

Resource Allocation in Multi-Target Attacker-Defender Games

by

İlknur Singin

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Resource Allocation in Multi-Target Attacker-Defender Games

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Graduate School of Sciences and Engineering

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İlknur Singin

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examining committee have been made.

Committee Members:

Assist. Prof. Pelin G. Canbolat (Advisor)

Prof. Fikri Karaesmen

Assoc. Prof. Emin Karagözoğlu

Date: _____

ABSTRACT

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İlknur Singin

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The optimal allocation of resources over the protection of several potential targets under a defense budget constraint is a challenging problem. In this thesis, we consider a game between an attacker and a defender who are competing over multiple targets. The defender has a limited amount of resources that need to be divided among multiple targets in an efficient manner. Our purpose is to identify the Nash equilibrium of this game, investigate the equilibrium investment strategy of the attacker and of the defender, gain insight into the allocation of the budget, and into the effects of changes in various model parameters, including the attacker's and the defender's valuations of the targets, the defense budget, and the efficiency of each party in their use of resources. While we consider also the sequential-move game where the defender decides first and the attacks follows, we mostly concentrate on the simultaneous-move game, where the parties decide simultaneously without observing each other's decisions. We solve the single-target problem explicitly, provide a characterization for the two-target problem, and propose a dynamic-programming formulation for the multi-target problem. We provide numerical examples to illustrate our model, results, and approach. Through these examples, we also make observations about the sensitivity of the equilibrium to changes in parameters, to the asymmetry of target valuations, and to the informational structure of the game.

ÖZETÇE

Çok Hedefli Savunma Oyunlarında Kaynak Dağılımı

İlknur Singin

Endüstri Mühendisliği, Yüksek Lisans

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Olası hedeflerin korunması amacıyla bütçe kısıtının var olduğu durumlarda kaynakların efektif bir şekilde dağılımı zorlayıcı bir problem olmuştur. Bu tezde saldırgan ve savunan arasında çok sayıda hedefi içeren bir oyunu inceliyoruz. Savunucunun efektif bir şekilde dağıtması gereken kısıtlı bir bütçesi olduğunu varsayıyoruz. Amacımız oyundaki Nash dengesini bulmak, saldırgan ve savunucunun yatırım stratejilerini incelemek, bütçe dağılımını kavramak ve modelde var olan parametrelerin -hedefin saldırgan ve savunucu için değeri, savunucunun bütçesi, yatırımın efektif kullanım katsayısı- değişiminin bütçe dağılımına etkisini incelemektir. Oyuncuların birbirlerinin stratejilerini gözlemlemeden karar verdiği eşzamanlı bir oyun modeli ile savunucunun yatırım stratejisine saldırgandan önce karar verdiği ardışık bir oyun modelini ele alıyoruz. Önce tek hedefli problemi çözümlüyoruz, ardından iki hedefli problem için Nash dengesini karakterize ederek analitik çözümler elde ediyoruz, son olarak çoklu hedef problemini çözümlmek adına dinamik programlama formülasyonu sunuyoruz. Modelimizi ve sonuçlarımızı sayısal örneklerle gösteriyoruz. Bu örnekler aracılığı ile parametrelerdeki değişimlerin, hedef değerlerindeki farklılıkların ve oyundaki bilgi akışının dengelerde oluşturduğu etkileri gözlemliyoruz.

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Chapter 1

INTRODUCTION

The terrorist attacks and the natural disasters that the world has suffered from over the last couple of decades made governments revisit their defense policies including the set of countermeasures against strategic and natural disasters and the budget devoted to this. Policy issues arising in this context have received attention also from the operations research, decision analysis, public policy and economics research communities. Numerous mathematical models have been developed to identify the strategic interactions between the defenders (governments) and the attackers (nature or terrorists). However, the efficient allocation of resources over multiple targets that need to be protected still remains as a challenging problem.

A lot of recent papers used game-theoretical methods to understand the strategies adopted by attackers and to propose defense strategies that would best counter these. However, most of this literature considers the competition of the attacker and the defender over one target and without a budgetary constraint. In reality, terror organizations (the attackers) can launch attacks at multiple targets and the government (the defender) is often faced by a budget limitation. In this thesis, we aim to analyze the equilibrium investment strategies of an attacker and a defender in a multi-target game under a defense-budget constraint. Even though we assume a single attacker, the decomposable nature of a single attacker's problem makes our results valid for multiple independent attackers.

In order to identify the best defense strategies in the presence of a budget constraint, we consider a game-theoretical model where the attacker makes investments to maximize the expected damage in a cost-efficient manner while the defender wishes to minimize the expected damage over multiple targets by choosing an allo-

cation of its defense budget over these targets. The limited nature of the resources enforces the defender to make decisions more strategically, by considering not only their potential effects on the decisions of the attacker, but also the tradeoffs between different targets.

The parameters of our model consist of the target valuations of the attacker and the defender, the budget of the defender, the disutility of investment for the attacker, the inherent defense levels at the targets, the efficiency factors for the attacker and the defender. We allow the target valuation to differ from each other and also to be different for the attacker and the defender. The disutility of investment for the attacker ensures that the attacker makes investments in a cost efficient way. In our analysis, we assume the disutility of investment as equal to the investment. Linear disutility functions can be treated similarly with an adjustment of the other problem parameters. The inherent defense levels at the targets represent the existing measures at the targets. The efficiency factors are used to convert monetary investments into defense units. We consider the simultaneous-move game where the attacker and the defender decide on their investments without observing the other's decision, and a sequential-move version where the defender decides first and the attacker decides after observing the defense strategy.

We address the following questions: How do the players allocate their resources in the Nash equilibrium? Do they concentrate their effort on certain targets or distribute it evenly over all? How sensitive is the Nash equilibrium to changes in model parameters? The answer to the latter question can be used in assessing the importance of parameters and identifying which parameters require most attention in estimating the equilibrium behavior. To answer these questions, we characterize the Nash equilibrium for the simultaneous-move game and the subgame-perfect Nash equilibrium for the sequential-move game. Using this characterization, we perform numerical experiments and investigate the effects of changes in parameters.

The remainder of this thesis is organized as follows. Chapter 2 presents a survey of the existing literature on game-theoretical models in the context of homeland security. Chapter 3 introduces our model and notation. Chapter 4 analyzes the single-target game, explicitly finds the equilibrium in the simultaneous- and

sequential-move games, and establishes the direction of change in the equilibrium investments and utilities as model parameters vary. Chapter 5 considers the two-target game, characterizes the Nash equilibrium and subgame-perfect Nash equilibrium, illustrates the results with numerical experiments, numerically compares the equilibrium outcomes in the simultaneous- and sequential-move games. Chapter 6 presents the multiple-target game, illustrates the calculation of subgame-perfect Nash equilibrium for the sequential-move game, and provides observations about the behavior of the equilibrium outcomes. Chapter 7 summarizes our work and discusses some directions for future research.



Chapter 2

LITERATURE REVIEW

Game theory has been widely used to study the interactions between the defender and the attacker. [Sandler and Arce, 2003] and [Sandler and Siqueira, 2009] provide extensive surveys on the literature applying game theory to analyze terrorism, the second one focusing on the allocation of a limited defense budget to protect potential targets. Other surveys include [Kardes and Hall, 2005] and [Wright et al., 2006], which investigate the operations-research applications in the context of homeland security.

While the interest in applications of operations research and game theory to the analysis of defense policies has grown in recent years, an early application in the military context was by [Dresher, 1961]. Most of the studies on allocation of defense resources assumes an infinite budget. Examples from this literature are [Jose and Zhuang, 2018] and [Payyappalli et al., 2017] analyzing the effects of risk sensitivity on the equilibrium outcomes, [Zhuang and Bier, 2011] studying deception and secrecy when one of the players has private information, [Wu et al., 2019] considering a game between one defender and two attackers, [Xu and Zhuang, 2016] deriving an optimal defense strategy with or without private information over discrete attacker choices. All of these papers study the single-target sequential-move game where defender moves first.

Papers studying the simultaneous-move game with multiple targets include [Bier et al., 2007], which examines a resource-allocation problem considering discrete attacker options, assuming that the defender allocates resources to locations (can be considered as potential targets) and the attacker must decide on which location to attack. [Samuel and Guikema, 2012] model the interactions between a single attacker and a defender consisting of multiple agencies. [Siqueira and Sandler, 2006] study the equilibrium in presence of supporters, sponsorship, and counter-terrorism

campaigns. [Nikoofal and Zhuang, 2015] compare sequential- and simultaneous-move game under symmetric and asymmetric valuation of targets. [Hausken et al., 2009] and [Hausken and Bier, 2011] also consider the simultaneous- and sequential-move games (two versions, one where the defender decides first, and the other where the attacker decides first). [Hausken et al., 2009] consider the choice among countermeasures against terrorism, natural disasters, and all hazards. [Hausken and Bier, 2011] consider a game where the defender faces multiple attackers. [Guan et al., 2017] explore the optimal resource allocation for an attacker and a defender, each subject to a budget constraint.

