DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS OF THE INFINITE SYMMETRIC GROUP

by
Cihan SAHİLLİOĞULLARI

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İZMİR

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We have read the thesis entitled "**CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS OF THE INFINITE SYMMETRIC GROUP**" completed by **CİHAN SAHİLLİOĞULLARI** under supervision of **PROF. DR. SELÇUK DEMİR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Selçuk DEMİR

Supervisor

Prof. Dr. Engin Büyükaşık

Jury Member

Dr. Öğr. Üyesi Haydar Göral

Jury Member

Prof. Dr. Latif SALUM
Director
Graduate School of Natural and Applied Sciences

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ABSTRACT

The characters of the infinite symmetric group were described by E. Thoma in 1964. E. Thoma's proof was difficult. Later, A. Okounkov found another proof based on the works of G. I. Olshanski. In thesis, we work out the details of A. Okounkov's proof. Related part of the work of G. I. Olshanski are either explained or given references.

Keywords: Unitary representations, infinite symmetric group, Brauer semigroups, spectral theory

SONSUZ SİMETRİK GRUBUN İNDİRGENEMEZ TEMSİLLERİNİN SINIFLANDIRILMASI

ÖZ

Sonsuz simetrik grubun karakterleri E. Thoma tarafından 1964 yılında tanımlandı. E. Thoma'nın kanıtı zordu. Daha sonra, A. Okounkov G. I. Olshanski'nin çalışmalarına dayanan başka bir kanıtını buldu. Bu tezde, A. Okounkov'un kanıtını detaylarıyla birlikte yazdık. G. I. Olshanski'nin çalışmasıyla bağlantılı olan kısımlar ya açıklandı ya da referanslar verildi.

Anahtar kelimeler: Üniter temsiller, sonsuz simetrik grup, Brauer yarıgrupları, spektral teori

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CHAPTER ONE
INTRODUCTION

Representation theory is an important subject for the area of the mathematical research. It allows us to reduce problems in abstract algebra to the problems in linear algebra. Firstly, Gauss used characters of finite abelian groups in the begining of 19th century. Representation theory has deep connections with areas such as number theory, homological algebra, mathematical physics and algebraic geometry. It still has unanswered problems and conjuctures. Moreover, the representaion theory of big groups is a devoloping part of modern mathematics and it has various applications in probability and mathematical physics (especially, the representations of $S(\infty)$).

In representation theory, characters have a prominent place. For a given infinite discrete group H, a character is a function from H to $\mathbb{C}$ which satisfies the folowing conditions,

(a) φ is central, that is, $\varphi(gh) = \varphi(hg)$ for all $h, g \in S(\infty)$,

(b) φ is positive definite, that is, for all $h_1, h_2, \cdots, h_n \in S(\infty)$ the matrix $\varphi(h_i h_j^{-1})$ is Hermition and non-negatively definite,

(c) φ is indecomposable, that is, it can not be written as the sum of two linearly independent functions satisfying (a) and (b),

(d) φ is normalized, that is, $\varphi(e) = 1$.

Characters contains lots of information about the unitary representation of infinite discrete groups. If one know the characters of the infinite symmetric group, then it is possible to construct its unitary representations.

In this thesis, we will work out the characters of irreducible representations of the infinite symmetric group which is defined by

$$S(\infty) = \bigcup_{n \geq 1} S(n)$$

where $S(n)$ is the symmetric group on n elements. One can observe that, $S(n)$'s have the form of an increasing chain of subgroups. There is a natural topology on the group $S(\infty)$ which is induced by the weak operator topology in the representation by permutations of the standard basis vectors

$$S(\infty) \longrightarrow U(l_2)$$

where $U(l_2)$ is the group of unitary operators in the coordinate Hilbert space l_2. A unitary representation of the group $S(\infty)$ is

$$\pi : S(\infty) \longrightarrow U(H)$$

where $U(H)$ is the group of unitary operators in the Hilbert space H and π is a group homomorphism. The above representation π is said to be tame if it is continuous according to the above topology on $S(\infty)$ and the weak topology on $U(H)$. We can give more useful definition about the tame representation. Denote by

$$H_n = H^{S_n(\infty)} \qquad n = 1, 2, \cdots ,$$

where H_n is the subspace of invariants with respect to the action of the group $S_n(\infty)$ in the above unitary representation and $S_n(\infty)$ is the subgroup of $S(\infty)$ which fixes the numbers $1, 2, \cdots , n$. The subgroups $S_n(\infty)$ is a fundamental neighbourhood base of identity in the topology on $S(\infty)$ which is defined above. π is the tame representation of $S(\infty)$ if and only if

$$H = \overline{\bigcup_{n \geq 1} H_n}.$$

E. Thoma found characters of infinite symmetric group and showed them in Thoma (1964). Thoma's proof was complicated. A. M. Vershik and S. V. Kerov used asymptotic methods to prove Thoma's theorem. A. Okounkov used the ideas and the method of Olshanski (1990) to give a new simpler proof of Thoma's theorem. The characters of the group $S(\infty)$ are the functions of the form

$$\varphi(\sigma) = \prod_{k\in[\sigma]} \left(\sum_{i=1}^{\infty} n(a_i)a_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} n(-b_i)b_i^k \right), \tag{1.1}$$

where

$$a_1 \geq a_2 \geq a_3 \geq \cdots > 0, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots > 0$$

and

$$\sum n(a_i)a_i + \sum n(b_i)b_i \leq 1.$$

Our main aim is to work out A. Okounkov's contribution to this theorem and we give the details for his work. Let us consider the group $G = S(\infty) \times S(\infty)$ and its diagonal subgroup $K = \text{diag}S(\infty)$. The group G is called the infinite bisymmetric group. A unitary representation π of the bisymmetric group G in a Hilbert space is called a spherical representation of the pair (G, K) if it is irreducible and it has K-invariant vector α with the length 1. This vector is called spherical vector and it is unique up to a scalar factor. G. I. Olshanski's ideas for the infinite groups was that it is necessary to produce a good representation theory. Therefore, if we study unitary representations of G, we actually need to study the representations of the pair (G, K) (see Ol'shanskij (1990)).

The spherical function of the spherical representation of the pair (G, K) is $\varphi(g) = (\pi(g, g)\alpha, \alpha)$. By virtue of the following theorem, the spherical function can be considered as a character of the group $S(\infty)$.

Theorem 1. *(Ol'shanskij (1990), p. 987) For any discrete group K there exists a natural bijective correspondence between the spherical representaions of the Gel'fand pair $(K \times K, \text{diag}\, K)$ and the finite factor representaitons of K.*

In Olshanski (1990) and Ol'shanskij (1990), G. I. Olshanski explained that E.

Thoma's theorem is equivalent to description of all spherical representations of the pair (G, K).

In chapter two, we are going to recall what we know about G. I Olshanski work. G. I. Olshanski introduced a semigroup which has an important role for the theory of admissible representations of the group G. Firstly, Brauer constructed a semigroup which is called Brauer semigroup by using ideas of diagrams. A diagram is constructed by putting the same points on the upper and lower horizontal axis respectively and connecting each point with a different point by a curve. Moreover, we assign a nonnegative number for each curve. S. V. Kerov obtained realizations of representations of a Brauer semigroup (see Kerov (1989)). G. I. Olshanski used these ideas to construct a semigroup. G. I. Olshanski considered elements of $g \in G$ as a diagram consisting of points $..., -2, -1, 1, 2, ...$ on the upper horizantal axis and the same points on the lower horizonral axis. All connected points have length 0. He also defined some elements A_i's, C_i's and P_i's. The semigroup defined by

$$\Gamma = \left\langle G^D, A_1, C_j, P_k \right\rangle_{j \geq 1, k \geq 0} / \langle C_1 = 1 \rangle.$$

is called Olshanski's semigroup . Elements A_i's, C_i's and P_i's have natural involution which is reflection in the horizontal axis and these elements remains invariant under this involution. G. I. Olshanski produced a theorem which is related to the semigroup Γ (see Chapter 2). We also give some identities of the elements of the semigroup Γ. These identities is very handy for our work. It is sufficint to know the irreducible representation of the pair (G, K) on the restriction of the subsemigroup $\Gamma(n) \subset P_n \Gamma P_n$. Because we have

$$(\pi(g)\alpha, \alpha) = (\pi(P_n g P_n)\alpha, \alpha), \tag{1.2}$$

where α is a spherical vector. The irreducibility of representation π_n on $\Gamma(n)$ due to following theorem.

Theorem 2. *(Ol'shanskij (1990), p. 991) Let π be an admissible representation of the pair (G, K). Let π_n be the representation of the semigroup $\Gamma(n)$ on the subspace*

$H(\pi)_n$. Then if the π is irreducible then the representation π_n is irreducible.

Finally, we mention about the semigroup structure on the double cosets $K_n^D \backslash G / K_n^D$ (see Chapter 2). This semigroup $\Gamma^{'}(n)$ is isomorphic to the subsemigroup $\Gamma(n) \subset P_n \Gamma P_n$ of Γ.

In chapter three, A. Okounkov's contribution has a vast effect in the proof of Thoma's theorem (see Okounkov (1994)). He used the spectral theory for the operators A_i's. A_i's are self-adjonit bounded operators with the norm $||A_i|| \leq 1$. Let π be a spherical representation of the pair (G, K). Let μ be a spectral measure which is supported on the interval $[-1, 1]$. Then the operator A_1 has a spectral representation as follows

$$\int_{[-1,1]} t^k d\mu = (A_1^k \alpha, \alpha) = (P_0 A_1^k P_0 \alpha, \alpha) = c_{k+1}, \tag{1.3}$$

where α is spherical vector and the numbers c_k are the moments of this measure. For a given element $\sigma \in S(\infty)$ which has one cycle of length k, we have

$$(\pi(\sigma)\alpha, \alpha) = c_k. \tag{1.4}$$

If σ has more than one cycle, we also have

$$(\pi(\sigma)\alpha, \alpha) = \prod_{k \in [\sigma]} c_k. \tag{1.5}$$

Therefore, one can say the moments of measure μ give the characters of the group $S(\infty)$. We will give explicit formula for the characters of the group $S(\infty)$ by using atoms of this measure and some auxiliary conclusions. At the end, we will obtain the Thoma's theorem which is our main purpose.

