Categories of Graphs and Operations on Graphs

by

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Abstract

This thesis is an exploratory study on relationships between graph theory, category theory and algebras. We investigate if there is a viable definition of a morphism for (finite) graphs that is compatible with the fundamental functor which sends a graph to the base-pointed monoid of all paths on that graph. We construct three different graph categories and study functors from these graph categories to base-pointed monoid categories and algebra categories. As a result, we explain how graph operations correspond to morphisms in defined graph categories.

ÖZET

Bu tezde, kategori teorisi,çizge teorisi ve cebirler arasındaki ilişkiler araştırılmaktadır. Sonlu çizgeler için, bir çizgeyi o çizgenin bütün yollarının oluşturduğu baz noktalı monoide taşıyan izleç ile uyumlu uygulanabilir bir dönüşüm olup olmadığı araştırılmıştır. Bunun için üç farklı çizge kategorisi kurulmuş ve bu kategoriler ile baz noktalı monoid kategorileri ve cebir kategorileri arasındaki izleçler çalışılmıştır. Sonuç olarak, çizge operasyonlarının tanımlanmış çizge kategorilerinde hangi dönüşümlere karşılık geldiği açıklanmıştır.

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Chapter 1

Introduction

This thesis is mostly an exploratory study on relationships between graph theory, category theory and algebra. The central problem we investigate is as follows:

> Is there a viable definition of a morphism for (finite) graphs that
>
> 1. encompass the combinatorial operations on graphs such as vertex deletions or insertions, edge deletions or insertions, or more interestingly, edge contractions, and
> 2. compatible with the fundamental functor which sends a graph to the base-pointed monoid of all paths on that graph?

Among other things, we construct and investigate the following diagram of categories and functors

$$\begin{tikzcd} Graphs \arrow[r, "Path"] \arrow[dr] & Mon_* \arrow[r, "\text{Span}"] \arrow[d, "\text{Inc}"] & Alg_k \arrow[d, "\text{Inc}"] \\ & \text{Set}_* \arrow[r, "\text{Span}"] & \text{Vect}_k \end{tikzcd}$$

and then play with different notions of a morphism on different categories of graphs to see if all, or any, of the structures depicted in the diagram above are preserved.

Surprisingly, contrary to the minimality in their structure, graphs are very rigid objects in admitting morphisms. The obvious notion of a morphism, namely a map on the set of vertices that also preserve connectivity information, is not sufficiently rich. After this observation, we took our cue from algebraic geometry: (affine) algebraic varieties, or schemes in general, are also very rigid objects in admitting a viable notion of a morphism. One successful strategy in overcoming this rigidity was to extend the notion of a morphism and use *algebraic correspondences*. From this point of view, analogous to cobordism, a morphism between two algebraic variety X and Y is a suitable subvariety of the product $X \times Y$. In choosing these morphisms, one uses the full power of the algebraic structure on the space of functions on varieties as well as any of the several topologies on the product variety readily available on the category of algebraic varieties.

Unfortunately, many of the algebraic and topological tools available in studying algebraic varieties do not exist for graphs. So, we turned to very basic set theory: there are three basic notions of (composable) correspondences between two sets X and Y:

1. functions from X to Y
2. partial functions from X to Y
3. relations between X and Y

We considered different notions of morphisms on the category of graphs using each of these correspondences we listed above. Each of these choices do give drastically different categories of graphs. We observe this by studying which of the basic combinatorial operations on graphs (insertions, deletions and contractions) are best expressed within these different categories of graphs.

1.1 Plan of the thesis

In Chapter 2, we give basic information and examples of categories and subcategories. We introduce categories of graphs, category of base-pointed sets , category of base-pointed monoids and category of algebras. There are three categories of graphs that are considered in this thesis; Gr_{rel} edge-preserving relations as morphisms, Gr_{par} edge-preserving partial functions as morphisms and Gr_{fun} edge-preserving functions as morphisms.
In Chapter 3, we study functors from graph categories to category of base-pointed monoids and from category base-pointed monoids to category of algebras.
In Chapter 4, we give definitions of graph operations, vertex deletions and insertions, edge deletions and insertions, edge contraction. Then we decide which of these operations gives morphisms in graph categories Gr_{rel}, Gr_{par} and Gr_{fun}.

Chapter 2

Categories

2.1 Preliminaries

Definition 2.1.1. *A **category** $\mathscr{C}$ is the following data:*

(i) a class of objects Obj($\mathscr{C}$),

(ii) a set of morphisms $Hom_{\mathscr{C}}(X, Y)$ for all X, Y in Obj($\mathscr{C}$),

(iii) a composition between morphisms for all X, Y, Z in Obj($\mathscr{C}$)

$$\circ : Hom_{\mathscr{C}}(Y, Z) \times Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{C}}(X, Z),$$

(iv) for all $f \in Hom_{\mathscr{C}}(X, Y)$, $g \in Hom_{\mathscr{C}}(Y, Z)$ and $h \in Hom_{\mathscr{C}}(Z, T)$, $h \circ (g \circ f) = (h \circ g) \circ f$ (i.e., composition is associative),

(v) for all X in Obj($\mathscr{C}$) there is an identity morphism $\mathbb{1}_X$ such that for all Y in Obj($\mathscr{C}$), for all $f \in Hom_{\mathscr{C}}(X, Y)$, $f \circ \mathbb{1}_X = f$ and for all $g \in Hom_{\mathscr{C}}(Y, X)$, $\mathbb{1}_X \circ g = g$.

Example 2.1.2. *Category of sets **Set**: objects are sets and morphisms are set functions. Composition is the function composition, which we know to be associative.*

Remark 2.1.3. *The above definition of a category considers the collection of objects as classes not sets. Since it is complicated to work on classes, we assume that there is a set U, the universe. If a set X is a member of the universe, we will call it small. Then consider **Set** to be the category of all small sets. Throughout this thesis all the categories defined are subcategories of this **Set**, so we are only studying with small categories.*

Example 2.1.4. *Category of groups **Gr**: objects are groups, morphisms are group homomorphisms.*

Example 2.1.5. *Category of vector spaces over a field k, **Vect**$_k$: objects are vector spaces over k, morphisms are k-linear maps.*

Example 2.1.6. *Category of rings: objects are rings and morphisms are morphisms of rings.*

Example 2.1.7. *Category of modules over a ring R: objects are left (or right) R-modules, morphisms are morphisms of left (respectively right) R-modules.*

Example 2.1.8. *Category of sets with based points* $\mathbf{Set}_*$*: objects are sets with base points* $(X, *)$ *and morphisms are set functions that map one base point to another, e.g.* $f : (X, *_X) \to (Y, *_Y)$ *such that* $f(*_X) = (*_Y)$.

Example 2.1.9. *Let* $G = (V, E)$ *be a directed graph. Let* $s : E \to V$ *and* $t : E \to V$ *be source and target functions, respectively. Let* $E^{(0)} = V$ *and* $E^{(1)} = E$*. Put*

$$E^{(2)} = \{(x, y) \in E \times E : s(x) = t(y)\},$$

$$E^{(3)} = \{(x, y, z) \in E \times E \times E : s(x) = t(y) \text{ and } s(y) = t(z)\}.$$

Note that $E^{(0)}$ *is the set of vertices, and we will consider them as paths of length zero. Also, note that* $E^{(1)}$ *is the set of all paths of length* 1 *in G,* $E^{(2)}$ *is the set of all paths of length* 2 *in G and so on. Define* $E^{(n)}$ *similarly and consider* $E^{(\infty)} = \bigsqcup_{n\geq 0} E^{(n)}$*. We can now view this G as a category* $\mathscr{C}(G)$.

Objects of $\mathscr{C}(G)$ *are the vertices of G (Obj(*$\mathscr{C}(G)$*)=V). For any* $v_1, v_2 \in V$,

$$Hom_{\mathscr{C}(G)}(v_1, v_2) = \{\alpha \in E^{(\infty)} : s(\alpha) = v_1, t(\alpha) = v_2\},$$

i.e., all paths from v_1 *to* v_2*. We have a composition law, which is just the concatenation of paths and it is clearly associative. For all* $v \in Obj(\mathscr{C}(G))$*, there is an identity morphism* e_v *in* $Hom_{\mathscr{C}(G)}(v, v)$ *which only consists of vertex v.*

Definition 2.1.10. *Let* $\mathscr{C}$ *be a category. A* ***subcategory*** $\mathscr{S}$ *of* $\mathscr{C}$ *is given by a subcollection of objects in* $\mathscr{C}$*, denoted* $Obj(\mathscr{S})$*, and a subset of morphisms of* $\mathscr{C}$*, denoted* $Hom(\mathscr{S})$ *where*

- *for any object X in* $\mathscr{S}$*, the identity morphism* $\mathbb{1}_X$ *is in* $Hom_{\mathscr{S}}(X, X)$,
- *for every morphism* $f : X \to Y$ *in* $Hom(\mathscr{S})$*,* X, Y *belong to* $Obj(\mathscr{S})$,
- *for every* f, g *in morphisms, the composition* $g \circ f$ *is a morphism whenever it is meaningful.*

Example 2.1.11. *The category* Set_f *of all finite sets is a subcategory of the category Set.*

Example 2.1.12. *The category Ab of abelian groups is a subcategory of the category Gr of groups.*

Example 2.1.13. *The category* Alg_k *of algebras over k is a subcategory of* $Vect_k$.

