COFINITELY SUPPLEMENTED MODULES

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

In this thesis, we defined Cofinitely Supplemented Modules and investigated main properties of Cofinitely Supplemented Modules, e.g. “arbitrary sum of cofinitely supplemented modules is cofinitely supplemented.” Or, “every maximal submodule of a module M has a supplement if and only if every cofinite submodule ha s asupplement. We also proved that the ring R is Semiperfect if and only if every R-module is cofinitely supplemented. For a non-local Dedekind Domain R, an R-module M is amply cofinitely supplemented if and only if M is injective or Torsion.

ÖZET

Bu tezde dual sonlu bütünleşmiş modül tanımlandı ve "Dual sonlu Bütünleşmiş Modüllerin keyfi toplamının yine Dual sonlu" olduğu veya, "bir M modülünün her maksimal alt modülünün bütünleştiricisi vardır ancak ve ancak her dual sonlu altmodülün bütünleştiricisi vardır." Yine " R yarımükemmeldir ancak ve ancak her R modülü sonlu bütünleşmiştir. Lokal olmayan Noetherian Dedekind bölgesi için, bir R modülü fazlasıyla sonlu bütünleşmiştir ancak ve ancak M inlektif veya burulmalıdır.

CONTENTS

CHAPTER ONE
INTRODUCTION

It is well known that every subspace of a vector space is a direct summand. This is not the case for modules in general. A submodule T of a module M is a direct summand if $T \cap N = 0$ and $T + N = M$ for some submodule N of M. In this case T is maximal with respect to the first condition and minimal with respect to the second condition. So direct summands are among submodules satisfiying the maximality and minimality conditions given above. A submodule K which is maximal in the set of submodules of M and disjoint from some submodule B called a complement of B in M. Complements always exist by Zorn's Lemma. Modules whose complements are direct summands are called CS Modules or Extending Modules. The main results on Extending Modules are collected in (Dung,1995).

The dual situations are more complicated. If $M = A + B$ and A is minimal with respect to this property, A is called a supplement of the submodule B in M. Supplements does not exist in general. For example subgroups of the abelian group Z does not have supplements. So we have two kinds of questions: Does the submodule have a supplement ? If yes, then is this supplement a direct summand ? The main results about supplements are collected in (Mohamed,1990) and (Wisbauer,1991).

This thesis is devoted to the existence problem of supplements. A Submodule N where M/N is finitely generated is called a cofinite submodule of M. The modules M, each of whose cofinite submodule has a supplement are called Cofinitely Supplemented Modules. Cofinitely Supplemented Modules from a general point of view and over a Dedekind Domain are studied in this thesis.

In chapter one, basic notions that we use throughout the thesis are given and especially properties of small submodules of modules are given in detail.

In chapter two, Supplemented modules, Hollow and Local modules are introduced with necassary properties. The definitions of Cofinitely Supplemented and Amply Cofinitely Supplemented modules are given and theorems over an arbitrary ring are derived. For instance it is proved that a ring R is Semiperfect if and only if every R-module is cofinitely supplemented if and only if every maximal submodule has a supplement.

In chapter three we researched properties of Cofinitely and Amply Cofinitely Supplemented Modules over Dedekind Domains. We know that over a Non-Local Dedekind Domain R, an R-module M is Amply Cofinitely Supplemented if and only if M is Injective or Torsion.

Throughout R will denote an associative ring with unity and M will stand for a unitary left R-module.

1-1 Preliminaries.

Lemma 1.1.1 (Zorn's Lemma) Let X be a non-empty partially ordered set. If every chain has an upper bound, then X has a maximal element.

Theorem 1.1.2 (Homomorphism Theorem) If $f: M \to N$ is a homomorphism and S is a submodule of an R-module M with $S \leq \mathrm{Ker} f$, then there is exactly one homomorphism $g: M/S \to N$ with $f = g \circ \pi$, where $\pi: M \to M/S$ is the canonical epimorphism. Furthermore, if $S = \mathrm{Ker} f$, then g is monomorphism. If f is epimorphism then g is also epimorphism.

Proof- Define $g: M/S \to N$ as, $g(m + S) = f(m)$ then $m_1 + S = m_2 + S$ implies that $m_1 - m_2 \in S \leq \mathrm{Ker} f$ so $0 = f(m_1 - m_2)$, then $f(m_1) = f(m_2)$ and thus g is well defined.

In addition to that $go\pi(m) = g(\pi(m)) = g(m + S) = f(m)$.

Suppose there exists another map h: $M/_S \to N$ such that $f = ho\pi$ which implies $ho\pi = go\pi$ and since π is epimorphism,
$h(m + S) = h(\pi(m)) = ho\pi(m) = go\pi(m) = g(m + S)$. Therefore h = g.

If S = Kerf then Kerg = $\{m + S \mid f(m) = 0\} = \{m + S \mid m \in Kerf\} = 0$. If f is an epimorphism for any $n \in N$, there exists $m \in M$ such that f(m) = n, g(m + S) = f(m) = n implies g is also epimorphism.

Proposition 1.1.3 Let N be a submodule of K, then there exists an epimorphism f: $M/_N \to M/_K$.

Proof- Define the map f: $M/_N \to M/_K$ as f(m + N) = m + K. Then suppose that $m_1 + N = m_2 + N$, so $m_1 - m_2 \in N \le K$ implies that $m_1 + K = m_2 + K$. Hence the map is properly defined. Surjectivity is straight forward.

Theorem 1.1.4 (First Isomorphism Theorem) Let M and N be R-modules and ϕ: M $\to$ N be a homomorphism, then $M/_{Kerf} \cong Im\phi$. In particular if ϕ is an epimorphism then $M/_{Kerf} \cong N$.

Proof- By 1.1.2, g: $M/_{Kerf} \to N$ is monomorphism and so g: $M/_{Kerf} \to Im\phi$ is an isomorphism with the map ϕ: M $\to$ N.

Theorem 1.1.5 (Second isomorphism theorem) Let M and N be R-modules then $(M+N)/_N \cong M/_{(M \cap N)}$.

Proof- Let's define f: M $\rightarrow {}^{(M+N)}\!/\!_{N}$ as f(m) = m + N. Then clearly f is an epimorphism. Kerf = {m∈M | m∈N } = M ∩ N. Therefore by 1.1.4,

$$ {}^{(M+N)}\!/\!_{N} \cong {}^{M}\!/\!_{(M\cap N)}. $$

Theorem 1.1.6 (Third Isomorphism Theorem) Let M and N be R-modules and V be a submodule of both M and N, then $\dfrac{M/V}{N/V} \cong {}^{M}\!/\!_{N}$.

Proof- Define the epimorphism as in 1.1.3 f: ${}^{M}\!/\!_{V} \rightarrow {}^{M}\!/\!_{N}$. Then

Kerf = {m + V | m∈N } = ${}^{N}\!/\!_{V}$. Therefore by 1.1.4 $\dfrac{M/V}{N/V} \cong {}^{M}\!/\!_{N}$.

Lemma 1.1.7 (Modular law) Let A,B and C be submodules of an R-module M with B is submodule of C, then (A + B) ∩ C = (A ∩ C) + B.

Proof- For any x from (A + B) ∩ C, it can be represented as x = a + b = c for some a, b, c from A,B,C respectively. This implies a = b - c. Since B ≤ C, a∈C. Thus x = a + b∈(A ∩ C) + B. Converse is straight forward.

Definition 1.1.8 We say that a module M is finitely generated, if there is a finite set $F_n = \{x_1,\dots,x_n\}$ with each x_i $(1 \le i \le n)$ is from M such that any element x from M can be written as $x = r_1x_1 + \dots + r_nx_n$ with each r_i $(1 \le i \le n)$ is from R.

Definition 1.1.9 An R-module M is called free, if there is a basis $\{x_i\}_{i\in I}$ for M. That is every element m of M can be written uniquely as $m = \sum_{i\in I} r_ix_i$ where $r_i \in R$. (i∈I).

Proposition 1.1.10 If a module M is finitely generated then any factor module of M is also finitely generated.

Proof- Since M is finitely generated any element of M can be written as $m = r_1x_1 + \ldots\ldots + r_nx_n$, where $\{x_1,\ldots.x_n\}$ is a spanning set for M. If we choose an element $m + N$ from M/N then $m + N = (r_1x_1 + N) + \ldots\ldots + (r_nx_n + N)$. Therefore $\{x_1 + N,\ldots\ldots,x_n + N\}$ becomes a spanning set for M/N.

