GIBBS MEASURES AND PHASE TRANSITIONS IN VARIOUS ONE-DIMENSIONAL MODELS

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By
Ahmet Şensoy
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Azer Kerimov (Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Alexander Goncharov

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Assist. Prof. Dr. Tarık Kara

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Prof. Dr. Atilla Erçelebi

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Prof. Dr. Emin Özçağ

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

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Ahmet Şensoy
Ph.D. in Mathematics
Supervisor: Assoc. Prof. Dr. Azer Kerimov
December, 2013

In the thesis, limiting Gibbs measures of some one dimensional models are investigated and various criterions for the uniqueness of limiting Gibbs states are considered. The criterion for models with unique ground state formulated in terms of percolation theory is presented and some applications of this criterion are discussed. A one-dimensional long range Widom-Rowlinson model with periodic and biased particle activities is explored. It is shown that if the spin interactions are sufficiently large versus particle activities then the Widom-Rowlinson model does not exhibit a phase transition at low temperatures. Finally, an interdisciplinary approach is followed. A financial application of the theory of phase transition is considered by applying the Ising model to understand the role of herd behavior on stock market crashes. Accordingly, model suggests a criteria to detect the existence of herd behavior in financial markets under certain assumptions.

Keywords: Gibbs measure, Phase transition, Ising model, Widom-Rowlinson model.

ÖZET

ÇEŞİTLİ BİR BOYUTLU MODELLERDE GIBBS ÖLÇÜMLERİ VE FAZ GEÇİŞLERİ

Ahmet Şensoy
Matematik, Doktora
Tez Yöneticisi: Doç. Dr. Azer Kerimov
Aralık, 2013

Bu tezde bazı bir boyutlu modellerde limit Gibbs ölçümleri incelenmiş ve bu ölçümlerin tekilliği için çeşitli kriterler ele alınmıştır. Tek taban durulu modeller için bir tekillik kriteri süzülme teorisi çerçevesinde sunulmuş ve bu kriterin çeşitli uygulamaları tartışılmıştır. Tanecik aktivite parametreleri periyodik ve yanlı olan bir boyutlu ve uzak etkileşimli Widom-Rowlinson modeli, bu kriter altnda incelenmiştir. Modeldeki dönüler arası etkileşimin tanecik aktivitelerine oranla yeterince büyük olduğu durumlarda, modelin düşük sıcaklıklarda faz geçişine sahip olmadığı gösterilmiştir. Son olarak disiplinler arası bir yaklaşım izlenmiştir. Genel bir Ising modelindeki faz geçişi, borsalarda yaşanan büyük düşüşlerde sürü psikolojisinin etkisini incelemek amacıyla değerlendirilmiştir. Model, belirli varsayımlar altında finans piyasalarında yaşanan büyük düşüşlerde sürü psikolojiinin varlığını tespit etmek için bir kriter önermektedir.

Anahtar sözcükler: Gibbs ölçümü, Faz geçişi, Ising modeli, Widom-Rowlinson modeli.

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Contents

Chapter 1

Introduction

This thesis analyzes the corporate behavior of random variables located on the sites of one dimensional lattice. The interaction between random variables is studied in terms of Gibbs measures and phase transitions. Being a branch of rigorous statistical mechanics, probability and stochastic processes, the rigorous theory of Gibbs measures dates back to Dobrushin [1, 2, 3, 4, 5] and Lanford and Ruelle [6] who proposed it as a natural mathematical description of an equilibrium state of a physical system which consists of a very large number of interacting components.

In terms of probability, a Gibbs measure is the distribution of a countably infinite family of random variables which admit some prescribed conditional probabilities. Since the 1970s, this notion has received considerable attention from both physicists and mathematicians and its significance, is now, widely accepted.

In this introduction we give an outline of a particular physical background which gives rise to the definition of Gibbs measures and then we will justify the interest in the theory mostly following [7, 8].

1.1 Background

The initial idea depends on a concept called *spin system* which was born around 1920 in an attempt to understand the phenomenon of ferromagnetism, that is the basic mechanism by which certain materials (such as iron, nickel and cobalt) form permanent magnets, or are attracted to magnets. At that time, three points were clearly understood: First, ferromagnetism should be due to the alignment of the elementary spins of the atoms that persists even after an external field is turned off. Second, it is temperature dependent in the sense that heating the material loses the coherent alignment. And third, the spins should exert an attractive ferromagnetic interaction among each others which is rather short range. However, there were unanswered questions, in particular, how such a short range interaction could sustain the observed very long range coherent behavior of the material, and why such an effect should depend on the temperature?

To understand the situation, Lenz had an idea of inventing a toy model for the ferromagnetic system which is based on the collective behavior of the many microscopic elements in the system and independent of the precise details of their interaction. The model was analyzed in the PhD thesis of Ernst Ising [9] who found (correctly) no sign of ferromagnetism and conjectured (wrongly) the same results for higher dimensions. This model is called the Ising model and it is one of the most investigated models in the history of statistical mechanics, in particular, in the theory of lattice spin systems.

Lenz's simplification assumes that atoms are placed on the sites of a regular lattice $\mathbb{Z}^d$ and represented by the simplest possible spin variables taking only the two values from the set $\{-1, 1\}$. Only the nearest neighboring spins would interact and this interaction would favor these spins to take the same values. There can be, in addition, an external magnetic field h favoring globally either the plus or the minus-sign. This interaction can be represented by a *Hamiltonian* function H that assigns to a spin configuration $\sigma \equiv \{\sigma_i\}_{i\in\mathbb{Z}^d}$ the energy

$$H(\sigma) \equiv -\sum_{\substack{i,j\in\mathbb{Z}^d \\ ||i-j||_1=1}} \sigma_i\sigma_j - h\sum_{i\in\mathbb{Z}^d} \sigma_i \tag{1.1}$$

Since the sum does not converge, the formula above makes no sense. A sensible interpretation would be the fact that we consider a spin configuration on an infinite lattice, and that since the magnets consist of a finite but very large number of atoms, we should always consider finite sets $\Lambda \subset \mathbb{Z}^d$ and spin configurations $\sigma_\Lambda \equiv \{\sigma_i\}_{i\in\Lambda}$ and compute the energy of such a configuration by restricting the sums in (1.1) to run over the set Λ only (can be thought as an informal axiom of the statistical mechanics). Now, it follows that the equilibrium properties of a system can be described by specifying a probability measure on the space $\{-1,1\}^{\mathbb{Z}^d}$. The proper choice of the probability measure is the Gibbs measure (can be considered as another axiom) which formally is given by

$$\mu_\beta(d\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} \rho(d\sigma) \tag{1.2}$$

where Z_β is a normalizing constant and ρ is the uniform measure on the configuration space. To make sense of (1.2) in the infinite volume, we start with the *a priori* measure ρ that describes the non-interacting system. In finite volumes, the uniform measure on the finite space $\{-1,1\}^\Lambda$ can be taken as

$$\rho_\Lambda(\sigma_\Lambda = s_\Lambda) = \prod_{i\in\Lambda} \rho_i(\sigma_i = s_i) \tag{1.3}$$

where $\rho_i(\sigma_i = +1) = \rho_i(\sigma_i = -1) = 1/2$. To extend this construction to the infinite volume, first we make $\{-1,1\}^{\mathbb{Z}^d}$ into a measure space with the product topology of the discrete topology on $\{-1,1\}$. The corresponding sigma-algebra $\mathcal{F}$ is then the product sigma-algebra. The measure ρ is then defined by specifying that for all cylinder events $\mathcal{A}_\Lambda$ i.e. events that for some finite set $\Lambda \subset \mathbb{Z}^d$ depend only on the values of the variables σ_i with $i \in \Lambda$,

$$\rho(\mathcal{A}_\Lambda) = \rho_\Lambda(\mathcal{A}_\Lambda) \tag{1.4}$$

with ρ_Λ defined in (1.3). Thus, we have set up an a-priori probability space $(\mathcal{S}, \mathcal{F}, \rho)$ describing a system of non-interacting spins.

To understand the new construction, we give a new interpretation to (1.1): Since the expression makes no sense in infinite-volume, one can ask what is the energy of an infinite-volume configuration within a finite-volume Λ. This quantity is naturally defined by

$$H_\Lambda(\sigma) \equiv - \sum_{\substack{i \vee j \in \Lambda \\ ||i-j||_1 = 1}} \sigma_i \sigma_j - h \sum_{i \in \Lambda} \sigma_i \tag{1.5}$$

which differs from the simple restriction of (1.1) to Λ by a term $2 \sum_{\substack{i \in \Lambda, j \notin \Lambda \\ ||i-j||_1=1}} \sigma_i \sigma_j$, which represents the interaction of the spins in Λ with those outside of it; and it actually involves only spins at the boundary of Λ. The finite-volume restriction given by (1.5) is compatible under iteration i.e. if $\Lambda' \supset \Lambda$ then

$$(H_{\Lambda'})_\Lambda(\sigma) = H_\Lambda(\sigma) \tag{1.6}$$

Now (1.5) and (1.6) allows us to define, for any fixed configuration $\eta \in \mathcal{S}$ and finite subset $\Lambda \subset \mathbb{Z}^d$, a probability measure

$$\mu_\Lambda^\eta(d\sigma_\Lambda) = \frac{1}{Z_{\beta,\Lambda}^\eta} e^{-\beta H_\Lambda((\sigma_\Lambda, \eta_{\Lambda^c}))} \rho_\Lambda(d\sigma_\Lambda) \tag{1.7}$$

The idea is that (1.7) defines the family of the conditional probabilities of some measures μ_β defined on the infinite volume space. They satisfy automatically the compatibility conditions required for conditional probabilities and so have a chance to be conditional probabilities of some infinite-volume measure. Dobrushin started from this observation to define the notion of the infinite-volume Gibbs measure (i.e. the proper definition for the (1.2)):

A probability measure μ_β on ($\mathcal{S}$,$\mathcal{F}$) is a Gibbs measure for the Hamiltonian H and inverse temperature β, if and only if its conditional distributions (conditioned on configurations in the complement of any finite set Λ) are given by (1.7).

which brings out two important questions: Does such a measure exist? and if it exists, is it unique? Before investigating answers for these questions, we will provide a more general and formal set up in the next section.

1.2 General Setup

1.2.1 Topological Background

Throughout this chapter, we will consider the lattice system $\mathbb{Z}^d$ and Λ will denote a finite subset of $\mathbb{Z}^d$. Spins will take values from the set $\mathcal{S}_0$ which is a complete separable metric space (to avoid discussions, $\mathcal{S}_0$ is assumed to be finite). $\mathcal{S}_0$ is equipped with its sigma-algebra $\mathcal{F}_0$ generated by the open sets in the metric topology to obtain a measure space $(\mathcal{S}_0,\ \mathcal{F}_0)$. Finally, we add a probability measure ρ_0 (a-priori distribution of the spin) to complete single-site probability space $(\mathcal{S}_0\ ,\mathcal{F}_0,\ \rho_0)$.

First aim is to extend the settings for infinitely many non-interacting spins. Thus, we consider the infinite product space

$$\mathcal{S} \equiv {\mathcal{S}_0}^{\mathbb{Z}^d} \tag{1.8}$$

$\mathcal{S}$ is turned into a complete separable metric space by equipping it with the product topology: Consider the open sets generated by the balls $B_{\epsilon,\Lambda}(\sigma)$ where

$$B_{\epsilon,\Lambda}(\sigma) \equiv \{\sigma' \in \mathcal{S} : \max_{i\in\Lambda} |\sigma_i - \sigma_i'| < \epsilon\} \tag{1.9}$$

where $\sigma \in \mathcal{S}$, $\Lambda \subset \mathbb{Z}^d$ and $\epsilon \in \mathbb{R}_+$. And the Borel sigma-algebra $\mathcal{F}$ of $\mathcal{S}$ is the product sigma-algebra

$$\mathcal{F} = {\mathcal{F}_0}^{\mathbb{Z}^d} \tag{1.10}$$

Note that in our context, product topology of a metric space is metrizable, and if $\mathcal{S}_0$ is complete separable metric space then so is $\mathcal{S}$.

The following theorem is an important fact to be used:

Theorem 1.2.1. (Tychonov's Theorem) *If $\mathcal{S}_0$ is compact then $\mathcal{S}$ defined in (1.8) equipped with the product topology is compact.*

We will use $\mathcal{S}_\Lambda \equiv \mathcal{S}_0^\Lambda$ for finite volume configuration space and $\mathcal{F}_\Lambda = \mathcal{F}_0^\Lambda$ for sigma-algebra of local events. We will call an event *local* or *cylinder* if it is measurable with respect to $\mathcal{F}_\Lambda$ for some finite Λ. A sequence of volumes $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_n \subset \cdots \subset \mathbb{Z}^d$ having the property that for any finite $\Lambda' \subset \mathbb{Z}^d$, there exist n such that $\Lambda' \subset \Lambda_n$ will be called an *increasing and absorbing sequence*. Then the family of sigma-algebras $\mathcal{F}_{\Lambda_n}$ forms a filtration of $\mathcal{F}$. Similarly, $\mathcal{S}_{\Lambda^c} \equiv {\mathcal{S}_0}^{\mathbb{Z}^d \setminus \Lambda}$ and $\mathcal{F}_{\Lambda^c} \equiv {\mathcal{F}_0}^{\mathbb{Z}^d \setminus \Lambda}$.

In the rest of the chapter,we will refer to several classes of real valued functions on $\mathcal{S}$. One of them is $B(\mathcal{S}, \mathcal{F})$ which is the space of bounded and measurable functions ($f : \mathcal{S} \to \mathbb{R}$ is measurable if for any Borel set $B \subset \mathcal{B}(\mathbb{R})$, $\mathcal{A} \equiv \{\sigma : f(\sigma) \in B\}$ is contained in $\mathcal{F}$). The corresponding bounded functions measurable with respect to $\mathcal{F}_\Lambda$ is denoted by $B(\mathcal{S}, \mathcal{F}_\Lambda)$. Functions belonging to some $B(\mathcal{S}, \mathcal{F}_\Lambda)$ is called *local* functions and their space is denoted by

$$B_{loc}(\mathcal{S}) \equiv \cup_{\Lambda \subset \mathbb{Z}^d} B(\mathcal{S}, \mathcal{F}_\Lambda) \tag{1.11}$$

The closure $B_{ql}(\mathcal{S})$ of the set of local functions under uniform convergence is called the *quasi-local* functions and characterized by the following property

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{\substack{\sigma, \sigma' \in \mathcal{S} \\ \sigma_\Lambda = \sigma'_\Lambda}} |f(\sigma) - f(\sigma')| = 0 \tag{1.12}$$

The spaces $C(\mathcal{S})$, $C_{loc}(\mathcal{S})$ and $C_{ql}(\mathcal{S})$ of continuous, local continuous and quasi-local continuous functions are defined in a similar way.

Lemma 1.2.2. *(a) If $\mathcal{S}_0$ is compact, then $C(\mathcal{S}) = C_{ql}(\mathcal{S}) \subset B_{ql}(\mathcal{S})$.*

(b) If $\mathcal{S}_0$ is discrete, then $B_{ql}(\mathcal{S}) = C_{ql}(\mathcal{S}) \subset C(\mathcal{S})$.