2.2 Graph Categories

2.2.1 Category of graphs with morphisms as relations : Gr_{rel}

(i) Objects : Directed graphs $G = (V, E)$,

(ii) Morphisms : Relations between vertex sets preserving edges;
Let $G = (V, E)$ and $H = (W, F)$ be two graphs. Then a morphism in $Hom_{Gr_{rel}}(G, H)$ is a relation $R \subseteq V \times W$ such that

$$\forall (x, z), (y, t) \in R, (x, y) \in E \Rightarrow (z, t) \in F.$$

(iii) Composition between morphisms ;
Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$ be directed graphs. Let $R \in Hom_{Gr_{rel}}(G_1, G_2)$ and $S \in Hom_{Gr_{rel}}(G_2, G_3)$. Define

$$\begin{aligned} S \circ R = \{(a, b) : a \in V_1, b \in V_3 \text{ and } \exists c \in V_2 \\ \text{such that } (a, c) \in R \text{ and } (c, b) \in S\}. \end{aligned}$$

We need to check that this composition gives a morphism, i.e. a relation preserving edges. Clearly, $S \circ R \subseteq V_1 \times V_3$. Let $(x, z), (y, t)$ be in $S \circ R$. Let's show that $(x, y) \in E_1 \Rightarrow (z, t) \in E_3$. Since

$$(x, z) \in S \circ R, \text{ there exists } c_1 \in V_2 \text{ such that } (x, c_1) \in R, (c_1, z) \in S,$$

$$(y, t) \in S \circ R, \text{ there exists } c_2 \in V_2 \text{ such that } (y, c_2) \in R, (c_2, t) \in S.$$

If $(x, y) \notin E_1$, there is nothing to check. If $(x, y) \in E_1$, $(x, c_1), (y, c_2) \in R$ implies that $(c_1, c_2) \in E_2$ because R is a morphism. We also have $(c_1, z), (c_2, t) \in S$, so we have $(z, t) \in E_3$ as $(c_1, c_2) \in E_2$ and S is a morphism.

(iv) Composition is associative ;
Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), G_3 = (V_3, E_3)$ and $G_4 = (V_4, E_4)$ be directed graphs. Let $R \in Hom_{Gr_{rel}}(G_1, G_2)$, $S \in Hom_{Gr_{rel}}(G_2, G_3)$ and $T \in Hom_{Gr_{rel}}(G_3, G_4)$. We will prove

that $T \circ (S \circ R) = (T \circ S) \circ R$.

$$\begin{aligned} T \circ (S \circ R) = \{(a,b) : a \in V_1, b \in V_4, \exists c \in V_3 \text{ such that} \\ (a,c) \in S \circ R \ \& \ (c,b) \in T\} \\ = \{(a,b) : a \in V_1, b \in V_4, \exists c \in V_3\ \exists c' \in V_2 \text{ such that} \\ (a,c') \in R, (c',c) \in S \ \& \ (c,b) \in T\} \end{aligned}$$

$$\begin{aligned} (T \circ S) \circ R = \{(a,b) : a \in V_1, b \in V_4, \exists c \in V_2 \text{ such that} \\ (a,c) \in R \ \& \ (c,b) \in T \circ S\} \\ = \{(a,b) : a \in V_1, b \in V_4, \exists c \in V_2\ \exists c' \in V_3 \text{ such that} \\ (a,c) \in R, (c,c') \in S \ \& \ (c',b) \in T\} \end{aligned}$$

Hence, $T \circ (S \circ R) = (T \circ S) \circ R$.

(v) For all $G = (V,E) \in Obj(Gr_{rel})$, there is an identity morphism $\mathbb{1}_G = \{(v,v) : v \in V\}$ in $Hom_{Gr_{rel}}(G,G)$ such that for all $H = (W,F) \in Obj(Gr_{rel})$, for all $R \in Hom_{Gr_{rel}}(G,H)$ and for all $S \in Hom_{Gr_{rel}}(H,G)$, $R \circ \mathbb{1}_G = R$ and $\mathbb{1}_G \circ S = S$.

$$\begin{aligned} R \circ \mathbb{1}_G &= \{(a,b) : a \in V, b \in W, \exists x \in V \text{ such that } (a,x) \in \mathbb{1}_G \ \& \ (x,b) \in R\} \\ &= \{(a,b) : a \in V, b \in W, (a,b) \in R\} = R. \end{aligned}$$

$$\begin{aligned} \mathbb{1}_G \circ S &= \{(a,b) : a \in W, b \in V, \exists x \in V \text{ such that } (a,x) \in S \ \& \ (x,b) \in \mathbb{1}_G\} \\ &= \{(a,b) : a \in W, b \in V, (a,b) \in S\} = S. \end{aligned}$$

So Gr_{rel} is a category.

2.2.2 Category of graphs with morphisms as partial functions : Gr_{par}

(i) Objects : Directed graphs $G = (V,E)$,

(ii) Morphisms : Partial functions between vertex sets preserving edges;
Let $G = (V,E)$ and $H = (W,F)$ be two graphs. Then a morphism in $Hom_{Gr_{par}}(G,H)$ is a partial function $f : V \to W$ such that

$$\forall x, y \in V, (x,y) \in E \Rightarrow (f(x), f(y)) \in F,$$

whenever both $f(x)$ and $f(y)$ are defined.

In fact, a morphism $f \in Hom_{Gr_{par}}(G, H)$ is a relation $f \subseteq V \times W$ satisfying the following properties;

$$\forall a, b \in W \ \forall c \in V, (c, a) \in f \ \& \ (c, b) \in f \Rightarrow a = b,$$

$$\forall (x, z), (y, t) \in f, (x, y) \in E \Rightarrow (z, t) \in F.$$

(iii) Composition between morphisms ;
Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$ be directed graphs. Let $f \in Hom_{Gr_{par}}(G_1, G_2)$ and $g \in Hom_{Gr_{par}}(G_2, G_3)$. Define

$$g \circ f = \{(a, b) : a \in V_1, b \in V_3 \text{ and } \exists c \in V_2 \text{ such that } (a, c) \in f \& (c, b) \in g\}.$$

We need to check that this composition gives a morphism, i.e. a partial function preserving edges. Clearly, $g \circ f \subseteq V_1 \times V_3$. Assume that $\forall a, b \in V_3, \forall c \in V_1$, $(c, a) \in g \circ f$ and $(c, b) \in g \circ f$. We need to prove that $a = b$. Since $(c, a) \in g \circ f$, there exists $d \in V_2$ such that $(c, d) \in f$ and $(d, a) \in g$. Also, $(c, b) \in g \circ f$ implies that there is a $d' \in V_2$ such that $(c, d') \in f$ and $(d', b) \in g$. Then we have $d = d'$ as $f \in Hom_{Gr_{par}}(G_1, G_2)$, $(c, d) \in f$ and $(c, d') \in f$. Similarly, since g is a morphism between G_2 and G_3, $(d', a) \in g$ and $(d', b) \in g$, we have $a = b$. So the first condition holds.
For the last one, we need to check that $\forall (x, z), (y, t) \in g \circ f$, $(x, y) \in E_1$ implies $(z, t) \in E_3$. As $(x, z), (y, t) \in g \circ f$,

$$\exists w \in V_2 \text{ such that } (x, w) \in f \text{ and } (w, z) \in g,$$

$$\exists w' \in V_2 \text{ such that } (y, w') \in f \text{ and } (w', t) \in g.$$

Since $(x, y) \in E_1$ and $(x, w), (y, w') \in f$, $(w, w') \in E_2$. So that $(z, t) \in E_3$ as $(w, w') \in E_2$ and $(w, z), (w', t) \in g$.

(iv) Composition is associative ;
Since any partial function is a relation and composition is associative on relations, composition is associative on partial functions preserving edges.