[Zhuang et al., 2010] consider a multi-period signaling game where only the defender has private information. [Hausken and Zhuang, 2012] examine the equilibrium strategies of players and deterrence of attacks in a multi-period game. [Jose and Zhuang, 2013] study the issue of timing taking into account new technologies. [Hausken and Zhuang, 2011b] analyze a problem where each of the attacker and the defender makes decisions about defending their own assets and attacking the other's. Similarly, [Hausken and Zhuang, 2011a] allow both players to defend and to attack, they assume that the attacker may accumulate resources as time progresses. [Hausken, 2008] develops a game-theoretic model in which the terrorist is the defender. These papers consider a single target at each time period and do not incorporate budget constraints. [Shan and Zhuang, 2018] explore resource allocation in a multi-period game with multiple targets, with budgetary constraints on both the defense and attack resources.

[Powell, 2007] uses game theory to investigate the strategies in defending critical infrastructure problems with budget constraint and considers both terrorism and natural disasters assuming discrete attacker choice. [Golany et al., 2009] study the protection against both strategic and non-strategic threats with a limited budget. [Zhuang and Bier, 2007] develop a model of resource allocation for protecting assets against both intelligent and non-intelligent actors. [Haphuriwat and Bier, 2011] study the balancing between target hardening and overarching protection. [Golalikhani and Zhuang, 2011] study overarching protection by allowing the defender to defend a set of targets against strategic attackers. [Shan and Zhuang, 2013b] develops a model to

study the defense strategies in which defender faces with either a strategic or non-strategic attacker. [Zhang and Zhuang, 2019] discusses the allocation of defensive resources to counter different attack types such as bombing and hostage taking.

A stream of the literature on game-theoretical analysis of terrorism is devoted to informational asymmetry, secrecy and uncertainty. [Nikoofal and Zhuang, 2012] study the case where the attacker's target valuation is unknown to the defender. [Bier et al., 2008] explore the impact of uncertainty about the attacker's target valuation and cost effectiveness of defense investment on the defender's resource allocation. [Zhai et al., 2019] investigate the uncertainty of attack utility, compare the attack and defense strategies in two cases depending on whether the defender has information on the attacker's utility or not. [Wang and Bier, 2011] develop a model to identify the optimal defense strategy dealing with the unobserved attributes in the attacker's utility function. [Wang and Bier, 2016] consider the situation where the defender is uncertain about the attacker's effort and target valuation. [Nikoofal and Gümüs, 2015] investigate the attacker's private information on the defense strategy in two cases: one where the attacker's target preference is known to the defender but the degree of rationality is unknown, and another where the attacker's degree of rationality is known to the defender but the attacker's target preference is not. [Wang and Bier, 2009] propose a two-stage dynamic game in which the attacker's valuation of the target is not observed by the defender. [Xu and Zhuang, 2019] develop a model where the defender faces multiple attackers. [Zhang et al., 2019] model the possibility of deception by the defender. [Dighe et al., 2009] propose a two-target secrecy game between a single attacker and either one defender or two defenders, assuming the number of defended resources are known to the attacker but which specific targets are defended is a secret.

[Liu et al., 2019] develop a game-theoretic model for studying the allocation of limited resources across a massive number of targets. [Yolmeh and Baykal-Gürsoy, 2019] study the interaction of strategic and operational decisions and present a two-stage game in which strategic decisions are followed by operational decisions. [Zhang et al., 2018] study the impact of risk preferences on the attacker's and the defender's equilibrium behavior. [Guan and Zhuang, 2016] investigate the warm-up effect on

players resource allocation in a sequential-move game. [Shan and Zhuang, 2013a] develop a model where a certain portion of defensive budget is reserved for equal distribution. [Valenzuela et al., 2015] consider a multi-resolution attacker-defender game by pairing resource allocation and optimal response. [Golany et al., 2015] and [Golany et al., 2017] study an attacker-defender zero-sum resource-allocation game where the defender has multiple resources. [Paulson et al., 2016] propose a sequential-move resource-allocation game in which first the defender chooses among countermeasures and targets, then the attacker chooses a target to attack and a type of threat. [Hausken and He, 2016] propose a simultaneous-move game where the defender seeks to reduce the threat score over multiple targets.

[Xu et al., 2016] explore protection strategy on military supply-chain risk management in a sequential-move attacker-defender game. They deduce that the defender is better off utilizing risk-management tools such as capacity back-up and inventory protection. [Cárceles-Poveda and Tauman, 2011] develop a two-stage game between a terrorist organization and an arbitrary number of countries. In the first stage, a subset of countries choose their investments that reduce the terrorist's resources, and in the second stage, the terrorist distributes the remaining resources to attack the countries and the countries choose their defensive measures simultaneously.

Attacker-defender games in the field of networked systems have also been studied in the literature. [Rao et al., 2018] consider the problem of defending infrastructures consisting of a network of systems from physical and cyber attacks that target their communications. [Li et al., 2019] propose a sequential-move game where a target and the payoffs of the players are evaluated in the whole network. [Nochenson and Heimann, 2012] study an aged-based network security investment game to protect a network of computers.

Our model is inspired by the model of [Zhuang and Bier, 2007], which does not include a budgetary constraint neither for the defender nor for the attacker. We introduce a limited budget for the defender and allow the efficiency factors of the attacker and the defender to differ. Also differently from them, we propose a dynamic-programming formulation for the multi-target problem. Our work is also

closely related to [Guan et al., 2017], who assume a budgetary constraint for both the attacker and the defender. Unlike them, we assume that cost efficiency of the attacker's investment is accounted for in the attacker's utility function. Our problem is also related to the literature on Colonel Blotto games introduced by [Borel, 1921]. A review of this literature is available in [Kovenock and Roberson, 2010]. Our problem differs from Colonel Blotto games as we do not assume that a target is destroyed or protected for sure when the attacker and the defender do not invest equal amounts.



Chapter 3

MODEL AND FORMULATION

3.1 Model

This thesis seeks to answer how two players, namely an attacker and a defender, allocate their resources in a competition involving multiple targets $1, \dots, N$. In this competition, the defender decides on how much to invest for the protection of each target while the attacker decides on how much to invest for the attack of each target. We denote the attacker's investment for target n as x_n and the defender's as y_n . We assume that each player invests a nonnegative amount for each target and the defender's total investment is also subject to a budgetary constraint. Accordingly, the defender has access to a finite budget B , but the attacker can invest an arbitrarily large amount in each target as long as the value of the target makes up for it. We let v_n be the attacker's valuation of target n and w_n the defender's. Each player may value different targets differently and the two players may have different valuations for the same target. As an example for the latter, some historical building that the defender considers as a primary asset may be of little value to the attacker.

We assume the probability $p_n(x_n, y_n)$ of damage at each target n is a contest-success function ([Tullock, 1980], [Skaperdas, 1996]). If the attacker invests x_n in the attack of target n and the defender invests y_n in the defense of this target, then

$$p_n(x_n, y_n) = \frac{k_A x_n}{k_A x_n + k_D y_n + c_n}, \quad (3.1)$$

where k_A and k_D are efficiency parameters for the attacker and the defender, respectively, and c_n represents the existing defense at target n . The efficiency factors k_A and k_D can be considered as the effectiveness of an attack and defense, respectively. Accordingly, the success of the attack or defense effort depends not only on the attacker's and the defender's investments, but also how effectively these investments are used, e.g., in training and technology.

Table 3.1: Notation & Explanation

<i>Parameters</i>	
N	Number of targets
n	Target index $n = 1, \dots, N$
v_n	Attacker's valuation of target n
w_n	Defender's valuation of target n
c_n	Inherent defense level at target n
B	Total budget of the defender
k_A	Efficiency factor for the attacker
k_D	Efficiency factor for the defender
<i>Decision Variables</i>	
x_n	Attacker's investment at target n
y_n	Defender's investment at target n
x^n	Attack strategy $(x_1, \dots, x_n)$
y^n	Defense strategy $(y_1, \dots, y_n)$
<i>Functions</i>	
$g_n(x_n)$	Disutility of attacker from investing x_n at target n
$p_n(x_n, y_n)$	Probability of damage at target n given x_n and y_n
$u_n^A(x_n, y_n)$	Attacker's expected utility at target n given x_n and y_n
$u_n^D(x_n, y_n)$	Defender's expected utility at target n given x_n and y_n
$J_A^n(x^n, y^n)$	Attacker's expected total utility at targets $1, \dots, n$
$J_D^n(x^n, y^n)$	Defender's expected total utility at targets $1, \dots, n$
$\hat{x}^n(y^n)$	Attacker's best response to defense strategy y^n
$\hat{y}^n(x^n)$	Defender's best response to attack strategy x^n
$U_D^n(b, x^n)$	Defender's maximum expected total utility at targets $1, \dots, n$ given defense budget b and x^n
$U_A^n(b)$	Attacker's equilibrium expected total utility at targets $1, \dots, n$ given defense budget b
$U_D^n(b)$	Defender's equilibrium expected total utility at targets $1, \dots, n$ given defense budget b

As shown in [Zhuang and Bier, 2007], these probability functions satisfy the following properties:

- The probability of damage at target n is increasing in the investment of the attacker with decreasing marginal returns, i.e.,

$$\frac{\partial p_n(x_n, y_n)}{\partial x_n} \geq 0 \quad \text{and} \quad \frac{\partial^2 p_n(x_n, y_n)}{\partial^2 x_n} \leq 0.$$

- The probability of damage at target n is decreasing in the investment of the defender with increasing marginal returns, i.e.,

$$\frac{\partial p_n(x_n, y_n)}{\partial y_n} \leq 0 \quad \text{and} \quad \frac{\partial^2 p_n(x_n, y_n)}{\partial^2 y_n} \geq 0.$$

- If the attacker does not invest on a target, then the probability of damage at this target is zero, i.e., $p_n(0, y_n) = 0$. On the other hand, even when the attacker invests a positive amount to a target, the probability of damage converges to zero as the defender allocates arbitrarily large amounts to the defense of this target, i.e., $\lim_{y_n \rightarrow \infty} p_n(x_n, y_n) = 0$.