CHAPTER TWO
OLSHANSKI'S SEMIGROUP

In this chapter, we will define the semigroups which are important for our study. These semigroups were defined by G. Olshanski. Before doing that we will give some useful facts.

We define some important tools which are necessary to make sense for the semigroup Γ. These tools have a relation with $G = S(\infty) \times S(\infty)$ and diag$S(\infty)$. Let $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a bijection. We call this bijection finite if the following set,

$$\{i \in \mathbb{Z} | g(i) \neq i\},$$

is finite. Let us define G^D to be the group of all finite bijections $\mathbb{Z} \longrightarrow \mathbb{Z}$ with the properties $g(0) = 0$ and $g(\mathbb{N}) = \mathbb{N}$. Namely, our group is

$$G^D = \{g | g(0) = 0, g(\mathbb{N}) = \mathbb{N}\}.$$

We will introduce the following terminology. Set

$$G^D(n) = \{g | g(i) = i, |i| > n\}$$

and

$$G^D_n = \{g | g(i) = i, |i| \leq n\}.$$

It is easy to see that, G^D is actually the union of the increasing chain of its subgroups $G^D(n)$,

$$\{e\} = G^D(0) \subseteq G^D(1) \subseteq G^D(2) \subseteq \cdots$$

$$\bigcup_n G^D(n) = G^D.$$

Similary, the subgroups of G^D_n is a decreasing chain of subgroups,

$$G^D_0 \supseteq G^D_1 \supseteq G^D_2 \supseteq \ldots$$

$$\bigcap_n G_n^D = \{e\}.$$

It is straightforward to check that the subgroups $G^D(n)$ and G_n^D commute with each other. We can define a subgroup of the group G^D,

$$K^D = \{g \in G^D | g(-i) = -g(i)\}.$$

K^D has an increasing and decreasing chain of subgroups,

$$K^D(n) = K^D \cap G^D(n)$$

and

$$K_n^D = K^D \cap G_n^D,$$

which are in K^D. We can consider elements of $S(\infty)$ on finite permutations of the set $\{1, 2, ...\}$. By this way and definiton of G^D , the group G^D is isomorphic to $G = S(\infty) \times S(\infty)$ and the group K^D is clearly isomorphic to the group $S(\infty)$. We also have $G^D(n) = S(n) \times S(n)$. Let π be a unitary representation of G^D in a Hilbert space H. H_n is the subspace of K_n^D-invariant vectors. It is obvious that, H_n is invariant under the action of the group $G^D(n)$. Moreover, the subspaces H_n has the property that

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq ...$$

The union $\underset{n}{\cup} H_n$ is an algebraically invariant subspace in H. The representation π is called an admissible representation of the pair (G^D, K^D) if

$$H = \overline{\bigcup_n H_n}.$$

Let us consider two disjoint finite sets P and $P^{'}$. An element of $B(P, P^{'})$ is an element which satisfies the following conditions:

- A partition of the set $P \cup P'$ into pairs with a nonnegative real number given to each pair, and

- an unordered finite collection of nonnegative real numbers.

It is useful to consider the element of P and P' on horizontal line. In order to understand easily, we will give an example of it. Let $|P| = 6$ and $|P'| = 6$. We can consider the elements of P and $P^{'}$ on the upper and lower horizontal axis respectively. Then we can describe an element of $B(P, P')$ in the following figure:

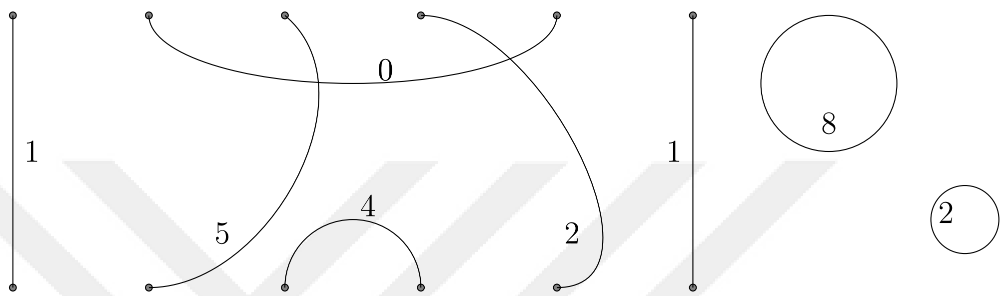
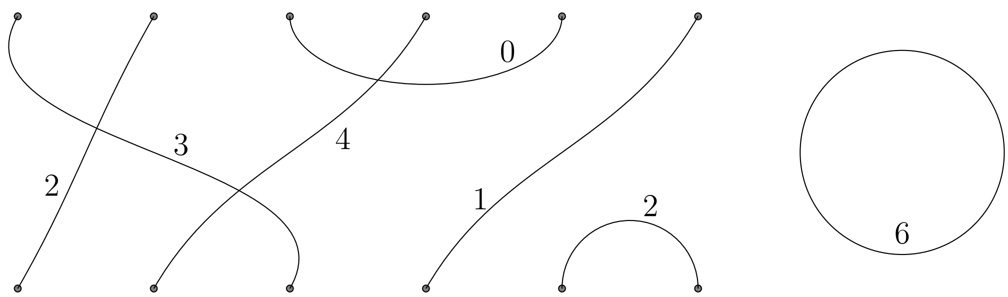


Figure 2.1 An element of $B(P, P')$

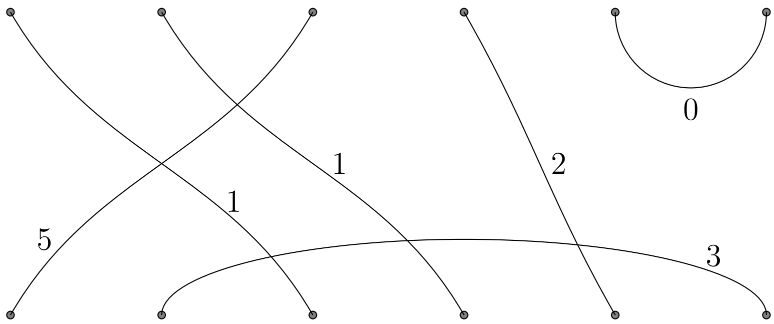
It can be easily observed that, there exists a natural map

$$B(P, P') \times B(P', P'') \longrightarrow B(P, P'')$$

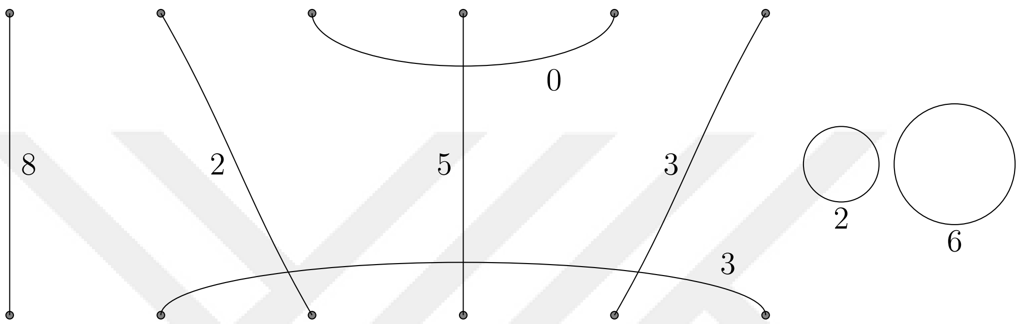
which connects $N_1 \in B(P, P')$ to $N_2 \in B(P', P'')$ via P'. The length of two connected components are added up. Notice that, after connecting two components we can obtain more cycles than N_1 and N_2 have. One can see it in the below example. We will now show how we can connect two components. Let $|P| = 6, |P^{'}| = 6$ and $|P^{''}| = 6$. Let $N_1 \in B(P, P')$ and $N_2 \in B(P', P'')$. Then



times



is equal to



We can consider an element of $B(P, P')$ like a wiring diagram which has inputs and outputs according to pairs of $P \cup P'$. We use the terminology a nonnegative resistance instead of the length. When we multiply two objects, we mean that, two wires are glued to each other. By a simple observation, one can notice that $Aut(P)$ is the subgroup of

$$Aut(P) \subset B(P, P)$$

which has no cycles and all components of it have the length zero. For a bijection,

$$g : P \longrightarrow P$$

every element $p \in P$ is connected to the element $g^{-1}(p)$ in the second copy of P.

Our aim is now to study the object $B(\mathbb{Z}, \mathbb{Z})$. By using $B(\mathbb{Z}, \mathbb{Z})$, we will construct our important semigroup Γ which is defined later. Notice that, the multiplication rule may fail for infinite diagrams, because we can obtain infinitely many loops which we do not want to have. Fortunately, we will study on specific set of infinite diagrams. Because of that, the multiplication will not be failed. Now, we define the elements which generates our semigroup Γ. The diagram A_k is illustrated in the following figure:

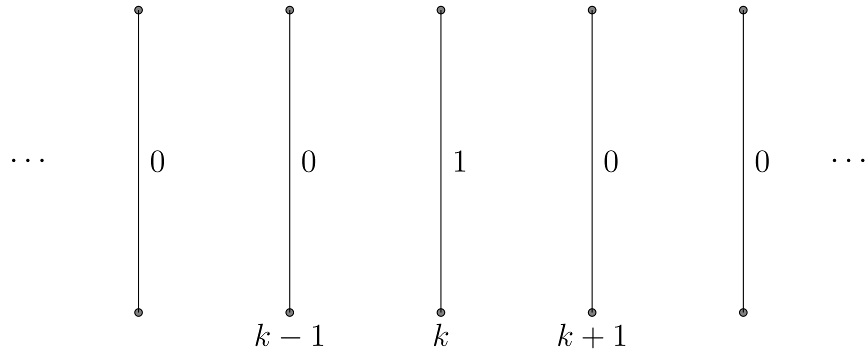


Figure 2.2 The diagram A_k

The diagram A_k has one segment of length one at $k \in \mathbb{Z}$ and the length of all other segments is zero. The diagram C_k is illustrated in the following figure:

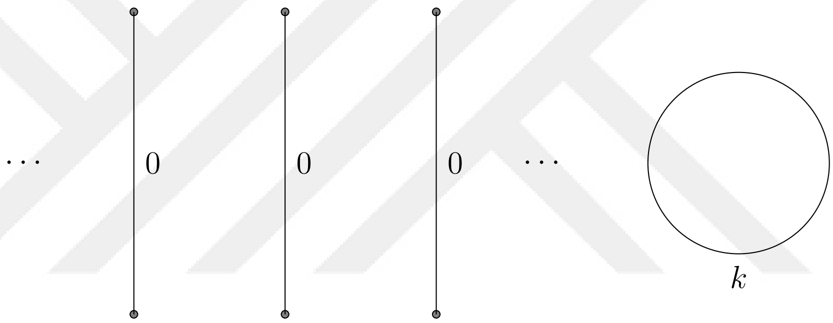


Figure 2.3 The diagram C_k

The diagram C_k, $k = 1, 2, 3, \dots$ has the segments which have zero length and it has one loop of length k. The diagram P_k, $k = 0, 1, 2, \dots$ is illustrated in the following figure:

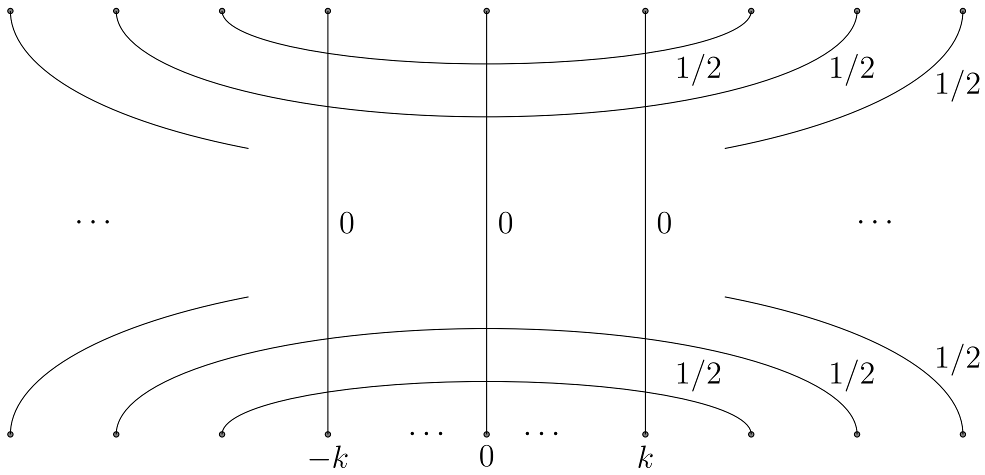


Figure 2.4 The diagram P_k

The diagram P_k has $2k+1$ vertical segments of length zero and all other components are arcs of length $1/2$. We want to make sense for the diagram P_k^2 by modding out $C_1 = 1$. Namely, $P_k^2 = P_k \bmod C_1$.

Definition 1. Γ *is the semigroup which is generated by the following diagrams*

$$\Gamma = \left\langle G^D, A_1, C_j, P_k \right\rangle_{j \geq 1, k \geq 0} / \langle C_1 = 1 \rangle$$

modulo the relation $C_1 = 1$.

Next, we will give significant properties of the defined diagrams. These diagrams are crucial for us. They will make easier to show some propositions and theorems later. We will give the following identities from (Okounkov (1994)) for our work.

Proposition 1.

$$A_i A_j = A_j A_i, \tag{2.1}$$

$$g A_i g^{-1} = A_{g(i)} \qquad g \in G, \tag{2.2}$$

$$A_i P_n = P_n A_i \qquad |i| \leq n, \tag{2.3}$$

$$A_i P_n = A_{-i} P_n \qquad |i| > n, \tag{2.4}$$

$$P_n A_i P_n = P_n (i, k) P_n \qquad |i| \leq n, |k| > n, \tag{2.5}$$

$$P_n A_{i_1}^{k_1} \dots A_{i_r}^{k_r} P_n = P_n \prod_{j=1}^{r} C_{k_j+1} \qquad n < i_j, i_m \neq i_l, \tag{2.6}$$

$$P_0 \sigma P_0 = P_0 \prod_{k \in [\sigma]} C_k, \tag{2.7}$$

where $[\sigma]$ consists of positive integers which are the length of cycles of σ.

Proof. (2.1): When we multiply A_i and A_j, we obtain $A_i A_j$ whose i-th and j-th segment have length 1 and all other segments have length 0. Similarly, $A_j A_i$ whose j-th and i-th segment have length 1 and all other segments have length 0. Therefore $A_i A_j = A_j A_i$.

(2.2): When we multiply gA_i, we obtain the element $g^{'}$ whose segment which connects $(i+1)$-th point to i-th point in the bottom has length 1 and $g^{'}$ has the same shape with g. Then $gg^{'}$ has the same shape with the identity element of G except for the segment $g(i)$ has length 1, that is, $g^{'}g^{-1} = A_{g^{-1}(i)}$. Thus $gA_i g^{-1} = g^{'}g^{-1} = A_{g(i)}$.

(2.3): Since $|i| \leq n$, the diagram $A_i P_n$ has the same shape with P_n whose i-th segment has length 1. Similarly, the diagram $P_n A_i$ has the same shape with P_n whose i-th segment has length 1. Therefore $A_i P_n = P_n A_i$.

(2.4): Since $|i| > n$, the diagram $A_i P_n$ has the same shape with P_n except for the length of upper arc which connects i-th point to $-i$-th point . This arc has length $3/2$. Similarly, the multiplication of $A_{-i} P_n$ gives the same result. Thus $A_i P_n = A_{-i} P_n$.

(2.5): Since $|i| \leq n$, by using (2.3) we have $P_n A_i P_n = P_n^2 A_i = \cdots C_1 C_1 P_n A_i$, for the other side of equality $P_n s_{i,k} P_n$, when we go from i-th point, it passes through two arcs and again connects with itself. Thus this segment has length 1. Hence $P_n s_{i,k} P_n = \cdots C_1 C_1 C_1 A_i P_n$. Since we work in our group Γ, we have $C_1 = 1$ therefore $P_n A_i P_n = P_n s_{i,k} P_n$.

(2.6): If we consider the simplest case $P_n A_{i_1}^{k_1} P_n = C_{k_1+1}$ for $n < i_1$ (One can see this visually by drawing the product $P_n A_{i_1}^{k_1} P_n$). For the case,

$$P_n A_{i_1}^{k_1} \cdots A_{i_r}^{k_r} P_n = P_n A_{(i_1,\cdots,i_r)}^{(k_1,\cdots,k_r)} P_n \tag{2.8}$$

where the element $A_{(i_1,\cdots,i_r)}^{(k_1,\cdots,k_r)}$ is the element whose i_j-th segment has length k_j for $j = 1, 2, \cdots, r$. Then, after multiplication, for each i_j-th segment of $A_{(i_1,\cdots,i_r)}^{(k_1,\cdots,k_r)}$, we have a loop whose length is $k_j + 1$. Therefore

$$P_n A_{(i_1,\cdots,i_r)}^{(k_1,\cdots,k_r)} P_n = P_n \prod_{j=1}^{r} (C_{k_j} + 1) C_1 C_1 \cdots$$

Since we work in our group Γ, we have $C_1 = 1$ thus

$$P_n A_{(i_1,\cdots,i_r)}^{(k_1,\cdots,k_r)} P_n = P_n \prod_{j=1}^{r} (C_{k_j} + 1).$$

(2.7): We will prove it by induction on the length of the nontrivial cycles of σ. If $\sigma = (h_1 h_2 \cdots h_n)$ has one cycle with the length n. When we multiply $P_0 \sigma P_0$, we go from h_1 to h_2, we draw one loop with the length 1. Because we pass two arcs that have lengths $1/2$. Continuing in this manner from h_1 to itself, we pass $2n$ arcs, so that we have a loop which has length n. Therefore $P_0 \sigma P_0 = P_0 C_n$. If $\sigma = \tau\alpha$ has two cycles with the lengths n_1 and n_2 respectively. Similarly, for the cycles τ and α we have two loops which have lengths n_1 and n_2. Namely, $P_0 \sigma P_0 = P_0 C_{n_1} C_{n_2}$. We now have $P_0 \tau P_0 = P_0 C_{n_1}$, $P_0 \sigma P_0 = P_0 C_{n_2}$ and $P_0 \tau\alpha P_0 = P_0 C_{n_1} C_{n_2}$. Thus,

$$P_0 \tau P_0 P_0 \alpha P_0 = P_0 \tau P_0 \alpha P_0 = P_0 C_{n_1} C_{n_2}. \tag{2.9}$$

Assume that it is true for n. Let $\sigma = \tau_1 \tau_2 \cdots \tau_n$ has n cycles with the lengths $k_1, k_2, ..., k_n$ respectively. Consider the element $P_0 \sigma P_0 = P_0 \tau_1 \tau_2 \cdots \tau_n \tau_{n+1} P_0$

by using the result (2.9)

$$\begin{aligned} P_0\sigma P_0 &= P_0\tau_1\tau_2\cdots\tau_{n+1}P_0 \\ &= P_0\tau_1P_0\tau_2P_0\cdots P_0\tau_{n+1}P_0 \qquad (2.10) \\ &= P_0C_{k_1}C_{k_2}\cdots C_{k_{n+1}} \end{aligned}$$

thus, we obtain

$$P_0\sigma P_0 = P_0\prod_{k\in[\sigma]} C_k. \qquad (2.11)$$

□

More generally, one can obtain the following identity by using the same way we use above.

$$P_n\sigma P_n = P_n\prod_{k\in[\sigma]} C_k. \qquad (2.12)$$

We will also use this identity in some cases. When we look into the group Γ, there exists a natural involution $*$ in the group Γ. This involution is a reflection with respect to the horizontal axis. It is easy to see that, under this involution the elements A's, C's and P's are invariant namely, $A^* = A$, $C^* = C$ and $P^* = P$. We can consider an element g of $S(\infty)$ as a diagram. It is a diagram with no cycles, no components of positive length. Thus, this involution sends a permutation g to its inverse permutation g^{-1}. The following theorem shows us how important Olshanski semigroup Γ for the theory of admissible representaion is.