(v) For all $G = (V, E) \in Obj(Gr_{par})$, there is an identity morphism $\mathbb{1}_G = \{(v, v) : v \in V\}$ in $Hom_{Gr_{par}}(G, G)$ such that for all $H = (W, F) \in Obj(Gr_{par})$, for all $f \in Hom_{Gr_{par}}(G, H)$ and for all $g \in Hom_{Gr_{par}}(H, G)$, $f \circ \mathbb{1}_G = f$ and $\mathbb{1}_G \circ g = g$.

$$\begin{aligned} f \circ \mathbb{1}_G &= \{(a, b) : a \in V, b \in W, \exists x \in V \text{ such that } (a, x) \in \mathbb{1}_G \ \& \ (x, b) \in f\} \\ &= \{(a, b) : a \in V, b \in W, (a, b) \in f\} = f. \end{aligned}$$

$$\begin{aligned}\mathbb{1}_G \circ g &= \{(a,b) : a \in W, b \in V, \exists x \in V \text{ such that } (a,x) \in g \ \& \ (x,b) \in \mathbb{1}_G\} \\ &= \{(a,b) : a \in W, b \in V, (a,b) \in g\} = g.\end{aligned}$$

2.2.3 Category of graphs with morphisms as functions : Gr_{fun}

(i) Objects : Directed graphs $G = (V, E)$,

(ii) Morphisms : Functions between vertex sets preserving edges;
Let $G = (V, E)$ and $H = (W, F)$ be two graphs. Then a morphism in $Hom_{Gr_{fun}}(G, H)$ is a function $f : V \to W$ such that

$$\forall x, y \in V, (x, y) \in E \Rightarrow (f(x), f(y)) \in F.$$

(iii) Composition between morphisms ;
Composition of morphisms in Gr_{fun} is just the function composition, so it is obvious that composition of two morphisms will give a function but we need to check that it preserves edges. Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$ be graphs. Let $f \in Hom_{Gr_{fun}}(G_1, G_2)$ and $g \in Hom_{Gr_{fun}}(G_2, G_3)$. We'll show that $g \circ f$ respects edge relations. Let $x, y \in V_1$. Assume that $(x, y) \in E_1$, let us show that $((g \circ f)(x), (g \circ f)(y)) \in E_3$. Since $(x, y) \in E_1$ and f is a morphism between G_1, G_2, we must have $(f(x), f(y)) \in E_2$ and we have $(g(f(x)), g(f(y))) \in E_3$ as g is also a morphism. Hence composition preserves edges.

(iv) Composition is clearly associative;

(v) For all $G = (V, E) \in Obj(Gr_{fun})$, there is an identity morphism $\mathbb{1}_G$ defined as $v \mapsto v$ for all $v \in V$ in $Hom_{Gr_{fun}}(G, G)$ such that for all $H = (W, F) \in Obj(Gr_{fun})$, for all $f \in Hom_{Gr_{fun}}(G, H)$ and for all $g \in Hom_{Gr_{fun}}(H, G)$, $f \circ \mathbb{1}_G = f$ and $\mathbb{1}_G \circ g = g$.

Gr_{fun} is a category.

2.3 Category of sets with base points Set$_*$

Definition 2.3.1. ***A set with a base point (S,*)*** *is a set S with a distinguished element $*$.*

Definition 2.3.2. *Let A, B be sets with base points. Define $A \wedge B = A \times B / \backsim$ where*

$$(a, b) \backsim (c, d) \textit{ iff } (a = * \textit{ or } b = *) \textit{ and } (c = * \textit{ or } d = *).$$

Let $(*, *) = * \in A \wedge B$.

Objects are sets with base points $(S, *)$, morphisms are functions $f : (S, *_S) \to (P, *_P)$ satisfying $f(*_S) = *_P$. Composition is the function composition. Let $f : (S, *_S) \to (P, *_P)$ and $g : (P, *_P) \to (R, *_R)$ be morphims. Then

$$g \circ f(*_S) = g(f(*_S)) = g(*_P) = *_R,$$

so composition gives a based function. Since function composition is associative, we have the associativity. For all $(S, *)$ in $Obj(\text{Set}_*)$ there is an identity $\mathbb{1}_{(S,*)}$ defined as $\mathbb{1}_{(S,*)}(s) = s$ and $\mathbb{1}_{(S,*)}(*) = *$ for all $s \in S$ satisfying the following conditions; for all $(P, *)$ in $Obj(\text{Set}_*)$, for all $f \in Hom_{\text{Set}_*}((S, *), (P, *))$,
$f \circ \mathbb{1}_{(S,*)} = f$ and for all $g \in Hom_{\text{Set}_*}((P, *), (S, *))$, $\mathbb{1}_X \circ g = g$. Hence Set_* is a category.

2.4 Category of monoids with base point Mon_*

Definition 2.4.1. *A* ***monoid*** *is a set M together with an associative binary operation which has a unit.* ***A monoid with a base point (M,*)*** *is a monoid M with a distinguished element $*$ such that $x \cdot * = * \cdot x = *$ for every $x \in M$.*

Objects of Mon_* are monoids with base points $(M, *)$.
Morphisms are based functions which respects monoid operations;
Let $(M, *), (N, *)$ be monoids with base points. A morphism f between $(M, *)$ and $(N, *)$ is a function satisfying;

$$f(*_M) = *_N,$$

$$\forall m, m' \in M, f(m \cdot_M m') = f(m) \cdot_N f(m').$$

Composition is the function composition but we need to check that composition gives a function satisfying the above properties.
Notation: We will only write $Hom(\, , \,)$ instead of $Hom_{Mon_*}(\, , \,)$ when the context is clear.
Let $(M, *), (N, *), (P, *)$ be objects of Mon_*. Let f be in $Hom((M, *)(N, *))$, and g in $Hom((N, *), (P, *))$.
Clearly $g \circ f(*_M) = *_P$. Let m, m' be elements of M. Then we have

$$g \circ f(m \cdot_M m') = g(f(m \cdot_M m')) = g(f(m) \cdot_N f(m')) = g(f(m)) \cdot_P g(f(m')).$$

So composition gives us a morphism between $(M, *), (P, *)$ in Mon_*.
Composition is associative as function composition is associative.
For all $(M, *) \in Obj(Mon_*)$, there is an identity morphism $\mathbb{1}_M$ such that $\mathbb{1}_M(m) = m$ for all $m \in M$. For all $(N, *) \in Obj(Mon_*)$, for all $f \in Hom_{Mon_*}((M, *)(N, *))$, $g \in Hom_{Mon_*}((N, *), (M, *))$, we have $f \circ \mathbb{1}_M = f$ and $\mathbb{1}_M \circ g = g$.
Hence Mon_* is a category.

2.5 Algebras

Definition 2.5.1. *An **algebra** is a vector space A over a field k with an associative unital multiplication $m : A \otimes A \to A$ and a map $u : k \to A$ such that $u(\lambda) = \lambda 1_A$ which makes the following diagrams commute;*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

i.e. denoting $m(x \otimes y)$ by $x \cdot y$, we have
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (multiplication is associative),

$$\begin{tikzcd} A \arrow[r, "id\otimes u"] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

i.e. $a \cdot 1 = a$,

$$\begin{tikzcd} A \arrow[r, "u\otimes id"] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

i.e. $1 \cdot a = a$ for all $x, y, z, a \in A$.

Example 2.5.2. *k **as a** k**-algebra:** Define $m : k \otimes k \to k$ by $m(x \otimes y) = xy$ for all $x, y \in k$. Then all three diagrams commute by field axioms.*

Example 2.5.3. *$k[x]$ **as a** k**-algebra:** Let $\mathcal{B} = \{x^i : i = 0, 1, \ldots\}$, then*
$span(\mathcal{B}) = k[x]$ and $span(\mathcal{B} \times \mathcal{B}) = k[x] \otimes k[x]$. Define the multiplication on basis elements $m(x^i \otimes x^j) = x^{i+j}$ and extend it linearly to $span(\mathcal{B} \times \mathcal{B}) = k[x] \otimes k[x]$. Define $u : k \to k[x]$ by $u(k) = k \cdot 1$. Look at the first diagram;

$$\begin{tikzcd} k[x] \otimes k[x] \otimes k[x] \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"] & k[x] \otimes k[x] \arrow[d, "m"] \\ k[x] \otimes k[x] \arrow[r, "m"] & k[x] \end{tikzcd}$$

Since we have;

$$\begin{aligned} m \circ (m \otimes id)(x^i \otimes x^j \otimes x^k) &= m(m(x^i \otimes x^j) \otimes x^k) \\ &= m(x^{i+j} \otimes x^k) = x^{i+j+k}, \end{aligned}$$

$$\begin{aligned} m \circ (id \otimes m)(x^i \otimes x^j \otimes x^k) &= m(x^i \otimes m(x^j \otimes x^k)) \\ &= m(x^i \otimes x^{j+k}) = x^{i+j+k}; \end{aligned}$$

the diagram commutes. The second diagram;

$$\begin{tikzcd} k[x] \arrow[r, "id\otimes u"] \arrow[dr, "id"'] & k[x] \otimes k[x] \arrow[d, "m"] \\ & k[x] \end{tikzcd}$$

also commutes since $m \circ (id \otimes u)(x^i) = m(x^i \otimes 1) = x^i = id(x^i)$. *Similarly the third diagram commutes, hence* $k[x]$ *is an algebra over* k.

