SUPPORT-BASED		
n	iter	t	n	iter	t
120	5	1.60	120	84	0.40
240	5	8.90	240	119	1.39
360	6	45.44	360	143	4.04
480	6	107.99	480	162	6.28

Table 3.2: CPU times (in seconds) and the number of iterations required by Algorithms 6 and 7 to compute the inner numerical radius of $A_n + iB_n$ for various n , where A_n is the Fiedler matrix and B_n is the Moler matrix of size $n \times n$.

by each algorithm. The reason why the level-set approach requires more time is that it computes all eigenvalues of matrices of size $2n \times 2n$, whereas the support based algorithm computes only the largest eigenvalues of $n \times n$ matrices.

The case when $\lambda_1(\theta_*)$ is not simple. The next example illustrates the convergence rates of Algorithms 6 and 7 when the largest eigenvalue is not simple at the global minimizer. Consider the tridiagonal matrix

$$\tilde{A} = \begin{bmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & a_3 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & a_n \end{bmatrix} + 0.5iI_n \quad \text{with} \quad a_j = 2 + \frac{j}{n} \quad (3.14)$$

for $n = 10$ and let $A = \tilde{A}e^{i\pi/6}$. The global minimum of $\lambda_1((Ae^{i\theta} + A^*e^{-i\theta})/2)$ is attained at $\theta_* = -2.6179938779914944$ and $\lambda_1(\theta_*)$ has multiplicity 2. The quadratic convergence of Algorithm 6 is no longer true for this example; this is illustrated in Table 3.3 where the last seven iterates of the algorithm are listed. On the other hand, we observe faster convergence for Algorithm 7 consistent with the rate-of-convergence result in Theorem 3.3.9. This is depicted in Table 3.4.

k	$r^{(k)}$
28	-0.9999999999999492
29	-0.9999999999999833
30	-0.9999999999999944
31	-0.9999999999999982
32	-0.9999999999999996
33	-0.9999999999999998
34	-1.0000000000000000

Table 3.3: The iterates $r^{(k)}$ of Algorithm 6 to compute the inner numerical radius of the matrix A defined as in (3.14) are listed.

Linear systems in saddle point form. This is another example where the eigenvalue function is nonsmooth at the optimizer. The matrix of a saddle point linear system is of the form

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}, \quad (3.15)$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m \times n}$ with $m \leq n$ and $C \in \mathbb{R}^{m \times m}$ is symmetric positive semidefinite. The matrix $\mathcal{A}$ is usually large and sparse, and can be reduced to the block diagonal matrix $\text{diag}(A, S)$ with $S = -(C + BA^{-1}B^T)$ by row operations. Now observe that S is symmetric negative semidefinite implying that $\mathcal{A}$ is indefinite with n positive eigenvalues and $\text{rank}(S)$ negative eigenvalues.

The indefiniteness of $\mathcal{A}$ is a hurdle for iterative solvers such as Krylov subspace methods; it slows down the convergence. It has been shown in [52] that, letting $\mathcal{J} := \text{diag}(I_n, -I_m)$, if there is a real scalar μ such that $\mathcal{M}(\mu) = \mathcal{A} - \mu\mathcal{J}$ is positive definite, then a conjugate gradient iteration that depends on this value of μ can be constructed to solve the linear system $\mathcal{J}\mathcal{A}x = \mathcal{J}b$. Clearly the positive definiteness of $\mathcal{M}(\mu)$ for some $\mu \in \mathbb{R}$ is equivalent to the positive definiteness of $-\mathcal{A} \sin \theta + \mathcal{J} \cos \theta$

k	$\ell^{(k+1)}$	$ \omega^{(k+1)} - \theta_* $
12	-1.000024850740653	0.000903071031368
13	-1.000002750999332	0.000041649882203
14	-1.000000001968154	0.000001828314169
15	-1.000000000011313	0.000000003906009
16	-1.000000000000000	0.000000000007542
17	-1.000000000000000	0.000000000000000

Table 3.4: The sequence $\{\ell^{(k+1)}\}$ and the error of the sequence $\{\omega^{(k+1)}\}$ by Algorithm 7 are listed with respect to k on the example involving the computation of the inner numerical radius of $A = \tilde{A}e^{i\pi/6}$, where $\tilde{A}$ is as in (3.14).

for some $\theta \in [0, 2\pi]$. Now Theorem 3.2.1 suggests testing the definiteness of the pair $(\mathcal{A}, \mathcal{J})$. If this pair is definite, then, by part (iii) of Theorem 3.2.1, the matrix $-\mathcal{A} \sin \varphi + \mathcal{J} \cos \varphi$ is positive definite for $\varphi := -\pi/2 + \phi$ where ϕ is the angle such that $\gamma(\mathcal{A}, \mathcal{J})e^{i\phi}$ is the point in $F(\mathcal{A}, \mathcal{J})$ closest to the origin. This in turn implies that $\mathcal{M}(\mu)$ is positive definite for $\mu := -\cos \varphi / \sin \varphi$.

We consider a linear system in saddle point form that arise from a stable discretization of a Stokes equation with the coefficient matrix $\mathcal{A}$ of the form (3.15). This linear system is generated by the MATLAB package “Incompressible Flow Iterative Solution Software (IFISS) version 3.5” [75]; more specifically, the sparse matrices A, B, C are constructed by running the script file `stokes_testproblem` with the default options, resulting in A of size $n = 578$ and C of size $m = 256$. For compatibility with the particular examples worked through in [52, 30], we shift A by $0.0764I_n$, and run support based algorithm to determine the definiteness of the pair $(\mathcal{A}, \mathcal{J})$. We detect that the pair is definite. Moreover computations yield $\mu = 0.0541$ for which $\mathcal{M}(\mu)$ is positive definite, indeed $\lambda_{\min}(\mathcal{M}(\mu)) = 0.0223$. The nonsmoothness in these computations is encountered in a strong fashion, as $\lambda_1(\mathcal{A} \cos \theta + \mathcal{J} \sin \theta)$ at the minimizing θ has

k	$\ell^{(k+1)}$	$ \omega^{(k+1)} - \theta_* $
11	-0.022231442152684	0.000008162291389
12	-0.022224901919276	0.000006757114932
13	-0.022224901723492	0.000000000136544
14	-0.022224901666671	0.000000000058712
15	-0.022224901666670	0.000000000000000

Table 3.5: This table is analogous to Table 3.4, but it concerns the example of the saddle point linear system arising from the Stokes equation, in particular the positive definiteness of $\mathcal{A} - \mu\mathcal{J}$ for some μ , where $\mathcal{A} \in \mathbb{R}^{(n+m) \times (n+m)}$ is as in (3.15) and $\mathcal{J} := \text{diag}(I_n, -I_m)$.

multiplicity three. Table 3.5 indicates rapid convergence for the sequences $\{\ell^{(k)}\}$ and $\{\omega^{(k)}\}$ by Algorithm 7. In particular, according to the table, the sequences $\{\ell^{(2k)}\}$, $\{\ell^{(2k+1)}\}$ appear to be converging at a quadratic rate. Additionally, the number of accurate decimal digits of $\omega^{(k)}$ is doubled at every two iterations. Once again, these are consistent with the assertions of Theorem 3.3.9.

3.4 Subspace Framework for Large-Scale Computation of Inner Numerical Radius

We now deal with the general univariate eigenvalue optimization problems of the form (3.4) when the Hermitian matrices $A_1, \dots, A_d$ involved are of large size. This setting encompasses the eigenvalue optimization characterization (3.3) for the inner numerical radius when the matrices A, B are large. Hence, the approach discussed in this section can be incorporated into Algorithm 5 to deal with large Hermitian matrix pairs. For instance, it can be used to determine the definiteness of a large-scale Hermitian pair (A, B) , or for such a large definite Hermitian pair, it can be used to find an angle φ such that $\tilde{A} + i\tilde{B} = e^{-i\varphi}(A + iB)$ with $\lambda_{\min}(\tilde{B}) = \gamma(A, B)$.

We rely on the subspace framework from Chapter 2, which is based on the conversion of the original problem into a small-scale one by means of orthogonal projections and restrictions to certain subspace. It is established in the literature that the framework converges globally and at least at a superlinear rate locally in Chapter 2 and [44] under the assumption that the objective eigenvalue function $\lambda_1(\omega)$ is simple at the converged global minimizer. Here we generalize the basic subspace procedure in Chapter 2 slightly, taking into account also the nonsmooth case when $\lambda_1(\omega)$ is multiple at the converged global minimizer. We prove that the convergence of the generalized subspace procedure occurs at a quadratic rate asymptotically in the nonsmooth case. This rate-of-convergence issue in the nonsmooth setting has been left open by the previous works.

3.4.1 Generalized Subspace Procedure

The subspace procedure is built around the reduced problems of the form

$$\min_{\omega \in \Omega} \lambda_1^{\mathcal{V}}(\omega), \quad (3.16)$$

where

$$\begin{aligned} A^{\mathcal{V}}(\omega) &:= V^* A(\omega) V \\ &= f_1(\omega) V^* A_1 V + \cdots + f_d(\omega) V^* A_d V \end{aligned}$$

for a given small dimensional subspace $\mathcal{V}$, say $\dim \mathcal{V} = k$, and an $n \times k$ matrix V whose columns form an orthonormal basis for $\mathcal{V}$. Note that (3.16) involves smaller $k \times k$ eigenvalue problems compared with (3.4), which involves $n \times n$ eigenvalue problems.

In the remaining part of this section, we shall describe a procedure to construct a small dimensional subspace $\mathcal{V}$ such that the global minimizers and the globally minimal values of $\lambda_1(\omega)$ and $\lambda_1^{\mathcal{V}}(\omega)$ are nearly the same. Here and throughout the rest of this section $\lambda_j(\omega)$ and $v_j(\omega)$ denote the j th largest eigenvalue and a corresponding unit eigenvector of $A(\omega)$.

The subspace procedure is presented formally in Algorithm 8. At every iteration, the subspace procedure first solves a projected small-scale problem for a given sub-

space $\mathcal{V}$. Then, denoting the global minimizer of this small problem with ω_* , the subspace is expanded with the inclusion of an eigenvector corresponding to $\lambda_1(\omega_*)$, as well as eigenvectors corresponding to other eigenvalues of $A(\omega_*)$ that are at most ϵ away from $\lambda_1(\omega_*)$. We denote $\mathcal{V}_k$ as the subspace at the k th iteration and $\lambda_1^{(k)}(\omega) := \lambda_J^{\mathcal{V}_k}(\omega)$. The following interpolation result between the eigenvalues of the full and projected problems generated by Algorithm 8 is an immediate corollary of Lemma 2.2.6.

Theorem 3.4.1. *The following are satisfied by the sequences $\{\omega^{(k)}\}$ and $\{\mathcal{V}_k\}$ generated by Algorithm 8 for each j , each $k = 1, \dots, j$ and each $p = 1, \dots, \ell_k$:*

- (i) $\lambda_p^{(j)}(\omega^{(k)}) = \lambda_p(\omega^{(k)})$.
- (ii) if $\lambda_p(\omega^{(k)})$ is simple, then the same holds for $\lambda_p^{(j)}(\omega^{(k)})$, and $(\lambda_p^{(j)})'(\omega^{(k)}) = \lambda_p'(\omega^{(k)})$.

In the case of inner numerical radius, the projected small-scale problems can be solved globally and efficiently by means of the algorithms in Section 3.3. The support based algorithm in Section 3.3 is applicable to solve the projected problems associated with various other eigenvalue optimization problems of the form (3.4), as long as a global lower bound γ on $\lambda_1''(\omega)$ at the points of differentiability is available. The main computational burden of the subspace procedure stems from lines 2, 3, 7, 8, which require the computation of the eigenvectors of the full problem.

The subsequent two subsections are devoted to analyses of the convergence properties of Algorithm 8. The next subsection provides formal arguments in support of the fact that every convergent subsequence of the sequence $\{\omega^{(k)}\}$ by the algorithm converges to a global minimizer of $\lambda_1(\omega)$. Then Section 3.4.3 addresses how quickly this convergence occurs.

3.4.2 Global Convergence

For the sake of rigor, here and in the next section, we consider the problem at our hands in the infinite dimensional setting. In particular, in this analysis the matrix-valued functions are replaced by self-adjoint compact operators $\mathbf{A}(\omega) : \ell^2(\mathbb{N}) \rightarrow$

Algorithm 8 The Generalized Subspace Procedure

Input: Matrix-valued function $A(\omega)$, closed interval $\Omega \subset \mathbb{R}$, and $\epsilon \in \mathbb{R}^+$.

Output: The sequence $\{\omega^{(k)}\}$.

- 1: $\omega^{(1)} \leftarrow$ a random point in Ω
 - 2: $v_1^{(1)} \leftarrow$ eigenvector corresponding to $\lambda_1(\omega^{(1)})$
 - 3: $v_2^{(1)}, \dots, v_{\ell_1}^{(1)} \leftarrow$ eigenvectors corresponding to eigenvalues $\lambda_2(\omega^{(1)}), \dots, \lambda_{\ell_1}(\omega^{(1)})$
of $A(\omega^{(1)})$ such that $\lambda_1(\omega^{(1)}) - \lambda_j(\omega^{(1)}) \leq \epsilon$ for $j = 2, \dots, \ell_1$
 - 4: $\mathcal{V}_1 \leftarrow \text{span}\{v_1^{(1)}, v_2^{(1)}, \dots, v_{\ell_1}^{(1)}\}$
 - 5: **for** $k = 1, 2, \dots$ **do**
 - 6: $\omega^{(k+1)} \leftarrow \arg \min_{\omega \in \Omega} \lambda_1^{(k)}(\omega)$
 - 7: $v_1^{(k+1)} \leftarrow$ eigenvector corresponding to $\lambda_1(\omega^{(k+1)})$
 - 8: $v_2^{(k+1)}, \dots, v_{\ell_{k+1}}^{(k+1)} \leftarrow$ eigenvectors corresponding to eigenvalues
 $\lambda_2(\omega^{(k+1)}), \dots, \lambda_{\ell_{k+1}}(\omega^{(k+1)})$ of $A(\omega^{(k+1)})$ such that $\lambda_1(\omega^{(k+1)}) - \lambda_j(\omega^{(k+1)}) \leq \epsilon$
for $j = 2, \dots, \ell_{k+1}$
 - 9: $\mathcal{V}_{k+1} \leftarrow \mathcal{V}_k \oplus \text{span}\{v_1^{(k+1)}, v_2^{(k+1)}, \dots, v_{\ell_{k+1}}^{(k+1)}\}$
 - 10: **end for**
-

$\ell^2(\mathbb{N})$, where $\ell^2(\mathbb{N})$ denotes the Hilbert space consisting of square summable infinite sequences of complex numbers equipped with the inner product $\langle v, w \rangle = \sum_{k=1}^{\infty} v_k w_k$ and the norm $\|v\| = \sqrt{\sum_{k=1}^{\infty} |v_k|^2}$. The compact self-adjoint operator $\mathbf{A}(\omega)$ dependent on the parameter ω is still assumed to be of the form specified in (3.4); only now $A_j : \ell^2(\mathbb{N}) \rightarrow \ell(\mathbb{N})$ are self-adjoint compact operators for $j = 1, \dots, d$. Intuitively $\mathbf{A}(\omega)$ for each ω , as well as $A_1, \dots, A_d$, can be considered as infinite dimensional Hermitian matrices.

The global convergence of the subspace procedure is a consequence of the monotonicity (Lemma 2.2.2) and interpolation (Lemma 3.4.1) properties, as well as the uniform Lipschitz continuity of the reduced eigenvalue function (Lemma 2.2.1).

Now let $\{\omega^{(\ell_k)}\}$ be a convergent subsequence of $\{\omega^{(k)}\}$. The interpolation property

(part (i) of Theorem 3.4.1) implies

$$\min_{\omega \in \Omega} \lambda_1(\omega) \leq \lambda_1(\omega^{(\ell_k)}) = \lambda_1^{(\ell_k)}(\omega^{(\ell_k)}), \quad (3.17)$$

while the monotonicity property (Lemma 2.2.2) implies

$$\begin{aligned} \min_{\omega \in \Omega} \lambda_1(\omega) &\geq \min_{\omega \in \Omega} \lambda_1^{(\ell_{k+1}-1)}(\omega) \\ &= \lambda_1^{(\ell_{k+1}-1)}(\omega^{(\ell_{k+1})}) \geq \lambda_1^{(\ell_k)}(\omega^{(\ell_{k+1})}). \end{aligned} \quad (3.18)$$

Hence, $\min_{\omega \in \Omega} \lambda_1(\omega)$ is squeezed in between $\lambda_1^{(\ell_k)}(\omega^{(\ell_k)})$, $\lambda_1^{(\ell_k)}(\omega^{(\ell_{k+1})})$, the gap between which is decaying to zero as $k \rightarrow \infty$ due to uniform Lipschitz continuity (Lemma 2.2.1). This leads to the global convergence result whose formal statement and proof are given in Theorem 2.3.1 of Chapter 2.

3.4.3 Rate-of-Convergence

In this section, as has been done earlier for the rate-of-convergence analysis of the support based algorithm, we again assume $\lambda_1(\omega)$ has a unique global minimizer over Ω , say at ω_* . It turns out that Theorem 3.3.7 that is established for the support based algorithm applies to this setting as well; this is a consequence of Theorem 2.3.1, in particular its assertion that every convergent subsequence of the sequence $\{\omega^{(k)}\}$ by the subspace framework is guaranteed to converge to a global minimizer. Hence, the sequence $\{\omega^{(k)}\}$ itself converges to the unique global minimizer $\omega_* = \arg \min_{\omega \in \Omega} \lambda_1(\omega)$.

Here we are concerned with how quickly $\{\omega^{(k)}\}$ by Algorithm 8 converges to ω_* . If the eigenvalue $\lambda_1(\omega_*)$ is simple, then the eigenvalue function $\lambda_1(\omega)$ is real analytic at ω_* . In this case, it has been shown in [44] that the precise R-order of convergence of $\{\omega^{(k)}\}$ to ω_* is $1 + \sqrt{2}$, i.e., there exists a sequence $\{\varepsilon^{(k)}\}$ converging to zero such that $|\omega^{(k)} - \omega_*| \leq \varepsilon^{(k)}$ for all k large enough and $\varepsilon^{(k+1)} = O((\varepsilon^{(k)})^{1+\sqrt{2}})$.

Otherwise, $\lambda_1(\omega_*)$ is multiple with algebraic multiplicity $K > 1$. Throughout this section, it is assumed that

$$0 \notin \text{bd } \partial \lambda_j(\omega_*), \quad \text{for } j = 1, \dots, K, \quad (3.19)$$

where $\lambda_j(\omega)$ denotes the j th largest eigenvalue of $\mathbf{A}(\omega)$. This assumption is equivalent to having non-zero one-sided derivatives of $\lambda_j(\omega)$ at ω_* for $j = 1, \dots, K$, and holds generically. As already discussed in Section 3.3.3, $\lambda_1(\omega)$ is continuous and piecewise real analytic at ω_* , but usually not differentiable. Furthermore, due to Theorem 1.1.2, there exist real analytic eigenvalue functions $\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_n(\omega)$ of $\mathbf{A}(\omega)$. Precisely K of these n real analytic functions at ω_* must be equal to $\lambda_1(\omega_*)$ with nonzero derivatives; a zero derivative for one of these K functions at ω_* contradicts (3.19). Hence, without loss of generality, suppose

$$\begin{aligned} \tilde{\lambda}_1(\omega_*) &= \dots = \tilde{\lambda}_\ell(\omega_*) \\ &= \tilde{\lambda}_{\ell+1}(\omega_*) = \dots = \tilde{\lambda}_K(\omega_*) = \lambda_1(\omega_*) \end{aligned}$$

are such that

$$\tilde{\lambda}'_1(\omega_*), \dots, \tilde{\lambda}'_\ell(\omega_*) > 0, \quad \tilde{\lambda}'_{\ell+1}(\omega_*), \dots, \tilde{\lambda}'_K(\omega_*) < 0.$$

Our arguments make use of the gap

$$\begin{aligned} \varphi &:= \lambda_1(\omega_*) - \lambda_{K+1}(\omega_*) \\ &= \lambda_1(\omega_*) - \max \left\{ \tilde{\lambda}_j(\omega_*) \mid j = K+1, \dots, n \right\} \end{aligned} \tag{3.20}$$

as well as the real analytic eigenvalue functions of $\mathbf{A}^{\mathcal{V}_k}(\omega)$ which we denote with $\tilde{\lambda}_1^{(k)}(\omega), \dots, \tilde{\lambda}_{d_k}^{(k)}(\omega)$, where $d_k := \dim \mathcal{V}_k$.

The Hermite interpolation property extends to $\tilde{\lambda}_j(\omega)$ and $\tilde{\lambda}_j^{(k)}(\omega)$ at the iterates $\omega^{(k)}$ of Algorithm 8 for large k in the way stated by Lemma 3.4.2 below. This result immediately follows from Lemma 2.2.6, as the set $\{\tilde{\lambda}_j(\omega) \mid j = 1, \dots, K\}$ correspond to the set of largest K eigenvalues of $\mathbf{A}(\omega)$ for all ω in an interval containing ω_* , say in $\mathcal{I} := (\omega_* - \delta, \omega_* + \delta)$ with $\delta > 0$. Furthermore, $\omega^{(k)} \in \mathcal{I}$ for large k , and the eigenvectors corresponding to $\tilde{\lambda}_j(\omega^{(k)})$ for $j = 1, \dots, K$ are included in the subspaces.

Lemma 3.4.2. *Let $\{\omega^{(k)}\}$ be the sequence generated by Algorithm 8, and ω_* denote the unique global minimizer of $\lambda_1(\omega)$ over Ω such that $\lambda_1(\omega_*)$ has algebraic multiplicity $K \geq 2$. For each $k \in \mathbb{Z}^+$ large enough, there exist $\ell_1, \dots, \ell_K \in \{1, \dots, \dim \mathcal{V}_k\}$*

satisfying

$$\tilde{\lambda}_j(\omega^{(k)}) = \tilde{\lambda}_{\ell_j}^{(k)}(\omega^{(k)}), \quad \tilde{\lambda}'_j(\omega^{(k)}) = \left(\tilde{\lambda}_{\ell_j}^{(k)} \right)'(\omega^{(k)}),$$

for $j = 1, \dots, K$.

It is a matter of convention how we label the real analytic eigenvalue functions of $\mathbf{A}^{\mathcal{V}_k}(\omega)$. For ease of notation, from here on, we relabel if necessary so that $\ell_j = j$ for $j = 1, \dots, K$. The next lemma gives a description of $\omega^{(k+1)}$, the global minimizer of $\lambda_1^{(k)}(\omega)$, in terms of the real analytic eigenvalues $\tilde{\lambda}_1^{(k)}(\omega), \dots, \tilde{\lambda}_K^{(k)}(\omega)$.

Lemma 3.4.3. *Suppose that $\lambda_1(\omega)$ has a unique global minimizer over Ω at ω_* . Furthermore, suppose $\lambda_1(\omega_*)$ has algebraic multiplicity $K \geq 2$, and that (3.19) holds. The following assertions are satisfied for all k large enough:*

- (i) *Letting $\delta_j := \tilde{\lambda}'_j(\omega_*) > 0$ for $j = 1, \dots, \ell$, there exists $\varepsilon_j > 0$ such that for all $\omega \in (\omega_* - \varepsilon_j, \omega_* + \varepsilon_j)$ we have*

$$\tilde{\lambda}'_j(\omega) \geq 3\delta_j/4 \quad \text{and} \quad \left(\tilde{\lambda}_j^{(k)} \right)'(\omega) \geq \delta_j/2.$$

- (ii) *Letting $\delta_j := \tilde{\lambda}'_j(\omega_*) < 0$ for $j = \ell + 1, \dots, K$, there exists $\varepsilon_j > 0$ such that for all $\omega \in (\omega_* - \varepsilon_j, \omega_* + \varepsilon_j)$ we have*

$$\tilde{\lambda}'_j(\omega) \leq 3\delta_j/4 \quad \text{and} \quad \left(\tilde{\lambda}_j^{(k)} \right)'(\omega) \leq \delta_j/2.$$

- (iii) *The point $\omega^{(k+1)}$ is such that*

$$\lambda_1^{(k)}(\omega^{(k+1)}) = \tilde{\lambda}_{j_1}^{(k)}(\omega^{(k+1)}) = \tilde{\lambda}_{j_2}^{(k)}(\omega^{(k+1)})$$

for some $j_1 \in \{1, \dots, \ell\}$ and some $j_2 \in \{\ell + 1, \dots, K\}$.