Theorem 3. *(Ol'shanskij (1990)) Every admissible representation π of a pair (G, K) extends canonically to a $*$-representation by contractions (that is, operators of norm $\leqslant 1$) of the corresponding semigroup Γ in $H(\pi)$. In this representation, the idempotent P_n is mapped to the orthogonal projection onto $H(\pi)_n$.*

The set of contractions has a structure of a semigroup under operator composition. A representation of a semigroup Γ is a morphism $\pi^{'} : \Gamma \longrightarrow \Gamma(H)$ such that $\pi^{'}(x_1x_2) = \pi^{'}(x_1)\pi^{'}(x_2)$ and $\pi^{'}(x^*) = \pi^{'}(x)^*$ for all $x_1, x_2, x \in \Gamma$ where $\Gamma(H)$ denotes the semigroup of contractions.

Let us explain some facts about the above theorem. $\pi^{'}$ denotes the canonical extension. For a given transposition $(i,n) \in G^O$ where G^O denotes the group of all finite bijections from $\mathbb{Z}$ to $\mathbb{Z}$. Moreover, all diagrams with no cycles and no components of positive length form a group isomorphic to G^O. Consider the following operator

$$\pi^{'}((i,n)),$$

has a limit in the weak operator topology. In order to show this, we need to show that the following limit

$$\lim_{n\to\infty}(\pi^{'}((i,n))\alpha,\beta), \tag{2.13}$$

exists for every α and β are in the subspace $\bigcup\limits_m H(\pi)_m$. We may assume that the vectors α and β are in $H(\pi)_m$ and we can find numbers n_1, n_2 such that $n_1, n_2 > m$. By the definition of K_m, the element $(n_1,n_2)(-n_1,-n_2)$ is in K_m.

$$\begin{aligned}(\pi'((i,n_1))\alpha,\beta) &= (\pi^{'}((n_1,n_2)(-n_1,-n_2)(i,n_2)(n_1,n_2)(-n_1,-n_2))\alpha,\beta) \\ &= (\pi^{'}((i,n_2))\pi^{'}((n_1,n_2)(-n_1,-n_2))\alpha, \\ &\qquad\qquad \pi^{'}((n_1,n_2)(-n_1,-n_2))\beta) \\ &= (\pi^{'}((i,n_2))\alpha,\beta). \end{aligned} \tag{2.14}$$

Since the element $(n_1,n_2)(-n_1,-n_2)$ is in K_m and the vectoes α,β are K_m-invariant vectors, we have $\pi^{'}((n_1,n_2)(-n_1,-n_2))(\alpha) = \alpha$ and $\pi^{'}((n_1,n_2)(-n_1,-n_2))(\beta) = \beta$. The above argument shows that, $(\pi^{'}((i,n))\alpha,\beta)$ is independent from the choice of n if

n is greater than m. Therefore

$$(\pi^{'}((i,n))\alpha,\beta) = \lim_{n\to\infty}(\pi^{'}((i,n))\alpha,\beta). \tag{2.15}$$

Now, let us investigate the relation between the operator $\pi^{'}(A_i)$ and the operator $\pi^{'}((i,n))$.

$$\begin{aligned}
\lim_{n\to\infty}\pi^{'}((i,n)) &= \lim_{n\to\infty}(\pi^{'}(P_m(i,n)P_m))\alpha,\beta) \\
&\overset{(2.5)}{=} (\pi^{'}(P_mA_iP_m)\alpha,\beta) \\
&= (\pi^{'}(A_i)\alpha,\pi^{'}(P_m^*)\beta) \\
&= (\pi^{'}(A_i)\alpha,\pi^{'}(P_m)\beta) \\
&= (\pi^{'}(A_i)\alpha,\beta).
\end{aligned} \tag{2.16}$$

The above argument shows that, the operator $\pi^{'}(A_i)$ can be defined as follows.

$$\pi^{'}(A_i) := \lim_{n\to\infty}\pi^{'}((i,n)).$$

In a similar way, we will show how to define the operator $\pi^{'}(C_k)$. The limit $\lim_{n_1,\cdots,n_k\to\infty}\pi^{'}((n_1,n_2,\cdots,n_k))$ exists if the limit

$$\lim_{n_1,\cdots,n_k\to\infty}(\pi^{'}((n_1,n_2,\ldots,n_k))\alpha,\beta)$$

exists for arbitrary vectors α and β are in $H(\pi)_m$. We can obtain the following,

$$\lim_{n_1,\ldots,n_k\to\infty}(\pi^{'}((n_1,n_2,\ldots,n_k))\alpha,\beta) \tag{2.17}$$

which is equivalent to

$$\begin{aligned}
\lim_{n_1,\ldots,n_k\to\infty}(\pi^{'}(P_m(n_1,n_2,\ldots,n_k)P_m)\alpha,\beta) &\overset{(2.12)}{=} (\pi^{'}(P_mC_k)\alpha,\beta) \\
&= (\pi^{'}(C_k)\alpha,\pi^{'}(P_m^*)\beta) \\
&= (\pi^{'}(C_k)\alpha,\pi^{'}(P_m)\beta) \\
&= (\pi^{'}(C_k)\alpha,\beta). \qquad (2.18)
\end{aligned}$$

By above argument, we have

$$\pi^{'}(C_k) := \lim_{n_1,\cdots,n_k\to\infty}\pi^{'}((n_1,n_2,\cdots,n_k)). \qquad (2.19)$$

Let us consider the following operator

$$\frac{1}{n!}\sum_{g\in K_m(n)}\pi^{'}(g)$$

where $K_m(n)=K_m\cap K(n)$. Let α be any vector in the space of $K_m(n)-$invariants. Then

$$\begin{aligned}
\frac{1}{n!}\sum_{g\in K_m(n)}\pi^{'}(g)\alpha &= \frac{1}{n!}\sum_{g\in K_m(n)}\alpha \\
&= \frac{1}{n!}.n!\alpha \qquad (2.20) \\
&= \alpha.
\end{aligned}$$

Since the group $K_m(n)$ has $n!$ elements. It is reasonable to write the operator $\pi^{'}(P_m)$ in the following form

$$\pi^{'}(P_m) = \lim_{n\to\infty}\frac{1}{n!}\sum_{g\in K_m(n)}\pi^{'}(g). \qquad (2.21)$$

The groups $K_m(n)$ are in the form of increasing chain of subgroups. Namely

$$K_m(0) \subseteq K_m(1) \subseteq K_m(2) \subseteq \cdots$$

Therefore, the subspaces of $K_m(n)-$invariants $n = 0, 1, 2, ...$ has decreasing chain of subspaces

$$H(\pi)_{m,0} \supseteq H(\pi)_{m,1} \supseteq H(\pi)_{m,2} \supseteq \cdots$$

and moreover, the subspace $H(m)_m$ is the intersection of above subspaces.

$$H(\pi)_m = \bigcap_n H(\pi)_{m,n}.$$

By (2.19) and previous theorem, we have $\pi(P_m)\alpha = \alpha$, So the projection onto the subspace $H(\pi)_m$ is the limit of projection operators onto the subspaces $H(\pi)_{m,n}$.

We now introduce a significant semigroup structure on double cosets $K_n^D \backslash G/K_n^D$, where $n = 1, 2, \cdots$. For $n = 0$ the set $\Gamma^{'}(0)$ is equal to the double coset $K\backslash G/K$. We will later show that, the semigroup $\Gamma^{'}(0)$ is isomorphic to Δ where Δ denotes the conjugacy classes of the infinite symmetric group . Let θ_n be the natural projection from G to the double coset $K_n^D\backslash G/K_n^D$. We also set $G(m) = S(m) \times S(m) \subset G$ and $K(m) = \operatorname{diag} S(m) \subset K$, $m = 1, 2, ...$ Let us consider the decreasing chain of subsets

$$S(\infty) = W \supset W_1 \supset \cdots$$

where $W_m = \{s \in S(\infty) | s(i) > m, i = 1, 2, \cdots , m\}$.

Lemma 1. *The subset W_m is equal to the set $S_m(\infty)w_mS_m(\infty)$ for any $m = 1, 2, ..$ where $w_m \in S(2m) \subset S(\infty)$ and*

$$w_m(i) = \begin{cases} i + m, & \mathit{if}\, i = 1, 2, \cdots , m, \\ i - m, & \mathit{if}\, i = m + 1, \cdots , 2m, \\ i, & \mathit{if}\, i > m. \end{cases} \tag{2.22}$$

Proof. Let s be any element of W_m. Then there exists $g \in S_m(\infty)$ such that

$g(s(i)) = m + i$, $\forall i \leq m$. By putting the element $g(s(i)) = m + i$ into w_m, we obtain $w_m(g(s(i)) = i, \forall i \leq m$. Then $w_m gs = h \in S_m(\infty)$. Since, $w_m^{-1} = w_m$ we have $s = gw_m h \in S_m(\infty)w_m S_m(\infty)$. Conversely, take any element x from $S_m(\infty)w_m S_m(\infty)$. Then $x = hw_m g$ where $h, g \in S_m(\infty)$. If $i \leq m$, then $x(i) = (hw_m)g(i) = h(w_m(i)) = h(i + m) > m$. Therefore $x \in W_m$. Hence, $W_m = g \in S_m(\infty)$.

□

Moreover, we consider the endormorphism $S(\infty) \longrightarrow S_n(\infty)$ induced by shifting i to $i + 1$. By this endomorphism, the images of the set W_m and the element w_m are the set $W_m^{(n)}$ and the element $w_m^{(n)}$. It is obvious that

$$W_m^{(n)} = S_{m+n}(\infty)w_m^{(n)} S_{m+n}(\infty). \tag{2.23}$$

In a similar way, the above set can be written as

$$W_m^{(n)} = \{s \in S(\infty) | s(i) = i \quad \forall i \leq n, s(i) > m + n \quad \forall i \in n + 1, \cdots, m + n\},$$

$$w_m(i) = \begin{cases} i + m, & \text{if } i = n + 1, \cdots, n + m, \\ i - m, & \text{if } i = n + m + 1, \cdots, n + 2m, \\ i, & \text{if } i \leq n \text{ or } i > n + 2m. \end{cases} \tag{2.24}$$

Next, we will talk about the structure of the subsemigroup $\Gamma'(n)$.

