FEEDBACK FLUID QUEUES WITH MULTIPLE THRESHOLDS

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By
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August 2006

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Unlike discrete or continuous time queuing systems fed with point processes, workload in fluid queues arrives at the system as a fluid flow rather than jobs or packets. The rate of the fluid flow is governed by a continuous time Markov chain in Markov fluid queues. In first order fluid queues, rates are deterministically determined by a background Markov chain whereas in second order fluid queues, a Brownian motion is additionally inserted to the queue content process. Each of those queues can either accommodate a single regime or multiple regimes (equivalently multiple thresholds) in which the rates and the infinitesimal generator might be different in different regimes but they should be fixed within a single regime. In this thesis, we first generalize the existing solution of first order feedback fluid queues with multiple thresholds for the steady state distribution function of queue occupancy by also allowing the existence of repulsive type boundaries and states with zero rates. Secondly, we complete the boundary conditions for not only the transient but also the steady state solution of second order feedback fluid queues with multiple thresholds. Finally, we apply the theory of feedback fluid queues with multiple thresholds as an effective approximation to the Markov modulated discrete time queueing model that arises in the performance evaluation of an adaptive MPEG video streaming system in UMTS environment. By doing so, we eliminate the state space explosion problem that arises in the original discrete model.

Keywords: Markov fluid queues, fluid queues with multiple thresholds, second order fluid queues, Brownian motion.

ÖZET

ÇOK EŞİKLİ GERİ BESLEMELİ AKIŞKAN KUYRUK SİSTEMLERİ

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Ayrık sıra sistemlerinde paket veya iş olarak erişen iş yükü, akışkan kuyruk sistemlerinde sıvı akışı gibi erişir. Bu akışın hızı bir sürekli zaman Markov zinciri tarafından belirlenir. Kabaca, akışkan kuyruk sistemleri birinci ve ikinci derece olmak üzere iki gruba ayrılır. İkinci derece akışkan kuyruk sistemleri içerik işlemine eklenmiş olan Brownian hareketi sayesinde değişinti içerdiği halde, birinci derece akışkan kuyruk sistemlerinde akışkan hızı yalnızca Markov zinciri tarafından belirlenir ve değişinti içermez. Her iki sistem de tek veya çok rejimli (eşlenik olarak çok eşikli) olabilir. Farklı rejimlerde farklı akışkan hızları ve geçiş matrislerine sahip olabildikleri halde tek bir rejim içerisinde bu değerler sabittir. Tez içerisinde birinci derece çok eşikli geribeslemeli akışkan kuyruk sistemlerinin kararlı zaman dağılım fonksiyonları çözümünü itici eşik ve sıfır akışkan hızına izin vererek genelleştirdik. İkinci olarak, ikinci derece çok eşikli geribeslemeli akışkan kuyruk sistemlerinin çözümünde hem geçici hem de kararlı zaman analizindeki sınır şartlarını tamamladık. Son olarak çok eşikli geribeslemeli akışkan kuyruk sistemleri teorisini Markov modüleli ayrık kuyruk sistemi ile modellenmiş bir UMTS içerisinde MPEG video aktarım sisteminin performans başarımına bir yaklaşım olarak uygulayarak; durum sayısı fazlalığı yüzünden sayısal sonuç alınamayan ayrık modele yaklaşık sonuçlar elde ettik.

Anahtar Kelimeler: Markov akışkan kuyruk sistemleri, çok eşikli geribeslemeli akışkan kuyruk sistemleri, ikinci derece akışkan kuyruk sistemleri, Brownian hareketi.

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Chapter 1

Introduction

One of the most prominent tools for performance evaluation in telecommunication networks is *queueing systems*. A queueing system consists of randomly arriving jobs with random service times; a server, and a waiting room for arriving jobs which are not under service. Three types of queueing systems are commonly used in the performance evaluation of telecommunication networks: renewal, Markovian, and Markov fluid queueing systems. In the first two, the arrival and service processes are point processes and in the third model we use fluid flows as opposed to using point processes.

Renewal Queueing Systems: Arrival process and service time of these queueing systems is modeled with a renewal process. The total workload (in terms of time required) in the queue (or buffer) at the arrival instant of $(n+1)^{st}$ job denoted by W_{n+1} can be found by the following Lindley's equation:

$$W_{n+1} = \max(W_n + B_n - A_n, 0), n \geq 0, \tag{1.1}$$

where A_n is the interarrival time between the n^{th} and the $(n+1)^{st}$ jobs and B_n is the service time of the n^{th} job. Examples include M/G/1 and GI/G/1 queues.

According to the independence assumption of interarrival times for a renewal process (see [3] for detailed information on renewal processes), there is no correlation among successive interarrival times; therefore they are not able to capture one of the most important traffic characteristics in today's networks which is the bursty traffic behavior. Informally, *burstiness* is described as the existence of autocorrelation in successive interarrival or service times in which case the arrival or the service process is said to be bursty. For bursty traffic, it is well known that short term traffic rates can very well exceed the average rates. According to [1] and [2], burstiness depends on mainly two factors: the shape of the marginal distribution and the autocorrelation function.

Markov-based Queueing Systems: In these systems, arrival or service process is driven (or modulated) by either a discrete or a continuous time Markov chain. The interarrival and service times are then made to depend on the instantanous state of the modulating chain. Markov-based queueing systems can effectively be used for capturing the burstiness. Moreover, there is a vast amount of literature on Markovian queueing systems and their use in the performance evaluation of telecommunication networks.

Markovian Fluid Flow Queueing Systems: In these systems rather than like jobs or packets, workload arrives at the buffer as a fluid flow. The rate of this flow is determined by a continuous time Markov chain. This property makes the system Markovian. We will use shortly the term "fluid queue" without indicating the Markovian property throughout the thesis. Fluid queues are roughly divided into two groups as first order and second order fluid queues. In first order fluid queues, rates are deterministically determined by a background Markov chain, whereas in the second order fluid queues a Brownian motion is additionally inserted to the buffer content process. Each of those queues can either accommodate a single regime or multiple regimes in which the rates and the infinitesimal generator might be different in different regimes but they should be constant within a single regime. It might either be the transient or the steady state queue occupancy distribution that the performance analysts seek a solution.

1.1 First Order Single Regime Fluid Queues

A first order fluid queue with a single regime is described by a joint Markov process $\{C(t), X(t); t \geq 0\}$, where $\{X(t); t \geq 0\}$ is a finite state continuous time Markov chain (CTMC) and does not depends on the content process $\{C(t), t \geq 0\}$ of the buffer. In each state, i, of the driving process $\{X(t); t \geq 0\}$, the content of the buffer changes according to an assigned constant net rate[1] r_i whenever the content is in between the upper and lower boundaries and it remains unchanged whenever the rate forces the content to overflow the upper boundary point or underflow the lower boundary point. It is not difficult to drive that the transient distribution function $F_i(x,t) = P(C(t) \le x, X(t) = i)$ satisfies the partial differential equation

$$\frac{\partial \mathbf{F}(x,t)}{\partial t} + \frac{\partial \mathbf{F}(x,t)}{\partial x} R = \mathbf{F}(x,t) Q, \tag{1.2}$$

where R is the diagonal matrix of net rates, Q is the infinitesimal generator of the CTMC and $\mathbf{F}$ is the row vector of $F_i(x,t)$'s i.e. $\mathbf{F}(x,t) = [F_1(x,t)F_2(x,t)\ldots F_N(x,t)]$ for an N state system. Eq. (1.2) emerges from forward Kolmogorov equations,

$$F_i(x,t+\Delta t) = F_i(x - r_i\Delta t, t)(1 + q_{ii}\Delta t) + \sum_{j \ne i} F_j(x - o(\Delta t), t) q_{ji} \Delta t + o(\Delta t). \tag{1.3}$$

Taking the limit of (1.2) as $t \to \infty$, we obtain the following ordinary matrix differential equation:

$$\frac{d\mathbf{F}(x)}{dx} R = \mathbf{F}(x) Q, \tag{1.4}$$

where

$$\mathbf{F}(x) = \lim_{t \to \infty} \mathbf{F}(x,t) \tag{1.5}$$

Obviously, the solution of (1.4) can be expressed as

$$\mathbf{F}(x) = \sum_{i=1}^{N} a_i e^{\lambda_i x} \phi_i, \tag{1.6}$$

where (λ_i, ϕ_i) is the i^th^ left eigenvalue-eigenvector pair of the matrix QR^{-1}, provided that eigenvalues are simple and R is invertible, i.e., there is no state with zero net rate. The unknown coefficients can be found by the corresponding sufficient number of boundary conditions which can be obtained by formulating

[1] What makes the system first order is having a constant net rate. If rates have an additional Brownian motion then the system becomes second order.

the nature of queue (we use buffer and queue interchangeably throughout the thesis) at the boundaries and additionally at infinity if the queue is of infinite size. For instance, if the queue size is infinite, since there will be no accumulation at zero for states with positive rate, the distribution function has to be zero at the zero boundary for these states. Moreover, for a stable system, the coefficients corresponding to eigenvalues with positive real parts will be zero. A more detailed analysis can be found in [5].

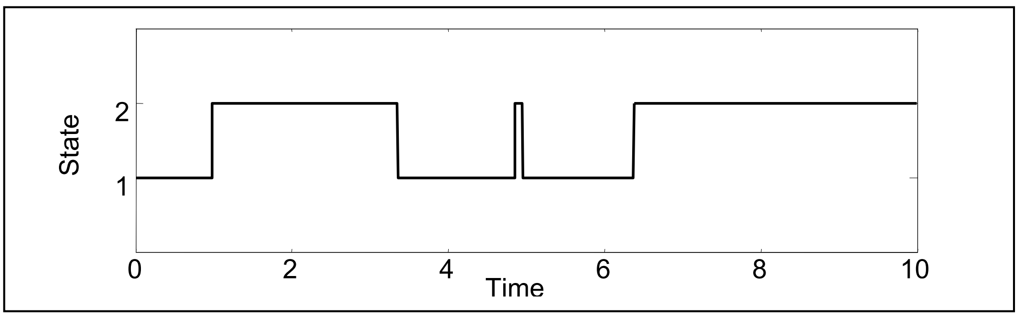
1.2 Second Order Single Regime Fluid Queues

Based on [6], second order fluid queues are described by the same joint Markov process $\{C(t), X(t); t \geq 0\}$ as the first order queues. The only difference is that the rate of change of buffer content is determined by Brownian motion whose variance and drift parameters change according to the background process. The PDF of the buffer content behavior, $\mathbf{f}(x,t)$, can then be found from the partial differential equation:

$$\frac{\partial \mathbf{f}(x,t)}{\partial t} - \frac{\partial^2 \mathbf{f}(x,t)}{\partial x^2}\Sigma + \frac{\partial \mathbf{f}(x,t)}{\partial x}R = \mathbf{f}(x,t)Q\,, \tag{1.7}$$

where Σ is the diagonal matrix of half of variances, i.e. $\{\Sigma\}_{i,i} = \sigma_i^2/2$ is the half of variance of state i and R is the diagonal matrix of mean drifts. The spectral solution and derivation of (1.7) are not as simple as the first order one and the corresponding analysis can be found in [6].

The difference of first order and second order singe regime fluid queues can be understood from 10 second sample paths depicted in Fig. 1.1.



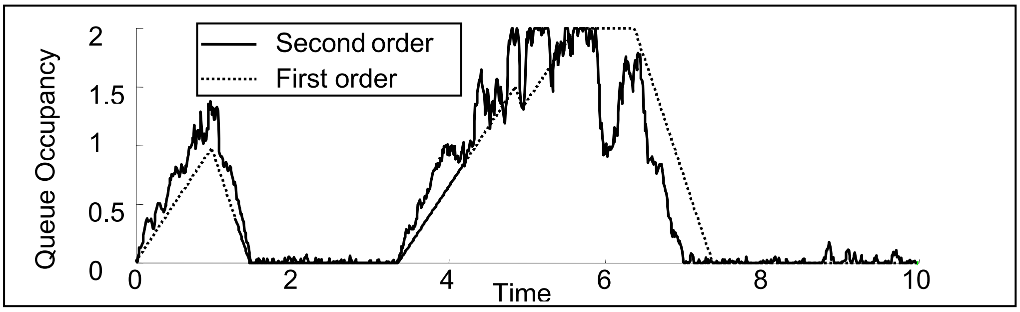


Figure 1.1 State transitions and corresponding sample paths for first and second order single regime fluid queues

In that figure, both first and second order systems have a queue length 2 and two states in the background process. In the first state net rate of change of queue content is 1 for the first order fluid queue and drift is selected as 1 with variance 0.1 for the second order queue. In the second state net rate of change of queue content is -2 for the first order fluid queue and drift is selected as -2 with variance 0.1 for the second order queue.

1.3 First Order Feedback Fluid Queues with Multiple Thresholds

First order single regime fluid queues are generalized in [7] by defining $K+1$ boundary points (we use the terms boundary point and threshold interchangeably) $0 = T^{(0)} < T^{(1)} < .... < T^{(K)} < \infty$ or equivalently assigning different infinitesimal generators and rates to the different regions between these boundary points. The queue is said to be in regime (or region) k if $T^{(k-1)} < C(t) < T^{(k)}$. Assuming there are N states and K regimes, this system and its parameters are shown in Fig. 1.2.

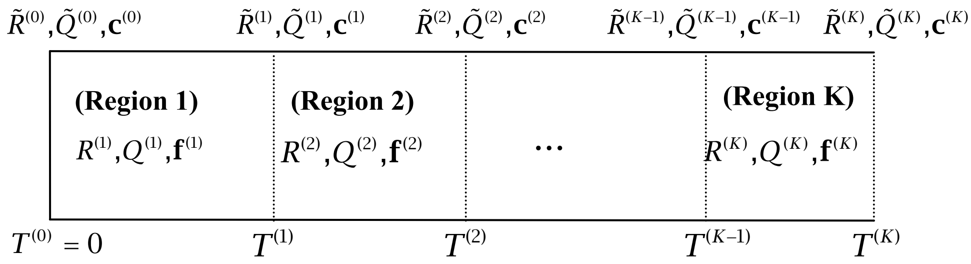


Figure 1.2 A First Order Feedback Fluid Model with Multiple Thresholds

In this figure, $R^{(k)}$ is the diagonal matrix of rates, $Q^{(k)}$ is the infinitesimal generator for the background process, $\mathbf{f}^{(k)}$ (abbreviated notation for $\mathbf{f}^{(k)}(x)$) is vector of PDFs for region k. $\tilde{R}^{(k)}$ is the diagonal matrix of rates, $\tilde{Q}^{(k)}$ is the infinitesimal generator for the background process, $\mathbf{c}^{(k)}$ is vector of probability mass accumulations, all at the boundary point $T^{(k)}$. Let $S_+^{(k)}$, $S_0^{(k)}$ and $S_-^{(k)}$ denote the set of states with positive, zero, and negative rates, respectively, in region k, and $\tilde{S}_+^{(k)}$, $\tilde{S}_0^{(k)}$ and $\tilde{S}_-^{(k)}$ denote the set of states with positive, zero, and negative rates, respectively, at the boundary point $T^{(k)}$, throughout the thesis. $S = S_+^{(k)} \cup S_-^{(k)} \cup S_0^{(k)}$ is the set of all states.

1.4 Second Order Feedback Fluid Queues with Multiple Thresholds

The idea of generalization of a second order single regime fluid queue to a second order feedback fluid queue with multiple thresholds is the same as the one made for first order fluid queues. As in the single regime case there still exists a finite state background process $\{X(t); t \geq 0\}$, in each state of which drift and variance parameters are distinct. However, contrary to the single regime case, the infinitesimal generator of the driving process, and the drift and variance parameters of each state are determined by piecewise constant functions of buffer occupancy. The parameters of this system are depicted in Fig. 1.3.

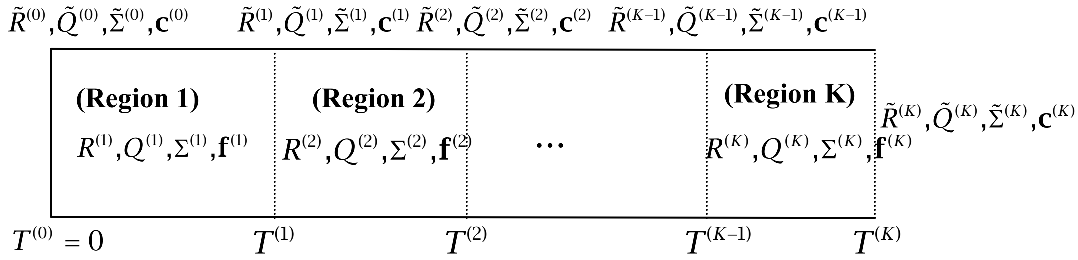


Figure 1.3 A Second Order Feedback Fluid Model with Multiple Thresholds

In this figure, notation is the same as the first order case. The additional parameter Σ represents the diagonal matrix of half of the variances and superscript in parenthesis represents the regime to which the parameters belong.

1.5 Motivation and Thesis Overview

The thesis has three main contributions:

- First order fluid queue models are studied in [5] and their multi-regime extension is studied in [7]. The reference [7] makes the following two main assumptions for the analysis of multi-regime fluid queues: for the background process (i) there is no state with a negative rate below the boundary and a positive rate above the boundary, (ii) there is no state with rate zero in any regime of the buffer. However, there are stochastic systems that naturally occur in practice that violate these two assumptions. We relax these assumptions in Chapter 2.
- Transient solution to second order fluid queues are studied in [8] from which steady state buffer occupancy distribution can be obtained. After a thorough study of [8], we find out that some of the boundary conditions in the solution given in [8] are absent. We obtain the missing boundary conditions in Chapter 3.

- Fluid queues play two important roles in queueing theory. Beyond their role in modeling continuous flows, they are also used as an approximate model to either continuos or discrete time Markov modulated queueing systems. This approximation becomes extremely useful whenever the underlying dimensionality of the original system is excessive making it very hard to solve the corresponding Markov chain if not impossible. However, we note that the computational complexity for the fluidized queue, i.e. fluid queue approximation, is independent of the queue length. Beyond the number of states, structure of the matrix constructed for the solution of discrete systems is also an essential criteria in the computational complexity. However, if that matrix is not well structured number of states become a crucial parameter. In Chapter 4, we use the fluid models as an approximation to a specific family of Markov modulated discrete time queues and finally we apply these results to calculate the steady state distribution of the buffer occupancy in an adaptive video streaming system.

The thesis is organized as follows. In Chapter 2, we study single and multi-regime first order Markov fluid queues. Chapter 3 addresses second order single and multi-regime fluid queues. Fluid queue approximations to Markov modulated discrete time queues are described in Chapter 4 along with an application in the context of the performance evaluation of an adaptive video streaming system in a UMTS environment. We conclude in the final chapter.

Chapter 2

Analysis of First Order Feedback Fluid Queues with Multiple Thresholds

In a single regime system, the rates and infinitesimal generator are independent of the queue occupancy; therefore this system does not accommodate any queue content management policy. Fluid queues in which the rates and infinitesimal generator change as a function of queue occupancy are called feedback fluid queues. When such changes are allowed to be piecewise continuous and queue capacity is finite, a solution is given in [9]. Furthermore, the same system with infinite queue length, is analyzed in [18] for a two-state system. However there are two limitations in the above mentioned studies. The first limitation is that the steady state solution to the queue occupancy is given in terms of a linear differential equation but with variable coefficients and it is generally hard to obtain numerical solutions for such systems. Moreover both studies assumed that the sign of the rate in each state is the same for all queue occupancies which limits the practicability of the proposed approaches. We note that this assumption is violated in some adaptive communication systems that we study in this thesis. In this thesis, we study a multi regime fluid queue in which the rate and the infinitesimal generator are fixed for each regime (or so called region) of queue occupancy. Equivalently, the rate and infinitesimal generators are piecewise constant (or as mathematicians call *simple*) functions of queue occupancy. Such a system is called *first order feedback fluid model with multiple thresholds* and it is solved in [7] mainly with four assumptions:

1) For all intermediate boundary points, if the rate in the region just below the boundary point is negative in a state then the rate in the region just above the boundary is also negative in that state.

2) Rates are nonzero everywhere except for some boundary points or equivalently rates are nonzero inside a region.

3) For all intermediate boundary points, if the rate in the region just below the boundary point is positive in a state and the rate in the region just above the boundary is negative in that state then the rate at that boundary point is zero.

4) In all regions and at all intermediate boundary points, there exist at least one state with a positive rate and another state with a negative rate.

However, it is possible to face real systems in which assumptions 1 and 2 are violated. For example, in the case study of Chapter 4 we can see those in some adaptive policies. In this thesis we relax the first two assumptions. We also note that the problem cannot be solved by the approach in [9] and [18], since in both these studies the rates are assumed to be nonzero everywhere on the queue.

2.1 Preliminaries

2.1.1 System Description

As described in section 1.3, first order single regime fluid queues are generalized in [7] by defining $K+1$ boundary points $0 = T^{(0)} < T^{(1)} < .... < T^{(K)} < \infty$ or equivalently assigning different infinitesimal generators and rates to the different regions between these boundary points. The queue is said to be in regime k if $T^{(k-1)} < C(t) < T^{(k)}$. Assuming there are N states and K regimes, this system and its parameters are shown in Fig. 2.1.

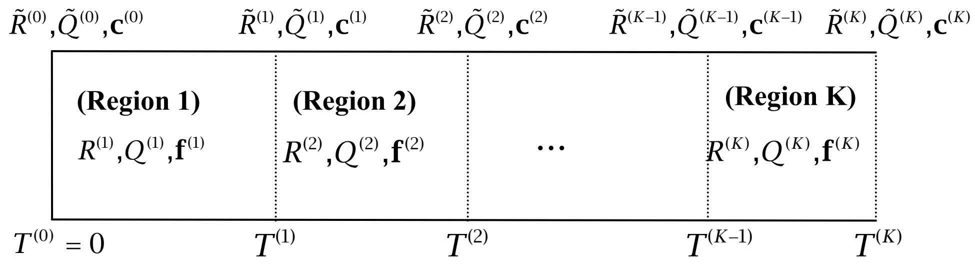


Figure 2.1 A First Order Feedback Fluid Model with Multiple Thresholds

In the figure, $R^{(k)}$ is the diagonal matrix of rates, $Q^{(k)}$ is the infinitesimal generator for the background process, $\mathbf{f}^{(k)}$ (abbreviated notation for $\mathbf{f}^{(k)}(x)$) is vector of PDFs for region k. $\tilde{R}^{(k)}$ is the diagonal matrix of rates, $\tilde{Q}^{(k)}$ is the infinitesimal generator for the background process, $\mathbf{c}^{(k)}$ is the vector of probability mass accumulations, all at the boundary point $T^{(k)}$. Let, $S_+^{(k)}$, $S_0^{(k)}$ and $S_-^{(k)}$ denote the set of states with positive, zero, and negative rates, respectively, in region k and $\tilde{S}_+^{(k)}$, $\tilde{S}_0^{(k)}$ and $\tilde{S}_-^{(k)}$ denote the set of states with positive, zero, and negative rates, respectively, at the boundary point $T^{(k)}$ throughout the thesis. $S = S_+^{(k)} \cup S_-^{(k)} \cup S_0^{(k)}$ is the set of all states.

2.1.2 Boundary Classification

The boundary points are classified according to their location and also the behavior both just at that point and the neighborhood of that point.

Boundary classification according to the location

We use the classification in [8] in which boundary points are divided into two groups according to their location.

1) Terminal Boundaries, i.e., boundaries at the edges of a queue: $T^{(0)}$ and $T^{(K)}$

2) Intermediate boundaries, i.e., other boundary points: $T^{(1)}, T^{(2)}, \ldots, T^{(K-1)}$ for a K region system.

Boundary classification according to the rates in neighbor regions

If the sign of the rates is the same at both sides of an intermediate boundary point in a state then we say that it is an emitting boundary for that state. For all emitting boundaries we assume that the sign of the rate is the same at that point as neighboring regions. This is a common assumption as used in [7].

If the rate just above a boundary point is positive and the rate below is negative in a state then we call that point a repulsive boundary point for that state, as was used in [10].

If the rate just above a boundary point is negative and rate below is positive in a state then we call it a sticky boundary in that state, as was used in [10]. It is important to note that the name "sticky" is also used for a specific type of boundary behavior, which is independent of the drift parameter at neighbor points; see [12] for more details.

Boundary classification according behavior at the boundary point

In the literature, there are a number of behavioral features assigned to the boundary point such as absorbing, sticky, reflecting, etc. For first order fluid queues, we use left and right continuity and absorbing behaviors. Left and right continuity of the behavior are clear. The boundary behaves as if it is in one of the neighbor regions depending on the direction of continuity. There is nothing special for this boundary point.

If the boundary point has an absorbing behavior then the queue stays the same until the background process transits to another state for which this particular point is not absorbing anymore. For first order fluid queues, absorbing behavior is equivalent to the situation of having a zero rate at that boundary point. For second order fluid queues, both the drift and the variance parameters are zero. In the analysis as was made in [7], all sticky boundaries are assumed to be absorbing, i.e. whenever the boundary point is reached, then the content process stays at that point till a change in the state of the background process. If fluid model is used for applications in which entities are continuous then this assumption is very meaningful because without this assumption, if drift at this point is left or right continuous then the system will make oscillations within an infinitesimal neighborhood of the sticky boundary point with an infinite speed.