Using these definitions, the attacker's expected utility at target n can be expressed as

$$u_n^A(x_n, y_n) = v_n p_n(x_n, y_n) - g_n(x_n) = \frac{v_n k_A x_n}{k_A x_n + k_D y_n + c_n} - x_n. \quad (3.2)$$

Here the function $g_n(x_n)$ corresponds to the disutility of investing x_n in the attack of target n . Hence the attacker's expected utility from investing x_n in the attack of target n given the investment y_n of the defender is the difference between the expected damage at target n and the disutility of the investment x_n for the attacker. The disutility function guarantees that the attacker makes an efficient investment decision in the absence of a budgetary constraint. In our analysis, we take it as equal to the investment itself, but this assumption can be relaxed to more general functions as long as the convexity of the resulting optimization problem remains valid.

The defender's expected utility is defined in terms of the expected damage as

$$u_n^D(x_n, y_n) = -w_n p_n(x_n, y_n) = -\frac{w_n k_A x_n}{k_A x_n + k_D y_n + c_n}. \quad (3.3)$$

Both the attacker and the defender aim to maximize their expected utilities for n targets. We assume that the expected total utility for each is the sum of expected utilities over the targets. Given the attacker's investment vector $x^N = (x_1, \dots, x_N)$ and the defender's investment vector $y^N = (y_1, \dots, y_N)$, their expected total utilities are

$$J_A^N(x^N, y^N) = \sum_{n=1}^N \frac{v_n k_A x_n}{k_A x_n + k_D y_n + c_n} - \sum_{n=1}^N x_n, \quad (3.4)$$

$$J_D^N(x^N, y^N) = -\sum_{n=1}^N \frac{w_n k_A x_n}{k_A x_n + k_D y_n + c_n}. \quad (3.5)$$

Consequently, their optimization problems can be stated as follows. The attacker wishes to maximize $J_A^N(x^N, y^N)$ over $x^N \geq 0$ given the defense strategy y^N of the defender, and the defender wishes to maximize $J_D^N(x^N, y^N)$ over $y^N \geq 0$ subject to $\sum_{n=1}^N y_n \leq B$ given the attack strategy x^N of the attacker. In other words, the attacker aims to inflict maximum damage at minimum investment cost whereas the defender aims to maximize protection given the limited budget. Given the defense strategy y^N of the defender, the attacker's optimization problem can be stated as follows:

$$\max_{x^N} \quad J_A^N(x^N, y^N) = \sum_{n=1}^N \frac{v_n k_A x_n}{k_A x_n + k_D y_n + c_n} - \sum_{n=1}^N x_n \quad (3.6)$$

$$\text{subject to} \quad x_n \geq 0 \quad \text{for all } n = 1, 2, \dots, N. \quad (3.7)$$

Given the attack strategy x^N of the attacker, the defender's optimization problem is as follows:

$$\max_{y^N} \quad J_D^N(x^N, y^N) = -\sum_{n=1}^N \frac{w_n k_A x_n}{k_A x_n + k_D y_n + c_n} \quad (3.8)$$

$$\text{subject to} \quad \sum_{n=1}^N y_n \leq B, \quad (3.9)$$

$$y_n \geq 0 \quad \text{for all } n = 1, 2, \dots, N. \quad (3.10)$$

The attacker and the defender are assumed to have common knowledge and complete information about the target valuations of both sides (v_n 's and w_n 's), the defender's available budget (B), the defender's inherent defense at all targets (c_n 's), and the efficiency factors (k_A and k_D). The thesis focuses on the simultaneous-move game where the attacker and the defender simultaneously decide on their investments, but considers also the sequential-move version where the defender first decides on the defense strategy and the attacker then follows with the attack strategy.

In the simultaneous-move game, an attack strategy $\hat{x}^N(y^N)$ is a *best response of the attacker* to the defense strategy y^N if it is a solution of the problem (3.6)-(3.7). Similarly, a defense strategy $\hat{y}^N(x^N)$ is a *best-response of the defender* to the attack strategy x^N if it is a solution of the problem (3.8)-(3.10). A *Nash equilibrium* is a pair of attack and defense strategy (x^{N*}, y^{N*}) such that x^{N*} is a best response of the attacker to y^{N*} and y^{N*} is a best response of the defender to x^{N*} .

Table 3.1 lists the parameters, decision variables and functions used throughout this thesis.

3.2 Dynamic-Programming Formulation

We use dynamic programming for finding the equilibrium of the game-theoretical model representing the competition between the attacker and the defender over multiple targets. In our formulation, targets correspond to the stages of the dynamic program, and for each stage n , the attacker and the defender decide on their investments (x_n, y_n) . The damage probabilities at target n is determined by these investments along with the inherent defense level c_n allocated to this target. The budgetary constraint of the defender ties together the problems over multiple targets. In the absence of such a constraint, the problem can be decomposed into N independent subproblems. If each subproblem has its own budget constraint and/or there is a disutility of investment included in the utility calculation of the defender, then there exists a unique Nash equilibrium for each subproblem and so for the original problem. [Canbolat et al., 2012] proposed an efficient method to calculate the unique Nash equilibrium for each subproblem under a linear disutility function

of investment. However, with an overall budget constraint, whatever the defender invests for the protection of a target reduces the amount available for investment for the protection of other targets; therefore, in some sense, the budget constraint induces an additional layer of competition, namely one among different targets. The limited budget of the defender affects also the decisions of the attacker, who is assumed to be rational and to act under the common-knowledge assumption. More specifically, the budget constraint of the defender affects the defense strategy, which, in turn, affects the attack strategy.

For an attack-defense strategy profile to form a Nash equilibrium, it is necessary that players choose their best response at each target given their available budgets. In particular, given a defense strategy y^N , the attacker's problem can be decomposed into N subproblems since the attacker is not subject to a budget constraint. Subproblem n of the attacker associated with target n is as follows:

$$\max_{x_n} \quad u_n^A(x_n, y_n) = \frac{v_n k_A x_n}{k_A x_n + k_D y_n + c_n} - x_n \quad (3.11)$$

$$\text{subject to} \quad x_n \geq 0. \quad (3.12)$$

On the other hand, given an attack strategy x^N , the defender's problem is similar to a knapsack problem, which can be formulated using dynamic programming. The corresponding recursions are as follows:

$$U_D^n(b, x^n) = \max_{0 \leq y_n \leq b} \{u_n^D(x_n, y_n) + U_D^{n-1}(b - y_n, x^{n-1})\} \quad \text{for } n = 1, 2, \dots, N, \quad (3.13)$$

with $U_D^0(b, x^0) \equiv 0$ for all $b \geq 0$ (where x^0 is indeed a vacuous vector). Accordingly, for finding a Nash equilibrium for the multi-target problem, one can start with target $n = 1$, identify the Nash equilibrium (x_1^*, y_1^*) for this target for all $b \in [0, B]$, and iterate with (3.11)-(3.13) until $n = N$ is reached. For $n = N$, one needs the solution of (3.13) only for $b = B$, as B is the total defense budget to be allocated among different targets.

In the following chapters, we investigate the solution of (3.11)-(3.13) analytically and numerically for a single target, two targets, and an arbitrary but finite number of targets.

Chapter 4

SINGLE-TARGET ATTACKER-DEFENDER GAME

In this chapter, we consider the single-target attacker-defender game formulated in Chapter 3, characterize the Nash equilibrium for this problem, describe how it changes with respect to various model parameters, and illustrate these results through numerical examples. While the solution to the single-target instance may be interesting on its own, it also provides the basis for solving the multi-target problem by using dynamic programming.

