Real Contact 3-Manifolds Through Surgery

by

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ABSTRACT

In this thesis, we study real contact 3-manifolds. A real 3-manifold is a smooth 3-manifold equipped with an orientation preserving involution with empty or 1-dimensional fixed point set. By a real contact manifold we mean a real manifold with an anti-symmetric contact structure. We prove that every real 3-manifold can be obtained from the standard real 3-sphere via equivariant surgeries and we give equivariant surgery descriptions for arbitrary real 3-manifolds. Then we define equivariant contact surgery which allows one to extend the real contact structure to the surgered manifold. Via equivariant contact surgery we show that any real manifold admits a real contact structure. We give examples to illustrate how our algorithms for finding an equivariant (contact) surgery diagram work. We determine that certain contact structures are real, obtaining them via equivariant contact surgeries.

ÖZET

Bu tezde, reel kontak 3-manifoldları çalışıyoruz. Bir 3-manifold; sabit kümesi boş küme ya da bir boyutlu bir altmanifold olan, oryantasyonu koruyan bir involüsyonu olduğunda, reel 3-manifold olarak adlandırılır. Bir reel 3-manifold; reel yapıya göre antisimetrik kontak yapı taşıdığında reel kontak 3-manifold olarak adlandırılır. Her reel 3-manifoldun standart reel 3 boyutlu küreden ahenkli cerrahi adlı bir operasyonla elde edildiğini gösteriyoruz ve bu süreci betimliyoruz. Daha sonra, reel kontak yapıyı cerrahi sonrası manifolda yayabilmek için ahenkli kontak cerrahi operasyonunu tanımlıyoruz. Ardından her reel 3-manifolda reel kontak bir yapı atanabileceğini ahenkli kontak cerrahi yardımı ile kanıtlıyoruz. Farklı tiplerde ahenkli kontak cerrahi şemaları üretmek için iki ayrı algoritma tanımlayıp örnekler üzerinde yaptığımız inşaayı algoritmik olarak gösteriyoruz. Son olarak belirli kontak yapıların aynı zamanda reel kontak olduğunu ahenkli kontak cerrahi ile elde ederek gösteriyoruz.

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INTRODUCTION

Complex manifolds are naturally equipped with an involution defined by complex conjugation which certainly provides a richness to the field of study. Real manifolds came out with the purpose of putting a structure on smooth manifolds resembling the complex conjugation of complex manifolds. A real structure is defined as a non-trivial involution with its fixed point set called the real part. On manifolds with dimension $2n$, it is required to be orientation preserving if n is even and orientation reversing if n is odd with n-dimensional real part (if non-empty). On a $(2n-1)$-manifold, a real structure is orientation preserving if n is even and orientation reversing otherwise and the fixed point set is $(n-1)$-dimensional.

Though there are many ways to construct real manifolds, detecting the existence of real structures on an arbitrary manifold is a major challenge. It is known that there are some cohomological obstructions for existence and most 3-manifolds do not accept a nontrivial involution [24]. On the other hand, the number of non-equivalent (non-isotopic) real structures on a given manifold is not known, except for a small number of manifolds.

In this thesis, we study 3 dimensional real manifolds. 3-manifolds are studied in various points of view introduced throughout the years and are highly approachable. You can approach a 3-manifold via surgery, handlebody decompositions, branched coverings, open books, etc. Another useful structure that every 3-manifold admits is the contact structures. Although contact structures are studied in the framework of geometry in early stages, recent studies brought a topological perspective to the field. E. Giroux established a correspondence between open books and contact structures [11]. Moreover, Giroux introduced convex surface theory [12] enabling cutting and pasting on contact 3-manifolds. Using the properties of convex surfaces, F. Ding and H. Geiges introduced contact surgery on contact 3-manifolds.[2].

Most of the classical structures on 3-manifolds can be redefined to be compatible with the real structure and the structural theorems can be adapted into an equivariant setting on a real 3-manifold. In [20], F. Öztürk and N.Salepci conjectured and partially proved a correspondence between real contact structures and real open book decompositions on a real 3-manifold in the sense of Giroux. A real contact structure is a contact structure where the oriented contact hyperplanes are mapped to the same hyperplanes with opposite orientation under the real structure. Inspired by their work, we started this study with the question: Does every real 3-manifold admit a real contact structure? With this question in mind, first we find an equivariant surgery description for a given real 3-manifold with the purpose of turning the surgery procedure into an equivariant contact surgery. Our result is as follows.

Theorem 3.3.2. *Let (M, c_M) be a real 3-manifold. There is an equivariant link $\mathcal{L} = L \cup L_S \cup \overline{L}$ in the standard real S^3 with L_S is equivariant unlink for c_{st} and $c_{\mathrm{st}}(L) = \overline{L}$ so that (M, c_M) is equivariantly diffeomorphic to the manifold $(S^3_\sigma(\mathcal{L}), c_{\mathcal{L}})$, where the surgery coefficients σ can be taken to be ± 1.*

This theorem with its proof provides us with an algorithm to find an equivariant surgery link in the standard real 3-sphere for a given real manifold. Then with the help of the facts from convex surface theory and their real versions, we equivariantly isotope the equivariant surgery link to be Legendrian and perform equivariant contact surgery, which resultsin the following theorem:

Theorem 4.3.6. *Every real 3-manifold admits a real contact structure.*

In his thesis, C. Evers studied real contact manifolds in various dimensions. He gave examples of real manifolds with no real contact structure in dimension 5. For real 3-manifolds, he noted our question as well[9]. Theorem 4.3.6 answers this question affirmatively.

Here is an outline of the present text. In Chapter 2, we give the basics of real

3-manifolds, contact geometry and convex surfaces. Chapter 3 is mainly about equivariant surgeries. We define equivariant surgeries and observe how each type of equivariant surgery alters the real part of a real 3-manifold. These surgeries along invariant knots allow us to extend the real structure to the surgered manifold. Then we prove Theorem 3.3.2 by constructing an equivariant surgery link explicitly in the standard real 3-sphere for any real 3-manifold.

In Chapter 4, we step into the real contact 3-manifolds. We start with basics of the real contact manifolds and real open book decompositions. We define equivariant contact surgeries for each type of invariant knot. These are surgeries on invariant Legendriand knots that allow us to extend a real tight contact structure to the surgered manifold. This is not doable for every contact surgery coefficient, contrary to the case of the usual contact surgery. For each type of invariant knot, we determine all contact surgery coefficients for which an equivariant contact surgery is possible and how the real contact structure extends to the surgered manifold. Then we prove Theorem 4.3.6 by finding an explicit equivariant contact surgery link in the standard real contact 3-sphere for an arbitrary real 3-manifold. We come up with two algorithms for producing two different types of equivariant contact surgery diagrams. In the final chapter, we give examples to illustrate how our algorithms work. On $S^1 \times S^2$ and some lens spaces, we show that certain contact structures are real.

Chapter 2

PRELIMINARIES

The purpose of this chapter is to recall the notion of real 3-manifolds and contact 3-manifolds. In the first section we will define real structures on 3-manifolds, set up the terminology and give some examples. We will follow by introducing real Heegaard splittings of real 3-manifolds, which is one of the main structures we build our work on. As we speak of real Heegaard splittings and more generally embedded invariant surfaces in 3-manifolds, for a better understanding, real structures on surfaces and their properties will be reviewed in the third section. In the last two sections, we will review the basics of contact topology and convex surface theory. We will assume the objects we consider -manifolds, functions, vector fields etc.- are smooth.

2.1 Real Manifolds

On every complex manifold, there is a concept of symmetry induced by the complex conjugation. Real structures on even dimensional manifolds are motivated by the complex conjugation on complex manifolds. Real structure is an involution that mimics the complex conjugation in terms of the fixed point set and the orientation reversing/preserving properties. For odd dimensional manifolds we can still define real structures once we view them as the boundary of even dimensional real manifolds.

Definition 2.1.1.

- A real structure on a $(2n)$-dimensional manifold M is an involution such that it has n-dimensional fixed point set, if not empty, and is orientation preserving if n is even and orientation reversing if n is odd.

- A real structure on a $(2n-1)$-manifold is an involution with $(n-1)$-dimensional fixed point set, if not empty, and is orientation preserving if n is even and orientation reversing if n is odd.

- An oriented k-manifold M with a real structure c_M is called a real k-manifold, denoted by (M, c_M).

In this thesis, we are particularly interested in 3-dimensional real manifolds. By a *real 3-manifold*, we mean a closed oriented 3-manifold M equipped with a real structure c_M and denote it by (M, c_M). The fixed point set will be called the *real part*, denoted by $\mathrm{Fix}(c_M)$. In this case, the real structure c_M is orientation preserving and the real part $\mathrm{Fix}(c_M)$ consists of disjoint simple closed curves in M.

Two real structures c_M and c'_M on M are *equivalent* if there is an orientation preserving diffeomorphism h of M such that $c'_M = h \circ c_M \circ h^{-1}$. If the diffeomorhism

h is isotopic to the identity, then the real structures c_M and c'_M are called *equivariantly isotopic* (isotopic through real structures). In general, a diffeomorphism $h : (M, c_M) \to (M', c'_M)$ with $c'_M = h \circ c_M \circ h^{-1}$ is called an *equivariant* map and in the presence of an equivariant diffeomorphism, we call the real manifolds (M, c_M) and (M', c'_M) *equivariantly diffeomorphic.*

Example 1 (Standard real 3-sphere)**.** *Consider $S^3 \subset \mathbb{C}^2$ identified with $\{|z_1|^2 + |z_2|^2 = 1\}$ where (z_1, z_2) are coordinates for $\mathbb{C}^2$. The complex conjugation given by $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ is a real structure on S^3 with the real part a circle. Alternatively, identifying S^3 with $\mathbb{R}^3 \cup \{\infty\}$ with coordinates (x, y, z), the π-rotation about z-axis is a real structure; the real part is z-axis $\cup\infty$.*

It is well-known that there is a unique real structure on S^3 with non-empty real part, up to equivariant isotopy [28]. We call this real structure the standard real structure and denote it by c_{st}.

There are many ways to construct 3-manifolds. Employing these methods respecting to the symmetry, one ends up with real 3-manifolds. Some of these methods would be the following.

- Let L be a link in S^3. The 2-fold cyclic branched covering of S^3 over L is a real 3-manifold where the real structure is the involution induced by the covering map.

- If the boundaries of two identical handlebodies are identified by an orientation reversing involution one gets a 3-manifold. On this 3-manifold, the involution defined by sending two handlebodies to each other is a real structure with real part lying on the common boundary. Moreover, this defines a real Heegaard splitting for the real 3-manifold obtained which will be discussed in the next section.

- Performing an equivariant surgery on a real 3-manifold on an invariant link and extending the real structure to the surgered manifold produces a new real 3-manifold. This method will be discussed in Chapter 3 in detail.

We will end this section with following example recalling the complete classification of real structures on the solid torus $S^1 \times D^2$ due to Hartley [13].

Example 2 (Real Solid Tori)**.** *We choose an oriented identification of* T^2 *with* $\mathbb{R}^2/\mathbb{Z}^2$*, where* x *direction corresponds to the meridional direction of* T^2*, and fix the coordinates of* $S^1 \times D^2$ *as* (y, x, t) *where* t *is the radius direction of* D^2*. In these coordinates, the four possible real structures on the solid torus up to equivariant isotopy are (Figure 2.1):*

(1) $c_1 : (y, (x, t)) \mapsto (-y, (-x, t))$*;*

(2) $c_2 : (y, (x, t)) \mapsto (y, (x + \frac{1}{2}, t))$*;*

(3) $c_3 : (y,(x,t)) \mapsto (y+\frac{1}{2},(x,t))$;

(4) $c_4 : (y,(x,t)) \mapsto (y+\frac{1}{2},(x+\frac{1}{2},t))$.

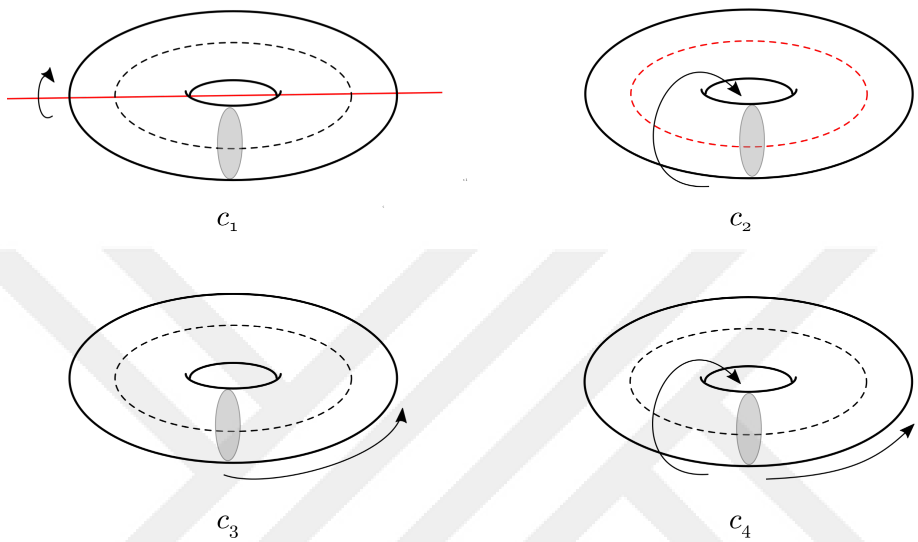

Figure 2.1: All possible real structures c_i on $S^1 \times D^2$

- *The real structure* c_1 *has a fixed point set consisting of two properly embedded arcs and the quotient space* $(S^1 \times D^2)/c_1$ *is a 3-ball.*

- *The real structure* c_2 *fixes the core of the solid torus and the quotient space* $(S^1 \times D^2)/c_1$ *is again a solid torus.*

- *The real structures* c_3 *and* c_4 *are both fixed point free. The corresponding quotient spaces are solid tori.* c_3 *and* c_4 *are equivalent, but not equivariantly isotopic.*

The restriction of c_i $(0 \leq i \leq 4)$ to the boundary torus $T^2 = S^1 \times S^1$ is an orientation preserving involution on the torus. Conversely, any orientation preserving involution on T^2 can be extended to an involution on $S^1 \times D^2$. Such an extension is unique up to isotopy and fixes a core of the solid torus setwise. With an abuse of notation we denote the restriction of c_i to the torus by c_i too.

Likewise there are three involutions on a circle up to isotopy through involutions: reflection, identity and rotation by π (antipodal map). We also denote them by c_1, c_2 (or id) and c_4 respectively.

Throughout the text, we will use the above classification and the labeling of the real structures, especially when we describe equivariant neighborhoods of knots and equivariant surgery in the next chapter.

2.2 Real Heegaard Splittings

A useful structure on a 3-manifold is its Heegaard splitting. An embedded Heegaard splitting of the 3-manifold M is given by a closed oriented surface Σ embedded in M bounding handlebodies U_1 and U_2 on both sides in M, i.e., $M = U_1 \cup_\Sigma U_2$ where $\Sigma = \partial(U_1) = \partial(U_2)$. On the other hand, let U be an oriented handlebody and $f : \partial U \to \partial U$ is an orientation reversing map. Then the pair (U, f) is an abstract Heegaard splitting of the manifold $(U \cup U)/(x \sim f(x))$.

Definition 2.2.1. Let (M, c_M) be a real 3-manifold. Let (M, Σ) be an embedded Heegaard splitting. If $c_M(U_1) = U_2$ and $c_M|_\Sigma$ is orientation reversing, we call the triple (M, c_M, Σ) a *real Heegaard splitting* for (M, c_M).

An *abstract real Heegaard splitting* is a triple (Σ, f, c) where Σ is the boundary of the handlebody U, $f : \Sigma \to \Sigma$ is the gluing map and $c : U \to U$ is an orientation preserving involution with $c|_{\partial U} \circ f = f \circ c|_{\partial U}$. Then c extends to a real structure on $(U \cup U)/(x \sim f(x))$ and the triple defines a real manifold.

Remark 1. An embedded Heegaard splitting gives rise to an abstract one and vice versa. This remains true for the real embedded and the abstract real Heegaard splittings too. See [20] for details. Furthermore, every real 3-manifold admits a real embedded Heegaard splitting [18].

Now let us examine some real Heegaard splitting constructions on (S^3, c_{st}). The simplest Heegaard splitting of S^3 with Heegaard surface S^2 and the gluing map the reflection through the equator is indeed real with the real part being the equator. This can be seen embeddedly in (S^3, c_{st}) as $S^3 = U_0 \cup V_0$ where $U_0 = (\mathbb{R}^3_{y\geq 0} \cup \{\infty\})$ and $V_0 = S^3 - U_0$. This Heegaard splitting can be stabilized by adding g pairs of equivariant handles to get a real Heegaard splitting for (S^3, c_{st}) of even genus $2g$. Here, note that we start with a real Heegaard splitting, where the real part separates the Heegaard surface. If we choose the attaching regions of each handle in those separated parts of the surface, we again get a real Heegaard splitting with separating real part. Alternatively, we can get this splitting from the real open book given in Example 7. This proves (1) of the following lemma. For (2), see the proof of Lemma 4.3.5.

Lemma 2.2.1. (S^3, c_{st}) *has a genus-g real Heegaard splitting* (Σ_g, s) *for every* $g > 0$. *Furthermore,*

(1) there is a real Heegaard splitting of every even genus with separating real part;

(2) there is a real Heegaard splitting of every genus with non-separating real part.

Lemma 2.2.2. *Let* (M, Σ) *be an embedded real Heegaard splitting for the real manifold* (M, c_M). *Then* $(\Sigma, c_M|_\Sigma, \mathrm{id})$ *is an abstract real Heegaard splitting for* (M, c_M).

Proof. Assume that Σ bounds handlebodies U_1 and U_2 on both sides in M. We can think of M as $M = U_1 \cup_{\mathrm{id}} U_2$. The real structure c_M maps U_1 to U_2 and vice versa. Also, by definition, $c_M|_\Sigma$ is an orientation reversing involution on Σ. We will show that the real 3-manifold $(\widetilde{M}, c_{\tilde{M}})$ given by the real Heegaard splitting $(\Sigma, c_M|_\Sigma, \mathrm{id})$ is equivariantly diffeomorphic to (M, c_M). Define the function $F : M \to \widetilde{M}$ as

$$F = \begin{cases} U_1 \to U_1, x \mapsto x \\ \\ U_2 \to U_1, x \mapsto c_M(x), \end{cases}$$

which is a diffeomorphism and as we can check that $F \circ c_M = c_{\tilde{M}} \circ F$, it is an equivariant diffeomorphism. □

Any real manifold (M, c_M) has a abstract real Heegaard splitting of the form $(\Sigma, c_M|_\Sigma, \mathrm{id})$ by the lemma above. We will denote this particular Heegaard splitting by (Σ, c), where $c = c_M|_\Sigma$ is the gluing map and the real structure on the Heegaard surface Σ.

Note. Recall that the Heegaard genus is the minimum genus of the Heegaard splitting among all the Heegaard splittings of a 3-manifold. One may define the real Heegaard genus to be the minimum genus of the Heegaard splitting among all the real Heegaard splittings of a real 3-manifold. Then one may wonder if the two are equal, or when the two are equal. The answer to the former is no. In fact, $L(p, q)$ with $q^2 \not\equiv 1 \pmod{p}$ has no real Heegaard splitting of genus-1 as will be discussed in Chapter 3.