Proposition 1.1.11 Let M be an R-module and N be its submodule. For the natural epimorphism $\phi\colon M \to M/N$, if B is a maximal submodule in M/N then $\phi^{-1}(B)$ is also maximal in M.

Proof- Suppose that there exists a proper submodule K of M such that $\phi^{-1}(B) \leq K$. Then $\phi(\phi^{-1}(B)) \leq \phi(K)$, since $\varnothing$ is an epimorphism $B \leq \phi(K)$. By the maximality of B, $\phi(K) = M/N$. This implies $\phi^{-1}(\phi(K) = \phi^{-1}(M/N)$ and then $K + \mathrm{Ker}\phi = M$. That is $M = K$.

Theorem 1.1.12 If M is a finitely generated module then for any submodule N of M, there exists a maximal submodule P of M such that $N \leq P$.

Proof- Denote the collection of all proper submodules of M containing N by Γ. Then Γ is non-empty because $N \in \Gamma$. We may partially order Γ by inclusion. Let Λ be a non-empty chain of Γ. Denote by N_0 the union of all elements of Λ. Then $N \leq N_0$, and it may be verified that N_0 is a submodule of M. Note that this verification uses the fact that Λ is totally ordered. N_0 is a proper submodule of M. For suppose otherwise, and let $\{m_1,\ldots..,m_n\}$ be a set of generators of M. Then each m_i belongs to N_0, therefore is contained in some member of Λ. So by total order property of Λ, there must be a member of Λ which contains every m_i, i.e. which is the whole of M, contradiction. Thus $N_0 \in \Gamma$ and is an upper bound of Λ. It follows by Zorn's Lemma that Γ possesses a maximal member P.

Corollary 1.1.13 For an R-module M, if M/N is finitely generated then N is contained in a maximal submodule of M.

Proof- By 1.1.12 M/N has a maximal submodule P/N. By 1.1.11, $\phi^{-1}(P/N)$ is a maximal submodule containing N.

1.2 Small Submodules

Definition 1.2.1 Let M be an R-module and N be a submodule of M. N is called superfluous(or small) in M, if for any proper submodule K of M whenever $N + K = M$ then $K = M$.

Notation: If N is small in M then we denote this with $N \ll M$.

1.2.2 Properties Let M be an R-module and N be a submodule of M.

1- N is small in M if and only if $N + K \neq M$ for any proper submodule K of M.

2- Let N be a small submodule of a module M, then any submodule of N is also small in M.

3- Let N be a module and A be a submodule of N. If A is small in N, then it is small in any module containing N.

4- Let X be small in A and Y be small in B, then $X + Y$ is small in $A + B$.

5- Let f: $M \to N$ be a homomorphism and let A be a submodule of M. If A is small in M, then f(A) is small in f(M).

6- Let $K \leq L \leq M$. Then $L << M$ if and only if $K << M$ and $L/K << M/K$.

7- $L_1 + L_2 + \ldots\ldots + L_n << M$ if and only if $L_i << M$ $(1 \leq i \leq n)$

8- If L is a direct summand of M, then $K << L$ if and only if $K << M$.

Proofs-

1- Suppose that $N + K = M$ then since N is small in M, $K = M$, contradiction. Conversely assume that $N + K = M$, but by assumption this is true for just the case $K = M$.

2- Suppose that K is a submodule of N and let $K + X = M$ for some submodule X of M, then $K + X \leq N + X = M$. Since N is small in M, $X = M$.

3- Let M be a module containing N and suppose that $A + X = M$ for some submodule X of M. Then by modular law, $A + (N \cap X) = N$. And since A is small in N, $N \cap X = N$. So $N \leq X$. Hence $A \leq X$. Therefore we get $X = A + X = M$.

4- Suppose that $(X + Y) + Z = A + B$ where Z is a subgroup of $A + B$.X is small in $A + B$ and so we get $Y + Z = A + B$. Similarly since Y is small in $A + B$, $Z = A + B$. Therefore $X + Y$ is small in $A + B$.

5- Suppose that $f(A) + X = f(M)$ with $X \leq f(M)$. Then $f^{-1}(f(A)) + f^{-1}(X) = f^{-1}(f(M)) = M$ and then $A + \mathrm{Ker}f + f^{-1}(X) = M$ implies $A + f^{-1}(X) = M$ (Since $\mathrm{Ker}f = f^{-1}(0) \leq f^{-1}(X)$). Since $A<<M$, $f^{-1}(X) = M$ and

then $f(f^{-1}(X)) = f(M)$ impliying, $X \cap Imf = f(M)$. Therefore $X = f(M)$.

6- ($\Rightarrow$) By 2, $K \ll M$. Suppose that $L/K + X/K = M/K$ where X/K is a submodule of M/K, then $L + X = M$ and by assumption $X = M$. That is $X/K = M/K$.

($\Leftarrow$) Let $L + X = M$ then $(L+X)/K = M/K$ that is $L/K + ((X+K)/K) = M/K$. Then by assumption $((X+K)/K) = M/K$. So $X + K = M$ and again by assumption $X = M$.

7- ($\Rightarrow$) By 2

($\Leftarrow$) If $L_1 + L_2 + ... + L_n + X = M$, then since $L_1 \ll M$, $L_2 + ... + L_n + X = M$, since $L_2 \ll M$, $L_3 + ... + L_n + X = M$ going on this way we get $X = M$ at the end.

8- ($\Rightarrow$) By 3

($\Leftarrow$) Let $K + X = L$, since L is a direct summand of M, there exists $L_0 \leq M$ such that $L \oplus L_0 = M$ so $K + X + L_0 = M$ then by assumption $X + L_0 = M$ then $X \oplus L_0 = M$. Therefore by modular law $L = L \cap (X + L_0) = X + (L \cap L_0) = X$.

1.3 The Radical

Definition 1.3.1 The sum(intersection) of all small(maximal) submodules of a module M is called the radical of M and denoted by RadM.

Proposition 1.3.2 Let $\{B_i\}_{i\in I}$ be a family of submodules of M containing submodule C of M. Then we have

$$\bigcap_{i\in I}(Bi/C) = (\bigcap_{i\in I} Bi)/C \ .$$

Proof- ($\subset$) Straight forward.

($\supset$) Take $b_i + C$ from $\bigcap_{i\in I} Bi/C$ then $b_i \in \bigcap_{i\in I} Bi$ implies that $b_i \in B_i$ for every $i \in I$. Then $b_i + C \in Bi/C$ result follows from here.

Definition 1.3.3 If an R-module is a sum of simple modules then it is called a semisimple module.

Theorem 1.3.4 For an R-module M, the following conditions are equivalent.

1- Every submodule of M is a sum of simple submodules.

2- M is a sum of simple submodules.

3- M is a direct sum of simple submodules

4- Every submodule of M is a direct summand of M.

Proof- See (Kasch,1981) 8.1.3

Proposition 1.3.5 If M is semisimple then every proper submodule of M is contained in a maximal submodule.

Proof- Let U be a submodule of M then U is a direct summand of M and $M = U \oplus (\bigoplus_{i\in I} U_i)$ for submodules U_i of M with $i \in I$. Then the factor module $M/(U \oplus (\bigoplus_{i\in I-\{j\}} U_i))$ is isomorphic to U_j. So, $U \oplus (\bigoplus_{i\in I} U_i)$ is maximal in M.

Proposition 1.3.6 Rad($M/RadM$) = 0, that is $M/RadM$ has no small submodules.

Proof- Rad($M/RadM$) is the intersection of all maximal submodules of $M/RadM$. Let $\left\{ X_i/RadM \right\}_{i\in I}$ be the collection of all maximal submodules of $M/RadM$. Then by 1.3.2 $\bigcap_{i\in I}(Xi/RadM) = (\bigcap_{i\in I} Xi)/RadM = RadM/RadM$ =0.

Proposition 1.3.7 If every submodule of M is contained in a maximal submodule then RadM<<M.

Proof- Suppose that RadM + X = M for some proper submodule X of M. Then X is contained in some maximal submodule P of M, but also Rad(M) ≤ P, that means Rad(M) + X = M ≤ P contradiction. Therefore X = M.

Corollary 1.3.8 If M is finitely generated then RadM<<M.

Proposition 1.3.9 If $M/RadM$ is semisimple and RadM << M, then every proper submodule of M is contained in a maximal submodule.

Proof- Let U be a proper submodule of M and σ: M → $M/RadM$ be the canonical projection. σ(U) = $(RadM + U)/RadM \neq M/RadM$ because RadM << M. Hence by 1.1.12 σ(U) is contained in a maximal submodule of $M/RadM$. Therefore by 1.1.11 U is contained in a maximal submodule of M.