Proof. (i) It follows from Proposition 2.2.9 in Chapter 2 that there exists an open interval $\mathcal{I}$ containing ω_* and a positive constant $\eta \in \mathbb{R}^+$ such that $\tilde{\lambda}''_j(\omega)$, as well as $\left(\tilde{\lambda}_j^{(k)} \right)''(\omega)$ for all k sufficiently large are bounded in absolute value by ν uniformly over all $\omega \in \mathcal{I}$ and over all such large k .

The uniform boundedness of $|\tilde{\lambda}_j''(\omega)|$ combined with $\tilde{\lambda}_j'(\omega_*) = \delta_j$ imply the existence of an open interval $\widehat{\mathcal{I}} := (\omega_* - \hat{\varepsilon}, \omega_* + \hat{\varepsilon}) \subseteq \mathcal{I}$ such that $\tilde{\lambda}_j'(\omega) \geq 3\delta_j/4 \quad \forall \omega \in \widehat{\mathcal{I}}$.

Since the global minimizer ω_* is assumed to be unique, by Lemma 3.3.7 we have $\omega^{(k)} \rightarrow \omega_*$ as $k \rightarrow \infty$. Hence, choose k large enough so that $\omega^{(k)} \in \widehat{\mathcal{I}}/2 := (\omega_* - \hat{\varepsilon}/2, \omega_* + \hat{\varepsilon}/2)$. Now we employ the Hermite interpolation property (Lemma 3.4.2) to deduce

$$\left(\tilde{\lambda}_j^{(k)}\right)'(\omega^{(k)}) = \tilde{\lambda}_j'(\omega^{(k)}) \geq 3\delta_j/4$$

for all such large k . Since $\left|\left(\tilde{\lambda}_j^{(k)}\right)''(\omega)\right|$ is also uniformly bounded in $\widehat{\mathcal{I}}$ by a constant independent of k , there exists $\widetilde{\mathcal{I}} := (\omega^{(k)} - \widetilde{\varepsilon}, \omega^{(k)} + \widetilde{\varepsilon}) \subseteq \widehat{\mathcal{I}}$ such that

$$\left(\tilde{\lambda}_j^{(k)}\right)'(\omega) \geq \delta_j/2 \quad \forall \omega \in \widetilde{\mathcal{I}}$$

for all such large k . By choosing k even larger if necessary, it can be ensured that $|\omega^{(k)} - \omega_*| < \widetilde{\varepsilon}/2$ so that $(\omega_* - \varepsilon_j, \omega_* + \varepsilon_j) \subseteq \widetilde{\mathcal{I}}$ for $\varepsilon_j := \widetilde{\varepsilon}/2$.

(ii) This can be proven in a way similar to part (i).

(iii) Setting $\varepsilon := \min\{\varepsilon_j \mid j = 1, \dots, K\}$ for ε_j as in part (i) and (ii), there exists an open interval $\mathcal{I} = (\omega_* - \varepsilon, \omega_* + \varepsilon)$ such that for all large k the following hold:

$$\begin{aligned} \left(\tilde{\lambda}_j^{(k)}\right)'(\omega) &= \delta_j/2 > 0 \quad \forall \omega \in \mathcal{I} \quad \text{for } j = 1, \dots, \ell, \\ \left(\tilde{\lambda}_j^{(k)}\right)'(\omega) &= \delta_j/2 < 0 \quad \forall \omega \in \mathcal{I} \quad \text{for } j = \ell + 1, \dots, K. \end{aligned}$$

Furthermore, let $\widetilde{\mathcal{I}} := (\omega_* - \widetilde{\varepsilon}, \omega_* + \widetilde{\varepsilon})$ for $\widetilde{\varepsilon} := \min\{\varepsilon, \varphi/(8\gamma)\}$, where γ is the uniform Lipschitz constant in Lemma 2.2.1, and φ is the eigenvalue gap as in (3.20). Without loss of generality, we make the following two assumptions. First, the set $\{\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_K(\omega)\}$ corresponds to the set of K largest eigenvalues of $\mathbf{A}(\omega)$ for all $\omega \in \widetilde{\mathcal{I}}$. Secondly, $\omega^{(k)}, \omega^{(k+1)} := \arg \min_{\omega \in \Omega} \lambda_1^{(k)}(\omega) \in \widetilde{\mathcal{I}}$, since $\omega^{(k)} \rightarrow \omega_*$ as $k \rightarrow \infty$.

We start by showing that $\lambda_1^{(k)}(\omega) > \tilde{\lambda}_{j'}^{(k)}(\omega)$ for all $\omega \in \widetilde{\mathcal{I}}$ and for $j' = K + 1, \dots, \dim \mathcal{V}_k$. To this end, first observe that

$$\lambda_K(\omega) - \lambda_{K+1}(\omega) \geq \lambda_K(\omega_*) - \lambda_{K+1}(\omega_*) - \varphi/4 = 3\varphi/4$$

for all $\omega \in \tilde{\mathcal{I}}$, where we employ the Lipschitz continuity of $\lambda_K(\omega)$ and $\lambda_{K+1}(\omega)$ with the Lipschitz constant γ (see Lemma 2.2.1). In particular, for each $j \in \{1, \dots, K\}$, we have

$$\begin{aligned} 3\varphi/4 &\leq \tilde{\lambda}_j(\omega^{(k)}) - \lambda_{K+1}(\omega^{(k)}) \\ &\leq \tilde{\lambda}_j^{(k)}(\omega^{(k)}) - \lambda_{K+1}^{(k)}(\omega^{(k)}) \end{aligned}$$

where in the second line we exploit the interpolation property (Lemma 3.4.2), and the monotonicity (i.e., $\lambda_{K+1}(\omega^{(k)}) \geq \lambda_{K+1}^{(k)}(\omega^{(k)})$) due to Lemma 2.2.2). Now the last inequality implies

$$\begin{aligned} \tilde{\lambda}_j^{(k)}(\omega) - \lambda_{K+1}^{(k)}(\omega) &\geq \\ \tilde{\lambda}_j^{(k)}(\omega^{(k)}) - \lambda_{K+1}^{(k)}(\omega^{(k)}) - \varphi/2 &= \varphi/4 \end{aligned}$$

for all $\omega \in \tilde{\mathcal{I}}$. Note that above we make use of the uniform Lipschitz continuity of $\tilde{\lambda}_j^{(k)}(\omega)$ and $\lambda_{K+1}^{(k)}(\omega)$ with the uniform Lipschitz constant γ independent of the subspace $\mathcal{V}_k$; the latter is immediate from Lemma 2.2.1, whereas the former can be seen from $|\left(\tilde{\lambda}_j^{(k)}\right)'(\omega)| \leq \|\mathbf{A}'(\omega)\|_2$. The last inequality means that $\lambda_{K+1}^{(k)}(\omega)$ is the largest of $\tilde{\lambda}_j^{(k)}(\omega)$ for $j = K+1, \dots, \dim \mathcal{V}_k$ for $\omega \in \tilde{\mathcal{I}}$, so for every $j = K+1, \dots, N$ for all $\omega \in \tilde{\mathcal{I}}$, we deduce

$$\lambda_1^{(k)}(\omega) - \tilde{\lambda}_{j'}^{(k)}(\omega) \geq \tilde{\lambda}_j^{(k)}(\omega) - \lambda_{K+1}^{(k)}(\omega) \geq \varphi/4.$$

It follows from the previous paragraph that $\lambda_1^{(k)}(\omega) = \tilde{\lambda}_j^{(k)}(\omega)$ for some $j \in \{1, \dots, K\}$ for all $\omega \in \tilde{\mathcal{I}}$, yet $\lambda_1^{(k)}(\omega) > \tilde{\lambda}_j^{(k)}(\omega)$ for $j \notin \{1, \dots, K\}$. Now consider a point $\tilde{\omega} \in \tilde{\mathcal{I}}$ such that

- $\lambda_1^{(k)}(\tilde{\omega}) = \tilde{\lambda}_{p_1}^{(k)}(\tilde{\omega}) = \dots = \tilde{\lambda}_{p_q}^{(k)}(\tilde{\omega})$
 $\exists p_1, \dots, p_q \in \{1, \dots, \ell\}$ for some $q \geq 1$,
yet $\lambda_1^{(k)}(\tilde{\omega}) > \tilde{\lambda}_j^{(k)}(\tilde{\omega})$ for $j \notin \{p_1, \dots, p_q\}$, or
- $\lambda_1^{(k)}(\tilde{\omega}) = \tilde{\lambda}_{n_1}^{(k)}(\tilde{\omega}) = \dots = \tilde{\lambda}_{n_s}^{(k)}(\tilde{\omega})$
 $\exists n_1, \dots, n_s \in \{\ell+1, \dots, K\}$ for some $s \geq 1$,
yet $\lambda_1^{(k)}(\tilde{\omega}) > \tilde{\lambda}_j^{(k)}(\tilde{\omega})$ for $j \notin \{n_1, \dots, n_s\}$.

We shall show that such a point cannot be a minimizer of $\lambda_1^{(k)}(\omega)$. For these two cases, we respectively have

$$\begin{aligned}\partial\lambda_1^{(k)}(\tilde{\omega}) &= \text{Co} \left\{ \left(\tilde{\lambda}_{p_j}^{(k)} \right)'(\tilde{\omega}) \mid j = 1, \dots, q \right\} \subseteq (\delta_{p,\min}/2, \infty), \\ \partial\lambda_1^{(k)}(\tilde{\omega}) &= \text{Co} \left\{ \left(\tilde{\lambda}_{n_j}^{(k)} \right)'(\tilde{\omega}) \mid j = 1, \dots, s \right\} \subseteq (-\infty, \delta_{n,\max}/2)\end{aligned}$$

where $\delta_{p,\min} := \min\{\delta_{p_j} \mid j = 1, \dots, q\} > 0$, and $\delta_{n,\max} := \max\{\delta_{n_j} \mid j = 1, \dots, s\} < 0$. In either case, $0 \notin \partial\lambda_1^{(k)}(\tilde{\omega})$ implying $\tilde{\omega}$ cannot be a minimizer of $\lambda_1^{(k)}(\omega)$. Therefore, since $\omega^{(k+1)} \in \tilde{\mathcal{I}}$ is a minimizer of $\lambda_1^{(k)}(\omega)$, we must have

$$\lambda_1^{(k)}(\omega^{(k+1)}) = \tilde{\lambda}_{j_1}^{(k)}(\omega^{(k+1)}) = \tilde{\lambda}_{j_2}^{(k)}(\omega^{(k+1)})$$

for some $j_1 \in \{1, \dots, \ell\}$ and $j_2 \in \{\ell + 1, \dots, K\}$. □

Now we are ready to present the main quadratic rate-of-convergence result in the nonsmooth setting; this result follows from the Hermite interpolation properties in Lemma 3.4.2, as well as Lemma 3.4.3.

Theorem 3.4.4 (Quadratic Convergence in the Nonsmooth Case). *Suppose that the global minimizer $\omega_* := \arg \min_{\omega \in \Omega} \lambda_1(\omega)$ is unique and such that the eigenvalue $\lambda_1(\omega_*)$ is multiple, condition (3.19) holds. The sequence $\{\omega^{(k)}\}$ generated by Algorithm 8 satisfies*

$$|\omega^{(k+1)} - \omega_*| = O((\omega^{(k)} - \omega_*)^2) \tag{3.21}$$

for all large k .

Proof. Part (i) and (ii) of Lemma 3.4.3 shows the existence of an open interval $\mathcal{I} := (\omega_* - \varepsilon, \omega_* + \varepsilon)$ where $\tilde{\lambda}'_j(\omega)$ for $j = 1, \dots, \ell$ is bounded from below uniformly by a positive real number and $\tilde{\lambda}'_j(\omega)$ for $j = \ell + 1, \dots, K$ is bounded from above uniformly by a negative real number. Let us consider k large enough so that $\omega^{(k)} \in \mathcal{I}$.

By part (iii) of Lemma 3.4.3, there exist $j_1 \in \{1, \dots, \ell\}$ and $j_2 \in \{\ell + 1, \dots, K\}$ such that $\tilde{\lambda}_{j_1}^{(k)}(\omega^{(k+1)}) = \tilde{\lambda}_{j_2}^{(k)}(\omega^{(k+1)})$. For such a pair of j_1, j_2 define the real

analytic functions

$$\begin{aligned}\lambda(\omega) &:= \lambda_{j_1}(\omega) - \lambda_{j_2}(\omega), \\ \lambda^{(k)}(\omega) &:= \lambda_{j_1}^{(k)}(\omega) - \lambda_{j_2}^{(k)}(\omega).\end{aligned}$$

Observe that there exists a constant $\zeta \in \mathbb{R}^+$ such that

$$\lambda'(\omega^{(k)}) = \tilde{\lambda}'_{j_1}(\omega^{(k)}) - \tilde{\lambda}'_{j_2}(\omega^{(k)}) \geq \zeta \implies \left| [\lambda'(\omega^{(k)})]^{-1} \right| \leq \zeta^{-1}$$

for all such large k . Note also that the real analyticity of $\tilde{\lambda}_{j_1}(\omega)$, $\tilde{\lambda}_{j_2}(\omega)$ implies the Lipschitz continuity of $\lambda'(\omega)$ on $\mathcal{I}$.

The proof manipulates the following equation, which is immediate from an application of Taylor's theorem with integral remainder:

$$0 = \lambda(\omega_*) = \lambda(\omega^{(k)}) + \int_0^1 \lambda'(\omega^{(k)} + t(\omega_* - \omega^{(k)}))(\omega_* - \omega^{(k)}) dt.$$

We employ $\lambda(\omega^{(k)}) = \lambda^{(k)}(\omega^{(k)})$ (due to Lemma 3.4.2) in this equation, then multiply both sides by $[\lambda'(\omega^{(k)})]^{-1}$ to obtain

$$\begin{aligned}0 &= [\lambda'(\omega^{(k)})]^{-1} \lambda^{(k)}(\omega^{(k)}) + (\omega_* - \omega^{(k)}) + [\lambda'(\omega^{(k)})]^{-1} \times \\ &\quad \int_0^1 [\lambda'(\omega^{(k)} + t(\omega_* - \omega^{(k)})) - \lambda'(\omega^{(k)})] (\omega_* - \omega^{(k)}) dt.\end{aligned}\tag{3.22}$$

An application of Taylor's theorem to $\lambda^{(k)}(\omega)$ about $\omega^{(k)}$ with second order remainder combined with the equalities $\lambda^{(k)}(\omega^{(k+1)}) = 0$ (part (iii) of Lemma 3.4.3) and $(\lambda^{(k)})'(\omega^{(k)}) = \lambda'(\omega^{(k)})$ (part (ii) of Lemma 3.4.2) leads us to

$$[\lambda'(\omega^{(k)})]^{-1} \lambda^{(k)}(\omega^{(k)}) = (\omega^{(k)} - \omega^{(k+1)}) + O((\omega^{(k)} - \omega^{(k+1)})^2).$$

Now using the last equality in (3.22) gives rise to

$$\begin{aligned}0 &= (\omega_* - \omega^{(k+1)}) + O((\omega^{(k)} - \omega^{(k+1)})^2) + [\lambda'(\omega^{(k)})]^{-1} \times \\ &\quad \int_0^1 [\lambda'(\omega^{(k)} + t(\omega_* - \omega^{(k)})) - \lambda'(\omega^{(k)})] (\omega_* - \omega^{(k)}) dt,\end{aligned}$$

implying

$$\begin{aligned}|\omega^{(k+1)} - \omega_*| &\leq O((\omega^{(k)} - \omega^{(k+1)})^2) + \left| [\lambda'(\omega^{(k)})]^{-1} \right| \times \\ &\quad \int_0^1 |\lambda'(\omega^{(k)} + t(\omega_* - \omega^{(k)})) - \lambda'(\omega^{(k)})| |\omega_* - \omega^{(k)}| dt.\end{aligned}\tag{3.23}$$

Now the desired equality (3.21) follows from (3.23) by employing the bound $\left| [\lambda'(\omega^{(k)})]^{-1} \right| \leq \zeta^{-1}$, the Lipschitz continuity of $\lambda'(\omega)$, as well as the inequality

$$(\omega^{(k)} - \omega^{(k+1)})^2 \leq 2 [(\omega^{(k+1)} - \omega_*)^2 + (\omega^{(k)} - \omega_*)^2].$$

□

3.4.4 Numerical Experiments

In this section we test Algorithm 8 on numerical examples that concern three applications, namely the computation of the distance to a nearest definite pair, determining whether a given quadratic eigenvalue problem is hyperbolic or not, and inferring the existence of a conjugate gradient iteration for the saddle point linear system. In particular, the quadratic convergence of the subspace procedure is illustrated in the nonsmooth case on examples arising from the latter two applications.

The subspace framework is terminated when the condition

$$\left| \lambda_1^{(k)}(\omega^{(k+1)}) - \lambda_1^{(k-1)}(\omega^{(k)}) \right| < \text{tol}$$

is satisfied, where `tol` is a prescribed tolerance; we always set `tol` = 10^{-12} unless otherwise specified. The reduced eigenvalue optimization problems are solved by means of the algorithms discussed in Section 3.3. Indeed, we typically employ the support based algorithm (see Section 3.3.2), in particular the MATLAB package `eigopt` [60, Section 10], which is an implementation of this algorithm. The package requires a box to be optimized over, and a global lower bound γ on the second derivatives of the eigenvalue function; we provide the natural and theoretically sound choices for these, namely $[0, 2\pi]$ for the box and $-\|A\|_2 - \|B\|_2$ for the lower bound on the second derivatives.

Distance to the nearest definite pair. Consider the Hermitian matrix pair (A, B) with

$$A = \frac{\tilde{G} + \tilde{G}^*}{2} \quad \text{and} \quad B = -i \frac{\tilde{G} - \tilde{G}^*}{2}$$

and $\tilde{G} := Ge^{i\pi/6}$, where G is the 640×640 Grcar matrix. An application of the subspace framework for the computation of the inner numerical radius of $A + iB$ yields the global minimizer $\theta_* = 2.617992877994$ of $\lambda_{\max}(A \cos \theta + B \sin \theta)$, as well as $\lambda_{\max}(A \cos \theta_* + B \sin \theta_*) = 0.634045490256$. Now Theorem 3.2.2, implies $0 \in F(A + iB)$; this is confirmed by the plot of $F(A + iB)$ given on the top left in Figure 3.2. Consequently, the pair (A, B) is indefinite.

We deduce from Theorem 3.2.3 that $d_\delta(A, B) = 0.644045490256$ for $\delta = 10^{-2}$. Figure 3.2 illustrates the field of values of $A + iB$, $(A + \Delta A) + i(B + \Delta B)$ and $\tilde{A} + i\tilde{B} = e^{-i\varphi}((A + \Delta A) + i(B + \Delta B))$ for the choice of $\delta = 10^{-2}$, where $\Delta A, \Delta B, \varphi$ are as in Algorithm 5. The diamonds in the figure correspond to the points where the inner numerical radii are attained. Although $\lambda_{\max}(A \cos \theta_* + B \sin \theta_*)$ is simple, it turns out to be very close to the second largest eigenvalue; these two eigenvalues differ by an amount on the order of 10^{-7} . Quadratic convergence is achieved by our subspace framework; this is depicted in Table 3.6, which lists the iterates of Algorithm 8 with $\epsilon = 10^{-6}$.

Testing hyperbolicity of QEP. We consider the quadratic eigenvalue problem $Q(\lambda) = \lambda^2 A + \lambda B + C$ that is linked to a damped-mass spring system, where A, B, C are 500×500 matrices such that $A = I$,

$$B = \beta \begin{bmatrix} 20 & -10 & & & \\ -10 & 30 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 30 & -10 \\ & & & -10 & 20 \end{bmatrix}, C = \begin{bmatrix} 15 & -5 & & & \\ -5 & 15 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & -5 & 15 \end{bmatrix} \quad (3.24)$$

for a given real number $\beta > 0$. We construct the coefficient matrices of this QEP by using the MATLAB toolbox NLEVP [4], and determine the hyperbolicity of the QEP for different β values by testing the definiteness of the associated 1000×1000 pair (A_1, B_1) as in (3.13). Table 3.7 indicates whether the pair (A_1, B_1) is definite or not for eight equally spaced β values in the interval $[0.500, 0.528]$. The number of iterations to compute the inner numerical radius of $A_1 + iB_1$ (up to the prescribed

k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\omega^{(k+1)}$
4	4	0.629840138568	2.618127739876
5	5	0.632130046510	2.617077493245
6	6	0.634045279755	2.617993771783
7	8	0.634045490256	2.617993877986
8	10	0.634045490256	2.617993877986

Table 3.6: This table concerns an application of the subspace framework to the example of Figure 3.2, which involves the minimization of the largest eigenvalue of $A(\omega) = A \cos \omega + B \sin \omega$ over ω for certain Hermitian matrices A, B defined in terms of the 640×640 Grcar matrix. The globally minimal value of the reduced eigenvalue function $\lambda_1^{(k)}(\omega^{(k+1)})$ and $\omega^{(k+1)} := \arg \min_{\omega} \lambda_1^{(k)}(\omega)$ by Algorithm 8 starting with $\omega^{(1)} = 0.45$ are listed along with the subspace dimension $p := \dim \mathcal{V}_k$ with respect to the iteration number k .

tolerance $\text{tol} = 10^{-12}$) is eight for each β value. This is a nonsmooth example; the largest eigenvalue of $A_1 \cos \theta + B_1 \sin \theta$ has multiplicity 2 at the minimizing θ for each β value, yet we observe the quadratic convergence of Algorithm 8 consistent with what is expected in theory. This is hinted by Table 3.8, which lists the iterates $\lambda_1^{(k)}(\omega^{(k+1)})$ and $\omega^{(k+1)}$ with respect to the iteration number k for $\beta = 0.512$ and $\beta = 0.524$.

Linear systems in saddle point form. We consider again the matrix pair $(\mathcal{A}, \mathcal{J})$ discussed in Section 3.3.4, where $\mathcal{A} \in \mathbb{R}^{(n+m) \times (n+m)}$ is the coefficient matrix of the form (3.15) that originates from a discretization of the Stokes equation, and $\mathcal{J} = \text{diag}(I_n, -I_m)$. We run our subspace procedure to minimize $\lambda_{\max}(\mathcal{A} \cos \theta + \mathcal{J} \sin \theta)$ over $\theta \in [0, 2\pi]$. The computed results coincide with the ones obtained from a direct application of Algorithm 7 in Section 3.3.4. As remarked before, $\lambda_{\max}(\mathcal{A} \cos \theta_* + \mathcal{J} \sin \theta_*)$ has multiplicity three at the global minimizer θ_* . Our subspace framework again exhibits a quadratic convergence, which is evident from Table 3.9.