In first order fluid models, behavior at terminal boundaries are selected in the following way: if the rate is positive in a state in the uppermost region then the uppermost boundary point is selected to be absorbing in this state likewise if it is negative in a state in the lowermost region then the lowermost boundary point is assumed to have absorbing behavior in this state and all other terminal boundary points are assumed to be the same as the behavior in the regions next to them.

2.2 Solution

With the terminology and notation described in subsections 2.2.1 and 2.2.2, the assumptions of [7] can be rewritten as:

1) There is no repulsive boundary, i.e. $S_+^{(k+1)} \cap S_-^{(k)} = \varnothing$, $1 \le k \le K-1$, (2.A.1)

2) All rates are nonzero, i.e. $S_0^{(k)} = \varnothing$, $1 \le k \le K$, (2.A.2)

3) If $j \in S_-^{(k+1)} \cap S_+^{(k)}$, then rate at $T^{(k)}$ is zero, (2.A.3)

4) $S_+^{(k)} \ne \varnothing$, $S_-^{(k)} \ne \varnothing$, $1 \le k \le K$, $\tilde{S}_+^{(k)} \ne \varnothing$, $\tilde{S}_-^{(k)} \ne \varnothing$, $1 \le k \le K-1$. (2.A.4)

Assumptions 2.A.1 and 2.A.2 are relaxed in this section. Assumption 2.A.4 can be removed easily but from applicational point of view it does not seem to be necessary to do so. Solution without assumption 2.A.3 is left as a future work.

2.2.1 Boundary conditions

The following theorem solves first order feedback fluid models with multiple thresholds without assumptions 1 and 2. Some of the equations in the theorem are described within the proof. In the theorem, $\mathbf{1}$ is used to indicate the column vector of ones.

Theorem 2.1: Under the assumptions below

1) Rates at the boundary points, bordering regions with zero rate, are absorbing, i.e. $\tilde{r}_i^{(k-1)} = \tilde{r}_i^{(k)} = 0$ if $r_i^{(k)} = 0$, (2.A.5)

2) All of the repulsive boundary points are either absorbing or left continuous or right continuous, (2.A.6)

3) All sticky boundary points are absorbing, (2.A.7)

4) $S_+^{(k)} \neq \emptyset$, $S_-^{(k)} \neq \emptyset$, $1 \le k \le K$, $\tilde{S}_+^{(k)} \neq \emptyset$, $\tilde{S}_-^{(k)} \neq \emptyset$, $1 \le k \le K-1$, (2.A.8)

the steady state solution $\mathbf{f}^{(k)}(x)$ of first order feedback fluid queues with multiple thresholds obeys

$$\frac{d}{dx}\mathbf{f}^{(k)}(x)R^{(k)} = \mathbf{f}^{(k)}(x)Q^{(k)} \text{ for } T^{(k-1)} < x < T^{(k)},\ 1 \le k \le K \tag{2.1}$$

along with the boundary conditions

1) $c_i^{(0)} = 0$ for all $i \in S_+^{(1)}$, (2.2)

2) $c_i^{(k)} = 0$ for all $i \in (S_+^{(k)} \cap S_+^{(k+1)}) \cup (S_-^{(k)} \cap S_-^{(k+1)})$, $1 \le k \le K-1$, (2.3)

3) $c_i^{(K)} = 0$ for all $i \in S_-^{(K)}$, (2.4)

4) $\sum_{i=1}^{N}\sum_{k=1}^{K}\int_{T^{(k-1)}+}^{T^{(k)}-} f_i^{(k)}(x)dx + \sum_{k=0}^{K}\mathbf{c}^{(k)}.\mathbf{1} = 1$, (2.5)

5) $\mathbf{f}^{(k+1)}(T^{(k)}+)R^{(k+1)} - \mathbf{f}^{(k)}(T^{(k)}-)R^{(k)} = \mathbf{c}^{(k)}\tilde{Q}^{(k)}$, $1 \le k \le K-1$, (2.6)

6) $\mathbf{f}^{(1)}(0+)R^{(1)} = \mathbf{c}^{(0)}\tilde{Q}^{(0)}$, (2.7)

7) $\mathbf{f}^{(K)}(T^{(K)}-)R^{(K)} = -\mathbf{c}^{(K)}\tilde{Q}^{(K)}$, (2.8)

8) $f_i^{(k)}(T^{(k)}-) = 0$, $\forall i \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)})$, $1 \le k \le K-1$, (2.9)

9) $f_i^{(k+1)}(T^{(k)}+) = 0$, $\forall i \in (\tilde{S}_0^{(k)} \cup \tilde{S}_-^{(k)}) \cap S_+^{(k+1)}$, $1 \le k \le K-1$, (2.10)

10) If repulsive boundary points are either left or right continuous then accumulations at the repulsive points are zero in the corresponding states, i.e.,

$$c_i^{(k)} = 0,\ \forall i \in (S_-^{(k)} \cap S_+^{(k+1)}) \cap (\tilde{S}_+^{(k)} \cup \tilde{S}_-^{(k)}),\ 1 \le k \le K-1. \tag{2.11}$$

<u>**Proof:**</u> In the following steps equations are proven and (or) described one by one.

Step 1: (2.1)

In [7], the ODE of the system is given as

$$\frac{d}{dx}\mathbf{F}^{(k)}(x)R^{(k)} = (\mathbf{F}^{(k)}(x) - \mathbf{F}^{(k)}(T^{(k-1)}))Q^{(k)} +$$

$$(\mathbf{F}^{(k)}(T^{(k-1)}) - \mathbf{F}^{(k-1)}(T^{(k-1)}-))\tilde{Q}^{(k-1)} + (\mathbf{F}^{(k-1)}(T^{(k-1)}) - \mathbf{F}^{(k-2)}(T^{(k-2)}-))Q^{(k)} +$$

$$.....$$

$$(\mathbf{F}^{(1)}(T^{(1)}-) - \mathbf{F}^{(1)}(T^{(0)}))Q^{(1)} + \mathbf{F}^{(1)}(T^{(0)})\tilde{Q}^{(0)}, \tag{2.12}$$

where

$$F_i^{(k)}(x,t) = P(C(t) \le x, X(t) = i) \text{ for } T^{(k-1)} < x < T^{(k)} \tag{2.13}$$

and

$$\mathbf{F}^{(k)}(x,t) = [F_1^{(k)}(x,t), F_2^{(k)}(x,t), ..., F_N^{(k)}(x,t)]. \tag{2.14}$$

By just differentiating both sides of (2.12) with respect to the space parameter x, (2.1) can be obtained.

Step 2: (2.2), (2.3), (2.4) and (2.11)

Since nothing forces the content process to stay at these points in this state in transient, Probability Mass Accumulations (PMAs) are zero at these points. These conditions are equal to the continuity conditions in the CDF formulation of [7].

Step 3: (2.5) Normalization

(2.5) is the normalization condition, i.e., total of all PMAs and integral of all PDFs is equal to 1.

Step 4: (2.6)

Equation (2.6) can be obtained by writing (2.12) for region k and equating the variable x to $T^{(k)}-$ in this equation, and then writing (2.12) for region

$k+1$ and equating the variable x to $T^{(k)}+$ in this equation, and then subtracting the former equation from the latter.

Step 5: (2.7)

Equation (2.7) can be obtained by just writing (2.12) for the first region and equating the variable x to $0+$ in this equation.

Step 6: (2.8)

Equation (2.8) can be obtained by writing (2.12) for the last region and equating the variable x to $T^{(K)}-$ in this equation and then subtracting this equation side by side from

$$
\begin{aligned}
&0 = \pi\tilde{Q}^{(K)} + \mathbf{F}^{(K)}(T^{(K)}-)(Q^{(K)} - \tilde{Q}^{(K)}) + \\
&\mathbf{F}^{(K)}(T^{(K-1)})(\tilde{Q}^{(K-1)} - Q^{(K)}) + \mathbf{F}^{(K-1)}(T^{(K-1)}-)(Q^{(K-1)} - \tilde{Q}^{(K-1)}) + \\
&\quad\cdots\cdots \\
&\mathbf{F}^{(1)}(T^{(1)}-)(Q^{(1)} - \tilde{Q}^{(1)}) + \mathbf{F}^{(1)}(T^{(0)}))(\tilde{Q}^{(0)} - Q^{(1)}),
\end{aligned}
\tag{2.15}
$$

where π is the vector of steady state probabilities at the uppermost boundary point of the queue.

Step 7: (2.9)

If $i \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)})$ then according to the assumptions made in the theorem, the rate in the regime $k+1$ is either zero or positive. Let's first define the variable,

$$\breve{P}_i^k(x,y,t) = P(X(t) = i, x \le C(t) < y), T^{(k-1)} < x < T^{(k)}. \tag{2.16}$$

While writing Kolmogorov difference equation, we will try to find $\breve{P}_i^k(T^{(k)} + r_i^{(k)}\Delta t, T^{(k)}, t+\Delta t)$, $i \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)})$ in terms of other variables with time index t for sufficiently small Δt (i.e. at most only one state transition is possible within Δt). If the content process is in between $T^{(k)} + r_i^{(k)}\Delta t$ and $T^{(k)}$ and state is i at $t+\Delta t$ then at time t it is impossible that content process is in state i if Δt is sufficiently small, because, if the content process were at a point within $[T^{(k)}, T^{(k+1)})$ in state i then it will stay at that point or go to a higher location in the queue depending on the rate at this point; if the content process were at such a point that $C(t) < T^{(k)}$ in state i then at time $t+\Delta t$ it is obvious that $C(t+\Delta t) < T^{(k)} + r_i^{(k)}\Delta t$. Moreover, if $j \in S_-^{(k)} \cap \tilde{S}_-^{(k)} \cap S_+^{(k+1)}$, $T^{(k)} + r_i^{(k)}\Delta t \le C(t+\Delta t) < T^{(k)}$, and $X(t) = j$ then $C(t)$ can only be within interval $T^{(k)} + o_{j,1}(\Delta t) \le C(t+\Delta t) < T^{(k)} + o_{j,2}(\Delta t)$ where $o_{j,1}(\Delta t)$ and $o_{j,2}(\Delta t)$ are in the order of Δt and $o_{j,1}(\Delta t) < o_{j,2}(\Delta t) < 0$ if $r_i^{(k)} < r_j^{(k)}$.

If $T^{(k)} + r_i^{(k)}\Delta t \le C(t+\Delta t) < T^{(k)}$ and $i \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)})$ then the background process can be in a state $j \in S_-^{(k)} \cap S_-^{(k+1)}$ at time t with a nonzero probability. The transition can occur in region k with a probability of order of Δt, $o(\Delta t)$, due to the exponential distribution time between the state transitions. Probability of being in state $j \in S_-^{(k)} \cap S_-^{(k+1)}$ and in a reachable point to $T^{(k)} + r_i^{(k)}\Delta t \le C(t+\Delta t) < T^{(k)}$ is in the form of $\bar{P}_j^k(T^{(k)} + o_{j,1}(\Delta t), T^{(k)} + o_{j,2}(\Delta t), t)$ where $o_{j,1}(\Delta t)$ and $o_{j,2}(\Delta t)$ are in the order of Δt and $o_{j,1}(\Delta t) < o_{j,2}(\Delta t)$. Moreover, if $T^{(k)} + r_i^{(k)}\Delta t \le C(t+\Delta t) < T^{(k)}$ and $i \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)})$ then the background process can be in a state $j \in S_+^{(k)} \cap S_+^{(k+1)}$ at time t with a nonzero probability. Here transition can only in region k and state transition probability is again $o(\Delta t)$. Again probability of being in state $j \in S_+^{(k)} \cap S_+^{(k+1)}$ and in a reachable point to $T^{(k)} + r_i^{(k)}\Delta t \le C(t+\Delta t) < T^{(k)}$ is in the form of $\bar{P}_j^k(T^{(k)} + o_{j,1}(\Delta t), T^{(k)} + o_{j,2}(\Delta t), t)$ where $o_{j,1}(\Delta t)$ and $o_{j,2}(\Delta t)$ are in the order of Δt and $o_{j,1}(\Delta t) < o_{j,2}(\Delta t) \le 0$. Finally, if $T^{(k)} + r_i^{(k)}\Delta t \le C(t+\Delta t) < T^{(k)}$ and $i \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)})$ then the background process can be in a state $j \in (S_+^{(k)} \cup S_0^{(k)}) \cap \tilde{S}_0^{(k)}$ at time t with a nonzero probability. This scenario is possible only if the background process transits its state to i before the content process reaches the boundary point from the region k. Clearly, transition probability is again $o(\Delta t)$, and probability of being in state $j \in S_+^{(k)} \cap \tilde{S}_0^{(k)}$ and in a reachable point to

$T^{(k)} + r_i^{(k)}\Delta t \le C(t+\Delta t) < T^{(k)}$ is in the form of $\bar{P}_j^k(T^{(k)} + o_{j,1}(\Delta t), T^{(k)} + o_{j,2}(\Delta t), t)$ where $o_{j,1}(\Delta t)$ and $o_{j,2}(\Delta t)$ are in the order of Δt and $o_{j,1}(\Delta t) < o_{j,2}(\Delta t) < 0$.

Then let's write the Kolmogorov difference equation,

$$\bar{P}_i^k(T^{(k)}, t+\Delta t) = \sum_{j \ne i} \bar{P}_j^k(T^{(k)} + o_{j,1}(\Delta t), T^{(k)} + o_{j,2}(\Delta t), t) o(\Delta t). \tag{2.17}$$

We note that if $j \in (S_+^{(k)} \cap S_+^{(k+1)}) \cup (S_-^{(k)} \cap S_-^{(k+1)})$ then due to (2.3)

$$\lim_{\Delta t \to 0} \bar{P}_j^k(T^{(k)} + o_{j,1}(\Delta t), T^{(k)} + o_{j,2}(\Delta t), t) = 0. \tag{2.18}$$

Moreover, if $j \in (S_+^{(k)} \cup S_0^{(k)}) \cap \tilde{S}_0^{(k)}$ then since $o_{j,1}(\Delta t) < o_{j,2}(\Delta t) < 0$,

$$\lim_{\Delta t \to 0} \bar{P}_j^k(T^{(k)} + o_{j,1}(\Delta t), T^{(k)} + o_{j,2}(\Delta t), t) = 0. \tag{2.19}$$

Furthermore, if $j \in S_-^{(k)} \cap \tilde{S}_-^{(k)} \cap S_+^{(k+1)}$ since $o_{j,1}(\Delta t) < o_{j,2}(\Delta t) < 0$,

$$\lim_{\Delta t \to 0} \bar{P}_j^k(T^{(k)} + o_{j,1}(\Delta t), T^{(k)} + o_{j,2}(\Delta t), t) = 0. \tag{2.20}$$

From the definition of PDF,

$$\lim_{\Delta t \to 0} \left. \frac{\bar{P}_i^k(x + r_i^{(k)}\Delta t, x, t)}{\Delta t} \right|_{x=T^{(k)}} = \lim_{\Delta t \to 0} \left. \frac{\bar{P}_i^k(x + r_i^{(k)}\Delta t, x, t+\Delta t)}{\Delta t} \right|_{x=T^{(k)}} = -r_i^{(k)} f_i(T^{(k)}-, t)$$

$$, i \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)}). \tag{2.21}$$

Therefore by dividing both sides of (2.17) by Δt and by taking the limit as $\Delta t \to 0$ (2.9) can be obtained.

Step 8: (2.10)

Distributions of $T^{(K)} - C(t)$ and $C(t)$ are the same provided that we replace the space parameter x with $T^{(K)} - x$. For the process $T^{(K)} - C(t)$ the rates can be found by multiplying the rates in the process $C(t)$ with -1. Therefore for that process (2.10) is equivalent to (2.9) for the content process $C(t)$.□

<u>Remark 2.1:</u> Assumption 4 is inserted in order to prevent the queue content to stick to a subregion of queue.

Remark 2.2: Without assumption 3, CDF of the queue content will include more than one probability mass accumulation within an infinitesimal neighborhood of the threshold.

Remark 2.3: Assumption 2 is inserted doe to the reason that although infinitely many types of behavioral feature are possible for a boundary point this theorem solves the system for those three.

Remark 2.4: Without assumption 1, CDF of the queue content can again include more than one probability mass accumulation within an infinitesimal neighborhood of the threshold.

2.2.2 Spectral Solution

When $S_0^{(k)} = \emptyset, \forall k$ general solution of (2.1) is:

$$\mathbf{f}^{(k)}(x) = \sum_{i \neq j} a_i^{(k)} \exp(\lambda_i^{(k)} x) \phi_i^{(k)} + a_j^{(k)} \phi_j^{(k)}, \tag{2.22}$$

where $(\lambda_i^{(k)}, \phi_i^{(k)})$ is the i$^{\text{th}}$ left eigenvalue-eigenvector pair of (2.1) and $\phi_j^{(k)}$ is the eigenvector corresponding to the zero eigenvalue. In this solution there are NK unknown "a" coefficients (in (2.22)) and $NK + N$ unknown "c" coefficients (PMAs) that should be found.

When $S_0^{(k)} \neq \emptyset, \exists k$ then it is easy to see that

$$f_i^{(k)}(x) = -\sum_{j \neq i} f_j^{(k)}(x) \frac{q_{ji}^{(k)}}{q_{ii}^{(k)}}, \forall i \in S_0^{(k)}, \forall k. \tag{2.23}$$

Then (2.1) and (2.23) can be rewritten as:

$$\dot{\mathbf{f}}_{nz}^{(k)} R_{nz}^{(k)} = \mathbf{f}_{nz}^{(k)} (Q_{nz}^{(k)} - Q_a^{(k)} (Q_b^{(k)})^{-1} Q_0^{(k)}), \forall i \in S_0^{(k)}, \forall k \text{ and} \tag{2.24}$$

$$\mathbf{f}_{nz}^{(k)} Q_a^{(k)} + \mathbf{f}_0^{(k)} Q_b^{(k)} = 0, \forall i \in S_0^{(k)}, \forall k, \tag{2.25}$$

respectively, with the partitions:

$$\mathbf{f}^{(k)} = [\mathbf{f}_{nz}^{(k)} \mid \mathbf{f}_0^{(k)}], \tag{2.26}$$

where $\mathbf{f}_{nz}^{(k)}$ and $\mathbf{f}_0^{(k)}$ refer to the PDFs of $|S_+^{(k)}|+|S_-^{(k)}|$ states with nonzero rates and $|S_0^{(k)}|$ states with zero rates, respectively, and

$$R^{(k)}=\begin{bmatrix} R_{nz}^{(k)} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.27}$$

$$Q^{(k)}=\begin{bmatrix} Q_{nz}^{(k)} & Q_a^{(k)} \\ Q_0^{(k)} & Q_b^{(k)} \end{bmatrix}, \tag{2.28}$$

where $R_{nz}^{(k)}$, $Q_{nz}^{(k)}$, $Q_0^{(k)}$, $Q_a^{(k)}$and $Q_b^{(k)}$ are $(|S_+^{(k)}|+|S_-^{(k)}|)x(|S_+^{(k)}|+|S_-^{(k)}|)$, $(|S_+^{(k)}|+|S_-^{(k)}|)x(|S_+^{(k)}|+|S_-^{(k)}|)$, $|S_0^{(k)}|x(|S_+^{(k)}|+|S_-^{(k)}|)$, $(|S_+^{(k)}|+|S_-^{(k)}|)x|S_0^{(k)}|$ and $|S_0^{(k)}|x|S_0^{(k)}|$matrices respectively.

General solution to (2.24) is in the same form as the solution of (2.1), which is (2.21), with $KN-\sum_k |S_0^{(k)}|$ unknown "a" coefficients instead of KN. In this case there are still $NK+N$ unknown "c" coefficients. It is important to note that the same solution can be found without reordering states and dividing the matrices and vectors into partitions.

(2.6), (2.7), and (2.8), provide totally $NK+N$ boundary coefficients. Now let's count the number of other boundary equations that are given in Theorem 2.1. The equations (2.2), (2.3), and (2.4) provide $|S_+^{(1)}|$, $\sum_{k=1}^{K-1}|(S_+^{(k)}\cap S_+^{(k+1)})\cup(S_-^{(k)}\cap S_-^{(k+1)})|$ and $|S_-^{(K)}|$ equations, respectively. (2.9) and (2.10) provide $\sum_{k=1}^{K-1}|S_-^{(k)}\cap(\tilde{S}_0^{(k)}\cup\tilde{S}_+^{(k)})|$ and $\sum_{k=1}^{K-1}|(\tilde{S}_0^{(k)}\cup\tilde{S}_-^{(k)})\cap S_+^{(k+1)}|$ equations respectively. Let's denote the number of repulsive boundaries as N_{rep}. If repulsive boundaries are selected as either left or right continuous then condition 10 of Theorem 2.1 gives us N_{rep} boundary conditions but if they are selected as absorbing it does not provide and conditions. Then (2.2), (2.3), (2.4), (2.8), and (2.9) totally provide

$$|S_+^{(1)}|+|S_-^{(K)}|+\sum_{k=1}^{K-1}|(S_+^{(k)}\cap S_+^{(k+1)})\cup(S_-^{(k)}\cap S_-^{(k+1)})|$$

$$+\sum_{k=1}^{K-1}|S_-^{(k)}\cap(\tilde{S}_0^{(k)}\cup\tilde{S}_+^{(k)})|+\sum_{k=1}^{K-1}|(\tilde{S}_0^{(k)}\cup\tilde{S}_-^{(k)})\cap S_+^{(k+1)}|$$

boundary conditions. If repulsive boundaries are selected as left continuous then

$$\sum_{k=1}^{K-1}|S_-^{(k)}\cap(\tilde{S}_0^{(k)}\cup\tilde{S}_+^{(k)})|=\sum_{k=1}^{K-1}|S_-^{(k)}\cap S_0^{(k+1)}|, \tag{2.29}$$

$$\sum_{k=1}^{K-1}|(\tilde{S}_0^{(k)}\cup\tilde{S}_-^{(k)})\cap S_+^{(k+1)}|=\sum_{k=1}^{K-1}|(S_0^{(k)}\cup S_-^{(k)})\cap S_+^{(k+1)}|. \tag{2.30}$$

Since

$$|S_+^{(1)}|+|S_-^{(K)}|+\sum_{k=1}^{K-1}|(S_+^{(k)}\cap S_+^{(k+1)})\cup(S_-^{(k)}\cap S_-^{(k+1)})|+\sum_{k=1}^{K-1}|S_-^{(k)}\cap S_0^{(k+1)}|$$

$$+\sum_{k=1}^{K-1}|(S_0^{(k)}\cup S_-^{(k)})\cap S_+^{(k+1)}|=|S_+^{(1)}|+|S_-^{(K)}|+\sum_{k=1}^{K-1}|S_-^{(k)}\cap(S_0^{(k+1)}\cup S_-^{(k+1)})|$$

$$+\sum_{k=1}^{K-1}|(S_0^{(k)}\cup S_-^{(k)}\cup S_+^{(k)})\cap S_+^{(k+1)}|$$

$$=|S_+^{(1)}|+|S_-^{(K)}|+\sum_{k=1}^{K-1}|S_-^{(k)}|-N_{rep}+\sum_{k=1}^{K-1}|S_+^{(k+1)}|$$

$$=\sum_{k=1}^{K}|S_-^{(k)}|+\sum_{k=1}^{K}|S_+^{(k)}|-N_{rep},$$

Theorem 2.1 provides $2KN+N-\sum_k|S_0^{(k)}|+1$ boundary conditions together with Condition 10 and the normalization condition. Similarly, if repulsive boundaries are selected as right continuous then

$$\sum_{k=1}^{K-1}|S_-^{(k)}\cap(\tilde{S}_0^{(k)}\cup\tilde{S}_+^{(k)})|=\sum_{k=1}^{K-1}|S_-^{(k)}\cap(S_0^{(k+1)}\cup S_+^{(k+1)})|, \tag{2.31}$$

$$\sum_{k=1}^{K-1}|(\tilde{S}_0^{(k)}\cup\tilde{S}_-^{(k)})\cap S_+^{(k+1)}|=\sum_{k=1}^{K-1}|S_0^{(k)}\cap S_+^{(k+1)}|. \tag{2.32}$$

Since

$$|S_+^{(1)}|+|S_-^{(K)}|+\sum_{k=1}^{K-1}|(S_+^{(k)}\cap S_+^{(k+1)})\cup(S_-^{(k)}\cap S_-^{(k+1)})|$$

$$+\sum_{k=1}^{K-1}|S_-^{(k)}\cap(S_0^{(k+1)}\cup S_+^{(k+1)})|+\sum_{k=1}^{K-1}|S_0^{(k)}\cap S_+^{(k+1)}|=|S_+^{(1)}|+|S_-^{(K)}|$$

$$+\sum_{k=1}^{K-1} | S_-^{(k)} \cap (S_0^{(k+1)} \cup S_-^{(k+1)} \cup S_+^{(k+1)}) | + \sum_{k=1}^{K-1} | (S_0^{(k)} \cup S_+^{(k)}) \cap S_+^{(k+1)} |$$

$$= | S_+^{(1)} | + | S_-^{(K)} | + \sum_{k=1}^{K-1} | S_-^{(k)} | + \sum_{k=1}^{K-1} | S_+^{(k+1)} | - N_{rep}$$

$$= \sum_{k=1}^{K} | S_-^{(k)} | + \sum_{k=1}^{K} | S_+^{(k)} | - N_{rep} ,$$

Theorem 2.1 provides $2KN + N - \sum_k | S_0^{(k)} | + 1$ boundary conditions together with Condition 10 and the normalization condition Finally, if repulsive boundaries are selected as absorbing then

$$\sum_{k=1}^{K-1} | S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)}) | = \sum_{k=1}^{K-1} | S_-^{(k)} \cap (S_0^{(k+1)} \cup S_+^{(k+1)}) | , \tag{2.33}$$

$$\sum_{k=1}^{K-1} | (\tilde{S}_0^{(k)} \cup \tilde{S}_-^{(k)}) \cap S_+^{(k+1)} | = \sum_{k=1}^{K-1} | (S_0^{(k)} \cup S_-^{(k)}) \cap S_+^{(k+1)} | . \tag{2.34}$$

Since

$$| S_+^{(1)} | + | S_-^{(K)} | + \sum_{k=1}^{K-1} | (S_+^{(k)} \cap S_+^{(k+1)}) \cup (S_-^{(k)} \cap S_-^{(k+1)}) |$$

$$+\sum_{k=1}^{K-1} | S_-^{(k)} \cap (S_0^{(k+1)} \cup S_+^{(k+1)}) | + \sum_{k=1}^{K-1} | (S_0^{(k)} \cup S_-^{(k)}) \cap S_+^{(k+1)} | = | S_+^{(1)} | + | S_-^{(K)} |$$

$$+\sum_{k=1}^{K-1} | S_-^{(k)} \cap (S_0^{(k+1)} \cup S_-^{(k+1)} \cup S_+^{(k+1)}) | + \sum_{k=1}^{K-1} | (S_0^{(k)} \cup S_+^{(k)} \cup S_-^{(k)}) \cap S_+^{(k+1)} |$$

$$= | S_+^{(1)} | + | S_-^{(K)} | + \sum_{k=1}^{K-1} | S_-^{(k)} | + \sum_{k=1}^{K-1} | S_+^{(k+1)} |$$

$$= \sum_{k=1}^{K} | S_-^{(k)} | + \sum_{k=1}^{K} | S_+^{(k)} | ,$$

Theorem 2.1 provides $2KN + N - \sum_k | S_0^{(k)} | + 1$ boundary conditions together with the normalization condition. The last point that should be discussed about the boundary conditions given in Theorem 2.1 is the uniqueness of the solution. Beyond the normalization condition, extra conditions found in (2.9) and (2.10) are only related to the equations in (2.6) but they offer the exact values of PDFs.