Example 2.5.4. $k[x_1, \ldots, x_n]$ ***as a*** k***-algebra:*** *Let* $A = k[x_1, \ldots, x_n]$ *and* $\mathcal{B} = \{x_1^{i_1} \cdots x_n^{i_n} : i_1, \ldots, i_n = 0, 1, \ldots\}$ *be a basis of* A. *Then* $A \otimes A$ *has basis* $\mathcal{B} \otimes \mathcal{B} = \{(x_1^{i_1} \cdots x_n^{i_n}, x_1^{j_1} \cdots x_n^{j_n}) : i_1, \ldots, i_n = 0, 1, \ldots, j_1, \ldots, j_n = 0, 1, \ldots\}$. *Define the multiplication by*

$$m(x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1} \cdots x_n^{j_n}) = x_1^{i_1+j_1} \cdots x_n^{i_n+j_n}.$$

Then

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

commutes since

$$\begin{aligned} m \circ (m \otimes id)(x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1} \cdots x_n^{j_n} \otimes x_1^{k_1} \cdots x_n^{k_n}) &= m(x_1^{i_1+j_1} \cdots x_n^{i_n+j_n} \otimes x_1^{k_1} \cdots x_n^{k_n}) \\ &= x_1^{i_1+j_1+k_1} \cdots x_n^{i_n+j_n+k_n}, \end{aligned}$$

$$\begin{aligned} m \circ (id \otimes m)(x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1} \cdots x_n^{j_n} \otimes x_1^{k_1} \cdots x_n^{k_n}) &= m(x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1+k_1} \cdots x_n^{j_n+k_n}) \\ &= x_1^{i_1+j_1+k_1} \cdots x_n^{i_n+j_n+k_n}. \end{aligned}$$

As we have

$$\begin{aligned} m \circ (id \otimes u)(x_1^{i_1} \cdots x_n^{i_n}) &= m(x_1^{i_1} \cdots x_n^{i_n} \otimes 1) \\ &= x_1^{i_1} \cdots x_n^{i_n} \\ &= id(x_1^{i_1} \cdots x_n^{i_n}), \end{aligned}$$

the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "id\otimes u"] \arrow[dr, "id"'] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

Similarly, since

$$\begin{aligned} m \circ (u \otimes id)(x_1^{i_1} \cdots x_n^{i_n}) &= m(1 \otimes x_1^{i_1} \cdots x_n^{i_n}) \\ &= x_1^{i_1} \cdots x_n^{i_n} \\ &= id(x_1^{i_1} \cdots x_n^{i_n}), \end{aligned}$$

we also have the commutativity of the third diagram.

$$\begin{tikzcd} A \arrow[r, "u\otimes id"] \arrow[dr, "id"'] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

Therefore, $k[x_1, \ldots, x_n]$ *is a k-algebra with the above defined multiplication.*

Example 2.5.5. *Let M be a monoid and k be a field. Then* $Span_k(M)$ *is called a monoid algebra. There is an associative multiplication on monoid elements (i.e. basis elements), extend this multiplication linearly on* $Span_k(M)$*. Then as monoid multiplication is associative the following diagram commutes;*

$$\begin{tikzcd} Span_k(M) \otimes Span_k(M) \otimes Span_k(M) \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"] & Span_k(M) \otimes Span_k(M) \arrow[d, "m"] \\ Span_k(M) \otimes Span_k(M) \arrow[r, "m"] & Span_k(M). \end{tikzcd}$$

$$\begin{tikzcd} Span_k(M) \arrow[r, "id\otimes u"] \arrow[dr, "id"'] & Span_k(M) \otimes Span_k(M) \arrow[d, "m"] \\ & Span_k(M) \end{tikzcd}$$

also commutes as for any element $\sum_{n\in M}\lambda_n n$ in $Span_k(M)$,

$$m\circ(id\otimes u)(\sum_{n\in M}\lambda_n n) = m(\sum_{n\in M}\lambda_n n\otimes 1) = \sum_{n\in M}\lambda_n n.$$

$$\begin{tikzcd} Span_k(M) \arrow[r, "u\otimes id"] \arrow[dr, "id"'] & Span_k(M)\otimes Span_k(M) \arrow[d, "m"] \\ & Span_k(M) \end{tikzcd}$$

commutes similarly. Hence $Span_k(M)$ is an algebra.

Example 2.5.6. *Let $(M,*)$ be a monoid with a based point. We will call $Span_k(M,*)$ a monoid with base point algebra. As the preceding example, we have an associative multiplication on $(M,*)$. Extending it linearly on $Span_k(M)$, we have the desired commutative diagrams. Hence $Span_k(M,*)$ is an algebra.*

Example 2.5.7. *Let $G=(V,E)$ be a directed graph and k be a field. Let $\mathscr{P}(G)$ be the set of all paths in G. Then $Span(\mathscr{P}(G))$ becomes an algebra with concatenation as multiplication.*

2.5.1 Category of algebras over a field k: Alg_k

(i) Objects : Algebras A over k,

(ii) Morphisms : $k-$linear maps respecting multiplication;
Let A and B be algebras over k. Then a morphism in $Hom_{Alg_k}(A,B)$ is a k-linear map $f:A\to B$ such that $f(xy)=f(x)f(y)$ for all x,y in A.

(iii) Composition is the function composition;
Let A,B,C be objects of Alg_k. Let $f:A\to B$ and $g:B\to C$ be morphisms. Then for any $\lambda\in k$, for any $x\in A$, we have

$$g\circ f(\lambda x)=g(\lambda f(x))=\lambda g(f(x)),$$

hence composition gives a $k-$linear map. Let x,y be in A. Then

$$g\circ f(xy)=g(f(xy))=g(f(x)f(y))=g(f(x))g(f(y)),$$

showing that composition respects multiplication.

(iv) Composition is associative since function composition is associative,

(v) For all $A\in Obj(Alg_k)$, there is an identity morphism $\mathbb{1}_A$ such that $\mathbb{1}_A(x)=x$ for all $x\in A$. For all $B\in Obj(Alg_k)$, for all $f\in Hom_{Alg_k}(A,B)$, $g\in Hom_{Mon_*}(B,A)$, we have $f\circ\mathbb{1}_A=f$ and $\mathbb{1}_A\circ g=g$.

Alg_k is a category.

Chapter 3

Functors

Definition 3.0.8. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A* ***functor*** *$F : \mathscr{C} \to \mathscr{D}$ is a morphism of categories, which gives ;*

(1) a correspondence between objects of $\mathscr{C}$ and $\mathscr{D}$,

$$F : Obj(\mathscr{C}) \to Obj(\mathscr{D}),$$

associating an object $F(X)$ in $Obj(\mathscr{D})$ for every X in $Obj(\mathscr{C})$,

(2) for all X, Y in $Obj(\mathscr{C})$, $F : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ (a correspondence between morphisms) satisfying;

- *for all X in $Obj(\mathscr{C})$, $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$,*
- *for all X, Y, Z in $Obj(\mathscr{C})$, f in $Hom_{\mathscr{C}}(X, Y)$ and g in $Hom_{\mathscr{C}}(Y, Z)$, $F(g \circ_{\mathscr{C}} f) = F(g) \circ_{\mathscr{D}} F(f)$.*