When the attacker and the defender are competing over a single target, i.e., when $N = 1$, their respective optimization problems become

$$\max_{x_1} u_1^A(x_1, y_1) = \frac{v_1 k_A x_1}{k_A x_1 + k_D y_1 + c_1} - x_1 \quad (4.1)$$

$$\text{subject to} \quad x_1 \geq 0, \quad (4.2)$$

and

$$\max_{y_1} u_1^D(x_1, y_1) = -\frac{w_1 k_A x_1}{k_A x_1 + k_D y_1 + c_1} \quad (4.3)$$

$$\text{subject to} \quad 0 \leq y_1 \leq B. \quad (4.4)$$

The next two sections characterize the solution of these problems individually.

4.1 Attacker's Best Response

Given the defense strategy $0 \leq y_1 \leq B$, the attacker's maximization problem (4.1)-(4.2) is a convex optimization problem and its unique solution has a closed-form representation as Lemma 1 and Proposition 1 establish.

Lemma 1. *Given $y_1 \geq 0$, $u_1^A(x_1, y_1)$ is concave in $x_1 \geq 0$.*

Proof. The first and second derivatives of $u_1^A(x_1, y_1)$ with respect to x_1 are

$$\frac{\partial u_1^A(x_1, y_1)}{\partial x_1} = \frac{v_1 k_A (k_D y_1 + c_1)}{(k_A x_1 + k_D y_1 + c_1)^2} - 1, \quad (4.5)$$

$$\frac{\partial^2 u_1^A(x_1, y_1)}{\partial x_1^2} = \frac{-2v_1 k_A^2 (k_D y_1 + c_1)}{(k_A x_1 + k_D y_1 + c_1)^3} < 0, \quad (4.6)$$

so $u_1^A(x_1, y_1)$ is concave in $x_1 \geq 0$ provided that $y_1 \geq 0$. $\square$

Since the objective function of the attacker is concave in the attack level x_1 , we can use the first-order condition to characterize the attacker's best response to y_1 .

Proposition 1. *The best response of the attacker to the defense level $y_1 \in [0, B]$ is as follows:*

$$\hat{x}_1(y_1) = \begin{cases} \frac{\sqrt{v_1 k_A (k_D y_1 + c_1)} - (k_D y_1 + c_1)}{k_A} & \text{if } v_1 k_A > k_D y_1 + c_1 \\ 0 & \text{if } v_1 k_A \leq k_D y_1 + c_1. \end{cases} \quad (4.7)$$

The corresponding expected utility of the attacker is

$$u_1^A(\hat{x}_1(y_1), y_1) = \begin{cases} \left(\sqrt{v_1} - \sqrt{\frac{k_D y_1 + c_1}{k_A}} \right)^2 & \text{if } v_1 k_A > k_D y_1 + c_1 \\ 0 & \text{if } v_1 k_A \leq k_D y_1 + c_1. \end{cases} \quad (4.8)$$

Proof. By Lemma 1, the maximizer in (4.1) is found by setting the first derivative (4.5) to zero, which gives

$$v_1 k_A (k_D y_1 + c_1) = (k_A x_1 + k_D y_1 + c_1)^2.$$

Solving this equation for x_1 gives

$$x_1 = \frac{\sqrt{v_1 k_A (k_D y_1 + c_1)} - (k_D y_1 + c_1)}{k_A},$$

which is the best response of the attacker if positive. Otherwise, $u_1^A(x_1, y_1)$ is decreasing in $x_1 \geq 0$, so the attacker's best response is zero. The expected utility of the attacker is found by replacing $x_1 = \hat{x}_1(y_1)$ in $u_1^A(x_1, y_1)$. $\square$

4.2 Defender's Best Response

The defender's maximization problem (4.3)-(4.4) for a single target has a rather trivial solution. For any given attack level $x_1 > 0$, the expected utility of the defender in (4.3) is increasing in the defense level $y_1 \geq 0$; therefore, the maximizer is attained at the boundary B .

Proposition 2. *The best response of the defender to the attack level $x_1 \geq 0$ is*

$$\hat{y}_1(x_1) = \begin{cases} B & \text{if } x_1 > 0 \\ [0, B] & \text{if } x_1 = 0. \end{cases} \quad (4.9)$$

The corresponding expected utility of the defender is

$$u_1^D(x_1, \hat{y}_1(x_1)) = -\frac{w_1 k_A x_1}{k_A x_1 + k_D B + c_1}. \quad (4.10)$$

Proof. The first derivative of $u_1^D(x_1, y_1)$ with respect to y_1 is

$$\frac{\partial u_1^D(x_1, y_1)}{\partial y_1} = \frac{w_1 k_A k_D x_1}{(k_A x_1 + k_D y_1 + c_1)^2}, \quad (4.11)$$

which is positive if $x_1 > 0$, so the maximizer is $\hat{y}_1(x_1) = B$. If $x_1 = 0$, $u_1^D(x_1, y_1) = 0$ for any $y_1 \geq 0$, so any feasible y_1 is a best response. The expected utility is found by replacing $y_1 = B$ in (4.3). $\square$

4.3 Nash Equilibrium

Having obtained the best-response correspondences, we are now ready to characterize the Nash equilibrium of the single-target attacker-defender game.

Theorem 1. *The single-target attacker-defender game has either a unique Nash equilibrium or infinitely many Nash equilibria. In particular, the following two cases arise.*

(i) *If $v_1 k_A > k_D B + c_1$, then (x_1^*, y_1^*) such that*

$$x_1^* = \frac{\sqrt{v_1 k_A (k_D B + c_1)} - (k_D B + c_1)}{k_A}, \quad (4.12)$$

$$y_1^* = B. \quad (4.13)$$

is the unique Nash equilibrium. The attacker's and the defender's expected equilibrium utilities in this case are

$$U_A^1(B) = \left(\sqrt{v_1} - \sqrt{\frac{k_D B + c_1}{k_A}} \right)^2, \quad (4.14)$$

$$U_D^1(B) = w_1 \left(\sqrt{\frac{k_D B + c_1}{v_1 k_A}} - 1 \right). \quad (4.15)$$

(ii) If $v_1 k_A \leq k_D B + c_1$, then any (x_1^*, y_1^*) such that

$$x_1^* = 0, \quad (4.16)$$

$$y_1^* \in [(v_1 k_A - c_1)^+ / k_D, B], \quad (4.17)$$

where $z^+ \equiv \max(z, 0)$, is a Nash equilibrium. The attacker's and the defender's expected equilibrium utilities in this case are zero.

Proof. By Proposition 2, the defender's best response to $x_1^* > 0$ is B , and by Proposition 1, the attacker's best response to B is (4.12) if $v_1 k_A > k_D B + c_1$. To show the uniqueness, it is enough to show $x_1 = 0$ cannot be an equilibrium strategy for the attacker. This is true because for any $y_1 \in [0, B]$,

$$u_1^A(0, y_1) = 0 < u_1^A \left(\frac{\sqrt{v_1 k_A (k_D y_1 + c_1)} - (k_D y_1 + c_1)}{k_A}, y_1 \right).$$

The resulting utilities (4.14)-(4.15) are calculated by replacing (4.12)-(4.13) in (4.1) and (4.3).

If $v_1 k_A \leq k_D B + c_1$, then by Proposition 1, the attacker's best response to $y_1^* \geq (v_1 k_A - c_1)^+ / k_D$ is $x_1^* = 0$. Since $B \geq (v_1 k_A - c_1)^+ / k_D$, any $y_1^* \in [(v_1 k_A - c_1)^+ / k_D, B]$ is a best response to $x_1^* = 0$. On the other hand, for any $y_1 < (v_1 k_A - c_1)^+ / k_D$, the attacker's best response (4.7) is positive, and in turn, the defender's best response must be equal B , showing that such y_1 cannot be an equilibrium strategy for the defender. The utilities (4.1) and (4.3) equal zero since $x_1^* = 0$. $\square$

As established by Theorem 1, there are two possible cases. In the first case, the target is sufficiently valuable to the attacker, more specifically $v_1 > (k_D B + c_1) / k_A$, so the attacker invests a positive amount to destroy it and the defender invests all

the budget to protect it. In the second one, the value of the target does not justify a positive investment for the attacker, who knows that a positive attack level will make the defender invest all the available budget for the defense. In this case, since the attacker does not put any effort to destroy the target, the defender can invest any feasible amount without creating an incentive that would make the attacker invest positively. The need to avoid such an incentive leads to the lower bound $(v_1 k_A - c_1)^+ / k_D$ on the equilibrium defense level.

4.4 Sensitivity Analysis of Equilibrium Investments and Utilities

4.4.1 Attacker's Equilibrium Investment and Utility

In this section, we describe the changes in the equilibrium investment and utility of the attacker as the model parameters change. Starting with the target's value v_1 for the attacker, we recall that by Theorem 1, the attacker will not invest to destroy a target that has small value. More precisely, if v_1 is less than or equal to the threshold $(k_D B + c_1) / k_A$, the equilibrium investment and utility of the attacker are zero. Once v_1 exceeds this threshold, the attacker invest a positive amount, namely

$$x_1^* = \frac{\sqrt{v_1 k_A (k_D B + c_1)} - (k_D B + c_1)}{k_A}, \quad (4.18)$$

which is increasing in v_1 . The corresponding equilibrium utility

$$U_A^1(B) = \left(\sqrt{v_1} - \sqrt{\frac{k_D B + c_1}{k_A}} \right)^2, \quad (4.19)$$

is also increasing in v_1 . Hence, as one would expect, the more valuable a target is, the more the attacker will invest in order to destroy it and the larger will be the attacker's expected utility. We note that the attacker's equilibrium investment and utility do not depend on the valuation w_1 of the defender.