Therefore they are linearly independent from all other equations and uniqueness of the solution of [7] is still valid.

2.3 Numerical Examples

In this section, two examples are given. In the first example, the system has a repulsive boundary and the rate is nonzero in all the states. All three cases (left continuity, right continuity and absorbing behavior) are analyzed for this example. In the second example, the system includes not only a repulsive boundary but also states with zero rates. The repulsive boundary is selected to be absorbing for the second example.

Example 2.1 Let's assume that length of queue is 2 units and it is divided into two equal length regimes. Each regime behaves differently in terms of the net rates in each state. There are four states in the background process. In state 1, there is a repulsive boundary, net rate below 1 is -2 and above 1 it is 1. In state 2, there is a sticky boundary, net rate below 1 is 1 and above 1 is -3. Both of the states 3 and 4 are emitting. State 3 has a net rate of 1 below 1 and 2 above 1. State 4 has a net rate of -1 below 1 and -2 above 1. The last parameter is the infinitesimal generator which is fixed to

$$Q = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix},$$

everywhere on the queue for this example. According to these values both state-wise and total CDF and PDF plots are obtained for all three cases in the following figures. In these figures, Case 1 represents absorbing behavior, Case 2 and Case 3 represent right and left continuity of the behaviors, respectively, for the repulsive boundary.

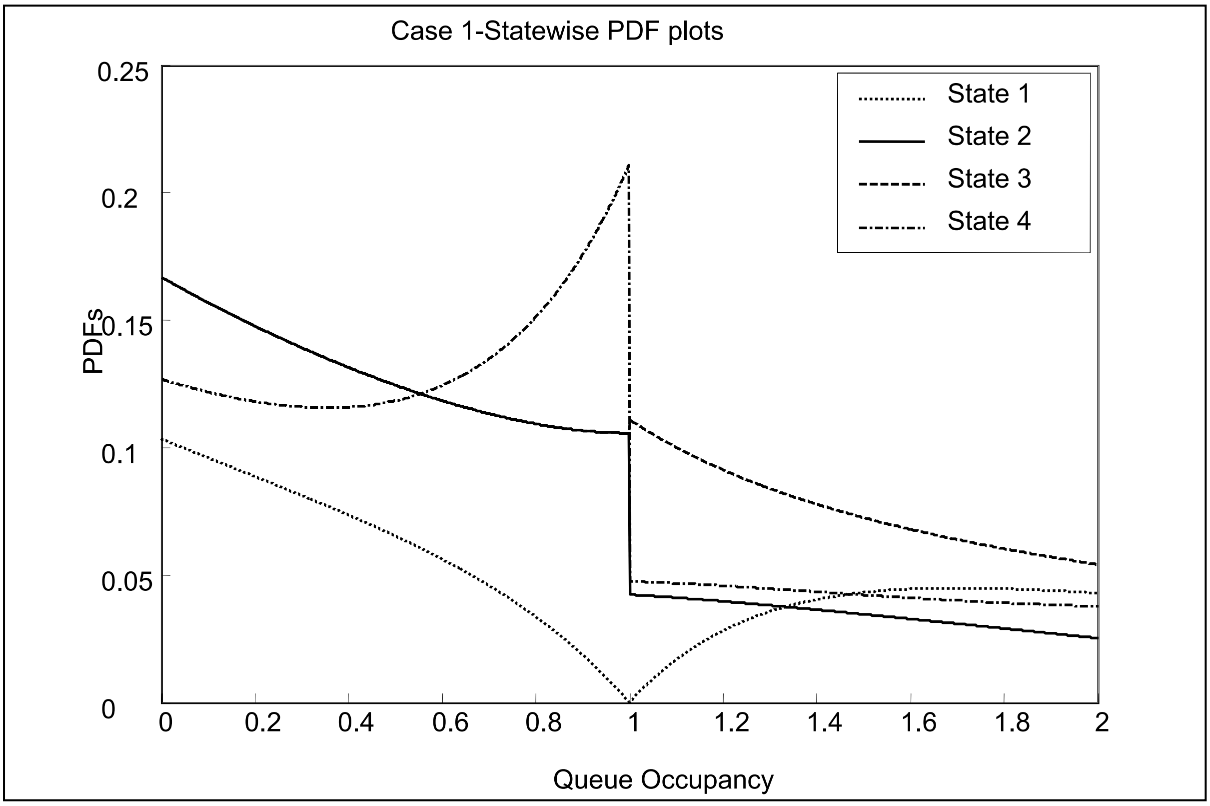


Figure 2.2 Case 1 Statewise PDF plots of Example 2.1

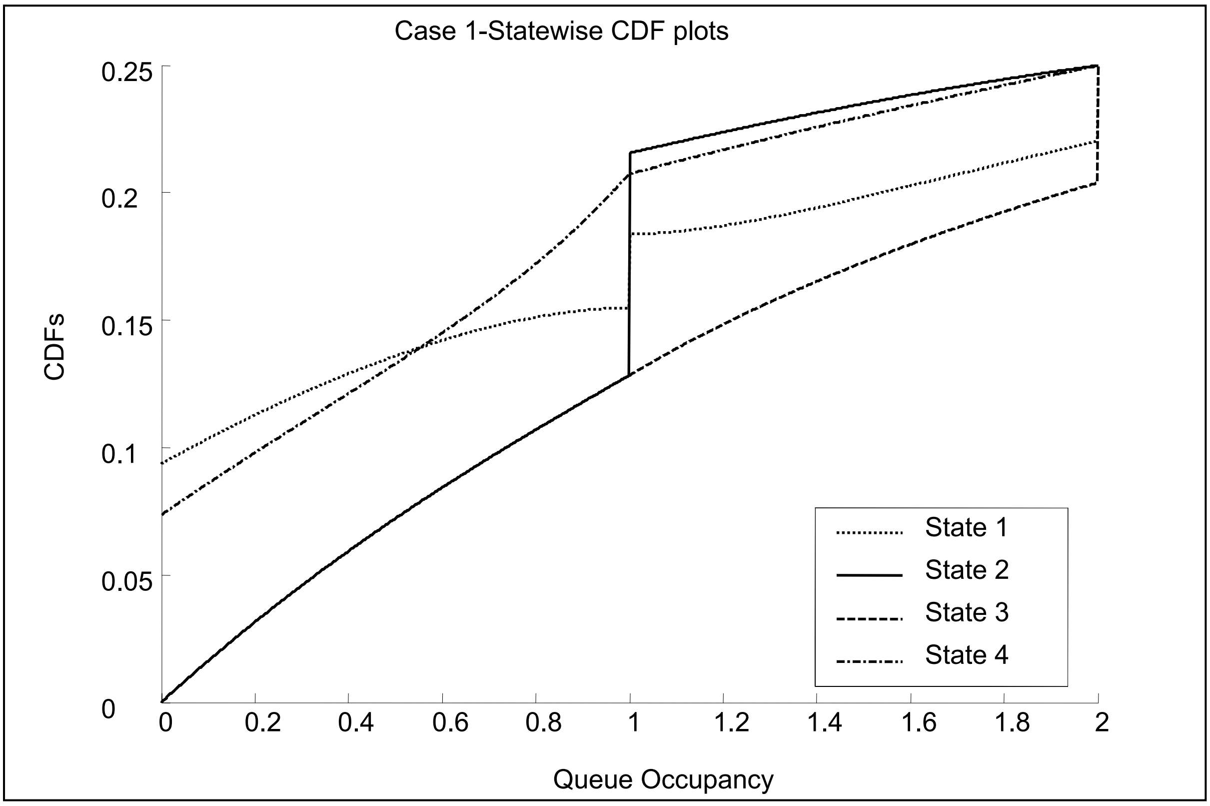


Figure 2.3 Case 1 Statewise CDF plots of Example 2.1

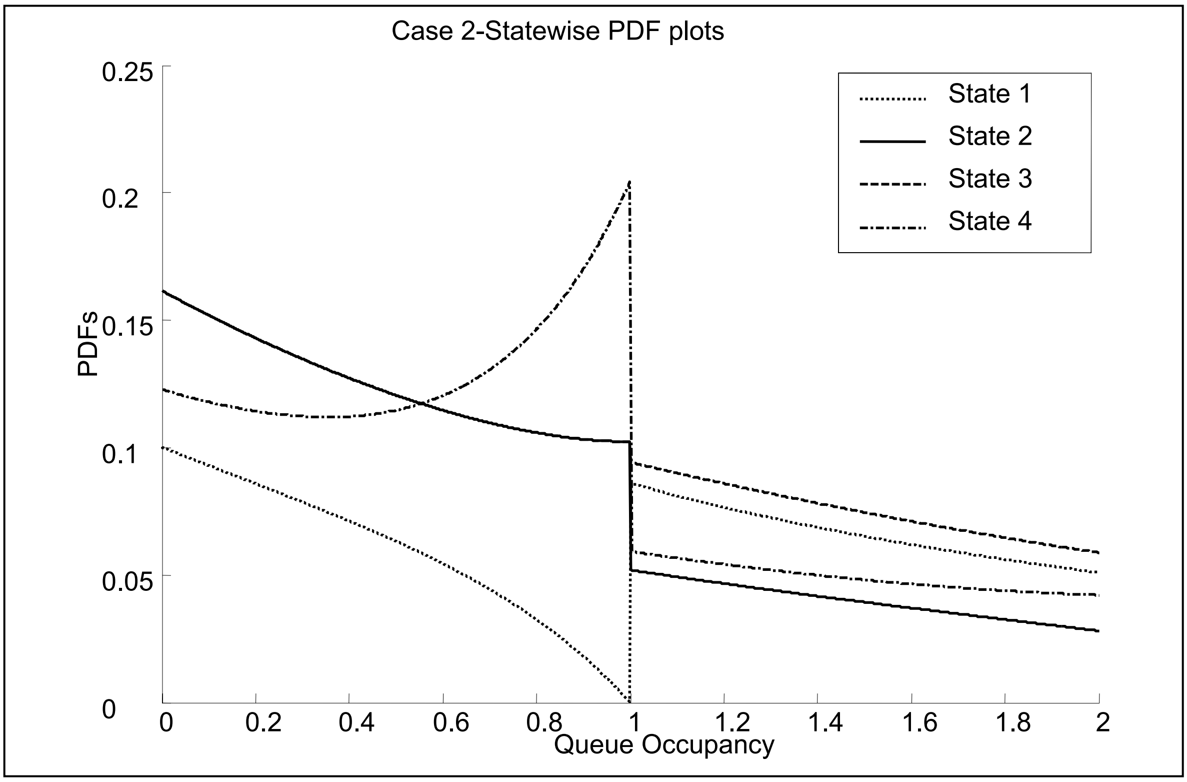


Figure 2.4 Case 2 Statewise PDF plots of Example 2.1

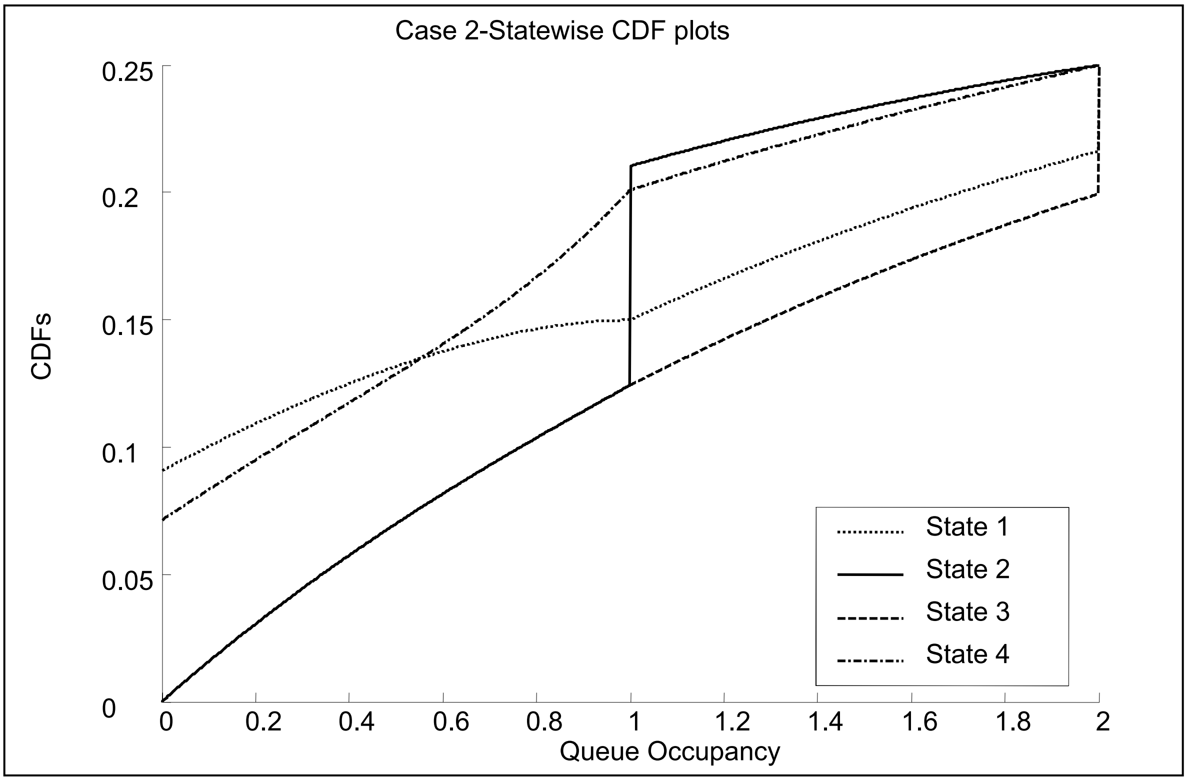


Figure 2.5 Case 2 Statewise CDF plots of Example 2.1

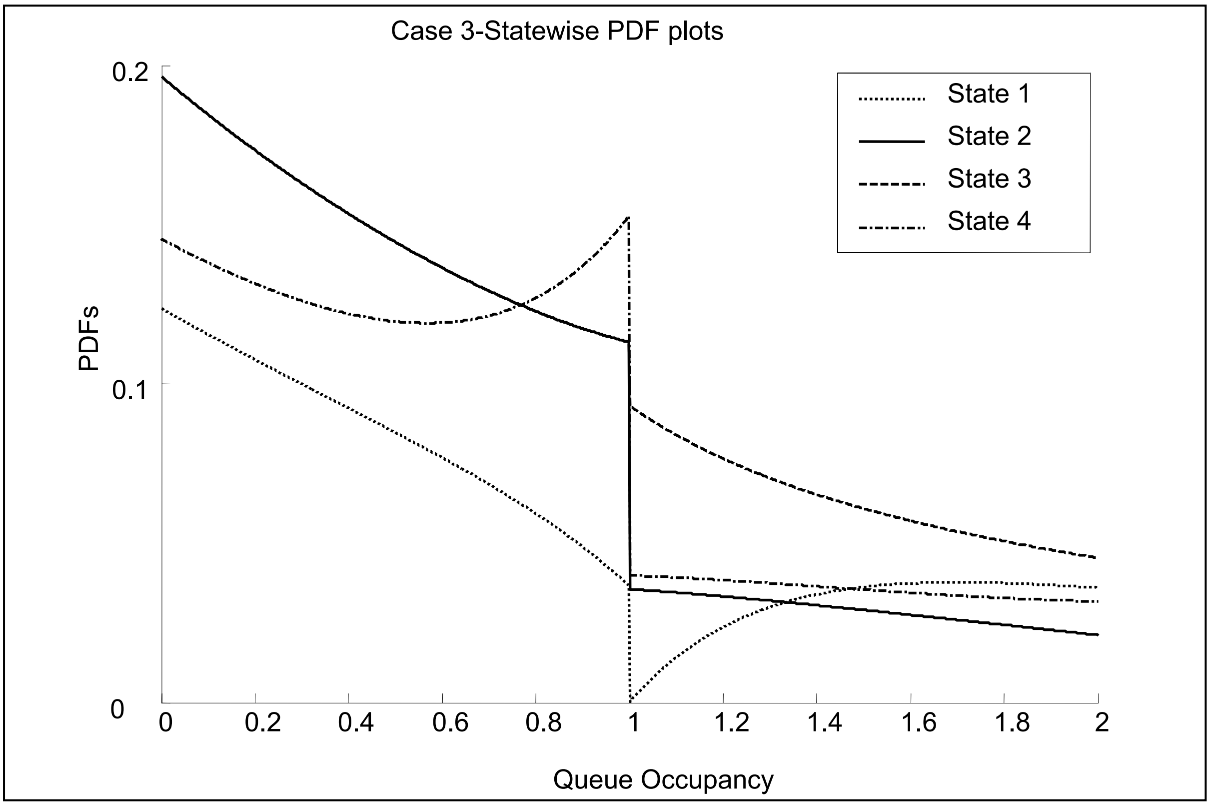


Figure 2.6 Case 3 Statewise PDF plots of Example 2.1

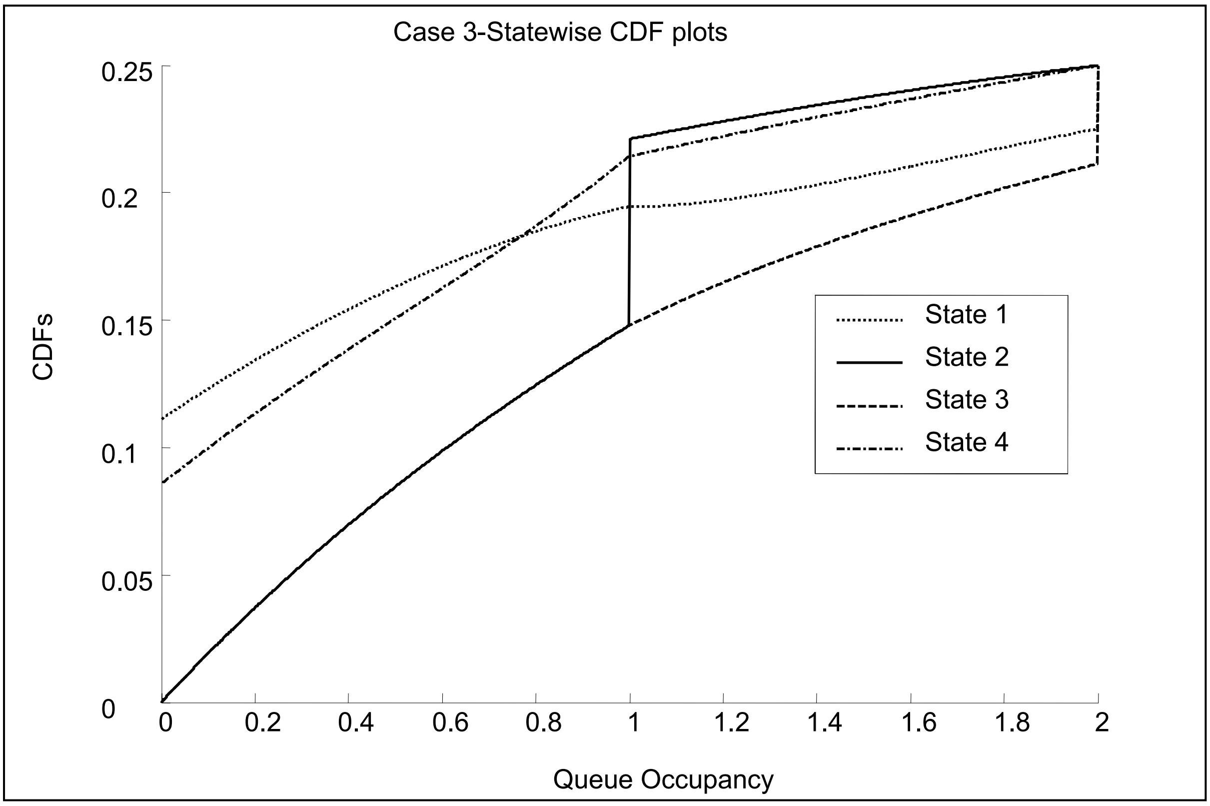


Figure 2.7 Case 3 Statewise CDF plots of Example 2.1

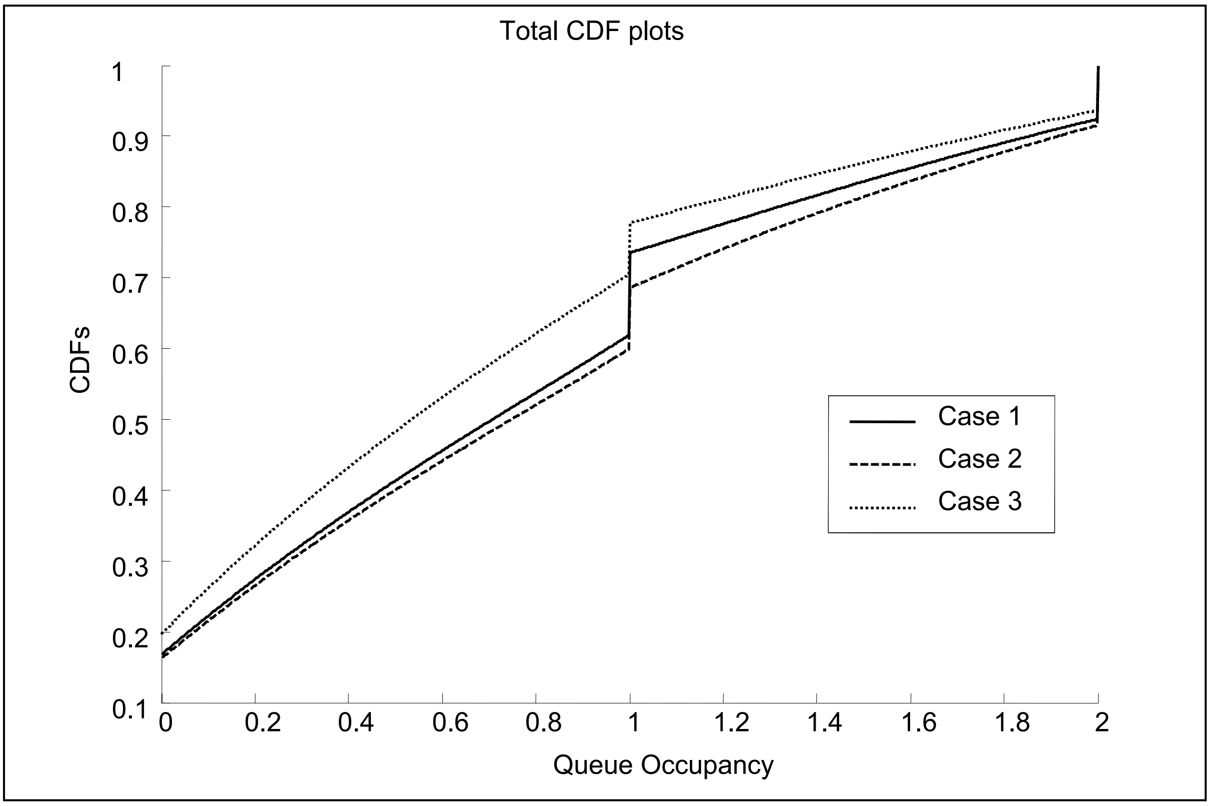


Figure 2.8 Total CDF plots of Example 2.1

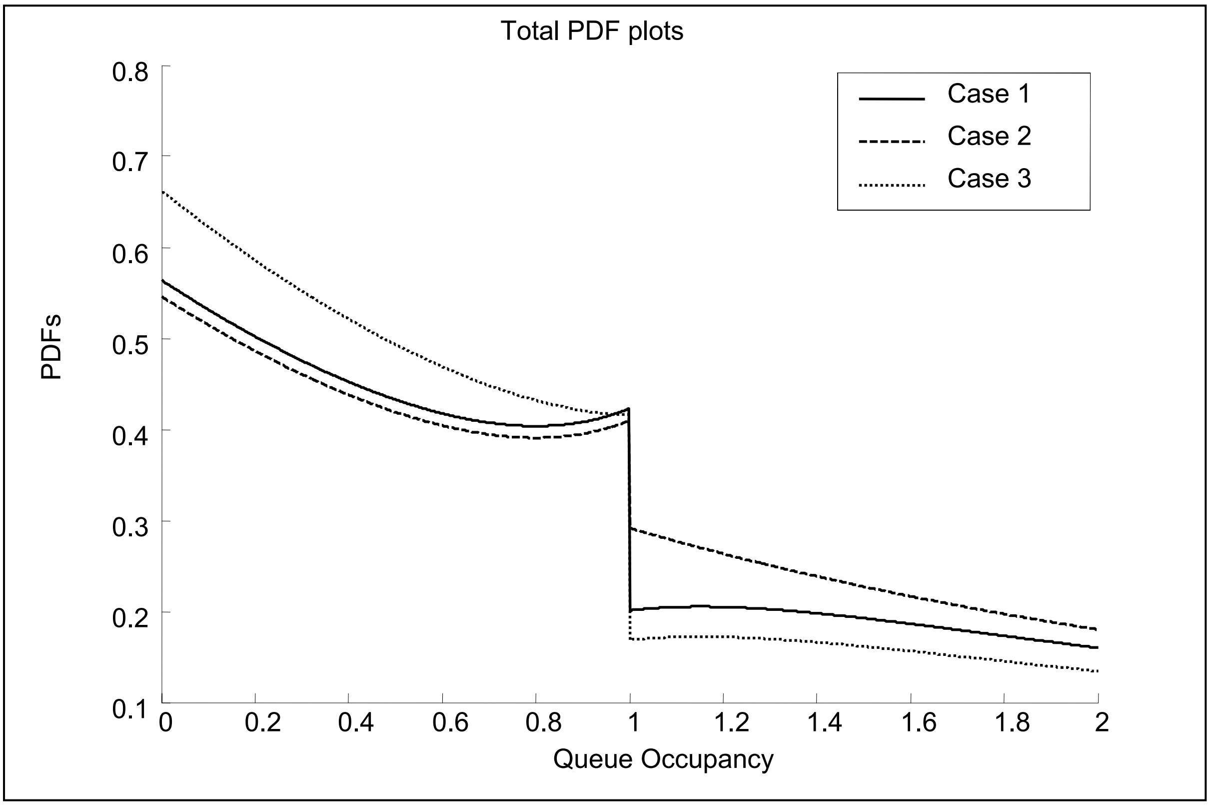


Figure 2.9 Total PDF plots of Example 2.1

Example 2.2: In this example there are 4 states and 3 regions. Boundary points are selected to be 0, 2, 4 and 6. Following rates are used in the system:

$r_1^{(1)} = -1$, $r_2^{(1)} = -1$, $r_3^{(1)} = 2$, $r_4^{(1)} = 1$,

$r_1^{(2)} = -1.5$, $r_2^{(2)} = 0$, $r_3^{(2)} = -1$, $r_4^{(2)} = 2$,

$r_1^{(3)} = 0$, $r_2^{(3)} = 1$, $r_3^{(3)} = -1.5$, $r_4^{(3)} = 2$.

Infinitesimal generators are as selected as following:

$$Q_1 = \tilde{Q}_0 = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1 \\ 0.1 & -0.3 & 0.1 & 0.1 \\ 0.1 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & -0.3 \end{bmatrix},$$

$$Q_2 = \tilde{Q}_1 = \begin{bmatrix} -0.9 & 0.3 & 0.3 & 0.3 \\ 0.3 & -0.9 & 0.3 & 0.3 \\ 0.3 & 0.3 & -0.9 & 0.3 \\ 0.3 & 0.3 & 0.3 & -0.9 \end{bmatrix}$$

$$Q_3 = \tilde{Q}_2 = \tilde{Q}_3 = \begin{bmatrix} -1.2 & 0.4 & 0.4 & 0.4 \\ 0.4 & -1.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & -1.2 & 0.4 \\ 0.4 & 0.4 & 0.4 & -1.2 \end{bmatrix}.$$

Using these parameters the following plots are obtained.

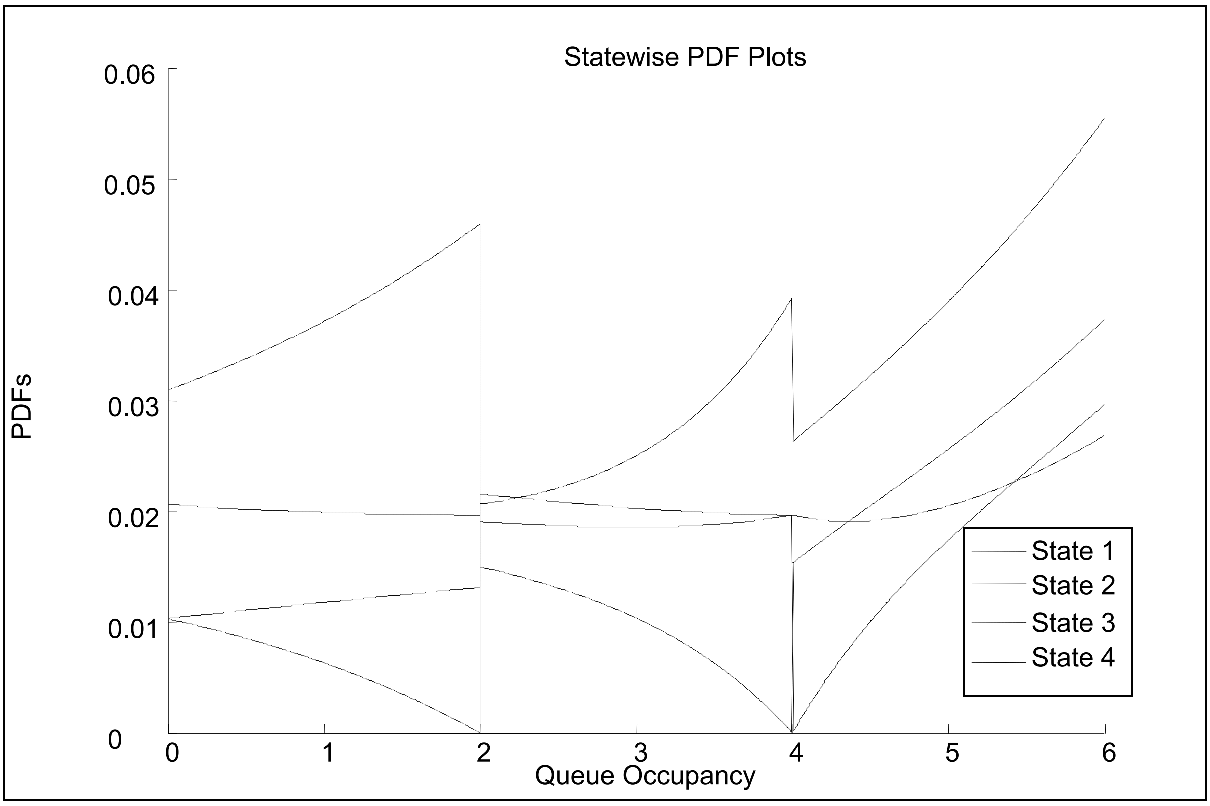


Figure 2.10 Statewise PDF plots of Example 2.2

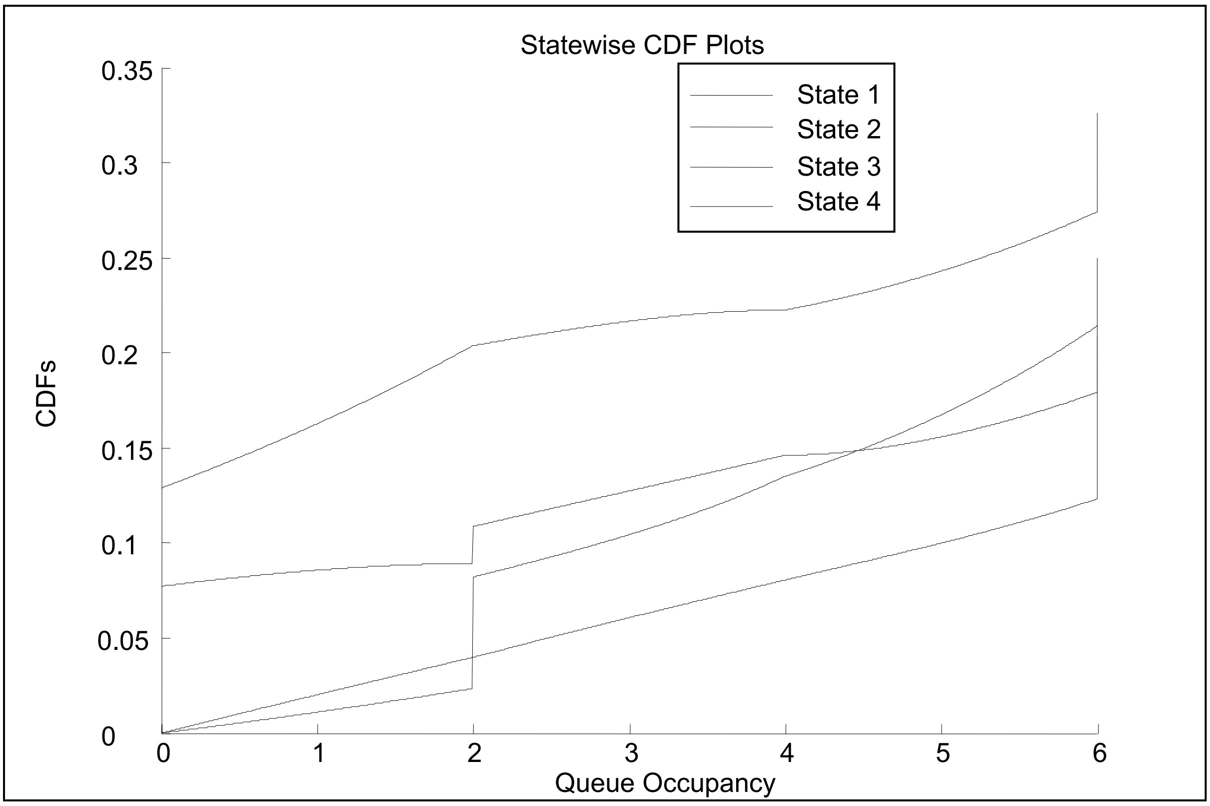


Figure 2.11 Statewise CDF plots of Example 2.2

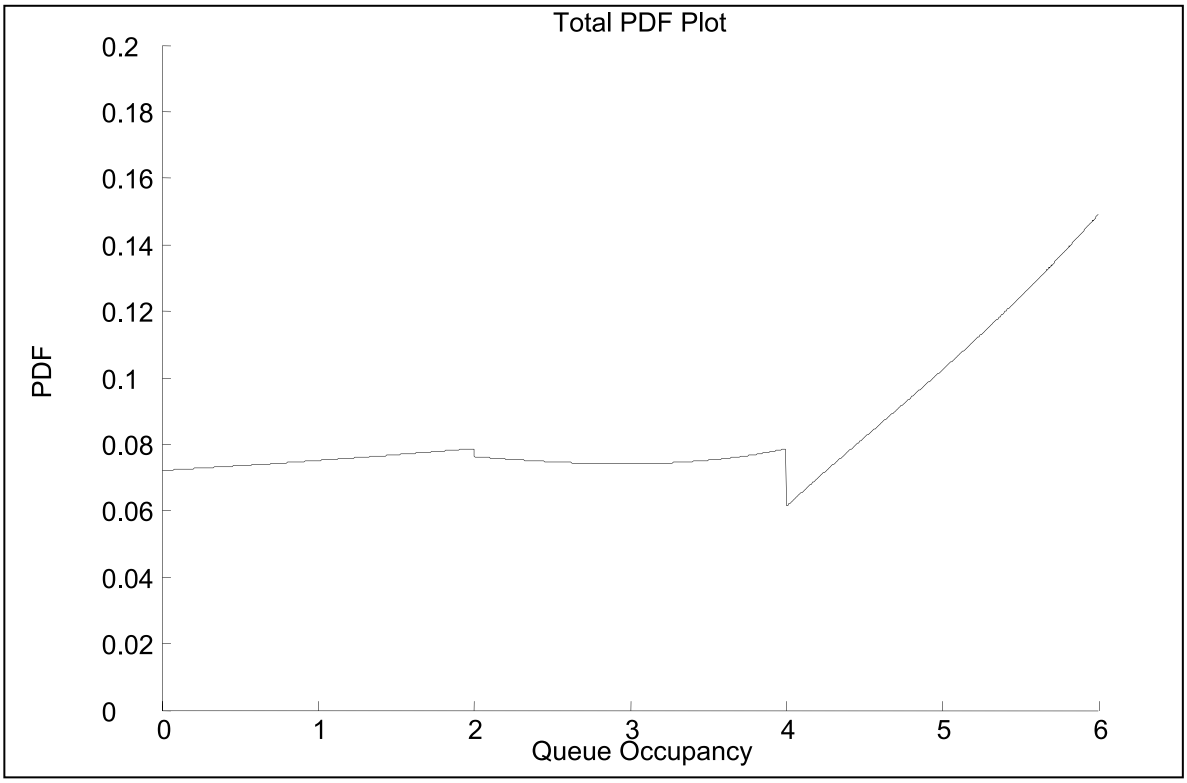


Figure 2.12 Total PDF plot of Example 2.2

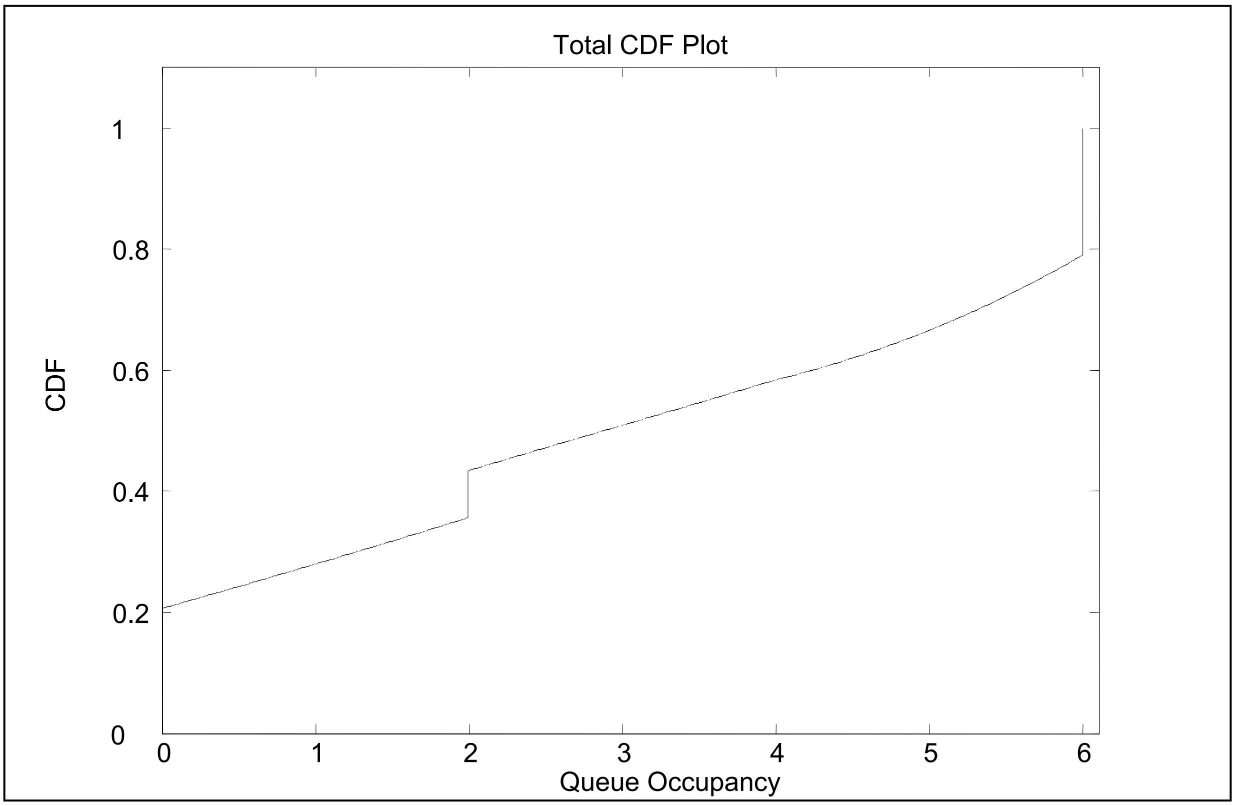


Figure 2.13 Total CDF plot of Example 2.2

Chapter 3

Analysis of Second Order Feedback Fluid Queues with Multiple Thresholds

In [6], second order single regime fluid queues are studied. Moreover, in [8], transient distribution of a second order feedback fluid queue with multiple thresholds is solved for sticky, repulsive and emitting intermediate boundaries and for absorbing, emitting and reflecting terminal boundaries. However, the number of boundary conditions for emitting intermediate boundaries is not enough to solve the system in this work. In this thesis, we complete those missing boundary conditions in this chapter. Also we find the steady state boundary conditions for second order feedback fluid queues with multiple thresholds for which the intermediate boundary points are emitting.

3.1 Preliminaries

3.1.1 Second Order Single Regime Fluid Queues

Similar to first order fluid queues, second order systems are driven by a background process $\{X(t); t \geq 0\}$ which is a finite state continuous time Markov process with an infinitesimal generator Q. In each state of the system, the net rate of the fluid flowing into the queue is determined by a Brownian motion with drift. In single regime, queue drift and the variance parameters of the Brownian motion depend on the current state of the background process. Behavior of the content process $\{C(t), t \geq 0\}$ at the terminal boundary points depends on how these boundary points are defined. In [6], two types of boundary points are considered: absorbing and reflecting boundary points. Absorption concept is the same as the one used in Chapter 2. Whenever the queue content approaches to the boundary point, at a state which is absorbing, then the content process sticks to that boundary point until the background process changes its state at which this boundary point isn't absorbing. Reflecting boundaries behave as if the queue were continuing behind the boundary point but the content process behaves as if the drift parameter of the Brownian motion is multiplied with -1.

Drift and variance in state i are denoted by μ_i and σ_i^2, respectively. When $\sigma_i^2 = 0$, the system behaves identically equivalent to a first order system at that state.

The state space of the system S is divided into four groups as

- S_+ : set of states with positive variance
- S_{0+} : set of states with zero variance and positive drift
- S_{0-} : set of states with zero variance and negative drift
- S_0 : set of states with zero variance and zero drift.

Also set of states with zero variance and nonzero drift is denoted as $S_* = S_{0+} \cup S_{0-}$.

3.1.2 Steady State Analysis of Second Order Single Regime Queues

The steady state analysis of a second order single regime system is carried out in [6]. Results will be given without proof since derivation of both transient and steady state equations and boundary conditions for multiple thresholds case are given in subsequent sections of this chapter. Transient PDF at point x at time t at state i provided that content process starts at point y at state j is denoted by

$f_i(x,t;y,j)$. Transient PMA at point 0 (T if T is the uppermost point of the queue) at time t at state i is denoted by $c_i(t;y,j)$ ($h_i(t;y,j)$), if we are given that the content process is y, the state is j.

The following theorem based on [6] offers us the partial differential equation satisfied at the intermediate points and the boundary conditions used to solve that equation provided that all terminal boundaries are reflecting.

Theorem 3.1:[6], The conditional densities $f_i(x,t;y,j)$ (Conditioned on $C(0) = y$ and $X(0) = j$) satisfy

$$\frac{\partial f_i(x,t;y,j)}{\partial t} = \frac{1}{2}\sigma_i^2 \frac{\partial^2 f_i(x,t;y,j)}{\partial x^2} - \mu_i \frac{\partial f_i(x,t;y,j)}{\partial x} + \sum_k f_k(x,t;y,j) q_{ki} \quad (3.1)$$

along with the boundary conditions

$$\frac{\partial c_i(t;y,j)}{\partial t} = \frac{1}{2}\sigma_i^2 \left.\frac{\partial f_i(x,t;y,j)}{\partial x}\right|_{x=0+} - \mu_i f_i(x+,t;y,j)\Big|_{x=0} + \sum_k c_k(t;y,j) q_{ki},$$

$$\forall i \quad (3.2)$$

$$\frac{\partial h_i(t;y,j)}{\partial t} = \frac{1}{2}\sigma_i^2 \left.\frac{\partial f_i(x,t;y,j)}{\partial x}\right|_{x=B-} - \mu_i f_i(x-,t;y,j)\Big|_{x=B} - \sum_k h_k(t;y,j) q_{ki},$$

$$\forall i \quad (3.3)$$

$$c_k(t;y,j) = 0,\ k \in S_+ \cup S_{0+} \quad (3.4)$$

$$h_k(t;y,j) = 0,\ k \in S_+ \cup S_{0-} \quad (3.5)$$

The following theorem, taken from [6], gives the steady state solution to the PDF of a second order single regime system provided that all terminal boundaries are reflecting. It can be obtained by just taking the limit of the equations in the previous theorem as the time parameter goes to infinity. Since the starting point and state of the content process become unimportant for steady state solution, the following notation is used for steady state analysis:

$$\lim_{t\to\infty} f_i(x,t;y,j) = f_i(x), \quad (3.6)$$

$$\lim_{t\to\infty} c_i(t;y,j) = c_i, \quad (3.7)$$

$$\lim_{t\to\infty} h_i(t;y,j) = h_i \,. \tag{3.8}$$

Theorem 3.2:[6], Densities $f_i(x)$ satisfy,

$$\frac{1}{2}\sigma_i^2 \frac{d^2 f_i(x)}{dx^2} - \mu_i \frac{df_i(x)}{dx} + \sum_k f_k(x) q_{ki} = 0 \tag{3.9}$$

along with the boundary conditions:

$$\frac{1}{2}\sigma_i^2 \left.\frac{df_i(x)}{dx}\right|_{x=0+} - \mu_i f_i(0+) + \sum_k c_k q_{ki} = 0 \,, \forall i \tag{3.10}$$

$$\frac{1}{2}\sigma_i^2 \left.\frac{df_i(x)}{dx}\right|_{x=B-} - \mu_i f_i(B-) - \sum_k h_k q_{ki} = 0 \,, \forall i \tag{3.11}$$

$$c_k = 0 \,,\ k \in S_+ \cup S_{0+} \,, \tag{3.12}$$

$$h_k = 0 \,,\ k \in S_+ \cup S_{0-} \,. \tag{3.13}$$

3.1.3 General Solution Method of the Second Order Differential Equation in the Steady State Analysis of Second Order Fluid Queues

By defining the matrices

$$R = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_N \end{bmatrix}, \tag{3.14}$$

$$\Sigma = \begin{bmatrix} \frac{(\sigma_1)^2}{2} & & & \\ & \frac{(\sigma_2)^2}{2} & & \\ & & \ddots & \\ & & & \frac{(\sigma_N)^2}{2} \end{bmatrix}, \tag{3.15}$$

the steady state equation (3.9) can be written in the following matrix form

$$\frac{d^2\mathbf{f}(x)}{dx^2}\Sigma - \frac{d\mathbf{f}(x)}{dx}R + \mathbf{f}(x)Q = \mathbf{0}. \tag{3.16}$$

If the states in the system are ordered as

$$\mathbf{f} = [\mathbf{f}_+ \quad \mathbf{f}_0 \quad \mathbf{f}_*], \tag{3.17}$$

then via the following partition of R, Σ and Q matrices

$$R = \begin{bmatrix} R_+ & & \\ & R_0 & \\ & & R_- \end{bmatrix}, \tag{3.18}$$

$$\Sigma = \begin{bmatrix} \Sigma_+ & & \\ & \Sigma_0 & \\ & & \Sigma_- \end{bmatrix}, \text{ and} \tag{3.19}$$

$$Q = \begin{bmatrix} Q_{++} & Q_{+0} & Q_{+*} \\ Q_{0+} & Q_{00} & Q_{0*} \\ Q_{*+} & Q_{*0} & Q_{**} \end{bmatrix}, \tag{3.20}$$

the general solution of the system can be written as

$$\mathbf{f}(x) = \sum_i a_i \exp(\lambda_i x)\phi_i \,. \tag{3.21}$$

Here λ_i is the left eigenvalue-eigenvector pair of the following matrix

$$B = \begin{bmatrix} 0 & M_{+*}(R_*)^{-1} & -M_{++}(\Sigma_+)^{-1} \\ 0 & M_{**}(R_*)^{-1} & -M_{*+}(\Sigma_+)^{-1} \\ I_+ & 0 & R_+(\Sigma_+)^{-1} \end{bmatrix}, \tag{3.22}$$

where

$$M_{+*} = Q_{+*} - Q_{+0}(Q_{00})^{-1}Q_{0*}, \tag{3.23}$$

$$M_{++} = Q_{++} - Q_{+0}(Q_{00})^{-1}Q_{0+}, \tag{3.24}$$

$$M_{**} = Q_{**} - Q_{*0}(Q_{00})^{-1}Q_{0*}, \tag{3.25}$$

$$M_{*+} = Q_{*+} - Q_{*0}(Q_{00})^{-1}Q_{0+}, \tag{3.26}$$

and I_+ is an identity matrix of size $|S_+|$ and $\phi_i = (\phi_{i,+}, \phi_{i,0}, \phi_{i,*})$ if $(\phi_{i,+}, \phi_{i,*}, \lambda_i\phi_{i,+})$ is the left eigenvector of the matrix B corresponding to the eigenvalue λ_i and

$$\phi_{i,0} = -\phi_{i,+} Q_{+0} (Q_{00})^{-1} - \phi_{i,*} Q_{*0} (Q_{00})^{-1}. \tag{3.27}$$

Since length of ϕ_i is $2|S_+| + |S_*|$ and number of PMAs is $|S_*| + 2|S_0|$, in order to find the unknowns of (3.21) besides normalization condition number of boundary conditions should be $2|S|$. Each of (3.10) and (3.11) provides $|S|$ equations that solve the system.