2.3 Real Structures on Closed Orientable Surfaces

This section will be about the classification of real structures on closed oriented surfaces and some of their properties. A real structure c on a closed connected oriented genus-g surface Σ_g is an orientation reversing involution where the real part is a set of n disjoint simple closed curves $B = \{r_1, ..., r_n\}$ with $n \leq g + 1$. The equivalence class of c up to conjugation by an orientation preserving diffeomorphism is determined by n and the connectedness of Σ_g- Fix (c). We call the real structure *separating* if the set Fix (c) separates the surface Σ_g into two disjoint subsurfaces and *non-separating* otherwise. If c is separating, then $\Sigma_g - B$ consists of two connected components S_1 and S_2 such that $\partial S_1 = \partial S_2 = B$ with $S_2 = c(S_1)$ and n and g are of opposite parities. This classification is due to Harnack and a generalized version of it can be found in [23].

Let $s_{\max}$ denote the real structure given by the reflection exchanging upper half of the surface with the lower half as shown in Figure 2.2.. Note that any real structure with $g+1$ real circles is separating and is equivalent to $s_{\max}$.

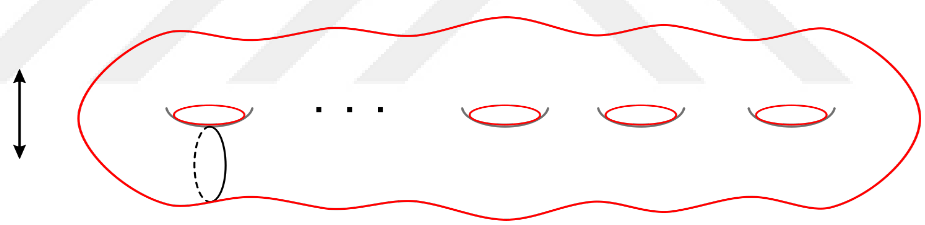

Figure 2.2: The maximal real structure $s_{\max}$ on Σ_g with $g+1$ real circles shown in red.

We denote by τ_α the Dehn twist about a simple closed curve α. We write $\tau_\alpha \cdot \tau_\beta$ or $\tau_\alpha \tau_\beta$ for composition (product) of Dehn twists.

Lemma 2.3.1. *Let f be a diffeomorphism on Σ_g and α be a simple closed curve on Σ_g. Then*

(i) $f \cdot \tau_\alpha \cdot f^{-1} = \tau_{f(\alpha)}$ *if f is orientation preserving;*

(ii) $f \cdot \tau_\alpha \cdot f^{-1} = \tau^{-1}_{f(\alpha)}$ *if f is orientation reversing.*

In particular, if f is an orientation reversing involution and α is an invariant curve we have

$$f \cdot \tau_\alpha = \tau_\alpha^{-1} \cdot f.$$

Lemma 2.3.2. *[26, Lemma 4.3.1] Let c be a real structure on Σ_g and α be a c-invariant simple closed curve. Then $c' = \tau_a^{\pm 1} \cdot c$ and $c'' = c \cdot \tau_a^{\pm 1}$ are also real structures on Σ_g. Moreover, α is a c_1-curve for c' and c'' if and only if α is a c_1-curve for c. On the other hand, α is a c_2-curve for c and c' if it is a c_3-curve for c and vice versa.*

On a surface Σ_g, the Dehn twists about the $2g+1$ curves shown in Figure 2.3 generate the mapping class group of Σ_g. The following lemma shows that we can find an invariant set of generators for a maximal real structure on Σ_g.

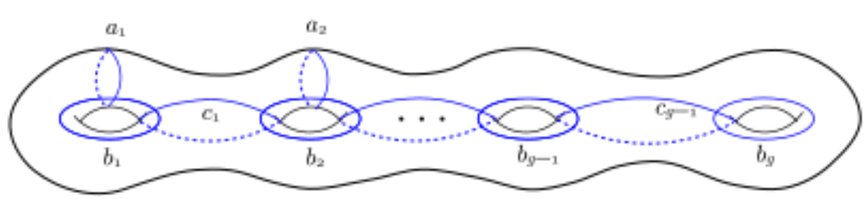

Figure 2.3: Dehn twists about the $2g+1$ curves in blue generate the mapping class group.

Lemma 2.3.3. *There is a set of invariant circles $B = \{a_1 \dots a_{2g+1}\}$ on $(\Sigma_g, s_{\max})$ such that B is a generating set for the mapping class group of Σ_g.*

Proof. Consider the $s_{\max}$ given by Figure 2.2). Obviously, $s_{\max}$ leaves all the generators given in figure 2.3 invariant. Let $s'_{\max}$ be another maximal real structure. Then by the classification, it is conjugate to $s_{\max}$ and there exists an orientation preserving diffeomorphism f such that $s'_{\max} = f \cdot s_{\max} \cdot f^{-1}$. Then the conjugates of the $2g+1$ generators by f are a new set of generators for the mapping class group and they are invariant under $s'_{\max}$. □

2.4 Contact Topology and Convex Surfaces

A contact structure ξ on a smooth 3-manifold M is a smooth hyperplane field which is maximally non-integrable, i.e., if α is a 1-form defining ξ locallyas $\ker \alpha = \xi$, then $\alpha \wedge d\alpha$ is nowhere zero.

If α defines ξ globally, then we say that the contact structure is coorientable. This implies that ξ, as a 2-plane field on M is orientable. Moreover, if the orientation of M agrees with the orientation given by $\alpha \wedge d\alpha$, we say that the contact structure ξ is positive. We assume ξ is cooriented and positive throughout this text. For more information on contact topology, see e.g. [10] or [21]. We will explicitly use the following contact 3-manifolds.

Example 3. *On $\mathbb{R}^3$ with coordinates (x, y, z), a contact structure can be given by the kernel of the 1-form $dz + xdy$. It is called the standard contact structure on $\mathbb{R}^3$.*

Example 4 (The standard contact 3-sphere)**.** *Let (x_1, y_1, x_2, y_2) be the cartesian coordinates on $\mathbb{R}^4$. On the unit sphere $S^3 \subset \mathbb{R}^4$, the 1-form given by $\sum_{i=1}^{2}(x_i dy_i - y_i dx_i)$ is a contact 1-form, and the kernel of this 1-form is called the standard contact struture on S^3.*

Alternatively, with complex coordinates (z_1, z_2) for $\mathbb{C}^2$, the standard contact structure is given by the 1-form

$$\alpha = \frac{1}{2}(z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2).$$

J. Martinet's theorem states that every 3-manifold admits a contact structure [17]. Two contact 3-manifolds (M, ξ) and (M', ξ') are called *contactomorphic* if there is a diffeomorphism $f : M \to M'$ such that $f_*(\xi) = \xi'$. Two contact structures ξ and ξ' on a manifold M are said to be *isotopic* if there is a contactomorphism $h : (M, \xi) \to (M, \xi')$ which is isotopic to the identity.

The contact structures fall into two disjoint categories: *overtwisted* or *tight*. An overtwisted disk in a contact 3-manifold is a disk with boundary tangent to the contact planes. As the name suggests, a contact structure is overtwisted if there is an overtwisted disk, and tight otherwise. The standard contact structure on $\mathbb{R}^3$ given in Example 3 is tight. The classification of the overtwisted contact structures reduces to the homotopy classes of oriented 2-plane fields [5].

The classification of the tight contact structures up to isotopy on arbitrary contact manifolds is by no means complete, though it is known for some 3-manifolds. The standard contact structure given in Example 4 is the unique tight contact structure on S^3 up to contact isotopy [6]. For lens spaces, the complete classification is due to K.Honda [14]. The tecniques employed there rely s on convex surface theory. Here we will briefly give some definitions and state some of the theorems in convex surface theory, as a reference for the next chapters. Details can be found in [7] or [21].

A surface S in a contact 3-manifold (M, ξ) is *convex* if there is a contact vector field v transverse to S. By a contact vector field, we mean a vector field whose flow ϕ_t preserves the contact planes. Equivalently, assuming α is a contact 1-form for ξ, a vector field v is a contact vector field if and only if $L_v\alpha = g\alpha$ for some $g : M \to \mathbb{R}$. This can be seen from the definition of the Lie derivative $L_v\alpha = \frac{\partial}{\partial t}\phi_t^*(\alpha)|_{t=0}$ and the fact that v is contact iff $\phi_t^*(\alpha) = g_t\alpha$. For example, the associated Reeb vector field X_α (which is given by the property $\alpha(X_\alpha) = 1, \iota_{X_\alpha}d\alpha = 0$) is a contact vector field, since $L_{X_\alpha}\alpha = 0$ by Cartan's formula.

Definition 2.4.1. Suppose v is a contact vector field making the surface S convex. The set $\Gamma = \{x \in S | v_x \in \xi_x\}$ is called the *dividing set* for S.

The dividing set of a convex surface S is a collection of disjoint simple closed curves and arcs with endpoints on ∂S.

As an example to convex surfaces and their dividing sets, we will examine the solid torus with convex boundary.

Example 5. *Consider* $N = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z}) \simeq \mathbb{R}^2 \times S^1$ *with coordinates* (x, y, z)*. The contact structure* ξ *defined as the kernel of*

$$\eta_1 = \cos(2\pi z)dx - \sin(2\pi z)dy$$

induces a contact structure on the solid torus $N_\delta = \{(x, y, z) \in N \mid x^2 + y^2 \leq \delta\}$*. The vector field* $v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ *is a contact vector field which is transverse to the boundary torus* $T_\delta = \partial N_\delta$*, making it convex.*

To find the dividing set Γ*, we make use of the definition itself and solve for* $x \in T_\delta$ *satisfying* $\alpha_x(v(x)) = 0$*. We find that it consists of two parallel copies of a simple closed curve and is given by*

$$\Gamma = \{\pm\delta \sin(2\pi z), \pm\delta \cos(2\pi z), z\}.$$

If T is a convex torus in a tight contact manifold its dividing set consists of $2n$ parallel copies of a simple closed curve. By fixing an identification of T with $\mathbb{R}^2/Z^2$, the slope of these curves is called the slope of the convex torus and denoted by $s(T)$. If T bounds a solid torus, the meridian will have slope 0 and the chosen longitude will have slope ∞. In the above example, a longitude can be chosen by pushing the core circle $C = \{(x, y, z) \in N_\delta | x = y = 0\}$ along a vector field (transverse to C) staying in the contact planes. This is called the contact longitude and is parallel to the dividing set. The meridian of N and the contact longitude provides and identification of T, making the slope $s(T_\delta) = \infty$. Honda classifies the tight contact structures on a solid torus with convex boundary in the following theorem.

Theorem 2.4.1 (Honda, [14]). *Consider the tight contact structures on $S^1 \times D^2$ with convex boundary T for which $\#\Gamma(T) = 2$ and $s(T) = -p/q,\ p \geq q > 0,\ (p,q) = 1$. Fix a characteristic foliation F which is adapted to Γ. There exist exactly $|(r_0+1)(r_1+1)\ldots(r_{k1}+1)(r_k)|$ tight contact structures on $S^1 \times D^2$ with this boundary condition, up to isotopy fixing T. Here, $r_0, \ldots, r_k$ are the coefficients of the continued fraction expansion of $-p/q$.*

2.5 Contact Surgery

Let K be a knot in M with a tubular neighborhood $N(K) \simeq S^1 \times D^2$ and the exterior $M - \mathring{N}(K)$. Choose a basis $\{\mu, \lambda\}$ for $H_1(\partial(M - \mathring{N}(K)); \mathbb{Z}) \simeq \mathbb{Z}^2$. The p/q-surgery along K yields the 3-manifold $M_{p/q}(K) := (M - \mathring{N}(K)) \cup (S^1 \times D^2)$, where the gluing map is characterized by $\{pt\} \times \partial D^2 \mapsto p\mu + q\lambda$.

The meridian μ can be chosen canonically as a curve which bounds a disk in $N(K)$ normal to K. The longitude λ, defined as a simple closed curve generating the kernel of the homomorphism on H_1 induced by the inclusion $\partial N(K) \to M - \mathring{N}(K)$. λ is also characterized by the property that it has linking number zero with K. If K is null-homologous it bounds an embedded surface in M called a Seifert surface for K. By pushing K along a Seifert surface, one can obtain a longitude λ. This particular choice of longitude defines a trivialization of the normal bundle of K and is called the *Seifert framing* of K. The resulting manifold is completely determined by the knot and the surgery coefficient.

Now we will define contact surgery, which is a procedure allowing us to extend a contact structure to the surgered manifold. By a rational contact surgery we mean a topological p/q-surgery where the surgery coefficient is measured relative to the contact framing. We will present the procedure below. For more details see [21].

A knot K (or link) in a contact manifold (M, ξ) is said to be *Legendrian* if it is tangent to ξ everywhere. A Legendrian knot comes with a canonical framing induced by a transverse vector field of ξ. This framing is called the *contact framing*. If K is null-homologous, the integer measuring the difference between the contact framing and the Seifert framing is called the *Thurston-Bennequin invariant* $\mathbf{tb}(K)$.

In general, when the knot is equipped with a certain framing then the difference between this framing and the contact framing will be called the *twisting number* $\mathrm{tw}(K)$. A useful theorem for computing the tw is given below, when the knot lies on a convex surface.

Theorem 2.5.1 ([7]). *Let K be a Legendrian curve on a convex surface S, then*

$$tw(K,S) = -\frac{1}{2}\#(K \cap \Gamma_S),$$

where # means the geometric intersection number and $tw(K,S)$ represents the twisting number of contact planes with respect to the surface S. If S is a Seifert surface, the the above formula computes the Thurston Bennequin number of K.

In [2], F. Ding and H. Geiges give the definition of contact surgery. Let K be a Legendrian knot in the contact 3-manifold (M,ξ) and $N(K)$ be the contact neighborhood with convex boundary as in Example 5. Let λ denote the longitude given by the contact framing and μ denote the meridian of $N(K)$. A contact (p/q)-surgery ($p \neq 0$) on K is performed by removing the neighborhood $N(K)$ form M and gluing a solid torus N_0 to $M - \mathring{N(K)}$ with the identifications

$$\mu_0 \mapsto p\mu + q\lambda, \quad \lambda_0 \mapsto p'\mu + q'\lambda$$

where μ_0 is a meridian and λ_0 is a longitude of N_0, and p', q' satisfy $pq' - qp' = 1$.

The neighborhood $N(K)$ is a tight contact solid torus with boundary slope ∞. To have a well defined contact structure on the surgered 3-manifold, the dividing sets of N_0 and $M - \mathring{N(K)}$ have to match. By the identification above, the curve $-p'\mu_0 + p\lambda_0$ maps to the longitude λ, so the solid torus N_0 should be a tight contact solid torus with boundary slope $-p/p'$. According to the Theorem 2.4.1, there is such a tight contact solid torus unless $p = 0$, and it is unique for $-p/p' = 1/k$ with $k \in \mathbb{Z}$. For the other slopes, the contact structure can be extended over the solid torus in more than one way, which may result in the same 3-manifold with non-isotopic contact structures. In [3], Ding and Geiges give an algorithm to turn any rational contact surgery into (± 1)-contact surgeries. They prove the following theorem.

Theorem 2.5.2. *Every closed, orientable contact 3-manifold* (M, ξ) *can be obtained via contact* (± 1)*-surgery on a Legendrian link in* (S^3, ξ_{st}).

Chapter 3

REAL 3-MANIFOLDS THROUGH SURGERY

3.1 Equivariant surgery

In this section, we will define all types of surgeries which allow us to extend the real structure to the surgered region. These surgeries will be called *equivariant surgeries.* Let (M, c_M) be a real 3-manifold and K be a c_M-invariant knot. Then K has a unique equivariant tubular neighborhood $N(K)$ [1]. Moreover, $N(K)$ is equivariantly isotopic to one of the real solid tori $(S^1 \times D^2, c_i)$ given in Example 2 and Figure 2.1. An invariant knot K is called a c_i-knot if it has an equivariant neighborhood of type $(S^1 \times D^2, c_i)$.

Let $N(K)$ be an equivariant neighborhood of K and (μ, λ) be a meridian and longitude pair for $N(K)$. In order to perform p/q-surgery along K, we remove the interior of the neighborhood and glue back a solid torus $S^1 \times D^2$. Let (μ_0, λ_0) denote the meridian and the standard longitude of $S^1 \times D^2$ and ϕ be the gluing map

$$\phi : \partial(S^1 \times D^2) \to M - \mathring{N}(K), \quad (\mu_0, \lambda_0) \mapsto (p\mu + q\lambda,\ p'\mu + q'\lambda)$$

where $pq' - qp' = 1$. In order to extend the real structure to the surgered region, the itwo nvolutions on the gluing region should match. If K is a c_i-knot, we must glue back a c_j-solid torus where c_j satisfies the condition $c_i \circ \phi = \phi \circ c_j$. Before examining each case, recall that

$$c_1(\mu,\lambda) = (-\mu,-\lambda) \qquad c_2(\mu,\lambda) = (\mu+\pi,\lambda)$$
$$c_3(\mu,\lambda) = (\mu,\lambda+\pi) \qquad c_4(\mu,\lambda) = (\mu+\pi,\lambda+\pi).$$

Case 1: K is a c_1-knot. Since $c_1 \circ \phi = \phi \circ c_1$ for any p/q, the real structure always extends as c_1 to the surgered region.

Case 2. K is a c_2-knot. The real structure extends as $\begin{cases} c_2 \text{ if } q \text{ is even, } q' \text{ is odd;} \\ c_3 \text{ if } q \text{ is odd, } q' \text{ is even;} \\ c_4 \text{ if } q \text{ is odd, } q' \text{ is odd.} \end{cases}$

Case 3. K is a c_3-knot. The real structure extends as $\begin{cases} c_2 \text{ if } p \text{ is even, } p' \text{ is odd;} \\ c_3 \text{ if } p \text{ is odd, } p' \text{ is even;} \\ c_4 \text{ if } p \text{ is odd, } p' \text{ is odd.} \end{cases}$

Case 4. K is a c_4-knot. The real structure extends as $\begin{cases} c_2 \text{ if } p+q \text{ is even, } p'+q' \text{ is odd;} \\ c_3 \text{ if } p+q \text{ is odd, } p'+q' \text{ is even;} \\ c_4 \text{ if } p+q \text{ is odd, } p'+q' \text{ is odd.} \end{cases}$

Above we gave equivariant surgery definitions for an arbitrary surgery coefficient p/q. Now we will restrict ourselves to (± 1)-surgeries, as our equivariant surgery algorithm presented in Chapter 3.3 involves only (± 1)-surgeries. We define these for each type of invariant knot below.

Type I: K is a c_1-knot: The real structure is extended as c_1 to the sewed back solid torus after (± 1)-surgery. The number of components of the real part decreases by 1 if the fixed points of K belong to different components of the real part, and increases by 1 if they belong to a connected component of the real part.