Theorem 4. *(Ol'shanskij (1990), p. 990) For any $n = 0, 1, \cdots$ the set $\Gamma'(n)$ has the structure of a semigroup uniquely defined as follows:*

$$g \in G(n + m) \quad \textit{and} \quad h \in G(n + m) \Rightarrow \theta_n(g)\theta_n(h) = \theta_n(gw_m^{(n)}h).$$

Proof. Firstly, we will show that the given semigroup multiplication is well-defined. Let p be the smallest positive integer such that g and h are in $G(n + p)$. If $m \geq p$ and

$w \in W_m^{(n)}$. Then $\theta_n(gwh) = \theta_n(gxw_m^{(n)}yh)$ for some $x, y \in S_{m+n}(\infty)$. Since $g, h \in G(n+p)$ and $x, y \in S_{m+n}(\infty)$, g and h commute with K_{m+n}. Then $\theta_n(xgw_m^{(n)}hy) \in K_n^D xgw_m^{(n)}hyK_n^D = K_n^D gw_m^{(n)}hK_n^D$. It shows that, the multiplication does not depend on the choice of w. SÍnce $m \geq p$, the element $w_m^{(n)} \in W_p^{(n)}$. Then $\theta_n(gw_m^{(n)}h) = \theta_n(gw_p^{(n)}h) \quad \forall m \geq n$. Thus the element $\theta_n(gw_m^n h)$ does not depend on m. If $m \geq p$, one can denote this element $\gamma(g, h)$. It is obvious that, $\gamma(ug, h) = \gamma(g, hu) = \gamma(g, h)$ for all $u \in K_n^D$. We also show that,

$$\gamma(gu, h) = \gamma(g, uh) = \gamma(g, h) \quad \text{for all} \quad u \in K_n^D. \tag{2.25}$$

It is sufficient to show $uW_m^{(n)} = W_m^{(n)}u = W_m^{(n)}$. Let us choose m such that $u \in K^D(n+m)$. For any $s \in W_m^{(n)}$, we have $su \in s \in W_m^{(n)}$. Thus $uW_m^{(n)} \subset W_m^{(n)}$. Converse of this is also clear, thus we have $uW_m^{(n)} = W_m^{(n)}$. Above arguments shows that, the element $\gamma(g, h)$ depends only on $\theta_n(g)$ and $\theta_n(h)$.

Secondly, we should show that the given multiplication is associative. Let us consider the elements of symmetric group as a matrices. Let k and l be positive integers. We can define a map $(s, t) \longmapsto st$ which is from $S(n+k) \times S(n+l)$ into $S(n+k+l)$. We can write elements $t \in S(n+l)$ as matrices of size $(n+l) \times (n+l)$ and entries of matrices consist of 0 and 1. We consider this matrices as block matrices such that the block t_{11} is $n \times n$, the block t_{22} is $l \times l$ and the blocks t_{12} and t_{21} are $n \times l$ and $n \times l$ respectively. Similarly, one can also construct a matrix for the element $s \in S(n+k)$ such that $[s_{ij}]$ are block matrices where $1 \leqslant i, j \leqslant 2$. For the elements $s \in (n+k)$ and $t \in S(n+l)$, we have

$$st = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_{11} & 0 & t_{12} \\ 0 & 1 & 0 \\ t_{21} & 0 & t_{22} \end{pmatrix} \tag{2.26}$$

We can write the element $w_p^{(n)}$ as a block matrix as follows

$$w_p^{(n)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in S(n+2p)$$

Now, let us take any element $g = (s,t) \in G(n+k)$ and $h = (s^{'}, t^{'}) \in G(n+l)$. Then we can embed elements s and $s^{'}$ into $G(n+2p)$ such that $\alpha(s)$ and $\alpha(s^{'})$ are in $G(n+2p)$. Let

$$\alpha(s) = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha(s) = \begin{pmatrix} a^{'} & b^{'} & 0 \\ c^{'} & d^{'} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then by multiplying following matrices, we obtain $ss^{'} = \alpha(s)\alpha(s^{'}) = sw_p^{(n)}s^{'}w_p^{(n)}$. Similary, for the elements t and $t^{'}$, we have $tt^{'} = tw_p^{(n)}t^{'}w_p^{(n)}$. Then

$$\begin{aligned} \theta_n(gh) &= \theta_n(ss^{'}, tt^{'}) \\ &= \theta_n(sw_p^{(n)}s^{'}w_p^{(n)}, tw_p^{(n)}t^{'}w_p^{(n)}) \\ &= \theta_n(sw_p^{(n)}s^{'}, tw_p^{(n)}t^{'}). \end{aligned} \tag{2.27}$$

On the other hand, we also have

$$\begin{aligned} \theta_n(g)\theta_n(h) &= \theta_n(s,t)\theta_n(s^{'}, t^{'}) \\ &= \theta_n((s,t)w_p^{(n)}(s^{'}, t^{'})) \\ &= \theta_n((s,t)w_p^{(n)}w_p^{(n)}(s^{'}, t^{'})) \\ &\overset{(w_p^{(n)})^2 = w_p^{(n)}}{=} \theta_n((sw_p^{(n)}, tw_p^{(n)})(w_p^{(n)}s^{'}, w_p^{(n)}t^{'})) \\ &= \theta_n((sw_p^{(n)}w_p^{(n)}s^{'}), (tw_p^{(n)}w_p^{(n)}t^{'})) \\ &= \theta_n(sw_p^{(n)}s^{'}, tw_p^{(n)}t^{'}). \end{aligned} \tag{2.28}$$

By using (2.26) and (2.27), we obtain $\theta_n(gh) = \theta_n(g)\theta_n(h)$. Since the multiplication (2.25) is associative, the multiplication in $\Gamma^{'}(n)$ is also associative. Therefore, the structure $\Gamma^{'}(n)$ is a semigroup. Moreover, there is an involution on this group defined by $\theta_n(g)^* = \theta_n(g^{-1}), g \in G$. $\square$

Let us fix any admissible representation π of the pair (G, K). The smallest positive integer that makes $H(\pi)_n \neq 0$. This integer is called the depth of the representation π. Assume that the depth of π is n. Let P_n be the orthogonal projection from $H(\pi)$ into the subspace $H(\pi)_n$. The operator $P_n\pi(g)P_n$ depends only on $\theta_n(g) \in \Gamma'(n)$. More precisely, we denote its restriction onto the subspace $H(\pi)_n$ by π_n. π_n is a map from $\Gamma'(n)$ into $\Gamma'(H(\pi)_n)$.

Theorem 5. *(Ol'shanskij (1990), p. 991) π_n is a representation of the semigroup $\Gamma'(n)$ in the space $H(\pi)_n$ for any n such that $H(\pi)_n \neq 0$.*

Let π be an irreducible admissible representation of the pair $(G^D, K^D) \cong (G, K)$. We know that, it is possible to reconstruct an irreducible representation if we know the at least one of matrix element $\varphi_\pi(g) = (g\alpha, \alpha)$ for all $\alpha \in G$. We define a subsemigroup of Γ as follows:

$$\Gamma(n) = P_n \Gamma P_n \subset \Gamma.$$

This subsemigroup $\Gamma(n)$ acts on the subspace $H(\pi)_n$. Let α be any vector in $H(\pi)_n$. Then

$$\begin{aligned}
(P_n g P_n \alpha, \alpha) &= (P_n(gP_n\alpha), \alpha) \\
&= (gP_n\alpha, P_n^*\alpha) \\
&= (gP_n\alpha, P_n\alpha) \\
&= (g\alpha, \alpha).
\end{aligned} \tag{2.29}$$

Since $(P_n g P_n \alpha, \alpha) = (g\alpha, \alpha)$ for all $\alpha \in H(\pi)_n$, it is enough to know representation of the subsemigroup $\Gamma(n)$ for one of values of n. Let π_n denotes the representation of $\Gamma(n)$ on $H(\pi)_n$. By previous theorem π_n is $*$-representation of the subsemigroup $\Gamma(n)$ on the subspace $H(\pi)_n$. The irreducibility of the representation π_n follows from the irreducibility of the representation π. Assume n is the depth of the representation. By the definition of depth, $H(\pi)_m = 0$ for all $m < n$. Then we have $\pi(P_m) = 0$ for all $m < n$. Let us consider the subsemigroup $\Gamma(n)^\circ$ of $\Gamma(n)$ defined by the elements

$$\begin{aligned}
&A_i P_n, \quad |i| \leq n,\\
&C_k P_n \quad k \geq 2,\\
&g P_n, \quad g \in G^D(n).
\end{aligned}$$

One can easily see that, the elements of $\Gamma(n)^\circ$ only can act on the subspace $H(\pi)_n$ as non-zero operators. The subsemigroup $\Gamma(n)^\circ$ is isomorphic to the semigroup $(S(n) \ltimes \mathbb{Z}_+^n) \times (S(n) \ltimes \mathbb{Z}_+^n) \times \mathbb{Z}_+^\infty$. In general theory, we have following results. Irreducible representations of $\Gamma(n)^\circ$ are easy to find. Assume $p = (p_1, \cdots, p_s)$ is a partition of n and the numbers $x_1, \cdots, x_s$ are the set of distinct points which are in the interval $[-1, 1]$. Let N be a representation of the semigroup $\mathbb{Z}_+^n$ that sends p_i generators into the multiplication by x_i for all $i = 1, \cdots, s$. One can see that, $S(p) = \prod_{i=1}^s S(p_i)$ is the centralizer of the representation N in the symmetric group $S(n)$. Let $\lambda_1, \cdots, \lambda_s$ be a set of Young diagrams such that each $|\lambda_i|$ equals to p_i. The representation of $S(n) \ltimes \mathbb{Z}_+^n$ is induced by the representation $(\otimes_{i=1}^s \pi_{\lambda_i}) \otimes N$ of the subsemigroup $S(p) \ltimes \mathbb{Z}_+^n \subset S(n) \ltimes \mathbb{Z}_+^n$ is irreducible. These representations contains all irreducible $*$-representations of $S(n) \ltimes \mathbb{Z}_+^n$. Moreover, their representaions are contractions. It is convenient to represent the collection of points $x_1, \cdots, x_s$ and $\lambda_1, \cdots, \lambda_s$ as a Young distribution. A young distrubition is a function Λ which sends to points of the interval $[-1, 1]$ to Young diagrams. Note that, $\Lambda(x) = \emptyset$ for all but infinitely many x. We can set

$$|\Lambda| = \sum_x |\Lambda(x)|$$

and

$$\text{supp}\Lambda = \{x | \Lambda(x) \neq \emptyset\}.$$

We can determine the representation of the semigroup $(S(n) \ltimes \mathbb{Z}_+^n) \times (S(n) \ltimes \mathbb{Z}_+^n))$ by a pair of Young distributions Λ, M such that the number of elements of Λ and M

are equal to n. Since the operators C_n's are $*$-stable and central, thus we have

$$\pi(C_n) = c_n, \tag{2.30}$$

where the points c_n's belongs to the interval $[-1, 1]$. Now, the classification problem for the admissible representation $G \cong G^D$ becomes the problem of description of all parameters M, Λ and $\{c_n\}$ if this representation exists.