To see the effects of changes in k_D , B , and c_1 , we let

$$f(z) = \frac{\sqrt{v_1 k_A z} - z}{k_A},$$

and differentiate it with respect to z . The derivative

$$\frac{df(z)}{dz} = \frac{\sqrt{v_1 k_A} - 2\sqrt{z}}{2k_A \sqrt{z}},$$

is positive for $0.25v_1k_A > z$ and negative for $0.25v_1k_A < z$. Accordingly, for $k_DB + c_1 < 0.25v_1k_A$, the attacker investment is increasing as $k_DB + c_1$, or equivalently any of the parameters k_D , B , or c_1 increases; for $0.25v_1k_A \leq k_DB + c_1 < v_1k_A$, the attacker investment is decreasing as $k_DB + c_1$, or equivalently any of the parameters k_D , B , or c_1 increases. For $k_DB + c_1 \geq v_1k_A$, the equilibrium attacker investment is fixed as zero. Hence when the efficiency factor k_D of the defender, the defense budget B , and the inherent defense level c_1 are small enough, an increase in any one of these leads to a larger attacker investment; however, when the total defense $k_DB + c_1$ exceeds the threshold $0.25v_1k_A$, the effect on the attacker's investment is reversed. While the effect on the investment is not monotone, the effect of changes in these parameters of the defender on the attacker's equilibrium utility is. The expression (4.19) is decreasing in k_D , B , and c_1 as long as $v_1k_A > k_DB + c_1$; otherwise, the attacker's equilibrium utility remains as zero.

Finally, in order to understand the effect of the attacker's efficiency factor, we differentiate (4.18) with respect to k_A and obtain the derivative

$$\frac{(k_DB + c_1) - 0.5\sqrt{v_1k_A(k_DB + c_1)}}{k_A^2}.$$

This expression is positive for $k_A < 4(k_DB + c_1)/v_1$ and negative for $k_A > 4(k_DB + c_1)/v_1$. We recall that for $k_A \leq (k_DB + c_1)/v_1$, the attacker invests zero. For $(k_DB + c_1)/v_1 < k_A < 4(k_DB + c_1)/v_1$, the attacker investment is increasing in k_A , and for $k_A \geq 4(k_DB + c_1)/v_1$, the attacker investment is decreasing in k_A . On the other hand, the attacker's equilibrium utility (4.19) is increasing in k_A as long as it is positive; otherwise, $k_A \leq (k_DB + c_1)/v_1$ and the attacker's equilibrium utility is null.

4.4.2 Defender's Equilibrium Investment and Utility

When there is only one target to be defended, the attacker mainly aims at minimizing the probability of damage at this target, so the equilibrium investment is independent of the defender's valuation w_1 and the equilibrium utility of the defender is decreasing in w_1 , as it represents the negative of the expected damage at the target. When the attacker does not invest, that is, when $v_1k_A \leq k_DB + c_1$, the

attacker's equilibrium investment and utility are both independent of w_1 , and the latter equals zero.

The defender's equilibrium utility remains as zero when v_1 or k_A increase without violating the condition $v_1 \leq (k_D B + c_1)/k_A$. Under this condition, any defense level between $(v_1 k_A - c_1)^+/k_D$ and B is an equilibrium strategy, and this interval shrinks as v_1 or k_A increases. Once $v_1 k_A$ exceeds the threshold $k_D B + c_1$, the defender's equilibrium investment is uniquely set as B and further increases in v_1 or k_A do not affect this amount. In this case, the defender's equilibrium utility (4.15) is decreasing in the attacker's valuation v_1 and the attacker's efficiency factor k_A . The more valuable a target is for the attacker, the larger investment the attacker will allocate for destroying it, which would cause a higher probability of damage and consequently yield a lower expected utility for the defender.

To see the effect of an increase in the parameters k_D , B , and c_1 , once more we observe that for $k_D B + c_1 \geq v_1 k_A$, the attacker invests zero, so any investment between $(v_1 k_A - c_1)^+/k_D$ and B is a best response for the defender. When this condition is satisfied, an increase in k_D , B , or c_1 will not change the defender's equilibrium utility, which is zero; however, these changes will expand the interval $[(v_1 k_A - c_1)^+/k_D, B]$ of the equilibrium strategies. For $k_D B + c_1 < v_1 k_A$, the unique defender strategy is to invest all. As long as this condition is preserved, the defender's equilibrium investment is independent of k_D and c_1 , but is increasing in the budget B . The defender's equilibrium utility (4.15) is increasing in k_D , B , and c_1 , as one would expect.

To conclude, when the target is highly valuable for the attacker, the only parameter that will affect the defender's equilibrium investment is the budget. The results discussed above are stated in the following corollary.

Corollary 1. *Suppose $v_1 k_A > k_D B + c_1$.*

- (i) *The equilibrium investment of the attacker is independent of the defender's target value w_1 , increasing in the attacker's target value v_1 , first increasing then decreasing in the efficiency factors k_A and k_D , the inherent defense level c_1 , and the defender's budget B .*

- (ii) The equilibrium investment of the defender is independent of the defender's target value w_1 , the attacker's target value v_1 , the efficiency factors k_A and k_D , and the inherent defense level c_1 , and increasing in B .
- (iii) The equilibrium utility of the attacker is independent of the defender's target value w_1 , increasing in the attacker's target value v_1 and efficiency factor k_A , and decreasing in the defender's efficiency factor k_D , the inherent defense level c_1 , and the defender's budget B .
- (iv) The equilibrium utility of the defender is decreasing in the defender's target value w_1 , the attacker's target value v_1 , and the attacker's efficiency factor k_A , and increasing in the defender's efficiency factor k_D , the inherent defense level c_1 , and the defender's budget B .

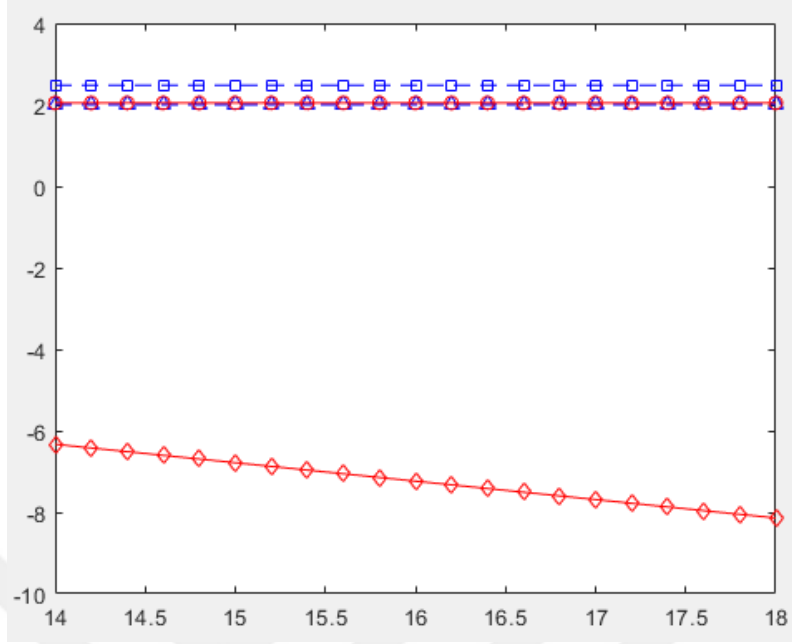
Table 4.1 summarizes the sensitivity results of Corollary 1. We note that the budget B is the only parameter that affects all the equilibrium investments and utilities.