β	0.500	0.504	0.508	0.512	0.516	0.520	0.524	0.528
definite	no	no	no	no	no	yes	yes	yes

Table 3.7: The definiteness of the pair (A_1, B_1) defined as in (3.13) associated with the quadratic eigenvalue problem $Q(\lambda) = \lambda^2 A + \lambda B + C$ for A, B, C as in (3.24) and for several values of β .

Performance of the Subspace Framework. Finally we test the performance of Algorithm 8 for the computation of the inner numerical radius of Hermitian pairs (A_n, B_n) of the form

$$A_n = \frac{C_n + C_n^*}{2} \quad \text{and} \quad B_n = -i \frac{C_n - C_n^*}{2}, \quad (3.25)$$

with $C_n = P_n + iR_n$, the matrix P_n denoting the $n \times n$ matrix obtained from the finite difference discretization of the Poisson operator by employing the five-point formula, and R_n denoting a random $n \times n$ sparse matrix generated by the Matlab command `spran(n,n,20/n)`. Table 3.10 lists the computed values of the inner numerical radius by the subspace framework, number of subspace iterations and run-times in seconds to reach the specified accuracy for the pairs (A_n, B_n) of sizes varying between 10000 and 90000. The number of subspace iterations, as well as the time to solve the reduced eigenvalue optimization problems, do not vary much with respect to n . However, the time required for the computation of the largest eigenvalue of the full problem at every iteration increases with respect to n . In essence the total runtime is determined by these large-scale eigenvalue computations for large values of n .

3.5 Conclusion

The algorithm in [60] based on piece-wise quadratic model functions appears to be quite effective in dealing with global minimization problems involving a nonconvex largest eigenvalue function of a Hermitian matrix depending on one parameter. On

the other hand, the subspace framework in Chapter 2 is quite effective to deal with such problems when the Hermitian matrix is large. It accurately reduces the large dimensionality by projecting the Hermitian matrix to small subspaces formed of eigenvectors. Here we have illustrated the efficiency of these algorithms on the computation of inner numerical radius.

As a by-product, we have generalized the subspace framework of Chapter 2 to better cope with nonsmoothness at the minimizer. The generalized subspace framework adds not only the eigenvector corresponding to the largest eigenvalue, but also the eigenvectors corresponding to nearby eigenvalues.

We have proven rapid convergence results for both algorithms in the nonsmooth case when the largest eigenvalue is not simple generically. The algorithm in [60] is shown to generate a sequence $\{\ell^{(k)}\}$ of lower bounds such that both $\{\ell^{(2k)}\}$ and $\{\ell^{(2k+1)}\}$ converge to the globally smallest value of the largest eigenvalue function at a quadratic rate. The generalized subspace framework is shown to generate a sequence of iterates $\{\omega^{(k)}\}$ that converge to the global minimizer at a quadratic rate. What we witness in practice is consistent with these theoretical findings. To this end, several numerical results concerning the inner numerical radius computation in the nonsmooth case are reported confirming the expected in theory.

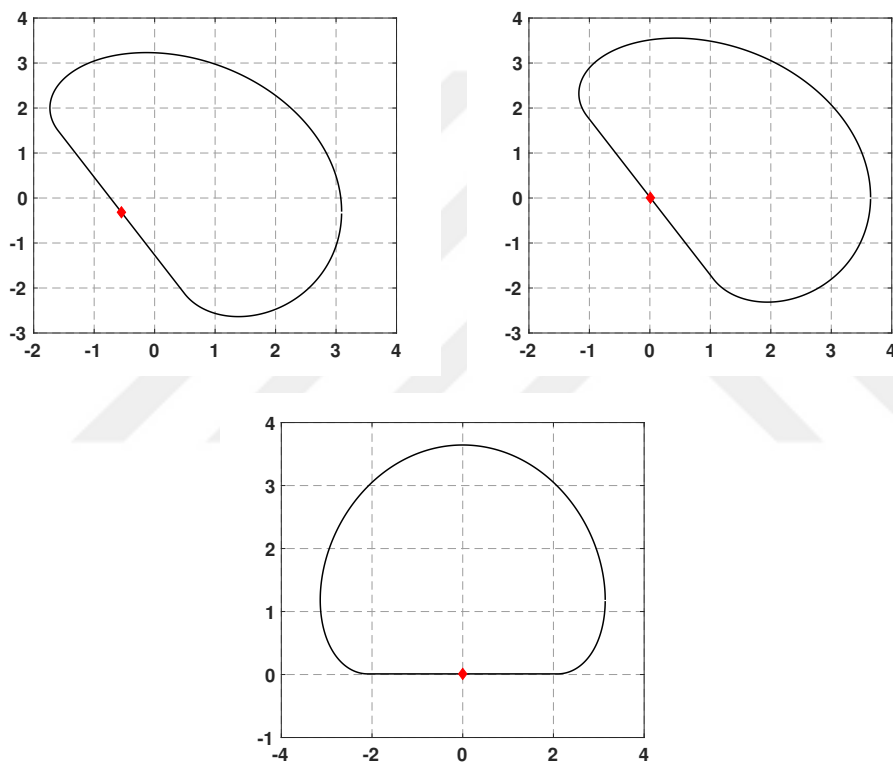


Figure 3.2: The figure depicts the field of values of $A+iB$ (top left), $A+\Delta A+i(B+\Delta B)$ (top right), $e^{-i\psi}(A + \Delta A + i(B + \Delta B))$ (bottom) for the choice of $A = (G + G^*)/2$ and $B = -i(G - G^*)/2$ with G denoting the 640×640 Gcar matrix. The minimal perturbation matrices $\Delta A, \Delta B$ and the quantity ψ are those returned by Algorithm 5 for $\delta = 10^{-2}$. The diamonds mark the points where the inner numerical radii are attained.

$\beta = 0.512$			
k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\omega^{(k+1)}$
4	6	0.218717671040	1.972911641774
5	8	0.006821928930	1.890604000858
6	10	0.008594146027	1.897161234772
7	12	0.008594402114	1.897151450236
8	14	0.008594402114	1.897151450823

$\beta = 0.524$			
k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\omega^{(k+1)}$
4	6	0.233188266820	1.985354587351
5	8	-0.006697302959	1.901962042436
6	10	-0.004923289259	1.908357152861
7	12	-0.004923056427	1.908348045018
8	14	-0.004923056427	1.908348041619

Table 3.8: The minimal value of the reduced eigenvalue function $\lambda_1^{(k)}(\omega^{(k+1)})$, the corresponding global minimizer $\omega^{(k+1)} := \arg \min_{\omega \in \Omega} \lambda_1^{(k)}(\omega)$, and the subspace dimensions $p := \dim \mathcal{V}_k$ are listed with respect to the iteration number k for the example concerning the hyperbolicity of a QEP.

k	p	$\lambda_1^{(k)}(\omega^{(k+1)})$	$\omega^{(k+1)}$
2	2	-0.876669786135	2.023688714623
3	4	-0.023285504705	3.088556373675
4	7	-0.022225058342	3.087503074213
5	10	-0.022224901666	3.087502918535
6	13	-0.022224901666	3.087502918535

Table 3.9: This table concerns the coefficient matrix $\mathcal{A}$ for the saddle point system arising from the Stokes equation, and the positive definiteness of $\mathcal{A} - \mu\mathcal{J}$ for some μ , where $\mathcal{J} := \text{diag}(I_n, -I_m)$. The minimal value of the reduced eigenvalue function $\lambda_1^{(k)}(\omega^{(k+1)})$, the global minimizer $\omega^{(k+1)}$ and the subspace dimension $p := \dim \mathcal{V}_k$ are listed with respect to the iteration number k , when the subspace procedure is applied to minimize $\lambda_1(\mathcal{A} \cos \omega + \mathcal{J} \sin \omega)$ over ω .

n	# iter	total time	reduced prob	eigval comp	$\zeta(A_n + iB_n)$
10000	21	22.62	4.41	17.16	653.69
22500	26	110.61	6.43	100.96	983.80
40000	24	168.44	5.91	156.91	1316.77
62500	20	315.34	13.18	291.97	1667.44
90000	21	594.99	11.12	569.67	1995.49

Table 3.10: The table concerns a Hermitian matrix pair (A_n, B_n) with $A_n, B_n \in \mathbb{R}^{n \times n}$ defined as in (3.25) in terms of $C_n = P_n + iR_n$, the Poisson matrix P_n and the sparse random matrix R_n . The computed values of the inner numerical radius $\zeta(A_n + iB_n)$ by Algorithm 8, the number of subspace iterations (2nd column) and the computation times (3rd-5th columns) are listed for various values of n . For each value of n , the total run-time, the time spent for the solution of the reduced eigenvalue optimization problems, and the time spent for the large-scale eigenvalue computations in seconds are given in the 3rd, 4th, 5th columns, respectively.

Chapter 4

MINIMAX OPTIMIZATION PROBLEMS INVOLVING LARGEST EIGENVALUES OF LARGE-SCALE MATRIX-VALUED FUNCTIONS

In this chapter, we focus on the minimax optimization problems with the largest eigenvalue of a Hermitian matrix-valued function as the objective. A particular application of this type of problems is the minimization of the numerical radius and it has significance on the transient behavior of a parameter dependent linear dynamical system. In the first part of this chapter, we deal with the problems when the matrix-valued function is of small size and depends on a few parameters. We propose a globally convergent algorithm, an adaptation of the algorithm in [60] based on piece-wise quadratic support functions. In the second part, inspired by the observations in Chapter 2, we propose a subspace framework to deal with the minimax eigenvalue optimization problems involving large matrices. As in Chapter 2, we again prove the global convergence of the proposed subspace framework, and establish a superlinear convergence result with respect to the subspace dimension. We confirm this global and quick convergence on the examples arising from the minimization of the numerical radius.

Although the rate-of-convergence analyses for the subspace frameworks resemble those in Chapter 2 for the optimization of the J th largest eigenvalue, minimax problems pose additional changes. For instance, we eliminate the inner maximization problem with the help of the implicit function theorem, but then carry out a careful analysis to show that the local maximizers of the inner problems defined implicitly as functions of the outer minimization parameters are indeed global maximizers.

4.1 Introduction

The asymptotic behavior of the discrete dynamical system

$$z_{k+1} = Az_k, \tag{4.1}$$

where $A \in \mathbb{C}^{n \times n}$ and $z_k \in \mathbb{C}^n$, depends on the spectrum of A . More precisely, z_k converges to 0 for all $z_0 \in \mathbb{C}^n$ if and only if A is asymptotically stable, i.e., its spectrum is contained strictly inside the unit circle in the complex plane. Indeed, this asymptotic behavior is determined by the spectral radius of A ,

$$\rho(A) := \max \{ |\lambda| \mid \lambda \in \Lambda(A) \},$$

where $\Lambda(A)$ denotes the spectrum of A . We have $\lim_{k \rightarrow \infty} z_k = 0 \ \forall z_0$ if and only if $\rho(A) < 1$. Additionally, when $\rho(A) < 1$, the smaller the spectral radius is the faster z_k decays to 0 asymptotically. However, the spectrum of A does not give enough information about the transient behaviour of the iterates z_k . This means that even though A is asymptotically stable, the norm $\|A^k\|_2$ can be arbitrarily large before the eventual decay as $k \rightarrow \infty$. To this end, the inequalities [37]

$$0 \leq \rho(A) \leq r(A) \leq \|A\|_2 \leq 2r(A) \tag{4.2}$$

are helpful, where $r(A)$ denotes the numerical radius of A defined in Section 1.1.3. Moreover, the power inequality $r(A^k) \leq r(A)^k$ [71] yields

$$\|A^k\|_2 \leq 2r(A)^k. \tag{4.3}$$

Inequalities (4.2) and (4.3) indicate that the 2-norms of A^k can be bounded from below and above in terms of $r(A)$ and $r(A^k)$, hence these numerical radii are relevant to the transient and asymptotic behaviors of (4.1).

Suppose that the matrix in (4.1) depends on parameters over a set of admissible parameter values, say Ω , and is of the form

$$A(x) = f_1(x)A_1 + \cdots + f_\kappa(x)A_\kappa \tag{4.4}$$

for given $A_1, \dots, A_\kappa \in \mathbb{C}^{n \times n}$ and $f_1, \dots, f_\kappa : \tilde{\Omega} \rightarrow \mathbb{R}$ that are real analytic on $\tilde{\Omega}$, which is an open subset of $\mathbb{R}^d$ such that $\Omega \subseteq \tilde{\Omega}$. In this situation, the spectral radius minimization problem, that is finding x that minimizes $\rho(A(x))$ globally, is desirable to maximize the asymptotic decay rate of the trajectories of $z_{k+1} = A(x)z_k$. This is a highly difficult nonconvex optimization problem; the objective $\rho(A(x))$ is not even Lipschitz continuous [10]. Instead, we consider the minimization of the numerical radius, formally the problem at our hand is

$$\min_{x \in \Omega} r(A(x)) = \min_{x \in \Omega} \max_{\omega \in [0, 2\pi)} \lambda_{\max} \left(\frac{A(x)e^{i\omega} + A(x)^*e^{-i\omega}}{2} \right). \quad (4.5)$$

It follows from (4.2) that we have

$$\min_{x \in \Omega} \rho(A(x)) \leq \min_{x \in \Omega} r(A(x)),$$

that is, if the minimal value of $r(A(x))$ is less than one, then the discrete system is asymptotically stable for some x and this minimal value of $r(A(x))$ provides also an upper bound for the maximal decay rate of $z_{k+1} = A(x)z_k$ over all permissible x .

Although our motivation comes from the minimization of the numerical radius, we deal with a more general setting. Formally, let $M(x, \omega) : \Omega \times \mathcal{B} \rightarrow \mathbb{C}^{n \times n}$ be a given Hermitian matrix-valued function of the form

$$M(x, \omega) := f_1(x, \omega)A_1 + \dots + f_\kappa(x, \omega)A_\kappa \quad (4.6)$$

for given Hermitian matrices $A_1, \dots, A_\kappa \in \mathbb{C}^{n \times n}$ and functions $f_1, \dots, f_\kappa : \tilde{\Omega} \times \tilde{\mathcal{B}} \rightarrow \mathbb{R}$ that are real analytic on their domain $\tilde{\Omega} \times \tilde{\mathcal{B}}$. The problem that we are concerned with in this chapter is

$$\min_{x \in \Omega} \max_{\omega \in \mathcal{B}} \lambda_1(x, \omega), \quad (4.7)$$

where the objective function $\lambda_1(x, \omega)$ denotes the largest eigenvalue of $M(x, \omega)$. The sets $\Omega, \tilde{\Omega}$ are assumed to be compact, open subsets of $\mathbb{R}^d$ such that $\Omega \subset \tilde{\Omega}$, whereas $\mathcal{B}, \tilde{\mathcal{B}}$ are closed, open intervals in $\mathbb{R}$ such that $\mathcal{B} \subset \tilde{\mathcal{B}}$. Observe, in particular, that the minimization of the numerical radius defined in (4.5) fits into the scope of (4.7).

To our knowledge, there are no general techniques to deal with minimax or maximin optimization problems of the form (4.7) involving the largest eigenvalue of neither a small-scale nor a large-scale matrix-valued function.

4.2 Small-Scale Problems

In this section, we discuss an algorithm that solves (4.7) globally, when we have a few optimization parameters, e.g. $d = 1, d = 2$. The problem in (4.7) can be expressed as

$$\min_{x \in \Omega} \lambda(x)$$

where $\lambda(x) := \max_{\omega \in \mathcal{B}} \lambda_1(x, \omega)$. Let us define $\omega(x)$ implicitly by

$$\omega(x) := \arg \max_{\omega \in \mathcal{B}} \lambda_1(x, \omega).$$

The function $\lambda_1(x, \omega)$ over ω can have more than one global maximizer, in this case $\omega(x)$ is defined as an arbitrary global maximizer. Hence the objective that we would like to minimize is

$$\lambda(x) = \lambda_1(x, \omega(x)).$$

The next result tells us about the differentiability of the functions $\lambda(x)$ and $\omega(x)$. Note that in this result and throughout the rest, the derivatives $D_x \lambda_1(\cdot)$, $D_{x\omega} \lambda_1(\cdot)$ and $\omega'(x)$ are row vectors and $\nabla_{xx}^2 \lambda_1(\cdot)$ denotes the $d \times d$ Hessian of $\lambda_1(\cdot)$ with respect to x only.

Lemma 4.2.1. *Let $\tilde{x} \in \Omega$ be a point such that the global maximizer $\omega(\tilde{x})$ of $\lambda_1(\tilde{x}, \omega)$ over all $\omega \in \mathcal{B}$ is unique and $\lambda_1(\tilde{x}, \omega(\tilde{x}))$ is simple. Furthermore, assume*

$$\left. \frac{\partial^2 \lambda_1(x, \omega)}{\partial \omega^2} \right|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \neq 0. \tag{4.8}$$

Then the following hold:

- (i) $\lambda_1(x, \omega)$ is twice continuously differentiable at $(x, \omega) = (\tilde{x}, \omega(\tilde{x}))$;

- (ii) There exists a neighborhood $\mathcal{N}$ of $\tilde{x}$ such that for all $x \in \mathcal{N}$, the global maximizer $\omega(x)$ is unique and twice continuously differentiable with respect to x at $x = \tilde{x}$;
- (iii) Consequently, the function $\lambda(x)$ is twice continuously differentiable at $x = \tilde{x}$ and its derivatives satisfy
- (iiia) $\Phi'_+(0) \geq \Phi'_-(0)$ where $\Phi(\alpha) := \lambda(\tilde{x} + \alpha p)$ for every $p \in \mathbb{R}^d$, and
- (iiib) $\nabla \lambda(\tilde{x}) = D_x \lambda_1(x, \omega)^T \big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))}$,
 $\nabla^2 \lambda(\tilde{x}) = \{ \nabla_{xx}^2 \lambda_1(x, \omega) + D_{\omega x} \lambda_1(x, \omega)^T \omega'(x) \} \big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))}$

Observe that under assumption (4.8), we must have

$$\frac{\partial^2 \lambda_1(x, \omega)}{\partial \omega^2} \bigg|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} < 0$$

since $\omega(\tilde{x})$ is a global maximizer of the eigenvalue function $\lambda_1(\tilde{x}, \omega)$ over all $\omega \in \mathcal{B}$. Part (i) follows from the uniqueness of $\omega(\tilde{x})$ and part (ii) follows from the implicit function theorem due to assumption (4.8). Throughout this section we assume that (4.8) holds.

The algorithm that we discuss in this section largely depends on the next result. It follows from part (iiia) of Lemma 4.2.1 and its proof is similar to that of Theorem 5.2 in [60].

Theorem 4.2.2 (Lower Support Function). *Let $x^{(k)} \in \Omega$ be such that the maximizer $\omega^{(k)}$ of $\lambda_1(x^{(k)}, \omega)$ over all $\omega \in \mathcal{B}$ is uniquely attained and $\lambda_1(x^{(k)}, \omega^{(k)})$ is simple. Furthermore, let γ satisfy $\lambda_{\min}(\nabla^2 \lambda_1(x, \omega)) \geq \gamma$ for all x such that the maximizer $\omega(x)$ is unique and $\lambda_1(x, \omega)$ is simple. For every $x \in \mathbb{R}^d$, we have*

$$\lambda(x) \geq q(x; x^{(k)}) := \lambda(x^{(k)}) + \nabla \lambda(x^{(k)})(x - x^{(k)}) + \frac{\gamma}{2} \|x - x^{(k)}\|^2. \quad (4.9)$$

The function $q(x; x^{(k)})$ is called the lower support function that underestimate $\lambda(x)$ about $x^{(k)}$. In order to solve eigenvalue optimization problems globally, an algorithm based on such support functions has been devised in [60], an adaptation of the algorithm due to Breimann and Cutler [8] for eigenvalue optimization. Here we follow that algorithm to minimize $\lambda(x)$.

The support functions that we define in (4.9) depend on the lower bound γ . Before giving the algorithm formally, we elaborate how one can obtain γ analytically in the following theorem.

Theorem 4.2.3. *The function $\lambda(x)$ satisfies*

$$\lambda_{\min}(\nabla^2 \lambda(\tilde{x})) \geq \lambda_{\min} \left(\nabla_{xx}^2 M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right)$$

for all $\tilde{x}$ such that the maximizer $\omega(\tilde{x})$ is unique and $\lambda_1(\tilde{x}, \omega(\tilde{x}))$ is simple, where

$$\nabla_{xx}^2 M(x, \omega) := \begin{bmatrix} \frac{\partial^2 M(x, \omega)}{\partial x_1^2} & \frac{\partial^2 M(x, \omega)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 M(x, \omega)}{\partial x_1 \partial x_d} \\ \frac{\partial^2 M(x, \omega)}{\partial x_2 \partial x_1} & \frac{\partial^2 M(x, \omega)}{\partial x_2^2} & \cdots & \frac{\partial^2 M(x, \omega)}{\partial x_2 \partial x_d} \\ & & \ddots & \\ \frac{\partial^2 M(x, \omega)}{\partial x_d \partial x_1} & \frac{\partial^2 M(x, \omega)}{\partial x_d \partial x_2} & \cdots & \frac{\partial^2 M(x, \omega)}{\partial x_d^2} \end{bmatrix}.$$

Proof. Let $\tilde{x} \in \Omega$ be given such that the maximizer $\omega(\tilde{x})$ of $\lambda_1(\tilde{x}, \omega)$ over all $\omega \in \mathcal{B}$ is unique and $\lambda_1(\tilde{x}, \omega(\tilde{x}))$ is simple. By part (iiib) of Lemma 4.2.1 we have

$$\nabla^2 \lambda(\tilde{x}) = \left\{ \nabla_{xx}^2 \lambda_1(x, \omega) + D_{\omega x} \lambda_1(x, \omega)^T \omega'(x) \right\} \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))}. \quad (4.10)$$

It also follows from part (ii) of Lemma 4.2.1 that there exists a neighborhood $\mathcal{N}$ of $\tilde{x}$ such that for all $x \in \mathcal{N}$ the implicitly defined function $\omega(x)$ satisfies

$$\frac{\partial \lambda_1(x, \omega(x))}{\partial \omega} = 0.$$

Taking derivatives of this equation with respect to x at $(x, \omega) = (\tilde{x}, \omega(\tilde{x}))$ yields

$$\omega'(\tilde{x}) = - \left[D_{\omega x} \lambda_1(x, \omega) / D_{\omega \omega}^2 \lambda_1(x, \omega) \right] \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))}. \quad (4.11)$$

Note that we know the denominator in (4.11) is negative due to (4.8). When we plug (4.11) for $\omega(\tilde{x})$ in (4.10), we obtain

$$\nabla^2 \lambda(\tilde{x}) = \left\{ \nabla_{xx}^2 \lambda_1(x, \omega) - \frac{D_{x\omega} \lambda_1(x, \omega) D_{x\omega} \lambda_1(x, \omega)^T}{\nabla_{\omega \omega}^2 \lambda_1(x, \omega)} \right\} \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))}.$$

As the denominator in the last expression for $\nabla^2 \lambda(\tilde{x})$ is negative, we have

$$\lambda_{\min}(\nabla^2 \lambda_1(\tilde{x}, \omega(\tilde{x}))) = \lambda_{\min}(\nabla^2 \lambda(\tilde{x})) \geq \lambda_{\min} \left(\nabla_{xx}^2 \lambda_1(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right)$$

$$= \lambda_{\min} \left(H(\tilde{x}, \omega(\tilde{x})) + 2 \sum_{j=2}^n \frac{1}{\lambda_1(\tilde{x}, \omega(\tilde{x})) - \lambda_j(\tilde{x}, \omega(\tilde{x}))} \Re(H_j(\tilde{x}, \omega(\tilde{x}))) \right)$$

where

$$\begin{aligned} [H(\tilde{x}, \omega(\tilde{x}))]_{k,\ell} &= v^* \left\{ \frac{\partial^2 M(x, \omega)}{\partial x_k \partial x_\ell} \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right\} v \\ [H_j(\tilde{x}, \omega(\tilde{x}))]_{k,\ell} &= \left(v^* \left\{ \frac{\partial M(x, \omega)}{\partial x_k} \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right\} v_j \right) \left(v_j^* \left\{ \frac{\partial M(x, \omega)}{\partial x_\ell} \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right\} v \right), \end{aligned}$$