Proposition 1.3.10 For any $f \in Hom(M,N)$, $f(RadM) \leq RadN$.

Proof- Considering the radical of M as sum of small submodules then result follows by 1.2.1 (5).

CHAPTER TWO

SUPPLEMENTED MODULES

2.1 Preliminaries

Definition 2.1.1 Let M be an R-module and U,V be submodules of M. We say that V is a supplement of U, if the following properties hold.

1) $U + V = M$

2) V is minimal with respect to this property. That is, if there is another submodule K of M with the property $M = U + K$, then V is a submodule of K.

Note that supplements does not exist always, for example let Z denote the ring of Rational Integers. F any non-zero free Z-module and n an integer with $n \geq 2$ then the submodule nF does not have a supplement in F because if $F = nF + K$ for some submodule K then $F = nF + pK$ for any prime p which does not divide n and $K \neq pK$.

Proposition 2.1.2 Let M be an R-module and U,V be submodules of M. Then V is a supplement of U if and only if $U + V = M$ and $U \cap V \ll V$.

Proof- Suppose that V is a supplement of U. Then $U + V = M$. Assume that $(U \cap V) + X = V$ for some submodule X of V. Then $U + (U \cap V) + X = M$, so $U + X = M$. But then $X = V$, because otherwise there is a contradiction with the minimality of V.

Conversely assume that $U + V = M$ and $U \cap V \ll V$. Then suppose there exists a submodule K of M such that $K \leq V$ and $U + K = M$. By the modular law, $(U \cap V) + K = V$ and by assumption $K = V$.

Proposition 2.1.3 Let M be an R-module with submodules U and V, and let V be a supplement of U. If W + V = M for some W ≤ U, then V is also a supplement for W.

Proof- $W \cap V \leq U \cap V \ll V$. By 1.2.2(2), $W \cap V \ll V$.

Proposition 2.1.4 Let M be an R-module with submodules U and V, and let V be a supplement of U. For a submodule L of U, $(V+L)/L$ is a supplement of U/L in M/L.

Proof- It is clear that, $(V+L)/L + U/L = M/L$ because U + V = M. Suppose that $((V+L)/L \cap U/L) + X/L = (V+L)/L$, for some submodule X/L of M/L this implies that $((V+L)\cap U)/L + X/L = (V+L)/L$ and thus

$((V + L) \cap U) + X = V + L$. By the modular law, $(V\cap U) + L + X = V + L$ implies $(V\cap U) + X = V + L$. By 1.2.2(3), $X = V + L$. Therefore $X/L = (V+L)/L$.

Definition 2.1.5 Let M be an R-Module. We say that M is supplemented if every submodule of M has a supplement in M.

Proposition 2.1.6 If M is supplemented then $M/RadM$ is semisimple.

Proof- By 1.3.6 $M/RadM$ has no small submodules. Let $N/RadM$ be a submodule of $M/RadM$. Then $\sigma^{-1}(N/RadM)$ is a submodule of M where $\sigma: M \to M/RadM$ is the canonical epimorphism. So by assumption there exists a submodule K of M with $\sigma^{-1}(N/RadM) \cap K \ll K$ and $\sigma^{-1}(N/RadM) + K = M$.

Hence $\sigma^{-1}(N/_{RadM}) \cap K << M$. Then by 1.2.2(5), $N/_{RadM} \cap \sigma(K) << M/_{RadM}$ and so $N/_{RadM} \cap \sigma(K) = 0$. Therefore $N/_{RadM}$ is a direct summand of $M/_{RadM}$.

Proposition 2.1.7 An Artinian module is supplemented.

Proof- Let M be an Artinian module and U be a submodule of M. There is a submodule V of M such that U + V = M. Suppose that there is a submodule V_0 of V with $U + V_0 = M$ (if there is no such module then V is supplement of U.)
$U + V_1 = M$ and $V_1 \subset V_0$...e.t.c. continuing in this way we have a descending chain of submodules $V \supset V_0 \supset V_1 \supset ...... \supset V_n$.... But since M is Artinian, this sequence must end somewhere, call the module at the end as V_n. Therefore V_n is the supplement of U in M.

Proposition 2.1.8 Let V be a supplement of U in M. Then if U is maximal in M, then V is cyclic and $U \cap V = RadV$ is the unique maximal submodule of V.

Proof- Take an element x from V - U, then $U \leq U + \langle x \rangle$ implies that by the maximality of U, $U + \langle x \rangle = M$. Thus since V is the supplement of U in M this implies that $\langle x \rangle = V$.

$U \cap V << V \leq M$, so $U \cap V$ is a submodule of RadV. Then
$M/_U = (U+V)/_U \cong V/_{(U \cap V)}$, so $U \cap V$ is maximal in V implies that RadV is a submodule of $U \cap V$. Therefore $U \cap V = RadU$.

2.2 Hollow, Local, Supplemented Modules

Definition 2.2.1 A module M is called **hollow** if every proper submodule of M is small and a module L is called **local** if it contains a largest proper submodule, that is a proper submodule containing all proper submodules of M.

Proposition 2.2.2 A local module M is cyclic generated by $x \in M$ - RadM.

Proof- Since M is local it has a largest submodule N. It is clear that RadM = N. For any x from X - RadM, Rx = M because otherwise Rx $\leq$ N and so N = M contradiction. Therefore M is cyclic generated by x.

Proposition 2.2.3 If M is local then the largest submodule N of M is small in M.

Proof- Suppose that for some submodule X of M, X + N = M. Then X = M because otherwise $X \leq N$ and so N = X + M = M.

Proposition 2.2.4 Every local module is hollow.

Proof- By 2.2.3, RadM is small in M and so any proper submodule N is also small in M by 1.2.2 (2).

Proposition 2.2.5 Let M be an R-module then M is hollow and RadM $\neq$ M if and only if M is local.

Proof- Since M is hollow and RadM $\neq$ M, RadM is small in M. Then the sum of all proper submodules of M which coincides with the sum of all small submodules of M is the largest submodule of M because RadM $\neq$ M. Converse is by 2.2.4

Definition 2.2.6 Let M be an R-module and U be a submodule of M. We say that U has ample supplements if whenever M = U + V for some submodule V of M then U has a supplement contained by V.

If every submodule of M has ample supplements then M is called amply supplemented.

Note that an amply supplemented module is supplemented. Since N + M = M for any submodule N of M.

Proposition 2.2.7 Every hollow module is supplemented.

Proof- Suppose that M is hollow. For any submodule N of M, N + M = M and $N \cap M \ll M$. Thus M is supplement of its each proper submodule. Therefore M is supplemented.

Definition 2.2.8 M is called finitely supplemented if every finitely generated submodule of M has a supplement in M.

2.3 Cofinitely Supplemented Modules

Definition 2.3.1 A submodule N of an R-module M is called cofinite if M/N is finitely generated.

Definition 2.3.2 M is called cofinitely supplemented if every cofinite submodule N of M has a supplement in M. And M is called amply cofinitely supplemented if every cofinite submodule has ample supplements.

Example 2.3.3 An example of a module which is cofinitely supplemented but not supplemented is the **Z**-module **Q**. **Q** is not torsion and hence it is not supplemented by (Zöschinger,1974).

Suppose that Q/A is finitely generated for some submodule A of **Q**. Then **Q** = A, because otherwise since **Q** is divisible so is Q/A. But this is impossible unless **Q** = A.

Lemma 2.3.4 Homomorphic image of a Cofinitely Supplemented Module is cofinitely supplemented.

Proof- Let f: M $\to$ N be a homomorphism and M be a Cofinitely Supplemented Module. Suppose that X is a cofinite submodule of f(M), then

$$M/f^{-1}(X) \cong \left(M/Kerf\right) \Big/ \left(f^{-1}(X)/Kerf\right) \quad \text{with } M/\mathrm{Kerf} \cong f(M) \text{ and } f^{-1}(X)/Kerf \cong X.$$

Thus we can say $M/f^{-1}(X)$ is finitely generated. Since M is cofinitely

supplemented, $f^{-1}(X)$ has a supplement U in M, then $f^{-1}(X) + U = M$ and $f^{-1}(X) \cap U \ll U$. So, $f(f^{-1}(X)) + f(U) = f(M)$ and since X is a submodule of f(M), $f(f^{-1}(X)) = X$ and so $X + f(U) = f(M)$. In addition to that $f(f^{-1}(X)) \cap f(U) \ll f(U)$ by 1.2.2(5). Therefore $X \cap f(U) \ll f(U)$.