3.1.4 Second Order Feedback Fluid Queues with Multiple Thresholds

As described in section 1.4, the idea of generalization of a second order single regime fluid queue to a second order feedback fluid queue with multiple thresholds is the same as the one made for first order fluid queues. As in the single regime case there still exists a finite state background process, $\{X(t); t \geq 0\}$, in each state of which drift and variance parameters are distinct. However, contrary to the single regime case infinitesimal generator of the driving process, drift and variance parameters of each state are determined by piecewise constant functions of buffer occupancy. The parameters of this system are depicted in Fig. 3-1.

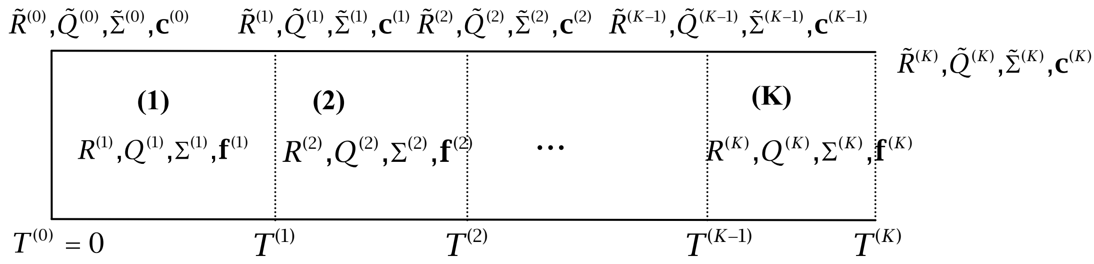


Figure 3.1 A Second Order Feedback Fluid Model with Multiple Thresholds

In this figure, notation is the same as the first order case. The additional parameter Σ represents the diagonal matrix of half of the variances and superscript in parenthesis represents the regime to which the parameters belong. $S_+^{(k)}$ denotes the set of states with positive variance in region k, $S_{0+}^{(k)}$ denotes the set of states with zero variance and positive drift in region k, $S_{0-}^{(k)}$ denotes the set of states with zero variance and negative drift in region k, and $S_0^{(k)}$ denotes the set of states with zero variance and zero drift in region k. Also set of states with zero variance and nonzero drift in region k is denoted by $S_*^{(k)} = S_{0+}^{(k)} \cup S_{0-}^{(k)}$.

3.1.5 Transient Analysis of Second Order Feedback Fluid Queues with Multiple Thresholds made in [8]

This subsection summarizes the theory given in [8]. Complete results and their proofs can be found in section 3.3.

At intermediate points, the PDF of the system satisfies

$$\frac{\partial f_i^{(k)}(x,t)}{\partial t} = \frac{1}{2}(\sigma_i^{(k)})^2 \frac{\partial^2 f_i^{(k)}(x,t)}{\partial x^2} - \mu_i^{(k)} \frac{\partial f_i^{(k)}(x,t)}{\partial x} + \sum_{j \in S} f_j^{(k)}(x,t) q_{ji}^{(k)},$$

$$1 \le k \le K-1,\ \forall i \in S. \tag{3.28}$$

At all intermediate boundary points, from $x = T^{(1)}$ to $x = T^{(K-1)}$, the following boundary condition is satisfied:

$$\frac{dc_i^{(k)}(t)}{dt} + f_i^{(k+1)}(T^{(k)}+,t) r_i^{(k+1)} - \frac{1}{2} \left. \frac{\partial (f_i^{(k+1)}(x,t)(\sigma_i^{(k+1)})^2)}{\partial x} \right|_{x=T^{(k)}+}$$

$$- f_i^{(k)}(T^{(k)}-,t) r_i^{(k)} + \frac{1}{2} \left. \frac{\partial (f_i^{(k)}(x,t)(\sigma_i^{(k)})^2)}{\partial x} \right|_{x=T^{(k)}-} = \sum_{j \in S} c_j^{(k)}(t) \tilde{q}_{ji}^{(k)},\ 1 \le k \le K-1,$$

$$\forall i \in S. \tag{3.29}$$

At $x = T^{(0)}$, (3.29) becomes

$$\frac{dc_i^{(0)}(t)}{dt} + f_i^{(1)}(0+,t) r_i^{(1)} - \frac{1}{2} \left. \frac{\partial (f_i^{(1)}(x,t)(\sigma_i^{(1)})^2)}{\partial x} \right|_{x=0+} = \sum_{j \in S} c_j^{(0)}(t) \tilde{q}_{ji}^{(0)},$$

$$\forall i \in S. \tag{3.30}$$

Also at $x = T^{(K)}$ (3.29) becomes

$$\frac{dc_i^{(K)}(t)}{dt} - f_i^{(k)}(T^{(K)}-,t)r_i^{(K)} + \frac{1}{2}\frac{\partial(f_i^{(k)}(x,t)(\sigma_i^{(K)})^2)}{\partial x}\bigg|_{x=T^{(K)}-} = \sum_{j\in S} c_j^{(K)}(t)\tilde{q}_{ji}^{(K)},$$

$$\forall i \in S. \tag{3.31}$$

a) Intermediate boundary points:

In [8], three types of intermediate boundary points, namely sticky, emitting and repulsive boundary points, are considered. Behaviors at boundary points are assigned due to the values of drifts and variances above and below the boundary points. Moreover, we assume that all drifts are nonzero at all states at all interior points of regions. Based on [8], boundary conditions for all three types of boundary points are given below.

1) Sticky boundary points

For sticky boundary points the drift above the boundary point is negative and below the boundary point is positive. In the original work, all sticky boundaries are selected as absorbing. For this type of boundary points besides (3.29) the following two boundary conditions are given:

$$f_i^{(k)}(T^{(k)}-,t) = 0,\ \forall i : r_i^{(k)} > 0 \wedge r_i^{(k+1)} < 0,\ 1 \le k \le K-1, \tag{3.32}$$

$$f_i^{(k)}(T^{(k)}+,t) = 0,\ \forall i : r_i^{(k)} > 0 \wedge r_i^{(k+1)} < 0,\ 1 \le k \le K-1. \tag{3.33}$$

2) Emitting boundary points

For emitting boundary points, PMA is zero, i.e.,

$$c_i^{(k)}(t) = 0,\ \forall i : r_i^{(k)} r_i^{(k+1)} > 0,\ 1 \le k \le K-1. \tag{3.34}$$

3) Repulsive boundary points

These boundaries have the property that the drift above the boundary point is positive and below the boundary point is negative. All repulsive boundaries are selected as absorbing. In the original work "Isolating" term is used instead of repulsive. For this type of boundary point, the following two boundary conditions are given besides (3.29):

$$f_i^{(k)}(T^{(k)}-,t) = 0,\ \forall i : r_i^{(k)} < 0 \wedge r_i^{(k+1)} > 0,\ 1 \le k \le K-1, \tag{3.35}$$

$$f_i^{(k)}(T^{(k)}+,t) = 0,\ \forall i : r_i^{(k)} < 0 \wedge r_i^{(k+1)} > 0,\ 1 \le k \le K-1. \tag{3.36}$$

b) Terminal Boundaries:

Three types of terminal boundaries- namely reflecting, absorbing and emitting- are analyzed in [8]. As in the intermediate boundaries, behaviors at terminal boundary points are assigned due values of the drifts and variances above or below the boundary points. For example all terminal boundary points with nonzero variance in its neighbor region are assumed to have the reflection property.

1) Reflecting boundary points

In addition to (3.30), at $x = T^{(0)}$ for reflecting terminal boundary points PMA is zero, i.e.,

$$c_i^{(0)}(t) = 0\,,\ \forall i \in S_+\,. \tag{3.37}$$

In addition to (3.31) at $x = T^{(K)}$ for reflecting terminal boundary points PMA is zero, i.e.,

$$c_i^{(K)}(t) = 0\,,\ \forall i \in S_+\,. \tag{3.38}$$

2) Absorbing boundary points

At $x = T^{(0)}$ if $i \in S_{0-}$ then this boundary point is defined as an absorbing boundary and (3.30) is given as a boundary condition.

At $x = T^{(K)}$ if $i \in S_{0+}$ then this boundary point is defined as an absorbing boundary and (3.31) is given as a boundary condition.

3) Emitting boundary points

At $x = T^{(0)}$ if $i \in S_{0+}$ then this boundary point is defined as an emitting boundary and

$$f_i^{(1)}(0+,t)r_i^{(1)} = \sum_{j\in S} c_j^{(0)}(t)\tilde{q}_{ji}^{(0)}\ ,\ \forall i \in S_{0+}\,, \tag{3.39}$$

$$c_i^{(0)}(t) = 0\,,\ \forall i \in S_{0+} \tag{3.40}$$

are given as a boundary conditions.

At $x = T^{(K)}$ if $i \in S_{0-}$ then it is defined as an emitting boundary and

$$-f_i^{(K)}(T^{(K)}-,t)r_i^{(K)} = \sum_{j\in S} c_j^{(K)}(t)\tilde{q}_{ji}^{(K)}\ ,\ \forall i \in S_{0-}\,, \tag{3.41}$$

$$c_i^{(K)}(t) = 0, \forall i \in S_{0-} \tag{3.42}$$

are given as a boundary conditions.

3.1.6 Discussion on Claimed Transient Solution

Let's check the number of equations and unknowns in the following simple system on Fig. 3.2:

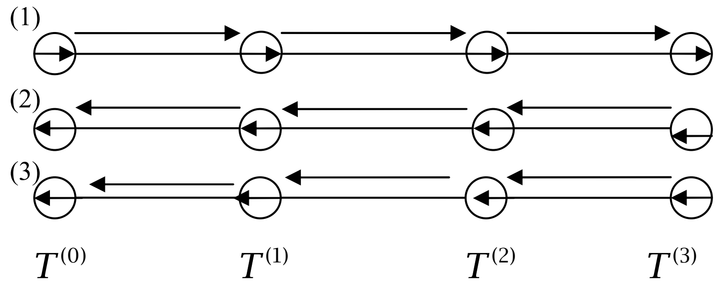


Figure 3.2 A Second Order Feedback Fluid Queue Example with Multiple Thresholds

In the figure, background process has 3 states each of which is depicted by a line. The queue is divided into 3 parts. Drift in state 1 is always positive and drifts of states 2 and 3 are negative everywhere on the queue. Let's assume that variance is positive everywhere, then according to the classification made in [8] all intermediate boundary points are emitting type then for each intermediate boundary point (3.29) is given as the boundary condition. Therefore we obtain 6 boundary conditions from intermediate boundary points. Also according to the classification of terminal boundary points in [8] here since variance is positive everywhere, all 6 terminal boundary points are reflecting, therefore 6 boundary conditions can be obtained from terminal boundary conditions. Totally we can obtain 12 boundary conditions for this system. These boundary conditions will be still valid if the limit of them is taken as time index goes to infinity, for the solution in steady state. Now let's count the number of unknowns. Again since variance is positive everywhere and there exists no repulsive or sticky type neighborhood, there exists no PMA neither at intermediate nor at intermediate points on the queue. Then unknowns of the system are just the coefficients in

(3.21) for each region for steady state solution. Since $|S_+|=3$ then there are 6 unknown coefficient for each region, then totally we need to find 18 unknowns in order to solve the system. However only 12 boundary conditions can be obtained from the analysis made in [8]. Then obviously some of the boundary conditions are absent and the analysis made in [8] is incomplete. In the following section this incompleteness is overcome and the missing boundary conditions are found.

3.2 Analysis of Second Order Feedback Fluid Queues With Multiple Thresholds with Emitting Intermediate Boundaries

<u>Theorem 3.3:</u> Second order feedback fluid queues with multiple thresholds satisfy

$$\frac{\partial}{\partial t} f_i^{(k)}(x,t) - \frac{1}{2}(\sigma_i^{(k)})^2 \frac{\partial^2 f_i^{(k)}(x,t)}{\partial x^2} + \mu_i^{(k)} \frac{\partial f_i^{(k)}(x,t)}{\partial x} = \sum_{l=1}^{N} q_{li}^{(k)} f_l(x,t), \quad (3.43)$$

along with the boundary conditions on intermediate emitting boundaries

$$\frac{1}{2}(\sigma_i^{(k)})^2 \left.\frac{\partial f_i^{(k)}(x,t)}{\partial x}\right|_{x=T^{(k)}-} - \frac{1}{2}(\sigma_i^{(k+1)})^2 \left.\frac{\partial f_i^{(k+1)}(x,t)}{\partial x}\right|_{x=T^{(k)}+} - \mu_i^{(k)} f_i^{(k)}(T^{(k)}-,t)$$

$$+\mu_i^{(k+1)} f_i^{(k+1)}(T^{(k)}+,t) = \sum_{l=1}^{N} \tilde{q}_{li}^{(k)} c_l^{(k)}(t), \quad (3.44)$$

$$(\sigma_i^{(k)})^2 f_i^{(k)}(T^{(k)}-,t) - (\sigma_i^{(k+1)})^2 f_i^{(k+1)}(T^{(k)}+,t) = 0, \quad (3.45)$$

and on terminal reflecting boundaries

$$-\frac{1}{2}(\sigma_i^{(1)})^2 \left.\frac{\partial f_i^{(1)}(x,t)}{\partial x}\right|_{x=0+} + \mu_i^{(1)} f_i^{(1)}(0+,t) = \sum_{l=1}^{N} \tilde{q}_{li}^{(0)} c_l^{(0)}(t), \quad (3.46)$$

$$c_i^{(0)}(t) = 0, \quad (3.47)$$

$$\frac{1}{2}(\sigma_i^{(K)})^2 \left.\frac{\partial f_i^{(K)}(x,t)}{\partial x}\right|_{x=T^{(K)}-} - \mu_i^{(K)} f_i^{(K)}(T^{(K)}-,t) = \sum_{l=1}^{N} \tilde{q}_{li}^{(K)} c_l^{(K)}(t), \tag{3.48}$$

$$c_i^{(K)}(t) = 0. \tag{3.49}$$

Proof:

In [6], generator of a single regime double Markov process $\{C(t), X(t); t \geq 0\}$ is given as,

$$(Lg)(x,i) = \lim_{t \downarrow 0} \frac{(U_t g)(x,i) - g(x,i)}{t}, \tag{3.50}$$

where g is a twice continuously differentiable function and

$$(U_t g)(x,i) = E[g(C(t), X(t)) \mid C(0) = x, X(0) = i]. \tag{3.51}$$

For a second order feedback fluid queue with multiple thresholds, the following expressions for the operator L can be obtained (proof can be found in Appendix).

$$(Lg)(x,i) = \frac{1}{2}(\sigma_i(x))^2 g''(x,i) + \mu_i(x) g'(x,i) + \sum_{l:l \neq i} q_{il}(x)(g(x,l) - g(x,i))$$

$$= \frac{1}{2}(\sigma_i(x))^2 g''(x,i) + \mu_i(x) g'(x,i) + \sum_{l=1}^{N} q_{il}(x) g(x,l), \tag{3.52}$$

where

$$\sigma_i(x) = \begin{cases} \sigma_i^{(k)}, T^{(k-1)} < x < T^{(k)} \\ \tilde{\sigma}_i^{(k)}, x = T^{(k)} \end{cases}, \tag{3.53}$$

$$\mu_i(x) = \begin{cases} \mu_i^{(k)}, T^{(k-1)} < x < T^{(k)} \\ \tilde{\mu}_i^{(k)}, x = T^{(k)} \end{cases}, \text{ and} \tag{3.54}$$

$$Q(x) = \begin{cases} Q^{(k)}, T^{(k-1)} < x < T^{(k)} \\ \tilde{Q}^{(k)}, x = T^{(k)} \end{cases}. \tag{3.55}$$

We know that

$$c_i^{(k)}(t;y,j) = F_i^{(k+1)}(T^{(k)}+,t;y,j) - F_i^{(k)}(T^{(k)}-,t;y,j) \text{ and} \tag{3.56}$$

$$f_i^{(k)}(x,t;y,j) = \frac{\partial F_i^{(k)}(x,t;y,j)}{\partial x}. \tag{3.57}$$

Based on [6] Kolmogorov Forward equation for F is:

$$\frac{d}{dt}[\sum_i \int g(x,i) d_x F_i^{(.)}(x,t;y,j)] = \sum_i \int (Lg)(x,i) d_x F_i^{(.)}(x,t;y,j) \tag{3.58}$$

By Substituting (3.52) into (3.58) and by omitting the parameters y and j, e.g. $F_i^{(.)}(x,t)$ instead of $F_i^{(.)}(x,t;y,j)$,

$$\frac{d}{dt}[\sum_{k=0}^{K}\sum_i g(T^{(k)},i)c_i^{(k)}(t) + \sum_{k=1}^{K}\sum_i \int_{T_{k-1}+}^{T_k-} g(x,i) f_i^{(k)}(x,t)dx]$$

$$= \sum_{k=0}^{K}\sum_i \{\frac{1}{2}(\tilde{\sigma}_i^{(k)})^2 g''(T^{(k)},i) + \tilde{\mu}_i^{(k)} g'(T^{(k)},i) + \sum_{l=1}^{N} \tilde{q}_{il}^{(k)} g(T^{(k)},l)\} c_i^{(k)}(t)$$

$$+\sum_{k=1}^{K}\sum_i \int_{T_{k-1}-}^{T_k+} \{\frac{1}{2}(\sigma_i^{(k)})^2 g''(x,i) + \mu_i^{(k)} g'(x,i) + \sum_{l=1}^{N} q_{il}^{(k)} g(x,l)\} f_i^{(k)}(x,t)dx \tag{3.59}$$

can be obtained. By substituting the relations in (3.60) and (3.61) obtained by integration by parts,

$$\int_{T^{(k-1)}+}^{T^{(k)}-} g''(x,i) f_i^{(k)}(x,t)dx = g'(x,i) f_i(x,t)\,|_{T^{(k-1)}+}^{T^{(k)}-} - \int_{T^{(k-1)}+}^{T^{(k)}-} g'(x,i)\frac{\partial f_i^{(k)}(x,t)}{\partial x}dx$$

$$= g'(T^{(k)}-,i) f_i^{(k)}(T^{(k)}-,t) - g'(T^{(k-1)}+,i) f_i^{(k)}(T^{(k-1)}+,t) - g(x,i)\frac{\partial f_i^{(k)}(x,t)}{\partial x}\,|_{T^{(k-1)}+}^{T^{(k)}-}$$

$$+\int_{T^{(k-1)}+}^{T^{(k)}-} g(x,i)\frac{\partial^2 f_i^{(k)}(x,t)}{\partial x^2}dx$$

$$= g'(T^{(k)}-,i) f_i^{(k)}(T^{(k)}-,t) - g'(T^{(k-1)}+,i) f_i^{(k)}(T^{(k-1)}+,t) - g(T^{(k)}-,i)\left.\frac{\partial f_i^{(k)}(x,t)}{\partial x}\right|_{x=T^{(k)}-}$$

$$+g(T^{(k-1)}+,i)\left.\frac{\partial f_i^{(k)}(x,t)}{\partial x}\right|_{x=T^{(k-1)}+} + \int_{T^{(k-1)}+}^{T^{(k)}-} g(x,i)\frac{\partial^2 f_i^{(k)}(x,t)}{\partial x^2}dx \tag{3.60}$$

$$\int_{T^{(k-1)}+}^{T^{(k)}-} g'(x,i) f_i^{(k)}(x,t)dx = g(x,i) f_i^{(k)}(x,t)\,|_{T^{(k-1)}+}^{T^{(k)}-} - \int_{T^{(k-1)}+}^{T^{(k)}-} g(x,i)\frac{\partial f_i^{(k)}(x,t)}{\partial x}dx$$

$$= g(T^{(k)}-,i) f_i^{(k)}(T^{(k)}-,t) - g(T^{(k-1)}+,i) f_i^{(k)}(T^{(k-1)}+,t) - \int_{T^{(k-1)}+}^{T^{(k)}-} g(x,i)\frac{\partial f_i^{(k)}(x,t)}{\partial x}dx \tag{3.61}$$

into (3.59) and by arranging the terms in (3.55) the following important relation can be found:

$$
\sum_{k=1}^{K-1}\sum_i g(T^{(k)},i)\left\{\begin{array}{l}\frac{d}{dt}c_i^{(k)}(t)+\frac{1}{2}(\sigma_i^{(k)})^2\left.\frac{\partial f_i^{(k)}(x,t)}{\partial x}\right|_{x=T^{(k)}-}-\frac{1}{2}(\sigma_i^{(k+1)})^2\left.\frac{\partial f_i^{(k+1)}(x,t)}{\partial x}\right|_{x=T^{(k)}+}\\ -\mu_i^{(k)}f_i^{(k)}(T^{(k)}-,t)+\mu_i^{(k+1)}f_i^{(k+1)}(T^{(k)}+,t)-\sum_{l=1}^{N}\tilde{q}_{li}^{(k)}c_l^{(k)}(t)\end{array}\right\}
$$

$$
-\sum_{k=1}^{K-1}\sum_i g'(T^{(k)},i)\{\tilde{\mu}_i^{(k)}c_i^{(k)}(t)+\frac{1}{2}(\sigma_i^{(k)})^2 f_i^{(k)}(T^{(k)}-,t)-\frac{1}{2}(\sigma_i^{(k+1)})^2 f_i^{(k+1)}(T^{(k)}+,t)\}
$$

$$
-\sum_{k=1}^{K-1}\sum_i g''(T^{(k)},i)\{\frac{1}{2}(\tilde{\sigma}_i^{(k)})^2 c_i^{(k)}(t)\}
$$

$$
+\sum_i g(0,i)\{\frac{d}{dt}c_i^{(0)}(t)-\frac{1}{2}(\sigma_i^{(1)})^2\left.\frac{\partial f_i^{(1)}(x,t)}{\partial x}\right|_{x=0+}+\mu_i^{(1)}f_i^{(1)}(0+,t)-\sum_{l=1}^{m}\tilde{q}_{li}^{(0)}c_l^{(0)}(t)\}
$$

$$
-\sum_i g'(0,i)\{\tilde{\mu}_i^{(0)}c_i^{(0)}(t)-\frac{1}{2}(\sigma_i^{(1)})^2 f_i^{(1)}(0+,t)\}
$$

$$
-\sum_i g''(0,i)\{\frac{1}{2}(\tilde{\sigma}_i^{(0)})^2 c_i^{(0)}(t)\}
$$

$$
+\sum_i g(T^{(K)},i)\{\frac{d}{dt}c_i^{(K)}(t)+\frac{1}{2}(\sigma_i^{(K)})^2\left.\frac{\partial f_i^{(K)}(x,t)}{\partial x}\right|_{x=T^{(K)}-}-\mu_i^{(K)}f_i^{(K)}(T^{(K)}-,t)-\sum_{l=1}^{m}\tilde{q}_{li}^{(K)}c_l^{(K)}(t)\}
$$

$$
-\sum_i g'(T^{(K)},i)\{\tilde{\mu}_i^{(K)}c_i^{(K)}(t)+\frac{1}{2}(\sigma_i^{(K)})^2 f_i^{(K)}(T^{(K)}-,t)\}
$$

$$
-\sum_i g''(T^{(K)},i)\{\frac{1}{2}(\tilde{\sigma}_i^{(K)})^2 c_i^{(K)}(t)\}
$$

$$
+\sum_{k=1}^{K}\sum_i\int_{T^{(k-1)}+}^{T^{(k)}-}\left\{\begin{array}{l}\frac{\partial}{\partial t}f_i^{(k)}(x,t)-\frac{1}{2}(\sigma_i^{(k)})^2\frac{\partial^2 f_i^{(k)}(x,t)}{\partial x^2}+\mu_i^{(k)}\frac{\partial f_i^{(k)}(x,t)}{\partial x}\\ -\sum_{l=1}^{m}q_{li}^{(k)}f_l^{(k)}(x,t)\end{array}\right\}g(x,i)dx=0
\tag{3.62}
$$

This equation yields both (3.43) and all boundary conditions.