Table 4.1: Sensitivity Analysis: $v_1 k_A > k_D B + c_1$

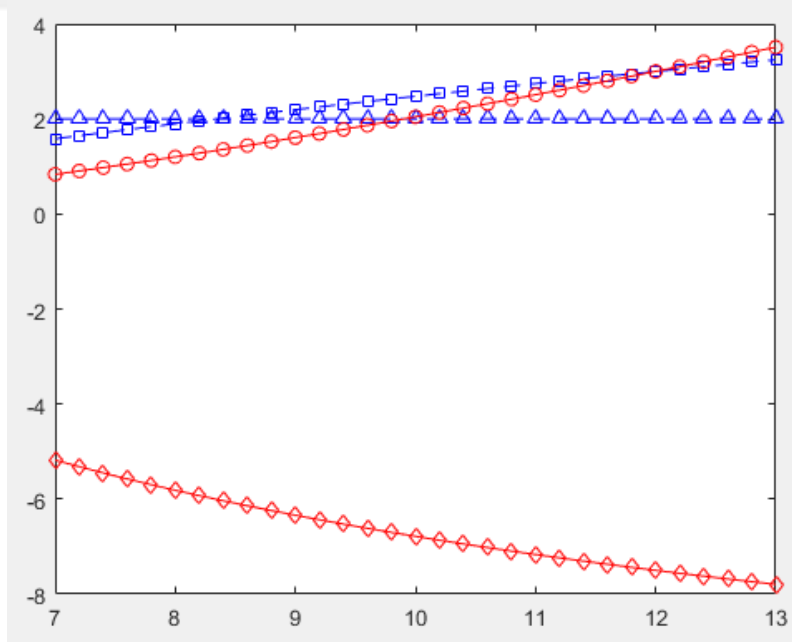
Parameter	x_1^*	y_1^*	$U_A^1(B)$	$U_D^1(B)$
$w_1 \uparrow$	=	=	=	↓
$v_1 \uparrow$	↑	=	↑	↓
$k_A \uparrow$	↗↘	=	↑	↓
$k_D \uparrow$	↗↘	=	↓	↑
$c_1 \uparrow$	↗↘	=	↓	↑
$B \uparrow$	↗↘	↑	↓	↑

= remains the same, ↓ decreases, ↑ increases, ↗↘ first increases, then decreases

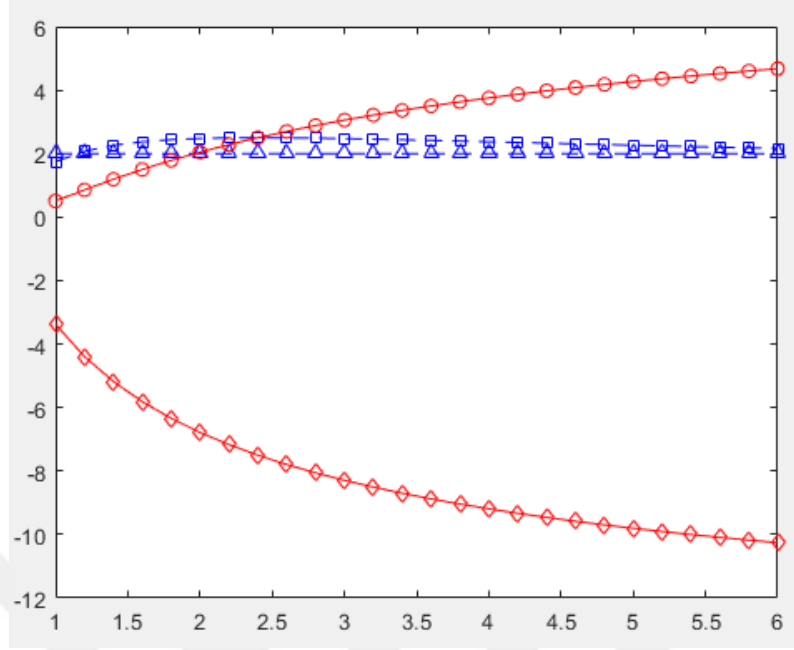
Figures 4.1-4.6 illustrate the sensitivity of equilibrium investments and utilities, with the baseline values $w_1 = 15$, $v_1 = 10$, and $k_A = k_D = c_1 = B = 2$.

Figure 4.1: Equilibrium Investments and Utilities vs. w_1

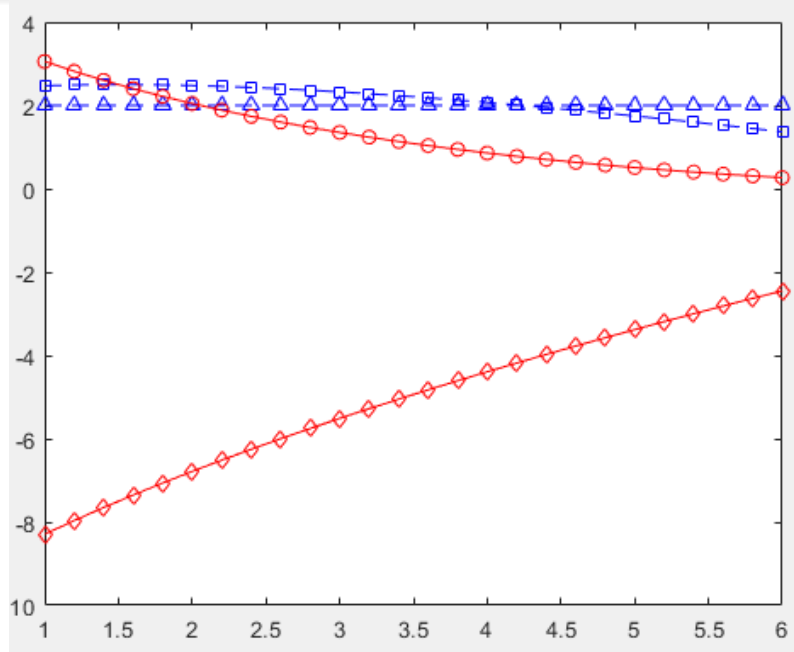
$\square x_1^*, \triangle y_1^*, \circ U_A^1(B), \diamond U_D^1(B)$

Figure 4.2: Equilibrium Investments and Utilities vs. v_1

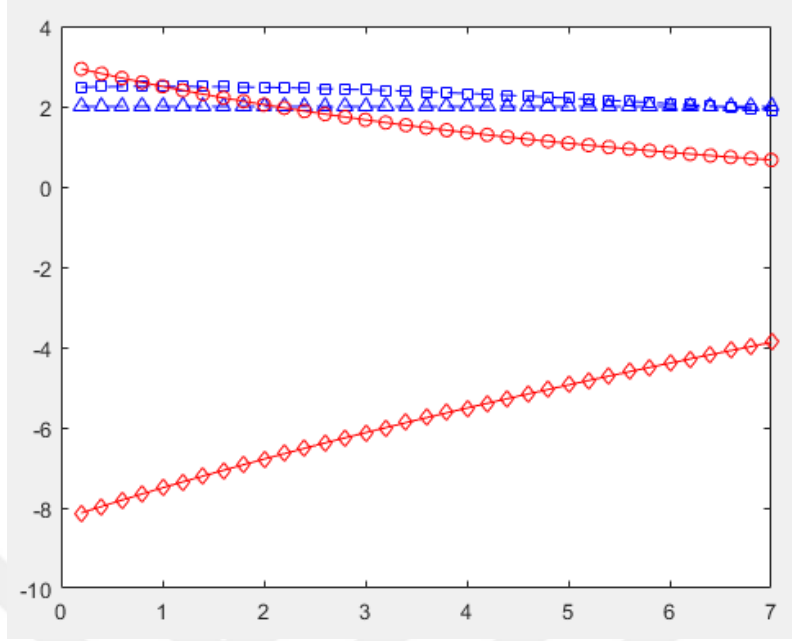
$\square x_1^*, \triangle y_1^*, \circ U_A^1(B), \diamond U_D^1(B)$

Figure 4.3: Equilibrium Investments and Utilities vs. k_A

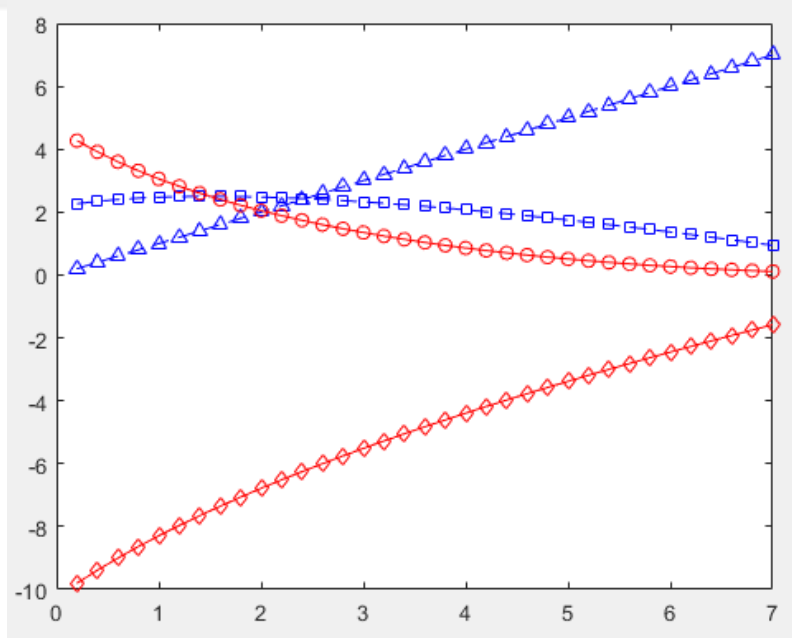
$\square x_1^*, \triangle y_1^*, \circ U_A^1(B), \diamond U_D^1(B)$

Figure 4.4: Equilibrium Investments and Utilities vs. k_D

$\square x_1^*, \triangle y_1^*, \circ U_A^1(B), \diamond U_D^1(B)$

Figure 4.5: Equilibrium Investments and Utilities vs. c_1

$\square x_1^*, \triangle y_1^*, \circ U_A^1(B), \diamond U_D^1(B)$

Figure 4.6: Equilibrium Investments and Utilities vs. B

$\square x_1^*, \triangle y_1^*, \circ U_A^1(B), \diamond U_D^1(B)$

Corollary 2 brings together our observations about the case where $v_1 k_A \leq k_D B + c_1$.