Corollary 2.3.5 Any factor module of a Cofinitely Supplemented Module is cofinitely supplemented.

Proof- Let M be a Cofinitely Supplemented Module. Consider a factor module M/N of M where N is a submodule of M. Consider the canonical epimorphism $\pi: M \to M/N$, then by 2.3.4, $\pi(M) = M/N$ is cofinitely supplemented.

Proposition 2.3.6 If M is cofinitely supplemented with finitely generated $M/Rad(M)$ then $M/Rad(M)$ is supplemented and semisimple.

Proof- By 2.3.5, $M/Rad(M)$ is cofinitely supplemented, let's take any submodule $N/Rad(M)$ of the module $M/Rad(M)$, then consider the factor module $\left(M/Rad(M)\right) \Big/ \left(N/Rad(M)\right)$. It is isomorphic to M/N. By 1.1.10, M/N is finitely generated and hence N has a supplement in M. By 2.1.4, $N/Rad(M)$ has a supplement.

Take any submodule $N/Rad(M)$ of the module $M/Rad(M)$. Since $M/Rad(M)$ is supplemented, $N/Rad(M)$ has a supplement $K/Rad(M)$ in $M/Rad(M)$. Hence $N/Rad(M) + K/Rad(M) = M/Rad(M)$. Further, $N/Rad(M) \cap K/Rad(M) \ll K/Rad(M) \leq M/Rad(M)$. But $M/Rad(M)$ has no small submodules hence $N/Rad(M) \cap K/Rad(M) = 0$. Therefore, $N/Rad(M) \oplus K/Rad(M) = M/Rad(M)$.

Lemma 2.3.7 Let N and U be submodules of M where N is supplemented. If there is a supplement for N + U in M then U also has a supplement in M.

Proof- Let X be a supplement of N + U in M and since N is supplemented, let Y be a supplement of (X + U) ∩ N in N.

<u>Claim:</u> Y is a supplement of X + U in M.
Proof of the claim: M = X + N + U = X + U + (((X + U) ∩ N) + Y). So M = X + U + Y Because ((X + U) ∩ N) is submodule of X + U. Also Y ∩ (X + U) = Y ∩ ((X + U) ∩ N) << Y.

Since Y + U ≤ N + U and X + Y + U = M, X is a supplement of Y + U. That is X ∩ (Y + U) << X. Also we can prove the following fact (X + Y) ∩ U ≤ (X ∩ (Y + U)) + ((Y ∩ (X + U)) as follows: Any z from (X + Y) ∩ U can be represented as z = x + y = u. From here x = u - y and thus x is an element of X ∩ (Y + U). Similarly y is an element of Y ∩ (X + U). Therefore z is an element of $(X \cap (Y+U)) + (Y \cap (X+U))$. Hence $(X+Y) \cap U$ is small in $X+Y$ and so $X+Y$ is a supplement of U.

Lemma 2.3.8 Let N and U be submodules of M with cofinitely supplemented N and M/U is finitely generated. If $N+U$ has a supplement in M, then U has also a supplement in M.

Proof- Let X be a supplement of N + U in M. Then we have

$$N/(N \cap (X+U)) \cong (N+(X+U))/(X+U) = M/(X+U) \cong \left. {M/U} \middle/ {(X+U/U)} \right.$$ the last

module is finitely generated since M/U is finitely generated. So $N/(N \cap (X+U))$ is also finitely generated. Since N is cofinitely supplemented $N \cap (X+U)$ has a supplement in N. Now let X be a supplement of N + U in M and Y be a supplement of $(X+U) \cap N$ in N. Then Y is a supplement of X + U in M. Because $M = X+N+U = X+U+(((X+U) \cap N)+Y) = X+U+Y$.

In addition to that $Y \cap (X+U) = Y \cap ((X+U) \cap N) \ll Y$. Since $Y+U \leq N+U$, X is a supplement of Y + U. That is $X \cap (Y+U) \ll X$. Also we can prove the following fact $(X+Y) \cap U \leq (X \cap (Y+U)) + ((Y \cap (X+U))$ as follows: Any z from $(X+Y) \cap U$ can be represented as $z = x+y = u$. From here $x = u - y$ and thus x is an element of $X \cap (Y+U)$. Similarly y is an element of $Y \cap (X+U)$. Therefore z is an element of $(X \cap (Y+U)) + (Y \cap (X+U))$. Hence $(X+Y) \cap U$ is small in X + Y and so X + Y is a supplement of U.

Proposition 2.3.9 Arbitrary sum of Cofinitely Supplemented Modules is cofinitely supplemented.

Proof- Let $(M_i)_{i \in I}$ be a collection of cofinitely supplemented submodules of M such that $M = \sum_{i \in I} M_i$. Suppose that N is a cofinite submodule of M. Then M/N has a generating set $\{x_1 + N, x_2 + N, \ldots\ldots, x_r + N\}$ so any x_i with $i \in I$ can be expressed as $x_i = m_{i1} + m_{i2} + \ldots\ldots + m_{is(i)}$ with each m_{ik} is from some M_{ik} where $i_k \in I$. Any element $m + N$ from M/N can be represented as: $m + N = r_1x_1 + \ldots\ldots + r_nx_n + N$. So, $m = r_1(m_{11} + \ldots\ldots + m_{1s(1)}) + \ldots\ldots + r_n(m_{n1} + \ldots\ldots + m_{ns(n)}) + n$, where $n \in N$. Thus $M = \sum_{i \in J} M_j + N$ with finite $J = \{1_1, \ldots 1_{s(1)}, 2_1, \ldots\ldots, n_{s(n)}\}$. Let's write

$M = \sum_{i \in J} Mj + N = M_{11} + \sum_{i \in J-\{11\}} Mj + N$. We know that M_{11} is cofinitely supplemented, and $M_{11} + \sum_{i \in J-\{11\}} Mj + N$ has trivially the supplement 0. Therefore $\sum_{i \in J-\{11\}} Mj + N$ has a supplement. Continuing in this way since the set J is finite at the end we can say N has a supplement.

Example 2.3.10 Consider the group $\bigoplus_{n \in N} Z(p^n)$, $Z(p^n)$ is local and by 2.2.4, it is hollow. Then any submodule of $Z(p^n)$ has supplement $Z(p^n)$ therefore by 2.3.10, $\bigoplus_{n \in N} Z(p^n)$ is cofinitely supplemented.(Note that it is also supplemented.)

Example 2.3.11 It is not true that any direct sum of supplemented modules is supplemented. Let R be any semiperfect ring (later will be seen) which is not right perfect, e.g. R is the subring of the field **Q** of Rational Numbers consisting of all elements m/n where $m,n \in Z$ and $n \notin Zp$, where p is any given prime in Z. Then the R-module R is supplemented but the R-module $R^{(N)}$ is not supplemented by (Wisbauer,1991) 43.9.

Definition 2.3.12 Let N be an R-module and if there is an epimorphism f: $\oplus M \rightarrow N$, then we say N is M-generated.

Corollary 2.3.13 If M is cofinitely supplemented then any M-generated module is cofinitely supplemented.

Proof- $\oplus M$ is cofinitely supplemented by 2.3.9, and by 2.3.4, the image Imf of the epimorphism f: $\oplus M \rightarrow N$ is cofinitely supplemented.

2.4 Open Submodules and Supplements

Definition 2.4.1 Let V be a supplement of U in an R-module M. If V has a supplement X contained by U. Then X is called interior of U.

Proposition 2.4.2 Let $U \leq M$ and X be an interior of U in M, then $U/X \ll M/X$.

Proof- Suppose that $U/X + Y/X = M/X$ for some submodule Y/X of M/X. Then $U + Y = M$. Also $X + V = M$ and by the modular law, $X + (Y \cap V) = Y$ impliying $U + X + (Y \cap V) = M$ and since $X \leq U$, $(Y \cap V) + U = M$, then since $V + U = M$, $Y \cap V = V$ implies $V \leq Y$. Therefore $M = X + V \leq Y$ then $M = Y$ impliying $Y/X = M/X$.

Definition 2.4.3 Let U be a submodule of the module M, then we say that U is open in M if there is no proper submodule Y of U such that $U/Y \ll M/Y$. In other words an open submodule U in M can not have a proper interior.

Proposition 2.4.4 Every supplement is open.