By taking the function g, its first and second derivatives as zero at every boundary point in all states except some state j we obtain (3.43).

By taking, first and second derivatives of the function g as zero at every boundary point and in all states and taking this function as zero at all boundary points except the point $T^{(k)}$ in all states except some state j we obtain (3.44).

By taking, second derivative of the function g as zero at every boundary point in all states and taking the first derivative function as zero at all boundary points except the point $T^{(k)}$ in all states except some state j we obtain (3.45).

By applying the same method as in (3.44) and (3.45) other boundary conditions can be found easily. □

The missing condition in [8] is (3.45).

Remark 3.1: If left or right continuity of the behavioral feature is under consideration then the boundary condition (3.45) is still valid.

Theorem 3.4: The steady state joint densities $f_i^{(.)}(x)$ of a second order feedback fluid queue satisfy

$$-\frac{1}{2}(\sigma_i^{(k)})^2 \frac{\partial^2 f_i^{(k)}(x)}{\partial x^2} + \mu_i^{(k)} \frac{\partial f_i^{(k)}(x)}{\partial x} = \sum_{l=1}^{m} q_{li}^{(k)} f_l^{(k)}(x) \ , \tag{3.63}$$

along with the boundary conditions on intermediate emitting boundaries,

$$\frac{1}{2}(\sigma_i^{(k)})^2 \left.\frac{\partial f_i^{(k)}(x)}{\partial x}\right|_{x=T^{(k)}-} - \frac{1}{2}(\sigma_i^{(k+1)})^2 \left.\frac{\partial f_i^{(k+1)}(x)}{\partial x}\right|_{x=T^{(k)}+} - \mu_i^{(k)} f_i^{(k)}(T^{(k)}-)$$

$$+\mu_i^{(k+1)} f_i^{(k+1)}(T^{(k)}+) = \sum_{l=1}^{m} \tilde{q}_{li}^{(k)} c_l^{(k)}, \tag{3.64}$$

$$(\sigma_i^{(k)})^2 f_i^{(k)}(T^{(k)}-) - (\sigma_i^{(k+1)})^2 f_i^{(k+1)}(T^{(k)}+) = 0\,, \tag{3.65}$$

and on terminal reflecting boundaries

$$-\frac{1}{2}(\sigma_i^{(1)})^2 \left.\frac{\partial f_i^{(1)}(x)}{\partial x}\right|_{x=0+} + \mu_i^{(1)} f_i^{(1)}(0+) = \sum_{l=1}^{m} \tilde{q}_{li}^{(0)} c_l^{(0)}\,, \tag{3.66}$$

$$c_i^{(0)} = 0\,, \tag{3.67}$$

$$\frac{1}{2}(\sigma_i^{(K)})^2 \left.\frac{\partial f_i^{(K)}(x)}{\partial x}\right|_{x=T^{(K)}-} - \mu_i^{(K)} f_i^{(K)}(T^{(K)}-) = \sum_{l=1}^{m} \tilde{q}_{li}^{(K)} c_l^{(K)}\,, \tag{3.68}$$

$$c_i^{(K)} = 0 . \tag{3.69}$$

Proof: Just take the limit of equations from (3.43) to (3.49) as $t \to \infty$.

□

3.3 Numerical Example

Example 3.1: For a quick example let's look at a two regime four state second order feedback fluid queue with emitting intermediate and reflecting terminal boundary points. Assume that size of the queue is 1 and intermediate boundary point is at 0.5. Variance is selected as 1 everywhere for all states. Other parameters are

$$r_1^{(1)} = 1,\ r_2^{(1)} = -1,\ r_3^{(1)} = 2,\ r_4^{(1)} = -3,$$

$$r_1^{(2)} = 2,\ r_2^{(2)} = -2,\ r_3^{(2)} = 2,\ r_4^{(2)} = -1,$$

$$\tilde{Q}^{(0)} = Q^{(1)} = \tilde{Q}^{(1)} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix},\ Q^{(2)} = \tilde{Q}^{(2)} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

Then resulting plots of this system are given in the following figures.

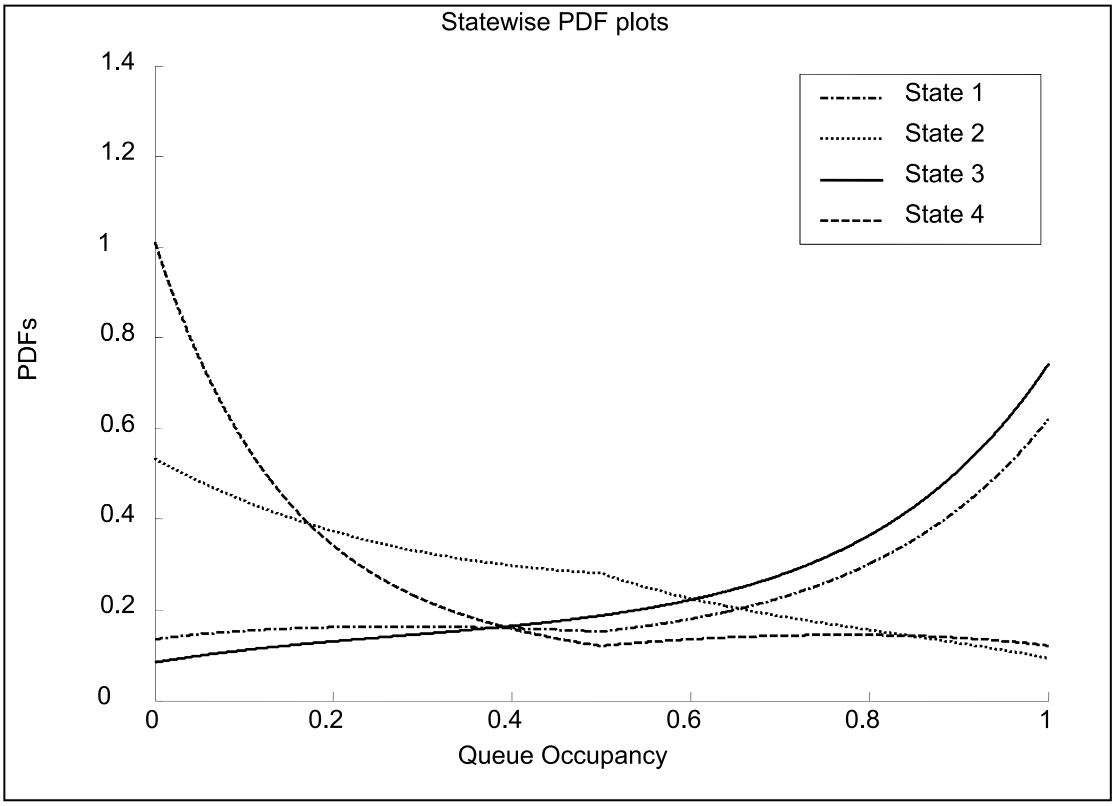


Figure 3.3 Statewise PDF plots of Example 3.1

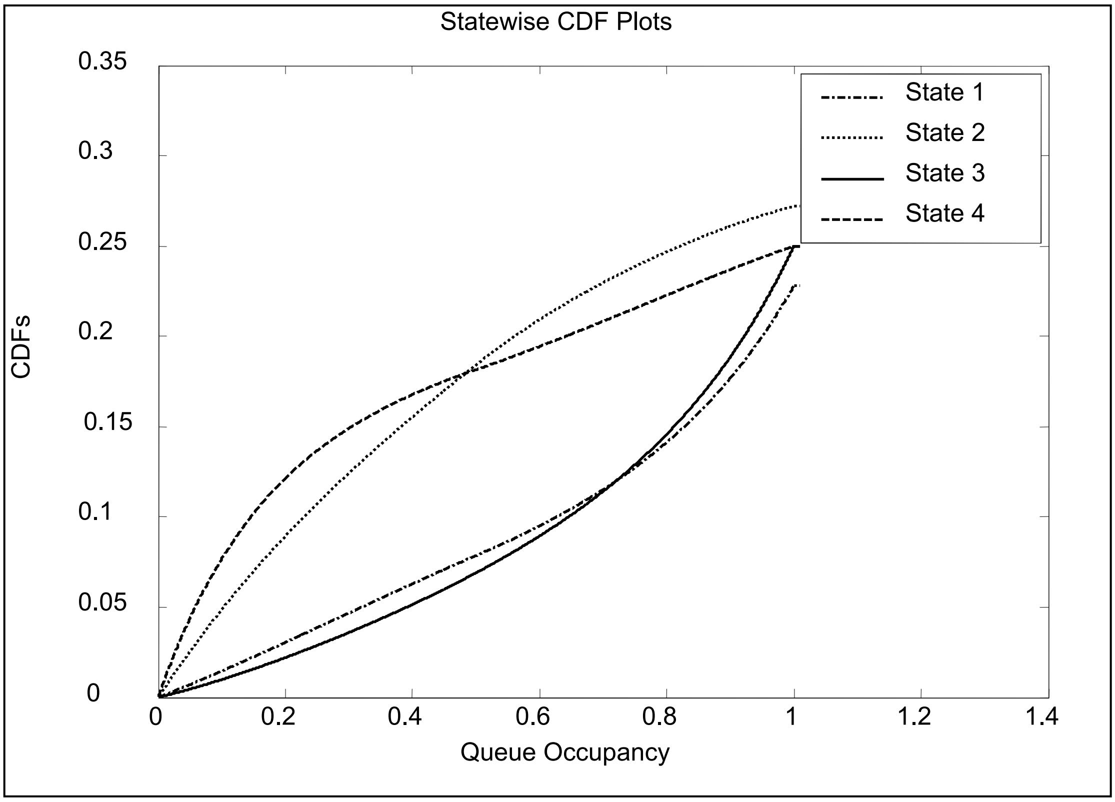


Figure 3.4 Statewise CDF plots of Example 3.1

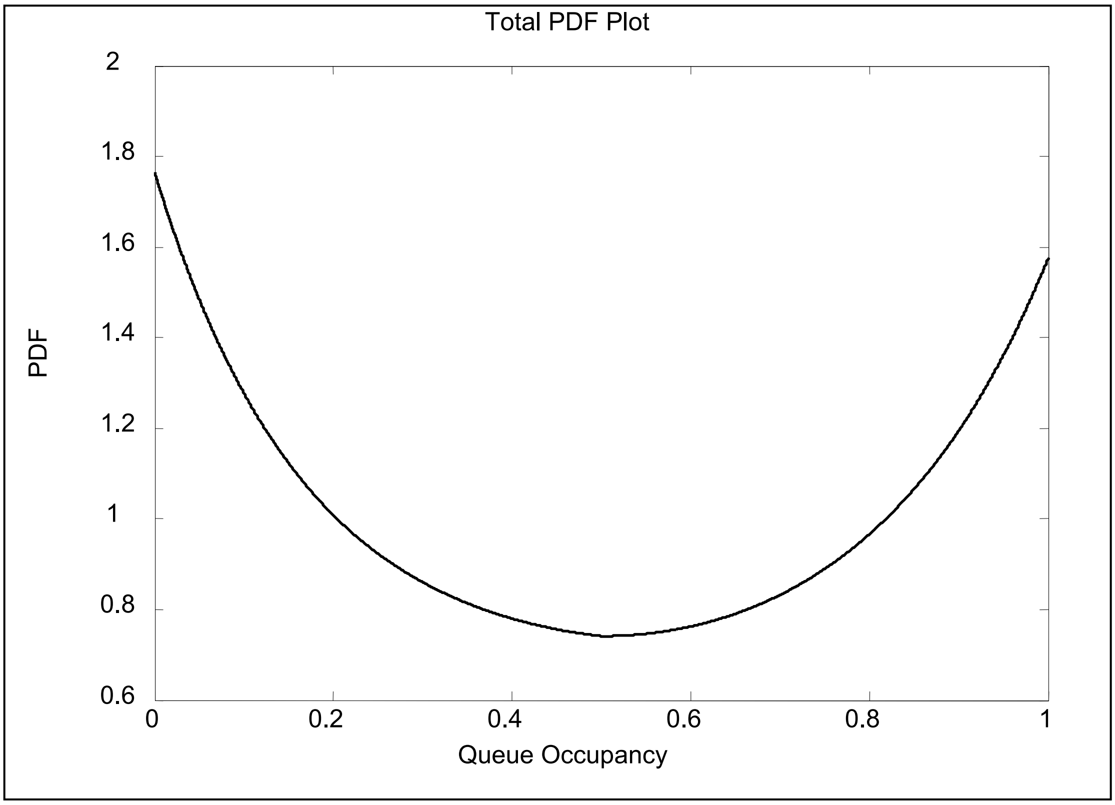


Figure 3.5 Total PDF plots of Example 3.1

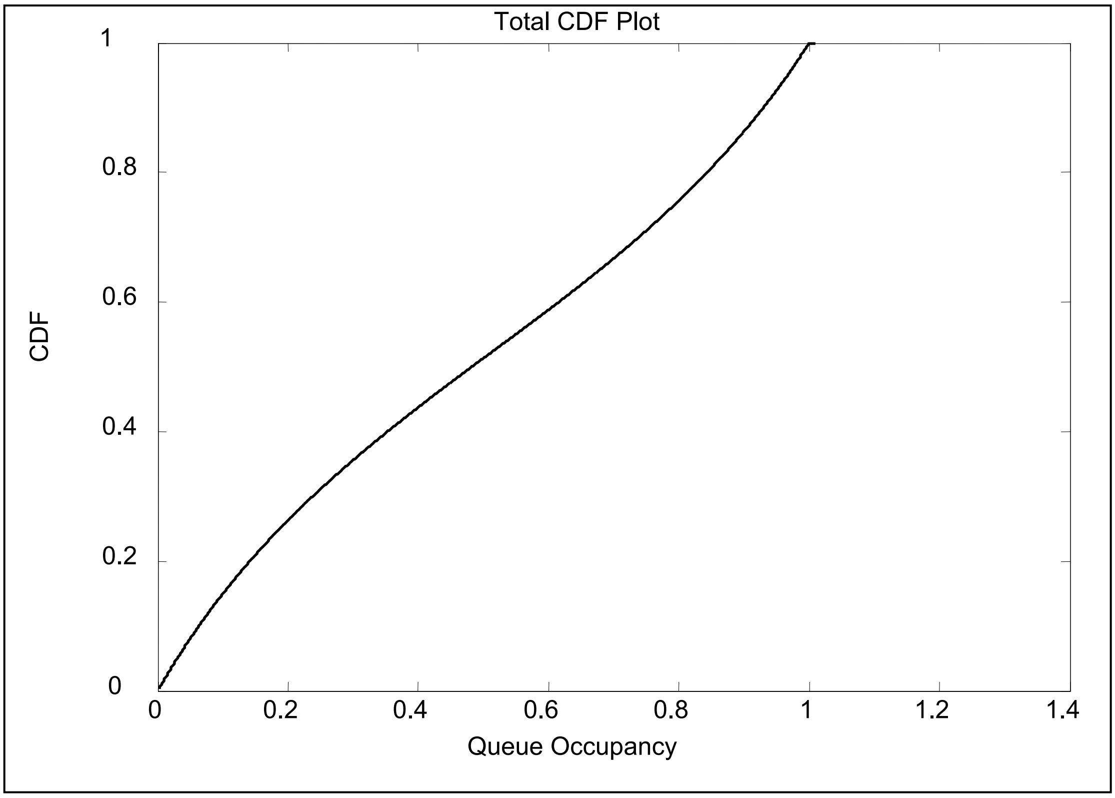


Figure 3.6 Total CDF plots of Example 3.1

Chapter 4

Fluid Model as an Approximation to Markov Modulated Discrete Models

State space of discrete Markov queueing models is linearly dependent on the queue size as far as the steady-state distribution of the queue occupancy is concerned. This dependence leads to the so-called state space explosion problem if the queue of the system is large. Unlike discrete models, the complexity of the fluid model solution is relatively insensitive to the queue size. Therefore, we conjecture that a variety of discrete models can be approximated with fluid models and they can be solved without having to face state space explosion. Among discrete models, we study discrete-time single server queues with Markov modulated arrivals and services.

In Section 4.1, a sketch of analysis of Markov modulated discrete time queueing systems is given. In sections 4.2 and 4.3, transformations from Markov modulated discrete time queues to first and second order Markov fluid queue models are given, respectively. In Section 4.4, we provide numerical examples in order to compare these two queueing models.

4.1 Markov Modulated Discrete Time Queue

A Markov modulated discrete time queueing system with deterministic constant (probabilistic) net rates in each state, is the discrete counterpart of the first (second) order single regime fluid model. More specifically, if the background process is in state i at the end of a time slot, the queue content will increase (or decrease depending on the sign of the rate) with rate $r_i^{(d)}$ (with a non-degenerate PMF for second order approximation) if no transition occurs and with rate $r_i^{(d)}$ (call this case 1, with PMF for second order approximation) or $r_j^{(d)}$ (call this case 2, with a PMF for second order), depending on the interpretation, if a transition occurs from state i to state j. If a transition happens within a time slot then for case 1 that transition occurs at the end of the time slot and for case 2 at the start of the time slot.

Assume that the discrete time Markov process, or the so-called background process, $\{X_n, n \geq 0, n \in Z\}$ has a finite state space $S^{(d)}$ with N states. Furthermore, the queue length is denoted by $K^{(d)}$ and therefore the content process $\{C_n, n \geq 0, n \in Z\}$ has a state space $\{0,1,...,K^{(d)}\}$ with $K^{(d)} + 1$ elements. Mathematically, the PMF function $f_{i,j}$ and CDF function $F_{i,j}$ are defined as,

$$f_{i,j} = \lim_{n \to \infty} P(C_n = j, X_n = i), \tag{4.1}$$

$$F_{i,j} = \lim_{n \to \infty} P(C_n \leq j, X_n = i), \tag{4.2}$$

which can be calculated by writing the forward Kolmogorov Equations. In these equations, for both of the cases, there are $N(K^{(d)} + 1)$ unknowns. Forward Kolmogorov equations also give us $N(K^{(d)} + 1)$ equations with one redundant equation. This deficiency in the rank can be completed by adding the normalization condition, i.e., the sum of all PMFs is one, to the equations. For case 1, $f_{i,0}$ and $f_{i,K}$ will be zero for states with positive and negative rates respectively. Therefore, the number of unknowns reduces to $NK^{(d)}$ unless there is a state with rate zero. Beyond that, this case is solved within the same framework as in the former case.

4.2 Conversion of Parameters from Discrete to Continuous for First Order Fluid Queues

In this section, continuous counterpart of discrete system parameters are given provided that the net flow rates are deterministic. For a DTMC, whenever the state is i, then a transition from this state to another will occur after a random amount time (or time slots) distributed according to a geometric distribution with parameter $\{P\}_{ii}$ and also the CTMC in state i is distributed according to exponential distribution with mean $\frac{1}{\{Q\}_{ii}}$,where Q is the infinitesimal generator of CTMC . Conversion from discrete to continuous time Markov chain can be done in two ways.

In the first method, we fit the complementary cumulative distribution functions (CCDFs) at integer multiples of Δt. Mathematically, defining the duration of time in which the discrete and continuous time Markov Chains stay in state i as X and Y, respectively, as

$$P(X \geq x\Delta t) = (\{P\}_{ii})^x, \tag{4.3}$$

$$P(Y \geq x\Delta t) = e^{-\{Q\}_{ii} x\Delta t}, \tag{4.4}$$

that yield the following equality

$$\{Q\}_{ii} = \frac{\ln(1/\{P\}_{ii})}{\Delta t}. \tag{4.5}$$

Then,

$$P(Y \geq x\Delta t) = e^{-\frac{\ln(1/\{P\}_{ii})}{\Delta t} x\Delta t} = (\{P\}_{ii})^x. \tag{4.6}$$

The off-diagonal elements can also be found using the same methodology as above. For a CTMC, an off-diagonal element in row i and in column j represents the rate of staying at state i times the probability of transition to state j if it is a-priori known that a transition occurs. For a DTMC, an off-diagonal element at row i and column j is the probability of having a transition to the state j after a slot time if the chain is currently in state i. Therefore, the following equality between the off-diagonal parameters appears to be an acceptable approximation,

$$\{Q\}_{ij} = \{Q\}_{ii} \frac{\{P\}_{ij}}{\sum_{j \neq i} \{P\}_{ij}} . \tag{4.7}$$

The approximation method used above breaks if there exists a zero element at the diagonal of the probability transition matrix of the discrete system. Then, equality (4.5) is not valid anymore. In order avoid this pitfall, we use the first order approximation of the logarithm function (the first component of the Taylor series expansion) instead of itself, as a second method. Both, the first and second methods lead to the same forward Kolmogorov differential equations in for the Markov fluid queues of interest if all the diagonal elements of one step transition matrix are nonzero. The first order approximation offers us the following simple relation between the parameters:

$$Q = \frac{(P - I)}{\Delta t}, \tag{4.8}$$

where P is the probability transition matrix of the DTMC and I is the identity matrix with an appropriate size.

Assuming that Δx is the counterpart of the size of an entity in fluid queues, continuous queue length K can be described in terms of the discrete one as,

$$K = K^{(d)} \Delta x . \tag{4.9}$$

In addition to the transition matrix, and queue length, rate parameters should be reconsidered as well. The amount of net workload which enters to the queue is $r_i^{(d)} \Delta x$, where Δx is the proportion of queue size of continuous queue to discrete queue, within a time slot, therefore the net rate will be,

$$r_i = \frac{r_i^{(d)} \Delta x}{\Delta t} . \tag{4.10}$$

Let's first write the transient equations for both of the cases (Case 1 and Case 2 respectively),

$$f_{i,k}(n) = f_{i,k-r_i^{(d)}}(n-1)\{P\}_{ii} + \sum_{j \neq i} f_{j,k-r_i^{(d)}}(n-1)\{P\}_{ij} , \tag{4.11}$$

$$f_{i,k}(n) = f_{i,k-r_i^{(d)}}(n-1)\{P\}_{ii} + \sum_{j \neq i} f_{j,k-r_j^{(d)}}(n-1)\{P\}_{ij} . \tag{4.12}$$

The only difference between equations (4.11) and (4.12) is the subscript of the f term in the summation. Because of the similarity of subsequent steps in both cases, we will follow the first case (Eq. (4.11)) and after obtaining the results, we will consider the second case. Subtracting $f_{i,k-r_i^{(d)}}(n-1)$ from both sides and dividing them by Δt lead us to obtain the following equality:

$$\frac{f_{i,k}(n)-f_{i,k-r_i^{(d)}}(n-1)}{\Delta t}=f_{i,k-r_i^{(d)}}(n-1)\frac{(1-\{P\}_{ii})}{\Delta t}+\sum_{j\neq i}f_{j,k-r_i^{(d)}}(n-1)\frac{\{P\}_{ij}}{\Delta t}. \tag{4.13}$$

Through the conversion to continuous case, space parameter k and time parameter n have to be replaced with $k\Delta x$ and $n\Delta t$ respectively,

$$\frac{f_i(k\Delta x,n\Delta t)-f_i(k\Delta x-r_i^{(d)}\Delta x,(n-1)\Delta t)}{\Delta t}=$$

$$f_i(k\Delta x-r_i^{(d)}\Delta x,(n-1)\Delta t)\frac{(1-\{P\}_{ii})}{\Delta t}+\sum_{j\neq i}f_j(k\Delta x-r_i^{(d)}\Delta x,(n-1)\Delta t)\frac{\{P\}_{ij}}{\Delta t}. \tag{4.14}$$

By using the approximations (4.5) and (4.7), we obtain the following equation,

$$\frac{f_i(k\Delta x,n\Delta t)-f_i(k\Delta x-r_i\Delta t,(n-1)\Delta t)}{\Delta t}=$$

$$f_i(k\Delta x-r_i\Delta t,(n-1)\Delta t)\{Q\}_{ii}+\sum_{j\neq i}f_j(k\Delta x-r_i\Delta t,(n-1)\Delta t)\{Q\}_{ij}\ . \tag{4.15}$$

Taking the limit of both sides as $\Delta t\to 0$, $n\to\infty$ such that $n\Delta t\to t$ and $\Delta x\to 0$, $k\to\infty$ such that $k\Delta x\to x$ results in,

$$\frac{\partial f_i(x,t)}{\partial t}+r_i\frac{\partial f_i(x,t)}{\partial x}=f_i(x,t)\{Q\}_{ii}+\sum_{j\neq i}f_j(x,t)\{Q\}_{ij}=\sum_j f_j(x,t)\{Q\}_{ij} \tag{4.16}$$

or in matrix form

$$\frac{\partial \mathbf{f}(x,t)}{\partial t}+\frac{\partial \mathbf{f}(x,t)}{\partial x}R=\mathbf{f}(x,t)Q \tag{4.17}$$

which is the space derivative of (1.3), the transient equation for fluid queues. Summing up both sides of (4.11) over the space parameter from 0 to k and continuing the same steps result in directly (1.3). We may obtain the steady state equations by taking the limit of the time parameter as $t \to \infty$. Also for the second case, continuing from equation (4.12), the limit of the last term on the RHS is

$$\lim_{\Delta t \to 0} \sum_{j \neq i} f_j(x - r_j \Delta t, t - \Delta t) \frac{\{P\}_{ij}}{\Delta t}. \tag{4.18}$$

Regardless of the multiplicity of Δt, as far as it is constant, $f_j(x - r_j \Delta t, t - \Delta t)$ converges to $f_j(x,t)$ as $\Delta t \to 0$. Therefore, both cases converge to the same model. Note that the transformation parameters are obtained through using only the first order approximation. Instead of this, inserting the power series expansion

$$\{P\}_{ii} = e^{-\{Q\}_{ii}\Delta t} = 1 - \{Q\}_{ii}\Delta t + \frac{(\{Q\}_{ii}\Delta t)^2}{2!} - \frac{(\{Q\}_{ii}\Delta t)^3}{3!} + \ldots.. \tag{4.19}$$

and the relation

$$\{Q\}_{ij} = \{Q\}_{ii} \frac{\{P\}_{ij}}{\sum_{j \neq i} \{P\}_{ij}} \tag{4.20}$$

in equation (4.14) lead us to obtain the same result in (4.16) since the terms except the first order approximation will be in the order of the powers of Δt which are larger than one.