$\lambda_j(\tilde{x}, \omega(\tilde{x}))$ denotes the j th largest eigenvalue of $M(\tilde{x}, \omega(\tilde{x}))$, whereas v, v_j represent unit eigenvectors corresponding to $\lambda_1(M(\tilde{x}, \omega(\tilde{x}))), \lambda_j(\tilde{x}, \omega(\tilde{x}))$. Now denoting

$$h_\ell^{(j)} := v_j^* \left\{ \frac{\partial M(x, \omega)}{\partial x_\ell} \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right\} v,$$

since for each $u \in \mathbb{R}^d$

$$u^T \Re(H_j(\tilde{x}, \omega(\tilde{x}))) u = u^T H_j(\tilde{x}, \omega(\tilde{x})) u = \left| \sum_{\ell=1}^d u_\ell h_\ell^{(j)} \right|^2 \geq 0,$$

we have $\Re(H_j(\tilde{x}, \omega(\tilde{x})))$ is positive semidefinite. Hence

$$\lambda_{\min}(\nabla^2 \lambda_1(\tilde{x}, \omega(\tilde{x}))) \geq \lambda_{\min}(H(\tilde{x}, \omega(\tilde{x}))) \geq \lambda_{\min} \left(\nabla_{xx}^2 M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right),$$

where the last inequality follows from the observations that

$$H(\tilde{x}, \omega(\tilde{x})) = [I_d \otimes v^*] \nabla_{xx} M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} [I_d \otimes v],$$

and for some unit vector $\tilde{v} \in \mathbb{C}^d$

$$\begin{aligned} \lambda_{\min}(H(\tilde{x}, \omega(\tilde{x}))) &= \tilde{v}^* H(\tilde{x}, \omega(\tilde{x})) \tilde{v} = [\tilde{v}^* \otimes v^*] \nabla_{xx}^2 M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} [\tilde{v} \otimes v] \\ &\geq \lambda_{\min} \left(\nabla_{xx}^2 M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right). \end{aligned}$$

□

Recall that $M(x, \omega)$ is of the form (4.6), and so

$$\nabla_{xx}^2 M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} = \sum_{j=1}^{\kappa} \nabla_{xx}^2 f_j(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \otimes A_j,$$

where $\otimes$ denotes the Kronecker product. This implies

$$\begin{aligned} \lambda_{\min} \left(\nabla_{xx}^2 M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right) &\geq - \left\| \sum_{j=1}^{\kappa} \nabla_{xx}^2 f_j(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \otimes A_j \right\|_2 \\ &\geq - \sum_{j=1}^{\kappa} \left\| \nabla_{xx}^2 f_j(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \otimes A_j \right\|_2 \quad (4.12) \\ &= - \sum_{j=1}^{\kappa} \left\| \nabla_{xx}^2 f_j(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right\|_2 \|A_j\|_2. \end{aligned}$$

4.2.1 The Algorithm

The algorithm to solve (4.7) for small-scale problems is borrowed from [60]. A formal description is given in Algorithm 9. It generates a sequence $\{x^{(k)}\}$ of estimates for a global minimizer of $\lambda(x)$ and a sequence $\{\bar{q}(x; x^{(k)})\}$ of piecewise quadratic models for $\lambda(x)$. At the k th iteration, $x^{(k+1)}$ is set to be a global minimizer of the piecewise quadratic function

$$\bar{q}(x; x^{(k)}) := \max \{q(x; x^{(j)}) \mid j = 1, \dots, k\}$$

and the quadratic function $q(x; x^{(k+1)})$ (defined as in (4.9)) constructed about $x^{(k+1)}$ is added to the piecewise quadratic function. The most challenging part in Algorithm 9 is the solution of the problem in line 5 which is the minimization of the largest of k quadratic functions with constant curvature. This problem can be posed as a bunch of nonconvex quadratic programming problems and it can be solved efficiently as long as d is small. Every convergent subsequence of the sequence $\{x^{(k)}\}$ is guaranteed to converge to a global minimizer of $\lambda(x)$ [60, Theorem 8.1] provided γ satisfies $\lambda_{\min}(\nabla^2 \lambda_1(x, \omega(x))) \geq \gamma$ at all x where $\lambda_1(x, \omega(x))$ is differentiable. In numerical experiments, we observe that Algorithm 9 converges at a linear rate.

4.2.2 Minimization of the Numerical Radius

Recall that for a given matrix-valued function $A(x)$ of the form (4.4), the problem of the minimization of the numerical radius of $A(x)$ is formally stated as

$$\min_{x \in \Omega} r(A(x)) = \min_{x \in \Omega} \max_{\omega \in [0, 2\pi)} \lambda_1(x, \omega)$$

Algorithm 9 Support Based Algorithm

Input: The matrix-valued function $M(x, \omega)$ of the form (4.6), the feasible region Ω ,
a closed interval $\mathcal{B}$.

Output: The sequence $\{x^{(k)}\}$.

- 1: $x^{(1)} \leftarrow$ an initial point in Ω .
 - 2: $q(x; x^{(1)}) \leftarrow$ the initial quadratic model about $x^{(1)}$.
 - 3: **for** $k = 1, 2, \dots$ **do**
 - 4: $\bar{q}(x; x^{(k)}) \leftarrow \max \{q(x, x^{(j)}) \mid j = 1, \dots, k\}$.
 - 5: $x^{(k+1)} \leftarrow \arg \min_{\omega \in \Omega} \bar{q}(x; x^{(k)})$.
 - 6: $q(x; x^{(k+1)}) \leftarrow$ the quadratic model about $x^{(k+1)}$.
 - 7: **end for**
-

where $M(x, \omega) = (A(x)e^{i\omega} + A(x)^*e^{-i\omega})/2$. This problem can be solved globally by means of the support function based approach. To do that, we have to determine a lower bound γ . For the particular choice $M(x, \omega) = (A(x)e^{i\omega} + A(x)^*e^{-i\omega})/2$, it follows from (4.12) that

$$\lambda_{\min} \left(\nabla_{xx}^2 M(x, \omega) \Big|_{(x, \omega) = (\tilde{x}, \omega(\tilde{x}))} \right) \geq - \sum_{j=1}^{\kappa} \left\| \nabla^2 f_j(\tilde{x}) \right\|_2 \|A_j\|_2.$$

Thus γ in Algorithm 9 can be chosen as any lower bound on

$$\min_{x \in \Omega} - \sum_{j=1}^{\kappa} \left\| \nabla^2 f_j(x) \right\|_2 \|A_j\|_2 \tag{4.13}$$

to minimize the numerical radius $r(A(x))$.

Now consider the affine case when $A(x) = A_0 + \sum_{j=1}^{\kappa} x_j A_j$, which also encompasses the matrix-valued function for the problem of stabilization by output feedback control (see Section 1.2.4 for a description of the stabilization by output feedback control problem). For the support function based algorithm, we conclude from (4.13) and Theorem 4.2.2 that, we can set $\gamma = 0$. This algorithm is efficient only when d is small, since it turns out to be difficult to solve the subproblems in line 5 of Algorithm 9 unless d is small. If the matrix-valued function $A(x)$ is in affine form, thanks to

the following representation of the numerical radius [55], we are able to optimize the numerical radius independent of the number of parameters.

Theorem 4.2.4. *The numerical radius of a matrix $A \in \mathbb{C}^{n \times n}$ is bounded by α if and only if there exists a Hermitian matrix $Z \in \mathbb{C}^{n \times n}$ such that the matrix*

$$\begin{bmatrix} \alpha I + Z & A \\ A^* & \alpha I - Z \end{bmatrix}$$

is positive semidefinite.

This representation allows us to minimize the numerical radius by expressing it as a semidefinite program in terms of the variables α, x , and Z . To be precise, the problem of the minimization of the numerical radius can alternatively be stated as

$$\begin{aligned} \min \quad & \alpha \\ \text{s.t.} \quad & \begin{bmatrix} \alpha I + Z & A(x) \\ A(x)^* & \alpha I - Z \end{bmatrix} \succeq 0 \\ & \alpha \in \mathbb{R}, x \in \mathbb{R}^d, Z \in H_n \end{aligned} \tag{4.14}$$

where H_n denotes the set of $n \times n$ Hermitian matrices.

4.2.3 Numerical Experiments

In this section, we present numerical results on the problem of minimizing the numerical radius. For all examples, the matrix-valued function $A(x)$ is in the affine form, so for Algorithm 9, we provide $\gamma = 0$. We obtain the results by an application of a Matlab implementation of Algorithm 9, `eigopt` and the convex optimization package in Matlab, `CVX`, that are publicly available [59, 28]. The results for the minimization of the spectral radius are obtained by the Matlab package `HANSO` [66].

Our first example considers the matrix valued function $A(x) = A_0 + xA_1$ with

given 5×5 random matrices

$$A_0 = \begin{bmatrix} 1.4126 & 0.7449 & -0.4259 & 1.5973 & 0.4431 \\ 1.5024 & -0.8282 & 0.6361 & 0.1124 & -0.1348 \\ 0.7304 & 0.5745 & 0.7932 & -0.3086 & -0.0183 \\ 0.4908 & 0.2818 & -0.8984 & 0.4567 & 0.4608 \\ -0.5861 & 1.1393 & 0.1562 & -0.2751 & 1.3623 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0.4519 & -0.8352 & 0.3953 & 0.3319 & 1.3396 \\ 1.6484 & -1.2760 & -0.8706 & 2.3652 & -0.9691 \\ -2.0284 & 0.6170 & -0.4977 & -0.4822 & 0.2087 \\ -0.4493 & 0.6127 & -0.1067 & 0.6474 & -0.6186 \\ 0.2360 & 0.2894 & -0.6878 & -1.0344 & 0.5120 \end{bmatrix}.$$

We minimize the numerical radius of $A(x)$,

$$r(A(x)) = \max_{\omega \in [0, 2\pi)} \lambda_1(x, \omega),$$

over $x \in [-3, 3]$ using Algorithm 9 and CVX, where $M(x, \omega) = \frac{A(x)e^{i\omega} + A(x)^*e^{-i\omega}}{2}$. The minimized numerical radius is given by $r(A(x_*)) = 2.2535$ for $x_* = -0.3912$. The numerical radius $r(A(x))$ is plotted as a function of x on the left in Figure 4.1 and we provide a plot of $\lambda_1(x_*, \omega)$ with respect to ω on the right in Figure 4.1. For this example, $r(A(x))$ is smooth at $x = x_*$, which is due to the fact that the numerical radius $r(A(x_*))$ is attained at a unique ω value as depicted on the right in Figure 3.1. At the global minimizer $x = x_*$, we compute $\rho(A(x_*)) = 1.8903$, whereas the computed global smallest value of $\rho(A(x))$ by HANSO is $\rho_* = 1.8459$.

The next example concerns the stabilization by output feedback control problem and also illustrates the nonsmoothness of the function $r(A(x))$. We minimize

- (i) $r(A + kb_1c_1^T)$ over $k \in [-3, 3]$
- (ii) $r(A + k_1b_1c_1^T + k_2b_2c_2^T)$ over $k_1, k_2 \in [-3, 3]$.

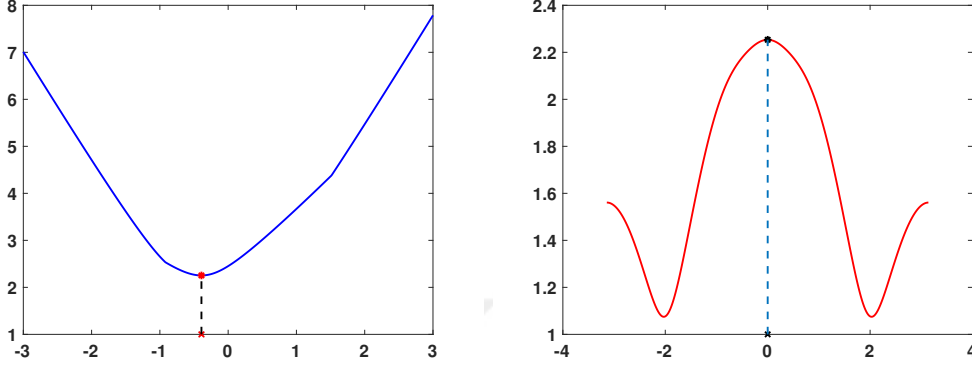


Figure 4.1: The figure depicts **(Left)** the plot of $r(A(x))$ as a function of x over $x \in [-3, 3]$ and **(Right)** the plot of $\lambda_1(x_*, \omega)$ as a function of ω . In the left plot, the cross at the horizontal axis marks the computed global minimizer $x_* = -0.3912$, whereas the dot marks $(x_*, r(A(x_*)))$. In the right plot, the cross at the horizontal axis marks $w_* = 0$, which is the value maximizing $\lambda_1(x_*, \omega)$ globally, while the dot marks $(\omega_*, \lambda_1(x_*, \omega_*))$.

for given 5×5 matrices

$$A = \begin{bmatrix} 0.3271 & -0.9444 & 0.9111 & 0.2398 & -0.0245 \\ 1.0826 & -1.3218 & 0.5946 & -0.6904 & -1.9488 \\ 1.0061 & 0.9248 & 0.3502 & -0.6516 & 1.0205 \\ -0.6509 & 0.0000 & 1.2503 & 1.1921 & 0.8617 \\ 0.2571 & -0.0549 & 0.9298 & -1.6118 & 0.0012 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.0708 & 2.3126 \\ -2.4863 & -1.7461 \\ 0.5812 & 1.5498 \\ -2.1924 & -2.3459 \\ -2.3193 & 1.7824 \end{bmatrix}, \quad C = \begin{bmatrix} 0.0799 & 0.2456 \\ -0.9485 & 1.6891 \\ 0.4115 & -1.2694 \\ 0.6770 & 2.9821 \\ 0.8577 & -0.2245 \end{bmatrix}$$

where b_i, c_i denote the i th columns of B, C . Both Algorithm 9 and CVX return the following results: $r(A + k^{(1)}b_1c_1^T) = 2.3257$ at the global minimizer $k^{(1)} = 0.1832$ and $r(A + k_1^{(2)}b_1c_1^T + k_2^{(2)}b_2c_2^T) = 2.2516$ at the global minimizer $(k_1^{(2)}, k_2^{(2)}) = (0.1288, -0.0608)$.

We display the results in Figure 4.2. In both cases, the nonsmoothness occurs at the global minimizers, which can be seen in the bottom plots in Figure 4.2. In the one parameter case, $r(A + k^{(1)}b_1c_1^T)$ is attained at four distinct ω values (two of them negative and two of them positive) and in the two parameter case, $r(A + k_1^{(2)}b_1c_1^T + k_2^{(2)}b_2c_2^T)$ is attained at six distinct ω values (three of them negative and three of them positive). Once again for this example we try to compute the global minimum of $\rho(A(x))$ by using HANSO. The computed global minima are $\rho^{(1)} = 1.8732$ and $\rho^{(2)} = 1.7891$ for the matrix-valued functions $A + kb_1c_1^T$ and $A + k_1b_1c_1^T + k_2b_2c_2^T$ respectively, whereas $\rho(A + k^{(1)}b_1c_1^T) = 1.8977$ and $\rho(A + k_1^{(2)}b_1c_1 + k_2^{(2)}b_2c_2^T) = 1.8217$.

As a final example, we compare the performances of the support-based algorithm and semidefinite programming to minimize the numerical radius $r(A_n + k_1b_1c_1^T + \cdots + k_db_dc_d^T)$ for various n and d , where A_n is a random matrix of size $n \times n$ and b_i, c_i are random vectors of size $n \times 1$ for $i = 1, \dots, d$. In Table 4.1, we provide CPU times required by each method and also the computed global minimizer and optimal numerical radius, when the number of variables is fixed $d = 1$, but the sizes of matrices and vectors vary. When $n \geq 150$, CVX can no longer minimize the numerical radius, since in this case the number of variables and the number of equality constraints that CVX has to deal with are very large. Table 4.2 depicts also CPU times required by each method and the computed optimal numerical radius when the size of matrices is fixed with $n = 75$, but the number of parameters is varied.

4.3 A Subspace Method for Large-Scale Problems

In this section, we deal with the same problem, defined in (4.7), except the Hermitian matrices $A_1, \dots, A_k$ involved are of large size. We are inspired by the idea in Chapter 2. But that chapter concerns the minimization or maximization of the J largest eigenvalue, whereas the problem here is a minimax problem. Although this looks challenging, it turns out that we can extend the ideas over there to our setting.

For a given matrix $V \in \mathbb{C}^{n \times k}$ with orthonormal columns satisfying $\mathcal{V} = \text{col}(V)$,

SUPPORT-BASED				
n	d	t	k_*	r_*
50	1	1.72	-0.0978543505325	9.4528710016636
100	1	6.63	0.0048364364953	13.7499921883642
150	1	23.20	0.0490513143848	17.0376149194624

SEMIDEFINITE PROGRAMMING				
n	d	t	k_*	r_*
50	1	4.06	-0.0978543799971	9.4528710016666
100	1	30.45	0.0048363446816	13.7499921883803
150	1	—	—	—

Table 4.1: This table concerns the minimization of $r(A_n + kb_1c_1^T)$ with respect to k for random A_n of size $n \times n$, random $b_1, c_1 \in \mathbb{R}^n$, and for varying sizes n . It lists the CPU times (in seconds) labeled with t , the minimal value r_* of $r(A_n + kb_1c_1^T)$ over k , and the corresponding global minimizer k_* by Algorithm 9 (top row) and CVX (bottom row). Here the number of parameters is fixed $d = 1$.

the reduced problem is of the form

$$\min_{\omega \in \Omega} \lambda^{\mathcal{V}}(x) = \min_{\omega \in \Omega} \max_{\omega \in \mathcal{B}} \lambda_1^{\mathcal{V}}(x, \omega) \quad (4.15)$$

where $\lambda_1^{\mathcal{V}}(x, \omega)$ is the largest eigenvalue of

$$M^{\mathcal{V}}(x, \omega) = V^* M(x, \omega) V = \sum_{j=1}^{\kappa} f_j(x, \omega) V^* A_j V.$$

We construct the subspace $\mathcal{V}$ by means of an iterative procedure. At every iteration, we first solve a reduced problem of the form (4.15). Then denoting the global minimizer of the reduced problem with x_* , we compute

$$\omega_* = \arg \max_{\omega \in \mathcal{B}} \lambda_1(x_*, \omega).$$

SUPPORT-BASED				SEMIDEFINITE PROGRAMMING			
n	d	t	r_*	n	d	t	r_*
75	3	14.93	12.1445542659945	75	3	9.86	12.1445542659982
75	4	44.21	12.1039035913584	75	4	16.35	12.1039035913605
75	5	278.38	12.0938462888240	75	5	9.68	12.0938462888247

Table 4.2: The table is similar to Table 4.1. Only now $r(A_n + k_1 b_1 c_1^T + \cdots + k_d b_d c_d^T)$ is minimized over $k_1, \dots, k_d$ for varying number of parameters d by Algorithm 9 and CVX. Now the sizes of the matrices are fixed, for all examples in this table $n = 75$.

The subspace $\mathcal{V}$ is expanded with the addition of an eigenvector corresponding to $\lambda_1(x_*, \omega_*)$, then the procedure is repeated with the expanded subspace. A formal description is given in Algorithm 10, where we use the notation $\mathcal{V}_\ell$ as the subspace at the ℓ th iteration and

$$\lambda^{(\ell)}(x) = \max_{\omega \in \mathcal{B}} \lambda_1^{(\ell)}(x, \omega), \quad \text{where} \quad \lambda_1^{(\ell)}(x, \omega) = \lambda_1^{\mathcal{V}_\ell}(x, \omega)$$

For the solution of the small-scale problem in line 6, we use the ideas of the previous section.

4.3.1 Theoretical Properties

We start this section with a result about the monotonicity of $\lambda^{\mathcal{V}}(x)$ with respect to the subspace $\mathcal{V}$. This monotonicity result will have an important role when we analyze the convergence of the subspace framework. Note that when $\mathcal{V} = \mathbb{C}^n$, $\lambda^{\mathcal{V}}(x) = \lambda(x)$.

Theorem 4.3.1 (Monotonicity). *Let $\mathcal{V}_1, \mathcal{V}_2$ be two subspaces of $\mathbb{C}^n$ such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. The following holds:*

$$\lambda^{\mathcal{V}_1}(x) \leq \lambda^{\mathcal{V}_2}(x) \leq \lambda(x) \quad \forall x \in \Omega.$$

Algorithm 10 The Subspace Framework

Input: The matrix-valued function $M(x, \omega)$ of the form (4.6), the feasible region Ω , a closed interval $\mathcal{B}$.

Output: The sequence $\{x^{(\ell)}\}, \{\omega^{(\ell)}\}$.

- 1: $x^{(1)} \leftarrow$ an initial point in Ω .
 - 2: $\omega^{(1)} \leftarrow \arg \max_{\omega \in \mathcal{B}} \lambda_1(x^{(1)}, \omega)$.
 - 3: $V_1 \leftarrow$ a unit eigenvector corresponding to $\lambda_1(x^{(1)}, \omega^{(1)})$.
 - 4: $\mathcal{V}_1 \leftarrow \text{span}\{V_1\}$
 - 5: **for** $\ell = 1, 2, \dots$ **do**
 - 6: $x^{(\ell+1)} \leftarrow \arg \min_{x \in \Omega} \lambda^{(\ell)}(x)$.
 - 7: $\omega^{(\ell+1)} \leftarrow \arg \max_{\omega \in \mathcal{B}} \lambda_1(x^{(\ell+1)}, \omega)$.
 - 8: $v_{\ell+1} \leftarrow$ a unit eigenvector corresponding to $\lambda_1(x^{(\ell+1)}, \omega^{(\ell+1)})$.
 - 9: $V_{\ell+1} \leftarrow \text{orth}[V_\ell \quad v_{\ell+1}]$ and $\mathcal{V}_{\ell+1} \leftarrow \text{col}(V_{\ell+1})$.
 - 10: **end for**
-

Proof. Observe that for all $\omega \in \mathcal{B}$

$$\begin{aligned} \lambda_1^{\mathcal{V}_1}(x, \omega) &= \max_{v \in \mathcal{V}_1, \|v\|_2=1} v^* M(x, \omega) v \\ &\leq \max_{v \in \mathcal{V}_2, \|v\|_2=1} v^* M(x, \omega) v = \lambda_1^{\mathcal{V}_2}(x, \omega) \end{aligned}$$

where the inequality is due to $\mathcal{V}_1 \subseteq \mathcal{V}_2$. This assertion in turn implies $\lambda^{\mathcal{V}_1}(x) \leq \lambda^{\mathcal{V}_2}(x)$ as $\lambda^{\mathcal{V}_1}(x) = \max_{\omega \in \mathcal{B}} \lambda_1^{\mathcal{V}_1}(x, \omega)$ and $\lambda^{\mathcal{V}_2}(x) = \max_{\omega \in \mathcal{B}} \lambda_1^{\mathcal{V}_2}(x, \omega)$. In a similar way, $\lambda^{\mathcal{V}_2}(x) \leq \lambda(x)$ due to $\mathcal{V}_2 \subseteq \mathbb{C}^n$. $\square$

In the next result, we present interpolation properties between $\lambda(x)$ and $\lambda^{\mathcal{V}}(x)$, regarding Algorithm 10. We later extend these interpolation properties to derivatives of these functions as well, but we give the details of that issue in the context of the rate-of-convergence analysis.