Type II: K is a c_2-knot: In this case, K is a real knot and the real structure is extended as c_4 to the sewed back solid torus after (± 1)-surgery. Note that in this case, we eliminate a component of the real part.

Type III: K is a c_3-knot: The real structure is extended as c_3 to the sewed back solid torus after (± 1)-surgery. Note that in this case the real part of the manifold is not affected, which makes this type of surgery not very interesting. We will not use this type of surgery in this study, since our fixed point free equivariant knots will be c_4-knots, as we discuss in next chapters.

Type IV: K is a c_4-knot: This is, in some sense, the reverse of the type-II surgery. The real structure is extended as c_2 to the sewed back solid torus after (± 1)-surgery. Note that, in this case, the core of the sewed back solid torus is pointwise fixed and we increase the number of components of the real part.

Another type of equivariant surgery emerges when we have a pair of knots K and $\overline{K}$ swapped by c_M.

Type V: Let K and $\overline{K} = c_M(K)$ be any pair of knots swapped by c_M and $K \cap \text{Fix}\,(c_M) = \emptyset$. An equivariant surgery on the pair of framed knots (K, ρ), $(\overline{K}, \rho)$ is done in the following way: We take out the neighborhoods $N(K)$ and $c_M(N(K))$ and glue back solid tori V and V' and extend the real structure equivariantly.

3.2 Recursively equivariant surgery

In this section we give an equivariant surgery description for a real 3-manifold, which is performed in a recursive manner.

Theorem 3.2.1. *Every real 3-manifold (M, c_M) can be obtained via a finite number of Dehn surgeries along an ordered collection of recursively invariant knots starting from the real 3-sphere (S^3, c_{st}). More precisely, there is a sequence $\{(M_j, s_j; K_j)\}_{j=0}^{k}$ of real 3-manifolds, a common Heegaard surface H and s_j-invariant knots $K_j \subset H \subset M_j$ such that $(M_0, s_0) = (S^3, c_{\text{st}})$ and (M_k, s_k) is equivariantly diffeomorphic to (M, c_M) and each (M_{j+1}, s_{j+1}), $0 \leq j \leq k$, is obtained from (M_j, s_j) via a ± 1 surgery along K_j where the framing along K_j is determined by H; at each step, the real structure is canonically extended to the surgered region.*

We will begin with proving two lemmas about real structures on surfaces. Then we are going to use them in the proof of Theorem 3.2.1. In Lemma 2.3.2, we saw that it is possible to produce new real structures from old ones by performing Dehn twists along invariant curves. Next lemma uses this fact to relate any real structure to the maximal real structure on Σ_g.

Lemma 3.2.2. *Any real structure s on Σ_g with $\#Fix\ (c) = k$ can be expressed in the form*

$$s = \beta^m \cdot \tau_{b_l} \dots \tau_{b_1} \cdot p \cdot s_{\max}, \tag{3.1}$$

where $p = \tau_{a_1}^{\sigma_1} \dots \tau_{a_t}^{\sigma_t} \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1}$ is an even palindrome (possibly empty) with a_j's invariant under $s_{\max}$; b_j's are in Fix $(p \cdot s_{\max})$; and $\beta \subset$ Fix (c) with $m = 1$ if s is separating but not maximal and $m = 0$ otherwise.

Proof. (i) Assume s is non-separating. Set $l = g + 1 - k$ and take $s_{\max}$-real circles $b_1, \dots, b_l$ on Σ_g. Then $\tau_{b_l} \dots \tau_{b_1} s_{\max}$ is a non-separating real structure with k real circles by Lemma 2.3.2 and it is conjugate to s by classification. Then there is an orientation preserving diffeomorphism f of Σ_g so that

$$\begin{aligned} s &= f \cdot \tau_{b_l} \dots \tau_{b_1} \cdot s_{\max} \cdot f^{-1} \\ &= f \cdot \tau_{b_l} f^{-1} \cdot f \dots f^{-1} \cdot f \cdot \tau_{b_1} \cdot f^{-1} \cdot f \cdot s_{\max} \cdot f^{-1} \\ &= \tau_{f(b_l)} \dots \tau_{f(b_1)} \cdot f \cdot s_{\max} \cdot f^{-1} \end{aligned}$$

Now by Lemma 2.3.3 we can write f as a product of Dehn twists about $s_{\max}$-invariant curves as $f = \tau_{a_1}^{\sigma_1} \dots \tau_{a_t}^{\sigma_t}$. Thus we have

$$f \cdot s_{\max} \cdot f^{-1} = \tau_{a_1}^{\sigma_1} \dots \tau_{a_t}^{\sigma_t} \cdot s_{\max} \cdot \tau_{a_t}^{-\sigma_t} \dots \tau_{a_1}^{-\sigma_1} = \tau_{a_1}^{\sigma_1} \dots \tau_{a_t}^{\sigma_t} \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot s_{\max}.$$

Note that since b_i's are $s_{\max}$-real, $f(b_i)$'s are $f \cdot s_{\max} \cdot f^{-1}$-real.

(ii) If s is separating but not maximal, take any circle $\beta \subset \text{Fix}\,(s)$. Then $\beta^{-1} \cdot s$ is non-separating and the proof follows from part (i).

(iii) Assume s is maximal. Then the equation holds with $m = 0$ and choosing no b_i's since $l = 0$, again following from part (i). □

Remark 2. Note that in the above lemma if $m = 0$, i.e. s is non-separating, β_i's are disjoint s-invariant circles on Σ_g. This can be interpreted as there are l disjoint s-invariant curves on Σ_g and a real maximal structure $s_{\max}$ so that β_i's are in $\text{Fix}(s_{\max})$ and $s = \tau_{\beta_l} \dots \tau_{\beta_1} \cdot s_{\max}$

Definition 3.2.1. On a real surface (Σ_g, c), we say that the product of Dehn twists $\tau_{r_m}^{\sigma_m} \dots \tau_{r_1}^{\sigma_1}$ *satisfies the recursive c-invariance condition* (or is *recursively c-invariant*) if for each $j = 1, \dots, m$,

$$\tau_{r_{j-1}}^{\sigma_{j-1}} \dots \tau_{r_1}^{\sigma_1} \cdot c(r_j) = r_j.$$

Lemma 3.2.3. *The even palindrome* p *in Lemma 3.1 has a recursively* $s_{\max}$*-invariant factorization in even powers.*

Proof. First, using the fact that p is an even palindrome, observe that it can be written as

$$p = (\tau_{a_1}^{\sigma_1} \dots \tau_{a_{t-1}}^{\sigma_{t-1}} \tau_{a_t}^{\sigma_t} \tau_{a_{t-1}}^{-\sigma_{t-1}} \dots \tau_{a_1}^{-\sigma_1})^2 \dots (\tau_{a_1}^{\sigma_1} \tau_{a_2}^{\sigma_2} \tau_{a_3}^{\sigma_3} \tau_{a_2}^{-\sigma_2} \tau_{a_1}^{-\sigma_1})^2 \cdot (\tau_{a_1}^{\sigma_1} \tau_{a_2}^{\sigma_2} \tau_{a_1}^{-\sigma_1})^2 \cdot \tau_{a_1}^{2\sigma_1}.$$

Now for each $1 < j < t$, set $f_j = \tau_{a_1}^{\sigma_1} \dots \tau_{a_j}^{\sigma_j}$, $\bar{f}_j = \tau_{a_j}^{\sigma_j} \dots \tau_{a_1}^{\sigma_1}$, and $r_1 = a_1$, $r_j = f_{j-1}(a_j)$. Then the factorization above becomes

$$p = \tau_{r_t}^{2\sigma_t} \dots \tau_{r_2}^{2\sigma_2} \tau_{r_1}^{2\sigma_1}.$$

We claim that this factorization is recursively s-invariant. In fact, $r_1 = a_1$ is invariant under s; furthermore, for each $1 < j < t$, (dropping the powers σ_j's for the sake of clarity)

$$\begin{aligned}
(\tau_{r_j}^2 \dots \tau_{r_1}^2 \cdot s_{\max})(r_{j+1}) &= f_j \cdot \bar{f}_j \cdot s_{\max}(f_j(a_{j+1})) \\
&= f_j \cdot (\bar{f}_j \cdot \bar{f}_j^{-1}) \cdot s_{\max}(a_{j+1}) \\
&= f_j(a_{j+1}) = r_{j+1}.
\end{aligned}$$

To finish the proof, we also note that $(\tau_{r_{j+1}} \tau_{r_j}^2 \dots \tau_{r_1}^2 \cdot s_{\max})(r_{j+1}) = r_{j+1}$. □

The proof above shows also that starting from $s_{\max}$ and composing with τ_{r_j}'s, we always get real structures. In fact, for every $1 \leq j < t$, (without writing σ_j's)

$$\begin{aligned}(\tau_{r_j}^2 \dots \tau_{r_1}^2 \cdot s_{\max}) \cdot (\tau_{r_j}^2 \dots \tau_{r_1}^2 \cdot s_{\max}) &= (f_j \cdot \bar{f}_j \cdot s_{\max}) \cdot (f_j \cdot \bar{f}_j \cdot s_{\max}) \\ &= f_j \bar{f}_j \bar{f}_j^{-1} f_j^{-1} s_{\max}^2 = \mathrm{id}\end{aligned}$$

and similarly

$$(\tau_{r_{j+1}} \tau_{r_j}^2 \dots \tau_{r_1}^2 \cdot s_{\max}) \cdot (\tau_{r_{j+1}} \tau_{r_j}^2 \dots \tau_{r_1}^2 \cdot s_{\max}) = \tau_{r_{j+1}} \tau_{r_{j+1}}^{-1} = \mathrm{id}.$$

This is true for every recursively c-invariant product $\tau_{r_m}^{\sigma_m} \dots \tau_{r_1}^{\sigma_1}$. If we set $s_0 = c$ and, for $j > 0$, $s_j = \tau_{r_j}^{\sigma_j} \cdot s_{j-1}$, $\{s_j\}_{j=0}^{j=m}$ is a sequence of real structures on Σ_g.

Proof of Theorem 3.2.1. We will follow the proof of the well-known Lickorish-Wallace Theorem (see e.g. [25]). Given the real 3-manifold (M, c_M), consider a real Heegaard splitting of M. Let the Heegaard surface H have genus g. Then we take a genus-g non-separating real splitting of (S^3, c_{st}) which exists for any $g > 0$ by Lemma2.2.1. Note that in these splittings, $c = c_M|_H$ and $s = c_{\mathrm{st}}|_H$ are gluing maps. In the usual proof of Lickorish-Wallace Theorem, the composition $c \cdot s$ is expressed as a product of Dehn twists and then each twist is extended over handlebodies. In our situation, we will express $c \cdot s$ as a recursively s-invariant product and show that that factorization describes an appropriate recursively equivariant sequence of (± 1)-surgeries. By the Remark 3 of Lemma 3.2.2, since s is non-separating with one real circle, there is an

$s_{\max}$ and s-invariant curves $\alpha_1, \ldots, \alpha_g$ so that $s = \tau_{\alpha_1} \ldots \tau_{\alpha_g} \cdot s_{\max}$. Moreover, using Equation 3.1 for c, the product $c \cdot s$ becomes

$$\begin{aligned} c \cdot s &= \tau_\gamma^m \cdot \tau_{\beta_l} \ldots \tau_{\beta_1} \cdot p \cdot s_{\max} \cdot \tau_{\alpha_1} \ldots \tau_{\alpha_g} \cdot s_{\max} \\ &= \tau_\gamma^m \cdot \tau_{\beta_l} \ldots \tau_{\beta_1} \cdot p \cdot \tau_{\alpha_1}^{-1} \ldots \tau_{\alpha_g}^{-1}, \end{aligned} \tag{3.2}$$

where $p = \tau_{a_1}^{\sigma_1} \ldots \tau_{a_t}^{\sigma_t} \tau_{a_t}^{\sigma_t} \ldots \tau_{a_1}^{\sigma_1}$ is an even palindrome (possibly empty) with a_j's invariant under $s_{\max}$; β_j's are in Fix $(p \cdot s_{\max})$; α_i's are s-invariant and $s_{\max}$-real; and $\gamma \in \mathrm{Fix}(c)$ with $m = 1$ if c is non-separating but not maximal and $m = 0$ otherwise.

Now we claim that the product (3.2) is recursively s-invariant. As α_i's are disjoint s-invariant curves, we have

$$\begin{aligned} s_0 &= s, \ r_1 = \alpha_g \\ s_1 &= \tau_{\alpha_g}^{-1} \cdot s, \ r_2 = \alpha_{g-1} \\ &\vdots \\ s_g &= \tau_{\alpha_1}^{-1} \ldots \tau_{\alpha_g}^{-1} \cdot s = s_{\max}, \ r_{g+1} = a_1 \end{aligned}$$

We know p is recursively $s_{\max}$-invariant by Lemma 3.2.3. We will not write that part again. Moving on with the remaining terms of the product with $s_{g+t+1} = p \cdot s_{\max}$,

$r_{g+t+2} = \beta_1$. Since β_j's are disjoint $p \cdot s_{\max}$-real curves, and γ is c-real, we have

$$\begin{aligned} s_{g+t+1} &= p \cdot s_{\max},\ r_{g+t+2} = \beta_1 \\ &\vdots \\ s_{g+t+l} &= \tau_{\beta_l} \dots \tau_{\beta_1} \cdot p \cdot s_{\max},\ r_{g+t+l+1} = \gamma \\ s_{g+t+l+1} &= c. \end{aligned}$$

By the above lines of equations, we start with the real structure $s_0 = s$ and an s_0 invariant curve r_0. Recursively we get real structures at each step by composing the real structure and a Dehn twist along the invariant curve and end up with $s_{g+t+l+1} = c$. In the proposition below we show that each of these steps determines an appropriate equivariant (± 1)-surgery with respect to the framing determined by H. Using that construction recursively the proof of Theorem 3.2.1 follows. □

Proposition 3.2.4. *Let X and Y be real 3-manifolds with real Heegaard splittings (H, c) and (H, c') respectively. Suppose $c' = \tau_\alpha^{\pm 1} \cdot c$ where α is a c-invariant curve on H. Then a (± 1)-surgery in X along α, with framing determined by H, followed by extending the real structure uniquely over the surgered region gives a real 3-manifold equivariantly diffeomorphic to (Y, c).*

Proof. We start with the following splitting of (X, c). Let H_1 and H_2 be the two handlebodies of the Heegaard splitting of (X, c). Now let X_0 be an equivariant neighborhood of H diffeomorphic to $H \times [-1,1]$ and set $X_1 = H_1 - X_0$ which we consider identical to $X_2 = H_2 - X_0$. Then X is equivariantly diffeomorphic to $X_1 \cup_{\mathrm{id}} X_0 \cup_{c|_{X_H}} X_2$ with the real structure $\tilde{c}$ defined as

$$\tilde{c} : \begin{cases} X_1 \to X_2, & x \mapsto x \\ X_2 \to X_1, & x \mapsto x \\ X_0 \to X_0, & x \mapsto c(x). \end{cases}$$

Similarly, we consider such a splitting $(Y_0, Y_1, Y_2; \tilde{c'})$ of Y. Here we consider X_1, X_2, Y_1, Y_2 identical and X_0, Y_0 identical.

Let $\nu(\alpha)$ be an annulus neighborhood of α in H and $N = \nu(\alpha) \times [-1,1]$ be a c-equivariant smooth neighborhood of α in X_0. Since α is both c- and c'-invariant, with a slight abuse, we will also denote by N a small c'-equivariant neighborhood of α in Y_0. We observe that the identity map (again with an abuse) is an equivariant homeomorphism from $X - N$ to $Y - N$, which can be made an equivariant diffeomorphism after smoothing. Now, with $T = S^1 \times D^2$ and φ a (± 1)-slope diffeomorphism on ∂N, $(X - N) \cup_\varphi T$ is diffeomorphic to Y. Here the previous diffeomorphism id_{X-N} extends trivially over T. Moreover the real structure $\tau_\alpha^{\pm 1} c|_{\partial T}$ (here $\tau_\alpha^{\pm 1}$ is considered to be a twist around a copy of α on $\partial T \sim_\varphi \partial N$) can be extended to a real structure over T uniquely. In fact,

(i) If $c|_\alpha = c_1$ then $c|_{\partial N} = c_1$ too. Then we should extend $\tau_\alpha^{\pm 1} c|_{\partial T}$ over T. Since it has 4 real points, it is equivariantly isotopic to c_1. The unique extension over T is c_1. This corresponds to a type-I equivariant surgery.

(ii) If $c|_\alpha = c_2$, i.e. α is real. Then $\tau_\alpha^{\pm 1} c|_{\partial T}$ is equivariantly isotopic to c_4 and the unique extension is c_4. This corresponds to a type-II equivariant surgery.

(iii) If $c|_\alpha = c_3$ then $c|_{\partial N} = c_3$ or $c|_{\partial N} = c_4$. The former is impossible on a real Heegaard surface , so it must have a c_4-neighborhood. Then $\tau_\alpha^{\pm 1} c|_{\partial T}$ is equivariantly isotopic to c_2. The unique extension over T is c_2. This is a type-IV equivariant surgery. □

3.3 Equivariant Lickorish-Wallace theorem

In this section, we will give an equivariant surgery description for a real 3-manifold. Recall that the Lickorish and Wallace theorem states that for any 3-manifold M, there is a link in S^3 such that (± 1)-surgeries on this link gives M. Certainly, even if the 3-manifold is real, its surgery link may not accept still any kind of symmetry in S^3. But it is possible to construct an equivariant surgery link by following the proof of Lickorish- Wallace theorem and adapting every step to the equivariant setting.

We will start with introducing some terminology. For an invariant knot K in (M, c_M) and a surgery coefficient σ, the real 3-manifold obtained by (σ)-equivariant surgery along K is denoted by $(M_\sigma(K), c_M(K))$ where $c_M(K)$ is the real structure on the surgered manifold.

Definition 3.3.1. A link L in a real manifold (M, c_M) is said to be c_M-*equivariant* if L is invariant under c_M and if c_M swaps any pair of knots (K, K') in L, then $K, K' \cap \mathrm{Fix}(c_M) = \emptyset$.

Remark 3. A c_1-knot and a c_2-knot (real knot) intersect by definition, hence they cannot be present at the same time in an equivariant link.

For an equivariant link $(L, \sigma) = (K_1, \sigma_1) \cup \cdots \cup (K_n, \sigma_n)$, $(M(L), c_M(L))$ denotes the real 3-manifold resulting from c_M-equivariant surgery along L. Though we make the definitions in general, we will always have $\sigma = \pm 1$, as they come from the powers of Dehn twists determined by the left or right handedness of the twist. So even if we drop the powers for brevity of the text, one may assume they are always ± 1.

Definition 3.3.2. On a real surface (Σ_g, c), we say that a product $\tau_{a_t}^{\sigma_t} \ldots \tau_{a_1}^{\sigma_1} \cdot \tau_{b_k}^{\pm 1} \ldots \tau_{b_1}^{\pm 1} \cdot \tau_{a_1'}^{\sigma_1} \ldots \tau_{a_t'}^{\sigma_t}$ of Dehn twists is called an *equivariant product* for c if $c(a_i) = a_i'$ and b_j are disjoint c-invariant curves for all $1 \leq i \leq t$ and $1 \leq j \leq k$.