4.3 Conversion from Discrete to Continuous Parameters for Second Order Fluid Queues

In a discrete system if change in the number of entities in the queue within a time slot has a non-degenerate PMF whenever the state of the background process at this duration of time is known, then we can decompose this change into two parts as,

$$r_i^{(d)} = r_m + r_r, \tag{4.21}$$

in which r_m is the mean of $r_i^{(d)}$ and r_r is a random variable with a mean zero and a PMF whose space index is shifted through the origin with an amount of r_m. One can then approximately describe the queue content process at a certain state *i* as a combination of two discrete processes: 1) A deterministic change due to r_m; 2) A probabilistic change with mean zero and variance $(\sigma_i^{(d)})^2$, equal to the variance of $r_i^{(d)}$.

Due to r_m, the queue content changes with rate

$$r_i = \frac{r_m \Delta x}{\Delta t} \tag{4.22}$$

deterministically beyond the change due to r_r. In [3], and also in most textbooks on stochastic processes, a zero mean Brownian motion with variance c^2 is described by a fair random walk with arbitrarily small time slot Δt and a step size of $\Delta x = c\sqrt{\Delta t}$. We will describe the change due to the r_r component, in a similar way. Assume that change of queue at the time slot n is X_n due to r_r. Define the following variable:

$$X(t) = \Delta x(X_1 + ... + X_{\lfloor t/\Delta t \rfloor}), \tag{4.23}$$

in which Δx is again the proportion of continuous queue length (selection of it depends on the model) to discrete queue length and Δt is the time slot duration. It is obvious that $X(t)$ is the continuous approximation of the discrete queue content change. Let's define $\tilde{X}_j = X_j / \sigma_i^{(d)}$ and the process $\tilde{X}(t) = X(t)/\sigma_i^{(d)}$. Then, since $E[X_i] = 0$ and $Var(X_i) = (\sigma_i^{(d)})^2$, $E[\tilde{X}(t)] = 0$ and $Var(\tilde{X}(t)) = (\Delta x)^2 \left\lfloor \frac{t}{\Delta t} \right\rfloor$. As $\Delta t \to 0$, with the relation $\Delta x = c\sqrt{\Delta t}$, the Central Limit Theorem (Theorem 4.1), $\tilde{X}(t)$ converges to Brownian motion with rate zero and variance c^2. Therefore, $X(t)$ converges to Brownian motion with rate zero and variance $c^2(\sigma_i^{(d)})^2$ in which

$$c = \frac{\Delta x}{\sqrt{\Delta t}} = \frac{K}{K^{(d)}\sqrt{\Delta t}}. \tag{4.24}$$

We then conclude that the overall discrete time Markov modulated process can be approximated by a second order fluid queue with rate

$$r_i = \frac{r_i^{(d)} K}{\Delta t K^{(d)}} \tag{4.25}$$

and variance

$$(\sigma_i)^2 = \frac{(K)^2}{(K^{(d)})^2 \Delta t} (\sigma_i^{(d)})^2. \tag{4.26}$$

Theorem 4.1:[3], If $X_1, X_2, \ldots$ are independent and identically distributed with mean μ and variance σ^2, then

$$\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \ldots + X_n - \mu n}{\sigma\sqrt{n}} \le a \right) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{4.27}$$

4.4 Numerical Examples

In this section, three numerical examples are given to show how these two models described above fit. Throughout the examples Δx is taken to be 1.

Example 4.1: For the first example, we assume that the background process is a DTMC with transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix},$$

with net rates $r_1 = -1$, $r_3 = 1$ and with queue length of 2.

For the first case, the probability transition diagram is depicted in Fig. 4.1.

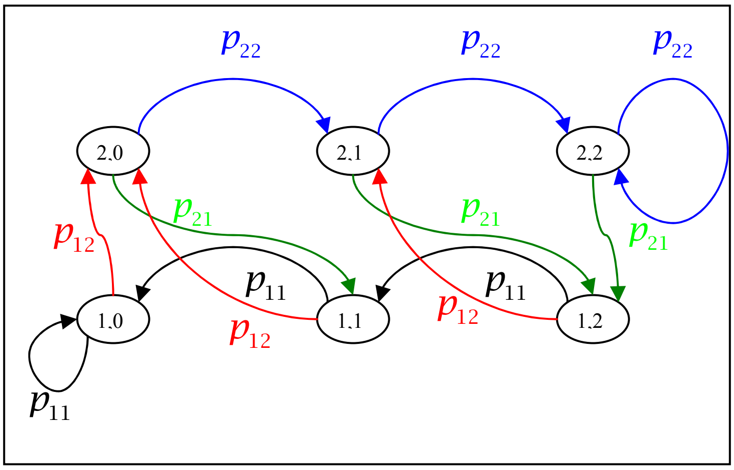


Figure 4.1 Transition Probability Diagram for Case 1

The following set of equations can be obtained for this case:

$$f_{1,0} = f_{1,0}p_{11} + f_{1,1}p_{11}$$

$$f_{1,1} = f_{2,0}p_{21} + f_{1,2}p_{11}$$

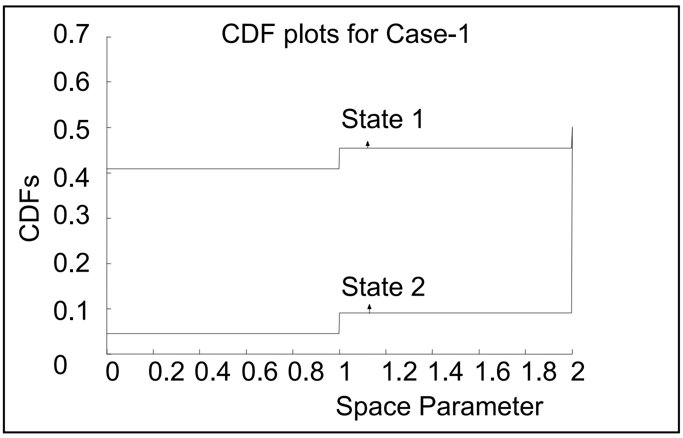
$$f_{1,2} = f_{2,1}p_{21} + f_{2,2}p_{21}$$

$$f_{2,0} = f_{1,0}p_{12} + f_{1,1}p_{12}$$

$$f_{2,1} = f_{2,0}p_{22} + f_{1,2}p_{12}$$

$$f_{2,2} = f_{2,2}p_{22} + f_{2,1}p_{22}$$

We provide the CDF plots in Fig. 4.2.



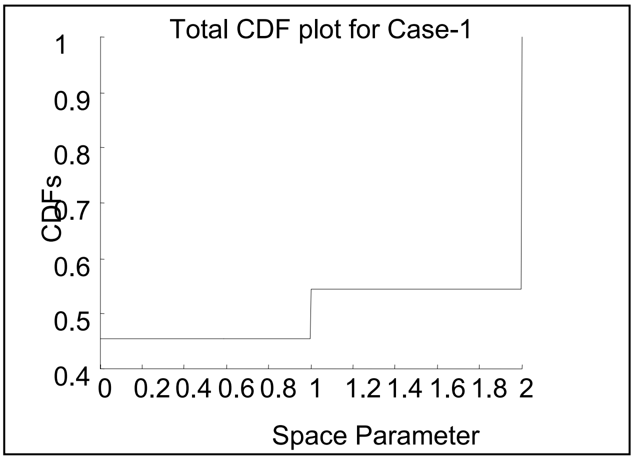


Figure 4.2 CDF plots for Case 1 for Example 4.1

For the second case, the probability transition diagram is shown in Fig. 4.3

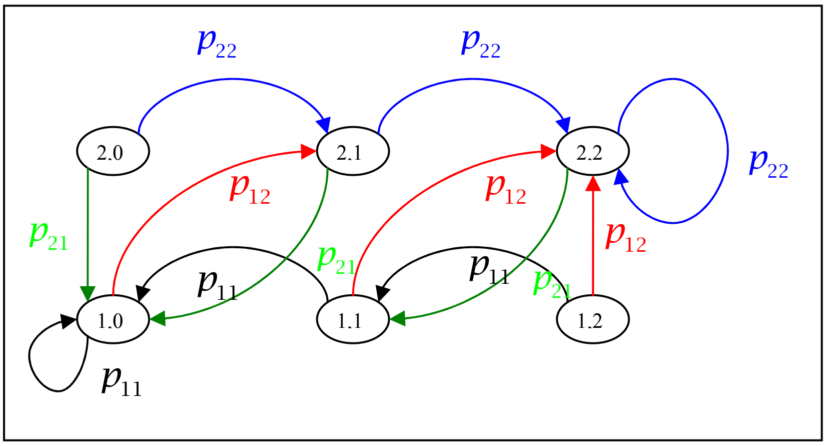


Figure 4.3 Transition Probability Diagram for Case 2

Similar to Case-1, the following set of equations may be obtained:

$$f_{1,0} = f_{1,0}p_{11} + f_{1,1}p_{11} + f_{2,0}p_{21} + f_{2,1}p_{21}$$

$$f_{1,1} = f_{1,2}p_{11} + f_{2,2}p_{21}$$

$$f_{1,2} = 0$$

$$f_{2,0} = 0$$

$$f_{2,1} = f_{2,0}p_{22} + f_{1,0}p_{12}$$

$$f_{2,2} = f_{2,1}p_{22} + f_{2,2}p_{22} + f_{1,1}p_{12} + f_{1,2}p_{12}$$

The solution to CDFs are depicted in Fig. 4.4.

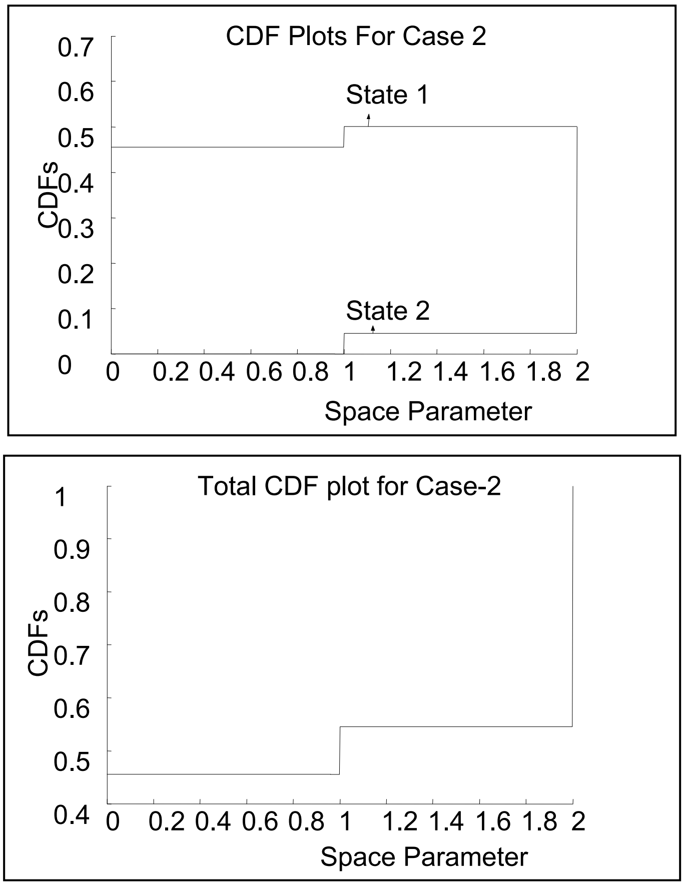


Figure 4.4 CDF plots for Case 2 for Example 4.1

After applying conversion of discrete parameters into continuous ones, plots of fluid approximation can be found as in the following figure.

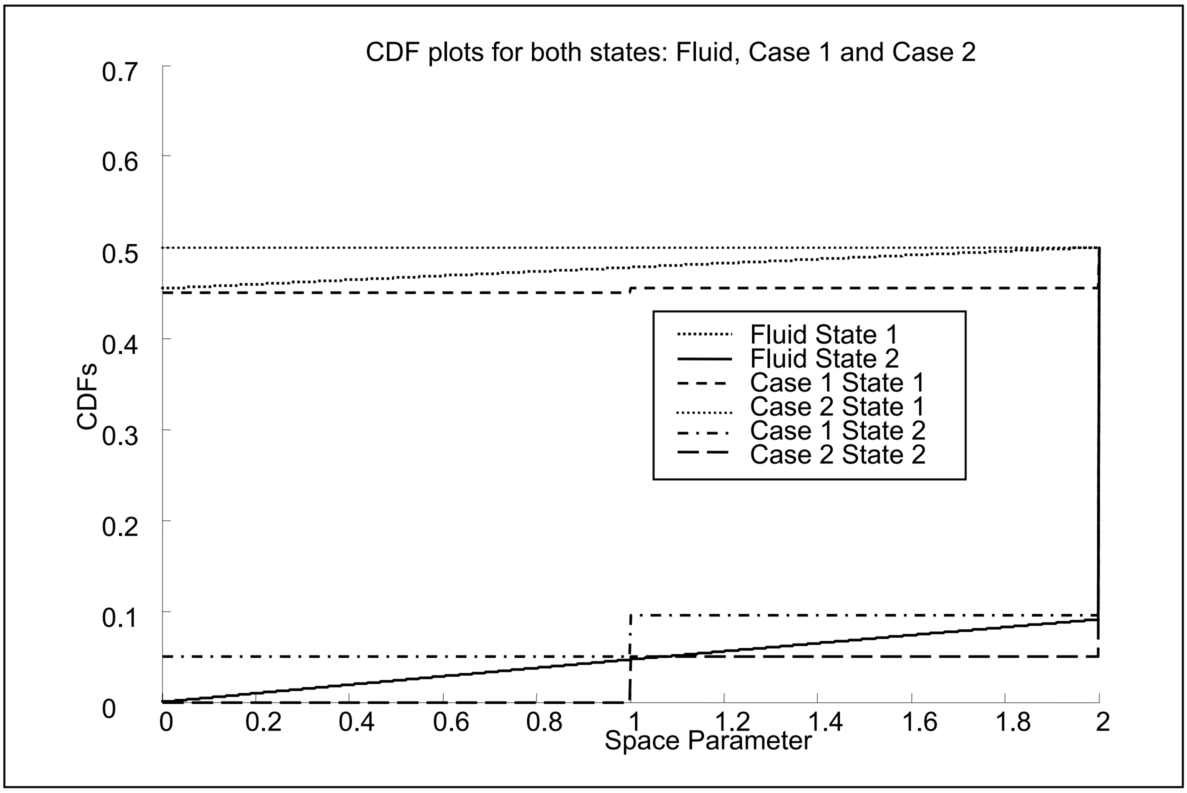


Figure 4.5 CDF plots for state-1 and state-2 (Fluid approximation, Case 1 and Case 2 together) for example 4.1

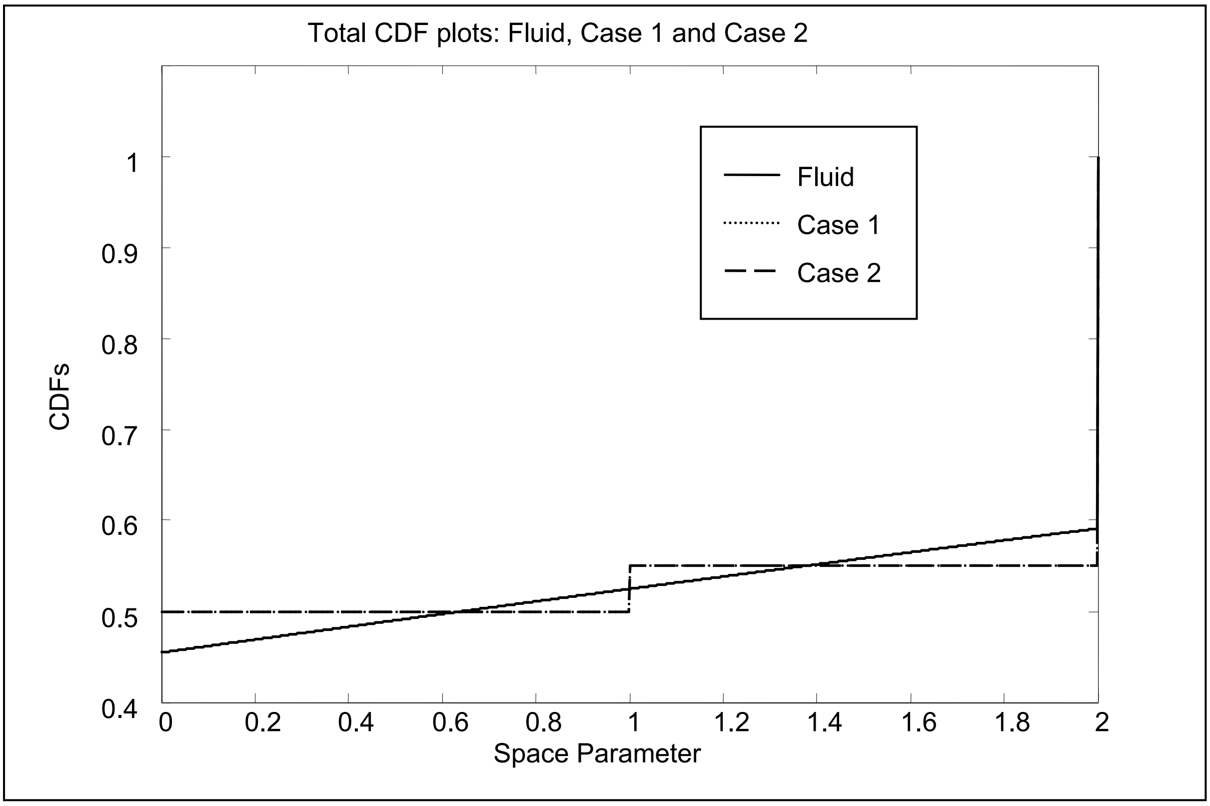


Figure 4.6 Total CDF plots (Fluid approximation, Case 1 and Case 2 together) for example 4.1

On Fig. 4.6 Case 1 and Case 2 plots coincide as expected.

Example 4.2: Now let's enlarge the queue size of the same system to 100. The resulting plots for fluid approximation, together with Case 1 and Case 2 discrete solutions are shown in the following figures.

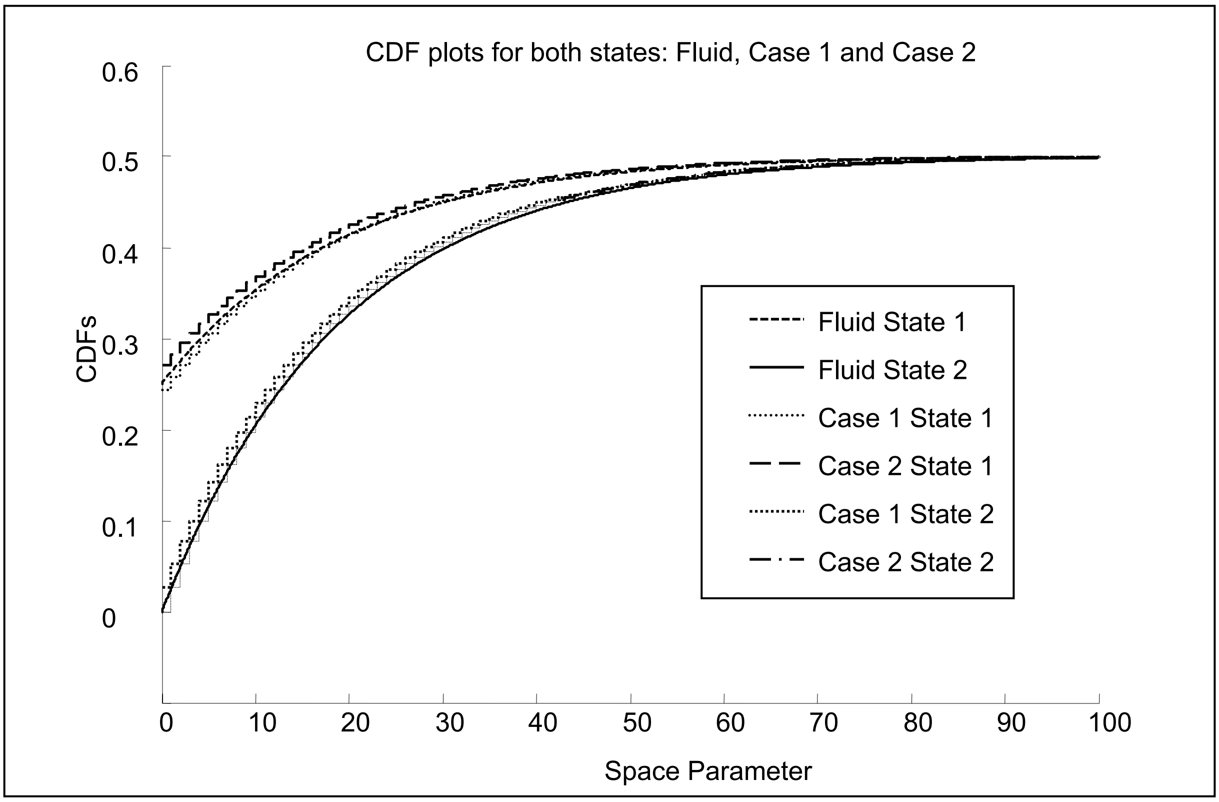


Figure 4.7 CDF plots for state-1 and state-2 (Fluid approximation, Case 1 and Case 2 together) for example 4.2

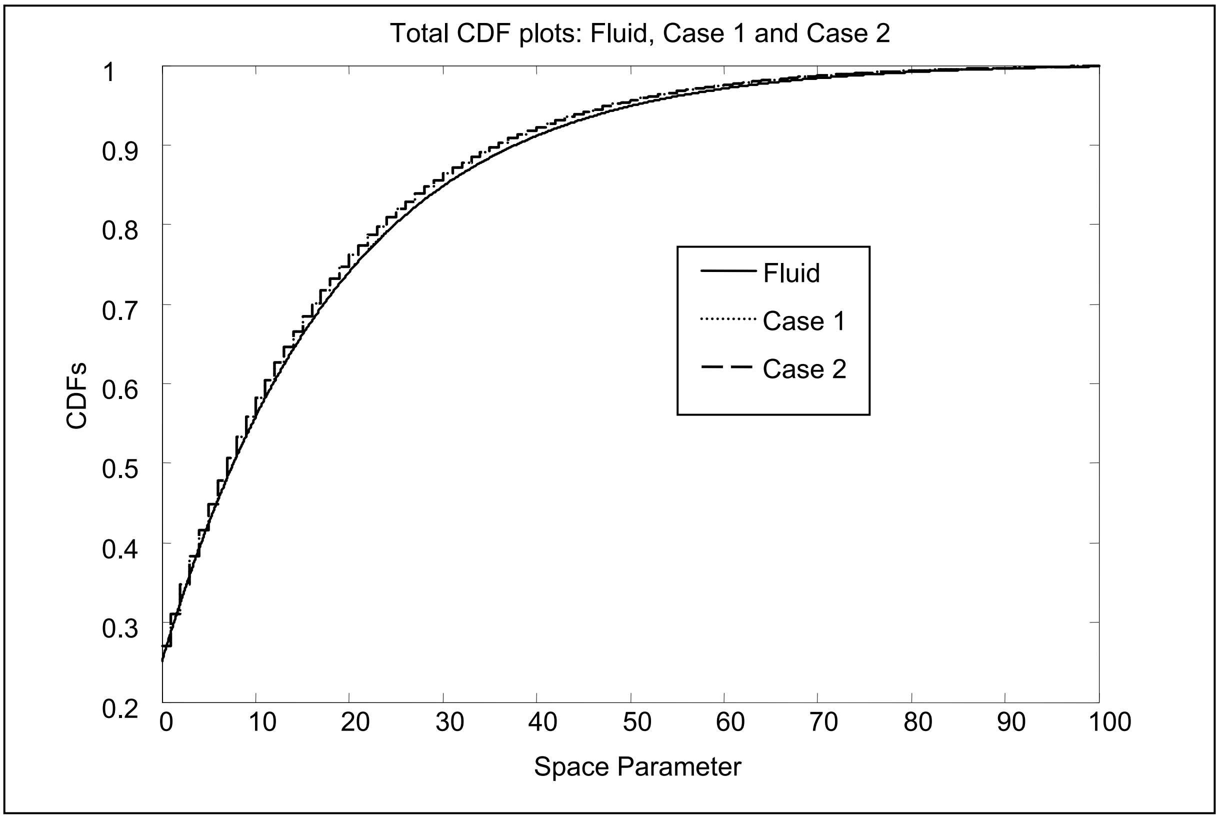


Figure 4.8 Total CDF plots (Fluid approximation, Case 1 and Case 2 together) for example 4.2

Again, Case 1 and Case 2 plots coincide as expected on Fig. 4.8.