Theorem 4.3.2 (Interpolation). *The following hold regarding Algorithm 10 for a*

given $\ell \in \mathbb{Z}^+$ and $j = 1, \dots, \ell$:

$$\begin{aligned}\lambda(x^{(j)}) &= \lambda_1(x^{(j)}, \omega^{(j)}) \\ &= \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)}) = \lambda^{(\ell)}(x^{(j)})\end{aligned}\tag{4.16}$$

Proof. From Theorem 4.3.1, we have $\lambda^{(\ell)}(x^{(j)}) \leq \lambda(x^{(j)})$. Additionally, lines 3, 4, 8 and 9 of Algorithm 10 imply $v_j \in \mathcal{V}_\ell$ for $j = 1, \dots, \ell$, that is there exists a unit vector α_j such that $v_j = V_\ell \alpha_j$. Hence,

$$\begin{aligned}\lambda(x^{(j)}) &= \lambda_1(x^{(j)}, \omega^{(j)}) = v_j^* M(x^{(j)}, \omega^{(j)}) v_j \\ &= \alpha_j^* V_\ell^* M(x^{(j)}, \omega^{(j)}) V_\ell \alpha_j \leq \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)}) \\ &\leq \lambda^{(\ell)}(x^{(j)}).\end{aligned}$$

This completes the proof as desired. $\square$

4.3.2 Convergence Analysis

This section is devoted to the convergence analysis of the subspace framework. We again consider the problem at our hands in the infinite dimensional setting. In particular, the matrices $A_1, \dots, A_\kappa$ in the definition of the matrix valued function $M(x, \omega)$ of the form (4.6) are now compact self-adjoint operators acting on $\ell^2(\mathbb{N})$, the Hilbert space consisting of square summable infinite sequences of complex numbers equipped with the inner product $\langle v, w \rangle = \sum_{k=1}^{\infty} \bar{v}_k w_k$ and the norm $\|v\|_2 = \sqrt{\sum_{k=1}^{\infty} |v_k|^2}$. Moreover, the operator norms defined by

$$\begin{aligned}\|A_\ell\|_2 &:= \sup \{ \|A_\ell v\|_2 \mid v \in \ell^2(\mathbb{N}) \text{ such that } \|v\|_2 = 1 \} \text{ for } \ell = 1, \dots, \kappa \quad \text{and} \\ \|M(x, \omega)\|_2 &:= \sup \{ \|M(x, \omega) v\|_2 \mid v \in \ell^2(\mathbb{N}) \text{ such that } \|v\|_2 = 1 \}\end{aligned}$$

are used in the arguments below.

Global Convergence

We first establish the global convergence of Algorithm 10 in Theorem 4.3.4 below. This result is a consequence of the monotonicity (Theorem 4.3.1) and the interpolation

(Theorem 4.3.2) properties, as well as the uniform Lipschitz continuity of the reduced problems which is formally stated and proven next. In the result, we use the notation

$$\delta_{\mathcal{I}}^{\mathcal{V}}(x) := \{ \lambda_1^{\mathcal{V}}(x, \omega) \mid \omega \in \mathcal{I} \}$$

for a given set $\mathcal{I} \subseteq \mathcal{B}$.

Lemma 4.3.3 (Uniform Lipschitz Continuity). *There exists a constant $\zeta > 0$ such that for all subspaces $\mathcal{V}$ the following hold:*

- (i) For all $\omega \in \mathcal{B}$, $|\lambda_1^{\mathcal{V}}(\tilde{x}, \omega) - \lambda_1^{\mathcal{V}}(\hat{x}, \omega)| \leq \zeta \|\tilde{x} - \hat{x}\|_2 \quad \forall \tilde{x}, \hat{x} \in \Omega;$
- (ii) For any given set $\mathcal{I} \subseteq \mathcal{B}$, $|\delta_{\mathcal{I}}^{\mathcal{V}}(\tilde{x}) - \delta_{\mathcal{I}}^{\mathcal{V}}(\hat{x})| \leq \zeta \|\tilde{x} - \hat{x}\|_2 \quad \forall \tilde{x}, \hat{x} \in \Omega$
- (iii) $|\lambda^{\mathcal{V}}(\tilde{x}) - \lambda^{\mathcal{V}}(\hat{x})| \leq \zeta \|\tilde{x} - \hat{x}\|_2 \quad \forall \tilde{x}, \hat{x} \in \Omega.$

Proof. (i) For a given $\omega \in \mathcal{B}$, it follows from Weyl's theorem [36, Theorem 4.3.1] that

$$\begin{aligned} |\lambda_1^{\mathcal{V}}(\tilde{x}, \omega) - \lambda_1^{\mathcal{V}}(\hat{x}, \omega)| &\leq \|M^{\mathcal{V}}(\tilde{x}, \omega) - M^{\mathcal{V}}(\hat{x}, \omega)\|_2 \\ &\leq \sum_{j=1}^{\kappa} |f_j(\tilde{x}, \omega) - f_j(\hat{x}, \omega)| \|A_j\|_2 \end{aligned}$$

for all $\tilde{x}, \hat{x} \in \Omega$. But the real analyticity of f_j on their domain $\tilde{\Omega} \times \tilde{\mathcal{B}}$ implies that each $f_j(x, \omega)$ is Lipschitz continuous implying to the existence of constants γ_j such that

$$|f_j(\tilde{x}, \omega) - f_j(\hat{x}, \omega)| \leq \gamma_j \|(\tilde{x}, \omega) - (\hat{x}, \omega)\|_2 = \gamma_j \|\tilde{x} - \hat{x}\|_2.$$

It follows that

$$|\lambda_1^{\mathcal{V}}(\tilde{x}, \omega) - \lambda_1^{\mathcal{V}}(\hat{x}, \omega)| \leq \left(\sum_{j=1}^{\kappa} \gamma_j \|A_j\|_2 \right) \|\tilde{x} - \hat{x}\|_2 \quad \forall \tilde{x}, \hat{x} \in \Omega$$

as desired.

(ii) First note that we can assume that the set $\mathcal{I}$ is closed, as otherwise replacing $\mathcal{I}$ with its closure $\text{Cl}(\mathcal{I})$ does not affect the argument below, as we have

$$\delta_{\mathcal{I}}^{\mathcal{V}}(x) = \sup_{\omega \in \mathcal{I}} \lambda_1^{\mathcal{V}}(x, \omega) = \sup_{\omega \in \text{Cl}(\mathcal{I})} \lambda_1^{\mathcal{V}}(x, \omega) = \delta_{\text{Cl}(\mathcal{I})}^{\mathcal{V}}(x).$$

This closedness assumption and the boundedness of $\lambda_1^\mathcal{V}(x, \omega)$ from above by its globally maximal value over $\mathcal{B}$ imply that $\delta_{\mathcal{I}}^\mathcal{V}(x)$ must be attained at some point $\omega \in \mathcal{I}$.

For any given two points $\tilde{x}, \hat{x} \in \Omega$, we have

$$\delta_{\mathcal{I}}^\mathcal{V}(\tilde{x}) = \sup_{\omega \in \mathcal{I}} \lambda_1^\mathcal{V}(\tilde{x}, \omega) \quad \text{and} \quad \delta_{\mathcal{I}}^\mathcal{V}(\hat{x}) = \sup_{\omega \in \mathcal{I}} \lambda_1^\mathcal{V}(\hat{x}, \omega).$$

Let us assume without loss of generality that $\delta_{\mathcal{I}}^\mathcal{V}(\tilde{x}) \geq \delta_{\mathcal{I}}^\mathcal{V}(\hat{x})$. Also let $\tilde{\omega}, \hat{\omega} \in \mathcal{I}$ be such that

$$\delta_{\mathcal{I}}^\mathcal{V}(\tilde{x}) = \lambda_1^\mathcal{V}(\tilde{x}, \tilde{\omega}) \quad \text{and} \quad \delta_{\mathcal{I}}^\mathcal{V}(\hat{x}) = \lambda_1^\mathcal{V}(\hat{x}, \hat{\omega}).$$

Finally,

$$\begin{aligned} |\delta_{\mathcal{I}}^\mathcal{V}(\tilde{x}) - \delta_{\mathcal{I}}^\mathcal{V}(\hat{x})| &= |\lambda_1^\mathcal{V}(\tilde{x}, \tilde{\omega}) - \lambda_1^\mathcal{V}(\hat{x}, \hat{\omega})| \\ &\leq |\lambda_1^\mathcal{V}(\tilde{x}, \tilde{\omega}) - \lambda_1^\mathcal{V}(\hat{x}, \tilde{\omega})| \leq \zeta \|\tilde{x} - \hat{x}\|_2, \end{aligned}$$

where the last inequality follows from part (i).

(iii) This is immediate due to part (ii) since $\lambda^\mathcal{V}(x) = \delta_{\mathcal{I}}^\mathcal{V}(x)$ for $\mathcal{I} = \mathcal{B}$. □

The next result states the global convergence of the subspace framework. The proof is omitted since it is identical to the proof of Theorem 2.3.1 of Chapter 2, but with $\lambda^{(\ell)}(x)$ here taking the role of $\lambda_j^{(\ell)}(\omega)$ in that proof.

Theorem 4.3.4 (Global Convergence). *The limit of every convergent subsequence of the sequence $\{x^{(\ell)}\}$ generated by Algorithm 10 in the infinite dimensional setting is a global minimizer of $\lambda(x)$ over all $x \in \Omega$. Furthermore*

$$\lim_{\ell \rightarrow \infty} \lambda^{(\ell)}(x^{(\ell+1)}) = \lim_{\ell \rightarrow \infty} \min_{x \in \Omega} \lambda^{(\ell)}(x) = \min_{x \in \Omega} \lambda(x). \quad (4.17)$$

Local Rate-of-Convergence

For the rest of the paper, we assume that the sequence $\{x^{(\ell)}\}$ itself is convergent. If this is the case, the global convergence result guarantees that $\{x^{(\ell)}\}$ converges to a global minimizer of $\lambda(x)$ over Ω . If $\lambda(x)$ has a unique global minimizer, then $\{x^{(\ell)}\}$ has to converge to this global minimizer. This subsection concerns the speed of this

convergence. We establish a local superlinear rate-of-convergence for Algorithm 10 in the one parameter case ($d = 1$) and for an extended version Algorithm 11 below, in the multi-parameter case. This analysis holds under the assumption that $\lambda(x)$ is smooth and the Hessian is invertible at the converged minimizer.

We first start with results that relate the first and second derivatives of $\lambda_1(x, \omega)$ with those of $\lambda_1^{(\ell)}(x, \omega)$ with respect to ω .

Theorem 4.3.5. *The following hold regarding Algorithm 10 for all $\ell \in \mathbb{Z}^+$ and $j = 1, \dots, \ell$:*

- (i) $\frac{\partial \lambda_1(x^{(j)}, \omega^{(j)})}{\partial \omega} = \frac{\partial \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})}{\partial \omega}$
- (ii) $\frac{\partial^2 \lambda_1(x^{(j)}, \omega^{(j)})}{\partial \omega^2} \geq \frac{\partial^2 \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})}{\partial \omega^2}$

Proof. We first remark that due to interpolation property, in particular equation (4.16), both of the functions $\lambda_1(x^{(j)}, \omega)$ and $\lambda_1^{(\ell)}(x^{(j)}, \omega)$ over $\omega \in \mathcal{B}$ have a global maximizer at $\omega^{(j)}$, so the eigenvalue functions $\lambda_1(x, \omega)$ and $\lambda_1^{(\ell)}(x, \omega)$ are twice continuously differentiable at $(x, \omega) = (x^{(j)}, \omega^{(j)})$ with respect to ω .

(i) Recall that from the proof of Theorem 4.3.2, a unit eigenvector α_j corresponding to $\lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})$ satisfies $v_j = V_\ell \alpha_j$. Using the analytical formulas for the derivatives of eigenvalue functions [47], we obtain

$$\begin{aligned} \frac{\partial \lambda_1(x^{(j)}, \omega^{(j)})}{\partial \omega} &= v_j^* \frac{\partial M(x^{(j)}, \omega^{(j)})}{\partial \omega} v_j = \alpha_j^* V_\ell^* \frac{\partial M(x^{(j)}, \omega^{(j)})}{\partial \omega} V_\ell \alpha_j \\ &= \alpha_j^* \frac{\partial M^{V_\ell}(x^{(j)}, \omega^{(j)})}{\partial \omega} \alpha_j = \frac{\partial \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})}{\partial \omega} \end{aligned}$$

as desired.

(ii) The second derivatives with respect to ω are related by

$$\begin{aligned} &\frac{\partial^2 \lambda_1(x^{(j)}, \omega^{(j)})}{\partial \omega^2} \\ &= \lim_{h \rightarrow 0} \frac{\lambda_1(x^{(j)}, (\omega^{(j)} + h)) - 2\lambda_1(x^{(j)}, \omega^{(j)}) + \lambda_1(x^{(j)}, (\omega^{(j)} - h))}{h^2} \\ &\geq \lim_{h \rightarrow 0} \frac{\lambda_1^{(\ell)}(x^{(j)}, (\omega^{(j)} + h)) - 2\lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)}) + \lambda_1^{(\ell)}(x^{(j)}, (\omega^{(j)} - h))}{h^2} \\ &= \frac{\partial^2 \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})}{\partial \omega^2} \end{aligned}$$

where the inequality follows from $\lambda_1(x^{(j)}, (\omega^{(j)} \pm h)) \geq \lambda_1^{(\ell)}(x^{(j)}, (\omega^{(j)} \pm h))$ due to monotonicity (Theorem 4.3.1) and $\lambda_1(x^{(j)}, \omega^{(j)}) = \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})$ due to the interpolation property (Theorem 4.3.2). $\square$

The reason why the subspace framework converges fast is the interpolation properties between $\lambda(x)$ and $\lambda^{(\ell)}(x)$ as well as the derivatives of $\lambda(x)$ and $\lambda^{(\ell)}(x)$. The next result extends the interpolation properties in Theorem 4.3.2 to first derivatives. For the rest of the section, we use the notation $\mathcal{B}(y, \varrho) = \{\tilde{y} \mid \|y - \tilde{y}\|_2 \leq \varrho\}$ as the closed ball (closed interval if y is scalar) of radius ϱ centered at y .

Theorem 4.3.6 (Hermite Interpolation). *The following are satisfied by Algorithm 10 for all $\ell \in \mathbb{Z}^+$ and $j = 1, \dots, \ell$:*

(i) *If $\lambda(x)$ is differentiable at $x = x^{(j)}$, $\lambda_1(x^{(j)}, \omega^{(j)})$ is simple and*

$$\frac{\partial^2 \lambda_1(x^{(j)}, \omega^{(j)})}{\partial \omega^2} \neq 0,$$

then $\lambda^{\mathcal{V}_\ell}(x)$ is differentiable at $x = x^{(j)}$.

(ii) *If $\lambda(x)$ and $\lambda^{(\ell)}(x)$ are differentiable at $x = x^{(j)}$, then*

$$\nabla \lambda(x^{(j)}) = \nabla \lambda^{(\ell)}(x^{(j)}).$$

Proof. (i) First the simplicity of $\lambda_1(x^{(j)}, \omega^{(j)})$ implies the simplicity of $\lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})$. Indeed, if $\lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})$ was not simple, then there would exist two mutually orthogonal unit vectors $\alpha, \tilde{\alpha}$ such that

$$\begin{aligned} \lambda_1(x^{(j)}, \omega^{(j)}) &= \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)}) = \alpha^* M^{\mathcal{V}_\ell}(x^{(j)}, \omega^{(j)}) \alpha \\ &= \tilde{\alpha}^* M^{\mathcal{V}_\ell}(x^{(j)}, \omega^{(j)}) \tilde{\alpha}. \end{aligned}$$

Setting $v := V_\ell \alpha$ and $\tilde{v} := V_\ell \tilde{\alpha}$, we get two mutually orthogonal unit vectors $v, \tilde{v}$ such that

$$\lambda_1(x^{(j)}, \omega^{(j)}) = v^* M(x^{(j)}, \omega^{(j)}) v = \tilde{v}^* M(x^{(j)}, \omega^{(j)}) \tilde{v}$$

which contradicts to the simplicity of $\lambda_1(x^{(j)}, \omega^{(j)})$. By the continuity of $\lambda_1^{(\ell)}(x, \omega)$ this simplicity property is preserved in a ball centered at $(x^{(j)}, \omega^{(j)})$. The eigenvalue function $\lambda_1^{(\ell)}(x, \omega)$ is twice continuously differentiable in this ball. Moreover, from Theorem 4.3.5 we have

$$\frac{\partial \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})}{\partial \omega} = \frac{\partial \lambda_1(x^{(j)}, \omega^{(j)})}{\partial \omega} = 0$$

and

$$0 > \frac{\partial^2 \lambda_1(x^{(j)}, \omega^{(j)})}{\partial \omega^2} \geq \frac{\partial^2 \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})}{\partial \omega^2},$$

where we also use the fact that $\omega^{(j)}$ is a maximizer of $\lambda_1(x^{(j)}, \omega)$ over $\omega \in \mathbb{R}$. Due to the implicit function theorem, there exists a unique analytic function $\omega(x) : \mathcal{B}(x^{(j)}, \varrho_1) \rightarrow \mathcal{B}(\omega^{(j)}, \varrho_2)$ such that $\omega(x^{(j)}) = \omega^{(j)}$ and

$$\frac{\partial \lambda_1^{(\ell)}(x, \omega(x))}{\partial \omega} = 0.$$

Now our claim is that $\lambda^{(\ell)}(x) = \lambda_1^{(\ell)}(x, \omega(x))$, hence $\lambda^{(\ell)}(x)$ is analytic for all x in a ball $\mathcal{B}(x^{(j)}, \varrho)$. We first show the uniqueness of the global maximizer $\omega^{(j)}$ of $\lambda^{(\ell)}(x^{(j)})$. Suppose on the contrary that there is another maximizer $\tilde{\omega} \neq \omega^{(j)}$ such that

$$\lambda^{(\ell)}(x^{(j)}) = \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)}) = \lambda_1^{(\ell)}(x^{(j)}, \tilde{\omega}).$$

But the inequality $\lambda_1^{(\ell)}(x^{(j)}, \tilde{\omega}) \leq \lambda_1(x^{(j)}, \tilde{\omega})$, a consequence of the monotonicity, yields

$$\lambda(x^{(j)}) = \lambda_1(x^{(j)}, \omega^{(j)}) = \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)}) \leq \lambda_1(x^{(j)}, \tilde{\omega})$$

which implies $\lambda(x)$ is attained at $\omega^{(j)}$ and $\tilde{\omega}$, contradicting the differentiability assumption on $\lambda(x)$ at $x = x^{(j)}$. Consequently, we have the following positive gap

$$\varphi := \lambda^{(\ell)}(x^{(j)}) - \sup \left\{ \lambda_1^{(\ell)}(x^{(j)}, \omega) \mid \omega \in \mathcal{B} \setminus \mathcal{B}(\omega^{(j)}, \varrho_2) \right\} > 0.$$

Our aim is to show that this positive gap is preserved for all $x \in \mathcal{B}(x^{(j)}, \varrho)$ for some ϱ . For this reason, we set $\varrho = \min\{\varrho_1, \varphi/4\zeta\}$. For each $x \in \mathcal{B}(x^{(j)}, \varrho)$, we define

$$\begin{aligned} \delta_1^{\mathcal{V}_\ell}(x) &:= \max \left\{ \lambda_1^{(\ell)}(x, \omega) \mid \omega \in \mathcal{B}(\omega^{(j)}, \varrho_2) \right\} \\ \delta_2^{\mathcal{V}_\ell}(x) &:= \sup \left\{ \lambda_1^{(\ell)}(x, \omega) \mid \omega \in \mathcal{B} \setminus \mathcal{B}(\omega^{(j)}, \varrho_2) \right\} \\ \delta^{\mathcal{V}_\ell}(x) &:= \delta_1^{\mathcal{V}_\ell}(x) - \delta_2^{\mathcal{V}_\ell}(x). \end{aligned}$$

Part (ii) of Lemma 4.3.3 implies that for every $x \in \mathcal{B}(x^{(j)}, \varrho)$, we have

$$\delta_1^{\mathcal{V}_\ell}(x) \geq \delta_1^{\mathcal{V}_\ell}(x^{(j)}) - \varphi/4 \quad \text{and} \quad \delta_2^{\mathcal{V}_\ell}(x^{(j)}) \geq \delta_2^{\mathcal{V}_\ell}(x) - \varphi/4.$$

Hence, as $\delta^{\mathcal{V}_\ell}(x^{(j)}) = \varphi$, we obtain

$$\begin{aligned} \delta^{\mathcal{V}_\ell}(x) &= \delta_1^{\mathcal{V}_\ell}(x) - \delta_2^{\mathcal{V}_\ell}(x) \geq \delta_1^{\mathcal{V}_\ell}(x^{(j)}) - \delta_2^{\mathcal{V}_\ell}(x^{(j)}) - \varphi/2 \\ &= \delta^{\mathcal{V}_\ell}(x^{(j)}) - \varphi/2 = \varphi/2. \end{aligned}$$

This means that $\lambda^{(\ell)}(x)$ is attained at some point $\tilde{\omega}(x)$ in the interior of $\mathcal{B}(\omega^{(j)}, \varrho_2)$ for each $x \in \mathcal{B}(x^{(j)}, \varrho)$. This global maximizer must satisfy the first order optimality condition

$$\frac{\partial \lambda_1^{(\ell)}(x, \tilde{\omega}(x))}{\partial \omega} = 0.$$

But as $\omega(x)$ is the unique point in $\mathcal{B}(\omega^{(j)}, \varrho_2)$ satisfying this condition, we must have $\tilde{\omega}(x) = \omega(x)$ and $\lambda^{(\ell)}(x) = \lambda_1^{(\ell)}(x, \omega(x))$ proving that $\lambda^{(\ell)}(x)$ is differentiable.

(ii) Recall from Theorem 4.3.2 the existence of a unit vector α_j which is an eigenvector corresponding to $\lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})$ and satisfies $v_j = V_\ell \alpha_j$. By employing the classical formulas for the derivative of an eigenvalue function [47], we obtain

$$\begin{aligned} \frac{\partial \lambda(x^{(j)})}{\partial x_s} &= \frac{\partial \lambda_1(x^{(j)}, \omega^{(j)})}{\partial x_s} = v_j^* \frac{\partial M(x^{(j)}, \omega^{(j)})}{\partial x_s} v_j \\ &= \alpha_j^* \frac{\partial M^{\mathcal{V}_\ell}(x^{(j)}, \omega^{(j)})}{\partial x_s} \alpha_j = \frac{\partial \lambda_1^{(\ell)}(x^{(j)}, \omega^{(j)})}{\partial x_s} = \frac{\partial \lambda^{(\ell)}(x^{(j)})}{\partial x_s} \end{aligned}$$

for $s = 1, \dots, d$. This completes the proof. $\square$

As we have mentioned before, the local superlinear rate-of-convergence in the multi-parameter case is established for an extended version. Before going into the details of the analysis, we shall describe the extended subspace framework, formally given by Algorithm 11. In this version, the subspace $\mathcal{V}_\ell$ is expanded by adding the eigenvectors not only at $x^{(\ell)}$, but also at the nearby points $x_{p,q}^{(\ell)} := x^{(\ell)} + h^{(\ell)} e_{p,q}$ for $p = 1, \dots, d$, $q = p, \dots, d$. Note in the description, letting e_p denote the p th column of the $d \times d$ identity matrix, $e_{p,q} := 1/\sqrt{2}(e_p + e_q)$ if $p \neq q$, and $e_{p,p} := e_p$. The drawback

of this version is that we additionally have to solve the inner maximization problem at these nearby points and compute the corresponding eigenvectors.