(c) If $\mathcal{S}_0$ is finite, then $C(\mathcal{S}) = B_{ql}(\mathcal{S}) = C_{ql}(\mathcal{S})$.

Next, we consider the space $\mathcal{M}_1(\mathcal{S}, \mathcal{F})$ of probability measures on $(\mathcal{S}, \mathcal{F})$. The most common topology equipped to this space is generated by the open balls

$$B_{f,\epsilon}(\mu) \equiv \{\mu' \in \mathcal{M}_1(\mathcal{S}, \mathcal{F}) : |\mu(f) - \mu(f')| < \epsilon\} \tag{1.13}$$

where $f \in C(\mathcal{S})$, $\epsilon \in \mathbb{R}_+$ and $\mu \in \mathcal{M}_1(\mathcal{S}, \mathcal{F})$. With this topology, $\mathcal{M}_1(\mathcal{S}, \mathcal{F})$ is a complete separable metric space and, if $\mathcal{S}_0$ is compact then $\mathcal{M}_1(\mathcal{S}, \mathcal{F})$ is compact.

1.2.2 Gibbs Measures

In this section, we will introduce the definitions, lemmas and theorems necessary to set up the Gibbsian theory.

Definition 1.2.3. *An interaction is a family* $\Phi \equiv \{\Phi_A\}_{A \subset \mathbb{Z}^d}$ *where* $\Phi_A \in B(\mathcal{S}, \mathcal{F}_A)$. *If all* $\Phi_A \in C(\mathcal{S}, \mathcal{F}_A)$, *the interaction is called continuous. Moreover, an interaction is called regular, if for all* $x \in \mathbb{Z}^d$, *there exists a constant* c *such that*

$$\sum_{A \ni x} ||\Phi_A||_\infty \leq c < \infty \tag{1.14}$$

A Hamiltonian can be constructed from a regular interaction in the following way

$$H_\Lambda(\sigma) \equiv - \sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\sigma) \tag{1.15}$$

for all finite $\Lambda \subset \mathbb{Z}^d$. If Φ is in $\mathcal{B}_0$ (Banach space equipped with the norm $|||\Phi||| = \sup_{x \in \mathbb{Z}^d} \sum_{A \ni x} ||\Phi_A||_\infty$), H_Λ satisfy the bound

$$||H_\Lambda||_\infty \leq C|\Lambda| \tag{1.16}$$

for some $C < \infty$. Here, H_Λ is quasi-local function. Moreover if Φ is continuous, it is a continuous quasi-local function for any finite Λ.

Definition 1.2.4. *A local specification is a family of probability kernels* $\{\mu_{\Lambda,\beta}^{(.)}\}_{\Lambda \subset \mathbb{Z}^d}$ *such that*

(a) For all Λ *and all* $\mathcal{A} \in \mathcal{F}$, $\mu_{\Lambda,\beta}^{(.)}(\mathcal{A})$ *is a* $\mathcal{F}_{\Lambda^c}$*-measurable function*

(b) For any $\eta \in \mathcal{S}$, $\mu_{\Lambda,\beta}^{\eta}$ *is a probability measure on* $(\mathcal{S}, \mathcal{F})$

(c) For any pair of volumes Λ, Λ' *with* $\Lambda \subset \Lambda'$ *and any measurable function* f

$$\int \mu_{\Lambda',\beta}^{\eta}(d\sigma')\mu_{\Lambda,\beta}^{(\eta_{\Lambda'^c}, \sigma'_{\Lambda'})}(d\sigma) f((\sigma_\Lambda, \sigma'_{\Lambda' \setminus \Lambda}, \eta_{\Lambda'^c})) = \int \mu_{\Lambda',\beta}^{\eta}(d\sigma') f((\sigma_{\Lambda'}, \eta_{\Lambda'^c})) \tag{1.17}$$

where the notation $(\sigma_\Lambda, \eta_{\Lambda^c})$ *is used to denote the configuration that equals* σ_x *if* $x \in \Lambda$, *and* η_x *if* $x \in \Lambda^c$.

Now, given a regular interaction, we can construct local specifications for the forthcoming Gibbs measures:

Lemma 1.2.5. *If* Φ *is regular interaction then*

$$\int \mu^{\eta}_{\Lambda,\beta}(d\sigma)f(\sigma) \equiv \int \rho_{\Lambda}(d\sigma_{\Lambda})\frac{e^{-\beta H_{\Lambda}((\sigma_{\Lambda},\eta_{\Lambda^c}))}}{Z^{\eta}_{\Lambda,\beta}}f((\sigma_{\Lambda},\eta_{\Lambda^c})) \tag{1.18}$$

defines a local specification called the Gibbs specification for the interaction Φ *at inverse temperature* β.

Then we can define the infinite-volume Gibbs measure as follows:

Definition 1.2.6. *Suppose that* $\{\mu^{(.)}_{\Lambda,\beta}\}$ *is a local specification. A measure* μ_{β} *is called compatible with this local specification if and only if for all* $\Lambda \subset \mathbb{Z}^d$ *and all* $\mathcal{A} \in \mathcal{F}$*, we have*

$$\mu_{\beta}(\mathcal{A}|\mathcal{F}_{\Lambda^c}) = \mu^{(.)}_{\Lambda,\beta}(\mathcal{A}), \qquad \mu_{\beta} - a.s. \tag{1.19}$$

A measure μ_{β} *that is compatible with the Gibbs specification for the interaction* Φ*, a-priori measure* ρ *at inverse temperature* β *is called a Gibbs measure corresponding to* Φ *and* ρ *at inverse temperature* β.

Theorem 1.2.7. (Dobrushin, Lanford and Ruelle equations) *A probability measure* μ_{β} *is a Gibbs measure for* Φ, ρ *and* β *if and only if, for all* $\Lambda \subset \mathbb{Z}^d$

$$\mu_{\beta}\mu^{(.)}_{\Lambda,\beta} = \mu_{\beta} \tag{1.20}$$

Definition 1.2.8. *The property of a specification to map continuous functions to continuous functions is called the Feller property.*

Lemma 1.2.9. *The local specifications of a continuous regular interaction have the Feller property.*

Proof. Let f be a continuous function. It is required to show that if $\eta_n \to \eta$ then $\mu^{\eta_n}_{\Lambda,\beta}(f) \to \mu^{\eta}_{\Lambda,\beta}(f)$. Since f is continuous, this property follows if

$$H_{\Lambda}(\sigma_{\Lambda}, \eta_{n,\Lambda^c}) \to H_{\Lambda}(\sigma_{\Lambda}, \eta_{\Lambda^c}) \tag{1.21}$$

Since H_{Λ} is a uniformly convergent sum of continuous functions by assumption, it is itself continuous. $\square$

Theorem 1.2.10. *Let Φ be a continuous regular interaction and $\mu_{\Lambda,\beta}^{(.)}$ be a corresponding local specification. Let Λ_n be an increasing and absorbing sequence of finite volumes. If for some $\eta \in \mathcal{S}$, the sequence $\mu_{\Lambda_n,\beta}^{\eta}$ of measures converges weakly to some probability measure ν, then ν is a Gibbs measure with respect to Φ, ρ and β.*

Proof. Let f be a continuous function. By assumption, we have

$$\mu_{\Lambda_n,\beta}^{\eta}(f) \to \nu(f), \quad \text{as } n \uparrow \infty \tag{1.22}$$

on the other hand, for all $\Lambda_n \supset \Lambda$,

$$\mu_{\Lambda_n,\beta}^{\eta}\mu_{\Lambda,\beta}^{(.)}(f) = \mu_{\Lambda_n,\beta}^{\eta}(f) \tag{1.23}$$

If we can make the assertion that $\mu_{\Lambda_n,\beta}^{\eta}\mu_{\Lambda,\beta}^{(.)}(f)$ converges to $\nu\mu_{\Lambda,\beta}^{(.)}(f)$, this implies ν satisfies (1.20) and so is a Gibbs measure. This assertion can be made if $\mu_{\Lambda,\beta}^{(.)}(f)$ is a continuous function which follows from Lemma 1.2.9. This method of taking increasing sequences of finite-volume measures is called *passing to the thermodynamic limit*. Theorem 1.2.10 plays a crucial role in the theory of Gibbs measures since it gives a way how to construct the infinite-volume Gibbs measures. □

Corollary 1.2.11. *Let $\mathcal{S}_0$ be compact and Φ be regular and continuous. Then there exists at least one Gibbs measure for any $0 \leq \beta < \infty$.*

Proof. $\mathcal{S}$ is compact by Tychonov's theorem and the set of probability measures on a compact space is compact with respect to the weak topology. Thus, any sequence $\mu_{\Lambda_n,\beta}^{\eta}$ must have convergent subsequences. By Theorem 1.2.10, any one of them provides a Gibbs measure. □

1.2.3 Phase Transitions

After establishing the concept of infinite-volume Gibbs measures and existence of them for a large class of systems, next thing to ask is under which circumstances such a Gibbs measure is unique or not. First, we start some results on the uniqueness conditions.

1.2.3.1 Dobrushin Uniqueness Criterion (High Temperatures)

One of the most elegant ways of obtaining a uniqueness condition belongs to Dobrushin and we will present it here mostly following the Simon's book [10].

Definition 1.2.12. *The total variation distance of two measures ν and μ is defined by*

$$||\nu - \mu|| \equiv 2 \sup_{\mathcal{A}\in\mathcal{F}} |\nu(A) - \mu(A)| \tag{1.24}$$

Theorem 1.2.13. *Let $\mu^{(.)}_{\Lambda,\beta}$ be a local specification having the Feller property. For $x, y \in \mathbb{Z}^d$, define the following*

$$\rho_{x,y} \equiv \frac{1}{2} \sup_{\substack{\eta,\eta' \\ \forall z\neq x\, \eta_z=\eta'_z}} ||\mu^{\eta}_{y,\beta} - \mu^{\eta'}_{y,\beta}|| \tag{1.25}$$

If $\sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \rho_{x,y} < 1$, then the local specification is compatible with at most one Gibbs measure.

Proof. *Variation* of a continuous function f at point x is defined by

$$\delta_x(f) = \sup_{\substack{\eta,\eta' \\ \forall z\neq x\, \eta_z=\eta'_z}} |f(\eta) - f(\eta')| \tag{1.26}$$

and its *total variation* is

$$\Delta(f) \equiv \sum_{x\in\mathbb{Z}^d} \delta_x(f) \tag{1.27}$$

then *the set of functions of finite total variation* is defined as $\mathcal{T} \equiv \{f \in C(\mathcal{S}) : \Delta(f) < \infty\}$ where $\mathcal{T}$ is a dense subset of $C(\mathcal{S})$. The proof consists of two steps:

(a) To show that Δ is semi-norm and if $\Delta(f) = 0$ then f is constant.

(b) To construct a contraction $\mathbb{T}$ with respect to Δ so that any solution of the DLR equations is $\mathbb{T}$-invariant.

If we can do these two steps then it holds that for any solution of the DLR equations, $\mu(f) = \mu(\mathbb{T}f) = \mu(\mathbb{T}^n f) \to c(f)$, independent of which one chosen. However, since the value on continuous functions determines μ, all solutions of the DLR equations are identical (In this proof section, β is dropped from the notification for simplification).

First, we start with the part (b): Let $x_1, x_2, ..., x_n, ...$ be an enumeration of all points in $\mathbb{Z}^d$. Set

$$\mathbb{T}f \equiv \lim_{n\uparrow\infty} \mu_{x_1}^{(.)} ... \mu_{x_n}^{(.)}(f) \tag{1.28}$$

For any continuous function, the limit in (1.28) exists in norm which implies that $\mathbb{T}$ maps continuous functions to continuous functions. By construction, if μ satisfies the DLR-equation with respect to the specification $\mu_\Lambda^{(.)}$ then $\mu(\mathbb{T}f) = \mu(f)$. Then, it remains to show that $\mathbb{T}$ is a contraction with respect to Δ if $\sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \rho_{x,y} \le \alpha < 1$.

To do that, we look at $\delta_x(\mu_y(f))$ where $x \neq y$ (otherwise it would be zero since $\mu_x(f)$ does not depend on η_x). Then

$$\begin{aligned}
\delta_x(\mu_y(f)) &\equiv \sup_{\substack{\eta,\eta' \\ \forall z\neq x\, \eta_z=\eta'_z}} ||\mu_y^\eta - \mu_y^{\eta'}|| \\
= \sup_{\substack{\eta,\eta' \\ \forall z\neq x\, \eta_z=\eta'_z}} | \int f(\sigma_y, \eta_{y'})\mu_y^\eta(d\sigma_y) &- \int f(\sigma_y, \eta'_{y'})\mu_y^\eta(d\sigma_y) \\
&+ \int f(\sigma_y, \eta'_{y'})(\mu_y^\eta(d\sigma_y) - \mu_y^\eta(d\sigma_y))| \\
\le \delta_x(f) + \sup_{\substack{\eta,\eta' \\ \forall z\neq y\, \eta_z=\eta'_z}} |f(\eta) - f(\eta')| \sup_{\substack{\eta,\eta' \\ \forall z\neq x\, \eta_z=\eta'_z}} & \sup_{\mathcal{A}\in\mathcal{F}} |\mu_y^\eta(\mathcal{A}) - \mu_y^{\eta'}(\mathcal{A})| \\
&= \delta_x(f) + \frac{1}{2}||\mu_y^\eta - \mu_y^{\eta'}||\delta_y(f) \\
&= \delta_x(f) + \rho_{x,y}\delta_y(f)
\end{aligned} \tag{1.29}$$

Lemma 1.2.14. *Let* $\sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \rho_{x,y} \le \alpha$. *Then, for all* $n \in \mathbb{N}$,

$$\Delta(\mu_{x_1}^{(.)} ... \mu_{x_n}^{(.)}(f)) \le \alpha \sum_{i=1}^n \delta_{x_i}(f) + \sum_{j\ge n+1} \delta_{x_j}(f) \tag{1.30}$$

Proof. We proceed by induction. If $n = 0$, (1.30) is the definition of Δ. Suppose that (1.29) holds for n. Then, since $\mu(\mathbb{T}f) = \mu(f)$, we have

$$\begin{aligned}
\Delta(\mu_{x_1}^{(.)}...\mu_{x_n}^{(.)}\mu_{x_{n+1}}^{(.)}(f)) \leq \alpha \sum_{i=1}^{n} \delta_{x_i}(\mu_{x_{n+1}}^{(.)}f) + \sum_{j \geq n+1} \delta_{x_j}(\mu_{x_{n+1}}^{(.)}f) \\
\leq \alpha \sum_{i=1}^{n}[\delta_{x_i}(f) + \rho_{x_i,x_{n+1}}\delta_{x_{n+1}}(f)] + \sum_{j \geq n+2} [\delta_{x_j}(f) + \rho_{x_j,x_{n+1}}\delta_{x_{n+1}}(f)] \\
= \alpha \sum_{i=1}^{n} \delta_{x_i}(f) + \sum_{i=1}^{\infty} \rho_{x_i,x_{n+1}}\delta_{x_{n+1}}(f) + \sum_{j \geq n+2} \delta_{x_j}(f) \\
\leq \alpha \sum_{i=1}^{n+1} \delta_{x_i}(f) + \sum_{j \geq n+2} \delta_{x_j}(f)
\end{aligned} \tag{1.31}$$

so the lemma is proved. $\square$

And passing to the limit $n \uparrow \infty$ brings the required estimate

$$\Delta(\mathbb{T}f) \leq \alpha\Delta(f) \tag{1.32}$$

It remains only to prove part (a): Now, f is continuous thus for any $\epsilon > 0$, there exists a finite Λ and configurations ω^+ and ω^- with $\omega^+_{\Lambda^c} = \omega^-_{\Lambda^c}$ such that

$$\begin{aligned}
\sup(f) \leq f(\omega^+) + \epsilon \\
\inf(f) \geq f(\omega^-) - \epsilon
\end{aligned} \tag{1.33}$$

using the following simple telescopic expansion

$$f(\omega^+) - f(\omega^-) \leq \sum_{x \in \Lambda} \delta_x(f) \leq \Delta(f) \tag{1.34}$$

we have $\sup(f) - \inf(f) \leq \Delta(f) + 2\epsilon$ for all ϵ which concludes the proof of the theorem. $\square$

Corollary 1.2.15. *For Gibbs specifications with respect to regular interactions, Dobrushin's uniqueness criterion becomes*

$$\sup_{x \in \mathbb{Z}^d} \sum_{A \ni x} (|A| - 1)||\Phi_A(\sigma)||_\infty < \beta^{-1} \tag{1.35}$$

Thus, if the temperature β^{-1} is "sufficiently high" then the Gibbs measure is unique.