Proof- Let V be a supplement of U in M. Then $U + V = M$, $U \cap V \ll V$. Suppose that there exists $X \leq V$ s.t. $V/X \ll M/X$, then $(U+V)/X = M/X$ implies that $(U+X)/X + V/X = M/X$ then $(U+X)/X = M/X$ and so $U + X = M$. Therefore by minimality of V, $X = V$.

Corollary 2.4.5 If X is interior of U, then this X is minimal with the property $U/X \ll M/X$.

Proof- Suppose there exists Y a submodule of X such that $U/Y \ll M/Y$. Then $U + V = M$ so, $U/Y + (V+Y)/Y = M/Y$ which implies that $V + Y = M$. Thus by the minimality $X = Y$.

Proposition 2.4.6 If M is amply supplemented, then every open submodule of M is a supplement.

Proof- Since M is amply supplemented, an open submodule U must have an interior X, but then U = X implies that U is supplement of V.

Corollary 2.4.7 In an Amply Supplemented Module, a submodule is open iff it is a supplement.

2.5 Maximal and Cofinitely Supplemented Modules

Definition 2.5.1 Let N be a submodule of a module M. A submodule K of M is called a complement of N in M if K is maximal in the collection of submodules Q of M with the property $Q \cap N = 0$.

Definition 2.5.2 Let M be a module and N is a submodule of M. We say that N is essential in M if $N \cap K \neq 0$ for any nonzero submodule K of M.

Proposition 2.5.3 Let K be a complement of N in M. Then $K \oplus N$ is essential in M.

Proof- Suppose that $(N \oplus K) \cap L = 0$ for a nontrivial submodule L of M. Then $N \cap (K + L) = 0$ because otherwise there exists a non-zero n in $N \cap (K + L)$ impliying $n = k + l$ for some n,k,l from N, K, L respectively. Then $n - k = l \in (N \oplus K) \cap L$ and so $n - k = 0$ by assumption. Thus $n = k$ contradiction with $N \cap K = 0$. But this contradicts with maximality of K. Therefore $(N \oplus K) \cap L \neq 0$.

Proposition 2.5.4 A module M is semisimple if and only if there are no essential proper submodules in M.

Proof- ($\Rightarrow$) Suppose that there is a proper essential submodule N of M. Then it is a direct summand by definition. Contradiction.

($\Leftarrow$) Conversely assume that A is a proper submodule of M. Then since it is not essential there exists a submodule B of M such that $A \cap B = 0$. By Zorn's lemma we can choose A maximal and proper with respect to the condition. So A can be considered as complement of B in M. By 2.5.3 $A \oplus B$ is essential in M. So by assumption $A \oplus B = M$.

Definition 2.5.5 Let M be an R-module then sum of all simple submodules of M or equivalently, intersection of all essential submodules of M is called the Socle of M and denoted by SocM.

Let's prove now the analogue of 2.1.6

Lemma 2.5.6 Let M be a cofinitely supplemented R-module. Then every cofinite submodule of $M/RadM$ is a direct summand.

Proof- Any cofinite submodule of $M/RadM$ has the form $N/RadM$ where N is a cofinite submodule of M. Then there exists a submodule K of M such that

M = N + K and N $\cap$ K << K, so N $\cap$ K << M. Hence M $\cap$ K is a subgroup of RadM. Thus $M/_{RadM} = N/_{RadM} \oplus ((K + RadM)/_{RadM})$ as required.

Theorem 2.5.7 The following statements are equivalent for a module M.

1- $M/_{Soc(M)}$ does not contain a maximal submodule.

2- Every maximal submodule is a direct summand of M.

3- Every cofinite submodule N of M is a direct summand of M.

Proof- (1) $\Rightarrow$ (2) Let P be a maximal submodule of M. If Soc(M) is a submodule of P then $P/_{Soc(M)}$ will be maximal in $M/_{Soc(M)}$ impossible. That's why Soc(M) is not a submodule of P. Thus there is a simple submodule S of M which is not in P. Then S $\cap$ P = 0, and also by maximality of P, M = P $\oplus$ S.

(2)$\Rightarrow$(3) Let $M/_N$ be finitely generated and K be a complement of N in M. Then N $\oplus$ K is essential in M. Suppose that M $\neq$ N $\oplus$ K, since there is an epimorphism f: $M/_N \rightarrow M/_{(N \oplus K)}$, $M/_{(N \oplus K)}$ is also finitely generated. Thus N $\oplus$ K is contained in a maximal submodule Q of M. By assumption Q is a direct summand of M. So, there exists a submodule T of M such that M = Q $\oplus$ T. Then (N $\oplus$ K) $\cap$ T = 0, and since N $\oplus$ K is essential in M, T = 0 a contradiction.

(3)$\Rightarrow$(1) Suppose that $M/_{Soc(M)}$ contains a maximal submodule $Q/_{Soc(M)}$. Then $M/_Q$ is simple and by assumption, Q is a direct summand of M. Hence M = Q $\oplus$ R for some submodule R of M. Since $M/_Q \cong$ R, R is also simple. Therefore R $\leq$ Soc(M) $\leq$ Q which is a contradiction.

Definition 2.5.8 If $M = \sum_{i \in I} M_i$, then this sum is called irredundant if, for every $j \in I$, $\sum_{i \neq j} M_i \neq M$ holds.

Theorem 2.5.9 For an R-module M, the following are equivalent:

1- M is a sum of hollow submodules and RadM << M.

2- Every proper submodule of M is contained in a maximal one and

a) Every maximal submodule has a supplement in M or

b) M is cofinitely supplemented.

3- M is an irredundant sum of local modules and RadM << M.

Proof- (1) $\Rightarrow$ (3) Let $M = \sum_{i \in I} L_i$ with hollow submodules $L_i \leq M$. Then

$M/RadM = \sum_{i \in I} L_i / RadM = \sum_{i \in I} (L_i + RadM)/RadM$. And $Rad(L_i) \leq L_i \cap RadM$.

Further $L_i/RadL_i$ is either zero or semisimple, because may be $L_i = RadL_i$ or since L_i is hollow then it is supplemented and by 2.1.6 $L_i/RadL_i$ is semisimple. thus by 1.1.3 $L_i/(L_i \cap RadM)$ is also semisimple. So by the isomorphism $(L_i + RadM)/RadM \cong L_i/(L_i \cap RadM)$, $M/RadM$ is either zero or sum of simple modules. So $M/RadM = \bigoplus_{i \in J} ((RadM + L_i)/RadM)$, where J is a subset of I.

$\sigma^{-1}(M/RadM) = \sigma^{-1}(\bigoplus_{i \in J} ((RadM + L_i)/RadM))$ implies that

$M = \bigoplus_{i \in J} (L_i + RadM) = \bigoplus_{i \in J} L_i$ by smallness of RadM in M. Clearly the sum is irredundant.

(c)⇒(b) $\sum_{i\in I} L_i$, L_i's are local.

$$M/_{RadM} = (\sum_{i\in I} L_i)/_{RadM} = \sum_{i\in I}(L_i + RadM)/_{RadM} \cong \sum_{i\in I} L_i/_{L_i \cap RadM}$$. L_i's are supplemented by 2.2.4 and 2.2.7. Then by 2.1.6 $L_i/_{RadL_i}$ is semisimple. Hence $L_i/_{(L_i \cap RadM)}$ is semisimple. Thus by 2.2.5 every submodule is contained in a maxımal one.

Suppose that K is a cofinite submodule of M. Then $M/_K$ is finitely generated. Thus there are finitely many local submodules $L_1,\ldots, L_n$ with $M = K + L_1+,\ldots,+L_n$ being a finite sum of supplemented modules. Then , by 2.3.9 K has a supplement in M.

(b)⇒(a) assume (a). Suppose that H is a sum of all hollow submodules of M and $H \neq M$. Then by assumption, there is a maximal submodule N of M containing H and again by assumption there is a supplement L of N in M. By 2.1.8 L is local i.e. hollow. But then L is a submodule of H contradiction. Therefore H = M. Since every submodule is contained in a maximal one then by 2.2.7 $RadM \ll M$.

Corollary 2.5.10 (1) If M is finitely generated then the following are equivalent:

(a)- M is supplemented.

(b)- Every maximal submodule of M has a supplement in M.

(c)- M is a sum of hollow submodules.

(d)- M is an irredundant sum of local submodules.

(2) If M is supplemented and $Rad(M) \ll M$, then M is an irredundant sum of local modules.

Proof- (1) (a)⇒(b) Straight forward

(b)⇒(c) Since M is finitely generated by 1.1.12 every proper submodule is contained in a maximal one. So by 2.5.9 M is a sum of hollow modules.