We can extend the interpolation and Hermite interpolation results of Theorem 4.3.2 and 4.3.6 to the nearby points $x_{p,q}^{(\ell)}$ for Algorithm 11. We state these results formally in the next theorem where the proof is omitted since the arguments are similar to those of Theorems 4.3.2 and 4.3.6. Moreover, the global convergence result of Theorem 4.3.4 holds for the sequence $\{x^{(\ell)}\}$ generated by Algorithm 11.

Theorem 4.3.7. *The assertions of Theorems 4.3.2 and 4.3.6 hold for Algorithm 11. Additionally, for all $\ell \in \mathbb{Z}_+$, $j = 1, \dots, \ell$, $p = 1, \dots, d$ and $q = p, \dots, d$ we have*

$$(i) \quad \lambda(x_{p,q}^{(j)}) = \lambda_1(x_{p,q}^{(j)}, \omega_{p,q}^{(j)}) = \lambda_1^{(\ell)}(x_{p,q}^{(j)}, \omega_{p,q}^{(j)}) = \lambda^{(\ell)}(x_{p,q}^{(j)}).$$

(ii) *If $\lambda(x)$ is differentiable at $x = x_{p,q}^{(j)}$, $\lambda_1(x_{p,q}^{(j)}, \omega_{p,q}^{(j)})$ is simple and*

$$\frac{\partial^2 \lambda_1(x_{p,q}^{(j)}, \omega_{p,q}^{(j)})}{\partial \omega^2} \neq 0,$$

then $\lambda^{(\ell)}(x)$ is differentiable at $x = x_{p,q}^{(j)}$.

(iii) *If $\lambda(x)$ and $\lambda^{(\ell)}(x)$ are differentiable at $x = x_{p,q}^{(j)}$, then*

$$\nabla \lambda(x_{p,q}^{(j)}) = \nabla \lambda^{(\ell)}(x_{p,q}^{(j)}).$$

Now, we give the details of the rate-of-convergence analysis, which is inspired by Chapter 2. This analysis depends on the Hermite interpolation property (Theorems 4.3.2 and 4.3.6). Note that $\lambda(x)$ and $\lambda^{(\ell)}(x)$ are defined in terms of the global maximizers of $\lambda_1(x, \omega)$ and $\lambda_1^{(\ell)}(x, \omega)$ over $\omega \in \mathcal{B}$ and this makes the analysis more complicated. We first show that $\lambda(x) = \lambda_1(x, \omega(x))$ and $\lambda^{(\ell)}(x) = \lambda_1^{(\ell)}(x, \omega^{(\ell)}(x))$ for all x close to the global minimizer x_* , where $\omega(x), \omega^{(\ell)}(x)$ are defined implicitly by

$$\frac{\partial \lambda_1(x, \omega(x))}{\partial \omega} = 0 \quad \text{and} \quad \frac{\partial \lambda_1^{(\ell)}(x, \omega^{(\ell)}(x))}{\partial \omega} = 0.$$

Throughout the section we use the following notation for $x \in \mathbb{R}^d, \omega \in \mathbb{R}$ and $\epsilon_1, \epsilon_2 > 0$

$$\mathcal{N}(x, \omega; \epsilon_1, \epsilon_2) := \{(\tilde{x}, \tilde{\omega}) \in \mathbb{R}^d \times \mathbb{R} \mid \|\tilde{x} - x\|_2 \leq \epsilon_1, |\tilde{\omega} - \omega| \leq \epsilon_2\}.$$

Algorithm 11 The Extended Subspace Framework

Input: The matrix-valued function $M(x, \omega)$ of the form (4.6), the feasible region Ω ,
a closed interval $\mathcal{B}$.

Output: The sequence $\{x^{(\ell)}\}, \{\omega^{(\ell)}\}$.

- 1: $x^{(1)} \leftarrow$ an initial point in Ω .
 - 2: $\omega^{(1)} \leftarrow \arg \max_{\omega \in \mathcal{B}} \lambda_1(x^{(1)}, \omega)$.
 - 3: $V_1 \leftarrow$ a unit eigenvector corresponding to $\lambda_1(x^{(1)}, \omega^{(1)})$.
 - 4: $\mathcal{V}_1 \leftarrow \text{span}\{V_1\}$.
 - 5: **for** $\ell = 1, 2, \dots$ **do**
 - 6: $x^{(\ell+1)} \leftarrow \arg \min_{x \in \Omega} \lambda^{(\ell)}(x)$.
 - 7: $\omega^{(\ell+1)} \leftarrow \arg \max_{\omega \in \mathcal{B}} \lambda_1(x^{(\ell+1)}, \omega)$.
 - 8: $v_{\ell+1} \leftarrow$ a unit eigenvector corresponding to $\lambda_1(x^{(\ell+1)}, \omega^{(\ell+1)})$.
 - 9: $h^{(\ell+1)} \leftarrow \|x^{(\ell+1)} - x^{(\ell)}\|_2$.
 - 10: **for** $p = 1, \dots, d$ **do**
 - 11: **for** $q = p, \dots, d$ **do**
 - 12: $x_{p,q}^{(\ell+1)} \leftarrow x^{(\ell+1)} + h^{(\ell+1)} e_{p,q}$.
 - 13: $\omega_{p,q}^{(\ell+1)} \leftarrow \arg \max_{\omega \in \mathcal{B}} \lambda_1(x_{p,q}^{(\ell+1)}, \omega)$.
 - 14: $v_{p,q}^{(\ell+1)} \leftarrow$ a unit eigenvector corresponding to $\lambda_1(x_{p,q}^{(\ell+1)}, \omega_{p,q}^{(\ell+1)})$.
 - 15: **end for**
 - 16: **end for**
 - 17: $V_{\ell+1} \leftarrow \text{orth} \begin{bmatrix} V_\ell & v_{\ell+1} & v_{1,1}^{(\ell+1)} & \dots & v_{1,d}^{(\ell+1)} & v_{2,2}^{(\ell+1)} & \dots & v_{d,d}^{(\ell+1)} \end{bmatrix}$.
 - 18: $\mathcal{V}_{\ell+1} \leftarrow \text{col}(V_{\ell+1})$.
 - 19: **end for**
-

Lemma 4.3.8. *Suppose that the global maximizer ω_* of $\lambda_1(x_*, \omega)$ over $\omega \in \mathcal{B}$ is unique, $\lambda_1(x_*, \omega_*)$ is simple and*

$$\frac{\partial^2 \lambda_1(x_*, \omega_*)}{\partial \omega^2} = \delta < 0. \quad (4.18)$$

Then there exists a neighborhood $\mathcal{N}(x_*, \omega_*; \epsilon_1, \epsilon_2)$ of (x_*, ω_*) such that

- (i) $\frac{\partial^2 \lambda_1(x, \omega)}{\partial \omega^2} \leq 3\delta/4 \quad \forall (x, \omega) \in \mathcal{N}(x_*, \omega_*; \epsilon_1, \epsilon_2).$
- (ii) *For all ℓ large enough, $\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial \omega^2} \leq \delta/2 \quad \forall (x, \omega) \in \mathcal{N}(x_*, \omega_*; \epsilon_1, \epsilon_2).$*

Proof. Proposition 2.2.9 in Chapter 2 establishes the existence of a neighborhood $\mathcal{N}$ of (x_*, ω_*) such that the eigenvalues $\lambda_1(x, \omega)$ and $\lambda_1^{(\ell)}(x, \omega)$ for all ℓ large enough remain simple and the first three derivatives are bounded above by constants uniformly for all $(x, \omega) \in \mathcal{N}$, where the constants are independent of ℓ . The boundedness of the third of derivative of $\lambda_1(x, \omega)$ and the assumption (4.18) yield a neighborhood $\widehat{\mathcal{N}} \subseteq \mathcal{N}$ such that for all $(x, \omega) \in \widehat{\mathcal{N}}$

$$\frac{\partial^2 \lambda_1(x, \omega)}{\partial \omega^2} \leq 3\delta/4.$$

Now, since the global maximizer ω_* is unique and

$$\lim_{\ell \rightarrow \infty} \lambda_1(x^{(\ell)}, \omega^{(\ell)}) = \lim_{\ell \rightarrow \infty} \lambda(x^{(\ell)}) = \lambda_1(x_*, \omega_*),$$

the continuity of $\lambda_1(x, \omega)$ implies $\omega^{(\ell)} \rightarrow \omega_*$. Choose ℓ large enough so that $(x^{(\ell)}, \omega^{(\ell)}) \in \widehat{\mathcal{N}}$ and so by part (ii) of Theorem 4.3.5

$$\frac{\partial^2 \lambda_1^{(\ell)}(x^{(\ell)}, \omega^{(\ell)})}{\partial \omega^2} \leq \frac{\partial^2 \lambda_1(x^{(\ell)}, \omega^{(\ell)})}{\partial \omega^2} \leq 3\delta/4$$

for all such ℓ large enough. Finally, since the third derivatives of $\lambda_1^{(\ell)}(x, \omega)$ are also bounded uniformly in $\widehat{\mathcal{N}}$ by a constant independent of ℓ , there must exist a neighborhood $\widetilde{\mathcal{N}} \subseteq \widehat{\mathcal{N}}$ of (x_*, ω_*) such that

$$\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial \omega^2} \leq \delta/2$$

for all $(x, \omega) \in \widetilde{\mathcal{N}}$ and all ℓ sufficiently large. □

The next result follows from the last lemma and establishes the existence of the functions $\omega(x)$ and $\omega^{(\ell)}(x)$ that are the maximizers of the functions $\lambda_1(x, \omega)$ and $\lambda_1^{(\ell)}(x, \omega)$ with respect to ω , and admit local smooth representations in terms of x for x near x_* .

Theorem 4.3.9. *Suppose (x_*, ω_*) is as in Lemma 4.3.8 and satisfies the assumptions of that lemma. For some $\varrho_1, \varrho_2 > 0$, the following hold:*

- (i) *There exists a unique three times differentiable function $\tilde{\omega}(x) : \mathcal{B}(x_*, \varrho_1) \rightarrow \mathcal{B}(\omega_*, \varrho_2)$ such that $\tilde{\omega}(x_*) = \omega_*$ and*

$$\frac{\partial \lambda_1(x, \tilde{\omega}(x))}{\partial \omega} = 0 \quad \text{and} \quad \frac{\partial^2 \lambda_1(x, \tilde{\omega}(x))}{\partial \omega^2} < \delta/2 \quad (4.19)$$

for all $x \in \mathcal{B}(x_*, \varrho_1)$. Furthermore $\tilde{\omega}(x)$ is the unique maximizer of $\lambda_1(x, \omega)$ over $\omega \in \mathcal{B}(\omega_*, \varrho_2)$ for all $x \in \mathcal{B}(x_*, \varrho_1)$.

- (ii) *For all ℓ large enough satisfying $(x^{(\ell)}, \omega^{(\ell)}) \in \mathcal{N}(x_*, \omega_*; \varrho_1, \varrho_2)$, there exists a unique three times differentiable function $\tilde{\omega}^{(\ell)}(x) : \mathcal{B}(x_*, \varrho_1) \rightarrow \mathcal{B}(\omega_*, \varrho_2)$ such that $\tilde{\omega}^{(\ell)}(x^{(\ell)}) = \omega^{(\ell)}$ and*

$$\frac{\partial \lambda_1^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x))}{\partial \omega} = 0 \quad \text{and} \quad \frac{\partial^2 \lambda_1^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x))}{\partial \omega^2} < \delta/2 \quad (4.20)$$

for all $x \in \mathcal{B}(x_*, \varrho_1)$. Furthermore $\tilde{\omega}^{(\ell)}(x)$ is the unique maximizer of $\lambda_1^{(\ell)}(x, \omega)$ over $\omega \in \mathcal{B}(\omega_*, \varrho_2)$ for all $x \in \mathcal{B}(x_*, \varrho_1)$.

Proof. We establish the existence of the neighborhood $\mathcal{N}(x_*, \omega_*; \epsilon_1, \epsilon_2)$ in Lemma 4.3.8 such that for all $(x, \omega) \in \mathcal{N}(x_*, \omega_*; \epsilon_1, \epsilon_2)$, the eigenvalue functions $\lambda_1(x, \omega)$ and $\lambda_1^{(\ell)l}(x, \omega)$ for large ℓ are simple and their second derivatives with respect to ω are bounded from above by $\delta/2$ uniformly. It follows from by implicit function theorem that for some $\varrho_1 < \epsilon_1$, $\varrho_2 < \epsilon_2$, there exist a unique function $\tilde{\omega}(x)$ satisfying (4.19) and a unique function $\tilde{\omega}^{(\ell)}(x)$ satisfying (4.20) for all ℓ large enough, in particular $(x^{(\ell)}, \omega^{(\ell)}) \in \mathcal{N}(x_*, \omega_*; \epsilon_1, \epsilon_2)$. Note that since $\lambda_1(x, \omega)$ and $\lambda_1^{(\ell)l}(x, \omega)$ are simple, hence real analytic in $\mathcal{N}(x_*, \omega_*; \epsilon_1, \epsilon_2)$, the functions $\tilde{\omega}(x)$ and $\tilde{\omega}^{(\ell)}(x)$ are three times differentiable. □

In the next result, we show that the unique maximizers $\omega(x)$ and $\omega^{(\ell)}(x)$ in Theorem 4.3.9 are indeed the global maximizers where $\lambda(x)$ and $\lambda^{(\ell)}(x)$ are attained for all x close to x_* . Furthermore, under the smoothness assumption on $\lambda(x)$ at x_* , we find a ball centered at x_* in which $\lambda(x)$ and $\lambda^{(\ell)}(x)$ for all ℓ large enough are smooth. Note that from here on the notations $\lambda_2(x, \omega), \lambda_2^{(\ell)}(x, \omega)$ stand for the second largest eigenvalues of $M(x, \omega), M^{\mathcal{V}_\ell}(x, \omega)$, respectively.

Theorem 4.3.10. *Suppose that the sequence $\{x^{(\ell)}\}$ by Algorithm 10 or Algorithm 11 converges to a point x_* that is strictly in the interior of Ω and that $\lambda(x_*)$ is attained at a unique ω_* , the eigenvalue $\lambda_1(x_*, \omega_*)$ is simple and $\partial^2 \lambda_1(x_*, \omega_*) / \partial \omega^2 < 0$. Let ϱ_1 , $\tilde{\omega}(x)$, and $\tilde{\omega}^{(\ell)}(x)$ for all ℓ large enough, say $\ell > \ell_1$, be as in Theorem 4.3.9. Then there exist an $\varrho < \varrho_1$ and an $\epsilon > 0$ satisfying the following:*

- (i) *We have $\lambda_1(x, \tilde{\omega}(x)) - \lambda_2(x, \tilde{\omega}(x)) \geq \epsilon$ and $\lambda(x)$ is uniquely attained at $\tilde{\omega}(x)$ for all $x \in \mathcal{B}(x_*, \varrho)$;*
- (ii) *For all ℓ large enough, say $\ell > \ell_2 > \ell_1$ and for all $x \in \mathcal{B}(x_*, \varrho)$, we have $\lambda_1^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x)) - \lambda_2^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x)) \geq \epsilon$ and $\lambda^{(\ell)}(x)$ is uniquely attained at $\tilde{\omega}^{(\ell)}(x)$.*

Proof. It follows from the arguments in Lemma 2.2.8 that due the simplicity of $\lambda_1(x_*, \omega_*)$ and the interpolation properties $\lambda_1(x^{(\ell)}, \omega^{(\ell)}) = \lambda_1^{(\ell)}(x^{(\ell)}, \omega^{(\ell)})$, there exists an $\epsilon > 0$ and a neighborhood $\mathcal{N}(x_*, \omega_*; \tilde{\varrho}_1, \tilde{\varrho}_2)$ such that

$$\lambda_1(x, \omega) - \lambda_2(x, \omega) \geq \epsilon \quad \forall x \in \mathcal{N}(x_*, \omega_*; \tilde{\varrho}_1, \tilde{\varrho}_2) \quad (4.21)$$

and, for all ℓ large enough,

$$\lambda_1^{(\ell)}(x, \omega) - \lambda_2^{(\ell)}(x, \omega) \geq \epsilon \quad \forall x \in \mathcal{N}(x_*, \omega_*; \tilde{\varrho}_1, \tilde{\varrho}_2). \quad (4.22)$$

Furthermore, by Theorem 4.3.9, there exist $\varrho_1 \leq \tilde{\varrho}_1$, $\varrho_2 \leq \tilde{\varrho}_2$ and $\tilde{\omega}(x)$ satisfying assertion (i) as well as $\tilde{\omega}^{(\ell)}(x)$ for ℓ large enough, say $\ell > \ell_1$, satisfying assertion (ii) of that theorem.

First, we prove that $\lambda(x)$ is maximized uniquely at $\tilde{\omega}(x)$ for all x in a ball centered at x_* . For each $x \in \mathcal{B}(x_*, \varrho_1)$, we define

$$\begin{aligned}\delta_1(x) &:= \max \{ \lambda_1(x, \omega) \mid \omega \in \mathcal{B}(\omega_*, \varrho_2) \}, \\ \delta_2(x) &:= \sup \{ \lambda_1(x, \omega) \mid \omega \in \mathcal{B} \setminus \mathcal{B}(\omega_*, \varrho_2) \}, \\ \delta(x) &:= \delta_1(x) - \delta_2(x).\end{aligned}$$

At the point x_* , we have $\delta_* := \delta(x_*) = \delta_1(x_*) - \delta_2(x_*) = \lambda(x_*) - \delta_2(x_*) > 0$. Let us consider the ball $\mathcal{B}(x_*, \delta_*/8\zeta)$, where the constant ζ is as in Lemma 4.3.3. Then for each $x \in \mathcal{B}(x_*, \varrho)$ with $\varrho := \min \{ \delta_*/8\zeta, \varrho_1 \}$ from part (ii) of Lemma 4.3.3 we have

$$\delta_1(x_*) - \delta_1(x) \leq \delta_*/8 \quad \text{and} \quad \delta_2(x) - \delta_2(x_*) \leq \delta_*/8,$$

implying

$$\delta(x) = \delta_1(x) - \delta_2(x) \geq \delta_1(x_*) - \delta_2(x_*) - \delta_*/4 = 3\delta_*/4. \quad (4.23)$$

This means that the global maximizer of $\lambda_1(x, \omega)$ over $\mathcal{B}$ must be some point in the interior of $\mathcal{B}(\omega_*, \varrho_2)$. Since by Theorem 4.3.9, the point $\tilde{\omega}(x)$ is the unique maximizer of $\lambda_1(x, \omega)$ over $\omega \in \mathcal{B}(\omega_*, \varrho_2)$ and $\lambda_1(x, \tilde{\omega}(x)) = \delta_1(x) \geq \delta_2(x) + 3\delta_*/4$ from (4.23), this global maximizer has to be $\tilde{\omega}(x)$. Moreover, since $\varrho \leq \varrho_1 \leq \tilde{\varrho}_1$ as well as $\tilde{\omega}(x) \in \mathcal{B}(\omega_*, \varrho_2)$ with $\varrho_2 \leq \tilde{\varrho}_2$, it follows from (4.21) that

$$\lambda_1(x, \tilde{\omega}(x)) - \lambda_2(x, \tilde{\omega}(x)) \geq \epsilon \quad \forall x \in \mathcal{B}(x_*, \varrho).$$

This completes the proof of assertion (i).

Now, the uniqueness of ω_* and the continuity of $\lambda_1(x, \omega)$ imply that $\omega^{(\ell)} \rightarrow \omega_*$. Hence, as $x^{(\ell)} \rightarrow x_*$, there exists $\ell_2 \geq \ell_1$ such that $x^{(\ell)} \in \mathcal{B}(x_*, \varrho)$ and $\omega^{(\ell)} \in \mathcal{B}(\omega_*, \varrho_2)$ for all $\ell \geq \ell_2$. We prove assertion (ii) for all $\ell \geq \ell_2$. For such an ℓ and for an $x \in \mathcal{B}(x_*, \varrho)$, we define

$$\begin{aligned}\delta_1^{\mathcal{V}_\ell}(x) &:= \max \{ \lambda_1^{(\ell)}(x, \omega) \mid \omega \in \mathcal{B}(\omega_*, \varrho_2) \}, \\ \delta_2^{\mathcal{V}_\ell}(x) &:= \sup \{ \lambda_1^{(\ell)}(x, \omega) \mid \omega \in \mathcal{B} \setminus \mathcal{B}(\omega_*, \varrho_2) \}, \\ \delta^{\mathcal{V}_\ell}(x) &:= \delta_1^{\mathcal{V}_\ell}(x) - \delta_2^{\mathcal{V}_\ell}(x).\end{aligned}$$

Due to the monotonicity property $\lambda_1(x, \omega) \geq \lambda_1^{(\ell)}(x, \omega)$ for all ω , we have $\delta_j(x) \geq \delta_j^{\mathcal{V}_\ell}(x)$ for $j = 1, 2$. Furthermore, due to Theorem 4.3.2 (in particular due to equation (4.16)).

$$\delta_1^{\mathcal{V}_\ell}(x^{(\ell)}) = \lambda_1^{(\ell)}(x^{(\ell)}, \omega^{(\ell)}) = \lambda_1(x^{(\ell)}, \omega^{(\ell)}) = \delta_1(x^{(\ell)})$$

These observations lead us to

$$\delta^{\mathcal{V}_\ell}(x^{(\ell)}) = \delta_1^{\mathcal{V}_\ell}(x^{(\ell)}) - \delta_2^{\mathcal{V}_\ell}(x^{(\ell)}) \geq \delta_1(x^{(\ell)}) - \delta_2(x^{(\ell)}) \geq 3\delta_*/4, \quad (4.24)$$

where the last bound is due to (4.23). Now for each $x \in \mathcal{B}(x_*, \varrho)$, recalling $\varrho \leq \delta_*/8\zeta$ by part (ii) of Lemma 4.3.3, we have

$$\begin{aligned} \delta_1^{\mathcal{V}_\ell}(x^{(\ell)}) - \delta_1^{\mathcal{V}_\ell}(x) &\leq \zeta \|x - x^{(\ell)}\|_2 \leq \zeta (\|x - x_*\|_2 + \|x_* - x^{(\ell)}\|_2) \leq \delta_*/4 \quad \text{and} \\ \delta_2^{\mathcal{V}_\ell}(x) - \delta_2^{\mathcal{V}_\ell}(x^{(\ell)}) &\leq \zeta \|x^{(\ell)} - x\|_2 \leq \zeta (\|x^{(\ell)} - x_*\|_2 + \|x_* - x\|_2) \leq \delta_*/4 \end{aligned}$$

which gives rise to

$$\delta^{\mathcal{V}_\ell}(x) = \delta_1^{\mathcal{V}_\ell}(x) - \delta_2^{\mathcal{V}_\ell}(x) \geq \delta_1^{\mathcal{V}_\ell}(x^{(\ell)}) - \delta_2^{\mathcal{V}_\ell}(x^{(\ell)}) - \delta_*/2 \geq \delta_*/4,$$

where the last lower bound follows from (4.24). Hence the global maximizer of $\lambda_1^{(\ell)}(x, \omega)$ over $\omega \in \mathcal{B}$ also lies strictly in the interior of $\mathcal{B}(\omega_*, \varrho_2)$. Once again due to assertion (ii) of Theorem 4.3.9 and the inequality $\delta_1^{\mathcal{V}_\ell}(x) \geq \delta_2^{\mathcal{V}_\ell}(x) + \delta_*/4$, this maximizer has to be $\tilde{\omega}^{(\ell)}(x)$. Note that since $\varrho \leq \tilde{\varrho}_1$ and $\tilde{\omega}^{(\ell)}(x) \in \mathcal{B}(\omega_*, \varrho_2)$ with $\varrho_2 \leq \tilde{\varrho}_2$, we obtain the following uniform gap from (4.22):

$$\lambda_1^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x)) - \lambda_2^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x)) \geq \epsilon \quad \forall x \in \mathcal{B}(x_*, \varrho).$$

This completes the proof of assertion (ii). □

The next result establishes the boundedness of higher order derivatives of $\lambda^{(\ell)}(x)$ in absolute value by a constant independent of ℓ for each x inside the ball $\mathcal{B}(x_*, \varrho)$ specified by Theorem 4.3.10.