In simple words, a product of Dehn twists is c-equivariant if the leftmost curve is mapped to the rightmost curve, the next leftmost curve is mapped to the next rightmost curve... and in the middle there are disjoint invariant curves.

Lemma 3.3.1. *Let (M, c_M) and (M', c_M') be two real manifolds with the associated real Heegaard splittings (H, c) and (H, c') respectively. Assume that the product*

$\tau_{a_t}^{\sigma_t} \ldots \tau_{a_1}^{\sigma_1} \cdot \tau_{b_k}^{\pm 1} \ldots \tau_{b_1}^{\pm 1} \cdot \tau_{a'_1}^{\sigma_1} \ldots \tau_{a'_t}^{\sigma_t}$ *is an equivariant product for* c. *Then there is an equivariant link* (L, σ) *in* (M, c_M) *with all the surgery coefficients given by* σ *being* ± 1 *and the real 3-manifold* $(M(L), c_L)$ *is equivariantly diffeomorphic to the manifold* $(M', c_{M'})$.

Proof. Topologically, the manifold M' can be obtained from M following the proof of Lickorish and Wallace's theorem [25]. Let $M = U_1 \cup_c U_2$ and $M' = U_1 \cup_{c'} U_2$ where U_1 and U_2 are two genus-g handlebodies bounded by H. We start with taking a neighborhood $H \times [0, 1]$ in M such that $H \times \{0\} \subset U_1$ and $H \times \{1\} \subset U_2$. Let $\epsilon_0 = 0 < \epsilon_1 < \ldots < \epsilon_{n+1} = 1$,$n = 2t + k$ be a sequence of numbers and H_i and H_{i+1} denote the boundary surfaces of the thickened surface $H \times [\epsilon_i, \epsilon_{i+1}]$. The 3 manifold $M' = U_1 \cup_{c'} U_2$ is obtained by identifying the boundary of U_2 with H_1 via c', the boundary H_n of $H \times [\epsilon_n, \epsilon_{n+1}]$ with the boundary H_n of $H \times [\epsilon_{n-1}, \epsilon_n]$ via $\tau_{a'_1}^{\sigma_1}$, . . . , and finally identifying H_0 with the boundary of U_1 via identity. In fact, each of the curves in the product is now lying on the distinct level surfaces H_i and they define a link in M. Equivalently, the 3-manifold M' is said to be obtained by surgery on the link $(a'_t, \sigma_t) \cup \ldots \cup (b_k, \pm 1) \ldots \cup (b_1, \pm 1) \cup \ldots \cup (a_t, \sigma_t)$ where the surgery coefficients $(\sigma_i = \pm 1)$ are given with respect to the framings determined by the surface they lie.

The above construction can be adapted into an equivariant setting. Consider the following splitting of (M, c_M): Take an equivariant neighborhood $H \times [-1, 1]$ of the Heegaard surface H. c_M sends $H \times [-1,0]$ to $H \times [0,1]$, sending H_{-1} to H_1 and $c_M|_{H_{-1}} = c$. Now take t copies, $H \times [i, i+1]$ of $H \times [0,1]$ and t copies, $H \times [-i-1, -i]$ of $H \times [-1, 0]$, $1 \leq i \leq t$. Then (M, c_M) is equivariantly diffeomorphic to the manifold

$$U_1 \cup_{\mathrm{id}} H \times [-t-1, -t] \cup_{\mathrm{id}} \ldots \cup_{\mathrm{id}} H \times [-1, 1] \cup_{\mathrm{id}} \ldots \cup_{\mathrm{id}} H \times [t, t+1] \cup_c U_2$$

with the real structure defined as

$$\tilde{c_M} = \begin{cases} U_1 \to U_2, \quad x \mapsto x \\ U_2 \to U_1, \quad x \mapsto x \\ H \times [i, i+1] \to H \times [-i-1, -i], \quad x \mapsto c_M(x). \end{cases} .$$

From now on, we will use the above splitting and real structure for (M, c_M), and write c_M for $\tilde{c_M}$, abusing the notation.

As above, we push the curves a'_i to the surface H_i, the curves m_j to H_0 and the curves a_i to H_{-i} to get the surgery link for the manifold M'. Labeling the knots as $K_i = a_i$, $C_j = b_j$, $\overline{K}_i = a'_i$, the link

$$(L, \sigma) = (\overline{K}_t, \sigma_t) \cup \ldots \cup (C_k, \pm 1) \cup \ldots \cup (C_1, \pm 1) \cup \ldots \cup (K_t, \sigma_t)$$

is a c_M-equivariant link, as the knot pair K_i and $\overline{K}_i$ are swapped by $\tilde{c_M}|_H = c_M|_H = c$ for each $1 \leq i \leq t$ and C_j is a c-invariant knot for all $1 \leq j \leq k$. From now on, we will drop the σ_i's in the proof. A right handed Dehn twist in the product corresponds to a $(+1)$-surgery and a left handed Dehn twist to a (-1)-surgery.

A little remark here: In the original proof, we push each curve in the product to distinct level of surfaces, in an order starting from the right. Here we push the invariant curves in the middle to the same surface H_0. This is possible because they must be disjoint by definition of an equivariant product.

We will prove that the resulting real manifold $(M(L), \tilde{c_M}(L))$ is equivariantly diffeomorphic to $(M', c_{M'})$ in two steps.

STEP 1: First we will show that with $(M(L), c_M(L))$ is equivariantly diffeomorphic to the real manifold $(\tilde{M'}, \tilde{c_{M'}})$ where

$$\tilde{M'} = U_1 \cup_{id} H\times[-t-1,-t] \cup_{\tau_{a_t}} \ldots \cup_{\tau_{a_1}} H\times[-1,0] \cup_{\tau_{b_k}\cdots\tau_{b_1}} H\times[0,1] \cup_{\tau_{a'_1}} \ldots \cup_{\tau_{a'_t}} H\times[t,t+1] \cup_c U_2$$

with real structure

$$\tilde{c_{M'}} = \begin{cases} U_1 \to U_2, \quad x \mapsto x \\ U_2 \to U_1, \quad x \mapsto x \\ H\times[i,i+1] \to H\times[-i-1,-i], \quad x \mapsto c_M(x). \end{cases}.$$

For simplicity, let us take $k = t = 1$ and drop the indices. Then we have a single invariant knot b lying on the surface H_0, and a pair of knots (a, a') swapped by the real structure in both manifolds. The general case where there are more curves is handled similarly. Now, take an equivariant neighborhood V_0 of the curve b, by first taking an equivariant annular neighborhood of b on H_0 and crossing with a small

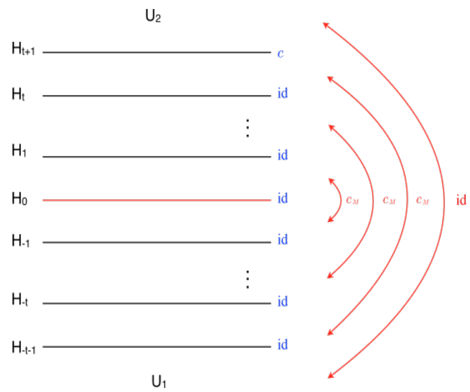

Figure 3.1: The real splitting associated to (M, c_M) is on the left and the real splitting associated to $(\tilde{M}', c_{\tilde{M}'})$ is on the right. Gluing maps between surfaces are shown in blue, real structures are shown in red.

interval I so that the neighborhood remains equivariant and between the surfaces H_1 and H_{-1} without touching them. Then take a pair of equivariant neighborhoods V' of a' between H_2 and H_0, and V of a between H_0 and H_{-2} in M in the similar way. Note that V_0, $V$and V' are also solid tori in $\tilde{M}'$. If you remove the interiors of those solid tori both from M and $\tilde{M}'$, we have identical pieces and this makes

$$\mathrm{id} : M - (\mathring{V}' \cup \mathring{V}_0 \cup \mathring{V}) \to \tilde{M}' - (\mathring{V}' \cup \mathring{V}_0 \cup \mathring{V})$$

an equivariant diffeomorphism. Note if the solid torus V in M has meridian and longitude (μ, λ), where λ is a copy of a, then the solid torus V_i in $\tilde{M}'$ has meridian $\mu \pm \lambda$. Similar arguments apply for V_0 and V'. Hence removing the solid tori from M and gluing solid tori with (± 1)-sloped diffeomorphisms gives $\tilde{M}'$.

We will now write the diffeomorphism between the surgered manifold $M(L)$ and $\tilde{M}'$ explicitly. The surgered manifold and the real structure c_L are constructed as follows: We can take a solid torus $T = S^1 \times D^2$ (T') diffeomorphic to V (V') in $\tilde{M}'$ via $\psi : T \to V$ $(\psi' : T' \to V')$ where ψ (ψ') sends the meridian of T (T') to the meridian $\mu \pm \lambda$ ($\mu' \pm \lambda'$) of V (V'). Also for V_0, as an equivariant solid torus in $\tilde{M}'$, ψ_0 can be taken an equivariant diffeomorphism between (T_0, c_*) and $(V_0, c_{\tilde{M}'}|_{V_0})$. Here we use c_* to indicate one of the real structures c_1, c_2 or c_4.

We will glue back T, V_0 and T' to $M - \mathring{V}' - \mathring{V}_0 - \mathring{V}$ and have

$$(M(L), c_L) = (M - \mathring{V}' - \mathring{V}_0 - \mathring{V}) \cup_{\phi'} T' \cup_{\phi_0} T_0 \cup_{\phi} T$$

where $\phi' : \partial T' \to \partial V'$, $\phi_0 : \partial T_0 \to \partial V_0$ and $\phi : \partial T \to \partial V$ are slope ± 1 diffeomorphisms and the real structure c_L is extended as given in the surgery descriptions:

$$c_L = \begin{cases} M - \mathring{V}' - V_0 - \mathring{V} \to S^3 - \mathring{V}' - \mathring{V}_0 - \mathring{V}, \quad x \mapsto c_{\text{st}}(x) \\ T' \to T, \quad x \mapsto x \\ T_0 \to T_0, \quad x \mapsto c_*(x) \\ T \to T', \quad x \mapsto x. \end{cases}$$

Now we define a diffeomorphism F from M_L to $\tilde{M}'$ as follows:

$$F = \begin{cases} M - \mathring{V}' - \mathring{V}_0 - \mathring{V} \to \tilde{M}' - \mathring{V}' - \mathring{V}_0 - \mathring{V}, \quad x \mapsto x \\ \\ T' \to V', \quad x \mapsto \psi'(x) \\ \\ T_0 \to V_0, \quad x \mapsto \psi_0(x) \\ \\ T \to Vx, \quad \mapsto \psi(x). \end{cases}$$

One can check that F is an equivariant diffeomorphism by showing $F \circ c_L = c_{\tilde{M}'} \circ F$.

$$F \circ c_L = \begin{cases} M - \mathring{V}' - \mathring{V}_0 - \mathring{V} \to \tilde{M}' - \mathring{V}' - \mathring{V}_0 - \mathring{V}, \quad x \mapsto c_M(x) \\ \\ T' \to V, \quad x \mapsto \psi(x) \\ \\ T_0 \to V_0, \quad x \mapsto \psi_0 \circ c_*(x) \\ \\ T \to V', \quad x \mapsto \psi'(x). \end{cases}$$

$$c_{\tilde{M}'} \circ F = \begin{cases} M - \mathring{V}' - V_0 - \mathring{V} \to \tilde{M}' - \mathring{V}' - V_0 - \mathring{V}, \quad x \mapsto c_M(x) \\ \\ T' \to V, \quad x \mapsto \psi(x) \\ \\ T_0 \to V_0, \quad x \mapsto c_{\tilde{M}'}|_{V_0} \circ \psi_0(x) = \psi_0 \circ c_*(x) \\ \\ T \to V', \quad x \mapsto \psi'(x). \end{cases}$$

STEP 2: $(\tilde{M}', c_{\tilde{M}'})$ is equivariantly diffeomorphic to $(M', c_{M'})$.

Take the following splitting of $(M', c_{M'})$:

$$U_1 \cup_{\text{id}} H \times [-t-1,-t] \cup_{\text{id}} \ldots \cup_{\text{id}} H \times [-1,0] \cup_{c'} H \times [0,1] \cup_{\text{id}} \ldots \cup_{\text{id}} H \times [t,t+1] \cup_{\text{id}} U_2$$

with the real structure

$$c_{M'} = \begin{cases} U_1 \to U_2, \quad x \mapsto x \\ U_2 \to U_1, \quad x \mapsto x \\ H \times [i,i+1] \to H \times [-i-1,-i], \quad x \mapsto x. \end{cases}$$

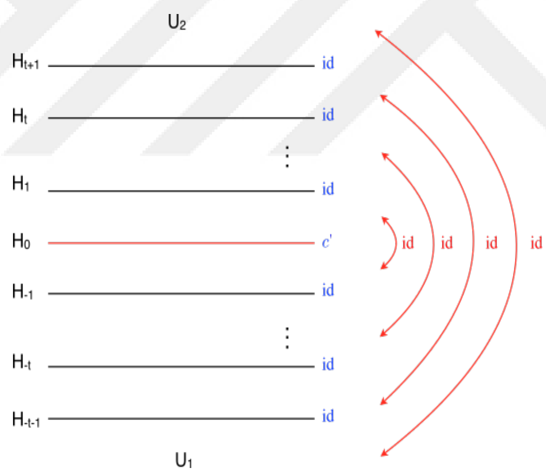

Figure 3.2: The splitting associated with $(M', c_{M'})$..

On the thickened surfaces, define the functions f_i and $\overline{f_i}$:

$f_i : H \times [i,i+1] \to H \times [i,i+1]$ is defined as $f_i = c_M|_{H\times[i,i+1]} \tau_{a'_t}^{-1} \ldots \tau_{a'_{i+1}}^{-1} \times \text{id}$;

$\overline{f_i} : H \times [-i-1,-i] \to H \times [-i-1,-i]$ is defined as $\overline{f_i} := \tau_{a_t} \ldots \tau_{a_{i+1}} \times \text{id}$. Now define a map from $(\tilde{M}', c_{\tilde{M}'})$ to $(M', c_{M'})$ using the above functions.

$$F = \begin{cases} U_1 \to U_1, \quad x \mapsto x \\ U_2 \to U_2, \quad x \mapsto x \\ H \times [i, i+1] \to H \times [i, i+1], \quad x \mapsto f_i(x) \\ H \times [-i-1, -i] \to H \times [-i-1, -i], \quad x \mapsto \overline{f_i}(x). \end{cases}$$

To prove that F is a well-defined diffeomorphism from $\tilde{M}'$ to M', we must show that it is well-defined under the identifications of the boundaries of thickened surfaces and handlebodies. We have

- on $\partial U_2 \sim_c H_{t+1}$ in $\tilde{M}'$, $x \mapsto x$ on $\partial U_2 \subset M'$, $c(x) \mapsto f_t(c(x)) = x$ on $H_{t+1} \subset M'$ and $\partial U_2 \sim_{\mathrm{id}} H_{t+1}$,

- on $H_i \sim_{\tau_{a'_i}} H_i$, $x \mapsto f_i(x)$, $\tau_{a'_i}(x) \mapsto f_{i-1}(\tau_{a'_i}(x)) = c\tau_{a'_t}^{-1} \dots \tau_{a'_i}^{-1}\tau_{a'_i}(x) = f_i(x)$,

- on $H_0 \sim_{\tau_{b_k} \dots \tau_{b_1}} H_0$, $x \mapsto f_0(x)$, $\tau_{b_k} \dots \tau_{b_1}(x) \mapsto \overline{f_0}\tau_{b_k} \dots \tau_{b_1}(x)$ and $c'(f_0(x)) = \overline{f_0}\tau_{b_k} \dots \tau_{b_1}(x)$,

- on $H_{-i} \sim_{\tau_{a_i}} H_{-i}$, $x \mapsto \overline{f_{i-1}}(x) = \tau_{a_t} \dots \tau_{a_i}(x)$, $\tau_{a_i}(x) \mapsto \overline{f_i}(\tau_{a_i}(x)) = \tau_{a_t} \dots \tau_{a_{i+1}}\tau_{a_i}(x)$,

- on $\partial U_1 \sim_{\mathrm{id}} H_{-t-1}$, $x \mapsto x$, $x \mapsto x$.

$$F^{-1} \cdot \tilde{c_{M'}} \cdot F = \begin{cases} U_1 \to U_2, \quad x \mapsto x \\ U_2 \to U_1, \quad x \mapsto x \\ H \times [i, i+1] \to H \times [-i-1, -i], \quad x \mapsto \overline{f_i}^{-1} c_M f_i(x) = x. \end{cases}$$

The fact that F is an equivariant diffeomorhism can be seen by checking $F^{-1} \cdot \tilde{c_{M'}} \cdot F = c'_M$. □

The above lemma proves that in the presence of an equivariant product of Dehn twists one can get an equivariant surgery description in a real 3-manifold. Now we will prove our main theorem by showing that given a real 3-manifold, one can find an equivariant product expression for the real structure in the standard real S^3, hence an equivariant surgery description.

Theorem 3.3.2 (Equivariant Lickorish-Wallace Theorem)**.** *Let (M, c_M) be a real 3-manifold. There is an equivariant link $\mathcal{L} = L \cup L_S \cup \overline{L}$ in the standard real S^3 with L_S is equivariant unlink for c_{st} and $c_{\mathrm{st}}(L) = \overline{L}$ so that (M, c_M) is equivariantly diffeomorphic to the manifold $(S^3_\sigma(\mathcal{L}), c_{\mathcal{L}})$, where the surgery coefficients σ can be taken to be ± 1.*

Proof. Let (M, c_M) be a real 3-manifold with the associated real Heegaard splitting (H, c). Assume the real part of M is non-empty and let $\{r_0, \ldots, r_k\} \subset H$ be the real part of c_M. If you consider the real structure $c' = \tau_{r_1} \ldots \tau_{r_k} \cdot c$ on H, it has a non-separating real part $\{r_0\}$ if $k \geq 1$ or if $k = 0$ and c is non-separating. We know there is a real Heegaard splitting (H, s) of (S^3, c_{st}) with a non-separating real part $\{r\} \subset H$. Note that if $k = 0$ and c is separating, then (S^3, c_{st}) has a real Heegaard splitting (H, s) with separating s by Example 2.2.1. By the classification of real structures on compact surfaces, c' and s are conjugate by an orientation preserving diffeomorphism f. Set $b_i = f(a_i)$ and note that $b_0 = r$. Then we have

$$
\begin{aligned}
f^{-1} \cdot c' \cdot f &= s \\
f^{-1} \cdot \tau_{r_1} \dots \tau_{r_k} \cdot c \cdot f &= s \\
(f^{-1} \cdot \tau_{r_1} \cdot f) \cdot f^{-1} \dots f \cdot (f^{-1} \cdot \tau_{r_k} \cdot f) \cdot f^{-1} \cdot c \cdot f &= s \\
\tau_{b_1} \dots \tau_{b_k} \cdot f^{-1} \cdot c \cdot f &= s \\
f^{-1} \cdot c \cdot f &= \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot s.
\end{aligned}
$$

The curves $b_1, \dots, b_k$ on H are now c_4-invariant with respect to s and we see that c and $\tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot s$ are conjugated by f. Let $f = \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1}$ be an expression for f in terms of generators of the mapping class group of H. Then

$$
\begin{aligned}
c =& f^{-1} \cdot \tau_{b_k}^{-1} \dots \tau_{m_1}^{-1} \cdot s \cdot f \\
=& \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot s \cdot \tau_{a_1}^{-\sigma_1} \dots \tau_{a_t}^{-\sigma_t} \\
=& \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot (s \cdot \tau_{a_1}^{-\sigma_1} \cdot s^{-1}) \cdot s \dots s^{-1} \cdot (s \cdot \tau_{a_t}^{-\sigma_t} \cdot s^{-1}) \cdot s \\
=& \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot \tau_{s(a_1)}^{\sigma_1} \dots \tau_{s(a_t)}^{\sigma_t} \cdot s.
\end{aligned}
$$

We will write $s(a_i) = a'_i$ for short. By the last line of the above equation, the product $c \cdot s^{-1}$ is an equivariant product for s. We complete the proof using the Lemma 3.3.1 and a get an equivariant link of the form $\mathcal{L} = L \cup L_S \cup \overline{L}$ so that L_S is an equivariant unlink consisting of c_4-knots and $c_{\text{st}}(L) = \overline{L}$.