Example 3.0.9. *The power set functor $\mathscr{P} : Set \to Set$ maps each set to its power set. Let X, Y be sets and $f : X \to Y$ be a function. Define $\mathscr{P}(f) : \mathscr{P}(X) \to \mathscr{P}(Y)$ such that $\mathscr{P}(f)(S) = f(S)$ for all $S \in \mathscr{P}(X)$. Then, it is clear that $\mathscr{P}(\mathbb{1}_X) = \mathbb{1}_{\mathscr{P}(X)}$ for all X in $Obj(Set)$. Let us check that for all X, Y, Z in $Obj(Set)$, f in $Hom(X, Y)$ and g in $Hom(Y, Z)$, $\mathscr{P}(g \circ f) = \mathscr{P}(g) \circ \mathscr{P}(f)$. Let $S \subseteq X$. Then $\mathscr{P}(g \circ f) = (g \circ f)(S)$ and $\mathscr{P}(g) \circ \mathscr{P}(f)(S) = \mathscr{P}(g)(f(S)) = g(f(S))$. Hence the equality holds.*

Example 3.0.10. *Let us define $Span : Set \to Vec_k$ which sends each set X to $Span(X)$ which is the k-vector space spanned by X. This is clearly a correspondence of objects. Now we will define the correspondence between morphisms. Let X, Y be sets. Define*

$$Span : Hom_{Set}(X, Y) \to Hom_{Vec_k}(Span(X), Span(Y))$$

in the following way, $f \mapsto Span(f)$ where $Span(f) : Span(X) \to Span(Y)$ is a map which takes any element $v = \sum_{a \in X} \lambda_a \cdot a$ of $Span(X)$ to an element $\sum_{a \in X} \lambda_a f(a)$ of $Span(Y)$. First we need to *check that this is a k-linear map. Let v, w be elements of $Span(X)$ and $\lambda \in k$. Check that*

$$Span(f)(v + w) = Span(f)(v) + Span(f)(w)$$

and

$$Span(f)(\lambda v) = \lambda Span(f)(v).$$

Write $v = \sum_{a \in X} \lambda_a a$ and $w = \sum_{a \in X} \mu_a a$. Then $v + w = \sum_{a \in X}(\lambda_a + \mu_a)a$.

$$\begin{aligned} Span(f)(v + w) &= Span(f)\left(\sum_{a \in X}(\lambda_a + \mu_a)a\right) \\ &= \sum_{a \in X}(\lambda_a + \mu_a)f(a) \\ &= \sum_{a \in X}\lambda_a f(a) + \sum_{a \in X}\mu_a f(a) \\ &= Span(f)\left(\sum_{a \in X}\lambda_a a\right) + Span(f)\left(\sum_{a \in X}\mu_a a\right) \\ &= Span(f)(v) + Span(f)(w). \end{aligned}$$

$$\begin{aligned} Span(f)(\lambda v) &= Span(f)\left(\lambda \sum_{a \in X}\lambda_a a\right) \\ &= Span(f)\left(\sum_{a \in X}\lambda\lambda_a a\right) \\ &= \sum_{a \in X}\lambda\lambda_a f(a) \\ &= \lambda\sum_{a \in X}\lambda_a f(a) \\ &= \lambda Span(f)(v). \end{aligned}$$

So this gives us a correspondence between morphisms. Let X be a set in $Obj(Set)$. Then $Span(\mathbb{1}_X) = \mathbb{1}_{Span(X)}$ because for any element
$v = \sum_{a \in X}\lambda_a a$ in $Span(X)$ we have

$$Span(\mathbb{1}_X)(v) = \sum_{a \in X}\lambda_a \mathbb{1}_X(a) = \sum_{a \in X}\lambda_a a = v = \mathbb{1}_{Span(X)}(v).$$

Let X, Y, Z be sets in $Obj(Set)$.

Let $f : X \to Y$ and $g : Y \to Z$. We will prove that

$$Span(g \circ f) = Span(g) \circ Span(f).$$

Let $v = \sum_{a \in X} \lambda_a a$ in $Span(X)$. Then

$$\begin{aligned} Span(g \circ f)(v) &= \sum_{a \in X} \lambda_a (g \circ f)(a) \\ &= \sum_{a \in X} \lambda_a g(f(a)) \\ &= Span(g) \left(\sum_{a \in X} \lambda_a f(a) \right) \\ &= (Span(g) \circ Span(f)) \left(\sum_{a \in X} \lambda_a a \right) \\ &= Span(g) \circ Span(f)(v). \end{aligned}$$

Thus concluding Span is a functor between Set and Vec_k.

Remark 3.0.11. *We have seen that functors are morphisms of categories. As the question of the composition of functors came naturally, so does the answer. Functors can be composed. Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories and $T : \mathscr{C} \to \mathscr{D}$, $S : \mathscr{D} \to \mathscr{E}$ be functors between them. The composite functor of S with T is defined as $c \mapsto S(T(c))$ on objects c and $f \mapsto S(T(f))$ on morphisms f, and this composite functor is denoted as $S \circ T$. This composition is associative. For every category $\mathscr{C}$, there is an identity functor $I_{\mathscr{C}}$, satisfying $I_{\mathscr{C}} \circ T = T$ and $S \circ I_{\mathscr{C}} = S$ for any category $\mathscr{D}$ and functors $T : \mathscr{D} \to \mathscr{C}$, $S : \mathscr{C} \to \mathscr{D}$. So, we have constructed the category of categories **CAT** which has categories as objects and functors as morphisms.*

3.1 Functors between graph categories and Mon_*

Let us try to define a functor between Mon_* and graph categories.
Let $F : Gr_{rel} \to Mon_*$ be such that

$$\text{for any } G, F(G) = (\{\text{ all paths in G including the empty path } \epsilon\}, *),$$

(i.e. correspondence between objects)

$$\text{for any } f \in Hom_{Gr_{rel}}, F(f) : F(G) \to F(H) \text{ is defined by the following rule:}$$

$$F(f)(v) = \begin{cases} w & \text{if } (v, w) \in f \text{ for some } w \in W \\ * & \text{otherwise} \end{cases}$$

for any $v \in V$. If α is a path in G, then it is a series of vertices, say $v_1, v_2, \ldots, v_n$. Then define $F(f)(\alpha) = F(f)(v_1) \cdots F(f)(v_n)$.
First, let us assume that $F(G)$ is an object in Mon_* (will be proven later). We need to prove that $F(f) \in Hom_{Mon_*}(F(G), F(H))$ in order to say that F is a functor. But as we will see in the next example, it does not give a morphism in Mon_*.

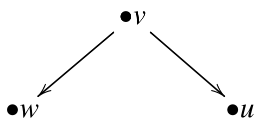


Figure 3.1: Graph 1

Figure 3.2: Graph 2

Let us name the first graph G and the other H. Let $R \in Hom_{Gr_{rel}}(G, H)$ be $\{(u, a), (u, b)\}$. Then we have $F(R)(u) = a$ and $F(R)(u) = b$, in other words $F(R)$ does not give a function. Thus, this F does not give a functor between Gr_{rel} and Mon_*.

Let $F : Gr_{par} \to Mon_*$ be such that

$$\text{for any } G, F(G) = (\{ \text{ all paths in G including the empty path } \epsilon\}, *).$$

We need to check that $F(G)$ is a monoid with a base point.

Definition 3.1.1. *Let A, B be monoids with base points. Define $A \wedge B = A \times B/ \backsim$ where*

$$(a, b) \backsim (c, d) \textit{ iff } (a = * \textit{ or } b = *) \textit{ and } (c = * \textit{ or } d = *).$$

Let $(*, *) = * \in A \wedge B$.

Note that for any $a \neq *$, we have $(a, *) \backsim (*, *) \backsim (*, a)$.
Define $\mu : F(G) \wedge F(G) \to F(G)$ as follows for all $\alpha, \beta \in F(G)$;

$$\mu(*, *) = *,$$

$$\mu(\alpha, \epsilon) = \mu(\epsilon, \alpha) = \alpha,$$

$$\mu(\alpha, \beta) = \begin{cases} \alpha\beta & ; \text{if } t(\beta) = s(\alpha) \\ * & ; \text{if } t(\beta) \neq s(\alpha) \end{cases}$$