Lemma 4.3.11. *Suppose that the sequence $\{x^{(\ell)}\}$ by Algorithm 10 or Algorithm 11 converges to a point x_* that satisfies the assumptions in Theorem 4.3.10. There exist $\varrho > 0$ and $\xi > 0$ such that for all ℓ large enough, we have*

$$\left| \frac{\partial^2 \lambda^{(\ell)}(x)}{\partial x_p \partial x_q} \right| \leq \xi \quad \text{and} \quad \left| \frac{\partial^3 \lambda^{(\ell)}(x)}{\partial x_p \partial x_q \partial x_s} \right| \leq \xi \quad \forall x \in \mathcal{B}(x_*, \varrho)$$

for $p, q, s = 1, \dots, d$.

Proof. By Theorems 4.3.9 and 4.3.10, the function $\lambda^{(\ell)}(x) = \lambda_1^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x))$ for all ℓ large enough are three times differentiable for all $x \in \mathcal{B}(x_*, \varrho)$. We first prove that

$$\left| \frac{\partial^2 \lambda^{(\ell)}(x)}{\partial x_p \partial x_q} \right| \leq \xi \quad \forall x \in \mathcal{B}(x_*, \varrho) \quad (4.25)$$

for $p, q = 1, \dots, d$. For a given $\tilde{x} \in \mathcal{B}(x_*, \varrho)$, observe that

$$\frac{\partial^2 \lambda^{(\ell)}(\tilde{x})}{\partial x_p \partial x_q} = \left[\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_p \partial x_q} + \frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_p \partial \omega} \frac{\partial \tilde{\omega}^{(\ell)}(x)}{\partial x_q} \right] \Big|_{(x, \omega) = (\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))}. \quad (4.26)$$

The implicitly defined function $\tilde{\omega}^{(\ell)}(x)$ satisfies

$$\frac{\partial \lambda_1^{(\ell)}(x, \tilde{\omega}^{(\ell)}(x))}{\partial \omega} = 0$$

for all $x \in \mathcal{B}(x_*, \varrho)$. Differentiating this equation with respect to x_q at $(x, \omega) = (\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))$ yields

$$\frac{\partial \tilde{\omega}^{(\ell)}(\tilde{x})}{\partial x_q} = - \left[\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_q \partial \omega} \Big/ \frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial \omega^2} \right] \Big|_{(x, \omega) = (\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))}.$$

By plugging this expression in (4.26), we obtain

$$\frac{\partial^2 \lambda^{(\ell)}(\tilde{x})}{\partial x_p \partial x_q} = \left[\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_p \partial x_q} - \frac{\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_p \partial \omega} \frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_q \partial \omega}}{\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial \omega^2}} \right] \Big|_{(x, \omega) = (\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))}, \quad (4.27)$$

where by part (ii) of Theorem 4.3.9, we have

$$\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial \omega^2} \leq \frac{\delta}{2}$$

for a negative δ that is independent of ℓ . Hence to obtain (4.25), it suffices to show that the mixed second derivatives of $\lambda_1^{(\ell)}(x, \omega)$ are bounded above in absolute value.

By employing the analytical formulas for the second derivatives of eigenvalue functions [47], we have

$$\frac{\partial^2 \lambda_1^{(\ell)}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))}{\partial x_p \partial x_q} = \left\{ v_1^* \frac{\partial^2 M^{\nu_\ell}(x, \omega)}{\partial x_p \partial x_q} v_1 + 2 \sum_{k=2}^n \frac{1}{\lambda_1^{(\ell)}(x, \omega) - \lambda_k^{(\ell)}(x, \omega)} \times \right. \\ \left. \mathcal{R} \left[\left(v_1^* \frac{\partial M^{\nu_\ell}(x, \omega)}{\partial x_p} v_k \right) \left(v_k^* \frac{\partial M^{\nu_\ell}(x, \omega)}{\partial x_q} v_1 \right) \right] \right\} \Big|_{(x, \omega) = (\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))}$$

where $\lambda_k^{(\ell)}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))$ denotes the k th largest eigenvalue of $M^{\nu_\ell}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))$ associated with a unit eigenvector v_k . This leads to

$$\left| \frac{\partial^2 \lambda_1^{(\ell)}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))}{\partial x_p \partial x_q} \right| \leq \left\{ \left\| \frac{\partial^2 M^{\nu_\ell}(x, \omega)}{\partial x_p \partial x_q} \right\|_2 + 2 \sum_{k=2}^n \frac{1}{\lambda_1^{(\ell)}(x, \omega) - \lambda_k^{(\ell)}(x, \omega)} \times \right. \\ \left. \left\| \frac{\partial M^{\nu_\ell}(x, \omega)}{\partial x_p} \right\|_2 \left\| \frac{\partial M^{\nu_\ell}(x, \omega)}{\partial x_q} \right\|_2 \right\} \Big|_{(x, \omega) = (\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))} \quad (4.28)$$

In (4.28), by part (ii) of Theorem 4.3.10, we have the bound

$$\lambda_1^{(\ell)l}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x})) - \lambda_k^{(\ell)}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x})) \geq \epsilon$$

for some $\epsilon > 0$ independent of ℓ . Additionally, the norms of $M^{\nu_\ell}(x, \omega)$ and its first two derivatives with respect to components x at $\omega = \tilde{\omega}^{(\ell)}(x)$ can be bounded from above by a constant in terms of $\|A_j\|_2$ and the maxima of the absolute values of $f_j(x, \omega)$ and its first two derivatives over all $x \in \mathcal{B}(x_*, \varrho)$ and $\omega \in \mathcal{B}(\omega_*, \varrho_2)$, where ϱ_2 is as in Theorem 4.3.9. Hence the right hand side of (4.28) can be bounded from above by a constant that does not depend on ℓ .

Similarly, we can obtain bounds for the mixed derivatives

$$\frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_p \partial \omega} \quad \text{and} \quad \frac{\partial^2 \lambda_1^{(\ell)}(x, \omega)}{\partial x_q \partial \omega}$$

in (4.27) at $(x, \omega) = (\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x}))$ independent of ℓ .

The third derivative of $\lambda^{(\ell)}(x)$ at $x = \tilde{x}$ can be obtained by differentiating (4.27) with respect to x_s . The resulting expression can be written in terms of the first three derivatives of $\lambda_1^{(\ell)}(x, \omega)$. As we showed for the second derivative of $\lambda^{(\ell)}(x)$, the third

derivative can be bounded from above in terms of a quantity that only depends on the uniform lower bound

$$\lambda_1^{(\ell)}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x})) - \lambda_k^{(\ell)}(\tilde{x}, \tilde{\omega}^{(\ell)}(\tilde{x})) \geq \epsilon,$$

as well as uniform upper bounds on the first three derivatives of $M^{\mathcal{V}_\ell}(x, \omega)$ that are independent of ℓ . $\square$

Note that there are no interpolation properties between the second and higher order derivatives of $\lambda(x)$ and $\lambda^{(\ell)}(x)$. However, by the Hermite interpolation property (Theorems 4.3.2, 4.3.6 and 4.3.7) and Lemma 4.3.11, we can obtain an upper bound for the gap between the second derivatives of $\lambda(x)$ and $\lambda^{(\ell)}(x)$. This is stated in the next result. For the proof, we refer to Lemma 2.2.8 in which $\lambda(x)$, $\lambda^{(\ell)}(x)$ take the role of $\lambda_J(\omega)$, $\lambda_J^\ell(\omega)$.

Lemma 4.3.12. *Suppose that the sequence $\{x^{(\ell)}\}$ by Algorithm 10 when $d = 1$ or Algorithm 11 converges to a point x_* that satisfies the assumptions in Theorem 4.3.10. Then there exists a constant $C > 0$ such that for all ℓ large enough*

$$\|\nabla^2 \lambda(x^{(\ell)}) - \nabla^2 \lambda^{(\ell)}(x^{(\ell)})\|_2 \leq Ch^{(\ell)}, \quad (4.29)$$

where we define $h^{(\ell)} := |x^{(\ell)} - x^{(\ell-1)}|$ for Algorithm 10 when $d = 1$. Furthermore, if $\nabla^2 \lambda(x_*)$ is invertible, then for all ℓ large enough

$$\left\| [\nabla^2 \lambda(x^{(\ell)})]^{-1} - [\nabla^2 \lambda^{(\ell)}(x^{(\ell)})]^{-1} \right\| \leq Ch^{(\ell)}. \quad (4.30)$$

Now, we are ready to prove the superlinear rate-of-convergence result which is a consequence of the Hermite interpolation results and Lemma 4.3.12 that relates the second derivatives of $\lambda(x)$ and $\lambda^{(\ell)}(x)$. A formal statement and its proof are given below.

Theorem 4.3.13. *Suppose that the sequence $\{x^{(\ell)}\}$ by Algorithm 10 when $d = 1$ or Algorithm 11 converges to a point x_* that is strictly in the interior of Ω such that $\lambda(x_*)$*

is attained at a unique ω_* , the eigenvalue $\lambda_1(x_*, \omega_*)$ is simple, $\partial^2 \lambda_1(x_*, \omega_*) / \partial \omega^2 < 0$ and $\nabla^2 \lambda(x_*)$ is invertible. Then there exists a constant $\mu > 0$ such that

$$\frac{\|x^{(\ell+1)} - x_*\|_2}{\|x^{(\ell)} - x_*\|_2 \max \{\|x^{(\ell)} - x_*\|_2, \|x^{(\ell-1)} - x_*\|_2\}} \leq \mu \quad \ell \geq 2. \quad (4.31)$$

Proof. By Theorem 4.3.10, we have a ball $\mathcal{B}(x_*, \varrho)$ such that $\lambda(x)$ and $\lambda^{(\ell)}(x)$ for all ℓ large enough are simple for all $x \in \mathcal{B}(x_*, \varrho)$. That means $\lambda(x)$ and $\lambda^{(\ell)}(x)$ are twice Lipschitz continuously differentiable. As $x^{(\ell)} \rightarrow x_*$ and $h^{(\ell)} := \|x^{(\ell)} - x^{(\ell-1)}\| \rightarrow 0$, for all ℓ large enough, say $\ell \geq \ell_1$, $\mathcal{B}(x^{(\ell)}, h^{(\ell)}) \subseteq \mathcal{B}(x_*, \varrho)$. Furthermore the invertibility of $\nabla^2 \lambda(x_*)$ ensures that $\nabla^2 \lambda(x^{(\ell)})$ and $\nabla^2 \lambda^{(\ell)}(x^{(\ell)})$ are invertible (indeed the norms of inverses of these Hessians are bounded above from above by a constant) for all ℓ large enough, say $\ell \geq \ell_2 \geq \ell_1$.

Now, for $\ell \geq \ell_2$, by Taylor's theorem with integral remainder

$$0 = \nabla \lambda(x_*) = \nabla \lambda(x^{(\ell)}) + \int_0^1 \nabla^2 \lambda(x^\ell + t(x_* - x^{(\ell)}))(x_* - x^{(\ell)}) dt.$$

Multiplying the both sides of this equation from left by $[\nabla^2 \lambda(x^{(\ell)})]^{-1}$ and employing the interpolation property $\nabla \lambda(x^{(\ell)}) = \nabla \lambda^{(\ell)}(x^{(\ell)})$ yield

$$\begin{aligned} 0 &= [\nabla^2 \lambda^{(\ell)}(x^{(\ell)})]^{-1} \nabla \lambda^{(\ell)}(x^{(\ell)}) + \\ &\quad \left\{ [\nabla^2 \lambda(x^{(\ell)})]^{-1} - [\nabla^2 \lambda^{(\ell)}(x^{(\ell)})]^{-1} \right\} + (x_* - x^{(\ell)}) + \\ &\quad [\nabla^2 \lambda(x^{(\ell)})]^{-1} \int_0^1 [\nabla^2 \lambda(x^\ell + t(x_* - x^{(\ell)})) - \nabla^2 \lambda(x^{(\ell)})] (x_* - x^{(\ell)}) dt \end{aligned} \quad (4.32)$$

As $x^{(\ell+1)}$ is a global minimizer of $\lambda^{(\ell)}(x)$, a second Taylor expansion of $\nabla \lambda^{(\ell)}(x)$ about $x^{(\ell)}$ implies

$$\begin{aligned} 0 &= \nabla \lambda^{(\ell)}(x^{(\ell+1)}) = \nabla \lambda^{(\ell)}(x^{(\ell)}) + \nabla^2 \lambda^{(\ell)}(x^{(\ell)})(x^{(\ell+1)} - x^{(\ell)}) + O([h^{(\ell+1)}]^2) \\ \implies &\quad [\nabla^2 \lambda^{(\ell)}(x^{(\ell)})]^{-1} \nabla \lambda^{(\ell)}(x^{(\ell)}) = -(x^{(\ell+1)} - x^{(\ell)}) + O([h^{(\ell+1)}]^2). \end{aligned}$$

Using this equality in (4.32) and taking 2-norms lead to

$$\begin{aligned} \|x^{(\ell+1)} - x_*\|_2 &\leq \left\| [\nabla^2 \lambda(x^{(\ell)})]^{-1} - [\nabla^2 \lambda^{(\ell)}(x^{(\ell)})]^{-1} \right\|_2 \|\nabla \lambda^{(\ell)}(x^{(\ell)})\|_2 \\ &\quad + O([h^{(\ell+1)}]^2) + \left\| [\nabla^2 \lambda(x^{(\ell)})]^{-1} \right\|_2 \\ &\quad \times \int_0^1 \left\| \nabla^2 \lambda(x^\ell + t(x_* - x^{(\ell)})) - \nabla^2 \lambda(x^{(\ell)}) \right\|_2 \|x_* - x^{(\ell)}\|_2 dt \end{aligned} \quad (4.33)$$

Now Lipschitz continuity of $\nabla^2 \lambda(x)$, the boundedness of $\left\| [\nabla^2 \lambda(x^{(\ell)})]^{-1} \right\|_2$ and Lemma 4.3.12 give rise to

$$\begin{aligned} \|x^{(\ell+1)} - x_*\|_2 &\leq C_1 \|x^{(\ell)} - x_*\|_2 \max \{ \|x^{(\ell)} - x_*\|_2, \|x^{(\ell-1)} - x_*\|_2 \} \\ &\quad + C_2 \|x_* - x^{(\ell)}\|_2^2 \end{aligned}$$

for some constants C_1, C_2 independent of ℓ . The desired result (4.31) follows from this inequality. $\square$

4.3.3 Numerical Experiments

In this section, we report numerical results obtained from several experiments to minimize the numerical radius $r(A(x))$ when $A(x)$ is affine and x is subject to box constraints. At each iteration of Algorithm 10, the numerical radius of the reduced matrix needs to be minimized and we need to compute the numerical radius of the large-scale matrix-valued function at the optimal point. The reduced optimization problems are solved by means of the support-based algorithm and semidefinite representation of numerical radius, in particular the Matlab packages `eigopt` [59] and `CVX` [28], respectively. We use the support based algorithm, in particular `eigopt` [59], to perform large-scale numerical radius computation.

The subspace framework is terminated, when the distance between $\lambda^{(\ell)}(x^{(\ell+1)})$ and $\lambda^{(\ell-1)}(x^{(\ell)})$ is less than a prescribed tolerance for some $\ell > 1$. Formally, we terminate when

$$|\lambda^{(\ell)}(x^{(\ell+1)}) - \lambda^{(\ell-1)}(x^{(\ell)})| < \text{tol}.$$

For the numerical experiments, we always set $\text{tol} = 10^{-12}$, unless otherwise specified.

Illustration of the Convergence of Subspace Framework: We first show the convergence of Algorithm 10 to illustrate the convergence results in Section 4.3.2 on the minimization of the numerical radius $r(A + kbc^T)$ with a random matrix $A \in \mathbb{R}^{200 \times 200}$ and random vectors $b, c \in \mathbb{R}^{200 \times 1}$. The computed minimal value by the subspace framework is $r_* = 7.0233612466$ and the corresponding global minimizer

ℓ	$x^{(\ell+1)}$	$ x^{(\ell+1)} - x_* $	$r(A^{\mathcal{V}_\ell}(x^{(\ell+1)}))$	$r_* - r(A^{\mathcal{V}_\ell}(x^{(\ell+1)}))$
5	0.0175986654	0.0017661097	7.0217876708	0.0015735758
6	0.0193825278	0.0000177525	7.0233598272	0.0000014194
7	0.0193612678	0.0000035073	7.0233612447	0.0000000019
8	0.0193647869	0.0000000117	7.0233612466	0.0000000000
9	0.0193647751	0.0000000001	7.0233612466	0.0000000000

Table 4.3: This table concerns an application of Algorithm 10 for the minimization of the numerical radius $r(A + kbc^T)$ over $k \in [-2, 2]$ for a random $A \in \mathbb{R}^{200 \times 200}$, random $b, c \in \mathbb{R}^{200}$. It lists the iterates $r(A^{\mathcal{V}_\ell}(x^{(\ell+1)}))$ and $x^{(\ell+1)}$ generated by Algorithm 10 and their errors. The notations r_* and x_* represent the minimal value of $r(A + kbc^T)$ over $k \in [-2, 2]$ and the corresponding global minimizer, respectively.

is $x_* = 0.0193647751$. In the left plot of Figure 4.3, we illustrate $r(A + kbc^T)$ as a function $k \in [-2, 2]$ and x_* is marked with the circle. Convergence to the global minimizer at a superlinear rate, consistent with what is expected in theory, is observed for this example. Table 4.3 lists the decay rate of the errors of the minimizers of the reduced numerical radius functions on the third column. In the same table, the errors of the minimal values of the reduced numerical radius functions are listed on the fifth column.

Performance of Subspace Framework: We test the performance of Algorithm 10 once again on minimization of $r(A + kbc^T)$ over $k \in [-2, 2]$ with random $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$ for $n = 200, 600, 1200, 1800$. In each example, we normalize the matrices A so that $\|A\|_2 = 10$. A superlinear rate-of-convergence is observed for each one of the examples. Figure 4.3 depicts the plots of the numerical radius functions for $n = 200$ (Left) and $n = 600$ (Right) with the computed global minimizers marked with dots. In these experiments, we aim to illustrate the factors determining the

efficiency of the subspace framework as the sizes of the matrices increase. In Table 4.4, we present the computed global minimizer and optimal numerical radius by the subspace framework, number of the subspace iterations and run-times in seconds to reach the specified accuracy. According to the table, the main factor that dominates the run-times as n increases is the computation of the large scale numerical radius a few times. The number of subspace iterations varies between 5 – 10, which is equal to the number of the numerical radius computation involving full problem. In essence, the solution of the reduced problem has negligible effect on the total time which is determined by these large-scale numerical radius computations.

n	# iter	total time	reduced prob	numer. rad.	k_*	r_*
200	9	23.684	0.8497	22.725	0.01936	7.02336
600	5	62.246	0.332	61.293	-0.00194	7.05362
1200	5	529.063	0.4901	525.047	-0.00019	7.10761
1800	10	4152.195	1.2861	4127.923	-0.00145	7.07701

Table 4.4: The performance of Algorithm 10 to minimize the numerical radius $r(A + kbc^T)$ with respect to the size n of a random square matrix A and random vectors b, c . The total run-time, time for the reduced problems, and the time for large-scale numerical radius computation are listed in seconds in the third, fourth, fifth columns. Additionally, the number of subspace iterations, the computed globally minimal value r_* of $r(A + kbc^T)$ over k , and the corresponding global minimizer k_* are reported in the second, seventh, and sixth columns, respectively.

4.4 Conclusion

In this chapter, we have studied the minimax optimization problems with the largest eigenvalue of a matrix-valued function appearing in the objective. While the outer minimization problem depends on several parameters, it is assumed that the inner

maximization problem depends on only one parameter. Such minimax type problems are abundant in literature, e.g. the minimization of the $\mathcal{H}_\infty$ -norm of a parameter dependent control system, the minimization of pseudospectral abscissa, the maximization of the distance to instability. Although our study is carried out in a general setting, we have also focused on a particular motivating application, namely the minimization of the numerical radius.

The algorithm in [60] based on piece-wise quadratic support functions is adapted to solve the problems at hand when the matrix-valued function is of small size. This approach enables us to reach a globally optimal solution. On the other hand, we solve the large-scale problems when the matrix-valued function is large by means of a subspace framework. This method projects the matrix-valued function into small subspaces and solves the resulting projected problem by employing the piece-wise quadratic support function based approach at every iteration. The subspace is chosen carefully in a way so that the framework converges globally at a superlinear rate with respect to the subspace dimension. We have presented numerical results for both small-scale and large-scale problems on examples arising from the minimization of the numerical radius. The numerical results confirm our findings in theory.

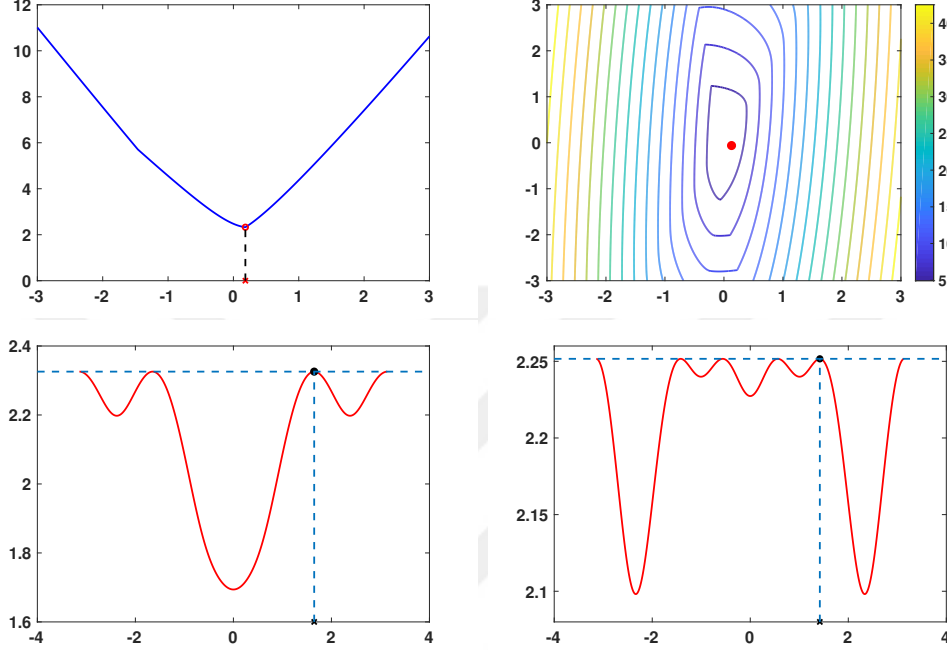


Figure 4.2: The plots concern the static output feedback control problem for one and two parameter cases. **(Top Left)** The plot of $r(A + kb_1c_1^T)$ as a function of $k \in [-3, 3]$. The computed global minimizer $k^{(1)} = 0.1832$ by Algorithm 9 is marked with the cross at the horizontal axis, whereas the dot marks $(k^{(1)}, r(A + k^{(1)}b_1c_1^T))$. **(Bottom Left)** The plot of $\lambda_1 \left\{ \left[(A + k^{(1)}b_1c_1^T)e^{i\omega} + (A + k^{(1)}b_1c_1^T)^*e^{-i\omega} \right] / 2 \right\}$ with respect to ω . The cross at the horizontal axis marks $\omega_* = 1.6483$, one of the ω values maximizing $\lambda_1 \left\{ \left[(A + k^{(1)}b_1c_1^T)e^{i\omega} + (A + k^{(1)}b_1c_1^T)^*e^{-i\omega} \right] / 2 \right\}$, while the dot marks $(\omega_*, \lambda_1 \left\{ \left[(A + k^{(1)}b_1c_1^T)e^{i\omega_*} + (A + k^{(1)}b_1c_1^T)^*e^{-i\omega_*} \right] / 2 \right\})$. The dashed horizontal line is the lowest horizontal line that bounds the graph of the eigenvalue function from above. **(Top Right)** The contour diagram of $r(A + k_1b_1c_1^T + k_2b_2c_2^T)$ with respect to $k_1 \in [-3, 3]$ in the horizontal axis and $k_2 \in [-3, 3]$ in the vertical axis. The global minimizer $(k_1^{(2)}, k_2^{(2)}) = (0.1288, -0.0608)$ is marked with a dot. **(Bottom Right)** The plot of the eigenvalue function $\lambda_1 \left\{ \left[(A + k_1^{(2)}b_1c_1^T + k_2^{(2)}b_2c_2^T)e^{i\omega} + (A + k_1^{(2)}b_1c_1^T + k_2^{(2)}b_2c_2^T)^*e^{-i\omega} \right] / 2 \right\}$ with respect to ω . The global maximizer marked with a cross in the horizontal axis is $\omega_* = 1.4192$. The dot and the dashed line stand for the quantities same as in bottom left.