1.2.3.2 Peierls Argument (Low Temperatures)

Previous section presents a condition for uniqueness of a Gibbs measure which naturally forces us to seek conditions where uniqueness does not hold. Contrary to the very general uniqueness criterion, the case where multiple Gibbs measures exist require a case by case study of respective interactions. Throughout the literature, several tools were introduced to investigate this problem and the basis of many of these tools is the *Peierls argument*.

In this part, we will explain the original derivation of the argument and later discuss the extensions. The intuitive idea is the following: For the large β (low temperature), the behavior of the Ising model is that the Gibbs measure should strongly favor the configurations with minimal H. If the external field $h \neq 0$, one can see that there is a unique such configuration of the system $\sigma_i = \text{sign}(h)$, whereas if $h = 0$ then there are two degenerate minima; $\sigma_i = +1$ and $\sigma_i = -1$. A natural idea is then to characterize a configuration by its deviation from such an optimal one. To move further, we introduce the following definition

Definition 1.2.16. *Let $< i, j >$ denote an edge of the $\mathbb{Z}^d$ and $< i, j >^\star$ denote the corresponding dual plaquette i.e. the unique d-1 dimensional facet that cuts the edge in the middle. We define*

$$\Gamma(\sigma) = \{< i, j >^\star\colon \sigma_i \sigma_j = -1\} \tag{1.36}$$

By definition, $\Gamma(\sigma)$ forms a surface in $\mathbb{R}^d$ and the following properties follow from the definition

Lemma 1.2.17. *Let Γ be the surface defined above, and let $\partial\Gamma$ denote its d-2 dimensional boundary.*

(a) $\partial\Gamma(\sigma) = 0$ *for all* $\sigma \in \mathcal{S}$. *Note that* $\Gamma(\sigma)$ *may have unbounded connected components.*

(b) Let Γ *be a surface in the dual lattice such that* $\partial\Gamma = \emptyset$. *Then there are exactly two configurations,* σ *and* $-\sigma$, *such that* $\Gamma(\sigma) = \Gamma(-\sigma) = \Gamma$.

(c) Any Γ can be decomposed into its connected components γ_i called contours (we use $\gamma \in \Gamma$ to state that "γ is a connected component of Γ").

(d) For any σ, any contour γ_i satisfies $\partial\gamma_i(\sigma) = \emptyset$. That is, each contour is either a finite and close, or an infinite and unbounded surface.

We denote by $\operatorname{int}\gamma$ the volume enclosed by γ, and by $|\gamma|$ the number of plaquettes in γ.

Theorem 1.2.18. (Peierls [11]) *Let μ_β be a Gibbs measure for the model (1.1) with $h = 0$ and ρ is the product measure defined in (1.3). For $d \geq 2$, there is $\beta_d < \infty$ such that $\beta > \beta_d$*

$$\mu_\beta[\exists_{\gamma\in\Gamma(\sigma):0\in\operatorname{int}\gamma}] < \frac{1}{2} \tag{1.37}$$

To prove the theorem, we need the following lemma:

Lemma 1.2.19. *Let μ_β be a Gibbs measure for the model (1.1) with $h = 0$ and γ be a finite contour. Then*

$$\mu_\beta[\gamma \in \Gamma(\sigma)] \leq e^{-2\beta|\gamma|} \tag{1.38}$$

Proof. The proof is an application of the DLR construction. Denote by γ^{in} and γ^{out} the layer of sites in $\mathbb{Z}^d$ adjacent to γ to the inferior of γ and exterior boundary of the contour γ. We have

$$\mu_\beta[\gamma \subset \Gamma(\sigma)] \equiv \mu_\beta[\sigma_{\gamma^{out}} = +1, \sigma_{\gamma^{in}} = -1] + \mu_\beta[\sigma_{\gamma^{out}} = -1, \sigma_{\gamma^{in}} = +1] \tag{1.39}$$

on the other hand,

$$\begin{aligned}
\mu^{+1}_{\operatorname{int}\gamma,\beta}[\sigma_{\gamma^{in}} = -1] &= \frac{\mathbb{E}_{\sigma_{\operatorname{int}(\gamma)\setminus\gamma^{in}}}\rho(\sigma_{\gamma^{in}} = -1)e^{-\beta H_{\operatorname{int}(\gamma)}(\sigma_{\operatorname{int}(\gamma)\setminus\gamma^{in}}, -1_{\gamma^{in}}, +1_{\gamma^{out}})}}{\mathbb{E}_{\sigma_{\gamma^{in}}}\mathbb{E}_{\sigma_{\operatorname{int}(\gamma)\setminus\gamma^{in}}}e^{-\beta H_{\operatorname{int}(\gamma)}(\sigma_{\operatorname{int}(\gamma)\setminus\gamma^{in}}, \sigma_{\gamma^{in}}, +1_{\gamma^{out}})}} \\
&= \frac{e^{-\beta|\gamma|}Z^{(-1)}_{\operatorname{int}(\gamma)\setminus\gamma^{in}}\rho(\sigma_{\gamma^{in}} = -1)}{\mathbb{E}_{\sigma_{\gamma^{in}}}e^{\beta\sum_{x\in\gamma^{in}, y\in\gamma^{out}}\sigma_y}Z^{\sigma_{\gamma^{in}}}_{\operatorname{int}(\gamma)\setminus\gamma^{in}}} \\
&\leq e^{-2\beta|\gamma|}\frac{Z^{(-1)}_{\operatorname{int}(\gamma)\setminus\gamma^{in}}}{Z^{(+1)}_{\operatorname{int}(\gamma)\setminus\gamma^{in}}} = e^{-2\beta|\gamma|}
\end{aligned} \tag{1.40}$$

where the last line follows from the symmetry of H_Λ under the global change $\sigma_x \to -\sigma_x$ (to replace the ratio of the two partition functions with spin-flip related boundary conditions by one). Similar argument is used for the second term in (1.39) and the lemma follows. $\square$

Proof. (Theorem 1.2.18) Proof follows from the trivial estimate

$$\mu_\beta[\exists_{\gamma\in\Gamma(\sigma):0\in\operatorname{int}\gamma}] \leq \sum_{\gamma\in\Gamma(\sigma):0\in\operatorname{int}\gamma} \mu_\beta[\gamma\in\Gamma(\sigma)] \tag{1.41}$$

and the number of contours of area k that enclose the region:

$$\{\gamma : 0 \in \operatorname{int}\gamma, |\gamma| = k\} \equiv C(d,k) \tag{1.42}$$

thus

$$\mu_\beta[\exists_{\gamma\in\Gamma(\sigma):0\in\operatorname{int}\gamma}] \leq \sum_{k=2d}^{\infty} k^{d/(d-1)} e^{-k(2\beta-\ln C_d)} \tag{1.43}$$

Ruelle shows that $C(d,k) \leq 3^k$ hence choosing $\beta > \frac{1}{2}\ln C_d$ gives the claimed estimate. $\square$

Theorem 1.2.18 intuitively implies that with probability greater than 1/2, the spin at the origin has the same sign with the spin at the infinity (could be +1 or −1) which establishes a long-range correlation. Note that Theorem 1.2.18 does not imply that there are no *infinite* contours with positive probability. However, in the next part we will show that μ_β can be decomposed into Gibbs measures containing infinite contours with probability zero and one, respectively. To do that, we first need to introduce the concept of *extremal Gibbs measures.* Due to the characterization of Gibbs measures through the DLR equations, it is clear that with any two Gibbs measures μ_β and μ'_β for the same local specification, their convex combinations $p\mu_\beta + (1-p)\mu'_\beta$ where $p \in [0,1]$, are also Gibbs measures. Hence, the set of Gibbs measures for a local specification forms a closed convex set.

Definition 1.2.20. *The extremal points of the closed convex set which is formed by Gibbs measures for a local specification are called extremal Gibbs measures or pure states (the name pure state is sometimes reserved to translation invariant extremal Gibbs measures).*

It can be shown that a Gibbs measure μ_β is extremal if and only if it is trivial on the tail sigma-field $\mathcal{F}^t \equiv \cap_{\Lambda\subset\mathbb{Z}^d}\mathcal{F}_{\Lambda^c}$, i.e. for all $\mathcal{A} \in \mathcal{F}^t$, $\mu_\beta(A) \in \{0,1\}$.

Now, we return to the investigation of the phase transition phenomena of the Ising model.

Theorem 1.2.21. *Consider the Ising model for parameters where the conclusion of the Theorem 1.2.18 holds. Then, there exists (at least) two extremal Gibbs measures μ_β^+ and μ_β^- satisfying $\mu^+(\sigma_0) = -\mu^-(\sigma_0) > 0$.*

Proof. We define the event $\mathcal{U} = \{\Gamma(\sigma)\ \mathit{contains\ no\ infinite\ contour}\}$ which is clearly a tail event. Then, if μ is any Gibbs measure, $\mu(.|\mathcal{U})$ is also a Gibbs measure provided $\mu(\mathcal{A}) > 0$. But such a μ exists: Take the local specifications with boundary conditions either $\eta = +1$ or $\eta = -1$. They are supported on $\mathcal{U}$ and so any weak limit $\mu^\pm$ of these sequences satisfies $\mu^\pm(\mathcal{U}) = 1$.

Now on $\mathcal{U}$, the set of points $x \in \mathbb{Z}^d$ that is not surrounded by a contour (the exterior of the contour) is connected and the spin configuration on this set is constant either $+1$ or -1. It is clear that the value of the spin on the exterior is a function of the tail sigma-algebra so if μ_β is extremal, it takes either one or the other value with probability one. Denote these measures by $\mu_\beta^\pm$ then

$$\mu_\beta^+(\sigma_0 = -1) = \mu_\beta^+[\exists_{\gamma\in\Gamma(\sigma):0\in\mathrm{int}\,\gamma}] < \frac{1}{2} \tag{1.44}$$

which implies the theorem. □

Chapter 2

A condition for the uniqueness of Gibbs states in one dimensional models

2.1 Introduction

As discussed in the previous chapter, the problem of the absence of phase transitions is one of the most central problems of statistical physics. Investigation of this problem stands on several different approaches. In the next two chapters, we will focus on several conditions of uniqueness of limiting Gibbs measures. One of the most popular uniqueness conditions in one-dimensional case come from [1, 2, 4, 12]. Accordingly, this condition states that the interaction between far located spins should decrease so speedily that the value of total interaction of the spins on any two complementary half-lines is finite. In this chapter, an alternative method for establishing the absence of phase transition covering the case when the value of total interaction of the spins on two complementary half-lines is infinite will be formulated.

This method reduces the problem of uniqueness of limiting Gibbs states to the problem of percolation of special clusters. On the one hand the method works only in models with unique ground state, on the other hand the method allows us to establish uniqueness for actual and strong long-range interactions. In two or more dimensional models most classical results are obtained for finite range potentials [13, 14, 15, 16, 17], but the formulated below method allows to obtain uniqueness theorems without complicated heavy cluster expansions in models with long-range interactions. The method is especially powerful in one-dimensional models with very slowly decreasing potentials (see the examples in applications in the end of this chapter, the classical methods mentioned above fail to work in this case). The origin of the main idea of this method goes back to [18], where the theorem of uniqueness of limiting Gibbs measures was was established for one-dimensional long-range anti-ferromagnetical models in which each spin struggles to alter differently oriented spins. In [19] very sophisticated zero-temperature phase diagram of this model was investigated and the hypothesis on the uniqueness of limit Gibbs states was formulated (since the potential of this model does not satisfy the strong decreasing conditions of [1, 2, 12, 4] the classical methods fail to prove the uniqueness).

We consider a model with the following Hamiltonian

$$H(\phi) = \sum_{B \subset \mathbf{Z}^\nu} U(\phi(B)) \tag{2.1}$$

where the spin variables $\phi(x)$ take values in some finite set Φ and $\phi(B)$ denotes the restriction of the configuration ϕ to the set B. We assume that the potential is a translationally invariant function: $U(\phi(B+v)) = U(\phi(B))$ for each vector v.

The following natural condition is necessary for the existence of the thermodynamic limit: for some constant C_0 not depending on the configuration ϕ

$$\sum_{B \subset \mathbf{Z}^\nu : x \in B} |U(\phi(B))| < C_0 \tag{2.2}$$

Definition 2.1.1. *We say that the configuration ϕ' is a finite perturbation of the configuration ϕ if there is a finite set A such that $\phi'(x) \neq \phi(x)$ for each $x \in A$ and $\phi'(x) = \phi(x)$ for all $x \in \mathbf{Z}^\nu - A$.*

Definition 2.1.2. *A configuration ϕ^{gr} is said to be a ground state, if for any finite perturbation ϕ' of the configuration ϕ^{gr} we have $H(\phi^{gr'}) - H(\phi^{gr}) \geq 0$.*

Below, we assume that the model (2.1) has a unique ground state. The main idea of the method is the following:

ν-cube with the center at the origin and with the length of edge $2N$ will be denoted by V_N: $V_N = \{x_1, x_2, \dots, x_\nu : -N \leq x_i \leq N,\ i = 1, 2, \dots, \nu\}$. The set of all configurations on V_N we denote by $\Phi(N)$. Suppose that the boundary conditions $\phi^i,\ i = 1, 2$ are fixed.