(c)⇒(d) M is finitely generated. So by 1.3.8 RadM << M then by 2.5.9 M is an irredundant sum of local submodules.

(d)⇒(a) Since M is finitely generated by 2.2.9 (3) ⇒ (1) result follows.

(2) Since M is supplemented then by 2.1.6 $M/_{RadM}$ is semisimple and so every proper submodule of M is contained in a maximal one. Therefore by 2.5.9 M is an irredundant sum of local submodules.

Next we characterize when a module is cofinitely supplemented. Let M be any R-module. Then Loc(M) will denote the sum of all local submodules of M and Cof(M) the sum of all cofinitely supplemented submodules of M. (Note that 0 is a local submodule and also a cofinitely supplemented submodule of M.) The next result generalizes 2.5.7.

Theorem 2.5.11 The following are equivalent for an R-module M.

1- M is cofinitely supplemented.
2- Every maximal submodule of M has a supplement in M.
3- The module $M/_{LocM}$ does not contain a maximal submodule.
4- The module $M/_{Cof(M)}$ does not contain a maximal submodule.

Proof- (1) ⇒ (2) Let X be a maximal submodule of M. Then $M/_X$ is simple, so by assumption X has a supplement.

(2) $\Rightarrow$ (3) Let X be a maximal submodule of M. Then there exists a submodule Y of M such that X + Y = M and $X \cap Y \ll Y$. Then $\frac{(X+Y)}{X} = \frac{M}{X}$ and so $\frac{(X+Y)}{X} \cong \frac{Y}{(X \cap Y)}$. Therefore $X \cap Y$ is a maximal submodule of Y. Then $X \cap Y = \mathrm{Rad}Y$. Hence Y is a local submodule of M. It follows that LocM is not a submodule of X. Hence $\frac{M}{LocM}$ does not contain a maximal submodule.

(3) $\Rightarrow$ (4) Suppose that $\frac{M}{CofM}$ contains a maximal submodule namely $\frac{X}{CofM}$. But then $\varnothing^{-1}(\frac{X}{CofM})$ is a maximal submodule of $\frac{M}{LocM}$, where $\varnothing: \frac{M}{LocM} \to \frac{M}{CofM}$ is epimorphism. Contradiction. Therefore $\frac{M}{CofM}$ does not contain a maximal submodule.

(4) $\Rightarrow$ (1) Let N be a cofinite submodule of M. Then N + CofM is a cofinite submodule of M and hence (4) gives that M = N + CofM. Since $\frac{M}{N}$ is finitely generated, it follows that $M = N + K_1 + ... + K_n$ for some positive integer n and cofinitely supplemented submodules $K_i (1 \le i \le n)$. By repeated use of 2.3.9 N has a supplement in M. It follows that M is cofinitely supplemented.

Definition 2.5.12 M is called semilocal if M is a finite sum of local modules. Note that semilocal modules are finitely generated.

Lemma 2.5.13 Let $L_i (1 \le i \le n)$ be a finite collection of local submodules of a module M and let N be a submodule of M such that $N + L_1 + ... + L_n$ has a supplement K in M. Then there exists a (possibly empty) subset I of $\{1,...,n\}$ such that $K + \sum_{i \in I} L_i$ is a supplement of N in M.

Proof- Suppose first that n = 1. Consider the submodule $H = (N + K) \cap L_1$ of L_1. If $H = L_1$ then 0 is a supplement of H in L_1 and the proof of 2.3.9 shows that

$K = K + 0$ is a supplement of N in M. If $H \neq L_1$ then L_1 is a supplement of H in L_1 and in this case $K + L_1$ is a supplement of N in M, again by the proof of 2.3.9. This proves the result when $n = 1$.

Suppose that $n > 1$. By induction on n, there exists a subset J of $\{2,\ldots,n\}$ such that $K + \sum_{i\in J} L_i$ is a supplement of $N + L_1$ in M. Now the case $n = 1$ shows that either

$K + \sum_{i\in J} L_i$ or $K + L_1 + \sum_{i\in J} L_i$ is a supplement of N in M.

Definitions 2.5.14 An abelian group M is called torsion if for every $a\in M$, there exists a non-zero $n\in \mathbf{N}$ with $na = 0$. It is called torsion-free if there is no such positive integer n with $na = 0$ for any element a of M. An abelian group G is called bounded if there exists a positive integer n such that $nG = 0$.

Consider an exact sequence E: $0 \to A \to B \to C \to 0$. B is said to be an extension of A by C. And such exact sequences form an abelian group Ext(C,A). An abelian group G is called **cotorsion**, if $\mathrm{Ext}(J,G) = 0$ for every torsion free group J. In other words, G is cotorsion if every extension of G by a torsion-free group splits.

Let G be a subgroup of an abelian group A. Then G is called pure in A if $nG = G \cap nA$ for every $n\in \mathbf{N}$, that is for any positive integer n, and for any element g from G, if $n \mid g$ in A then also $n \mid g$ in G.

Example 2.5.15 Let's give an abelian group which is cofinitely supplemented but not amply cofinitely supplemented. Let $A = \bigoplus_{n=1}^{\infty} \mathbf{Z}(p^n)$ then clearly A is torsion but not bounded and by (Fuchs,1970) 54.4, A is not cotorsion. That is Ext(Q,A) is non-zero. Hence there exists an exact sequence E: $0 \to A \to B \to Q \to 0$ which is not splitting. $B/A \cong Q$ is torsion-free and that's why A is pure in B again by (Fuchs,1970). Since Z(p) is a direct summand of A and A is pure in B by (Fuchs,1970)27.1, Z(p) is a direct summand in B. Thus there exists $K \leq B$ such that $B = Z(p) \oplus K$. Choose

$1 \in Z(p)$ and $k \in K$ with $o(k) = \infty$. If we denote cyclic group generated by $(1 + k)$ with X, then clearly $X + K = B$. If B was amply cofinitely supplemented then there would be a subgroup Y of X with $Y + K = B$ and $Y \cap K << Y$. But since $Y \cong Z$, $Y \cap K = 0$, so $B = Y \oplus K$ impliying that $Z(p) \cong B/K \cong Z$ contradiction. Now let's prove that B is cofinitely supplemented. Suppose that B/X is finitely generated. Then $B/(X+A)$ is also finitely generated

$X/(A \cap X) \cong (X+A)/A \leq B/A \cong Q$. And $(B/A)/(X+A/A) \cong B/(X+A)$. Since Q has no finitely generated non-zero factors, $B/(X+A) = 0$. Then $B = X + A$. A is cofinitely supplemented because it is a sum of cofinitely supplemented modules. Since A is cofinitely supplemented and $X + A$ has supplement then by 2.3.9, X has a supplement.

Lemma 2.5.16 Let M be a finitely generated module such that every maximal submodule is a direct summand of M. Then M is semisimple.

Proof- Let N be an essential submodule of M. If N is a proper submodule of M then N is contained in a maximal submodule P of M. But $M = P \oplus K$ for some submodule K of M. Hence P is not essential in M, a contradiction. Thus $N = M$. It follows by Lemma 2.5.4 that M is semisimple.

Let P(N) denote the collection of maximal submodules of M containing N. In particular, P(M) is the empty set and P(0) is the collection of all maximal submodules of M (which could also be the empty set). Define a relation $\mathfrak{R}$ on the lattice of submodules of M as follows: Given submodules N, L of M then $N \mathfrak{R} L$ if and only if $P(N) = P(L)$. Clearly, $\mathfrak{R}$ is an equivalence relation on the lattice of submodules of M. Note that every maximal submodule has a supplement in M if and only if $M \mathfrak{R} \mathrm{Loc}(M)$.

Theorem 2.5.17 The following statements are equivalent for a module M.

(1) M is amply cofinitely supplemented.

(2) Every submodule N of M with M/N is cyclic has ample supplements in M.

(3) Every maximal submodule has ample supplements in M.

(4) N $\mathfrak{R}$ Loc(N) for every submodule N of M.

(5) mR $\mathfrak{R}$ Loc(mR) for every m in M but not in Rad(M).