To prove that $F(G)$ is a monoid, we need to show that μ is associative and ϵ is a unit. Let $\alpha, \beta, \gamma \in F(G)$. Look at

$$\mu(\mu(\alpha,\beta),\gamma) = \begin{cases} \mu(\alpha\beta,\gamma) & ;\text{if } \mu(\alpha,\beta) = \alpha\beta \text{ i.e. if } t(\beta) = s(\alpha) \\ \mu(*,\gamma) & ;\text{if } \mu(\alpha,\beta) = * \text{ if } t(\beta) \neq s(\alpha) \end{cases}$$

$$\mu(\mu(\alpha,\beta),\gamma) = \begin{cases} (\alpha\beta)\gamma & ;\ \text{if } t(\beta) = s(\alpha) \text{ and } t(\gamma) = s(\alpha\beta) = s(\beta) \\ * & ;\text{if } t(\beta) = s(\alpha) \text{ and } t(\gamma) \neq s(\alpha\beta) \\ * & ;\text{otherwise.} \end{cases}$$

$$\mu(\alpha,\mu(\beta,\gamma)) = \begin{cases} \mu(\alpha,\beta\gamma) & ;\text{if } \mu(\beta,\gamma) = \beta\gamma \text{ i.e. if } t(\gamma) = s(\beta) \\ \mu(\alpha,*) & ;\text{if } \mu(\beta,\gamma) = * \text{ if } t(\gamma) \neq s(\beta) \end{cases}$$

$$\mu(\alpha,\mu(\beta,\gamma)) = \begin{cases} \alpha(\beta\gamma) & ;\ \text{if } t(\gamma) = s(\beta) \text{ and } t(\beta\gamma) = t(\beta) = s(\alpha) \\ * & ;\text{if } t(\gamma) = s(\beta) \text{ and } t(\beta\gamma) \neq s(\alpha) \\ * & ;\text{otherwise.} \end{cases}$$

Since we know that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ when $t(\beta) = s(\alpha)$ and $t(\gamma) = s(\beta)$, we have $\mu(\mu(\alpha,\beta),\gamma) = \mu(\alpha,\mu(\beta,\gamma))$. Hence μ is associative.
Let us prove that ϵ is a unit. Let $\alpha \in F(G)$ be different from $*$. Then by definition, $\mu(\alpha,\epsilon) = \alpha = \mu(\epsilon,\alpha)$. Also $\mu(*,\epsilon) = \mu(\epsilon,*) = *$ as $\mu(*,*) = *$ and $(\epsilon,*) \backsim (*,*) \backsim (*,\epsilon)$. So we showed that $F(G)$ is a monoid with a base point, this asserts that our functor candidate F works well as a correspondence between objects.

Now, let us define the correspondence between morphisms. Let $G = (V,E)$, $H = (W,F)$ be objects in Gr_{par}. Let $f \in Hom_{Gr_{par}}(G,H)$, i.e. $f : V \to W$ is a partial function satisfying edge relations. Note that any α in $F(G)$, which is not equal to ϵ and $*$, is a series of vertices $(a_n \cdots a_1)$ where $a_i \in V$. Define $F(f) : F(G) \to F(H)$ as follows;

$$F(f)(\epsilon) = \epsilon,$$

$$F(f)(*) = *,$$

$$F(f)(\alpha) = \begin{cases} f(a_n)\cdots f(a_1) & ;\text{if } f(a_1),\ldots,f(a_n) \text{ are defined} \\ * & ;\ \text{otherwise.} \end{cases}$$

We need to check if this $F(f)$ is a based function respecting monoid operation. For any $\alpha \in F(G)$, $F(f)(\alpha)$ is defined. It is a based function by definition. Let $w, w' \in F(G)$. Show that

$$F(f)(w \cdot_{F(G)} w') = F(f)(w) \cdot_{F(H)} F(f)(w').$$

Let $w = a_n \cdots a_1$ and $w' = b_m \cdots b_1$ where $a_i, b_j \in V$. Then

$$w \cdot_{F(G)} w' = a_n \cdots a_1 b_{m-1} \cdots b_1$$

if $t(w') = b_m = a_1 = s(w)$ and $w \cdot_{F(G)} w' = *$ otherwise. Then

$$F(f)(w \cdot_{F(G)} w') = \begin{cases} F(f)(a_n \cdots a_1 b_{m-1} \cdots b_1) & ;\text{if } a_1 = b_m \\ F(f)(*) & ;\ \text{otherwise.} \end{cases}$$

So we have $F(f)(w \cdot_{F(G)} w')$ equal to $f(a_n) \cdots f(a_1) f(b_{m-1}) \cdots f(b_1)$ if $a_1 = b_m$ and all $f(a_i)$, $f(b_j)$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$ are defined, and $*$ otherwise.

We also have

$$F(f)(w) = \begin{cases} f(a_n) \cdots f(a_1) & \text{if } f(a_i) \text{ are defined} \\ * & \text{otherwise} \end{cases}$$

$$F(f)(w') = \begin{cases} f(b_m) \cdots f(b_1) & \text{if } f(b_i) \text{ are defined} \\ * & \text{otherwise.} \end{cases}$$

So $F(f)(w) \cdot_{F(H)} F(f)(w')$ becomes $f(a_n) \cdots f(a_1) f(b_{m-1}) \cdots f(b_1)$ if $f(a_1) = f(b_m)$ and $f(a_i)$, $f(b_j)$ are defined for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and $*$ otherwise.

Hence, to have the equality $F(f)(w \cdot_{F(G)} w') = F(f)(w) \cdot_{F(H)} F(f)(w')$, $f(a_1) = f(b_m)$ must imply $a_1 = b_m$, which means that f must be one-to-one. So our attempt fails, this F does not give a functor between Gr_{par} and Mon_*, but if we take the subcategory Gr_{par1}, which has the same objects as Gr_{par} and one-to-one partial functions preserving edges as morphisms, this F will work as a functor.

To prove that Gr_{par1} is a subcategory of Gr_{par}, it is enough to show that composition of two one-to-one partial functions is one-to-one. Let f be a morphism between G and H, g be a morphism between H and K. We want to prove $g \circ f$ is a one-to-one partial function. Let v, w be vertices of G and assume that $g \circ f(v) = g \circ f(w)$. Then since g is one-to-one, we have $f(v) = f(w)$. As f is also one-to-one, $v = w$, hence $g \circ f$ is one-to-one. Now that we have Gr_{par1}, F as defined above gives a functor between Gr_{par1} and Mon_* if we prove two conditions;

$$\text{for all } G \text{ in } Obj(Gr_{par1}), F(\mathbb{1}_G) = \mathbb{1}_{F(G)},$$

$$F(g \circ f) = F(g) \circ F(f),$$

for all G, H, K in $Obj(Gr_{par1})$, for all f in $Hom_{Gr_{par1}}(G, H)$ and for all g in $Hom_{Gr_{par1}}(H, K)$. The first one is clear, since

$$F(\mathbb{1}_G)(\alpha) = \mathbb{1}_G(a_n) \cdots \mathbb{1}_G(a_1) = a_n \cdots a_1 = \alpha = \mathbb{1}_{F(G)}(\alpha)$$

for any $\alpha = a_n \cdots a_1$ in $F(G)$. Let G, H, K be objects of Gr_{par1} and f, g be morphisms between G and H, H and K, respectively. Let $\alpha = a_n \cdots a_1$ be in $F(G)$. Then;

$$F(g) \circ F(f)(\alpha) = \begin{cases} F(g)(f(a_n) \cdots f(a_1)) & \text{if } f(a_i) \text{ are defined} \\ F(g)(*) & \text{otherwise} \end{cases}$$

$$F(g) \circ F(f)(\alpha) = \begin{cases} g(f(a_n)) \cdots g(f(a_1)) & \text{if } f(a_i) \text{ and } g(f(a_i)) \text{ are defined} \\ * & \text{otherwise} \end{cases}$$

$$F(g \circ f)(\alpha) = \begin{cases} g \circ f(a_n) \cdots g \circ f(a_1) & \text{if } g \circ f(a_i) \text{ are defined} \\ * & \text{otherwise} \end{cases}$$

$$F(g \circ f)(\alpha) = \begin{cases} g(f(a_n)) \cdots g(f(a_1)) & \text{if } g \circ f(a_i) \text{ and } f(a_i) \text{ are defined} \\ * & \text{otherwise.} \end{cases}$$

So the second condition $F(g \circ f) = F(g) \circ F(f)$ is satisfied. Hence we have a functor between Gr_{par1} and Mon_*.

Now let us define a functor from Gr_{fun} to Mon_*. The correspondence between objects will be the same as defined before in Gr_{rel} and Gr_{par} because only the morphisms are different in the mentioned categories.