Let $r_k(s)$ denote the number of cycles of the length of the permutation $s \in S(\infty)$ for $k = 2, 3, \cdots$. We can characterize class of arbitrary permutation s by the sequence of nonnegative integers defined by $\{r_k(s)|k = 2, 3, \cdots\}$. We consider this sequence as a element of the semigroup $\oplus_{k=2}^{\infty}\mathbb{Z}_+$ where $\mathbb{Z}_+$ is the semigroup of nonnegative integers under addition.

Lemma 2. *The semigroup* $\Gamma^{'}(0)$ *is isomorphic to the semigroup* Δ.

Proof. Define a map from G to K such that $f(s, t) = st^{-1}$. Obviously, this map well defined, homomorphism and onto. Consider the restriction of f on $\Gamma^{'}(0)$. It is a one to one map from $\Gamma^{'}(0)$ to Δ. To show this, take any elements st^{-1} and xy^{-1} from the same conjugacy classes. Then there exists $k \in K$ such that $kxy^{-1}k^{-1} = st^{-1}$. Then $kxy^{-1}k^{-1}t = s$ and $kyy^{-1}k^{-1}t = t$. It shows that x, y are in the same conjugacy class and also y, t are in the same conjugacy class. Therefore, the restriction map is one to one. Hence , $\Gamma^{'}(0)$ is isomorphic to Δ. □

Observe that, the semigroup $\Gamma(0)$ is constant on the elements which have the same cycle type because of (2.7). Therefore, $\Gamma(0)$ is isomorphic to Δ as well. By this observation and previous lemma, we have $\Gamma^{'}(0) \cong \Gamma(0)$.

CHAPTER THREE
THOMA'S THEOREM

In this chapter, we will mention about spectral properties of the operators A_i and what the characters of the group $S(\infty)$ are. First of all, let π be a spherical representation of the pair (G^D, K^D). It is an irreducible representation of G^D with the property $H(\pi)_0 \neq 0$. By using Olshanski's theorem which we defined in previous chapter, π can be canonically extended a $*$-representation of the semigroup $\Gamma(0)$. For simplicity, we denote the extension of π with the same letter. This semigroup contains the elements of the form

$$\prod_i C_i^{k_i} P_0.$$

It is easy to see that, the semigroup above is commutative. Thus it has only 1-dimensional irreducible representations. Namely,

$$\dim H(\pi)_0 = 1.$$

Since $H(\pi)_0 \neq 0$, there is a unique K^D-invariant vector α up to a scalar with the norm $||\alpha|| = 1$. α is called the spherical vector. The spherical function

$$\varphi_\pi(g) = (\pi(g)\alpha, \alpha) \tag{3.1}$$

is independent from the choice of a spherical vector. Because, the spherical vector is unique up to a scalar. By a simple observation, the spherical function is constant on the double cosets $K^D \backslash G^D / K^D$.

$$\begin{aligned}
\varphi_\pi(kgk') &= (\pi(kgk')\alpha, \alpha) \\
&= (\pi(k)\pi(g)\pi(k')\alpha, \alpha) \\
&= (\pi(g)\pi(k')\alpha, \pi^*(k)\alpha) \\
&= (\pi(g)\pi(k')\alpha, \pi^{-1}(k)\alpha) \\
&= (\pi(g)\pi(k')\alpha, \pi(k^{-1})\alpha) \\
&= (\pi(g)\alpha, \alpha) \\
&= \varphi_\pi(g)
\end{aligned} \tag{3.2}$$

where k and k' are in K^D. Let σ be an element of $S(\infty)$ which has only one nontrivial cycle of length n.

$$\begin{aligned}
\varphi_\pi(\sigma) &= (\pi(\sigma)\alpha, \alpha) \\
&= (\pi(P_0 \sigma P_0)\alpha, \alpha) \\
&\overset{(2.7)}{=} (\pi(P_0 C_n)\alpha, \alpha) \\
&= (\pi(C_n)\alpha, \pi(P_0^*)\alpha) \\
&= (\pi(C_n)\alpha, \pi(P_0)\alpha) \\
&= (\pi(C_n)\alpha, \alpha) \\
&= c_n.
\end{aligned} \tag{3.3}$$

If σ has more than one cycle, then we have

$$\begin{aligned}
\varphi_\pi(\sigma) &= (\pi(\sigma)\alpha, \alpha) \\
&= (\pi(P_0\sigma P_0)\alpha, \alpha) \\
&\overset{(2.7)}{=} \left(\prod_{n\in[\sigma]} \pi(C_nP_0)\alpha, \alpha\right) \\
&= \left(\prod_{n\in[\sigma]} \pi(C_n)\alpha, \alpha\right) \\
&= \prod_{n\in[\sigma]} c_n.
\end{aligned} \tag{3.4}$$

3.1 Spectral Properties of A_i's in Spherical Representations of (G^D, K^D)

Let π be a spherical representation of (G^D, K^D). For simlicity, we will write g instead of $\pi(g)$. For the operator A_1 and the spherical vector α, μ denotes the spherical measure. Since $A_1^* = A_1$ and $||A_1|| \leq 1$, the spectrum of A_1 takes values between -1 and 1. By the properties of the operator A_1, it has spectral representation by the virtue of the ninth chapter of Kreyszig (1978),

$$\int_{[-1,1]} t^k d\mu = (A_1^k\alpha, \alpha) = (P_0A_1^kP_0\alpha, \alpha) = c_{k+1}, \tag{3.5}$$

where c_{k+1} is the moment of the measure.

Lemma 3. *(Okounkov (1994), Lemma 1)* $\lambda = \mu^{\otimes\infty}$.

Proof. In order to show that, it is enough to verify the given equality for the integrals of all monomials. It is possible since, $||A_i|| \leq 1$ for all i.

$$\begin{aligned}
\int_{[-1,1]^\infty} t_1^{k_1}.t_2^{k_2}\cdots t_l^{k_l}\,d\lambda &\overset{(3.5)}{=} (A_1^{k_1}A_2^{k_2}...A_l^{k_l}\alpha,\alpha) \\
&= (P_0A_1^{k_1}A_2^{k_2}...A_l^{k_l}P_0\alpha,\alpha) \\
&\overset{(2.6)}{=} \left(P_0\prod_i C_{k_i+1}\alpha,\alpha\right) \\
&= \left(\prod_i C_{k_i+1}\alpha,\alpha\right) \\
&= \prod_i c_{k_i+1} \\
&= \prod_i \int_{[-1,1]} t^{k_i}\,d\mu.
\end{aligned} \tag{3.6}$$

□

The spherical function of the pair (G^D, K^D) can be uniquely represented by the measure μ. Let σ be an element of $S(\infty)$. By above lemma, we have

$$\varphi^\mu(\sigma) \overset{(3.4)}{=} \prod_{k\in[\sigma]} c_k = \prod_{k\in[\sigma]} \int_{[-1,1]} t^{k-1}\mu(dt) \tag{3.7}$$

We now show some important properties of the measure μ.

Theorem 6. *(Okounkov (1994), Theorem 1) The measure μ is discrete, and its atoms only accumulated at the point zero.*

Proof. Firstly, we will show the first part of the theorem. Let us consider any Borel subset S of $[\varepsilon, 1]$ where $\varepsilon > 0$ and χ_S denotes the characteristic function of the set S. It is sufficient to show the inequality $\varepsilon\mu(S) \leq \mu(S)^2$. For this purpose, we will show the following two sided inequality

$$\varepsilon\mu(S) \leq (s\chi_S(A_1)\alpha, \chi_S(A_1)\alpha) \leq \mu(S)^2 \tag{3.8}$$

where $s = (12) \in S(\infty)$ and α is the spherical vector. The expression which is in the middle is real, because

$$\begin{aligned}
(s\chi_S(A_1)\alpha, \chi_S(A_1)\alpha) &= (s\chi_S(A_1)\alpha, \chi_S(A_1)\alpha) \\
&= (\chi_S(A_1)\alpha, s^{-1}\chi_S(A_1)\alpha) \qquad (3.9) \\
&= (\chi_S(A_1)\alpha, s\chi_S(A_1)\alpha).
\end{aligned}$$

By (3.9) and the property of inner product, we have

$$\begin{aligned}
(s\chi_S(A_1)\alpha, \chi_S(A_1)\alpha) &= \overline{(\chi_S(A_1)\alpha, s\chi_S(A_1)\alpha)}, \\
&= (\chi_S(A_1)\alpha, s\chi_S(A_1)\alpha).
\end{aligned} \tag{3.10}$$

For the left side of the inequality, we have

$$\begin{aligned}
(s\chi_S(A_1)\alpha, \chi_S(A_1)\alpha) &= (s\chi_S(A_1)P_1\alpha, \chi_S(A_1)P_1\alpha) \\
&= ((\chi_S(A_1)P_1)^* s\chi_S(A_1)P_1\alpha, \alpha) \\
&\overset{(2.3)}{=} (\chi_S(A_1)P_1 s P_1 \chi_S(A_1)\alpha, \alpha) \\
&\overset{(2.5)}{=} (\chi_S(A_1)P_1 A_1 P_1 \chi_S(A_1)\alpha, \alpha) \\
&= (A_1\chi_S(A_1)^2\alpha, \alpha) \\
&= (A_1\chi_S(A_1)\alpha, \alpha) \\
&= \int_S t d\mu \\
&\geq \int_S \varepsilon d\mu = \varepsilon\mu(S).
\end{aligned} \tag{3.11}$$