For the case where the real manifold (M, c_M) has empty real part, by Lemma 2.3.2, $\tau_r \cdot s$ is a real structure with empty real part. We follow exactly the same steps and end up with an equivariant link $\mathcal{L} = L \cup L_S \cup \overline{L}$ so that L_S is a c_2-knot (the only real knot) in S^3 and $c_{\text{st}}(L) = \overline{L}$. □

3.4 Equivariant surgery diagrams

In this section, we are going to define equivariant surgery diagrams for real 3-manifolds and the possible types of equivariant link diagrams emerging from our main theorem.

A link diagram L in S^3 or $\mathbb{R}^3$ represents an equivariant link if it is invariant under the rotation about the z -axis. An equivariant link consists of c_1, c_3, c_4-knots or equivariant knot pairs. Note that the only c_2-knot in S^3 is the unknot which is the axis of the rotation and it will not be a part of an equivariant link diagram for two reasons. A c_2-knot and a c_1-knot intersect at two fixed points and they can not be present in an equivariant link diagram at the same time. On the other hand, a type-II equivariant surgery on a c_2-knot destroys that real component and we do not need to do a type-II surgery on S^3 unless we want to obtain a real 3-manifold with empty real part. Here we will give equivariant surgery diagrams for real 3-manifolds with non-empty real part.

Now we will present the particular type of equivariant link diagrams that our main Theorem 3.3.2 requires.

In the most general situation (in an arbitrary real 3-manifold), from an equivariant product $\tau_{a_t} \dots \tau_{a_1} \cdot \tau_{b_k} \dots \tau_{b_1} \cdot \tau_{a'_1} \dots \tau_{a'_t}$ of Dehn twists with ± 1 powers, where the twist curves are simple closed curves on a genus-g Heegaard surface Σ_g, we obtain an equivariant link $\mathcal{L} = L \cup L_S \cup L'$ as follows. Push the pairs (a_i, a'_i) equivariantly into the handlebodies: push a_i in an increasing order to deeper levels of Heegaard surface in one handlebody, and push a'_i equivariantly in the other handlebody. These pairs of curves, now on copies of surfaces on distinct levels, form the sublink $L \cup L'$. Since b_i are disjoint and invariant by definition, they will form the equivariant sublink L_S on the Heegaard surface H. The surgery coefficients will be ± 1, given in terms of the framing coming from the Heegaard surface, this is called the *Heegaard framing.*

Chapter 4

REAL CONTACT STRUCTURES AND THEIR EXISTENCE

This chapter is dedicated to answer whether a real contact structure exists for any real 3-manifold. It is known that one can put a contact structure on any 3-manifold. In the presence of a real structure, can one find a real contact structure? We answer this positively in Section 4.3, using equivariant contact surgery.

4.1 Real open book decompositions

The correspondence between open book decompositions of 3manifolds and contact structures, by E.Giroux [11] provides a ground to explore topological aspects of a geometric structure- a contact structure-. On real 3-manifolds, Öztürk and Salepci introduce the concept of real open books and partially prove the conjecture called the Real Giroux Correspondence [20]. In this section, we review the real open books and real stabilizations and construct some real open books for (c_{st}, S^3).

Definition 4.1.1. Let (M, c_M) be a real 3-manifold and (B, π) be an open book decomposition for M. We say (B, π) is a *real open book* if $c_M(B) = B$ and $\rho \circ \pi = \pi \circ c_M$ where $\rho : S^1 \to S^1$ is a reflection.

In fact, ρ can be fixed to be $z \mapsto \bar{z}$ on $S^1 = \{|z| = 1\}$ since orientation reversing involutions on S^1 form a single conjugacy class. Then as ρ has two fixed points $\{-1, +1\}$, the real open book (B, π) has two invariant (under c_M) pages $S_- = \pi^{-1}(-1)$ and $S_+ = \pi^{-1}(+1)$. Equipped with the real structures induced by the restrictions $c_\pm = c_M|_{S_\pm}$, we call the pages $(S_\pm, c_\pm)$ the *real pages.*

Example 6. *A real open book for the standard real S^3 (identified with $\mathbb{R}^3 \cup \{\infty\}$) can be visualized by rotating Figure 4.1 around the vertical x-axis. The dots connecting the upper and lower arcs will be the binding, upper and lower arcs connecting the dots will be disk pages. Reflection through y-axis induces a real structure on S^3, satisfying the properties of a real open book. It leaves the binding invariant, acts as complex conjugation on the two real pages (seen as two arcs of y-axis divided by the dots). The real part is the y-axis which is a circle in S^3.*

Sometimes, for example when handling stabilizations, it is more convenient to use abstract open books rather than usual open books. An abstract open book decompositon is a pair (S, f) where S is a compact oriented surface with boundary and f is the monodromy: an orientation preserving diffeomorphism : $S \to S$ which is

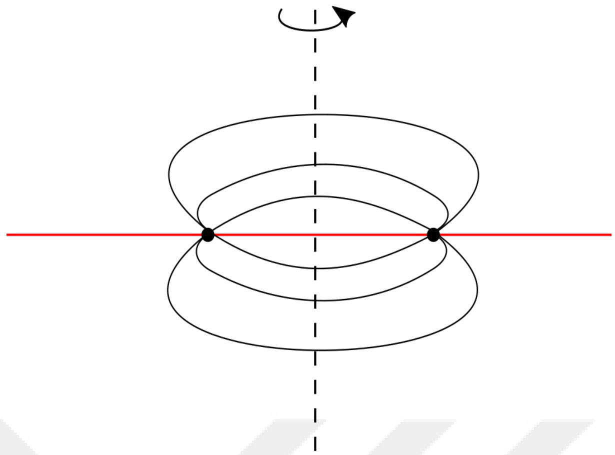

Figure 4.1: The real open book for the standard real S^3.

identity on the boundary ∂S. The abstract open book (S, f) gives rise to a 3-manifold

$$M = S_f \cup_{\text{id}} (\coprod S^1 \times D^2)$$

where S_f is the mapping torus $(S \times [0,1])/\sim$ with the equivalence relation $\sim$ defined by $(\phi(x), 0) \sim (x, 1)$.

It is possible to define abstract real open books for real 3-manifolds.

Definition 4.1.2. A abstract real open book decomposition is a triple (S, f, c) where (S, f) is an abstract open book and c is an orientation reversing involution on S such that $f \circ c = c \circ f^{-1}$.

It is well known that any open book decomposition (B, π) gives rise to an abstract open book (S, f) and this remains true for real version too. A real open book (B, π)

gives rise to a triple $(S_\pm, f, c_\pm)$ where $c_\pm$ is the inherited real structure on the real page $S_\pm$ and f is the monodromy. A fundamental property of such a monodromy is that $f = c_- \circ c_+$. Equivalently, $f^{-1} = c_\pm \circ f \circ c_\pm$. We denote this abstract real open book by (S, f, c) where S is one of the real pages and c is the corresponding real structure. By definition, c acts on ∂S and it either acts as reflection on a connected component or swaps a pair of boundary components.

Stabilizations are handy for constructing a new open book from an old one by adding a handle to the page and adjusting the monodromy.

A positive stabilization of an (abstract) open book (S, f) is an open book with page $S' = S \cup (1\text{-handle})$ and monodromy $f \circ \tau_a$ where a is a simple closed curve on S' intersecting the co-core of the 1-handle exactly once. Positive stabilizations give diffeomorphic 3-manifolds, i.e., the associated 3-manifolds to (S, f) and (S', f') are diffeomorphic. (Same for negative stabilizations where the monodromy is taken to be $f \circ \tau_a^{-1}$).

It is possible to speak of real positive stabilization if we choose the attaching region of the handles to be invariant under the real structure.

Definition 4.1.3. Let (S, f, c) be an abstract real open book. Let $S = S \cup H$ where

- either H is a 1-handle with its attaching region c-invariant, (in particular if the attaching region includes a pair of real points, then we impose the condition that the real points belong to the same real component);

- or $H = H_1 \cup H_2$ where H_1 and H_2 are 1-handles with their attaching regions interchanged by c.

Once the conditions above are satisfied, the real structure c extends uniquely to the handle(s) as $\tilde{c}$ and the monodromy extends as $\tilde{f}$ with $\tilde{f}|_H = \mathrm{id}$. In case where there is a single handle attached, we take a simple closed curve such that $\tilde{c}(a) = a$ intersecting the co-core of the handle once. For the case where a pair of handles is attached, we take a pair of curves $(a, \tilde{c}(a))$ such that a intersects the co-core of one handle once and hence $\tilde{c}(a)$ intersects the co-core of the other handle once. Let σ denote τ_a or the composition $\tau_a \circ \tau_{\tilde{c}(a)}$ depending on the case. Then a positive real stabilization of (S, f, c) is given by the open book $(S', \tilde{f} \circ \sigma, \tilde{c} \circ \sigma)$.

The unique extension $\tilde{c}$ of c to the handles is determined by the invariance type of the attaching region. For example, if the attaching region contains a pair of real points, then on the handle, $\tilde{c}$ acts as a reflection with respect to the core of the handle. This is a type-I attachment in Figure 4.2. All possible equivariant handle attachments and extensions are illustrated in Figure 4.2.

Now let us give some real open books of (S^3, c_{st}) using positive real stabilizations. We will use these open books in next chapter while proving Theorem 4.3.6.

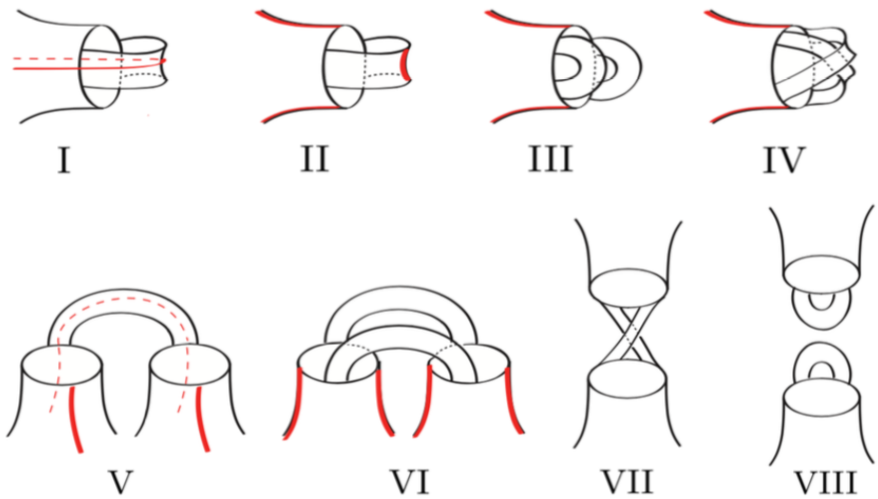

Figure 4.2: Possible equivariant handle attachments. Figure taken form [20].

Example 7. *The simplest open book decomposition of S^3 is the one with disk pages, where the binding is z-axis $\cup\{\infty\}$. Abstractly, the surface S_0 is a disk and the monodromy is $f_0 = \mathrm{id}$. The standard real structure on S^3 is a rotation about an unknot which can be taken to be y-axis $\cup\{\infty\}$. Then this open book is real with 2 real disk pages where the induced real structure is reflection about the y-axis. (S_0, f_0, ρ_0) is the corresponding abstract real open book, where ρ_0 is the reflection on the disk S_0. Succesive handle attachments of type-III when done g times produce a planar open book decomposition (S', f', c') where S' is a disk with $2g$-holes, the monodromy f' is the product of Dehn twists $\tau_{a_1}\tau_{a_1'}\ldots\tau_{a_g}\tau_{a_g'}$ and the real structure c' is $\rho\circ\tau_{a_1}\tau_{a_1'}\ldots\tau_{a_g}\tau_{a_g'}$ where ρ is the reflection about x-axis in Figure 4.3.*

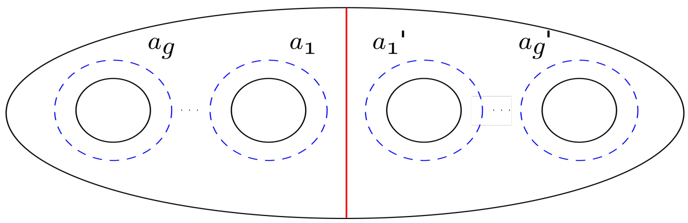

Figure 4.3: A real open book of S^3 after stabilizations of type-III.

A real open book decomposition (B, π) of a real 3-manifold (M, c_M) induces a real Heegaard splitting as follows: let a_1 and a_2 be the upper and lower semicircles of S^1, intersecting at two points such that $\rho(a_1) = a_2$ where ρ is the reflection on S^1 with the property $\pi \circ c_M = \rho \circ \pi$. Then $U_1 = \pi^{-1}(a_1) \cup B$ and $U_2 = \pi^{-1}(a_2) \cup B$ are handlebodies with $c_M(U_1) = U_2$ and $M = U_1 \cup_H U_2$ with the Heegaard surface $H = S_- \cup S_+$. An abstract real open book (S, f, c) induces the abstract real Heegaard splitting $(S \cup S, c \cup f \circ c)$.

For example, the real open book decomposition for S^3 given in the example above induces the abstract Heegaard splitting $(\Sigma_{2g}, \rho \circ \tau_{a_1}\tau_{a_1'} \ldots \tau_{a_g}\tau_{a_g'})$. Note that the real part is separating on the Heegaard surface.

4.2 *Real contact structures*

Definition 4.2.1. Let (M, c_M) be a real 3-manifold. A contact structure ξ on M is said to be real if $c_{M*}(\xi) = -\xi$, where $-\xi$ is the contact structure with planes oppositely oriented. The triple (M, ξ, c_M) will be called a real contact 3-manifold.

For a real contact 3-manifold (M, ξ, c_M), there is a 1-form α for ξ which is anti-symmetric, i.e., satisfying ${c_M}^*(\alpha) = -\alpha$. Recall that we assume our contact structures to be positive and co-oriented. In fact, if $\tilde{\alpha}$ is a 1-form globally defining ξ, we can set $\alpha = \frac{1}{2}(\tilde{\alpha} - c_M^*\tilde{\alpha})$. Conversely, in a real manifold (M, c_M), if a 1-form α defining ξ satisfies ${c_M}^*(\alpha) = -\alpha$, then $c_{M*}(\xi) = -\xi$. Hence a real contact 3-manifold can be given by the triple (M, α, c_M) too.

Example 8 (The standard real contact 3-sphere)**.** *On $S^3 \subset \mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2), the involution $(x_1, y_1, x_2, y_2) \mapsto (x_1, -y_1, x_2, -y_2)$ is a real structure with real part the circle $\{x_1^2 + x_2^2 = 1\}$. Because of the uniqueness of the real structure on S^3 with non-empty real part, we can call it the standard real structure c_{st}. The contact structure given as the kernel of the 1-form*

$$\alpha = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

is real with respect to the standard real structure. This contact structure is called the standard real (tight) contact structure on (S^3, c_{st}).

In complex coordinates (z_1, z_2), the standard real structure is the complex conjugation $(z_1, z_2) \mapsto (\overline{z}_1, \overline{z}_2)$ and the standard real contact structure is given by the contact form

$$\alpha = \frac{1}{2}(z_1 d\overline{z}_1 - \overline{z}_1 dz_1 + z_2 d\overline{z}_2 - \overline{z}_2 dz_2).$$

The natural examples of real contact manifolds are links of real algebraic singularities, where the complex conjugation induces a real structure. The canonical contact structure is real with respect to the real structure induced by the complex conjugation. We restrict ourselves to Brieskorn manifolds in the following example.

Example 9 (Brieskorn manifolds). *Let $V(p,q,r) \subset \mathbb{C}^3$ be the set of zeros of the polynomial $z_1^p + z_2^q + z_3^r$. The Brieskorn manifold $M(p,q,r)$ is defined as*

$$M(p,q,r) = V(p,q,r) \cap S^5_\epsilon.$$

A contact form on $M(p,q,r)$ can be given by

$$\alpha = \frac{i}{2} \sum (z_j d\bar{z}_i - \bar{z}_j dz_j).$$

The complex conjugation restricts to a real structure c on $M(p,q,r)$, making the contact form α a c-real contact form.

A contact structure ξ on a 3-manifold M is said to be supported by an open book (B,π) if there is a 1 form α for ξ with the property that $d\alpha$ is a positive area form on each page and $\alpha > 0$ on the binding B. Any open book supports a contact structure. Giroux proved that this contact structure is unique up to isotopy. In fact, he proved that there is a one to one correspondence between the set of open book decompositions up to positive stabilizations and the set of oriented contact structures up to isotopy. This theorem has an almost analogy for real open books and real contact structures on a real 3-manifold, conjectured and partly proven in [19]. (See the op. cit. for the missing part of the proof.)

Conjecture 4.2.1 (Real Giroux Correspondence)**.** *Let (M, c_M) be a real 3-manifold. Then there is a one to one correspondence between the real contact structures on M up to equivariant contact isotopy and the real open books on M up to positive real stabilization and equivariant isotopy.*

Let (B, π) be an open book supporting the contact structure ξ in M. Then the Heegaard splitting induced by the open book is a convex surface for ξ. This remains true in the real case. The Heegaard surface H is the union of the two real pages S_- and S_+ along their common boundary B. If the open book supports the contact structure ξ and α is a contact form for ξ, choose a volume form Ω on the Heegaard surface H such that Ω defines the same orientation as $d\alpha$ on S_- and the opposite one on S_+. The characteristic foliation is given by the vector field X satisfying $\iota_X \Omega = \alpha|_{TH}$ and is divided by B. Hence, the Heegaard surface $S_- \cup S_+$ is a convex surface with the binding being the dividing set. (For details see [10, Ex 4.8.4(4)].)

We will end this section with the versions of two lemmas from convex surface theory which we shortly discussed in Section 2.4. For more details of equivariant versions of structural theorems in contact topology and convex surfaces, see [20].