Let $G = (V, E)$ and $H = (W, F)$ be objects in Gr_{fun}. Let f be a morphism from G to H. Define $F(f) : F(G) \to F(H)$ as follows;

$$F(f)(\epsilon) = \epsilon,$$

$$F(f)(*) = *,$$

$$F(f)(\alpha) = f(a_n) \cdots f(a_1),$$

where $\alpha = a_n \cdots a_1$ is any element of $F(G)$, $a_n, \ldots, a_1$ in V. We need to prove that $F(f)$ is a based function respecting monoid operations, i.e. $F(f)$ is a morphism in Mon_* category. By definition, $F(f)$ is a based function. Let $\alpha = a_n \cdots a_1$ and $\alpha' = b_m \cdots b_1$ be elements of $F(G)$. We will show that

$$F(f)(\alpha \cdot_{F(G)} \alpha') = F(f)(\alpha) \cdot_{F(H)} F(f)(\alpha').$$

Firstly, we have $\alpha \cdot_{F(G)} \alpha' = a_n \cdots a_1 b_{m-1} \cdots b_1$ if $t(\alpha') = b_m = s(\alpha) = a_1$ and $\alpha \cdot_{F(G)} \alpha' = *$ otherwise. Then

$$F(f)(\alpha \cdot_{F(G)} \alpha') = \begin{cases} F(f)(a_n \cdots a_1 b_{m-1} \cdots b_1) & ; \text{if } a_1 = b_m \\ F(f)(*) & ;\ \text{otherwise.} \end{cases}$$

$$F(f)(\alpha \cdot_{F(G)} \alpha') = \begin{cases} f(a_n)\cdots f(a_1)f(b_{m-1})\cdots f(b_1) & ;\text{if } a_1 = b_m \\ * & ;\ \text{otherwise.} \end{cases}$$

Let us look at $F(f)(\alpha) \cdot_{F(H)} F(f)(\alpha')$.

$$F(f)(\alpha) = f(a_n)\cdots f(a_1),$$

$$F(f)(\alpha') = f(b_m)\cdots f(b_1),$$

then we have;

$$F(f)(\alpha) \cdot_{F(H)} F(f)(\alpha') = \begin{cases} f(a_n)\cdots f(a_1)f(b_{m-1})\cdots f(b_1) & ;\text{if } f(a_1) = f(b_m) \\ * & ;\ \text{otherwise.} \end{cases}$$

Similar as before, we need to have $f(b_m) = f(a_1)$ implies $b_m = a_1$ to have the desired equality, so that we need to have one-to-one functions as morphisms. Let us consider the subcategory Gr_{fun1}, having the same objects as Gr_{fun} and morphisms as one-to-one functions between vertex sets preserving edges. Then, we will have a functor from Gr_{fun1} to Mon_* as defined above.

Gr_{fun1} is a subcategory of Gr_{fun} if we prove that composition of one-to-one functions preserving edges is also one-to-one as preserving edges condition have been checked in Gr_{fun} category. Let f be a morphism between G and H, g be a morphism between H and K. Let v, w be vertices of G and assume that

$$g \circ f(v) = g \circ f(w).$$

So $f(v) = f(w)$ since g is injective. But as f is also injective, we have $v = w$, implying that $g \circ f$ is a one-to-one function. Hence, we have the subcategory Gr_{fun1}. Then only two conditions remain to show that F as defined between Gr_{fun} and Mon_* is a functor from Gr_{fun1} to Mon_*;

$$\text{for all } G \text{ in } Obj(Gr_{fun1}), F(\mathbb{1}_G) = \mathbb{1}_{F(G)},$$

$$F(g \circ f) = F(g) \circ F(f),$$

for all G, H, K in $Obj(Gr_{fun1})$, for all f in $Hom_{Gr_{fun1}}(G, H)$ and for all g in $Hom_{Gr_{fun1}}(H, K)$.

Since

$$F(\mathbb{1}_G)(\alpha) = \mathbb{1}_G(a_n)\cdots \mathbb{1}_G(a_1) = a_n \cdots a_1 = \alpha = \mathbb{1}_{F(G)}(\alpha)$$

for any $\alpha = a_n \cdots a_1$ in $F(G)$, the former condition holds. Let G, H, K be objects of Gr_{fun1} and f, g be morphisms between G and H, H and K, respectively. Let $\alpha = a_n \cdots a_1$ be in $F(G)$.

Then we have;

$$\begin{aligned} F(g) \circ F(f)(\alpha) &= F(g)(f(a_n) \cdots f(a_1)) \\ &= g(f(a_n)) \cdots g(f(a_1)), \end{aligned}$$

$$\begin{aligned} F(g \circ f)(\alpha) &= (g \circ f)(a_n) \cdots (g \circ f)(a_1) \\ &= g(f(a_n)) \cdots g(f(a_1)). \end{aligned}$$

So the latter condition $F(g \circ f) = F(g) \circ F(f)$ also holds. Therefore we have a functor between Gr_{fun1} and Mon_*.

3.2 Functor between Mon_* and Alg_k

Let us define Span : $Mon_* \to Alg_k$. For any object $(M, *)$ in Mon_*, let $\mathrm{Span}_k(M, *)$ be the vector space over k with basis $(M, *)$. Then we have seen in 2.5.6 that $\mathrm{Span}_k(M, *)$ is an algebra. Let $(M, *)$,$(N, *)$ be objects of Mon_* and $f : (M, *) \to (N, *)$ be a morphism. Define

$$\mathrm{Span}(f) : \mathrm{Span}_k(M, *) \to \mathrm{Span}_k(N, *)$$

in the following way;

$$\mathrm{Span}(f)\left(\sum_{m\in(M,*)} \lambda_m m\right) = \sum_{m\in(M,*)} \lambda_m f(m)$$

for any $\sum_{m\in(M,*)} \lambda_m m$ in $\mathrm{Span}_k(M, *)$. We need to check that this is a morphism in Alg_k. This gives us a k-linear map since

$$\begin{aligned} \mathrm{Span}(f)(v + w) &= \mathrm{Span}(f)\left(\sum_{m\in(M,*)} (\lambda_m + \mu_m)m\right) \\ &= \sum_{m\in(M,*)} (\lambda_m + \mu_m) f(m) \\ &= \sum_{m\in(M,*)} \lambda_m f(m) + \sum_{m\in(M,*)} \mu_m f(m) \\ &= \mathrm{Span}(f)\left(\sum_{m\in(M,*)} \lambda_m m\right) + \mathrm{Span}(f)\left(\sum_{m\in(M,*)} \mu_m m\right) \\ &= \mathrm{Span}(f)(v) + \mathrm{Span}(f)(w), \end{aligned}$$

$$
\begin{aligned}
\mathrm{Span}(f)(\lambda v) &= \mathrm{Span}(f)\left(\lambda \sum_{m\in(M,*)} \lambda_m m\right)\\
&= \mathrm{Span}(f)\left(\sum_{m\in(M,*)} \lambda\lambda_m m\right)\\
&= \sum_{m\in(M,*)} \lambda\lambda_m f(m)\\
&= \lambda \sum_{m\in(M,*)} \lambda_m f(m)\\
&= \lambda \mathrm{Span}(f)(v)
\end{aligned}
$$

where $v = \sum_{m\in(M,*)} \lambda_m m$, $w = \sum_{m\in(M,*)} \mu_m m$ are elements of $\mathrm{Span}_k(M,*)$, $\lambda \in k$. Now we need to prove that $\mathrm{Span}(f)$ respects multiplication. Let $v = \sum_{m\in(M,*)} \lambda_m m$ and $w = \sum_{n\in(M,*)} \lambda_n n$ be in $\mathrm{Span}_k(M,*)$. Then

$$
\begin{aligned}
\mathrm{Span}(f)(vw) &= \mathrm{Span}(f)\left(\sum_{m\in(M,*)} \lambda_m m \sum_{n\in(M,*)} \lambda_n n\right)\\
&= \mathrm{Span}(f)\left(\sum_{m\in(M,*)} \sum_{n\in(M,*)} \lambda_m\lambda_n mn\right)\\
&= \sum_{m\in(M,*)} \sum_{n\in(M,*)} \lambda_m\lambda_n f(mn)\\
&= \sum_{m\in(M,*)} \sum_{n\in(M,*)} \lambda_m\lambda_n f(m)f(n)\\
&= \sum_{m\in(M,*)} \lambda_m f(m) \sum_{n\in(M,*)} \lambda_n f(n)\\
&= \mathrm{Span}(f)(v) \cdot \mathrm{Span}(f)(w).
\end{aligned}
$$

So $\mathrm{Span}(f)$ is a morphism in Alg_k.

For all $(M,*)$ in $Obj(Mon_*)$, we have $\mathrm{Span}(\mathbb{1}_{(M,*)}) = \mathbb{1}_{\mathrm{Span}_k(M,*)}$, as

$$
\mathrm{Span}(\mathbb{1}_{(M,*)})\left(\sum_{m\in(M,*)} \lambda_m m\right) = \sum_{m\in(M,*)} \lambda_m m = \mathbb{1}_{\mathrm{Span}_k(M,*)}
$$

for any $\sum_{m\in(M,*)} \lambda_m m$ in $\mathrm{Span}_k(M,*)$.