Let $\mathbf{P^i_N}$ be the Gibbs distribution on $\Phi(N)$ corresponding to the boundary conditions $\phi^i,\ i = 1, 2$. Take $M < N$ and let $\mathbf{P^i_N}(\phi'(V_M)$ be the probability of the event that the restriction of the configuration $\phi(V_N)$ to V_M coincides with $\phi'(V_M)$.

The concatenation of the configurations $\phi(V_N)$ and $\phi^i(\mathbf{Z}^\nu - V_N)$ we denote by χ: $\chi(x) = \phi(x)$, if $x \in V_N$ and $\chi(x) = \phi^i(x)$, if $x \in \mathbf{Z}^\nu - V_N$.

$$\chi(x) = \begin{cases} \phi(x) & \text{if } x \in V_N \\ \phi^i(x) & \text{if } x \in \mathbf{Z}^\nu - V_N \end{cases}$$

Define

$$H_N(\phi|\phi^i) = \sum_{B \subset \mathbf{Z}^\nu : B \cap V_N \neq \emptyset} U(\chi(B))$$

At fixed N and fixed boundary conditions ϕ^i, the set of all configurations with minimal energy will be denoted by $\Phi^{\min}(N, \phi^i)$.

Now, define

$$H_N(\phi_N^{\min,i}|\phi^i) = \min_{\phi \in \Phi(N)} H_N(\phi|\phi^i)$$

where $\phi_N^{\min,i}$ is a configuration with the minimal energy (if the set $\Phi^{\min}(N, \phi^i)$ contains more than one element we arbitrarily choose any configuration with the minimal energy, it will be seen below that it is not essential).

The relative energy of a configuration ϕ with respect to $\phi_N^{\min,i}$ will be denoted by $H_N(\phi|\phi^i, \phi_N^{\min,i})$ which is defined as

$$H_N(\phi|\phi^i, \phi_N^{\min,i}) = H_N(\phi|\phi^i) - H_N(\phi_N^{\min,i}|\phi^i)$$

Note that since the ground state of the model (2.1) is unique, the configuration $\phi_N^{\min,i}$ almost coincides with the ground state φ^{gr} (see Lemma 2.3.1).

Let $\mathbf{P_N^i}$ be Gibbs distributions on $\Phi(N)$ corresponding to the boundary conditions ϕ^i, $i = 1, 2$ defined by using of relative energies of configurations. Take $M < N$ and let $\mathbf{P_N^i}(\phi'(V_M)$ be the probability of the event that the restriction of the configuration $\phi(V_N)$ to V_M coincides with $\phi'(V_M)$.

Suppose that $\mathbf{P^1}$ and $\mathbf{P^2}$ are two extreme limiting Gibbs states of the model (2.1). It is known that two extreme limit Gibbs states are either singular or coincide, the uniqueness of the limit Gibbs states of model (2.1) will be proven by showing that $\mathbf{P^1}$ and $\mathbf{P^2}$ are not singular: there exist two positive constants C_1 and C_2, such that for any M and $\phi'(V_M)$ there exists a number $N_0(M)$ such that for any $N > N_0$

$$C_1 < \mathbf{P_N^1}(\phi'(V_M))/\mathbf{P_N^2}(\phi'(V_M)) < C_2$$

The important point is the introduction of the contour model common for boundary conditions ϕ^i, $i = 1, 2$ (a *contour* is a connected sub-configuration not coinciding with the ground state). After that, by using of a well-known trick [20] we transfer interacting contours into "non-interacting" clusters (a *cluster* is a collection of contours connected by interaction bonds).

The geometrical-combinatorial Lemma 2.3.4 reduces the dependence of the expression $\mathbf{P_N^1}(\phi(V_M))/\mathbf{P_N^2}(\phi(V_M))$ on the boundary conditions ϕ^1 and ϕ^2 to the sum of statistical weights of 2-clusters connecting V_M with the boundary. The important point is that the statistical weight of 2-clusters are not necessarily positive and consequently we estimate the sum of absolute values of these weights. Thus, the problem of uniqueness of limiting Gibbs states reduces to the percolation type problem of estimation of the sum of some clusters connecting V_M and the boundary.

The formulated criterion works at all dimensions, and for models with very long-range interaction. Since in low dimensions the percolation is more rarely observed phenomenon, the criterion is especially powerful in one-dimensional case.

The decreasing conditions imposed on the potential in uniqueness Theorem 2.4.1 are most general; the results of [1, 2, 12, 4, 18] are obtained under more strong decreasing condition on the potential.

2.2 Main criterion of uniqueness

Let the boundary conditions ϕ^1 be fixed. Consider the $\mathbf{P^1_N}$ probability of the event that the restriction of the configuration $\phi(V_N)$ to V_M coincides with $\phi'(V_M)$:

$$\mathbf{P^1_N}(\phi'(V_M)) = \frac{\sum_{\phi(V_N):\phi(V_M)=\phi'(V_M)} \exp(-\beta H_N(\phi(V_N)|\phi^1, \phi_N^{\min,1}))}{\sum_{\phi(V_N)} \exp(-\beta H_N(\phi(V_M)|\phi^1, \phi_N^{\min,1}))}$$

$$= \frac{\exp(-\beta H^{in}_M(\phi'(V_M)))\, Y(\phi'(V_M), V_N, \phi^1)\, \Xi(V_N - V_M|\phi^1, \phi'(V_M), \phi_N^{\min,1})}{\sum_{\phi''(V_M)} \exp(-\beta H^{in}_M(\phi''(V_M)))\, Y(\phi''(V_M), V_N, \phi^1)\, \Xi(V_N - V_M|\phi^1, \phi''(V_M), \phi_N^{\min,1})}$$

$$= \frac{\exp(-\beta H^{in}_M(\phi'(V_M)))\, Y(\phi'(V_M), V_N, \phi^1)\, \Xi^{\phi^1,\phi'}}{\sum_{\phi''(V_M)} \exp(-\beta H^{in}_M(\phi''(V_M)))\, Y(\phi''(V_M), V_N, \phi^1)\, \Xi^{\phi^1,\phi''}} \tag{2.3}$$

where the summation in $\sum_{\phi''(V_M)}$ is taken over all possible configurations $\phi''(V_M)$, $H^{in}_M(\phi'(V_M)) = \sum_{B\subset V_M} U(\phi'(B)) - U(\phi_N^{\min,1})$ and $H^{in}_M(\phi''(V_M)) = \sum_{B\subset V_M} U(\phi''(B)) - U(\phi_N^{\min,1})$ are interior relative energies of $\phi'(V_M)$ and $\phi''(V_M)$.

$\Xi^{\phi^1,\phi'}$ and the partition functions corresponding to the boundary conditions $\phi^1(\mathbf{Z}^\nu - V_N)$, $\phi'(V_M)$, $\phi''(V_M)$ are denoted by $\Xi^{\phi^1,\phi''}$:

$$\begin{aligned} \Xi^{\phi^1,\phi'} &= \Xi(V_N - V_M|\phi^1, \phi'(V_M), \phi_N^{\min,1}), \\ \Xi^{\phi^1,\phi''} &= \Xi(V_N - V_M|\phi^1, \phi''(V_M), \phi_N^{\min,1}) \end{aligned} \tag{2.4}$$

The expression $Y(\phi(V_M), V_N, \phi^1)$ is defined as

$$Y(\phi(V_M), V_N, \phi^1) = \prod_{\substack{A\subset \mathbf{Z}^\nu : A\cap V_M \neq\emptyset; \\ A\cap \mathbf{Z}^\nu - V_N \neq \emptyset; \\ A\cap V_N - V_M = \emptyset}} \exp(-\beta(U(\phi(A)) - U(\phi_N^{\min,1}(A)))) \tag{2.5}$$

where ϕ in (2.5) is equal to ϕ' for $x \in V_M$ and is equal to ϕ^1 for $x \in \mathbf{Z}^\nu - V_N$.

The expression (2.5) gives the "lineal" interaction of $\phi(V_M)$ with the boundary conditions $\phi^1(\mathbf{Z}^\nu - V_N)$.

Let us consider the partition functions $\Xi^{\phi^1,\phi''} = \Xi(V_N - V_M|\phi^1, \phi''(V_M), \phi_N^{\min,1})$ corresponding to the boundary conditions $\phi^1(\mathbf{Z}^\nu - V_N)$, $\phi''(V_M)$ and $\Xi^{\phi^2,\phi'} = \Xi(V_N - V_M|\phi^2, \phi'(V_M), \phi_N^{\min,2})$ corresponding to the boundary conditions $\phi^2(\mathbf{Z}^\nu - V_N)$, $\phi'(V_M)$ as in (2.4).

Now define a super partition function

$$(\Xi^{\phi^1,\phi''} \ \Xi^{\phi^2,\phi'})$$
$$= \sum \exp(-\beta H_N(\phi^3(V_N)|\phi^1, \phi'', \phi_V^{\min,1})) \exp(-\beta H_N(\phi^4(V_N)|\phi^2, \phi', \phi_N^{\min,2}))$$

where the summation is taken over all configuration pairs $\phi^3(V_N)$ and $\phi^4(V_N)$, such that $\phi^3(V_M) = \phi''(V_M)$, $\phi^4(V_M) = \phi'(V_M)$.

Consider the partition of $\mathbf{Z}^\nu$ into ν-cubes $V_R(x)$, where $V_R(x)$ is a cube with the length of edge R and with the center at $x = (x_1, \ldots, x_\nu)$, where $x_i = R/2 + k_i R$; $i = 1, 2, \ldots, \nu$; and k_i is an integer number.

Definition 2.2.1. *Consider an arbitrary configuration ϕ. If $\phi(V_R(x)) \neq \phi^{gr}(V_R(x))$ the cube $V_R(x)$ will be called non regular. Two non regular cubes are connected if their intersection is nonempty. The connected components of non regular segments defined in such a way are called supports of contours and will be denoted by $suppK$. A contour is pair $K = (suppK, \phi(suppK))$.*

It can be readily shown that for each contour K, there exists a corresponding configuration ψ_K such that the only contour of the configuration ψ_K is K (ψ_K on $\mathbf{Z}^\nu - suppK$ coincides with ϕ^{gr}).

Definition 2.2.2. *The weight of contour K will be calculated by the following formula:*

$$\gamma(K) = H(\psi_K) - H(\phi^{gr}) \tag{2.6}$$

The statistical weight of a contour is

$$w(K_i) = \exp(-\beta\gamma(K_i)) \tag{2.7}$$

The formulas (2.6) and (2.7) yield:

$$\exp(-\beta H_N(\phi|\phi^1, \phi_N^{\min,1}) = \prod_{i=1}^{n} w(K_i) \exp(-\beta G(K_1, \ldots, K_n)) \tag{2.8}$$

where the multiplier $G(K_1, \ldots, K_n)$ corresponds to the interaction between contours and the boundary conditions ϕ^1.

$$G(K_1, \ldots, K_n) = \sum_{k=2}^{n} \sum_{i_1,\ldots,i_k} G(K_{i_1}, \ldots, K_{i_k}) \tag{2.9}$$

The summation above is taken over all possible non-ordered collections $i_1, \ldots, i_k$ at each fixed k.

The origin of the interaction between $K_{i_1}, \ldots, K_{i_k}$ is due to the fact that the weight of the contour $K_{i_j},\ \ j = 1, \ldots, k$ is calculated under the assumption that the configuration outside $supp(K_{i_j})$ coincides with the ground state.

The set of all interaction terms in the double sum (2.9) will be denoted by IG. (2.8) can be written as:

$$\begin{aligned} \exp(-\beta H_N(\phi|\phi^1, \phi_N^{\min,1})) &= \prod_{i=1}^{n} w(K_i) \prod_{B \in IG} (\exp(-\beta G(K_{i_1}, \ldots, K_{i_k}))) \\ &= \prod_{i=1}^{n} w(K_i) \prod_{G \in IG} (1 + \exp(-\beta G(K_{i_1}, \ldots, K_{i_k}) - 1)) \end{aligned} \tag{2.10}$$

From (2.10) we get

$$\exp(-\beta H(\phi|\phi^1, \phi_N^{\min,1})) = \sum_{IG' \subset IG} \prod_{i \in \mathbf{I}} w(K_i) \prod_{G \in IG'} g(G) \tag{2.11}$$

where the summation is taken over all subsets IG' (including the empty set) of the set IG, and $g(G) = \exp(-\beta G) - 1$.

Consider an arbitrary term of the sum (2.11), which corresponds to the subset $IG' \subset IG$. Let the interaction element $G \in IG'$.

Consider the set $\mathbf{K}$ of all contours such that for each contour $K \subset \mathbf{K}$, the set $suppK \bigcap G$ is nonempty. We call any two contours from $\mathbf{K}$ neighbors in IG'

interaction. The set of contours K' is called *connected* in IG' interaction if for any two contours K_p and K_q there exists a collection $(K_1 = K_p, \dots, K_n = K_q)$ such that any two contours K_i and K_{i+1}, $i = 1, \dots, n-1$, are neighbors.

The pair $D = [(K_i,\ i = 1, \dots, s); IG']$, where IG' is some set of interaction elements, is called a *cluster* provided there exists a configuration ϕ containing all K_i; $i = 1, \dots, s$; $IG' \subset IG$; and the set $(K_i,\ i = 1, \dots, s)$ is connected in IG' interaction. The statistical weight of a cluster D is defined by the formula

$$w(D) = \prod_{i=1}^{s} w(K_i) \prod_{(x,y) \in IG'} g(G)$$

Unfortunately the weight $w(D)$ is not necessarily positive, it will cause some non-crucial trouble below.

Two clusters D_1 and D_2 are called *compatible* if any two contours K_1 and K_2 belonging to D_1 and D_2, respectively, are compatible. A set of clusters is called *compatible* if any two clusters of it are compatible.

If $D = [(K_i,\ i = 1, \dots, s);\ IG']$, then we say that $K_i \in D$; $i = 1, \dots, s$.

If $[D_1, \dots, D_m]$ is a compatible set of clusters and $\bigcup_{i=1}^{m} suppD_i \subset V_N$, then there exists a configuration ϕ which contains this set of clusters. For each configuration ϕ we have

$$\exp(-\beta H_N(\phi|\phi^1, \phi_N^{\min,1})) = \sum_{IG' \subset IG} \prod w(D_i)$$

where the clusters D_i are completely determined by the set IG'. The partition function is

$$\Xi(\phi^1) = \sum w(D_1) \dots w(D_m)$$

where the summation is taken over all non-ordered compatible collections of clusters. In this way we come to non-interacting at distance clusters from interacting contours [20].

The following generalization of the definition of compatibility allows us to represent $(\Xi^{\phi^1,\phi''}\ \Xi^{2,'})$ as a single partition function.