Example 4.3: In this example a second order model is considered. A two-state background process with one step transition probability matrix

$$P = \begin{bmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{bmatrix}$$

is used. In states 1 and 2, net rates of change in the queue are

$$r_1 = \begin{cases} -3, \text{ with probability } 0.2 \\ -1, \text{ with probability } 0.6, \\ 1, \text{ with probability } 0.2 \end{cases}$$

$$r_2 = \begin{cases} -1, \text{ with probability } 0.2 \\ 1, \text{ with probability } 0.6 \\ 4, \text{ with probability } 0.2 \end{cases}.$$

Also, queue length is selected as 50. This problem is solved for both discrete and fluid queues, firstly, by assuming that all boundary points are reflecting for fluid and discrete models (Case 1) and secondly, by assuming that boundary points 0 and 50 are absorbing for states 1 and 2, respectively, and other boundary points are reflecting for the fluid queue (Case 2). The resulting CDF plots are given in the following figure.

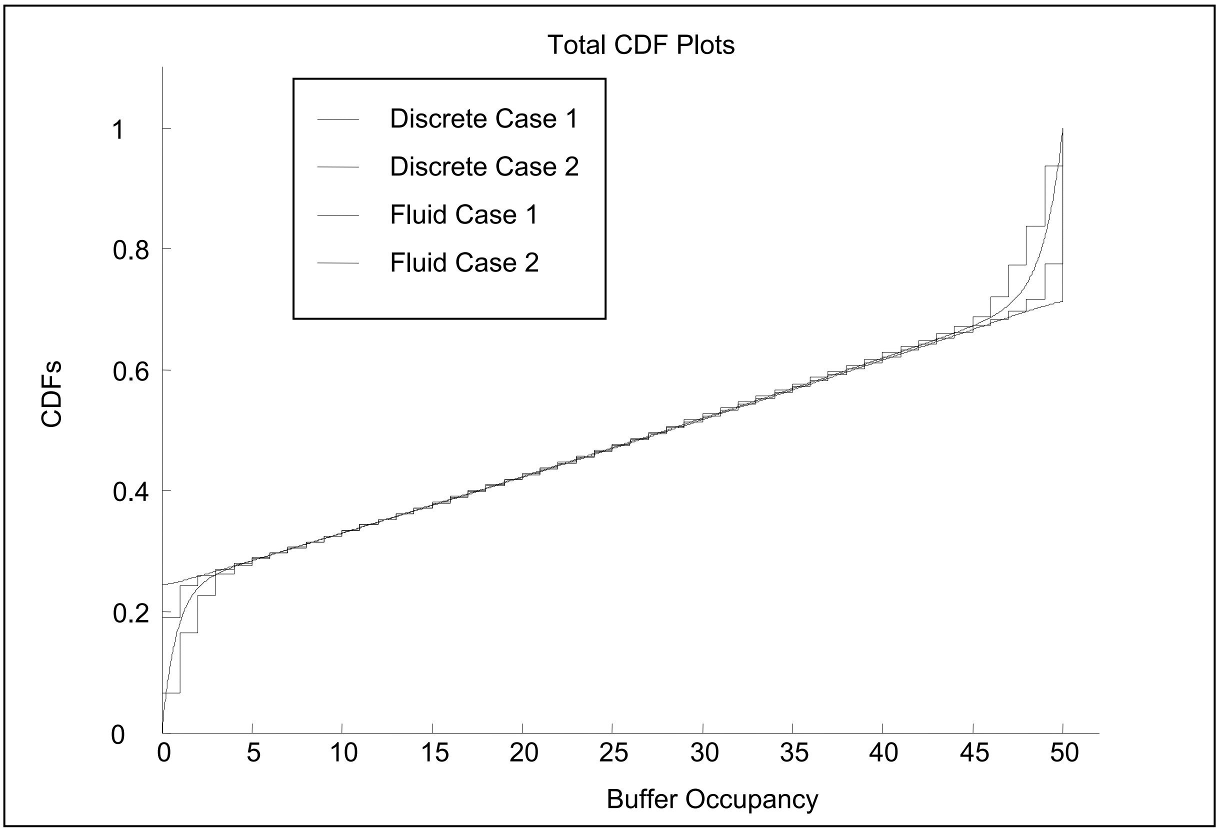


Figure4.9 Total CDF plots for example 4.3

4.5 A Case Study

In this section, a generic UMTS network loaded by MPEG video sources is analyzed with the fluid queue approximation to a discrete time queueing model called the SBBP (Switched Batched Bernoulli Process) model. This model is based on an unpublished work [20].

4.5.1 Discrete Model

Based on [20], transmission of video over UMTS channel is modeled by two SBBPs; one for emission process $\tilde{Y}(n)$, and one for service process $\tilde{N}(n)$. In other words, number of arrivals and departures in time slot n is determined by $\tilde{Y}(n)$ and $\tilde{N}(n)$, respectively. Each has its own background process. $Q^{(\tilde{Y})}$ denotes the transition probability matrix of $\tilde{Y}(n)$ which has 24 states, which correspond to six different frame types: I, B1, B2, P, B3, B4, and four different activity levels 1,2,3, and 4. The frames types I, B1, B2, P, B3, B4 are always sent in a cyclic order. With a high probability the system stays in the same activity level. $Q^{(\tilde{N})}$ denotes the transition probability matrix of $\tilde{N}(n)$ which has 20 states. Each of those states corresponds to the number of users in the system 1, 2,..,20. Since, in the analysis, one of the users is monitored, this user is assumed to be always on and these states may be interpreted to correspond as 0,1,...,19 additional users. Steady state probability mass distributions of $\tilde{Y}(n)$ depend on the Quantization Scale Parameter (QSP), $q = 1,2,...,31$, therefore in order to indicate this dependence, emission process can be denoted as $\tilde{Y}_q(n)$.

There are two proposed versions of this model:

1) Non-controlled video source

2) Rate-controlled video source.

In the non-controlled video source model, QSP is fixed and is this is independent of the buffer occupancy. In the rate-controlled video source model, QSP changes according to a feedback law which depends on the current activity level, frame to be encoded, buffer occupancy, and number of current users in the system. Change in QSP yields a change in the current emission rate, therefore providing a potential for effective buffer management.

Together with the emission and service processes, the overall background process may be viewed as a 20x24=480 state Markov process. Solution in discrete space is sensitive to the buffer length (200 in our case); therefore we need to solve a 480x201=96.480 state DTMC which can be very difficult. Instead, in the current thesis, we use the Markov fluid model as an approximation since their solution is relatively insensitive to the buffer length and we need to solve a 480 state system in continuous space.

4.5.2 First Order Fluid Approximation

First order fluid approximation of the SBBP model can be done through the analysis made in this chapter via taking the means of net rates and using them in the analysis as if they are deterministic.

Example 4.4: According to the real data we obtain five CDF plots for this example. Four of them correspond to the non-controlled video source for which QSP is selected as 1, 6, 21, and 31 for all buffer occupancies. The fifth one corresponds to a controlled video source. In that adaptive buffer management the policy QSP is selected as 1, 6, 21, and 31 for regions (0, 50), (50,100), (100, 150), and (150,200) respectively. The goal is to use an adaptive policy in order to steer the buffer occupancy from the two boundaries. Note that at the left boundary the capacity is not fully used and at the right boundary uncontrolled losses are inevitable. An adaptive system in which these two conditions are avoided would definitely be desirable.

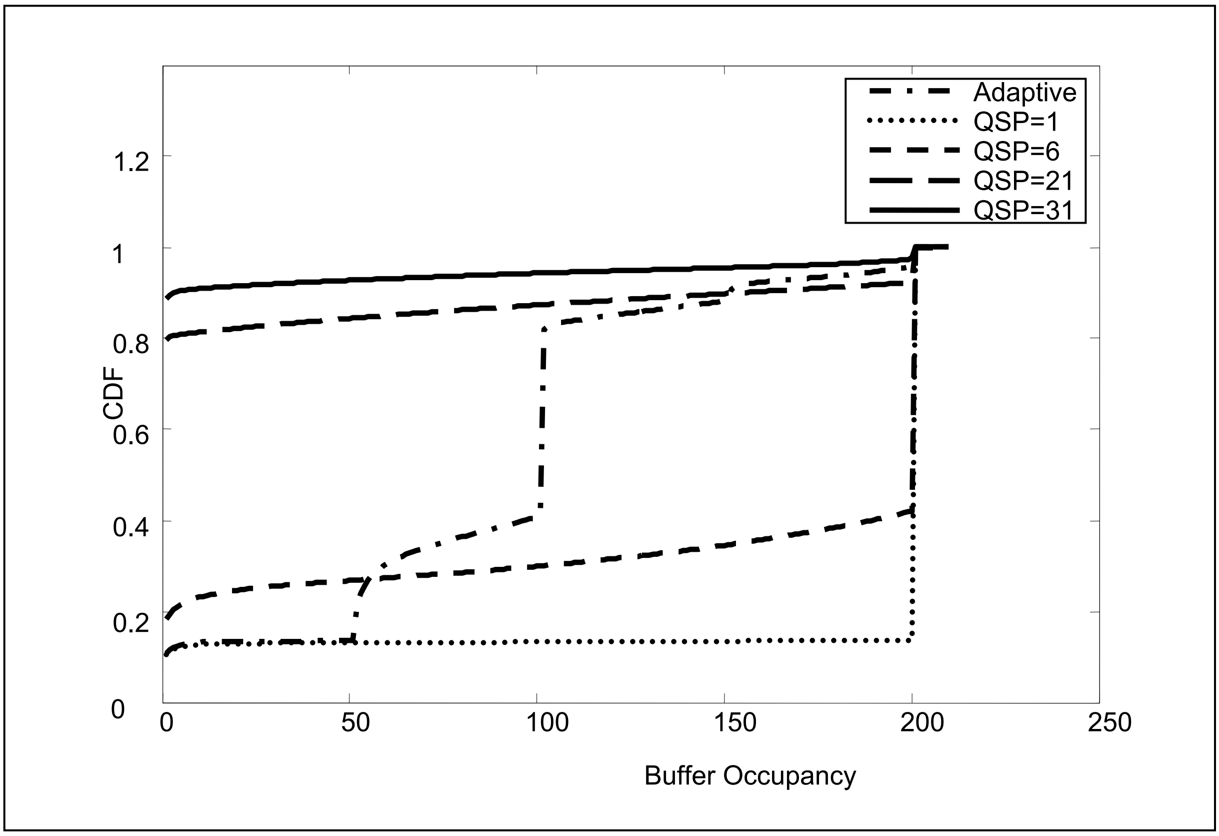


Figure 4.10 Total CDF plots for example 4.4

In Fig. 4.10, loss probabilities for the choices of QSP =1 and 6 are high since in these cases, the traffic injection rate is larger than what the network can support and uncontrolled frame losses occur. On the other hand, the buffer stays empty with a significant probability for the choices of QSP=21 and 31 which is equivalent to not using the capacity fully. Finally, with the adaptive policy, these two boundaries are avoided and better video streaming performance is achievable. We plan on extending our results so as to be able to find optimal or suboptimal policies for video streaming based on certain criteria, e.g., minimal rate distortion, using the results of this thesis.

Chapter 5

Conclusions

We have mainly three contributions in this thesis. Firstly, the analysis carried out in Chapter 2 generalizes the solution in [7] for the steady state distribution function of queue occupancy of first order feedback fluid queues with multiple thresholds by allowing the existence of repulsive type boundaries and the states with zero rates. Secondly, by the analysis carried out in Chapter 3 we complete the boundary conditions for not only the transient but also steady state solution of second order feedback fluid queues with multiple thresholds. Finally, we apply the theory of feedback fluid queues with multiple thresholds as an effective approximation to the Markov modulated discrete time queueing model that arises in the performance evaluation of an adaptive MPEG video streaming system in UMTS environment. By doing so, we eliminate the state space explosion problem that arises in discrete model.

Appendix

Generator for Second Order Feedback Fluid Queue with Multiple Thresholds

This part of the thesis shows the way to obtain the generator for Second Order Feedback Fluid Queue with multiple thresholds, which is given in equation (3.51). In fact this appendix needs a bit measure theory and stochastic calculus as prerequisite. Necessary (but not sufficient [sufficiency is a subjective measure in terms of knowledge]) information can be found in Preliminaries part. Also information about measure theory and its relationship with probability can be gained from [11]. [13] and [16] are good references to link measure theory and stochastic processes and stochastic calculus. Diffusion processes can be studied from [17]. Also [14] and [15] can be followed to improve knowledge on Stochastic Calculus.

A.1 Preliminaries

We start with the definitions and important theorems used in the subsequent sections.

Definition A.1 (M_e^2): [13] M_e^2 is a subset of linear span of elementary processes, where each h_i is square integrable.

Definition A.2(M^2): [13] The class of stochastic processes $f(t), t \geq 0$ such that

$$E\left[\int_0^\infty | f(t) |^2 \, dt\right] < \infty$$

and there is a sequence $f_1, f_2, f_3 ... \in M_e^2$ such that

$$\lim_{n\to\infty} E\left[\int_0^\infty | f(t) - f_n(t) |^2 \, dt\right] = 0 .$$

Definition A.3 (M_T^2 **): [**13] The space of all stochastic processes $f(t), t \geq 0$ such that $1_{[0,T)} f \in M^2$ for some finite T.

A.2 Solution

Proposition A.1: Under the assumption that $X(t) = i, \forall t \geq 0$ (i.e. the background process is stuck to a state) the following equation is satisfied by the content process $C(t)$ at intermediate points of a single regime second order fluid queue:

$$dC(t) = \mu_i dt + \sigma_i dW(t), \tag{A.1}$$

where $W(t)$ is standard Brownian motion.

Proof:

Without loss of generality we can assume that content process does not touch to the boundaries up to time t, then $C(t)$ may be represented as following:

$$C(t) = C(0) + \mu_i t + W_{\sigma_i^2}(t), \tag{A.2}$$

where $W_{\sigma_i^2}(t)$ denote non-standard Brownian motion with variance coefficient σ_i^2. Obviously

$$W_{\sigma_i^2}(t) = \sigma_i W(t), \tag{A.3}$$

then

$$C(t) = C(0) + \mu_i t + \sigma_i W(t). \tag{A.4}$$

Therefore,

$$C(t+h)-C(t)=\mu_i h+\sigma_i(W(t+h)-W(t)). \quad \text{(A.5)}$$

Now Let's define the partition $0=t_0^n<t_1^n<...<t_n^n=T$ where $t_j^n=\dfrac{Tj}{n}$. By using (A.5) the following expression can be obtained,

$$C(t_{j+1}^n)-C(t_j^n)=\mu_i\frac{T}{n}+\sigma_i(W(t_{j+1}^n)-W(t_j^n)). \quad \text{(A.6)}$$

By summing up the terms on both sides,

$$\sum_{j=0}^{n-1}(C(t_{j+1}^n)-C(t_j^n))=\sum_{j=0}^{n-1}(\mu_i\frac{T}{n})+\sum_{j=0}^{n-1}(\sigma_i(W(t_{j+1}^n)-W(t_j^n))) \quad \text{(A.7)}$$

and taking limit of both sides as n goes to infinity, we can see that left hand side converges to $X(T)-X(0)$. Also the first term on the right hand side is simply a Riemann integral $\int_0^T \mu_i dt$.

By defining the sequence of processes, $f_1,f_2,f_3...$ as $f_k=\sigma_i 1_{[0,T]}$, obviously we can see that $f_1,f_2,f_3...\in M_e^2$ and $f_1=f_2=....=f=\sigma_i 1_{[0,T]}$, therefore $f\in M_T^2$, thus

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}(\sigma_i(W(t_{j+1}^n)-W(t_j^n)))=\int_0^T\sigma_i dW. \quad \text{(A.8)}$$

Thus by just taking the derivative of both sides, the result follows.□

Now we will relax the assumption in Proposition A.1.

<u>Proposition A.2</u> Content process of a single regime second order fluid queue satisfies the following relation at intermediate points:

$dC(t)=\mu(X(t))dt+\sigma(X(t))dW(t)$ almost surely. (A.9)

<u>Proof:</u>

$C(t)$ may be represented as following:

$$C(t)=C(0)+\tilde{\mu}(t)t+\tilde{W}(t), \quad \text{(A.10)}$$

where $\tilde{\mu}(t)$ is the rate process driven a CTMC and it is constant for each state over time; $\tilde{W}(t)$ is a Brownian motion whose variance parameter is driven by the same CTMC. Here we again assume that the content process does not touch to the boundary points up to time t. Then

$$C(t+h) - C(t) = \tilde{\mu}(t+h) - \tilde{\mu}(t) + \tilde{W}(t+h) - \tilde{W}(t). \tag{A.11}$$

By defining the partition $0 = t_0^n < t_1^n < \ldots < t_n^n = T$ where $t_j^n = \frac{Tj}{n}$ the following expression can be obtained,

$$C(t_{j+1}^n) - C(t_j^n) = \tilde{\mu}(t_{j+1}^n) - \tilde{\mu}(t_j^n) + \tilde{W}(t_{j+1}^n) - \tilde{W}(t_j^n). \tag{A.12}$$

We can rewrite this equation as:

$$C(t_{j+1}^n) - C(t_j^n) = \mu_{M(t_j^n)}\frac{T}{n} + (\mu_{M(t_{j+1}^n)} - \mu_{M(t_j^n)})\frac{\tau}{n} + \sigma_{M(t_j^n)} W(\frac{T}{n})$$

$$+(\sigma_{M(t_{j+1}^n)} - \sigma_{M(t_j^n)})W(\tau), \tag{A.13}$$

where τ is a random variable greater than zero and less than T. Second term on the right hand side emerges from the possibility of a state in the background process.

By summing up the terms on both sides,

$$\sum_{j=0}^{n-1}(C(t_{j+1}^n) - C(t_j^n)) = \sum_{j=0}^{n-1}(\mu_{M(t_j^n)}\frac{T}{n}) + \sum_{j=0}^{n-1}(\mu_{M(t_{j+1}^n)} - \mu_{M(t_j^n)})\frac{\tau}{n}$$

$$+\sum_{j=0}^{n-1}\sigma_{M(t_j^n)}(W(t_{j+1}^n) - W(t_j^n)) + \sum_{j=0}^{n-1}(\sigma_{M(t_{j+1}^n)} - \sigma_{M(t_j^n)})W(\tau) \tag{A.14}$$

and taking limit as n goes to infinity, we can see that left hand side again converges to $X(T) - X(0)$; the first term on the right hand side is simply a integral of a process in Riemann sense $\int_0^T \mu_{M(t)} dt$. The second term on the right hand side vanishes because τ is a bounded random variable and the Markov chain changes its state finitely many times within a finite time interval almost surely, thus τ is multiplied by a constant and added finite times, although the denominator (n) goes to infinity. Let's look at the third term on the right hand side now by defining the sequence of processes, $f_1, f_2, f_3 \ldots$ as $f_k = \sum_{i=0}^{k-1} \sigma_{M(t_i^k)} 1_{[t_i^k, t_{i+1}^k)}$, we can see that $f_1, f_2, f_3 \ldots \in M_e^2$, and $f \in M_T^2$, thus

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} (\sigma_{M(t_j^n)} (W(t_{j+1}^n) - W(t_j^n))) = \int_0^T \sigma_{M(t)} dW . \tag{A.15}$$

Fourth term clearly goes to zero. Thus, if we take the derivative of both sides, the result follows. □

Proposition A.3 Content process of a second order feedback fluid queue with multiple thresholds satisfies the following relation at intermediate points:

$$dC(t) = \mu(X(t), C(t))dt + \sigma(X(t), C(t))dW(t) \text{ almost surely.} \tag{A.16}$$

Proof:

By selecting the time parameter that content process enters to a regime as the origin and assuming that the content process stay in this regime up to time t and following the same steps in the proof of Proposition A.2 the result can be obtained. □

Remark A.1: In the setup used in Chapter 3 intermediate boundary points are assumed to be either absorbing ;or left or right continuous. If an intermediate boundary point is selected as absorbing then derivative of the content process is zero. Since both drift and variance parameters are zero for an absorbing boundary, right hand side of (A.16) is also zero. For left and right continuity of parameters it is clear that (A.16) is also valid. Therefore for a second order feedback fluid queue with multiple thresholds proposition A.3 is valid not only for intermediate points but also for boundary points.

Lemma A.1: [17], Let $Y_t = Y_t^x$ be a stochastic integral in $\mathbf{R^n}$ of the form

$$Y_t^x(w) = x + \int_0^t u(s,w)ds + \int_0^t v(s,w)dW(w) , \tag{A.17}$$

where u is absolutely integrable almost surely and v is square integrable almost surely. Let g is a twice continuously differentiable function in $\mathbf{R^n}$ and m is a Markov Time. Assume that u and v are bounded on the set of (t,w) such that $Y(t,w)$ belongs to the support of g. Then

$$E^x[g(Y_m)] = g(x) + E[\int_0^m (\sum_i u_i(s,w)\frac{\partial g}{\partial x_i}(Y_s) + \frac{1}{2}\sum_{i,j}(vv^T)_{ij}\frac{\partial^2 g}{\partial x_i \partial x_j}(Y_s))ds]. \tag{A.18}$$

Theorem A.1 The generator of the process is

$$(Lg)(x,i) = \frac{1}{2}\sigma^2(x,i)\partial_{xx}g(x,i) + \mu(x,i)\partial_x g(x,i) + \sum_{j\neq i} q_{ij}g(x,j). \tag{A.19}$$

Proof:

According to the Lemma A.1 and the result obtained in Proposition A.3 following equation is satisfied:

$$E[g_i(C(t)) \mid C(0) = x, X(s) = i, 0 \le s \le t] = g_i(x)$$

$$+E[\int_0^t (u_i(x)\frac{\partial g_i}{\partial x} + \frac{1}{2}\sigma_i^2(x)\frac{\partial^2 g_i}{\partial x^2})ds \mid C(0) = x, X(s) = i, 0 \le s \le t]. \tag{A.20}$$

Then

$$E[g(C(t),X(t)) \mid C(0) = x, X(s) = i, 0 \le s \le t] = g(x,i)$$

$$+E[\int_0^t (u(x,X(t))\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(x,X(t))\frac{\partial^2 g}{\partial x^2})ds \mid C(0) = x, X(s) = i, 0 \le s \le t],$$

(A.21)

for $g(C(t),X(t)) = g_i(C(t))$, $u(x,X(t)) = u_i(x)$, $\sigma^2(x,X(t)) = \sigma_i^2(x)$ for $X(t) = i$.

For a small time interval Δt,

$$E[g(C(\Delta t),X(\Delta t)) \mid C(0) = x, X(0) = i] = [g(x,i)$$

$$+E[\int_0^{\Delta t} (u(x,X(\Delta t))\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(x,X(\Delta t))\frac{\partial^2 g}{\partial x^2})ds \mid C(0) = x, X(0) = i]](1 + q_{ii}\Delta t)$$

$$+\sum_{j\neq i}\left[\begin{array}{l} g(x+o(\Delta t),j)q_{ji}\Delta t + \\ E_{\tau}[E[\int_{\tau}^{\Delta t}(u(x+o(\Delta t),X(\Delta t))\frac{\partial g}{\partial x}+\frac{1}{2}\sigma^2(x+o(\Delta t),X(\Delta t))\frac{\partial^2 g}{\partial x^2})ds \\ \mid C(\tau)=x+o(\Delta t),X(\tau)=j]]xq_{ji}\Delta t \end{array}\right].$$

(A.22)

Then

$$(Lg)(x,i)=\lim_{\Delta t\to 0}\frac{E[g(C(\Delta t),X(\Delta t))\mid C(0)=x,X(0)=i]-g(x,i)}{\Delta t}=$$

$$\lim_{\Delta t\to 0}E[\frac{1}{\Delta t}\int_{0}^{\Delta t}(u(x,X(\Delta t))\frac{\partial g}{\partial x}+\frac{1}{2}\sigma^2(x,X(\Delta t))\frac{\partial^2 g}{\partial x^2})ds\mid C(0)=x,X(0)=i]](1+q_{ii}\Delta t)$$

$$\lim_{\Delta t\to 0}\sum_{j\neq i}\left[\begin{array}{l} g(x+o(\Delta t),j)q_{ij} + \\ E_{\tau}[E[\int_{\tau}^{\Delta t}(u(x+o(\Delta t),X(\Delta t))\frac{\partial g}{\partial x}+\frac{1}{2}\sigma^2(x+o(\Delta t),X(\Delta t))\frac{\partial^2 g}{\partial x^2})ds \\ \mid C(\tau)=x+o(\Delta t),X(\tau)=j]]xq_{ij} \end{array}\right] \quad \text{(A.23)}$$

and

$$(Lg)(x,i)=u(x,i)\frac{\partial g}{\partial x}+\frac{1}{2}\sigma^2(x,i)\frac{\partial^2 g}{\partial x^2}+\sum_{j\neq i}\left[g(x,j)q_{ij}\right].\square \quad \text{(A.24)}$$

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