Definition 4.2.2. In a real 3-manifold (M, c_M), an invariant (oriented) submanifold

N is called symmetric if c_M preserves the orientation of N, and called antisymmetric if c_M reverses the orientation of N. We write $c_M(N) = N$ for symmetric submanifolds and $c_M(N) = -N$ for antisymmetric submanifolds.

The following lemma is an adaptation of the Giroux's Flexibility Theorem [12].The real version tells us that on an invariant convex surface, the surface can be equivariantly perturbed so that the characteristic foliation becomes symmetric (or antisymmetric).

Lemma 4.2.2 (Real version of Giroux's Flexibility Theorem [19])**.** *Suppose S is an invariant closed convex surface in (M, ξ, c_M); S_ξ the (oriented) characteristic foliation and Γ_S an antisymmetric oriented dividing set. If F is another oriented symmetric (or antisymmetric) singular foliation with the same oriented dividing set, and if the orientations of F and S_ξ agree in a neighborhood of Γ_S, then there is an isotopy ϕ_s $(s \in [0, 1])$ of S satisfying:*

(1) $\phi_0 = \mathrm{id}$,

(2) ϕ_s is equivariant;

(3) $\phi_s|_{\Gamma_S} = \mathrm{id}$*;*

(4) $\phi_s(S)$ is convex and $\xi|_{\phi_1(S)} = F$.

Legendrian realization principle [14, Theorem 3.7] gives a criterion for making a collection C of non-isolating curves and arcs on a convex surface Legendrian after a perturbation of the convex surface. C is said to be non-isolating if C is transverse to Γ_S, every arc in C begins and ends on Γ_S, and every component of $S-(\Gamma_S \cup C)$ has a boundary component which intersects Γ_S . We are going to prove a weaker version in the real setting, for an invariant set of curves on an anti-symmetric convex surface, imposing an additional condition on the set C.

Lemma 4.2.3. *Let S be an anti-symmetric closed convex surface in (M, ξ, c_M) such that Γ_S is an antisymmetric oriented dividing set, $c_M|_S = c$ is an orientation reversing involution. Let C be a disjoint, non-isolating c-invariant set of curves. Assume $S-(\Gamma_S \cup C \cup Fix(c))$ consists only of components $S_1, S_1', \dots S_k, S_K'$ with $c(S_i) = S_i'$ for $1 \leq i \leq k$. Then there exists an equivariant isotopy ϕ_s $(s \in [0,1])$ of S:*

(1) $\phi_0 = \mathrm{id}$*;*

(2) $\phi_s(S)$ *is convex;*

(3) $\phi_1(\Gamma_S) = \Gamma_{\phi_1(S)}$*;*

(4) $\phi_1(C)$ *is Legendrian and* $\phi_1 \circ c \circ \phi_1^{-1}$*-invariant.*

Proof. In the original proof of Legendrian realization principle, a singular foliation F containing the set C (or an isotopic copy) which is also divided by Γ_S is constructed. Then using the Flexibility Theorem, this foliation can be made a characteristic foliation for Γ and since the leaves of the characteristic foliation are Legendrian, so is the set C. We will not repeat the proof here, instead point out how we can do it equivariantly. For a detailed proof see [14] or [7]. The singular foliation F is constructed around a boundary collar of each component of $S - (\Gamma_S \cup C)$ and then extended to the interior. The assumption that C is non-isolating makes it possible to extend C to a singular foliation.

In our case, we consider the set $C \cup \mathrm{Fix}(c)$. Γ_S and $\mathrm{Fix}(c)$ intersect transversally and nontrivially since Γ_S is anti-symmetric, which guarantees that $C \cup \mathrm{Fix}(c)$ is non-isolating provided C is non-isolating. Since $\mathrm{Fix}(c)$ is Legendrian by definition, the components of $S - (\Gamma_S \cup C \cup \mathrm{Fix}(c))$ may be treated as if they are components of $S - (\Gamma_S \cup C)$ with Legendrian boundary $\mathrm{Fix}(c)$. Now we can construct a singular foliation on each piece S_i exactly the same way in the original proof, then extend it to S_i' equivariantly. Then we use the real version of Flexibility to conclude. $\square$

Note that if c is separating, then $S - (\Gamma_S \cup C \cup \mathrm{Fix}(c))$ already consists of equivariant pair of components. The assumption is needed when c is non-separating.

4.3 Equivariant contact surgery

In this section, we will determine the possible cases where an equivariant surgery can be transformed into an equivariant contact surgery. We will consider each equivariant surgery type separately, starting with the equivariant surgery on an invariant knot. The following lemma classifies equivariant tight contact neighborhoods of invariant knots with convex boundary.

Lemma 4.3.1 (Real Contact Neighborhood Theorem for Equivariant Legendrian Knots [19]). *Let K be an invariant Legendrian knot in a real tight contact manifold (M, ξ, c_M). Let N be a sufficiently small real contact neighborhood of K with convex boundary. Then N is isotopic through real tight structures to one of the following:*

(1) $c_M|K = c_1 : N = (S^1 \times D^2, \eta_{tw(K)}, c_1)$ *if* $tw(K) \neq 0$*; and* $N = (S^1 \times D^2, dz + xdy, c_1)$ *if* $tw(K) = 0$.

(2) $c_M|K = \mathrm{id} : K$ *is real and* $N = (S^1 \times D^2, \eta_{tw(K)}, c_2)$.

(3) $c_M|K = c_3 : N = (S^1 \times D^2, \eta_{tw(K)}, c_3)$ *if* $tw(K)$ *is odd;*

$N = (S^1 \times D^2, \eta_{tw(K)}, c_4)$ *if* $tw(K)$ *is even.*

Here the contact structure η_n is given by the 1-form $\cos(2\pi nz)dx + \sin(2\pi nz)dy$ when $n \neq 0$ and $\eta_0 = xdz + dy$.. The z-direction is the direction of the core K.

Before discussing the equivariant contact surgery definition, we shall state the theorems regarding the real tight contact solid tori. The concept of contact surgery (for any nonzero rational coefficient) and its validity relies on the existence of the tight contact solid tori for each rational (nonzero) slope. In case we have an equivariant knot, if we want to perform an equivariant contact surgery, we need to make sure that there are real tight contact solid tori with needed slopes. However real tight solid tori do not exist for some rational slope.

Theorem 4.3.2 (Real Tight Solid Tori ,[22]). *We have the following slope restrictions on real tight solid tori:*

- *A c_1-real tight solid torus can have any rational slope $p/q \neq 0$;*
- *A c_2-real tight solid torus can have only slopes $\frac{1}{k}$, $k \in \mathbb{Z}$, and there is a unique one for each k;*
- *A c_3-real tight solid torus can have only slopes $\frac{1}{2k+1}$, $k \in \mathbb{Z}$, and there is a unique one for each k;*
- *A c_4-real tight solid torus can have only slopes $\frac{1}{2k}$, $k \in \mathbb{Z}$, and there is a unique one for each k.*

Now let K be an equivariant Legendrian knot in a real tight contact manifold (M, ξ, c_M). Let N be an equivariant contact neighborhood of K. On the convex boundary $\partial N = T$ denote the meridian by μ and the contact longitude by λ_c. As we explained in Chapter 2.5, to perform a p/q-contact surgery along K, we remove the interior of N and glue back a tight solid torus with slope $-p/p'$, where p', q' are integers satisfying $pq' - qp' = 1$. Recalling definitions of equivariant surgeries for arbitrary coefficients in Chapter 3.1, we will define equivariant contact surgeries for each knot type below.

Case 1. K is a c_1-knot. After surgery, we need to glue back a c_1-real tight solid torus with slope $-p/p'$, which exists for any slope. Hence, contact surgery can be made c_1-equivariantly for any nonzero surgery coefficient

Case 2. K is a c_2-knot. The contact surgery coefficient should be $1/q$ to be able to glue back a real tight solid torus. That makes $p = 1$.

If q and q' are both odd, then p' must be even and the real contact structure extends uniquely as a c_4-tight solid torus with slope $-1/p'$.

If q is odd and q' is even, then p' must be odd and the real contact structure extends uniquely as a c_3-tight solid torus with slope $-1/p'$.

If q is even and q' is odd, then the real contact structure extends as a c_2-tight solid torus with slope $-1/p'$.

Case 3. K is a c_3-knot. The contact surgery coefficient should be $1/q$ to be able to glue back a real tight solid torus. That makes $p = 1$.

If p' is odd, then real structures extends as c_4 but no c_4-real tight solid torus exists with slope $-1/p'$ for odd p'.

If p' is even, then p' must be odd and the real structure extends as c_3 but no c_3-real tight solid torus exists with slope $-1/p'$ for even p'.

Case 4. K is a c_4-knot. The contact surgery coefficient should be $1/q$ to be able to glue back a real tight solid torus. That makes $p = 1$.

If $q+1$ and $q'+p'$ are both odd, then p' must be even and the real contact structure extends uniquely as a c_4-tight solid torus with slope $-1/p'$.

If $q+1$ is odd and $q'+p'$ is even, then p' must be odd and the real contact structure extends uniquely as a c_3-tight solid torus with slope $-1/p'$.

If $q+1$ is even and $q'+p'$ is odd, then the real contact structure extends uniquely as a c_2-tight solid torus with slope $-1/p'$.

Let us summarize the above discussions in the following theorem.

Theorem 4.3.3 (Equivariant Contact Surgeries for Equivariant Legendrian Knots)**.**

(i) Along a Legendrian c_1-knot, equivariant contact surgery can be performed for any contact surgery coefficient. The real contact structure can be extended as a c_1-real tight structure over the solid torus (not uniquely).

(ii) Along a Legendrian c_2-knot, equivariant contact surgery can be performed for contact surgery coefficient $1/q$. The real contact structure can be extended as c_2-, c_3- or c_4-real tight structure over the solid torus. How to extend the real contact structure uniquely is given in Table 4.1.

(iii) Equivariant contact surgery along a Legendrian c_3-knot does not exist.

(iv) Along a Legendrian c_4-knot, equivariant contact surgery can be performed for contact surgery coefficient $1/q$. The real contact structure can be extended as c_2-, c_3- or c_4-real tight structure over the solid torus. How to extend the real contact

structure uniquely is given in Table 4.1.

Case	c_2-knot	c_4-knot
q and q' are odd	c_4-real tight	c_2-real tight
q odd and q' even	c_3-real tight	c_2-real tight
q even and p', q' odd	c_2-real tight	c_3-real tight
q, p' even and q' odd	c_2-real tight	c_4-real tight

Table 4.1: The extensions of the real contact structure for c_2- and c_4-knots after contact $1/q$-surgery.

Again, we are particularly interested in (± 1)-surgeries given with the Heegaard framing as the procedure in Theorem 3.3.2 suggests. So we will restrict ourselves to that case and determine the conditions allowing us to perform equivariant contact surgeries. Let tw denote the difference between Heegaard framing and the contact framing. A $(+1)$-surgery amounts to a $(1 - \mathrm{tw})$-contact surgery. Then we apply Theorem 4.3.3 with $p = (1-\mathrm{tw})$, $q = 1$, $p' = -\mathrm{tw}$, $q' = 1$. For c_1-knots, we glue back a c_1-real tight solid torus with slope $\frac{1-\mathrm{tw}}{\mathrm{tw}}$. For c_2-knots, equivariant contact surgery is possible only when $\mathrm{tw} = 2$ and we glue back a c_4-real tight solid torus with slope $-1/2$. Finally for c_4-knots, equivariant contact surgery is possible only when $\mathrm{tw} = 2$ and we glue back a c_2-real tight solid torus with slope $-1/2$.

Similarly treating the case of (-1)-surgery with respect to the Heegaard framing, we determine conditions for equivariant contact surgery. For c_1-knots, we glue back a c_1-real tight solid torus with slope $\frac{1+\mathrm{tw}}{\mathrm{tw}}$. For c_2-knots, equivariant contact surgery is possible only when $\mathrm{tw} = -2$ and we glue back a c_4-real tight solid torus with slope $-1/2$. For c_4-knots, equivariant contact surgery is possible only when $\mathrm{tw} = -2$ and we glue back a c_2-real tight solid torus with slope $-1/2$.

Finally, in case we have an equivariant knot pair (K, K'), we can put them in a Legendrian position without spoiling the equivariance. Now after we do a suitable contact surgery on K gluing back a contact solid torus $(S^1 \times D^2, \xi)$, all we need to do is to glue back the contact solid torus $(S^1 \times D^2, -\xi)$ when doing contact surgery along K'.

The ongoing discussion proves the following proposition.

Proposition 4.3.4. *The twisting conditions for an equivariant contact $(\pm 1 - tw)$-surgery and how you extend the real contact structure are given below.*

For a c_1-knot: $tw \neq \pm 1$, you glue back in a c_1-real tight solid torus with slope $\frac{(1 \pm tw)}{tw}$.

For a c_2-knot: $tw = \pm 2$, you glue back in a c_4-real tight solid torus with slope $-1/2$.

For a c_4-knot: $tw = \pm 2$, you glue back in a c_2-real tight solid torus with slope $-1/2$.

Before moving to Theorem 4.3.6, which is the main result of this chapter, we prove a lemma regarding convex Heegaard splittings of the real tight S^3.

Lemma 4.3.5. *For any genus $g \geq 1$, $(S^3, \xi_{std}, c_{\mathrm{st}})$ has a real Heegaard splitting where the Heegaard surface H is convex. Moreover, there are g disjoint c_4-circles $m_1, \dots, m_g$ on H such that $|m_i \cap \Gamma| = 4$ where Γ is the dividing set of H.*

Proof. We will start with a real open book decomposition of (S^3, c_{st}) with disk pages and by positive real stabilizations we will increase the genus of the induced Heegaard splitting. As the Heegaard surface induced by an open book decomposition is convex, the first part will be proven.

Now let $(S_0 = D_1, f_0 = \mathrm{id}, c_0 = \rho_0)$ be the real open book decomposition of (S^3, c_{st}) where ρ_0 is the reflection with respect to the y-axis. If we perform a real positive stabilization of type-II, the resulting real pages are annuli with the real structures shown in Figure 4.4.

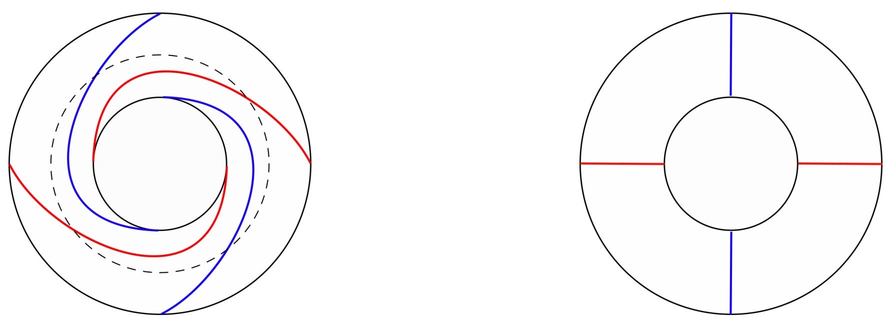

Figure 4.4: The real pages (S_-, c_-) (left) and (S_+, c_+)(right) after a real positive stabilization of type-II.

Performing real positive stabilizations of type-II g times successively, the real pages (S_-, c_-) and (S_+, c_+) become disks with g holes with $c_- = \rho_g \circ \tau_{a_1} \circ \cdots \circ \tau_{a_g}$

and $c_+ = \rho_g$ where ρ_g is the reflection having $g+1$ real arcs, see Figure 4.6.

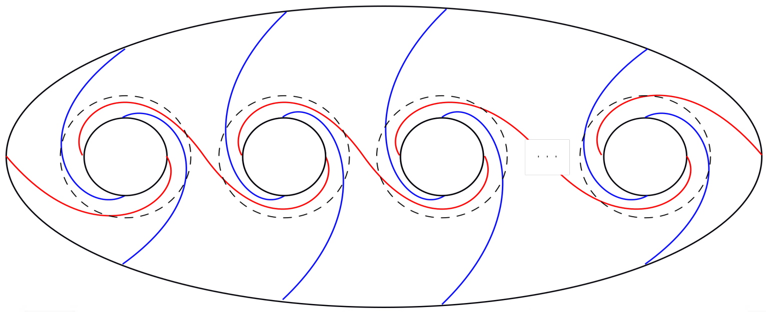

Figure 4.5: The real page (S_-, c_-). The red arcs are real, blue arcs are invariant.

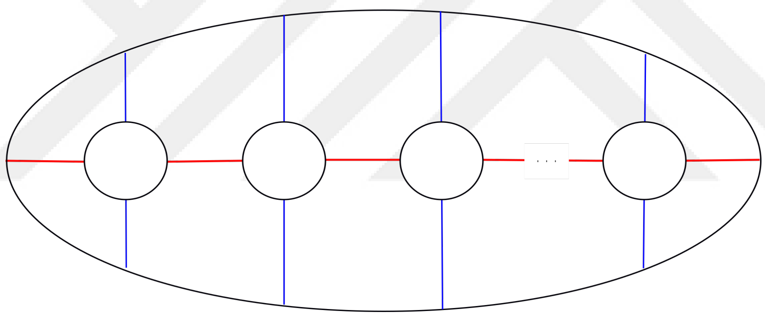

Figure 4.6: The real page (S_+, c_+). The red arcs are real, blue arcs are invariant.

We obtain a real Heegaard splitting with convex surface $H = S_- \cup S_+$ of genus g. The dividing set is the binding and the gluing map is $s = c_- \cup c_+$. The blue curves shown in Figure 4.5 are c_4-curves with respect to s and satisfy the conditions of the theorem. $\square$

Theorem 4.3.6. *Every real 3-manifold admits a real contact structure.*

Proof. Let (M, c_M) be a real 3-manifold with the real Heegaard splitting (H, c). By Theorem 3.3.2 there is an equivariant link $\mathcal{L} = L \cup L_S \cup \overline{L}$ in (S^3, c_{st}) so that $(S^3_{\mathcal{L}}, c_{\mathcal{L}})$ is equivariantly diffeomorphic to (M, c_M). Here the link L_S consists of unlinked c_{st}-invariant knots and L and $\overline{L}$ are swapped by c_{st}. To turn this equivariant surgery into an equivariant contact surgery, we must put L_S into an equivariant Legendrian position so that the components of L_S satisfy the twisting condition in Proposition 4.3.4. For L and $\overline{L}$, it is enough to make them Legendrian respecting the equivariance. Using the set up in the proof of Theorem 3.3.2, let Fix(c) consists of $k+1$ real circles $r_0, \ldots, r_k$. Consider (S^3, c_{st}) with the real contact Heegaard splitting (H, s) induced by the real open book decomposition given in Lemma 4.3.5. Let $r \subset H$ denote the real circle of s. Also, there are k c_4-circles (with respect to s) $m_1, \ldots m_k$ on H by Lemma 4.3.5. Then the $c' = \tau_{m_1}^{-1} \ldots \tau_{m_k}^{-1} \cdot s$ is a real structure with $\text{Fix}(c') = \{r, m_1, \ldots, m_k\}$. Now that c and c' are conjugate, there is an orientation preserving diffeomorphism f of H given by the product $\tau_{a_1}^{\sigma_1} \ldots \tau_{a_t}^{\sigma_t}$ such that $c = f \circ c' \circ f^{-1}$. Following the steps in the proof of Lemma 3.3.1, we get an equivariant surgery link $L \cup L_S \cup \overline{L}$ where L_S consists of the knots $m_1, \ldots, m_k$ on H. On each m_i, we perform a type-IV (-1)-equivariant surgery. Note that since $\{m_1, \ldots, m_k\}$ is a non-isolating set of curves transverse to the dividing set, we can make each m_i Legendrian with an equivariant perturbation of H by Lemma 4.2.3.