Definition 2.2.3. *A set of clusters is called 2-compatible provided any of its* *two parts coming from two Hamiltonians is compatible. In other words, in 2-compatibility an intersection of supports of two clusters coming from different partition functions is allowed.*

If $[D_1, \ldots, D_m]$ is a 2-compatible set of clusters and $\bigcup_{i=1}^{m} suppD_i \subset V_N - V_M$, then there exist two configurations ϕ^3 and ϕ^4 which contain this set of clusters. For each pair of configurations ϕ^3 and ϕ^4 we have

$$\exp(-\beta H_N(\phi^3|\phi^1, \phi_N^{\min,1})) \exp(-\beta H_N(\phi^4|\phi^2, \phi_N^{\min,2})) = \sum_{\substack{IG' \subset IG, \\ IG'' \subset IG}} \prod w(D_i)$$

where the clusters D_i are completely determined by the sets IG' and IG''.

The two-fold or double partition function is

$$\Xi^{\phi^1, \phi'', \phi^2, \phi'} = \Xi^{\phi^1, \phi''} \ \Xi^{\phi^2, \phi'} = \sum w(D_1) \ldots w(D_m)$$

where the summation is taken over all non-ordered 2-compatible collections of clusters.

Let $w(D_1) \ldots w(D_m)$ be a term of the double partition function $\Xi^{\phi^1, \phi'', \phi^2, \phi'}$. The connected components of the collection $[supp(D_1), \ldots, supp(D_m)]$ are the supports of the general clusters. A general cluster SD is a pair $(supp(SD), \phi(supp(SD)))$.

Instead of the expression *"generally compatible collection of clusters"* we will use the expression *"compatible collection of 2-clusters"*.

Definition 2.2.4. *A 2-cluster* $SD = [(D_i, \ i = 1, \ldots, m); \ IG', \ IG'']$ *is said to be long if the intersection of the set* $(\bigcup_{i=1}^{m} suppD_i) \bigcup IG' \bigcup IG''$ *with both* V_M *and* $\mathbf{Z}^\nu - V_N$ *is nonempty. In other words, a long 2-cluster by using of its contours and bonds connects the boundary with the cube* V_M.

A set of 2-clusters is called compatible provided the set of all clusters belonging to these 2-clusters are 2-compatible.

Definition 2.2.5. *We say that the model (2.1) has not-long 2-clusters property, if there exists a number* $\epsilon, 0 < \epsilon < 1$ *such that for each fixed cube* V_M, *there exists* *a number $N_0 = N_0(M)$ depending only on M, so that for all $N > N_0$ we have*

$$(1-\epsilon)\ \Xi^{\phi^1,\phi',\phi^2,\phi''} < \Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} = \sum w(SD_1)\dots w(SD_m) < (1+\epsilon)\ \Xi^{\phi^1,\phi',\phi^2,\phi''} \tag{2.12}$$

where the summation is taken over all non-long, non-ordered compatible collections of 2-clusters $[SD_1,\dots,SD_m]$, $\bigcup_{i=1}^m supp(SD_i) \subset V_N - V_M$ corresponding to the boundary conditions $\{\phi^1(\mathbf{Z}^\nu - V_N),\ \phi^2(\mathbf{Z}^\nu - V_N);\ \phi'(V_M)$ and $\phi''(V_M)\}$.

It means that if a model has a not-long 2-clusters property then the statistical weights of long 2-clusters are negligible.

Now, let us formulate the uniqueness criterion:

Theorem 2.2.6. *Any model (2.1) having not-long 2-clusters property has at most one limit Gibbs state.*

Define a partition function $\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)}$ as $\sum w(SD_1)\dots w(SD_m)$, where the summation is taken over all non-ordered compatible collections of 2-clusters $[SD_1,\dots,SD_m]$ containing at least one long 2-cluster, $\bigcup_{i=1}^m suppD_i \subset V_N - V_M$ corresponding to the boundary conditions $\phi^1(\mathbf{Z}^\nu - V_N)$, $\phi^2(\mathbf{Z}^\nu - V_N)$; $\phi'(V_M)$ and $\phi''(V_M)$.

Let us also define a partition function $\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.)}$ as $\sum w(SD_1)\dots w(SD_m)$ where the summation is taken over all terms of $\Xi^{\phi^1,\phi',\phi^2,\phi''}$, which are not included into $\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)}$.

Dividing of both sides of the equality

$$\Xi^{\phi^1,\phi',\phi^2,\phi''} = \Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} + \Xi^{\phi^1,\phi',\phi^2,\phi'',(l)}$$

by $\Xi^{\phi^1,\phi',\phi^2,\phi''}$, we get

$$1 = \frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)}}{\Xi^{\phi^1,\phi',\phi^2,\phi''}} + \frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l)}}{\Xi^{\phi^1,\phi',\phi^2,\phi''}}$$

By definitions (2.12) for any model having not long 2-clusters property the absolute value of the second term of the last equality (which is not necessarily positive) is less then ϵ.

Consider

$$\frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l)}}{\Xi^{\phi^1,\phi',\phi^2,\phi''}} = \frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l)}}{\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} + \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.)}}$$

If we replace each term belonging to $\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.)}$ by its absolute value, then $\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.)}$ transfers into $\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}$.

Since the sign of $\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.)}$ is not definite, we have (under assumption that $\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} > \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}$, which will follow below from (2.13)):

$$-\frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}}{\left(\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} - \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}\right)} \le \frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.)}}{\left(\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} + \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.)}\right)}$$
$$\le \frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}}{\left(\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} + \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}\right)}$$

Simple calculations show that the inequality (2.12) follows from the following inequality:

$$\frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}}{\left(\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} + \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}\right)} < \epsilon/2 \qquad (2.13)$$

Below the expression $\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} + \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}$ will be denoted by $\Xi^{\phi^1,\phi',\phi^2,\phi''(abs)}$.

The expression $\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.abs)}/\Xi^{\phi^1,\phi',\phi^2,\phi''(abs)}$ can be paraphrased as an *"absolute probability"* $P^{abs}(Long)$ of the event that there is at least one long 2-cluster.

Definition 2.2.7. *We say that in model (2.1) 2-cluster percolation does not take place if there exists a number $\epsilon, 0 < \epsilon < 1$ such that for each fixed cube V_M, there exists a number $N_0 = N_0(M)$, which depends on M only, such that if $N > N_0$ then (2.13) is held.*

Note that by definitions any model in which 2-cluster percolation does not take place has not-long super clusters property.

Along with Kolmogorov's "0-1 Law", it can be easily shown that for any model in which super cluster percolation does not take place for any V_M

$$\lim_{N\to\infty} \frac{\Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}}{\left(\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)} + \Xi^{\phi^1,\phi',\phi^2,\phi'',(l.,abs.)}\right)} = 0$$

Thus, for all models for which 2-cluster percolation does not take place, the probability of the event that starting at any cube V_M we can reach the infinity distanced boundary by 2-clusters is zero.

Now we formulate the main uniqueness criterion:

Theorem 2.2.8. *Any model (2.1) in which 2-cluster percolation does not take place has at most one limit Gibbs state.*

2.3 Proof of results

In this section we prove Theorem 2.2.6, Theorem 2.2.8 is a consequence of Theorem 2.2.6 since any model in which super cluster percolation does not take place has not-long 2-clusters property.

Let $\phi_N^{\min,1} \in \Phi(N)$ be a configuration with the minimal energy. The following lemma describes the structure of the configuration $\phi_N^{\min,1}$.

Lemma 2.3.1. *For arbitrary fixed boundary conditions ϕ^1 there exist positive constant N_b not depending on the boundary conditions ϕ^1 and N, such that the restriction of the configuration $\phi_N^{\min,1}$ to the cube V_{N-N_b} coincides with the ground state ϕ^{gr}.*

Proof. Obviously, for each value of N there is a number $N_b = N_b(N, \phi^1)$, $(0 \leq N_b \leq N)$ satisfying the lemma, thus, the restriction of the configuration $\phi_N^{\min,1}$ to the set V_{N-N_b} coincides with the ground state ϕ^{gr}.

Let $N_b((N, \phi^1)$ be minimal. Define $N_b(N) = \max_{\phi^1} N_b(N, \phi^1)$ where the maximum is taken over all possible boundary conditions ϕ^1. In order to prove the Lemma 2.3.1, we show that $\max_N N_b(N)$ is bounded.

Indeed, suppose that $\max_N N_b(N)$ is not bounded. Then there exist a sequence of numbers $N(k)$, a sequence of boundary conditions $\phi^k(x)$; $x \in \mathbf{Z}^\nu - V_{N(k)}$ and corresponding sequence of configurations with minimal energy

$\phi^{\min,k}_{N(k)}(x), \quad k = 1, 2, \ldots$ such that $\lim_{k\to\infty} N(k) = \infty$ and $\lim_{k\to\infty} N_b(N(k), \phi^k) = \infty$.

For each $N(k)$ and ϕ^k, define a point $z \in \mathbf{Z}^\nu$ maximally distanced from the boundary such that $\phi^{\min,k}_{N(k)}(x) \neq \phi^{gr}$.

Let us define a configuration $\psi_{N(k')}(x) = \phi^{\min,k}_{N(k)}(x - z)$. Now, note that the restriction of the configurations $\psi_{N(k')}$ to any cube V_N does not coincide with the ground state.

We say that a sequence of configurations $\psi_{V(k)}(x)$ *point-wisely converges* to the configuration $\psi(x)$, if for each $x \in \mathbf{Z}^\nu$, there exists k_1, such that $\psi_{N(k)}(x) = \psi(x)$, if $k > k_1$.

After this natural definition, by using a diagonal argument we can show that the sequence $\psi_{N(k')}(x), \ k' = 1, 2, \ldots$ has at least one limit point, say $\psi^{\min}(x) \neq \phi^{gr}$. Indeed, suppose that $x_1, x_2, x_3, \ldots$ is some ordering of all points of $\mathbf{Z}^\nu$. Then there exists a subsequence $\psi^{x_1}_{N(k')}$ of $\psi_{N(k')}$, such that $\psi^{x_1}_{N(k')}(x_1)$ is a constant. There exists a subsequence $\psi^{x_1,x_2}_{N(k')}$ of $\psi^{x_1}_{N(k')}$, such that $\psi^{x_1,x_2}_{N(k')}(x_2)$ is a constant. There exists a subsequence $\psi^{x_1,x_2,x_3}_{N(k')}(x)$ of $\psi^{x_1,x_2}_{N(k')}$ such that $\psi^{x_1,x_2,x_3}_{N(k')}(x_3)$ is a constant.

By continuing this process we obtain a subsequence $\psi^{x_1,x_2,x_3,\cdots}_{N(k')}(x)$ of $\psi_{N(k)}$ which converges to some configuration $\psi^{\min}$.

Now, note that $\psi^{\min}$ is a ground state. In fact, suppose that $\bar{\psi}$ is an arbitrary perturbation of $\psi^{\min}$ on some finite set W.

$$H(\bar{\psi}) - H(\psi^{\min}) \geq H_N(\bar{\phi}|\phi^{k'}) - H_N(\phi^{\min}|\phi^{k'}) - \epsilon(W, N(k), \phi^{k'})$$

where $\bar{\phi}$ is the same perturbation of $\phi^{\min}$ on the set $W - z$.

For each fixed W, the term $\epsilon(W, N(k'), \phi^{k'})$ tends to zero uniformly with respect to $\phi^{k'}$ while $N(k')$ tends to infinity. But, by construction $H_N(\bar{\phi}|\phi^{k'}) - H_N(\phi^{\min,k'}|\phi^{k'}) \geq 0$. Therefore, $H(\bar{\phi}) - H(\phi^{\min} \geq 0$ and $\psi^{\min}$ is a ground state.

Now, note that the configuration $\psi^{\min}(x) \neq \phi^{gr}(x)$. In fact, since the configuration $\psi_{V(k')}(x)$, which is just a shift of $\varphi^{\min,k'}_{V(k')}$, the ground state φ^{gr} can not coincide with $\psi_{N(k')}(x)$ on the cube V_N. And $\psi^{\min}$ is a limit of configurations $\psi_{V(k')}(x)$.

This contradicts the assumption that $\max_N N_b(N)$ is not bounded. Lemma 2.3.1 is proved. $\square$

Let $\mathbf{P^1}$ and $\mathbf{P^2}$ be two extreme limit Gibbs states corresponding to the boundary conditions ϕ^1 and ϕ^2 [21, 7], and $\mathbf{P^1_N}$ and $\mathbf{P^2_N}$ be Gibbs distributions on $\Phi(N)$ corresponding to the boundary conditions ϕ^1 and ϕ^2.

Theorem 2.3.2. *$\mathbf{P^1}$ and $\mathbf{P^2}$ are singular or coincide ([21, 7]).*

We prove the uniqueness of the limiting Gibbs states of model (2.1) by showing that $\mathbf{P^1}$ and $\mathbf{P^2}$ are not singular.

Lemma 2.3.3. *Limit Gibbs states $\mathbf{P^1}$ and $\mathbf{P^2}$ are absolutely continuous with respect to each other.*

Proof. In order to prove the Lemma 2.3.3, we show that for any V_M and arbitrary $\phi'(V_M)$ there exist two positive constants s_0 and S_0 not depending on V_M, ϕ^1, ϕ^2 and $\phi'(V_M)$, such that

$$s_0 \leq \mathbf{P^1}(\phi'(V_M))/\mathbf{P^2}(\phi'(V_M)) \leq S_0 \tag{2.14}$$

Let $\mathbf{P^1_N}$ and $\mathbf{P^2_N}$ be Gibbs distributions on $\Phi(N)$ corresponding to the boundary conditions ϕ^1 and ϕ^2, thus, $\lim_{N\to\infty} \mathbf{P^1_N} = \mathbf{P^1}$ and $\lim_{N\to\infty} \mathbf{P^2_N} = \mathbf{P^2}$ where by convergence we mean weak convergence of probability measures.

For establishing the inequality (2.14) we prove that for each fixed cube V_M, there exists a number $N_0(M)$, depending on M only, such that for $N > N_0$

$$s_0 \leq \mathbf{P^1_V}(\phi'(V_M))/\mathbf{P^2_V}(\phi'(V_M)) \leq S_0 \tag{2.15}$$

The probability $\mathbf{P}^{\mathbf{1}}_{\mathbf{V}}(\varphi'(V_M))$ is given by (2.3). $\mathbf{P}^{\mathbf{2}}_{\mathbf{V}}(\varphi'(V_M))$ has a similar representation.

In order to prove the inequality (2.15) it is enough to establish inequalities (2.16) and (2.17):

$$0.9 < Y(\varphi(I), V, \varphi^i) < 1.1; \quad i = 1,2 \tag{2.16}$$

and

$$1/S \leq (\frac{\Xi^{\phi^1,\phi''}}{\Xi^{\phi^1,\phi'}})/(\frac{\Xi^{\phi^2,\phi''}}{\Xi^{\phi^2,\phi'}}) \leq 1/s \tag{2.17}$$

for arbitrary $\varphi''(V_M)$, where $S = (1.1/0.9)^2 S_0$ and $s = (0.9/1.1)^2 s_0$.