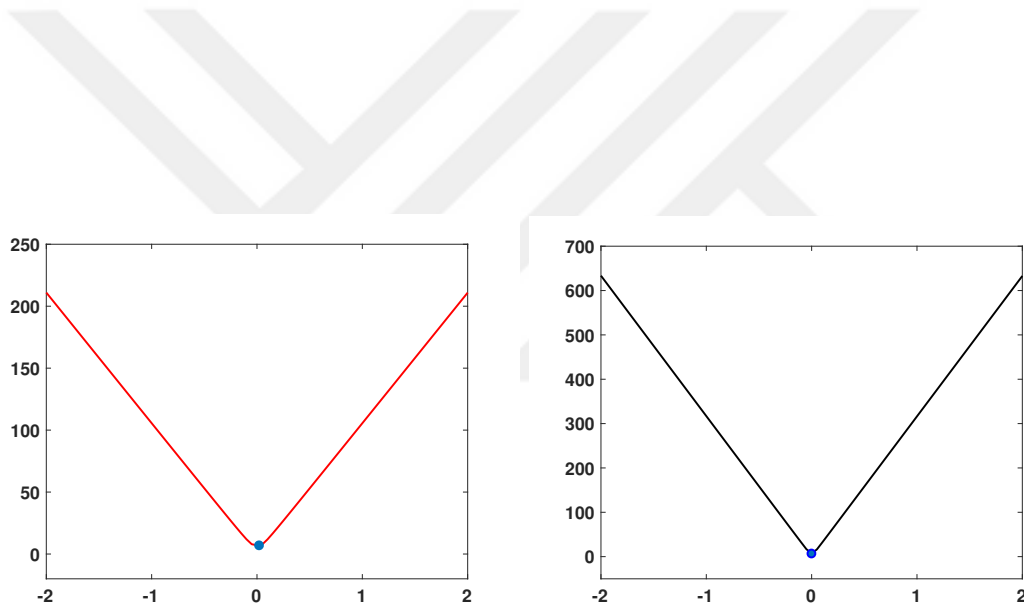


Figure 4.3: The figure depicts the plot of $r(A + kbc^T)$ as a function of $k \in [-2, 2]$ for **(Left)** $n = 200$ and **(Right)** $n = 600$. In each plot, the dot marks the computed global minimum.

Chapter 5

CONCLUSION AND FUTURE WORK

We have developed and analyzed algorithms for the optimization of a prescribed eigenvalue of a Hermitian and analytic matrix-valued function $A(\omega)$ dependent on several parameters over a set of permissible parameter values.

In the first part subspace procedures have been introduced to minimize and maximize the J th largest eigenvalue of $A(\omega)$ over ω in large-scale setting when $A(\omega)$ is of large size. The main idea of these procedures is to project $A(\omega)$ into small subspaces leading to small-scale eigenvalue optimization problems, which then can be solved by the existing algorithms. At every step of the subspace procedure, the subspace is expanded with addition of eigenvectors at the global optimizers of the small-scale projected problems, as well as, possibly, at nearby points in a way so that Hermite interpolation properties hold between the projected problem and the original large-scale problem. For the minimization of the J th largest eigenvalue, we have shown that the minimizers of the projected problems converge to a global minimizer of the original problem as the subspace dimension goes to infinity. This global convergence property does not hold for the maximization of the J th largest eigenvalue, for which we observe convergence to a local maximizer that is not necessarily a global maximizer. Additionally, we have proven that a superlinear rate-of-convergence occurs to the optimizers of the original problem with respect to the subspace dimension both for the minimization and for the maximization of the J th largest eigenvalue, but under the assumption that the eigenvalue function is simple at the converged optimizer. This superlinear rate-of-convergence result has an important practical consequence, namely that the original large-scale problem (for instance with size on the order of tens of thousands) can be approximated by a small-scale problem (with size on the

order of tens) very accurately. The convergence properties proven are also verified to hold in practice on various numerical experiments.

The second part has focused on the rate-of-convergence of two algorithms for the minimization of the largest eigenvalue of $A(\omega)$ depending on one parameter when nonsmoothness is present at the converged optimizer. The first one is the globally convergent algorithm in [60] based on piece-wise quadratic support functions that lie underneath the eigenvalue function globally, and that is effective especially for nonconvex eigenvalue optimization. The other one is a generalized version of the subspace procedure introduced in the first part in which potential nonsmoothness at the global optimizer (due to the non-simplicity of the eigenvalue at this point) is carefully taken care of. In this generalized subspace procedure, we expand the subspace by adding not only the eigenvector corresponding to the largest eigenvalue but also the eigenvectors corresponding to nearby eigenvalues. We have analyzed the rate-of-convergence of these two algorithms in the nonsmooth setting. Assuming the largest eigenvalue of $A(\omega)$ is multiple at the converged minimizer, and zero is strictly inside the generalized Clark derivative, for the former algorithm, we have proved that if the maximum of the errors of last two iterates generated by the algorithm is h , then the error of the next iterate is $O(h^2)$. Furthermore, in this nonsmooth setting, we have deduced that the iterates of the generalized subspace procedure converge to the minimizer at a quadratic rate. The validity of these quick rate-of-convergence results in practice are illustrated on several real examples concerning the computation of the inner numerical radius and that exhibit nonsmoothness at the minimizers.

The final part concerns the minimax optimization problems with the largest eigenvalue of a Hermitian matrix-valued function in the objective. We have proposed a globally convergent algorithm intended for small- up to medium-scale problems, which in essence is an adaptation of the algorithm in [60] based on piece-wise quadratic model functions. Additionally, the subspace procedure in the first part has been extended to deal with these minimax problems when the matrix-valued function has large size. Reminiscent of the approach in the first part, at every iteration, we first

solve a projected minimax problem, then expand the subspace with the addition of the eigenvectors corresponding to the largest eigenvalue at the point where the globally optimal value of the projected minimax problem is attained, and at nearby points. Once again, we have established a result in support of global convergence, i.e., the optimizers of the projected problems converge to the optimizer of the original problem as the subspaces are expanded. Furthermore, the rate of this convergence is again shown to be superlinear in the smooth case. These theoretical findings regarding the minimax problems have been confirmed in practice by numerical experiments that involve the minimization of the numerical radius.

Future Research. There are a few properties of the subspace procedures that we observe in practice, yet could not prove rigorously at this point. We list the most important ones below, and believe that these open problems are worth investigating.

1. The first issue indicated in Section 2.3.2 is the local convergence of the subspace procedures for the maximization of the J th largest eigenvalue. Although, we always observe the convergence of the subspace procedures to a local maximizer that is not necessarily a global maximizer in practice, a formal proof of this observation is open.
2. The second problem left open is a formal proof of the rate-of-convergence of the basic greedy subspace procedure, that is Algorithm 1, when the number of parameters is greater than one ($d > 1$). Even though numerical experiments indicate a superlinear rate-of-convergence of Algorithm 1 in this multi-variate case, this is not formally proven yet. A related issue is that how the number of parameters affects the rate-of-convergence of the basic greedy subspace procedure. A good understanding of this issue may give rise to an efficient subspace procedure for large-scale linear semidefinite programs, which is partly discussed in the subsequent paragraph.
3. We have shown that the generalized subspace procedure outlined in Algorithm 8 has a quadratic rate-of-convergence in the nonsmooth setting when there is

only one parameter. Extension of this rate-of-convergence analysis to the multi-variate case when there are more than one parameter, as well as extension to the J th largest eigenvalue (rather than the largest eigenvalue), are left open.

4. The rapid convergence result for the piece-wise quadratic support function based algorithm in [60] has been established in Chapter 3 in the nonsmooth setting in the context of the minimization of the largest eigenvalue dependent on one parameter. We could not so far generalize this result to the multi-variate case. Furthermore, in the smooth setting (when the eigenvalue is simple at the converged minimizer), we nearly almost always observe a linear rate-of-convergence, but leave a thorough analysis of this rate-of-convergence observation for future.

Another topic related to this thesis work is large-scale linear SDPs, in particular devising subspace frameworks for these problems. The current-state-of-art numerical algorithms for linear SDPs are still not capable of large problems, for instance those involving large matrix variables. In what follows we present a brief discussion of a connection between a linear SDP and eigenvalue optimization. Let S_n denote the space of $n \times n$ symmetric matrices with the inner product $\langle A, B \rangle := \text{tr}(B^T A)$ for $A, B \in S_n$. We denote the set of positive semidefinite matrices by S_n^+ and $A \succeq 0$ means that A is in S_n^+ . The primal-dual standard form of a linear semidefinite program are given by

$$\begin{array}{ll}
 \max & \langle C, X \rangle \\
 \text{(P)} \quad \text{s.t.} & \mathcal{A}(X) = b, \\
 & X \succeq 0
 \end{array}
 \qquad
 \begin{array}{ll}
 \min & b^T y \\
 \text{(D)} \quad \text{s.t.} & Z = \mathcal{A}^T(y) - C \\
 & Z \succeq 0
 \end{array}$$

where $C \in S_n, b \in \mathbb{R}^m$ are the cost matrix, the right-hand side vector, respectively, and $X \in S_n^+$ is the primal matrix variable, $y \in \mathbb{R}^m$ is the dual variable. Above the linear operator $\mathcal{A} : S_n \rightarrow \mathbb{R}^m$ and its adjoint operator $\mathcal{A}^T : \mathbb{R}^m \rightarrow S_n$ are defined by

$$\mathcal{A}(X) = \begin{bmatrix} \langle A_1, X \rangle \\ \vdots \\ \langle A_m, X \rangle \end{bmatrix}, \quad \mathcal{A}^T(y) = \sum_{j=1}^m y_j A_j \tag{5.1}$$

for given $A_1, \dots, A_m \in S_n$. Noting that

$$Z \succeq 0 \Leftrightarrow \lambda_{\min}(Z) \geq 0 \Leftrightarrow -\lambda_{\max}(-Z) \geq 0 \Leftrightarrow \lambda_{\max}(-Z) \leq 0,$$

and lifting this constraint to the objective function in the dual problem by means of a Lagrange multiplier $a \geq 0$, we obtain the following eigenvalue optimization problem

$$(\text{EOP}) \quad \min_{y \in \mathbb{R}^m} a\lambda_{\max}(C - \mathcal{A}^T(y)) + b^T y.$$

For the notational convenience, we denote the objective function by

$$f(y) := a\lambda_{\max}(C - \mathcal{A}^T(y)) + b^T y.$$

Now suppose that the identity matrix I_n is in the range of $\mathcal{A}^T$, i.e., $\exists \tilde{y} \in \mathbb{R}^m$ such that $\mathcal{A}^T(\tilde{y}) = I_n$. This property implies that the matrix $X \in S_n$ satisfying the constraints of (P) has constant trace, that is

$$\mathcal{A}(X) = b \Rightarrow \text{tr}(X) = \langle I_n, X \rangle = \langle \mathcal{A}^T(\tilde{y}), X \rangle = \langle \tilde{y}, \mathcal{A}(X) \rangle = b^T \tilde{y}.$$

Under this constant trace property, the dual semidefinite program (D) is equivalent to the eigenvalue optimization problem (EOP). This result is stated and proven formally in [33, Proposition 5.1.1]. We include the formal statement and a proof below for the sake of completeness.

Proposition 5.0.1. *Assume there exists $\tilde{y} \in \mathbb{R}^m$ such that $\mathcal{A}^T(\tilde{y}) = I_n$. Then (D) is equivalent to (EOP) for $a = \max\{0, b^T \tilde{y}\}$.*

Proof. Let $\tilde{y} \in \mathbb{R}^m$ satisfy $\mathcal{A}^T(\tilde{y}) = I_n$. Then for any $y \in \mathbb{R}^m$ and $\lambda \geq \lambda_{\max}(C - \mathcal{A}^T(y))$, the point $y + \lambda \tilde{y}$ is a feasible solution for the dual problem, as indeed,

$$\mathcal{A}^T(y + \lambda \tilde{y}) - C = \lambda \mathcal{A}^T(\tilde{y}) + \mathcal{A}^T(y) - C = \lambda I_n - (C - \mathcal{A}^T(y)) \succeq 0.$$

We consider two cases, namely $b^T \tilde{y} < 0$ and $b^T \tilde{y} \geq 0$.

If $b^T \tilde{y} < 0$, that is $a = 0$, then the objective values of (D) and (EOP) tend to $-\infty$ along $y + \lambda \tilde{y}$ as $\lambda \rightarrow \infty$.

Suppose that $b^T \tilde{y} = a \geq 0$. For any $y \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}
 f(y + \lambda \tilde{y}) &= a \lambda_{\max}(C - \mathcal{A}^T(y + \lambda \tilde{y})) + b^T(y + \lambda \tilde{y}) \\
 &= a \lambda_{\max}(C - \mathcal{A}^T(y) - \lambda I_n) + b^T y + \lambda a \\
 &= a \lambda_{\max}(C - \mathcal{A}^T(y)) - \lambda a + b^T y + \lambda a \\
 &= a \lambda_{\max}(C - \mathcal{A}^T(y)) + b^T y \\
 &= f(y).
 \end{aligned}$$

That is the objective of the problem (EOP) is constant along the direction $\lambda \tilde{y}$. If we set $\hat{\lambda} := \lambda_{\max}(C - \mathcal{A}^T(y))$, then $\lambda_{\max}(C - \mathcal{A}^T(y + \hat{\lambda} \tilde{y}))$ vanishes. Hence, the optimal point y_* of (EOP) satisfies

$$\begin{aligned}
 f(y_*) &= a \lambda_{\max}(C - \mathcal{A}^T(y_*)) + b^T y_* \\
 &= a \lambda_{\max}(C - \mathcal{A}^T(y_* + \hat{\lambda}_* \tilde{y})) + b^T(y_* + \hat{\lambda}_* \tilde{y}) \\
 &= b^T(y_* + \hat{\lambda}_* \tilde{y}) \\
 &\geq d_*
 \end{aligned}$$

where $\hat{\lambda}_* := \lambda_{\max}(C - \mathcal{A}^T(y_*))$ and d_* is the optimal objective value of (D). Conversely, let y_* be the optimal point of (D). Then since y_* is feasible for (D),

$$d_* = b^T y_* \geq b^T y_* + a \lambda_{\max}(C - \mathcal{A}^T(y_*)) \geq f_*$$

which proves the equivalence of (D) and (EOP). $\square$

Primal-dual interior point methods work well on semidefinite programs, but if the program is defined over large matrix variables or large number of constraints, the methods are very slow and require large memory. In [34], an iterative algorithm, called “the spectral bundle method”, was developed for solving large constrained linear semidefinite programs based on their characterization in (EOP). This method allows the solutions of high dimensional problems to a degree. However, its practical use is limited by the fact that it is only a first order method amounting to a slow convergence. A systematic adaptation of the subspace procedures proposed in Chapter 2 for

the solution of large-scale linear SDPs may be possible by exploiting the eigenvalue optimization characterization in (EOP). This is especially motivated by the good theoretical convergence properties of the subspace procedures. It is a problem that we would like to tackle in near future.



NOMENCLATURE

$\mathbb{C}$	: the set of complex numbers
$\mathbb{R}$	: the set of real numbers
$\mathbb{Z}$	: the set of integers
$\mathbb{N}$	: the set of nonnegative integers
$\mathbb{C}^n$	: the set of complex vectors with n components
$\mathbb{R}^n$	: the set of real vectors with n components
$\mathbb{C}^{n \times m}$	: the set of complex $n \times m$ matrices
$\mathbb{R}^{n \times m}$	: the set of real $n \times m$ matrices
$\sigma_{\min}(A)$	: the smallest singular value of the matrix A
$\sigma_{\max}(A)$	: the largest singular value of the matrix A
$\sigma_J(\omega)$	: the J th largest singular value of a matrix-valued function $A(\omega)$
$\sigma_{-J}(\omega)$	: the J th smallest singular value of a matrix-valued function $A(\omega)$
$\lambda_{\min}(A)$	: the smallest eigenvalue of the Hermitian matrix A
$\lambda_{\max}(A)$	: the largest eigenvalue of the Hermitian matrix A
$\lambda_J(\omega)$	: the J th largest eigenvalue of a Hermitian matrix-valued function $A(\omega)$
$\lambda_{-J}(\omega)$	: the J th smallest eigenvalue of a Hermitian matrix-value function $A(\omega)$
$\ A\ _2$	: the 2-norm or operator norm of the matrix A
$\det(A)$	: the determinant of the matrix A
$F(A)$	: the field of values of the matrix A
$r(A)$	: the numerical radius of the matrix A
$\zeta(A)$	: the inner numerical radius of the matrix A
$\mathcal{D}(A)$	: the distance to instability of the matrix A
I_n	: the $n \times n$ identity matrix
A^T	: the transpose of the matrix A

A^*	: the adjoint or complex conjugate transpose of the matrix A
A^{-1}	: the inverse of the matrix A
$\text{tr}(A)$	: the trace of the matrix A
S_n	: the vector space of real symmetric $n \times n$ matrices
S_n^+	: the cone of real symmetric positive semidefinite $n \times n$ matrices
$\ H\ _{\mathcal{H}_\infty}$	: the $\mathcal{H}_\infty$ -norm of the transfer function H
$\text{Col}(A)$	: the column space of the matrix A
$\text{orth}(A)$	: a matrix whose columns form an orthonormal basis for $\text{Col}(A)$
$\ v\ _2$	: the Euclidean norm of a vector v in $\mathbb{R}^n$ or $\mathbb{C}^n$
$\sup$	: supremum
$\inf$	: infimum
$\ell_2(\mathbb{N})$	: the vector space consisting of square summable infinite sequences of complex numbers

Appendix A

PROOF OF QUADRATIC CONVERGENCE FOR THE LEVEL-SET ALGORITHM

We assume $\lambda_1(\theta)$ is not constant near θ_* , as otherwise $r^{(j)} = \lambda_*$ for all j large enough and there is nothing to prove. Consequently, there exists an interval $[\theta_* - \delta, \theta_* + \delta]$ such that $\lambda_1(\theta)$ is strictly decreasing on $[\theta_* - \delta, \theta_*]$, and is strictly increasing on $[\theta_*, \theta_* + \delta]$. Furthermore, for j large enough, there exists an open interval $I_\ell^{(j)} \subset [\theta_* - \delta, \theta_* + \delta]$ satisfying $\lambda_1(\theta) < r^{(j)}$ for all $\theta \in I_\ell^{(j)}$ and $\lambda_1(\theta)$ attains the value $r^{(j)}$ at the endpoints of this interval, as well as $r^{(j+1)} = \lambda_1(\phi_\ell^{(j+1)})$ where $\phi_\ell^{(j+1)}$ is the midpoint of $I_\ell^{(j)}$. We assume $I_\ell^{(j)}$ is of the form $I_\ell^{(j)} = (\ell_\ell^{(j)}, u_\ell^{(j)})$ with $\ell_\ell^{(j)} < u_\ell^{(j)}$. If $I_\ell^{(j)}$ is not of this form, the argument below applies to the interval $(\ell_\ell^{(j)}, u_\ell^{(j)} + 2\pi)$.

Now we can choose $\delta > 0$ small enough if necessary so that $\lambda_1(\theta)$ has the Taylor expansion

$$\lambda_1(\theta) = \lambda_1(\theta_*) + \beta(\theta - \theta_*)^2 + O((\theta - \theta_*)^3) \quad (\text{A.1})$$

for all $\theta \in [\theta_* - \delta, \theta_* + \delta]$ where $\beta > 0$ due to the fact that θ_* is a smooth minimizer of $\lambda_1(\theta)$. The eigenvalue function $\lambda_1(\theta)$ is one-to-one on $[\theta_* - \delta, \theta_*]$ and on $[\theta_*, \theta_* + \delta]$, which imply the existence of the inverse functions

$$\theta_- : [\lambda_1(\theta_*), \lambda_1(\theta_* - \delta)] \rightarrow [\theta_* - \delta, \theta_*],$$

$$\theta_-(\lambda) = \theta \iff \lambda_1(\theta) = \lambda,$$

and

$$\theta_+ : [\lambda_1(\theta_*), \lambda_1(\theta_* + \delta)] \rightarrow [\theta_*, \theta_* + \delta],$$

$$\theta_+(\lambda) = \theta \iff \lambda_1(\theta) = \lambda.$$

These functions have Puiseux series expansions [20, page 246] of the form

$$\theta_-(\lambda) = \theta_* + c_-(\lambda - \lambda_1(\theta_*))^{1/2} + O(\lambda - \lambda_1(\theta_*)) \quad (\text{A.2})$$

$$\theta_+(\lambda) = \theta_* + c_+(\lambda - \lambda_1(\theta_*))^{1/2} + O(\lambda - \lambda_1(\theta_*)) \quad (\text{A.3})$$

and satisfy $\theta_-(r^{(j)}) = \ell^{(j)}$, $\theta_+(r^{(j)}) = u^{(j)}$. Exploiting $\lambda_1(\ell^{(j)}) = r^{(j)}$, we have

$$r^{(j)} - \lambda_1(\theta_*) = \beta(\ell^{(j)} - \theta_*)^2 + O((\ell^{(j)} - \theta_*)^3)$$

from the Taylor expansion (A.1), and

$$(\ell^{(j)} - \theta_*) = c_-(r^{(j)} - \lambda_1(\theta_*))^{1/2} + O(r^{(j)} - \lambda_1(\theta_*))$$

from (A.2), from which we infer $c_- = -\beta^{-1/2}$. In a similar way, by exploiting $\lambda_1(u^{(j)}) = r^{(j)}$ in equations (A.1) and (A.3), we obtain $c_+ = \beta^{-1/2}$. Hence, summing up (A.2) and (A.3) and evaluating the resulting expression at $\lambda = r^{(j)}$ give rise to

$$\frac{\ell^{(j)} + u^{(j)}}{2} = \theta_* + O(r^{(j)} - \lambda_1(\theta_*)). \quad (\text{A.4})$$

Finally, we substitute (A.4) in (A.1) and use $\lambda_1((\ell^{(j)} + u^{(j)})/2) = r^{(j+1)}$ to obtain

$$r^{(j+1)} - \lambda_1(\theta_*) = O((r^{(j)} - \lambda_1(\theta_*))^2)$$

as desired.

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