Moreover, by Theorem 2.5.1, tw $= -2$ since $|m_i \cap \Gamma| = 4$. Hence, the condition in order to perform an equivariant contact surgery on a c_4-knot is satisfied and we perform an equivariant $(+1)$-contact surgery on each m_i.

Note that if $k = 0$, r_0 is separating, we cannot use the real Heegaard splitting given by Lemma 4.3.5 since r is non-separating there and so that c and s are not conjugate. In that case we use the real Heegaard splitting induced by the real open books given in Example 7 and Figure 4.3. The real circle is separating in this case and c and s are conjugate. The rest follows as above. □

In the proof of Theorem 3.3.2, the purpose of performing type-IV surgeries is to increase the number of real components. Another way to increase the number of real circles is by doing type-I surgery on a c_1-knot with the fixed points belonging to the same real component. Now we are going to present this procedure.

Lemma 4.3.7. *For any genus $g \geq 1$, $(S^3, \xi_{std}, c_{\mathrm{st}})$ has a real Heegaard splitting where the Heegaard surface H is convex. Moreover, there are g disjoint c_1-circles $n_1, \ldots, n_g$ on H such that $|n_i \cap \Gamma| = 0$ where Γ is the dividing set of H.*

In fact, one only needs to choose the c_1-circles $n_1, \ldots, n_g$ circles as depicted in Figure 4.8 on the contact Heegaard splitting given in Lemma 4.3.5.

The curves $n_1, \ldots, n_g$ are already Legendrian since they are the Legendrian divides of the convex surface H. By 2.5.1, tw $= 0$ for each n_i, hence their Heegaard

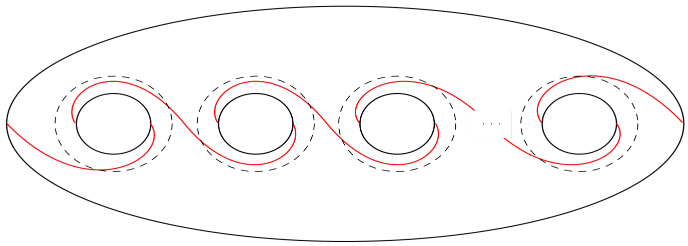

Figure 4.7: The real page (S_-, c_-). The red arcs are real.

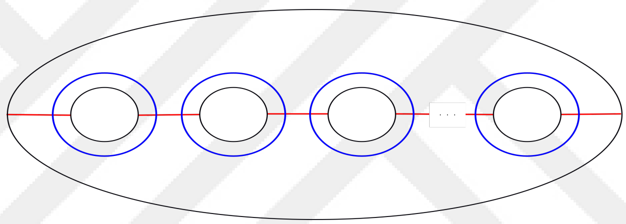

Figure 4.8: The real page (S_+, c_+). The red arcs are real, blue curves are c_1.

framing and contact framing are the same. We can repeat the proof of Theorem 4.3.6 using these curves, which presents at the end a different type of equivariant surgery link.

We have offered two different ways to prove Theorem 4.3.6. Each way presents a different type of equivariant surgery link. For further reference, let us recapitulate these approaches as two special algorithms.

Algorithm 1:

- Let (M, c_M) be a real 3-manifold with $|\text{Fix}(c_M)| = k$ and a real Heegaard splitting (H, c).

- We take a real Heegaard splitting (H, s) of $(S^3, c_{\mathrm{st}}, \xi_{st})$ with a convex Heegaard surface H.

- For L_S, choose $(k-1)$ disjoint Legendrian c_4-knots m_i with $\mathrm{tw} = -2$ and perform equivariant contact $(+1)$-surgery on each.

- Write $c \circ s^{-1} = \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_{m_{k-1}}^{-1} \dots \tau_{m_1}^{-1} \cdot \tau_{a'_1}^{\sigma_1} \dots \tau_{a'_t}^{\sigma_t}$

- Form the link L and $\overline{L}$ from the pairs (a_i, a'_i)'s. Then perform an appropriate equivariant contact surgery along L and $\overline{L}$.

Algorithm 2:

- Let (M, c_M) be a real 3-manifold with $|\mathrm{Fix}(c_M)| = k$ and a real Heegaard splitting (H, c).

- We take a real Heegaard splitting (H, s) of $(S^3, c_{\mathrm{st}}, \xi_{st})$ with a convex Heegaard surface H.

- Take $(k-1)$ disjoint Legendrian c_1-knots n_i with $\mathrm{tw} = 0$ and perform equivariant contact (± 1)-surgery on each.

- Write $c \circ s^{-1} = \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_{n_{k-1}}^{\pm 1} \dots \tau_{n_1}^{\pm 1} \cdot \tau_{a'_1}^{\sigma_1} \dots \tau_{a'_t}^{\sigma_t}$

- Form the link L and $\overline{L}$ from the pairs (a_i, a_i')'s. Then perform an appropriate equivariant contact surgery along L and $\overline{L}$.

4.4 Equivariant contact surgery diagrams

A contact surgery diagram is represented by the front projection of a Legendrian link in (S^3, ξ_{st}) with surgery coefficients given with respect to the contact framing. A contact surgery diagram defines a contact 3-manifold where the contact structure of the surgered manifold is not necessarily unique unless the contact surgery coefficients are of the form $1/k$, $k \in \mathbb{Z}$. Recall the standard real contact 3-sphere in Example 8. An equivariant contact surgery diagram in $(S^3, \xi_{st}, c_{\mathrm{st}})$ is a contact surgery diagram (on yz-plane) where

1) either it is invariant under the reflection about the y-axis or under the symmetry with respect to the the origin of the yz-plane;

2) any c_1-knot has contact surgery coefficient of the form $p/q \neq 0$;

3) any c_4-knot has contact surgery coefficient of the form $1/q$, $q \in \mathbb{Z}$;

4) any two knots as equivariant pair have the same contact surgery coefficient.

Note that in an equivariant contact surgery diagram we cannot have any c_2-knots since there is only one real circle in (S^3, c_{st}) and it is the symmetry axis. Moreover, there is no need for a type-II surgery unless you want to destroy the real part and end up with a real 3-manifold with empty real part. Finally, c_1- knots and c_4-knots cannot be present at the same time in an equivariant contact surgery diagram. If the diagram is invariant under the reflection through y-axis, there is no c_4-knots. On the other hand, if it is invariant under the symmetry with respect to the origin, there is no c_1-knot as there are no real points on the plane. Examples will be given in the next the chapter.

Remark 4. We gave two algorithms producing equivariant contact surgery diagrams in the previous section. Note that they only produce a small subset of equivariant contact surgery diagrams. Algorithm I produces a diagram with unlinked c_4-knots with $+1$-surgery coefficients and equivariant knot pairs with integer surgery coefficients. Algorithm II produces a diagram with unlinked c_1-knots with (± 1)-surgery coefficients and equivariant knot pairs with integer surgery coefficients.

Chapter 5

EXAMPLES

This chapter is a display for the applications of the results we obtained in Chapter 3 and 4. For some real 3-manifolds, we will give equivariant surgery descriptions and provide equivariant contact surgery diagrams.

5.1 $S^1 \times S^2$

The classification of all involutions on $S^1 \times S^2$ up to conjugation by a diffeomorphism of $S^1 \times S^2$ is due to Tollefson [27]. There are 13 classes in total and 4 of them are orientation preserving with one dimensional fixed point set. Hence there are 4 real structures on $S^1 \times S^2$ up to equivalence. If we identify $S^1 \times S^2$ with $\{(\theta, (x, y, z)) : x^2 + y^2 + z^2 = 1\} \subset S^1 \times \mathbb{R}^3$, there are 2 real structures with two real circles up to conjugation:

(1) $s_1 : (\theta, (x, y, z)) \mapsto (\theta, (-x, -y, z))$;

(2) $s_2 : (\theta, (x, y, z)) \mapsto (-\theta, (x, y, -z))$;

Now consider $S^1 \times S^2$ as the identification space obtained from $[-1,1] \times S^2$ by identifying $(-1,(x,y,z))$ with $(1,(-x,-y,z))$. The involutions with one real circle up to conjugation are:

(3) $s_3 : (t,(x,y,z)) \mapsto (t,(x,-y,-z))$;

(4) $s_4 : (t,(x,y,z)) \mapsto \begin{cases} (1-t,(-x,-y,-z)) \text{ if } 0 \leq t \leq 1 \\ (-1-t,(x,y,-z)) \text{ if } -1 \leq t < 1 \end{cases}$.

Each of the real structures above can be associated with a real genus-1 Heegaard splitting of $S^1 \times S^2$.

Let V be a solid torus with $\partial V = T$. Identifying $H_1(T;\mathbb{Z})$ with $\mathrm{SL}(2,\mathbb{Z})$, let $a = [1 \ \ 0]^T$ denote the meridian and $b = [0 \ \ 1]^T$ denote the longitude of V. Any simple closed curve on T can be represented as $xa + yb$ where $(x,y) = 1$, $x,y \in Z$. Note that the right-handed Dehn twists τ_a and τ_b generate the mapping class group of T where

$$\tau_a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tau_b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

By the classification of orientation reversing involutions on the torus T, any real structure with non-empty real part is conjugate to $s_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ if it has two real circles and conjugate to $c_{\text{st}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if it has only one real circle. We label these matrices as s_1 and c_{st} since we associate these real structures with the standard genus-1 real Heegaard splittings of $(S^1 \times S^2, s_1)$ and (S^3, c_{st}).

Consider the real structures on $T = S^1 \times S^1$ given by the matrices

$$s_1 = \begin{bmatrix} -1 & 2k \\ 0 & 1 \end{bmatrix}, \ s_2 = \begin{bmatrix} 1 & 2k \\ 0 & -1 \end{bmatrix}, \ s_3 = \begin{bmatrix} -1 & 2k+1 \\ 0 & 1 \end{bmatrix}, \ s_4 = \begin{bmatrix} 1 & 2k+1 \\ 0 & -1 \end{bmatrix}.$$

The abstract real Heegaard splitting (T, s_i) is associated to the real 3-manifold $(S^1 \times S^2, s_i)$. This can be seen by computing the quotients and the number of real components.

Now that we have an abstract real Heegard splitting for each real $S^1 \times S^2$, we will demonstrate how to obtain them via equivariant surgery from the standard real 3-sphere via Theorem 3.3.2

(1) The gluing involution s_1 satisfies $s_1 = \tau_{a+b}^{-1} c_{\text{st}}$ where $(a+b)$ is a c_4-knot for c_{st}. Then $(S^1 \times S^2, s_1)$ can be obtained from (S^3, c_{st}) by a an equivariant (-1)-surgery of type-IV on the unknot represented by $(a+b)$. Remember that the equivariant surgery coefficient is given with respect to the Heegaard framing. In this case, since the Seifert framing and theHeegaard framing of $(a+b)$ differ by 1, (-1)-surgery with

respect to the Heegard framing corresponds to a 0-topological surgery on the unknot, which gives $S^1 \times S^2$.

(2) We have $s_2 = \tau_{a-b} c_{\text{st}}$ where $(a-b)$ is a c_1-knot for c_{st}. In this case $(S^1 \times S^2, s_2)$ is given by an equivariant $(+1)$-surgery of type-I on the unknot represented by $(a-b)$ in (S^3, c_{st}). Here, the Seifert framing and the Heegaard framing of $(a-b)$ differ by -1 and $(+1)$-surgery with respect to the Heegard framing corresponds to a 0-topological surgery on the unknot.

(3) For $(S^1 \times S^2, s_3)$, we have $s_3 = \tau_b \tau_a c_{\text{st}}$, and $c_{\text{st}}(a) = b$. Hence $(S^1 \times S^2, s_3)$ is given by an equivariant $(+1)$-surgery of type-V on the Hopf link formed by the knot pair (a, b). Here Seifert framing and Heegaard framing of both a and b coincide.

(4) For $(S^1 \times S^2, s_4)$, we have $s_3 = \tau_b^{-1} \tau_a^{-1} c_{\text{st}}$, and $c_{\text{st}}(a) = b$. Hence$(S^1 \times S^2, s_3)$ is given by an equivariant (-1)-surgery of type-V on the Hopf link formed by the knot pair (a, b).

The tight contact structure on $S^1 \times S^2$ unique up to isotopy is given by

$$\xi = ker(xd\theta + ydz - zdy).$$

Apparently, the contact structure ξ is real with respect to s_1 and s_2. To this point, we do not know whether it is real with respect to s_3 and s_4. We will try to find that out using equivariant contact surgery.

The unique tight contact structure ξ of $S^1 \times S^2$ is obtained from (S^3, ξ_{st}) through contact $(+1)$-surgery on the unknot with tb$= -1$, rot$= 0$ [4] (see Figure 5.1).

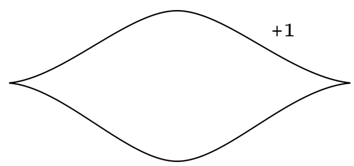

Figure 5.1: The contact surgery diagram for the unique tight contact structure on $S^1 \times S^2$

Indeed the contact surgery diagram in Figure 5.1 can be made equivariant in two ways as in Figure 5.2. Now let us see that our two algorithms produce both diagrams and give an equivariant contact surgery descriptions for $(S^1 \times S^2, \xi, s_1)$ and $(S^1 \times S^2, \xi, s_2)$.

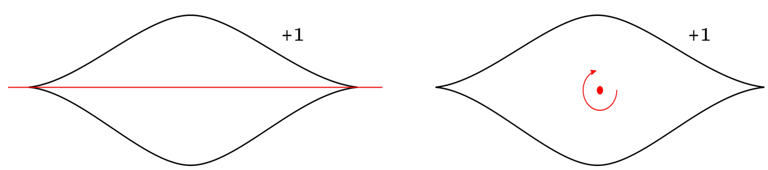

Figure 5.2: Two ways that the contact surgery diagram can be made equivariant.

According to the equivariant surgery description above, $(S^1 \times S^2, s_1)$ is given by an equivariant (-1)-surgery on the c_4-knot $(a+b)$. For the real contact part of the surgery, the abstract real Heegaard splitting (T, c_{st}) of (S^3, c_{st}) can be taken to be a contact Heegard splitting of (S^3, ξ_{st}) where T is a convex torus with two parallel dividing curves of slope -1, each represented by $(a-b)$. Then the curve $(a+b)$ can be made Legendrian equivariantly according to the Legendrian realization principle, by Lemma 4.2.3. We can compute the twisting number

$$\mathrm{tw} = -\frac{1}{2}|(a+b) \cap \{(a-b) \cup (a-b)\}| = -2$$

by Theorem 2.5.1. Now take out a standard c_4-real contact neighborhood of $(a+b)$ and glue a c_2-real contact solid torus with boundary slope $\frac{1+\mathrm{tw}}{\mathrm{tw}} = 1/2$. The resulting s_1-real contact structure on $S^1 \times S^2$ is the unique tight contact structure ξ. This is because $\mathrm{tw} - \mathbf{tb} = -1$ and a -1 surgery on the unknot $a+b$ with $\mathrm{tw} = -2$ corresponds to a $+1$-contact surgery on the unknot with $\mathbf{tb} = -1$. The corresponding equivariant contact surgery diagram is the one on the right in Figure 5.2.

$(S^1 \times S^2, s_2)$ is given by an equivariant $(+1)$-surgery on the c_1-knot $(a-b)$. As in the above case we take the same real Heegaard splitting with convex Heegaard torus. Then we have $\mathrm{tw} = 0$ as $(a-b)$ is parallel to the dividing set. The rest follows similarly except we glue a c_1-real contact solid torus with boundary slope $\frac{1-\mathrm{tw}}{\mathrm{tw}} = \infty$. The corresponding equivariant contact surgery diagram is the one on the left inFigure 5.2.

$(S^1 \times S^2, s_3)$ is obtained by doing a type-V $(+1)$-surgery on the knot pair formed by the curves (a, b). Note that Heegaard framing and Seifert framing coincide for both unknots a and b and they form a Hopf link in S^3. For contact part, we can make a Legendrian so that it is an unknot with $\mathrm{tw} = \mathbf{tb} = -1$, and an equivariant copy represents a Legendrian unknot b. The equivariant contact surgery diagram for an s_3-real contact structure is given in Figure 5.3. Indeed, this diagram produces a tight contact structure and since ξ is the unique tight contact structure on $S^1 \times S^2$, we can say that ξ is also s_3-real.

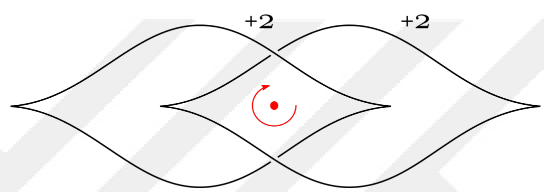

Figure 5.3: An equivariant contact surgery diagram for a s_3-real contact structure.

$(S^1 \times S^2, s_4)$ is given by a type-V (-1)-surgery on the same Hopf link as above and the corresponding equivariant contact surgery diagram is shown in Figure 5.4. Unlike the situation with s_3, this s_4-real contact structure is overtwisted. So the attempt to find out whether ξ is s_4-real or not is inconclusive with this particular surgery.

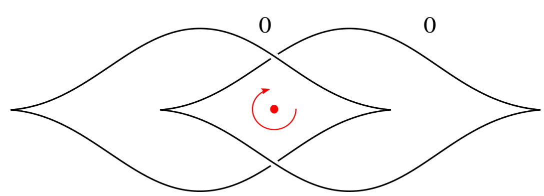

Figure 5.4: An equivariant contact surgery diagram for a s_4-real contact structure.

5.2 Lens spaces

Classification of real structures with non-empty real part on lens spaces up to equivariant isotopy is due to Hodgson and Rubinstein, presented in [15]. Since every lens space has a genus-1 Heegaard splitting, using the classification of real structures on solid tori, one can classify the real structures on lens spaces. Let $L(p,q)$ be a lens space and g be a real structure on $L(p,q)$. By [15, Theorem 4.2] any Heegaard torus in $L(p,q)$ can be isotoped to a g-equivariant Heegaard torus. Assume T is a g-equivariant torus in $L(p,q)$ bounding solid tori V and V'. There are two options for g:

(i) g is orientation preserving on T so that it extends to a real structure on each solid torus V and V';

(ii) g is orientation reversing on T and it sends V to V'.

Note that the latter case corresponds to a real genus-1 Heegaard splitting of $(L(p,q),g)$. Assume g is orientation preserving on T. Then there are three cases, each giving distinct real structures up to equivariant isotopy.

1) Type A: $g|_V = c_1$ and $g|_{V'} = c_1$.