For all $(M,*)$, $(N,*)$,$(P,*)$ in $Obj(Mon_*)$, $f : (M,*) \to (N,*)$ and

$g : (N, *) \to (P, *)$, for any $\sum_{m \in (M,*)} \lambda_m m$ in $\mathrm{Span}_k(M, *)$,

$$
\begin{aligned}
\mathrm{Span}(g \circ f)\left(\sum_{m \in (M,*)} \lambda_m m\right) &= \sum_{m \in (M,*)} \lambda_m (g \circ f)(m) \\
&= \sum_{m \in (M,*)} \lambda_m g(f(m)) \\
&= \mathrm{Span}(g)\left(\sum_{m \in (M,*)} \lambda_m f(m)\right) \\
&= \mathrm{Span}(g)\left(\mathrm{Span}(f) \sum_{m \in (M,*)} \lambda_m m\right) \\
&= \mathrm{Span}(g) \circ \mathrm{Span}(f)\left(\sum_{m \in (M,*)} \lambda_m m\right),
\end{aligned}
$$

hence concluding $\mathrm{Span}(g \circ f) = \mathrm{Span}(g) \circ \mathrm{Span}(f)$ and completing the proof that Span is a functor from (Mon_*) to Alg_k.

Chapter 4

Graph Operations

4.1 Vertex Deletion

Let G be a graph with vertex set V and edge set E. Let $v \in V$. We will delete vertex v and all edges adjacent to it, this process corresponds to a one-to-one partial function between vertex sets. We will denote the resulting graph as $G \setminus v$ which has vertex set $V \setminus \{v\}$ and edge set $E' = E \setminus \{\text{all edges adjacent to } v\}$. Define $f : V \to V \setminus \{v\}$ such that $f(x) = x$ for any $x \in V$ if $x \neq v$, and undefined for v. Does this operation corresponds to a morphism in Gr_{par}? We need to check that

$$\forall x, y \in V, (x, y) \in E \Rightarrow (f(x), f(y)) \in E',$$

whenever both $f(x)$ and $f(y)$ are defined. The answer is yes since for any vertex different from v, the edge relations are preserved and for edges starting or ending at v since $f(v)$ is not defined, it is clear that this also satisfies the above property.

Example 4.1.1. *H:*

$$u\bullet \longrightarrow v\bullet \longrightarrow w\bullet$$

$H \setminus v$:

$$u\bullet \qquad w\bullet$$

Example 4.1.2. *Let us look at graphs G and $G \setminus a$, $G \setminus b$.*

G:

$$\begin{tikzcd}
a\bullet \arrow[r] & \bullet b \\
c\bullet \arrow[u] \arrow[ur] \arrow[r] & \bullet d \arrow[u]
\end{tikzcd}$$

$G \setminus a$:

$$\begin{tikzcd}
& \bullet b \\
c\bullet \arrow[ur] \arrow[r] & \bullet d \arrow[u]
\end{tikzcd}$$

$G \setminus b$:

$$\begin{tikzcd}
a\bullet & \\
c\bullet \arrow[u] \arrow[r] & \bullet d
\end{tikzcd}$$

4.2 Edge Deletion

Let $G = (V, E)$ be a directed graph. Let e be an edge of G. We will denote the graph which is captured by deleting edge e by $G - e$. Then we can consider edge deletion as a function f from G to $G - e$ which is the identity function between vertex sets V. (Note that G and $G - e$ has the same vertex sets.) This operation does not give a morphism in three graph categories we defined above, because it does not preserve edges.

Example 4.2.1. G :

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d] & \bullet & \bullet \arrow[dl] \\
\bullet & \bullet \arrow[u] &
\end{tikzcd}$$

$G - e$:

$$\begin{tikzcd}
\bullet \arrow[d] & \bullet & \bullet \arrow[dl] \\
\bullet & \bullet \arrow[u] &
\end{tikzcd}$$

Example 4.2.2. G :

$$\begin{tikzcd}
& b\bullet \arrow[dr, "e"] & \\
a\bullet \arrow[ur] & & c\bullet
\end{tikzcd}$$

$G - e$:

$$\begin{tikzcd}
& b\bullet & \\
a\bullet \arrow[ur] & & c\bullet
\end{tikzcd}$$

4.3 Edge Addition

Let $G = (V, E)$ be a directed graph. Edge addition is just adding an edge between two non-adjacent vertices. (We do not want multiple edges between vertices.) We will denote the new graph as $G + e = (V, E \cup \{e\})$. Define $f : V \to V$ as $f(v) = v$ for all v. So this is clearly a one-to-one function. Since for all $x, y \in V$, $(x, y) \in E$ implies $(f(x), f(y)) \in E \cup \{e\}$, this operation is a morphism in Gr_{fun}.

Example 4.3.1. G :

$$\begin{tikzcd} a\bullet & b\bullet \\ c\bullet & \end{tikzcd}$$

$G + e$:

$$\begin{tikzcd} a\bullet & b\bullet \arrow[dl] \\ c\bullet & \end{tikzcd}$$

Example 4.3.2. H :

$$\begin{tikzcd} x\bullet & y\bullet \arrow[d] & \bullet z \arrow[l] \arrow[dl] \\ & t\bullet & \end{tikzcd}$$

$H + e$:

$$\begin{tikzcd} x\bullet \arrow[r] & y\bullet \arrow[d] & \bullet z \arrow[l] \arrow[dl] \\ & t\bullet & \end{tikzcd}$$

4.4 Edge Contraction

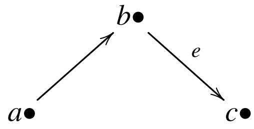
Let $G = (V, E)$ be a directed graph. Edge contraction is just taking an edge and identifying its end points. Let $e \in E$ be an edge with source vertex v and target vertex w. Let us denote the graph obtained by contracting e as G/e. Then edge contraction is a function $f : V \to V \setminus \{w\}$ (from the vertex set of G to the vertex set of G/e) defined as follows:

$$f(x) = x \text{ if } x \neq v \text{ and } x \neq w,$$

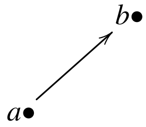
$$f(x) = v \text{ if } x = v \text{ or } x = w.$$

It is obvious that this function is not one-to-one. As we only contract an edge and identify its endpoints, this function preserves edges, so this is a morphism in Gr_{fun}.

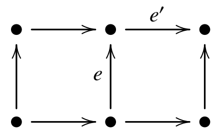
Example 4.4.1. G :



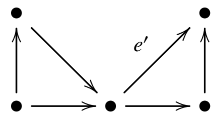
G/e :



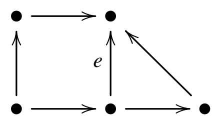
Example 4.4.2. H :



H/e :



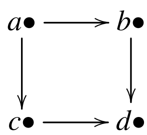
H/e' :



4.5 Vertex Expansion

Let $G = (V, E)$ be a directed graph. Vertex expansion is taking a vertex v and expanding it to two vertices v, v' and adding edges if necessary. For instance if there is an edge between x and v in G, we will put an edge between x and v'. Let us denote the new graph as $G * v$. Define $f : V \to V \bigcup \{v'\}$ as $f(x) = x$ if $x \neq v$ and $f(v) = v$ and $f(v) = v'$. This is clearly not a function but it gives a relation. Assume (x, v) is an edge in G. Then our relation takes x to x, v to v and v'. Since (x, v) is an edge, (x, v) and (x, v') are edges in $G * v$. So this relation preserves edges, hence a morphism in Gr_{rel}.

Example 4.5.1. G :



$G * a$:

$$\begin{tikzcd}
& a'\bullet \arrow[d] \arrow[dl] \\
a\bullet \arrow[r] \arrow[d] & b\bullet \arrow[d] \\
c\bullet \arrow[r] & d\bullet
\end{tikzcd}$$

Example 4.5.2. H :

$$\begin{tikzcd}
x\bullet & y\bullet \arrow[d] & \bullet z \arrow[l] \arrow[dl] \\
& t\bullet &
\end{tikzcd}$$

$H * x$:

$$\begin{tikzcd}
x\bullet & y\bullet \arrow[d] & \bullet z \arrow[l] \arrow[dl] \\
x'\bullet & t\bullet &
\end{tikzcd}$$

$H * z$:

$$\begin{tikzcd}
x\bullet & y\bullet \arrow[d] & \bullet z \arrow[l] \arrow[dl] \\
& t\bullet & \bullet z' \arrow[l] \arrow[ul]
\end{tikzcd}$$

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