Corollary 2. *Suppose $v_1 k_A \leq k_D B + c_1$.*

- (i) *The equilibrium investment and utility of the attacker are equal to zero, regardless of the defender's and the attacker's target values w_1 and v_1 , the efficiency factors k_A and k_D , the inherent defense level c_1 , and the defender's budget B .*
- (ii) *The interval of the defender's equilibrium investments shrinks as the attacker's target value v_1 or efficiency factor k_A increases, and expands as the defender efficiency factor k_D , the inherent defense level c_1 , or the budget B increases. The defender's equilibrium utility equals zero regardless of these parameters.*

The following figures show the equilibrium investments of the defender with baseline values $v_1 = 16$, $k_A = 1$, $k_D = 2$, $c_1 = 6$, and $B = 8$.

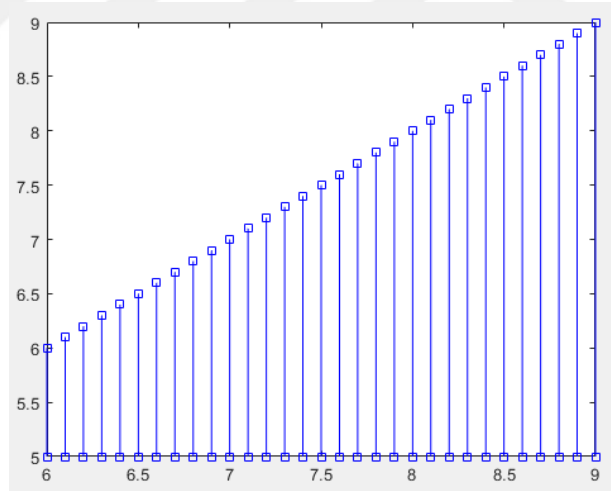
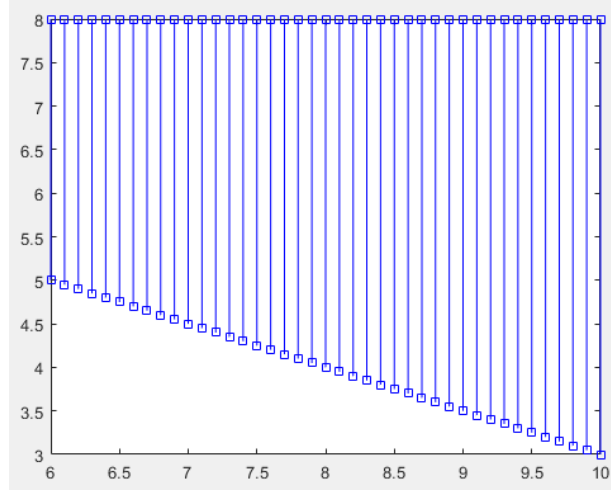
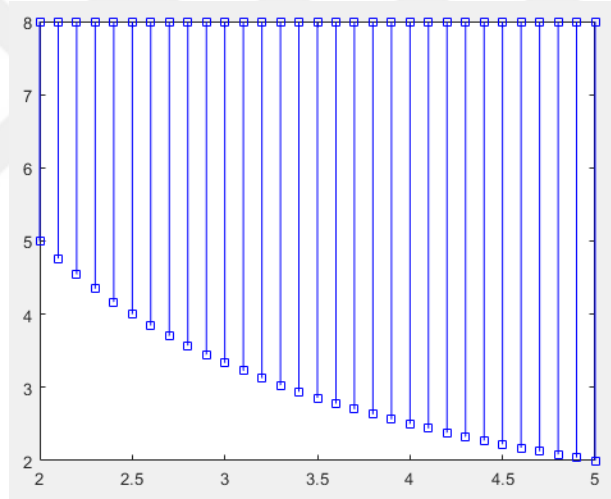


Figure 4.7: Defender's Equilibrium Investments vs. B

Figure 4.8: Defender's Equilibrium Investments vs. c_1 Figure 4.9: Defender's Equilibrium Investments vs. k_D

4.5 Subgame-Perfect Nash Equilibrium in the Sequential-Move Game

The rest of this chapter considers a sequential-move version of the single-target attacker-defender game, where the defender first chooses a defense level and the attacker determines a best strategy after observing the defender's choice. For any given defense level y_1 , the attacker will invest a nonnegative amount that maximizes the attacker's expected utility (4.1) and we already identified the maximizer, namely

the attacker's best response $\hat{x}_1(y_1)$ in Section 4.1 as

$$\hat{x}_1(y_1) = \begin{cases} \frac{\sqrt{v_1 k_A (k_D y_1 + c_1)} - (k_D y_1 + c_1)}{k_A} & \text{if } v_1 k_A > k_D y_1 + c_1 \\ 0 & \text{if } v_1 k_A \leq k_D y_1 + c_1. \end{cases} \quad (4.20)$$

Under the common-knowledge and rationality assumptions, the defender will predict this behavior of the attacker, so will solve the following problem:

$$\max_{y_1} u_1^D(\hat{x}_1(y_1), y_1) \quad (4.21)$$

$$\text{subject to } 0 \leq y_1 \leq B. \quad (4.22)$$

We first show that the expression in (4.21) is increasing in y_1 , then use this result to obtain the equilibrium investments and utilities.

Theorem 2. *The single-target attacker-defender game where the attacker decides after observing the investment of the defender has either a unique subgame-perfect Nash equilibrium or infinitely many subgame-perfect Nash equilibria. The resulting equilibrium investments and utilities coincide with those in the simultaneous-move game.*

Proof. Plugging (4.20) in $u_1^D(\hat{x}_1(y_1), y_1)$ gives

$$u_1^D(y_1) \equiv u_1^D(\hat{x}_1(y_1), y_1) = \begin{cases} w_1 \left(\sqrt{\frac{k_D y_1 + c_1}{v_1 k_A}} - 1 \right) & \text{if } v_1 k_A > k_D y_1 + c_1 \\ 0 & \text{if } v_1 k_A \leq k_D y_1 + c_1. \end{cases} \quad (4.23)$$

This function is increasing in $0 \leq y_1 < (v_1 k_A - c_1)/k_D$ and equals zero for $y_1 \geq (v_1 k_A - c_1)/k_D$. If $(v_1 k_A - c_1)/k_D > B$, then the unique maximizer is $\bar{y}_1 = B$, and by (4.20), the corresponding best-response of the attacker is

$$\bar{x}_1 = \hat{x}_1(B) = \frac{\sqrt{v_1 k_A (k_D B + c_1)} - (k_D B + c_1)}{k_A}.$$

If $(v_1 k_A - c_1)/k_D \leq B$, any $\bar{y}_1 \in [(v_1 k_A - c_1)^+/k_D, B]$ is a maximizer, and by (4.20), the corresponding best-response of the attacker is $\bar{x}_1 = 0$. The resulting utilities are calculated by replacing $(\bar{x}_1, \bar{y}_1)$ in $u_1^A(x_1, y_1)$ and $u_1^D(x_1, y_1)$. $\square$

Since the equilibria in the sequential- and simultaneous-move games are the same, the sensitivity results stated in Corollaries 1 and 2 apply to the subgame-perfect Nash equilibrium outcomes. Also the overlap of the equilibria in the two settings suggests that the defender does not experience any advantage from being the first mover.



Chapter 5

TWO-TARGET ATTACKER-DEFENDER GAME

This chapter is devoted to solving the two-target attacker-defender game by using the results of the previous chapter. Throughout, we assume that the two targets are independent and the success of an attack at each target is determined by the inherent defense level, the attacker's and defender's investments at this target.

For two targets, the dynamic-programming approach described in Chapter 3 leads to a two-stage program. First, we solve for the Nash equilibrium of the single-target game for target 1 treating the budget as a variable, then we use this to find the equilibrium investments for target 2 given the total budget B .

One insight from the analysis of the single-target problem is that for a target with $v_1 k_A \leq k_D B + c_1$, not investing is a Nash equilibrium strategy for the attacker. With this insight, for the two-target problem, we label the two targets in decreasing order of $v_n k_A - c_n$ so that $v_1 k_A - c_1 \geq v_2 k_A - c_2$. Accordingly, at target 1, either the attacker valuation is larger and/or the inherent defense at the first target is lower as compared to target 2.