Proof- (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let N be any submodule of M such that M/N is finitely generated. If N = M then N has ample supplements in M. Suppose that N does not equal to M. Let L be the proper submodule of M containing N such that L/N = Rad(M/N), i.e. L is the intersection of all maximal submodules of M containing N. Note that L/N is small in M/N, because M/N is finitely generated. Let P be any maximal submodule of M such that L is contained in P. By assumption there exists a submodule T of M such that M = P + T and P $\cap$ T << T. It follows that $M/L = P/L \oplus (T+L)/L$ because, $P/L \cap (T+L)/L << (T+L)/L$ and hence also in M/L, i.e. $P/L \cap (T+L)/L = 0$. Thus, M/L is finitely generated and semisimple. Therefore, L is a finite intersection of maximal submodules of M. Thus by 41.8 in (Wisbauer,1991), L has ample supplements in M.

Let K be any submodule of M such that M = N + K. Then M = L + K. There exists a submodule S of K such that M = L + S and L $\cap$ S << S. Now $M/N = L/N + (S+N)/N$, so that $M/N = (S+N)/N$ and hence M = S + N. Clearly N $\cap$ S is contained in L $\cap$ S so that N $\cap$ S << S. Thus S is a supplement of N in M. It follows that N has ample supplements.

(3)⇒(4) Let F be any submodule of M. Let Q be a maximal submodule of M such that F is not contained in Q. Then $M = F + Q$. There exists a submodule T of F such that $M = Q + T$ and $Q \cap T \ll T$. It follows that T is a local submodule of F and T is not contained in Q. Thus Loc(F) is not contained in Q. Therefore $F \mathfrak{R} \mathrm{Loc}(F)$.

(4)⇒(5) Clear.

(5)⇒(3) Let P be any maximal submodule of M and let G be a submodule of M such that $M = P + G$. There exists g in G such that g does not belong to P. Hence gR is not contained in P, so that Loc(gR) is not contained in P because $gR\ \mathfrak{R}\ \mathrm{Loc}(gR)$. Let L be a local submodule of gR and hence also of G, such that L is not contained in P. Then $M = P + L$ and $P \cap L$ is small in L. Thus L is a supplement of P in M. It follows that P has ample supplements.

Corollary 2.5.18 Let M be a module such that $N = \mathrm{Loc}(N)$ for every submodule N of M. Then every maximal submodule of M has ample supplements in M.

Lemma 2.5.19 Let M be an R-module with $M = U_1 + U_2$. If the submodules U_1, U_2 have ample supplements in M, then $U_1 \cap U_2$ has also ample supplements in M.

Proof- Let V be a submodule of M with $(U_1 \cap U_2) + V = M$. Then with the help of modular law, $M = U_1 + U_2 = U_1 + (U_2 \cap M) = U_1 + (U_2 \cap ((U_1 \cap U_2) + V))$ $= U_1 + (U_1 \cap U_2) + (U_2 \cap V) = U_1 + (U_2 \cap V)$. Similarly, $U_2 + (U_1 \cap V) = M$ also holds. Therefore there is a supplement X of U_1 in M with $X \leq U_2 \cap V$, and a supplement Y of U_2 with $Y \leq U_1 \cap V$. Also $(U_1 \cap U_2) + (X + Y) = M$ and $(X + Y) \cap (U_1 \cap U_2) = (Y \cap U_2) + (X \cap U_1) \ll X + Y$.

Proposition 2.5.20 A finitely generated amply cofinitely supplemented module is supplemented.

Proof- Let M be a finitely generated Amply Cofinitely Supplemented Module and $U \leq M$. By 2.3.7, $M/Rad(M)$ is semisimple. $M/(U + Rad(M))$ is also semisimple because it is epimorphic image of the module $M/Rad(M)$.

$M/(U + Rad(M)) = S_1 \oplus S_2 \oplus \ldots\ldots \oplus S_n$ for some simple modules S_i.

Let $S_1 \oplus S_2 \oplus \ldots \oplus S_n = S$, then let's say

$S_2 \oplus S_3 \oplus \ldots\ldots \oplus S_n = \mathbf{S}_1$, $S_1 \oplus S_3 \oplus \ldots\ldots \oplus S_n = \mathbf{S}_2$, ..., $S_1 \oplus S_2 \oplus \ldots\ldots \oplus S_{n-1} \oplus S_n = \mathbf{S}_n$.

Then it is clear that $\bigcap_{i=1}^{n} \mathbf{S}_i = U + Rad(M)$. Therefore U + Rad(M) is the intersection of finitely many maximal submodules and by 2.5.9 every maximal submodule has ample supplements. By 2.5.19 U + Rad(M) has ample supplements and so it has supplements. Since M is finitely generated $Rad(M) \ll M$, and hence by 2.3.9 U has a supplement.

Definition 2.5.21 We say that an R-module M is semiperfect if every factor module of M has a projective cover.

If the ring R as a left R-module is semiperfect then we say that the ring R is semiperfect.

Theorem 2.5.22 The following statements are equivalent for a ring R.

1- R is semiperfect

2- Every right R-module is amply cofinitely supplemented.

3- Every right R-module is cofinitely supplemented.

4- Every right R-module is amply cofinitely supplemented.

5- Every left R-module is cofinitely supplemented.

Proof- (1) $\Rightarrow$ (2) Suppose that R is a semiperfect ring. Let M be any right R-module. Let $m \in M$. By (Wisbauer,1991) 42.6 mR is supplemented and hence $mR/Loc(mR)$ does not contain a maximal submodule by 2.5.10. i.e. mR = Loc(mR). By 2.5.16. M is amply cofinitely supplemented.

(2)⇒(3) Clear.

(3)⇒(1) By (3) the right R-module R is supplemented and R is semiperfect by (Wisbauer,1991) 42.6

(1)⇔(4)⇔(5) By symmetry.

Definition 2.5.23 A module M is called π-projective if whenever N and L are submodules of M such that $M = N + L$, there exists an endomorphism ρ of M such that $\rho(M) \leq M$ and $(1-\rho)(M) \leq L$.

Proposition 2.5.24 Let M be a π-projective cofinitely supplemented module. Then M is amply cofinitely supplemented.

Proof- Let N be a cofinite submodule of M and let L be a submodule of M such that $M = N + L$. There exists an endomorphism ρ of M such that $\rho(M) \leq N$ and $(1-\rho)(M) \leq L$. Note that $(1-\rho)(N) \leq N$. Let K be a supplement of N in M. Then $M = \rho(M) + (1-\rho)(M) = \rho(M) + (1-\rho)(N + K) \leq N + (1-\rho)(K) \leq M$, so that $M = N + (1-\rho)(K)$. Note that $(1-\rho)(K)$ is a submodule of L.

Let $y \in N \cap (1-\rho)(K)$. Then $y \in N$ and $y = (1-\rho)(x) = x-\rho(x)$, for some $x \in K$. Next $x = y + \rho(x) \in N$, so that $y \in (1-\rho)(N \cap K)$. But $N \cap K << K$ gives that $N \cap (1-\rho)(K) = (1-\rho)(N \cap K) << (1-\rho)(K)$. Thus $N \cap (1-\rho)(K) << (1-\rho)(K)$ and $(1-\rho)(K)$ is a supplement of N in M. It follows that M is amply cofinitely supplemented.

Lemma 2.5.25 Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules $M_i (i \in I)$. Then $Loc(M) = \bigoplus_{i \in I} LocM_i$.

Proof- For each $i \in I$, let $\pi_i: M \to M_i$ be the canonical projection. Let L be a local submodule of M. For each $i \in I$, $\pi_i(L)$ is a homomorphic image of L and hence is a local submodule of M_i so that $\pi_i(L) \leq Loc(M_i)$. Hence $L \leq \bigoplus_{i \in I} LocM_i$. It follows that $Loc(M) \leq \bigoplus_{i \in I} LocM_i$. Conversely, for each $i \in I$, clearly $Loc(M_i) \leq Loc(M)$. Hence $\bigoplus_{i \in I} LocM_i \leq LocM$.

CHAPTER THREE

COFINITELY SUPPLEMENTED MODULES OVER COMMUTATIVE DOMAINS.

3.1 Cofinitely supplemented modules over Dedekind domains.

Throughout this section, let R be a Non-Local Commutative Noetherian Domain. Let Art(M) denote the sum of all Artinian submodules of M.

Lemma 3.1.1 Let M be an R-module. Then

1- Art(M) is the unique maximal locally Artinian submodule of M.

2- Art(M) = $\{m \in M: mP_1....P_n = 0$ for some positive integer n and maximal ideals $P_i (1 \leq i \leq n)\}$

3- Art($M/_{Art(M)}$) = 0.

4- Art(M) $\leq$ Loc(M) $\leq$ Tor(M).