Indeed, if the inequalities (2.16) and (2.17) hold, then

$$1/(1/s) \leq \mathbf{P}^{\mathbf{1}}_{\mathbf{V}}(\varphi'(V_M))/\mathbf{P}^{\mathbf{2}}_{\mathbf{V}}(\varphi'(V_M)) \leq 1/(1/S)$$

since the quotient of $(\sum_{i=1}^n a_i)/(\sum_{i=1}^n b_i)$ lies between $\min(a_i/b_i)$ and $\max(a_i/b_i)$.

Now we prove the inequalities (2.16) and (2.17):

The inequality (2.16) is a direct consequence of the condition that the potential is a decreasing function: for each fixed M there exists N_0, such that if $N > N_0$, then $0.9 < Y(\phi(I), N, \phi^i) < 1.1$; $i = 1, 2$.

So, in order to complete the proof of Lemma 2.3.3, we have to establish the following inequality (which is just the transformed inequality (2.17)):

$$1/S \leq \frac{\Xi^{\phi^1,\phi''} \ \Xi^{\phi^2,\phi'}}{\Xi^{\phi^2,\phi''} \ \Xi^{\phi^1,\phi'}} \leq 1/s \tag{2.18}$$

Now, we show that for each fixed cube V_M, there exists a number $N_0(M)$, which depends on M only, such that if $N > N_0(M)$

$$s \leq (\Xi^{\phi^1,\phi'}\Xi^{\phi^2,\phi''})/(\Xi^{\phi^1,\phi''}\Xi^{\phi^2,\phi'}) \leq S \tag{2.19}$$

for two positive constants s and S not depending on M, ϕ^1, ϕ^2, ϕ' and ϕ''.

Partition functions including only non-long super clusters satisfy the following key lemma which has geometrical-combinatorial explanation.

Lemma 2.3.4. *[18]*

$$\Xi^{\phi^1,\phi'',\phi^2,\phi',(n.l.)} = Q\,\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)}$$

where the factor $Q = Q(\phi^1(\mathbf{Z}^\nu - V_N), \phi^2(\mathbf{Z}^\nu - V_N), \phi'(V_M), \phi''(V_M))$ *is uniformly bounded:* $0 < const_1 < Q < const_2$.

Note that the factor Q appears due to the fact that the configurations with minimal energies corresponding to the different boundary conditions do not coincide everywhere (due to Lemma 2.3.1 they differs on some finite set and due to the condition (2.2) Q is finite).

Proof. Due to the factor Q without loss of generality we suppose that the configurations with minimal energies corresponding to the different boundary conditions coincide with φ^{gr}.

The summations in $\Xi^{\phi^1,\phi'',\phi^2,\phi',(n.l.)} = \Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)}$ are taken over all non-long, non-ordered compatible collections of 2-clusters.

We set a one-to-one correspondence between the terms of these two double partition functions: To the term

$$w(D_1^{1,''})w(D_2^{1,''})w(D_3^{1,''})w(D_4^{1,''})w(D_5^{2,'})w(D_6^{2,'})w(D_7^{2,'})w(D_8^{2,'})$$

(i.e. the first four factors of this term came from the partition function $\Xi^{\phi^1,\phi''}$ and the last four factors of this term came from the partition function $\Xi^{\phi^2,\phi'}$) of the super partition function $\Xi^{\phi^1,\phi'',\phi^2,\phi',(n.l.)}$, we correspond the term

$$w(D_1^{1,'})w(D_6^{1,'})w(D_7^{1,'})w(D_4^{1,'})w(D_5^{2,''})w(D_2^{2,''})w(D_3^{2,''})w(D_8^{2,''})$$

(i.e. the first four factors of this term came from the partition function $\Xi^{\phi^1,\phi'}$ and the last four factors of this term came from the partition function $\Xi^{\phi^2,\phi''}$) of the super partition function $\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)}$.

It can be readily shown that this one-to-one correspondence is correctly defined and works: if some term from $\Xi^{\phi^1,\phi',\phi^2,\phi'',(n.l.)}$ corresponding to the term from $\Xi^{\phi^1,\phi'',\phi^2,\phi',(n.l.)}$ does not exist (in other words, the corresponding clusters from $\Xi^{\phi^1,\phi'}$ or $\Xi^{\phi^2,\phi''}$ are overlapped) then the term from $\Xi^{\phi^1,\phi'',\phi^2,\phi',(n.l.)}$ is long super cluster, which is impossible. Thus, Lemma 2.3.4 is proved. $\square$

The inequality (2.19) is a direct consequence of (2.12) and Lemma 2.3.4. Lemma 2.3.3 is proved. $\square$

Proof. (*of Theorem 2.2.6*). Let $\mathbf{P^1}$ and $\mathbf{P^2}$ be two different extreme limit Gibbs states of the model (2.1) corresponding to the boundary conditions ϕ^1 and ϕ^2 respectively. Due to Lemma 2.3.3, $\mathbf{P^1}$ and $\mathbf{P^2}$ are not singular. Therefore, by Theorem 2.3.2, $\mathbf{P^1}$ and $\mathbf{P^2}$ coincide, which contradicts the assumption. Theorem 2.2.6 is proved. $\square$

The proof of uniqueness criterion stands on two main points. The most important point is an introduction of the contour model common for all boundary conditions. After that, by using of a well-known trick [20] we come to "non-interacting" clusters from interacting contours.

The combinatorial Lemma 2.3.4, which allows us to reduce the dependence of the expression $\mathbf{P^1_N}(\phi(V_M))/\mathbf{P^2_N}(\phi(V_M))$ on the boundary conditions ϕ^1 and ϕ^2 to the sum of statistical weights of 2-clusters connecting the cube V_M with the boundary (so called "long 2-clusters").

Theorem 2.2.6 and Theorem 2.2.8 have generalizations for non translation-invariant potentials.

2.4 Applications

2.4.1 One-dimensional models

2.4.1.1 First application

The problem of phase transitions in one-dimensional models with long range interaction has attracted the interest of many authors [20, 21, 7, 22, 23, 24]. It is well known that the condition $\sum_{r\in\mathbf{Z^1},r>0} r|U(r)| < \infty$ ($U(r)$ is a pair potential of long range) implies uniqueness of limit Gibbs states [1, 2, 12, 4]. Below we consider one-dimensional model under very natural regularity conditions and obtain uniqueness result without this strong restriction on potential of the model.

Condition 1. We say that the ground state ϕ^{gr} of the model (2.1) satisfies the *Peierls stability condition*, if there exists a constant t such that for any finite set $A \subset \mathbf{Z^1}$ $H(\phi') - H(\phi^{gr}) \geq t|A|$, where $|A|$ denotes the number of sites of A and ϕ' is a perturbation of ϕ^{gr} on the set A.

Condition 2. There exists a constant $\gamma < 1$, such that for any number L and any interval $I = [a, b]$ with the length n and for any configuration $\phi(I)$

$$\sum_{B\subset Z^1;B\cap I\neq\emptyset,B\cap(Z^1-[a-L,b+L])\neq\emptyset} |U(\phi(B))| \leq const\, n^\gamma\, L^{\gamma-1}$$

Condition 2 is very natural and particularly is held in models with pair potential $U(r) \sim 1/r^{1+\delta}$, as $r \to \infty, \delta > 0$.

Theorem 2.4.1. *Suppose that $\nu = 1$ and the model (2.1) satisfies Conditions 1 and 2. Then there exists a value of the inverse temperature β_{cr} such that if $\beta > \beta_{cr}$ then the model (2.1) has at most one limit Gibbs state.*

Conditions of Theorem 2.4.1 are very natural. Phase transition takes place if some of these conditions are absent [24, 25, 26, 27, 28].

Proof. In order to prove Theorem 2.4.1, we show that for any model (2.1) there exists β_{cr} such that if $\beta > \beta_{cr}$ then in the model (2.1) 2-cluster percolation does not take place (Theorem 2.3.2). In other words, we show that at low temperatures there exists a number ϵ such that for each fixed V_M (in our case interval $[-M, M]$) there exists a number N_0, which depends on M only such that if $N > N_0$ then the absolute probability (2.13) of long 2-clusters is less then ϵ. Long 2-clusters can connect the interval $\phi'(V_M)$ or $\phi''(V_M)$ with ϕ^1 or ϕ^2.

It can be easily shown that in order to prove Theorem 2.4.1, it is sufficient to show that the probability that there is at least one 2-cluster connecting $\phi(-\infty, -N)$ and $\phi'[-M, M]$ is less then ϵ_1, for some $\epsilon_1 < 0$ at $\beta > \beta_{cr}$.

By definition, the support of any 2-cluster is the union (connected by interaction elements) of contours or heap of intersected contours some sitting on others. Below, we call these contours and heaps of contours by 2-contours and denote them by SK.

We prove more strong result asserting that the absolute probability of the event that there is a 2-contour connected to $\phi(-\infty, -N)$ by interaction elements is less then ϵ_2 for some $\epsilon_2 < 0$ at $\beta > \beta_{cr}$.

For each 2-contour SK, we define the notion of essential support $ess\ suppK$. We say that an interval $[k, k+1]$ belongs to the essential support of SK if for at least one contour $K' = (suppK', \phi'(suppK'))$ belonging to SK, $\phi'[k, k+1] \neq \phi^{gr}[k, k+1]$. By $|ess\ suppSK|$ we denote the number of unit $[k, k+1]$ intervals belonging to $ess\ suppSK$.

Suppose that the support of 2-cluster SD consists of only 2-contour SK (without interaction elements). Then the statistical weight $w(SK)$ of this 2-cluster SK is equal to $w(SK) = \exp(-\beta\ s\ |ess\ suppSK|)$ and by straightforward applying of Peierls argument it can be easily shown that the absolute probability of this 2-cluster

$$P^{abs}(SD) < \exp(-\beta\ s\ |ess\ suppSK|) \tag{2.20}$$

where $s > 0$ is a constant (actually $s = 1 - (1-t)(1-t)$ where t is the Peierls constant, defined in Condition 1).

Now we are going to estimate the absolute probability of the event that there is at least one 2-cluster connecting $\phi(-\infty, -N)$ and $\phi'[-M, M]$.

Suppose that the 2-cluster SD is connected to $\phi(-\infty, -N)$. Let SK be the leftmost 2-contour belonging to SD. We say that a 2-contour K' is a *neighbor of the first order of* SK and write $SK \leftrightarrow SK'$ if SK and SK' are connected by interaction element. A 2-contour SK'' is called a *neighbor of qth order of* SK provided $SK \leftrightarrow SK_1 \leftrightarrow SK_2 \leftrightarrow \cdots \leftrightarrow SK_{q-1} \leftrightarrow SK''$ and there is no such diagram with fever arrows.

We are going to estimate $P^{abs}(SD)$ by using of the following method: in the first step we fix all 2-contours of order $q-1$ and take the summation over all 2-contours of order q, in the second step we fix all 2-contours of order $q-2$ and take the summation over all 2-contours of order $q-1$, and so on. We repeat this summation $q-1$ times.

Proposition 2.4.2. *Let SK_0 be a 2-contour of order k and suppose that for all 2-contours of order $k+1$, $w(SK) < \exp(-\frac{1}{2}\beta\, s\, |ess\ suppSK|)$. Then*

$$\sum_{SD:\ SD=(SK_0,SK,IG',IG'')} w(SD) < \exp(-\frac{1}{2}\beta\, s\, |ess\ suppSK_0|)$$

at sufficiently large values of β.

Proposition 2.4.2 states that if we fix a 2-contour and take the summation over all its neighbors then the constant s in statistical weight of this 2-cluster worsens at most to $s/2$. The proof is very standard and is based on the technique of restriction of entropy terms at low temperatures. We omit details (for detailed proof in special case see [18]).

Now we are ready to estimate the absolute probability of the event that there is a 2-contour SK_0 is connected to $\phi(-\infty, -N)$. If we fix a 2-contour SK_0 and consider the set of all 2-clusters containing SK_0 as its leftmost 2-contour, then by applying proposition we obtain the estimation:

$$w(SK_0) \leq \exp(-\frac{1}{2}\beta\, s\, |ess\ suppSK_0|)$$

Suppose that $|ess\ supp SK_0| = n$. By Condition 2 the absolute probability of the event that SK_0 is connected to $\phi(-\infty, -N)$ is less then

$$\sum_{L=0}^{\infty} \sum_{n=1}^{\infty} \exp(-\frac{1}{2}\beta\ s\ n)(\exp(\beta\ const\ n^{\gamma}\ L^{\gamma-1}) - 1)$$

which in turn is less then any given ϵ at sufficiently large values of β.

Finally, since the absolute probability of percolation is less then the absolute probability of the event that SK_0 is connected to $\phi(-\infty, -N)$, the 2-cluster percolation does not take place. Now Theorem 2.4.1 follows from Theorem 2.2.8.

□

Below an application of Theorem 2.2.8 is presented.

Example. Consider one-dimensional anti-ferromagnetical model with the Hamiltonian:

$$H(\phi) = \sum_{x,y \in \mathbf{Z}^1} |x-y|^{-1-\alpha} \phi(x)\phi(y) + h \sum_{x \in \mathbf{Z}^1} \phi(x) \tag{2.21}$$

where the spin variables $\phi(x)$ take 0 and 1, $0 < \alpha < 1$ and $h > 0$.

One can easily show that the constant configuration $\phi = 0$ is a unique ground state of the model (2.21) and the model satisfies the Conditions 1 and 2 of this subsection. Thus, we can apply Theorem 2.4.1 (or Theorem 2.2.8) and prove that the model (2.21) has a unique limit Gibbs states at low temperatures.

Now note that due to the fact that the model (2.21) has very long-range interaction, obtained result is rather non-trivial and is not a consequence of classical methods and results: Since

$$\sum_{r \in \mathbf{Z}^1, r>0} r|U(r)| = \sum_{r \in \mathbf{Z}^1, r>0} r\ r^{-1-\alpha} = \infty$$

the methods of [1] and [12] are not applicable.

2.4.2 Two and more dimensional models

The analogue of Theorem 2.4.1 can be proved for two dimensional models. In two-dimensional models standard cluster expansion method allows to obtain the same result under restriction that the potential has short interaction range ($U(\phi(B)) = 0$ when $|B|$ is greater then some constant). Note that the value of β_{cr} for two dimensional case must be greater then the value of the critical inverse temperature for site percolation.

Theorem 2.4.3. *Suppose that in model (2.1) site percolation does not take place and the value of $|g(G)|$ is uniformly less then 1. Then the model (2.1) has at most one limit Gibbs state.*

The proof can be carried out by using of Theorem 2.2.8. Actually, since there is no site percolation, for any long super cluster the number of interaction bonds G uniformly tends to infinity when the volume V_N increases. Now the absence of percolation by super clusters follows from the fact that $|g(G)|$ is uniformly less then 1.