2) Type B: $g|_V = c_2$ and $g|_{V'} = c_3$ or c_4

3) Type B': $g|_V = c_3$ or c_4 and $g|_{V'} = c_2$.

If g is orientation reversing on T, then it is represented by one of the matrices

$\pm \begin{bmatrix} -q & p' \\ p & q \end{bmatrix}$ where $q^2 + pp' = 1$. These two cases give us two more distinct real structures up to equivariant isotopy:

4) Type C: $g|_T = \begin{bmatrix} -q & p' \\ p & q \end{bmatrix}$

5) Type C': $g|_T = \begin{bmatrix} q & -p' \\ -p & -q \end{bmatrix}$. For $p = 2$, $L(p, q) \simeq \mathbb{RP}^3$ and all of the classes are isotopic. Hence, there is a unique real structure on $\mathbb{RP}^3$ with non-empty real part.

For $p > 2$, all of the classes are listed in Table 5.1.

Case	Equivariant isotopy classes
$q \equiv 1 \pmod{p}$	$A \simeq C, B \simeq B' \simeq C'$
$q \equiv -1 \pmod{p}$	$A \simeq C', B \simeq B' \simeq C$
$q^2 \equiv 1, q \not\equiv \pm 1 \pmod{p}$	A, B, B', C, C'
$q^2 \equiv -1 \pmod{p}$	A, B, B'
$q^2 \not\equiv \pm 1,\ \pmod{p}, p$ odd	A, B, B'
$q^2 \not\equiv \pm 1,\ \pmod{p}, p$ even	$A, B \simeq B'$

Table 5.1: Classification of real structures up to equivariant isotopy on $L(p, q)$.

5.2.1 Real Heegaard genus-1 case: the real structures C and C'

The real 3-manifolds accepting a genus-1 real Heegaard splitting are exactly (S^3, c_{st}), $(S^1 \times S^2, s_i)$ and the lens spaces $(L(p,q), C)$ and $(L(p,q), C')$ with $q^2 \equiv 1(\mod p)$. In Section 5.1, we gave equivariant surgery descriptions of $(S^1 \times S^2, s_i)$ and presented some equivariant contact diagrams. In this section, we are going to give equivariant surgery descriptions for the two non-isotopic real structures C and C' on $L(p,q)$ and illustrate how to turn them into equivariant contact surgery diagrams.

First note that C and C' are represented by

$$C = \begin{bmatrix} -q & p' \\ p & q \end{bmatrix}, C' = \begin{bmatrix} q & -p' \\ -p & -q \end{bmatrix}.$$

The real part of C (and C') consists of two circles if p and p' are both even. In this case we will write C as in the proof of Theorem 3.3.2 of the form

$$C = \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_b^{\pm 1} \cdot \tau_{a_1'}^{\sigma_1} \dots \tau_{a_t'}^{\sigma_t} \cdot c_{\mathrm{st}} \tag{5.1}$$

where b is a c_{st}-invariant curve and $c_{\mathrm{st}}(a_i) = a_i'$ for all $1 \leq i \leq t$.

The real part will consist of one circle if one of p or p' is odd and in this case we will write

$$C = \tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_{a_1'}^{\sigma_1} \dots \tau_{a_t'}^{\sigma_t} \cdot c_{\mathrm{st}}. \tag{5.2}$$

Let $[r_1, \ldots, r_n]$ denote the continued fraction $r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_n}}}$ for p/q, $r_i \geq 2$. We have the following identity:

$$\begin{bmatrix} r_1 & 1 \\ -1 & 0 \end{bmatrix} \ldots \begin{bmatrix} r_n & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} p & q' \\ -q & p' \end{bmatrix}.$$

Using this identity, we write C as:

$$\begin{aligned} C = \begin{bmatrix} -q & p' \\ p & q \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p & q' \\ -q & p' \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 & 1 \\ -1 & 0 \end{bmatrix} \ldots \begin{bmatrix} r_n & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & r_1 \end{bmatrix} \ldots \begin{bmatrix} 0 & -1 \\ 1 & r_n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{5.3}$$

We also know that if $q^2 \equiv 1(\mod p)$, then the continued fraction $[r_1, \ldots, r_n]$ is a palindrome: $r_i = r_{n-i+1}$ for $1 \leq i \leq n$. For the sake of simplicity, let us label the matrices as follows:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, X_{r_i} = \begin{bmatrix} 0 & -1 \\ 1 & r_i \end{bmatrix}, \tau_a^{r_i} = \begin{bmatrix} 1 & r_i \\ 0 & 1 \end{bmatrix}, \tau_b^{r_i} = \begin{bmatrix} 1 & 0 \\ -r_i & 1 \end{bmatrix}.$$

Then we have the following equalities:

$$X_{r_i} = A \cdot \tau_a^{r_i}, \qquad X_{r_i} = \tau_b^{r_i} \cdot A,$$

$$A^2 = -I, \quad A = \tau_b^{-1}\tau_a^{-1}\tau_b^{-1} = \tau_a^{-1}\tau_b^{-1}\tau_a^{-1},$$

$$\tau_{a+b}^n = \tau_a\tau_b^n\tau_a^{-1} = \tau_b^{-1}\tau_a^n\tau_b,$$

$$\tau_{a-b}^n = \tau_a^{-1}\tau_b^n\tau_a = \tau_b\tau_a^n\tau_b^{-1}.$$

Now writing Equation 5.3 in terms of the above matrices, we have

$$C = X_{r_1} \dots X_{r_n} \cdot c_{\text{st}}. \tag{5.4}$$

Recall that we represent the standard real structure c_{st} on S^3 by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Using the above equalities in (5.4) we will obtain an equivariant product of the form (5.1) or (5.2) for C. We basically substitute $A \cdot \tau_a^{r_i}$ or $\tau_b^{r_i} \cdot A$ for X_{r_i} in the following manner: start with substituting $\tau_b^{r_1} \cdot A$ for X_{r_1} on the left. Then alternately substitute $A \cdot \tau_a^{r_2}$ for X_{r_2}, $\tau_b^{r_3} \cdot A$ for X_{r_3} and go on from left to right until the middle term. From the right, start with substituting $A \cdot \tau_a^{r_1}$ for X_{r_1} and alternately substitute $\tau_b^{r_2} \cdot A$ for X_{r_2} and so on. There are four cases:

$$\begin{aligned} 1)\ n = 4k: \quad C &= \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \dots A^2 \cdot \tau_a^{r_{2k}} \cdot \tau_b^{r_{2k}} \cdot A^2 \dots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\text{st}} \\ &= \tau_b^{r_1}\tau_a^{r_2} \dots \tau_a^{r_2}\tau_b^{r_{2k}} \dots \tau_b^{r_2}\tau_a^{r_1} \cdot c_{\text{st}} \end{aligned}$$

Note that A^2 commutes and even number of A^2's cancel in the product, giving us the equivariant product above.

$$
\begin{aligned}
2)\ n = 4k+1: \quad C &= \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \ldots A^2 \cdot \tau_a^{r_{2k}} \cdot X_{r_{2k+1}} \cdot \tau_b^{r_{2k}} \cdot A^2 \ldots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \ldots \tau_a^{r_{2k}-1} \cdot \tau_{(a+b)}^{r_{2k+1}-1} \cdot \tau_b^{r_{2k}-1} \ldots \tau_a^{r_1} \cdot c_{\mathrm{st}}
\end{aligned}
$$

Here in the first line after we cancel out the A^2's we are left with $X_{r_{2k+1}}$ in the middle. We now use

$$
X_{r_{2k+1}} = A \cdot \tau_a^{r_{2k+1}} = \tau_a^{-1}(\tau_b^{-1}\tau_a^{-1}\tau_a^{r_{2k+1}}\tau_b)\tau_b^{-1} = \tau_a^{-1} \cdot \tau_{(a+b)}^{r_{2k+1}-1} \cdot \tau_b^{-1},
$$

which gives us the equivariant product we are looking for.

$$
\begin{aligned}
3)\ n = 4k+2: \quad C &= \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \ldots \tau_b^{r_{2k+1}} \cdot A^2 \cdot \tau_a^{r_{2k+1}} \ldots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1}\tau_a^{r_2} \ldots \tau_b^{r_{2k+1}} \cdot A^2 \cdot \tau_a^{r_{2k+1}} \ldots \tau_b^{r_2}\tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1}\tau_a^{r_2} \ldots \tau_b^{r_{2k+1}}\tau_b^{-1}\tau_a^{-1}\tau_b^{-1}\tau_a^{-1}\tau_b^{-1}\tau_a^{-1}\tau_a^{r_{2k+1}} \ldots \tau_b^{r_2}\tau_a^{r_1} \cdot c_{\mathrm{st}}
\end{aligned}
$$

In this case we have A^2 in the middle, which can be made equivariant by its composition itself as $A = \tau_b^{-1}\tau_a^{-1}\tau_b^{-1} = \tau_a^{-1}\tau_b^{-1}\tau_a^{-1}$.

$$
\begin{aligned}
4)\ n = 4k+3: \quad C &= \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \ldots \tau_b^{r_{2k+1}} \cdot A \cdot X_{r_{2k+2}} \cdot A \cdot \tau_a^{r_{2k+1}} \ldots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \ldots \tau_b^{r_{2k+1}+1} \cdot \tau_{(a+b)}^{r_{2k+2}+1} \cdot \tau_a^{r_{2k+1}+1} \ldots \tau_a^{r_1} \cdot c_{\mathrm{st}}
\end{aligned}
$$

In this case, the middle term $A \cdot X_{r_{2k+2}} \cdot A$ is treated similarly with the case $n = 4k+1$ and the equivariant product is obtained.

For C', we can derive the following equivariant products similarly, noting that

$$C' = -I \cdot C = A^2 \cdot C.$$

$$\begin{aligned}
1')\ n = 4k: \quad C' &= A^2 \cdot \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \dots \tau_a^{r_{2k}} \cdot \tau_b^{r_{2k}} \cdot A^2 \dots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \tau_a^{r_2} \dots \tau_a^{r_{2k}} \cdot A^2 \cdot \tau_b^{r_{2k}} \dots \tau_b^{r_2} \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \tau_a^{r_2} \dots \tau_a^{r_{2k}} \tau_a^{-1} \tau_b^{-1} \tau_a^{-1} \tau_b^{-1} \tau_a^{-1} \tau_b^{-1} \tau_b^{r_{2k}} \dots \tau_b^{r_2} \tau_a^{r_1} \cdot c_{\mathrm{st}}
\end{aligned}$$

$$\begin{aligned}
2')\ n = 4k+1: \quad C' &= A^2 \cdot \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \dots A^2 \cdot \tau_a^{r_{2k}} \cdot X_{r_{2k+1}} \cdot \tau_b^{r_{2k}} \cdot A^2 \dots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \dots \tau_a^{r_{2k}} \cdot A^2 \cdot X_{r_{2k+1}} \cdot \tau_b^{r_{2k}} \dots \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \dots \tau_a^{r_{2k}+1} \cdot \tau_{(a-b)}^{r_{2k+1}+1} \cdot \tau_b^{r_{2k}+1} \dots \tau_a^{r_1} \cdot c_{\mathrm{st}}
\end{aligned}$$

$$\begin{aligned}
3')\ n = 4k+2: \quad C' &= A^2 \cdot \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \dots \tau_b^{r_{2k+1}} \cdot A^2 \cdot \tau_a^{r_{2k+1}} \dots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \tau_a^{r_2} \dots \tau_b^{r_{2k+1}} \tau_a^{r_{2k+1}} \dots \tau_b^{r_2} \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \tau_a^{r_2} \dots \tau_b^{r_{2k+1}} \tau_a^{r_{2k+1}} \dots \tau_b^{r_2} \tau_a^{r_1} \cdot c_{\mathrm{st}}
\end{aligned}$$

$$\begin{aligned}
4')\ n = 4k+3: \quad C' &= A^2 \cdot \tau_b^{r_1} \cdot A^2 \cdot \tau_a^{r_2} \dots \tau_b^{r_{2k+1}} \cdot A \cdot X_{r_{2k+2}} \cdot A \cdot \tau_a^{r_{2k+1}} \dots \tau_b^{r_2} \cdot A^2 \cdot \tau_a^{r_1} \cdot c_{\mathrm{st}} \\
&= \tau_b^{r_1} \dots \tau_b^{r_{2k+1}-1} \cdot \tau_{(a-b)}^{r_{2k+2}-1} \cdot \tau_a^{r_{2k+1}-1} \dots \tau_a^{r_1} \cdot c_{\mathrm{st}}
\end{aligned}$$

Note that all of the products above are c_{st}-equivariant products in the sense that $a = c_{\mathrm{st}}(b)$ and $(a+b)$ and $(a-b)$ are c_{st}-invariant. Hence the products give equivariant surgery links in the standard real sphere (S^3, c_{st}).

Because of the restrictions on slopes of real tight solid tori, not all of these diagrams produce equivariant contact surgery diagrams. If we want to turn all of the equivariant surgery diagrams above into equivariant contact diagrams, we have to make sure the equivariant knot in the middle satisfies the twisting condition for equivariant Legendrian knots (the rest of the diagram is a collection of equivariant knot pairs, which cause no problem). The equivariant knots in the middle are: either the c_4-knot $(a+b)$ with $\mathrm{tw} = -2$, or the c_1-knot $(a-b)$ with $\mathrm{tw} = 0$. The latter does not violate our restrictions. For $(a+b)$, you can turn (-1)-equivariant surgery into an equivariant contact one. However this does not work for $(+1)$ surgery coefficient. The way to make it possible in the above equations is to replace $\tau_{(a+b)}$ with $\tau_{(a+b)}\tau_{(a+b)}^{-1}\tau_{(a+b)}$. Now we have (-1)-surgery on $(a+b)$, which can be turned into an equivariant contact surgery.

Let us illustrate with an example what is going on in the previous paragraph.

For $p = 2$, we have $L(2,1) \simeq \mathbb{RP}^3$ where there is a unique real structure. We can take it to be C and $p = 2$ falls into the case 2) with $n = 1$, $r_1 = 2$. We have the product $\tau_a^{-1}\tau_{(a+b)}^{+1}\tau_b^{-1}$ which gives us an equivariant surgery diagram. Now to turn it into an equivariant contact diagram, we rewrite it as $\tau_a^{-1}\tau_{(a+b)}^{+1}\tau_{(a+b)}^{-1}\tau_{(a+b)}^{+1}\tau_b^{-1}$. Now it gives an equivariant surgery diagram, see Figure 5.5. This gives an overtwisted real tight contact structure because of the contact 0-surgeries. We can drop the unknot pair as they will not affect the topology and the real structure. Of course,

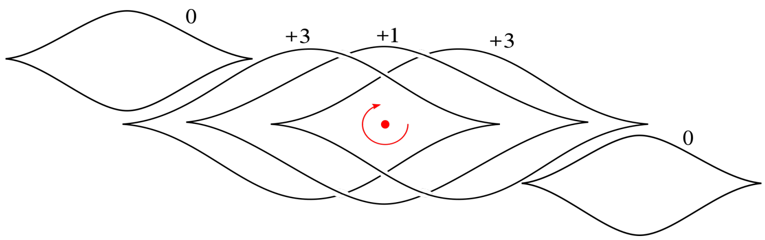

Figure 5.5: An equivariant contact surgery diagram for $\mathbb{RP}^3$

as long as we have a product of the form (5.2), which is not unique, we will have an equivariant contact structure. Here is another expression for the real structure C of $\mathbb{RP}^3$: $C = \tau_{(a-b)}^{-1} \cdot c_{\mathrm{st}}$. This product gives the equivariant contact surgery diagram in Figure 5.6 and shows that the unique tight contact structure on $\mathbb{RP}^3$ is a real contact structure. Note that our method of finding equivariant contact surgery diagrams produce both tight and overtwisted real contact structures.

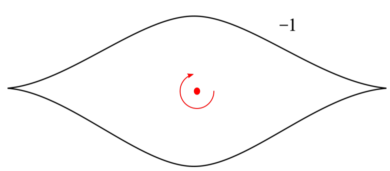

Figure 5.6: An equivariant contact surgery diagram for the tight $\mathbb{RP}^3$

5.2.2 The real structure A

Among the 5 classes of real structures on lens spaces, A is special in the sense that it can be described in many ways and studied in different concepts. Moreover, $L(p, q)$ admits a real structure of type A for any p, q. Some of the ways we can define it are as follows

1) It is an involution on $L(p, q)$ acting as a c_1-real structure on each Heegaard torus.
2) Considering $L(p, q)$ as a cyclic double branched cover of S^3 along a 2-bridge knot, it is the involution induced by the covering map.
3) If you consider $L(p, q)$ as the link of an isolated complex surface singularity, then the real structure induced by the complex conjugation is type A.

Let $p > q > 0$ and $[r_1, \ldots, r_n]$, $r_i \leq -2$ be the continued fraction for $-p/q$. Honda proved that on $L(p, q)$, there exists exactly $|(r_1+1)\ldots(r_k+1)|$ tight contact structures up to isotopy. Moreover, each tight contact structure on L(p, q) can be obtained from Legendrian ((−1)-contact) surgery on a link in the standard tight 3-sphere [14]. Topologically, surgery link is given by a chain of unknots with (ordered) coefficients r_i as in Figure 5.7. This surgery diagram in fact is an equivariant surgery diagram where all the unknots are c_1-knots. Performing equivariant surgery produces $(L(p, q), A)$. This can be seen by looking at the quotient. The quotient of an equivariant neighborhood of a c_1-knot under c_{st} is a 3-ball. Hence, the quotient of the surgered manifold is obtained by removing 3-balls and gluing back 3-balls to S^3, which is again S^3. Since the only real structure on $L(p,q)$ having quotient S^3 is A [15], the real structure on the surgered manifold is type A.

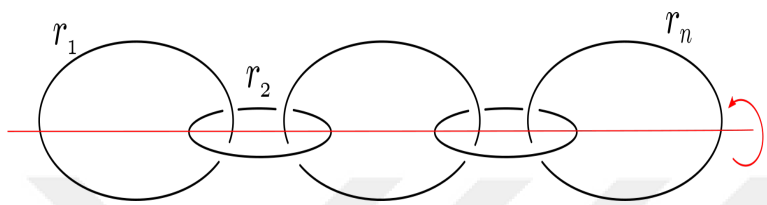

Figure 5.7: A surgery diagram for $L(p,q)$, which can also be taken to be equivariant.

There are exactly $|(r_i+1)|$ ways to stabilize an unknot to perform contact (-1) surgery on it. Moreover, each choice of stabilization will represent a Legendrian c_1-unknot. Hence, each contact surgery diagram describing a tight contact structure on $L(p,q)$ is also an equivariant contact surgery diagram (consisting of Legendrian c_1-knots)describing a A-real tight contact structure. Now we can state the following theorem.

Theorem 5.2.1. *Every tight contact structure on $L(p,q)$ is A-real.*

Remark 5. The equivariant contact surgery diagram we mentioned in the previous paragraph cannot be obtained by Algorithm 1 or 2 since Algorithm 1 does not have any c_1-knots and Algorithm 2 only produces equivariant contact surgery diagrams with unlinked c_1-knots.

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