Proof- (1) Let A = Art(M). Let $m \in A$. Then $m \in \sum_{i \in I} Li$ where $\{L_i\}_{i \in I}$ is the collection of all Artinian submodules of M. There exists a finite subset F of I such that $m \in \sum_{i \in F} Li$. But $\sum_{i \in I} Li$ is Artinian, so that mR is Artinian. Thus A is a locally Artinian submodule of M. Let B be a locally Artinian submodule of M. Then B = $\sum_{b \in B} bR \leq$ A. This proves (1).

(2) Let $m \in$ Art(M). By (1) mR is Artinian and hence mR has finite composition length. It follows that $mP_{1.....}P_n = 0$ and some (not necassarily distinct) maximal ideals $P_i(1 \leq i \leq n)$. Conversely, suppose that $x \in M$ and $xQ_1.....Q_k = 0$ for some positive integer k and maximal ideals $Q_i(1 \leq i \leq k)$. Then the descanding chain $xR \geq xQ_1 \geq ...... \geq xQ_1....Q_k = 0$ has Artinian factors, so that xR is Artinian. Thus $x \in A$.

(3)- Let C be a submodule of M containing A such that C/A is Artinian. Let $c \in C$. Then cR is Noetherian and hence $cR \cap A$ is Artinian by (1). But

$cR/(cR \cap A) \cong (cR + A)/A$ is also Artinian. Thus cR is Artinian and hence $c \in A$. It follows that C = A.

(4)- Let $m \in Art(M)$. By (1) mR is Artinian. Now $mR \cong R/J$ where J is the ideal $\{r \in R: mr = 0\}$ of R. The ideal R/J is Artinian and is a direct sum of local submodules and that $m \in Loc(M)$. Hence $Art(M) \leq Loc(M)$.

Let L be a local submodule of M. Then L is cyclic and, because R is not a local ring, L is not isomorphic to R. It follows that $L \leq Tor(M)$. Hence $Loc(M) \leq Tor(M)$.

Definition 3.1.2 A ring R is called one dimensional if every non-zero prime ideal is maximal.

Lemma 3.1.3 The following statements are equivalent for a Commutative Noetherian Domain R.
1- R is one dimensional.
2- Art(M) = Loc(M) = Tor(M) for every R-module M.

Proof- (1) $\Rightarrow$ (2) Let M be any R-module. Let $m \in Tor(M)$ and let $A=\{r \in R: mr = 0\}$. Then A is a non-zero ideal of R. By (1) every prime ideal of the ring R/A is maximal and R/A is Artinian by (Sharp,1972), Theorem4.6 Hence mR is Artinian and $m \in Art(M)$. Thus $Tor(M) \leq Art(M)$. By 3.1.1(4) (2) follows.

(2)$\Rightarrow$(1) Let P be any non-zero prime ideal of R. Appliying (2) to the R-module R/P we see that R/P is Artinian and hence P is a maximal ideal. Thus R is one dimensional.

Definition 3.1.4 A ring R is called a Jacobson ring if every prime ideal is an intersection of maximal ideals.

Lemma 3.1.5 Let R be a Jacobson ring and let M be an R-module. Then Art(M) = Loc(M).

Proof- It is sufficient to prove that every local R-module is Artinian. Let L be a local R-module. Then L is cyclic and we can suppose without loss of generality that L = R/A for some non-zero ideal A of R. Note that the ring R/A is local. Let P be any prime ideal of R containing A. Since R/P is local it follows that P is a maximal ideal. Hence every prime ideal of R/A is maximal and R/A is Artinian, as required.

Lemma 3.1.6 The following statements are equivalent for a Commutative Noetherian Domain R.

1- Loc(M) = Tor(M) for every R-module M.

2- R/A is a semiperfect ring for every non-zero ideal A of R.

Proof- (1) $\Rightarrow$ (2) Let A be any nonzero ideal of R. Let X be any R/A-module. Then X is an R-module such that XA = 0. Hence X is a torsion R-module and by hypothesis X = Loc(M). By 2.5.11 the R-module(hence the R/A-module) X is cofinitely supplemented. Now 2.5.22 gives that R/A is semiperfect.

(2)$\Rightarrow$(1) Let M be any R-module. By 3.1.1 (4), Loc(M) $\leq$ Tor(M). Let m$\in$Tor(M) and let B = {r$\in$R: mr = 0}. Then B is a non-zero ideal of R so that mR is a cyclic module over the semiperfect ring R/B. By 2.5.22 and 2.5.11, mR = Loc(mR) $\leq$ Loc(M). Thus Tor(M) $\leq$ Loc(M). It follows that Loc(M) = Tor(M).

Theorem 3.1.7 Let R be a Commutative Noetherian Non-local one dimensional domain. Then an R-module M is cofinitely supplemented if and only if $M/Tor(M)$ has no maximal submodule.

Proof- Let M_0 be the R-module $M/Tor(M)$. Suppose that M_0 has no maximal submodule. Then $M_0 = M_0P$ for every maximal ideal P of R. Let $0 \neq c \in R$. By (Sharp,1972), Theorem 4.6 R/Rc is an Artinian ring and hence $P_1 \ldots P_n \leq Rc$ for some positive integer n and maximal ideals $P_i(1 \leq i \leq n)$. Now $M_0 = M_0P_n = M_0P_{n-1}P_n = \ldots\ldots = M_0P_1 \ldots P_n \leq M_0R_c = M_0c \leq M_0$. Thus $M_0 = M_0c$ for every non zero element c of R. By (Sharp,1972) Proposition 2.7, M_0 is injective. On the other hand, because R is not a field, any injective R-module does not contain a maximal submodule by (Sharp,1972), Proposition 2.6. This completes the proof.

Lemma 3.1.8 Any Locally Artinian R-module is amply cofinitely supplemented.

Proof- Let M be a locally Artinian R-module. Let $m \in M$. Then mR is Artinian and mR is a direct sum of local submodules by the proof of 2.1.1 (4). Hence $mR = Loc(mR)$ for every $m \in M$. By 2.5.17 , M is amply cofinitely supplemented.

Corollary 3.1.9 Let R be a one dimensional domain. Then any torsion R-module is amply cofinitely supplemented.

Proof- By 3.1.3 and 3.1.8.

Lemma 3.1.10 Let an amply cofinitely supplemented R-module $M = N \oplus S$ be a direct sum of submodules N, S such that N has a maximal submodule. Then S is a torsion module.

Proof- Let K be a maximal submodule of N and let T be the submodule K ⊕ 0 of M. Then $M/N \cong N/K \oplus S$ is amply cofinitely supplemented. Thus, without loss of generality, we can suppose that N is simple. Let $0 \neq n \in N$ and note that N = nR. Let $x \in S$. Then S is a maximal submodule of M, because $M/S \cong N$, and

M = (n + x)R + S. By hypothesis there exists a local submodule L of (n + x)R such that M = L + S. Since the domain R is not a local ring it follows that $\mathrm{Loc}({}_R R) = 0$ and hence (n + x)R is not isomorphic to R. There exists $0 \neq c \in R$ such that (n + x)c = 0, so that xc = 0. It follows that S is a torsion module.

Theorem 3.1.11 Let R be a non-local Dedekind Domain. Then an R-module M is amply cofinitely supplemented if and only if M is injective or torsion.

Proof- The sufficiency follows from 3.1.9 and the fact that injective R-modules have no proper cofinite submodules. Conversely, suppose that M is an amply cofinitely supplemented R-module. Suppose that M is not injective. Let M = Tor(M). By 3.1.7 the factor module M/T is injective. Now T is not injective for otherwise M would be injective. By (Sharp,1972), Theorem 4.25 there exists a maximal ideal P of R such that $T \neq TP$. The module M/TP is amply cofinitely supplemented and is not injective because $\mathrm{Tor}(M/TP) = M/TP$. Moreover M is a torsion module if and only if M/TP is a torsion module. Hence we can suppose without loss of generality that TP = 0. (Sharp,1972), Proposition 2.6, $M/T = (M/T)P$ and hence M = MP + T. This implies that $MP = MP^2 + TP = MP^2$, i.e. MP = (MO)P. For any maximal ideal $Q \neq P$, M = MQ + T and T = TQ so that M = MQ and also MP = (MQ)P = (MP)Q. It follows that MP is a divisible R-module and hence MP is injective by (Sharp,1972), Theorem 4.25.

There exists a submodule N of M such that $M = N \oplus MP$. Note that $N \cong M/MP$ so that N is semisimple. Also N is non-zero since M is not injective. It follows that N has a maximal submodule. By 3.1.9, MP is a torsion module and hence so also is M.

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