Chapter 3

One dimensional long range Widom-Rowlinson model with periodic particle activities

3.1 Introduction

A gain in mixing entropy forces many multicomponent systems to a single phase. The system may pass to phases of prevailing particles of particular kind if some thermodynamical variables change. One of the basic models explaining this kind of phase separations lies in the relative strengths of repulsion between like and unlike particles. If the unlike particles experience a stronger repulsion than the like ones, at least at high density demixing phases are likely. The archetype for analogous systems is the Widom-Rowlinson model. The two particle Widom-Rowlinson model is a lattice gas model with two types of particles, allowed to share neighboring sites only if they are of the same type. The model was introduced [29] as a continuum model of particles in space. The lattice variant was studied firstly in [30]. The spin variables $\phi(x)$ belong to the spin space $\{-1, 0, +1\}$, where 0 corresponds to empty sites. The Hamiltonian of the model is defined as

$$H_0(\phi) = \sum_{x \in \mathbf{Z}^d} U_0(\phi(x)) + \sum_{x,y \in \mathbf{Z}^d} U_1(\phi(x), \phi(y))$$

where the chemical potential is the following

$$U_0(\phi(x)) = \begin{cases} -\ln \lambda_- & \text{if } \phi(x) = -1 \\ \quad 0 & \text{if } \phi(x) = 0 \\ -\ln \lambda_+ & \text{if } \phi(x) = +1 \end{cases}$$

and $\lambda_- > 0$ and $\lambda_+ > 0$ are the activity parameters of particles -1 and +1.

The hard-core pair interaction is given by

$$U_1(\phi(x), \phi(y)) = \begin{cases} \infty & \text{if } \phi(x)\phi(y) = -1 \text{ and } |x - y| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

This hard-core model exhibits so-called hard constraints, i.e. their properties arise by forbidding certain configurations. For small values of $\beta\lambda_- = \beta\lambda_+$, there is a unique Gibbs state on which the overall densities of +1 and -1 particles are almost surely equal. At $d \geq 2$ and for sufficiently large values of $\beta\lambda_- = \beta\lambda_+$, the symmetry of -1 and +1 particles is broken: there are limiting Gibbs states with overwhelming densities of -1 and +1 particles. In non-symmetric case $\lambda_- \neq \lambda_+$, most likely limiting Gibbs state is unique in $d \geq 2$, but rigorous proof is not known. The non-symmetric case in $d = 1$ is considered in [31].

In this paper, the results of [31] is extended by considering the case when particle activities depend also on lattice sites.

Consider the one dimensional long range Widom-Rowlinson model with the Hamiltonian

$$H(\phi) = \sum_{x \in \mathbf{Z}^1} U_0(\phi(x)) + \sum_{x,y \in \mathbf{Z}^1} U_1(\phi(x), \phi(y)) + \sum_{x,y \in \mathbf{Z}^1} U_2(\phi(x), \phi(y)) \tag{3.2}$$

where

$$U_0(\phi(x)) = \begin{cases} -\ln \lambda_-^x & \text{if } \phi(x) = -1 \\ \quad 0 & \text{if } \phi(x) = 0 \\ -\ln \lambda_+^x & \text{if } \phi(x) = +1 \end{cases}$$

$\lambda_-^x > 0$ and $\lambda_+^x > 0$ are the activity parameters of particles -1 and +1, that are periodic and depend on lattice sites $x \in \mathbf{Z}^1$: there is a positive integer p such that $\lambda_-^{x+p} = \lambda_-^x$ and $\lambda_+^{x+p} = \lambda_+^x$.

U_1 is defined as in (3.1) and U_2 is as the following

$$U_2(\phi(x), \phi(y)) = \begin{cases} -C|x-y|^{-\alpha} & \text{if } \phi(x)\phi(y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We impose a condition $\alpha > 1$ for the existence of the thermodynamic limit.

Let V_N be an interval with the center at the origin and with the length of $2N$, and $\Phi(N)$ denote the set of all configurations $\phi(V_N)$. We denote the concatenation of the configurations $\phi(V_N)$ and $\phi^i(\mathbf{Z^1} - V_N)$ by χ i.e. $\chi(x) = \phi(x)$, if $x \in V_N$ and $\chi(x) = \phi^i(x)$, if $x \in \mathbf{Z^1} - V_N$.

Define

$$H_N(\phi|\phi^i) = \sum_{\substack{x\in\mathbf{Z}^1\\ x\in V_N}} U_0(\chi(x)) + \sum_{\substack{x,y\in\mathbf{Z}^1\\ x>y\\ \{x,y\}\cap V_N\neq\emptyset}} (U_1(\chi(x),\chi(y)) + U_2(\chi(x),\chi(y)))$$

The finite-volume Gibbs distribution corresponding to the boundary conditions ϕ^i is

$$\mathbf{P^i_N}(\phi|\phi^i) = \frac{\exp(-\beta H_N(\phi|\phi^i))}{\Xi(N,\phi^i)}$$

where β is the inverse temperature and the partition function $\Xi(N,\phi^i) = \sum_{\phi\in V_N} \exp(-\beta H_N(\phi|\phi^i))$.

We say that a probability measure $\mathbf{P}$ on the configuration space $\{-1,0,1\}^{\mathbf{Z^1}}$ is an infinite-volume Gibbs state if for each N and for $\mathbf{P}$ almost all ϕ^i in $\{-1,0,1\}^{\mathbf{Z^1}}$, we have

$$\mathbf{P}(\phi(V_N) = \varphi(V_N)|\phi(\mathbf{Z^1} - V_N) = \phi^i(\mathbf{Z^1} - V_N)) = \mathbf{P^i_N}(\varphi|\phi^i)$$

In this paper we investigate the problem of uniqueness of Gibbs states of the model (3.2). The case $\alpha > 2$ is well known: since the interactions between distant spins decrease rapidly, the total interaction of complementary half-lines is finite and the phase transition is absent [32, 2, 4]. The case $2 > \alpha > 1$ is open for different possibilities. In the homogeneous and symmetric case $\lambda_-^x = \lambda_+^x,\ x \in \mathbf{Z}^1$, most likely the model exhibits a phase transition at sufficiently low temperatures as in ferromagnetic Ising model with long range interaction [24, 33].

We will treat the model (3.2) by a special method [18, 34] developed for the case when the interactions between distant spins decrease not rapidly. This low temperature regime method mixes two independent realizations of Gibbs fields and reduces the problem of phase transition to percolation type problems of special clusters connecting fixed segments with the boundary. The procedure of mixing of two independent realizations in other words "coupling" have had successful effects in numerous different cases [35, 36, 37, 38, 39]. In pursuance of [18, 34] for investigation of Gibbs states of model (3.2) we explore stability properties of ground states and by applying of uniqueness criterion from [34] (see Theorem 3.1.1 below) and we prove the following theorem.

Theorem 3.1.1. *Let $\sum_{x=1}^{p}(\ln\lambda_+^x - \ln\lambda_-^x) \neq 0$ and the interaction constant C is sufficiently large. Then the inverse temperature β_{cr} exists such that if $\beta > \beta_{cr}$ then the model (3.2) has at most one limiting Gibbs state.*

As it was mentioned above for weak interaction potentials $U_2(\phi(x), \phi(y))$, the model (3.2) does not exhibit phase transition. Nevertheless, the condition on constant C is necessary in order to avoid cases when in some part of the period local clusters of similar particles may withstand the influence of remaining particles leading to possible phase coexistence [40]. The structure of ground states of one dimensional Ising model with long range interaction and additional non-constant external field was investigated in [41].

3.2 Proofs

$\phi(B)$ denotes the restriction of the configuration ϕ to the set B. We say that the ground state ϕ^{gr} of the model (3.2) satisfies the Peierls stability condition with positive constant t, if $H(\phi') - H(\phi^{gr}) \geq t|A|$ for any finite set $A \subset \mathbf{Z^1}$ ($|A|$ denotes the number of sites of A and ϕ' is a perturbation of ϕ^{gr} on the set A).

Without loss of generality, we suppose that $\sum_{x=1}^{p}(\ln\lambda_+^x - \ln\lambda_-^x) = \Delta > 0$.

Lemma 3.2.1. *The model (3.2) has a unique ground state $\phi^{gr} \equiv 1$.*

Proof. Let ϕ' be a perturbation of $\phi^{gr} \equiv 1$ on a set A such that $\phi'(x)\phi'(y) \neq -1$ for all adjacent spins. Let $I_k = [1+kp, p+kp] \cap \mathbf{Z^1}$, then readily $\mathbf{Z^1} = \cup_{-\infty}^{\infty} I_k$. Suppose that all indices for which $I_{l_i} \cap A \neq \emptyset$ are $\{l_1, \dots, l_s\}$, then

$$H(\phi') - H(\phi^{gr}) = \sum_{i=1}^{s}(H(\phi'(I_{l_i})) - H(\phi^{gr}(I_{l_i}))) + \sum_{*}(U_1(\phi'(x), \phi'(y)) - U_1(\phi(x), \phi(y)))$$

where the summation in $\sum_*$ is taken over all pairs (x, y) not belonging to the same I_k. Since the long range interaction is ferromagnetic, we readily get the following

$$H(\phi') - H(\phi^{gr}) \geq \sum_{i=1}^{s}(H(\phi'(I_{l_i})) - H(\phi^{gr}(I_{l_i}))) \tag{3.3}$$

Consider $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j}))$ for some l_j. If $\phi'(I_{l_j})$ consists of only -1 particles then $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j})) \geq \Delta > 0$. If not then $\phi'(I_{l_j})$ is a union of spin blocks -1, 0 and +1 particles and since in each merger between distinct blocks we loose at least $C\dot{U}(1)$, we readily get $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j})) \geq (C \cdot U(1) - \sum_{i=1}^{p}\max(\ln\lambda_-^x, \ln\lambda_+^x)) > 0$ for sufficiently large values of C. Thus, in both cases $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j})) > 0$. $\square$

Lemma 3.2.2. *The unique ground state ϕ^{gr} of the model (3.2) satisfies the Peierls stability condition.*

Proof. Let ϕ' be a perturbation of $\phi^{gr} \equiv 1$ on a set A. Let us choose the constant C such that $(C \cdot U(1) - \sum_{i=1}^{p}\max(\ln\lambda_-^x, \ln\lambda_+^x)) > \Delta$. Then by (3.3)

$$H(\phi') - H(\phi^{gr}) \geq \sum_{i=1}^{s}(H(\phi'(I_{l_i})) - H(\phi^{gr}(I_{l_i}))) \geq \Delta \cdot s$$

and for $t = \frac{\Delta}{p}$, we readily get the required inequality

$$H(\phi') - H(\phi^{gr}) \geq t \cdot |A|$$

$\square$

We are going to prove uniqueness of limiting Gibbs states by applying the method installing strong relationship between stable ground states and limiting Gibbs states at low temperature regime. The method is developed for the case when the Hamiltonian of the model is translationally invariant function. Although the Hamiltonian of our model is not necessarily translationally invariant and is only periodic, this problem can be melted by using of the following approach:

Let the Hamiltonian be a periodic function with period p. Let us partition the lattice into disjoint intervals $[kp+1, (k+1)p]$ and replace the spin space $\{\Phi\}$ by $\{\Phi\}^{[1,p]}$ including $|\Phi|^p$ elements, then the model from translationally periodic with period p transfers to translationally invariant model. Thus, results of Chapter 2 are held also for periodic models with period p and in this way without loss of generality starting now we will suppose that the methods of uniqueness of limiting Gibbs states also work for periodic Hamiltonians. The following theorem installs a strong relationship between stable ground states and Gibbs states at low temperatures:

Theorem 3.2.3. *(see [34]). Suppose that a one dimensional model has a unique ground state satisfying Peierls stability condition and a constant $\gamma < 1$ exists such that for any number L and any interval $I = [a, b]$ with length n and for any configuration $\phi(I)$, we have the following inequality*

$$\sum_{\substack{B \subset Z^1 \\ B \cap I \neq \emptyset \\ B \cap (Z^1 - [a-L, b+L]) \neq \emptyset}} |U(\phi(B))| \leq (const)\, n^{\gamma}\, L^{\gamma - 1} \tag{3.4}$$

In that case, a value of the inverse temperature β_{cr} exists such that if $\beta > \beta_{cr}$ then the model has at most one limiting Gibbs state.

Now the Theorem 3.1.1 follows from the Theorem 3.2.3: Indeed, by

Lemma 3.2.1 and Lemma 3.2.2, the ground state ϕ^{gr} is unique and stable, and the condition $|U_2(\phi(x), \phi(y))| \leq C|x-y|^{-\alpha}$ with $\alpha > 1$ readily implies (3.4).

3.3 Final notes

Theorem 3.1.1 shows that if parameters of particle activities are periodic and biased in the Widom-Rowlinson model, the ferromagnetic influence of the boundary particles on like particles inside the volume vanishes when volume infinitely grows: in spite of strong long range attraction potential between similar particles, the phase in sufficiently large volume is almost independent on the configuration outside the volume.

We think that the Theorem 3.1.1 is held at all values of the temperature. Since the main method [34] used in this paper stands on low temperature estimations of configurations differing on ground states, we are stick to low temperature region.

Chapter 4

A financial application of the Ising model

4.1 Introduction

Traditional economic theory is based on a representative agent model in which an individual behaves in order to maximize his utility and this behavior produces the main characteristics of aggregate variables of the market. This approach has been severely criticized as it ignores two important facts of real markets: interaction between market participants and differences in their behavioral beliefs and intentions (heterogeneity).

In theory of complex systems with interacting units, it is well known that aggregate behavior of a system usually arise from the interactions among its units, not from complexity of extraneous factors or that of the units themselves and this interaction can bring out aggregate behavior which is very different from individual one. Many financial economists have realized the importance of this concept therefore modeling economic and financial systems with interacting agents has become very popular lately. In constructing these interactive models, different kinds of popular models in fundamental sciences such as Ising models, coupled map lattice models, sandpile models, noise trader models and etc. have been widely used [42, 43, 44, 45, 46, 47, 48, 49, 50, 51].

In this chapter, we use theory of the Ising model to construct an analogy between market participants and interacting particles, and show that how such an interaction effect can create severe outcomes. A reason why Ising model is of interest is that, it is a system made up of many subunits. The subunits in an Ising model are the interacting spins, and the subunits in the economy are market participants-buyers and sellers. During any time interval, these subunits of the economy may be either positive or negative as regards perceived market opportunities. People interact with each other and this fact produces what economists call the herd effect. The orientation of whether they buy or sell is influenced not only by neighbours but also by news usually realized by a global external field. If we hear bad news, we may be tempted to sell. In addition, we may naturally assume that the habit of individual's own also plays an important role in such a decision so the decision of any agent may be written as a function of the other agents' decisions and parameters for public news (externality) and idiosyncrasy.

By following the approach mentioned above, to understand the role of herd behavior on stock market crashes, an Ising model inspired by the work of Dahmen and Sethna [52] will be considered.

4.2 Modelling

Consider a network of agents buying and selling a single financial asset. Each agent is indexed by an integer $i = 1, \ldots, N$ and N is assumed to be very large. The agents who are directly connected to agent i are called neighbors of i. We assume that agent i can have two decisions: $\varphi_i \in \{+1, -1\}$ where +1 and -1 refers to buy and sell respectively. The decision of agent i is determined by;

$$\varphi_i = \mathrm{sign}(\sum_j I_{ij}\varphi_j + P + \varepsilon_i) \tag{4.1}$$

where $I_{ij} = I/d$ (I is a positive constant and d is the number of neighbors of agent i), denotes the tendency towards imitation. It shows how strongly agent j

has influence on agent and the sum runs over neighbors j of i (i.e. only neighbors have direct effect on an agent's decisions). Such a construction is analogous to random field Ising model.