For the right side of the inequality, we know A_1 and A_2 commutes each other. $\chi_S(A_1)$

and $\chi_S(A_2)$ are orthogonal projections. Thus,

$$\begin{aligned}
\chi_S(A_1)s\chi_S(A_1) &= \chi_S(A_1)^2 s\chi_S(A_1)^2 \\
&= \chi_S(A_1)^2 s\chi_S(A_1)s^{-1}s\chi_S(A_1) \\
&\overset{(2.2)}{=} \chi_S(A_1)^2\chi_S(A_2)s\chi_S(A_1) \\
&= \chi_S(A_1)^2\chi_S(A_2)s\chi_S(A_1)s^{-1}s \\
&= \chi_S(A_1)^2\chi_S(A_2)^2 s \\
&= \chi_S(A_1)\chi_S(A_2)s\chi_S(A_1)\chi_S(A_2)
\end{aligned} \tag{3.12}$$

we now have,

$$\begin{aligned}
(s\chi_S(A_1)\alpha, \chi_S(A_1)\alpha) &= (\chi_S(A_1)s\chi_S(A_1)\alpha, \alpha) \\
&= (\chi_S(A_1)\chi_S(A_2)s\chi_S(A_1)\chi_S(A_2)\alpha, \alpha) \\
&= (s\chi_S(A_1)\chi_S(A_2)\alpha, \chi_S(A_1)\chi_S(A_2)\alpha).
\end{aligned} \tag{3.13}$$

Say $\chi_S(A_1)\chi_S(A_2)\alpha = \beta$ and since $||s|| \leq 1$, we have

$$(s\beta, \beta) \leq ||s||.||\beta||^2 \leq ||\beta||^2 = \left(\int_S d\mu\right)^2 = \mu(S)^2. \tag{3.14}$$

Therefore, we obtain $\varepsilon\mu(S) \leq \mu(S)^2$. Then $\mu(S)$ is either equal to 0 or greater than equal to ε. In a similar way, this inequality holds for $S \subset [-1, \varepsilon]$. For any borel set $E \subset [\varepsilon, 1]$, if we divide E in the middle we have a new set $E^{'}$, then $\mu(E^{'}) \geqslant \varepsilon$ or $\mu(E^{'}) = 0$. Continuing in this way, we see that the measure μ consists of singletons thus μ is discrete. Let $B_1, B_2, ..., B_n$ be Borel subsets of the interval $[\varepsilon, 1]$. Then

$$\mu(B_1 \cup B_2 \cup ... \cup B_n) = \sum_{k=1}^{n} \mu(B_k) \leqslant 1. \tag{3.15}$$

On the other hand, we have

$$\varepsilon.n = \sum_{k=1}^{n} \varepsilon \leqslant \sum_{k=1}^{n} \mu(B_k) \leqslant 1. \tag{3.16}$$

Therefore, we obtain $n \leqslant 1/\varepsilon$. This shows there are at most $1/\varepsilon$ of its atoms in the interval $[\varepsilon, 1]$. Hence, its atoms only accumulate at zero. $\square$

Let π be an irreducible representaion of the pair (G, K) which has depth n and π is determined by the parameters Λ, M and c_k. Its is now convenient to consider that, π is not defined by the numbers c_k. It is defined by the spectral measure μ of the operator A_{n+1} and an arbitrary unit vector α which is in the subspace $H(\pi)_n$. Because

$$\begin{aligned} \int t^k d\mu = (A_{n+1}\alpha, \alpha) &= (P_n A_{n+1}^k P_n \alpha, \alpha) \\ &\overset{(2.6)}{=} (C_{k+1}\alpha, \alpha) \\ &= c_{k+1}. \end{aligned} \tag{3.17}$$

This measure corresponds to a spherical representation of the pair $(G, K) \cong (G_n, K_n)$. Therefore, it is a discrete measure. Since the spectrum of the operators A_i's, for $i = 1, 2, ..., n$ in the subspace $H(\pi)_n$ is supported in the interval $[-1, 1]$ and by the definition of Young distribution Λ. The spectrum of A_i's coincides with suppΛ. Let V denote the set suppμ of the measure μ.

Lemma 4. *(Okounkov (1994), Lemma 2) supp*$\Lambda \subset V \cup \{0\}$ *and supp*$M \subset V \cup \{0\}$.

Proof. Let x be any element of supp$\Lambda \setminus \{0\}$ and let α be a vector in the subspace $H(\pi)_n$ such that $A_1\alpha = x\alpha$ and $||\alpha|| = 1$. δ_x denotes the function which is equal to 1 at the point x and to 0 except for the point x. Let $s = s_{1,n+1}$. By the sixth chapter of Birman & Solomjak (2012), we have

$$\delta_x(A_1)\alpha = \alpha. \tag{3.18}$$

Then

$$
\begin{aligned}
0<|x|=|(x\alpha,\alpha)| &= |(A_1\alpha,\alpha)| \\
&\overset{(2.5)}{=} |(P_n s P_n\alpha,\alpha)| \\
&= |(s\alpha,\alpha)| \\
&= |(s\delta_x(A_1)\alpha,\delta_x(A_1)\alpha)| \\
&= |(\delta_x(A_1)s\delta_x(A_1)\alpha,\alpha)| \\
&\overset{(2.2)}{=} |(s\delta_x(A_{n+1})\alpha,\delta_x(A_{n+1})\alpha)|.
\end{aligned}
\tag{3.19}
$$

Since $||s|| \leq 1$, we have

$$
\begin{aligned}
|(s\delta_x(A_{n+1})\alpha,\delta_x(A_{n+1})\alpha)| &\leq |(\delta_x(A_{n+1}^2)\alpha,\alpha)| \\
&= |(\delta_x(A_{n+1})\alpha,\alpha)| \\
&= \int \delta_x d\mu(x) = \mu(x).
\end{aligned}
\tag{3.20}
$$

By (3.19) and (3.20), we have $0<|x|\leq \mu(x)$. If x is not in suppΛ. Then there is an open set X containing x such that $\mu(X)=0$. Since $\{x\}\subset X$, we have $\mu(x)=0$. But, it contradicts with the fact $\mu(x)>0$. Therefore $x\in V$. Similar argument is valid for M. $\square$

3.2 Thoma's Theorem

Let π be an spherical representation of the pair (G,K). μ denotes the spectral measure we have defined earlier. Let $B=\{a_i\}\cup\{b_i\}$ where a_i and b_i are all nonzero elements of V. We set $\nu(x) = \mu(x)/|x|$ for $|x| \neq 0$. Then we can write

$$\mu = \sum_{i=1}^{\infty} \mu(a_i)\delta_{a_i} + \sum_{i=1}^{\infty} \mu(-b_i)\delta_{-b_i}. \tag{3.21}$$

By using (3.5) and (3.21), we have

$$\begin{aligned}
c_k = \int t^{k-1} d\mu &= \int t^{k-1} d\left(\sum_{i=1}^{\infty} \mu(a_i)\delta_{a_i} + \sum_{i=1}^{\infty} \mu(-b_i)\delta_{-b_i}\right) \\
&= \int t^{k-1} d\left(\sum_{i=1}^{\infty} \mu(a_i)\delta_{a_i}\right) + \int t^{k-1} d\left(\sum_{i=1}^{\infty} \mu(-b_i)\delta_{-b_i}\right) \\
&= \sum_{i=1}^{\infty} \mu(a_i)a_i^{k-1} + \sum_{i=1}^{\infty} \mu(-b_i)(-b_i)^{k-1} \qquad (3.22) \\
&= \sum_{i=1}^{\infty} \frac{\mu(a_i)}{|a_i|} a_i^k + \sum_{i=1}^{\infty} \frac{\mu(-b_i)}{|-b_i|}(-b_i)^k \\
&= \sum_{i=1}^{\infty} \nu(a_i)a_i^k + \sum_{i=1}^{\infty} \nu(-b_i)(-b_i)^k
\end{aligned}$$

for every $k > 1$.

Theorem 7. *(Okounkov (1994), Theorem 2) All the numbers $\nu(a_i)$ and $\nu(-b_i)$ are integers.*

Before we give the proof of the above theorem, we need some identities. Let $O(\sigma)$ denote the set of orbits of σ on the set $\mathbb{N}$.

Lemma 5. *Let σ be an element of $S(\infty)$ and for $p \in O(\sigma)$, set $\Sigma(p) = \sum_{i \in p} k_i$. Then the following identity*

$$P_0 \sigma \prod A_i^{k_i} P_0 = P_0 \prod_{p \in O(\sigma)} C_{|p| + \Sigma(p)} \tag{3.23}$$

holds in the semigroup Γ.

Proof. We will not give the complete proof of the lemma. In order to understand how above equality works. Let us consider a simple example. Let $\sigma = (1234)$ then we illustrate the product $P_0\sigma A_1 A_2^2 A_3^3 A_4^4 P_0$ as follows:

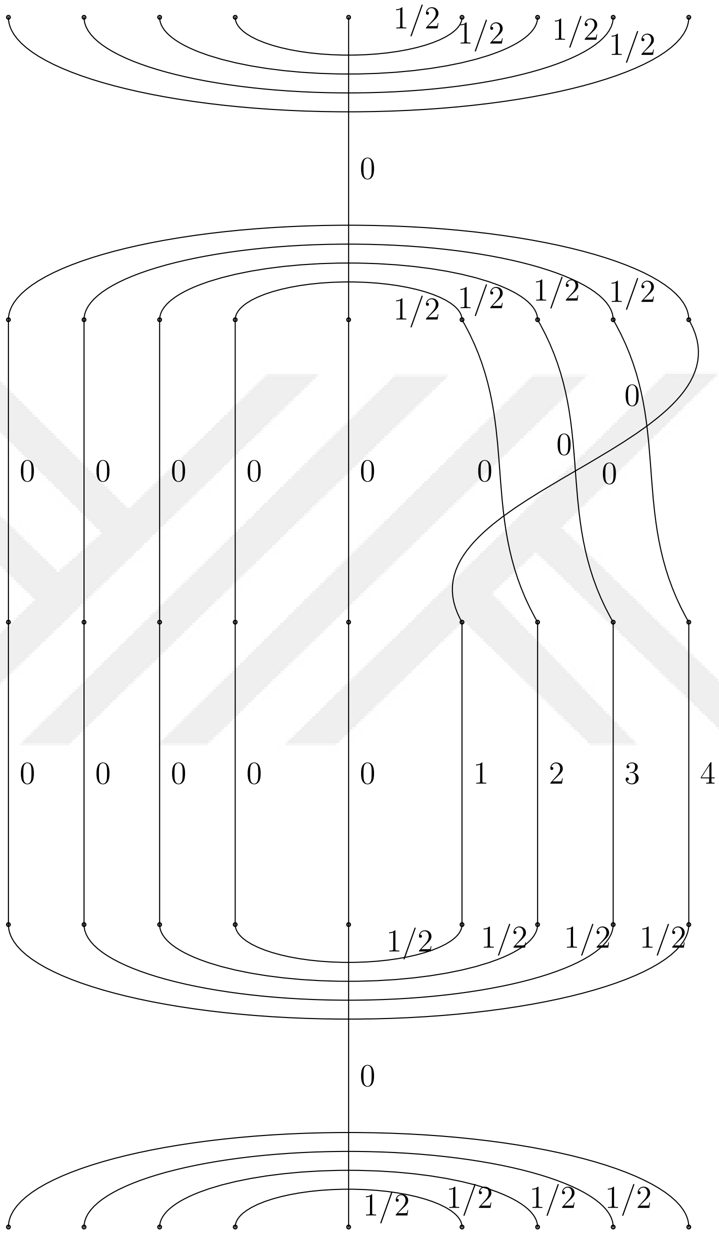


Figure 3.1 The product $P_0\sigma A_1 A_2^2 A_3^3 A_4^4 P_0$

Thus, the product $P_0\sigma A_1 A_2^2 A_3^3 A_4^4 P_0$ equals to $\prod C_{|p|+\Sigma(p)}$ where $|p| = |\{1,2,3,4\}| = 4$ and $\Sigma(p) = 10$. Since we work in Γ, loops whose lengths are one does not appear in the product. If σ had two disjoint cycles then the product would have two disjoint loops and we would reach the result in a similar way. $\square$

Lemma 6. *(Okounkov (1994), Lemma 3) Let $f_i(t)$ and $g_i(t)$, $i =; 1, 2, ...,$ be continuous functions on the interval $[-1, 1]$ such that all of them except for a finite number, be identically equal to 1. Let α be a spherical vector with the length $||\alpha|| = 1$. Then for any $\sigma \in S(\infty)$, we have*

(a)

$$\left(\sigma \prod_{i=1}^{\infty} f_i(A_i)\alpha, \alpha\right) = \prod_{p \in O(\sigma)} \int t^{|p|-1} \prod_{i \in p} f_i(t) d\mu, \tag{3.24}$$

(b)

$$\left(\sigma \prod_{i=1}^{\infty} f_i(A_i)\alpha, \prod_{i=1}^{\infty} g_i(A_i)\alpha\right) = \prod_{p \in O(\sigma)} \int t^{|p|-1} \prod_{i \in p} f_i(t)\overline{g_i(t)} d\mu. \tag{3.25}$$

Proof. (a) Let $f_i(t)$ be continuous function defined by $f_i(t) = t^{k_i}$ for $i = 1, 2, ...,$ such that $k_i = 0$. Then

$$\begin{aligned}
\left(\sigma \prod_{i=1}^{\infty} f_i(A_i)\alpha, \alpha\right) &= \left(P_0 \sigma \prod_{p \in O(\sigma)} A_i^{k_i} P_0 \alpha, \alpha\right) \\
&\overset{(3.23)}{=} \left(P_0 \prod_{p \in O(\sigma)} C_{|p|+\Sigma(p)} \alpha, \alpha\right) \\
&= \left(\prod_{p \in O(\sigma)} C_{|p|+\Sigma(p)} \alpha, \alpha\right) \\
&\overset{(3.6)}{=} \prod_{p \in O(\sigma)} \int t^{|p|+\Sigma(p)-1} d\mu \\
&= \prod_{p \in O(\sigma)} \int t^{|p|-1} \prod_{i \in p} t^{k_i} d\mu \\
&= \prod_{p \in O(\sigma)} \int t^{|p|-1} \prod_{i \in p} f_i(t) d\mu.
\end{aligned} \tag{3.26}$$

(b) Let $f_i(t)$,$g_i(t)$ be continious functions on the interval $[-1,1]$ and they have properties as we define in proposition. Then

$$\begin{aligned}
\left(\sigma\prod_{i=1}^{\infty} f_i(A_i)\alpha, \prod_{i=1}^{\infty} g_i(A_i)\alpha\right) &= \left(\prod_{i=1}^{\infty}(g_i(A_i))^*\sigma\prod_{i=1}^{\infty} f_i(A_i)\alpha, \alpha\right) \\
&= \left(\sigma\sigma^{-1}\prod_{i=1}^{\infty}(g_i(A_i))^*\sigma\prod_{i=1}^{\infty} f_i(A_i)\alpha, \alpha\right) \\
&\overset{(2.2)}{=} \left(\sigma\prod_{i=1}^{\infty}(g_{\sigma^{-1}(i)}(A_{\sigma^{-1}(i)}))^*\prod_{i=1}^{\infty} f_i(A_i)\alpha, \alpha\right) \\
&= \left(\sigma\prod_{i=1}^{\infty}(g_i(A_i))^*\prod_{i=1}^{\infty} f_i(A_i)\alpha, \alpha\right) \\
&= \left(\sigma\prod_{i=1}^{\infty}(g_i(A_i))^* f_i(A_i)\alpha, \alpha\right) \\
&\overset{(a)}{=} \prod_{p\in O(\sigma)}\int t^{|p|-1}\prod_{i\in p} f_i(t)\overline{g_i(t)}d\mu.
\end{aligned} \tag{3.27}$$

We have used the fact $\prod_{i\in p} g_i(t) = \prod_{i\in p} g_{\sigma^{-1}(i)}(t)$. □

We now are capable of proving the previous theorem.

Proof of theorem 11. Fixing $a = a_i$ and setting $\nu = \nu(a_i)$. Let us consider the following vector

$$\zeta^{(m)} = \prod_{i=1}^{m}\delta_a(A_i)\alpha \tag{3.28}$$

where α is a spherical vector with $||\alpha|| = 1$. Then for $\sigma \in S(m)$, we have

$$\begin{aligned}(\sigma\zeta^{(m)}, \zeta^{(m)}) &= \left(\sigma \prod_{i=1}^{m} \delta_a(A_i)\alpha, \prod_{i=1}^{m} \delta_a(A_i)\alpha\right) \\ &= \left(\sigma \prod_{i=1}^{m} (\delta_a(A_i))^* \delta_a(A_i)\alpha, \alpha\right) \\ &\overset{(3.25)}{=} \prod_{p \in O(\sigma)} \int t^{|p|-1} \prod_{i \in p} \overline{\delta_a(t)} \delta_a(t) d\mu.\end{aligned} \tag{3.29}$$

Then

$$\begin{aligned}\prod_{p \in O(\sigma)} \int t^{|p|-1} \prod_{i \in p} \overline{\delta_a(t)} \delta_a(t) d\mu &= \prod_{p \in O(\sigma)} \int t^{|p|-1} \delta_a(t)^2 d\mu \\ &= \prod_{p \in O(\sigma)} \int t^{|p|-1} \delta_a(t) d\mu \\ &= \prod_{p \in O(\sigma)} a^{|p|-1} \mu(a) \\ &= a^{m-\iota(\sigma)} \mu(a)^{\iota(\sigma)} \\ &= a^m (\mu(a)/|a|)^{\iota(\sigma)} = a^m \nu^{\iota(\sigma)}\end{aligned} \tag{3.30}$$

where $\iota(\sigma)$ denotes the number of cycles (with singletons) in the permutation $\sigma \in S(m)$. Let Alt(m) be antisymmetrization operator for the symmetric group $S(\infty)$.

Then

$$\begin{aligned}
(\mathrm{Alt}(m)\zeta^{(m)}, \zeta^{(m)}) &= \frac{1}{m!}\sum_{\sigma\in S(m)} \mathrm{sgn}(\sigma)(\sigma\zeta^{(m)}, \zeta^{(m)}) \\
&\overset{(3.30)}{=} \frac{1}{m!}\sum_{\sigma\in S(m)} \mathrm{sgn}(\sigma)a^m \nu^{\iota(\sigma)} \\
&= \frac{a^m}{m!}\sum_{\sigma\in S(m)} \mathrm{sgn}(\sigma)\nu^{\iota(\sigma)} \\
&= \frac{a^m}{m!}\nu(\nu-1)\cdots(\nu+m-1).
\end{aligned} \tag{3.31}$$

We have used basic relation $\mathrm{sgn}(\sigma) = (-1)^{m+\iota(\sigma)}$ and the identity

$$\sum_{\sigma\in S(m)} y^{\iota(\sigma)} = y(y+1)\cdots(y+m-1). \tag{3.32}$$

We know that the operator $\mathrm{Alt}(m)$ is a projection, thus the last step of the above equality is nonnegative for all m. In order to make it nonnegative, it must terminate at some step. Therefore $\nu = \nu(a)$ must be a natural number. On the other hand, $\nu(-b_i)$ must be a natural number as well. Because, if we multiply the spherical representation π by the representation sgn $\otimes$ sgn, our measure $\mu(x)$ turns into $\mu(-x)$ and the same argument gives the result. $\square$

We write $n(x)$ instead of $\nu(x)$ to stress it is an nonnegative integer. By virtue of (3.22), (3.7) and above result, the characters of the infinite symmetric group $S(\infty)$ will be given the following Thoma's theorem we already proved.

Theorem 8. *The characters of the infinite symmetric group are exactly the functions of the form*

$$\varphi(\sigma) = \prod_{k\in[\sigma]}\left(\sum_{i=1}^{\infty} n(a_i)a_i^k + (-1)^{k-1}\sum_{i=1}^{\infty} n(-b_i)b_i^k\right), \tag{3.33}$$

where

$$a_1 \geq a_2 \geq a_3 \geq \cdots > 0 \qquad b_1 \geq b_2 \geq b_3 \geq \cdots > 0$$

and

$$\sum n(a_i)a_i + \sum n(b_i)b_i \leq 1.$$

CHAPTER FOUR
CONCLUSION

In this thesis, we have worked out A. Okounkov's paper. We have used Olshanski's some theorems and results. Moreover, we have used spectral theory for bounded self-adjoint operators to reach our purpose. As a consequence we have given details for Okounkov's work about Thoma's theorem.

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