$P \in (-\infty, +\infty)$ is the term for public news which has a global influence on all agents. For example if $P > 0$ (denoting good news), agents will tend to buy.

ε_i is the personal judgment of agent i. We may assume a normal distribution (due to the central limit theorem) for ε_i with mean zero and variance σ^2,

$$\rho(\varepsilon_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}\frac{\varepsilon_i^2}{\sigma^2})$$

σ can be considered as a heterogeneity parameter. For larger σ, agents' decisions become more irregular and unpredictable. By this modelling, 4.1 becomes analogous to random field Ising model.

Note that 4.1 describes the decision of an agent at a given point in time. In the next instant, new ε_i's are drawn, neighborhood interaction creates new influences on agents, and agents may change their decisions. Based on these decisions, agents' average opinion about the market at a specific time t is formed in the following way;

$$A(t) = \frac{\sum_i \varphi_i(t)}{N}$$

One approach to solve this model is to replace the interaction of an agent with his/her neighbors by an interaction with the average opinion in the market. This is mean-field approximation where every agent is influenced by remaining others with equal strength (in real life, there is no natural topology for the interaction between agents, markets are assumed to have large number of agents and each agent has possibly many neighbors so a mean field approach seems appropriate). In this case, new neighborhood influence is of size $I_{ij} = I/N$ (every agent is a neighbor of i now) then new decision criteria is the following;

$$\varphi_i = \mathrm{sign}(IA + P + \varepsilon_i) \tag{4.2}$$

In our scenario, we start with a situation where P is very large i.e. public news is all good about the market hence everyone is optimistic and all agents have a *priori* decision to buy. Now suppose things are getting worse and P is decreasing progressively.

Every agent i changes his/her decision when $IA + P + \varepsilon_i$ changes sign. At any P level, all agents with $IA + P + \varepsilon_i > 0$ will still give a decision to buy and agents with $IA + P + \varepsilon_i < 0$ will give a decision to sell.

The probability that the personal judgment of agent i lies between ε_i and $\varepsilon_i + d\varepsilon_i$ is $\rho(\varepsilon_i)d\varepsilon_i$. As mentioned, at any P level, all agents i with $IA+P+\varepsilon_i > 0$ will still give a decision to buy; the probability of this is,

$$\int_{-IA-P}^{\infty} \rho(\varepsilon_i)d\varepsilon_i$$

and similarly, agents with $IA + P + \varepsilon_i < 0$ will give a decision to sell, which has the probability

$$\int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i$$

then the average opinion is,

$$\begin{aligned} A &= -1\{\int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i\} + 1\{\int_{-IA-P}^{\infty} \rho(\varepsilon_i)d\varepsilon_i\} \\ &= 1 - 2\int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i = 1 - 2F(-IA - P) \end{aligned} \tag{4.3}$$

which gives the self consistency relation $A = 1 - 2F(-IA - P)$ where F is the cumulative distribution of ρ. Observe that if there was not interaction among agents, we would have $A = 1-2F(-P)$ hence average opinion would be decreasing smoothly for any P.

If neighborhood influence is weak enough (i.e. for small enough I), expanding

R.H.S. of (4.3) in powers of I leads to

$$\begin{aligned} A = 1 - 2F(-IA - P) &= 1 - 2(F(-P) + (-IA)F'(-P) + \ldots) \\ &\approx 1 - 2(F(-P) - IA\rho(-P)) = 1 - 2F(-P) + 2IA\rho(P) \\ &\Rightarrow A(1 - 2I\rho(P)) \approx 1 - 2F(-P) \\ &\Rightarrow A \approx \frac{1 - 2F(-P)}{1 - 2I\rho(P)} \end{aligned} \tag{4.4}$$

so, in opposition to no interaction case, as $\rho(P)$ gets close to its maximum value, neighborhood influence I leads to an over-reaction, and as it gets stronger and exceeds a critical value I_c; slope of average opinion diverges when P reaches a critical level $P_c(I)$, i.e. if neighborhood influence is strong enough (I is sufficiently large), as P is decreased progressively, average opinion (which is positive at that moment) will first decrease smoothly then at a point, it will jump down to a negative value (happens when P reaches its critical level $P_c(I)$ for given I), then keep on decreasing.

Thus, in the presence of strong enough neighborhood influence, as public news gets worse, considerable amount of agents suddenly change their decisions from buy to sell whereas this transition would be smooth if neighborhood influence was weak or did not exist at all.

4.2.1 The critical interaction level

Determination of the critical interaction level is as the following. Note that since

$$A(P) = 1 - 2 \int_{-\infty}^{-IA-P} \rho(\varepsilon_i) d\varepsilon_i \tag{4.5}$$

and ρ is symmetric distribution for Eq. (4.5), $A(0) = 0$ is the trivial solution at $P = 0$. If $A(0) = 0$ is the only solution at $P = 0$ then there is no jump in the average opinion. To have non-trivial solutions for $A(0)$, the slope of the R.H.S. of Eq. (4.5) (as a function of $A(0)$) must be larger than 1 at $A(0) = 0$.

At $P = 0$ and near $A(0) = 0$, we can approximate R.H.S. of Eq. (4.5) in the following way;

$$\begin{aligned} 1-2F(-IA) &\approx 1-2[F(0)+(-IA(0))F'(0)+\ldots] \\ &= 1-2[1/2+(-IA(0))\rho(0)+\ldots] \\ &\approx 2I\rho(0)A(0) \end{aligned}$$

then Eq. (4.5) has multivalued solution if we have $2I\rho(0) \geq 1$. Then combining with the fact that $\rho(0) = \frac{1}{\sigma\sqrt{2\pi}}$, we have the critical interaction level

$$I_c = \sqrt{\frac{\pi}{2}}\sigma$$

which is not surprising since as agents heterogeneity gets stronger, in order to behave as a collective group having the same decision, they should have stronger influence on each other.

4.3 A criterion for detecting herding behavior

When neighborhood influence is close to its critical value, there is a universal scaling law taking the form;

$$\frac{dA}{dP} = \frac{1}{I_c - I} \quad G\left(\frac{P-P_c}{(I_c-I)^{3/2}}\right) \tag{4.6}$$

where $G(y=0)$ is constant and $G(y \to \infty \sim y^{-2/3})$, implying that if we approximate G with a Gaussian function of the form

$$f(x) = c + h\exp\left(-\frac{1}{2}\frac{(x-\gamma)^2}{w^2}\right)$$

then when neighborhood influence is strong enough, the peak of the slope of the average opinion has an height h and width w related by the scaling law of the form $h \sim w^{-2/3}$ whereas in case of no interaction among agents, this would be $h \sim w^{-1}$.

It is clear to see the latter since in case of no interaction, we would have

$$A = 1-2F(-P) \Rightarrow \frac{dA}{dP} = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\exp\left(-\frac{1}{2}\frac{P^2}{\sigma^2}\right)$$

which relates $h \sim \sigma^{-1}$ and $w \sim \sigma$ hence $h \sim w^{-1}$.

Dahmen and Sethna [52] derive the first relation as the following. Observe that

$$A = 1 - 2 \int_{-\infty}^{-IA-P} \rho(\varepsilon_i) d\varepsilon_i = 1 - 2F(-IA-P)$$

$$\begin{aligned} \frac{dA}{dP} = -2[\rho(-IA-P)\frac{d}{dP}(-IA-P)] &= 2\rho(-IA-P)(I\frac{dA}{dP}+1) \\ \Rightarrow \frac{dA}{dP} &= 2\rho(-IA-P) + \frac{dA}{dP}[2I\rho(-IA-P)] \\ \Rightarrow \frac{dA}{dP} = \frac{2\rho(x)}{1-2I\rho(x)} &\qquad where \quad x = -IA-P \end{aligned} \tag{4.7}$$

hence slope of the average opinion diverges if $1 - 2I\rho(x_c) = 0$ which also defines $x_c \equiv -IA(P_c) - P_c$.

Expanding around such a critical point gives

$$\begin{aligned} 1 - 2I\rho(x) &= 1 - 2I[\rho(x_c) + \rho'(x_c)(x-x_c) + \frac{1}{2}\rho''(x_c)(x-x_c)^2 + \ldots] \\ &= -2I[\rho'(x_c)(x-x_c) + \frac{1}{2}\rho''(x_c)(x-x_c)^2 + \ldots] \end{aligned} \tag{4.8}$$

then combining with the last line of (4.7), we have

$$\frac{dA}{dP} = \frac{\rho(x_c)}{-I[\rho'(x_c)(x-x_c) + \frac{1}{2}\rho''(x_c)(x-x_c)^2 + \ldots]} \tag{4.9}$$

ρ is normal distribution, hence it is analytic with one maximum and $\rho'' \neq 0$ so Eq. (4.9) gives two cases to consider: $\rho'(x_c) = 0$ and $\rho'(x_c) \neq 0$.

For $\rho'(x_c) = 0$, it follows that $x_c = 0$ hence $\rho(x_c = 0) = \frac{1}{\sigma\sqrt{2\pi}}$ and also by (4.8) we have $\rho(x_c) = \frac{1}{2I}$ implying $I = \sqrt{\frac{\pi}{2}}\sigma$ which is indeed equal to I_c. So, $\rho'(x_c) = 0$ occurs when "I is close to its critical value" (the case $\rho'(x_c) \neq 0$ is found for $I > I_c$) and in this situation integrating Eq.(4.9) leads to the following leading order scaling behavior

$$A(P) - A(P_c(I_c)) \sim (I_c - I)^{\beta} \Lambda(\frac{P-P_c}{(I_c-I)^{\beta\delta}}) \tag{4.10}$$

for small $(P - P_c)$ and $I_c - I$. Λ is the universal scaling function and the mean field exponents are $\beta = 1/2$ and $\beta\delta = 3/2$.

To get rid of $A(P_c(I_c))$, we take derivative of the average opinion with respect to P;

$$\begin{aligned}\frac{dA}{dP} &\sim (I_c - I)^{\beta-\beta\delta}\dot{\Lambda}(\frac{(P-P_c)}{(I_c-I)^{\beta\delta}}) \\ &= \frac{1}{(I_c-I)}\dot{\Lambda}(\frac{(P-P_c)}{(I_c-I)^{3/2}})\end{aligned}$$

where $\dot{\Lambda}$ is the derivative of Λ with respect to its argument $\frac{P-P_c}{(I_c-I)^{\beta\delta}}$.

In this case, approximating G with a Gaussian form relates $h \sim (I_c - I)^{-1}$ and $w \sim (I_c - I)^{3/2}$ hence the term h behaves like $h \sim w^{-2/3}$.

In the model studied in this work, if necessary time is given, the decreasing P (which is an exogenous factor) will enforce the system to be in an average state close to -1 whether neighborhood interaction exists or not. But in the given scenario, strong enough interaction among the agents (which is an endogenous factor) will serve as an accelerator in drops of the average opinion which also creates the affect of faster decay in the average opinion' slope just before this slope reaches its negative peak level. So approximating G with a Gaussian form allows us to compare the heights and widths of the bells in both cases.

4.4 Discussion and conclusion

An Ising model of interacting agents in a market, buying and selling a single financial asset is studied in this work. In this model, agents give their decisions to buy or sell according to a combination of neighborhood influence, public news and personal judgments. Based on these decisions, agents' average opinion about the market is formed. It is found that for small neighborhood influence or strong diversities in agents' judgments, average opinion decreases continuously as public news get worse. In the same scenario, if neighborhood influence is strong enough, a jump occurs in average opinion around a critical time when a group of agents suddenly change their decisions from buy to sell which causes a crash. Around this time, the peak of the slope of the average opinion, has an height h and width

w related by $h \sim w^{-2/3}$ whereas in the case of no interaction among agents, this relation is $h \sim w^{-1}$.

If we want to get an answer in an analytical way then mean field approximation is the way to go. Basically, replacing the decision criteria (4.1) by (4.2) takes into account the heterogeneity of agents and the influence of φ_j on φ_i but not the fact that this φ_i again influences φ_j creating a feedback so mean field approximation is more accurate if agents have large number of neighbors. Although this model has strong assumptions, it may at least help us to understand the importance of interaction and heterogeneity of market participants in stock market crashes. For future work, a few improvements in the model is considered: Interaction among agents can be time dependent; evolving subject to some criteria, and agents can have biased personal judgments characterizing them as *optimists* and *pessimists*.

Chapter 5

Conclusion

In this thesis, initially we focused on the theory of phase transitions in one dimensional models. In particular;

In Chapter 2, we present a criterion for the uniqueness of limit Gibbs states in classical models with unique ground state. Various applications of this criterion presented in the terminology of percolation theory are discussed.

In Chapter 3 (which is published as [53]), we consider a special model under additional external field; one dimensional long range Widom-Rowlinson model when particle activity parameters are periodic and biased. We show that if the interaction is sufficiently large versus particle activities then the model does not exhibit a phase transition at low temperatures.

In Chapter 4, different from the previous chapters, we followed an interdisciplinary approach. We considered a financial application of phase transition in a general Ising model to understand the role of herd behavior on stock market crashes. In particular, a model of interacting agents in a market, buying and selling a single financial asset is studied where agents give their decisions to buy or sell according to a combination of neighborhood influence, public news and personal judgments. Assuming public news gets worse progressively, the evolution of the agents' average opinion (based on their decisions) is investigated in the presence of weak and strong neighborhood influence. Accordingly, we suggest a criteria for detecting the existence of herd behavior under such an assumption.

I finally want to mention that the applications of phase transitions and the usage of tools from statistical mechanics on interdisciplinary (especially financial) concepts are limitless. Although they are not included in this thesis, the examples include the published manuscripts [54, 55, 56, 57] which are all written by the author of this thesis. In particular, in [54], the time-varying efficiency of the Federation of Euro-Asian Stock Exchanges is studied by generalized Hurst exponent with a rolling window technique. The results reveal that all FEAS members exhibit different degrees of long range dependence varying over time. For the federation members, strong positive relationship between efficiency and market liquidity is revealed. In the light of this fact, alternatives are suggested to improve market efficiency. In [55], the long range dependence in Middle East and North African stock markets' returns is investigated. Accordingly, these markets exhibit different degrees of long-range dependence. The least inefficient market is found to be Turkey. Moreover, Turkey and Israel show characteristics of developed financial markets. In [56], Random Matrix Theory is used to analyze the cross-correlations between worldwide stock markets. The majority of the cross-correlation coefficients are found to arise from randomness. Furthermore, the connection structure of markets before and after the crisis are displayed using network theory, and key financial markets are revealed. And last, in [57], the presence of long memory in a variety of interest rates in Turkey is studied by time varying generalized Hurst exponent. Analysis shows that adopting inflation targeting cause a sudden and considerable decrease in the long memory in interest rates. Moreover, degree of long memory is found to increase with interest rate maturity which is in contrast to economic theory.

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