When the attacker and the defender are competing over two targets, i.e., when $N = 2$, their optimization problems become

$$\max_{x_1, x_2} \quad J_A^2(x^2, y^2) = u_1^A(x_1, y_1) + u_2^A(x_2, y_2) \quad (5.1)$$

$$= \left(\frac{v_1 k_A x_1}{k_A x_1 + k_D y_1 + c_1} - x_1 \right) + \left(\frac{v_2 k_A x_2}{k_A x_2 + k_D y_2 + c_2} - x_2 \right)$$

subject to $x_1, x_2 \geq 0,$ (5.2)

and

$$\max_{y_1, y_2} J_D^2(x^2, y^2) = u_1^D(x_1, y_1) + u_2^D(x_2, y_2) \quad (5.3)$$

$$= -\frac{w_1 k_A x_1}{k_A x_1 + k_D y_1 + c_1} - \frac{w_2 k_A x_2}{k_A x_2 + k_D y_2 + c_2}$$

$$\text{subject to } y_1 + y_2 \leq B, \quad (5.4)$$

$$y_1, y_2 \geq 0, \quad (5.5)$$

respectively. Similarly to our analysis of the single-target problem, we first find the best-response correspondence for the attacker and the defender, then bring these together to identify the Nash equilibrium if there exists any.

5.1 Attacker's Best Response

The attacker's best response to the defense strategy $y^2 = (y_1, y_2)$ of the defender must solve the optimization problem (5.1)-(5.2). This problem involves no constraint tying the investments of the attacker at the two targets; indeed the only constraint in this problem is the nonnegativity of the investments. Therefore, for given $y^2 = (y_1, y_2)$, the attacker's best response is found by solving two single-target problems separately. The first subproblem is to find the best attack strategy at target 1 with value v_1 for the attacker given the defender's investment y_1 and the inherent defense level c_1 , and the second one is to find the best attack strategy at target 2 with value v_2 for the attacker given the defender's investment y_2 and the inherent defense level c_2 . By Lemma 1 of Chapter 4, each subproblem has a concave objective function, and Proposition 1 provides the best response for each target. The two-target best response of the attacker is a combination of the two best responses, as stated in the following proposition.

Proposition 3. *The best response of the attacker to the defense strategy $y^2 = (y_1, y_2) \geq 0$ is $\hat{x}^2(y^2) = (\hat{x}_1(y_1), \hat{x}_2(y_2))$ with*

$$\hat{x}_n(y_n) = \begin{cases} \frac{\sqrt{v_n k_A (k_D y_n + c_n)} - (k_D y_n + c_n)}{k_A} & \text{if } v_n k_A > k_D y_n + c_n \\ 0 & \text{if } v_n k_A \leq k_D y_n + c_n, \end{cases} \quad (5.6)$$

for $n = 1, 2$. The attacker's expected utility is then

$$J_A^2(\hat{x}^2(y^2), y^2) = \begin{cases} \left(\sqrt{v_1} - \sqrt{\frac{k_D y_1 + c_1}{k_A}} \right)^2 + \left(\sqrt{v_2} - \sqrt{\frac{k_D y_2 + c_2}{k_A}} \right)^2 & \text{if } v_1 k_A > k_D y_1 + c_1 \text{ and } v_2 k_A > k_D y_2 + c_2 \\ \left(\sqrt{v_1} - \sqrt{\frac{k_D y_1 + c_1}{k_A}} \right)^2 & \text{if } v_1 k_A > k_D y_1 + c_1 \text{ and } v_2 k_A \leq k_D y_2 + c_2 \\ \left(\sqrt{v_2} - \sqrt{\frac{k_D y_2 + c_2}{k_A}} \right)^2 & \text{if } v_1 k_A \leq k_D y_1 + c_1 \text{ and } v_2 k_A > k_D y_2 + c_2 \\ 0 & \text{if } v_1 k_A \leq k_D y_1 + c_1 \text{ and } v_2 k_A \leq k_D y_2 + c_2. \end{cases} \quad (5.7)$$

Proof. The proposition follows from the decomposition of the attacker's problem into two subproblems, one for each target, and applying Proposition 1 to each subproblem. The corresponding utilities are calculated by considering the four possible cases, namely one where the attacker invests positively for the attack of both targets, two cases where the attacker invests positively for the attack of only one target, and the remaining case where the attacker does not invest at all. $\square$

Accordingly, for any fixed defense strategy (y_1, y_2) , the attacker's optimal investment strategy consists of investing optimally at each target individually. In other words, the attacker decides on the investment at each target so as to maximize the expected utility associated with that target, without considering other targets. Having said that, as clear from (5.6), the optimal attacker investment each target depends on the defense level at that target, which in turn depends on the defense level at the other target because of the defender's budgetary constraint.

5.2 Defender's Best Response

As discussed in Section 3.2 of Chapter 3, this thesis explores the dynamic-programming formulation of the defender's problem to select an optimal investment strategy to counter the attacks at various targets under the budget constraint. In the same section, we established the dynamic-programming recursion (3.13) of the defender.

Writing these explicitly for two targets, we get

$$U_D^1(b, x^1) = \max_{0 \leq y_1 \leq b} \{u_1^D(x_1, y_1)\} \quad \text{for } 0 \leq b \leq B, \quad (5.8)$$

$$U_D^2(B, x^2) = \max_{0 \leq y_2 \leq B} \{u_2^D(x_2, y_2) + U_D^1(B - y_2, x^1)\}. \quad (5.9)$$

By Proposition 2, the solution of (5.9) can be written as a function of b as follows:

$$\hat{y}_1(b, x_1) = \begin{cases} b & \text{if } x_1 > 0 \\ [0, b] & \text{if } x_1 = 0. \end{cases} \quad (5.10)$$

The maximum expected utility $U_D^1(b, x^1)$ of the defender at target 1 is then

$$U_D^1(b, x^1) = -\frac{w_1 k_A x_1}{k_A x_1 + k_D b + c_1}. \quad (5.11)$$

Replacing this in (5.9) gives

$$U_D^2(B, x^2) = \max_{0 \leq y_2 \leq B} \left\{ u_2^D(x_2, y_2) + -\frac{w_1 k_A x_1}{k_A x_1 + k_D (B - y_2) + c_1} \right\}. \quad (5.12)$$

We define

$$\begin{aligned} V_D^2(x^2, y_2) &\equiv u_2^D(x_2, y_2) + -\frac{w_1 k_A x_1}{k_A x_1 + k_D (B - y_2) + c_1} \\ &= -\frac{w_2 k_A x_2}{k_A x_2 + k_D y_2 + c_2} - \frac{w_1 k_A x_1}{k_A x_1 + k_D (B - y_2) + c_1}, \end{aligned} \quad (5.13)$$

and show that it is concave in y_2 in order to find the best response of the defender.

Lemma 2. *Given $x^2 = (x_1, x_2) \geq 0$ such that $x^2 \neq 0$, $V_D^2(x^2, y_2)$ is concave in $0 \leq y_2 \leq B$.*

Proof. The first and second derivatives of the $V_D^2(x^2, y_2)$ with respect to y_2 are

$$\frac{\partial V_D^2(x^2, y_2)}{\partial y_2} = \frac{k_A k_D w_2 x_2}{(k_A x_2 + k_D y_2 + c_2)^2} - \frac{k_A k_D w_1 x_1}{[k_A x_1 + k_D (B - y_2) + c_1]^2}, \quad (5.14)$$

$$\frac{\partial^2 V_D^2(x^2, y_2)}{\partial y_2^2} = \frac{-2k_A k_D^2 w_2 x_2}{(k_A x_2 + k_D y_2 + c_2)^3} + \frac{-2k_A k_D^2 w_1 x_1}{[k_A x_1 + k_D (B - y_2) + c_1]^3}. \quad (5.15)$$

The second derivative is negative for $x^2 \geq 0$ such that $x^2 \neq 0$, so $V_D^2(x^2, y_2)$ is concave in $0 \leq y_2 \leq B$. $\square$

Proposition 4. *The best response of the defender $\hat{y}^2(x^2)$ to the attack strategy $x^2 = (x_1, x_2) \geq 0$ is as follows.*

(i) If $x^2 = 0$, then any $(y_1, y_2) \geq 0$ such that $y_1 + y_2 \leq B$ is a best response. The defender's expected utility is then $J_D^2(x^2, \hat{y}^2(x^2)) = 0$.

(ii) If $x^2 \neq 0$,

$$\hat{y}_2(x^2) = \begin{cases} 0 & \text{if } \tilde{y}_2 \leq 0 \\ \tilde{y}_2 & \text{if } 0 < \tilde{y}_2 < B \\ B & \text{if } \tilde{y}_2 \geq B, \end{cases} \quad (5.16)$$

and

$$\hat{y}_1(x^2) = B - \hat{y}_2(x^2), \quad (5.17)$$

where

$$\tilde{y}_2 = \frac{\sqrt{w_2 x_2}(k_A x_1 + k_D B + c_1) - \sqrt{w_1 x_1}(k_A x_2 + c_2)}{k_D(\sqrt{w_1 x_1} + \sqrt{w_2 x_2})}. \quad (5.18)$$

The defender's expected utility is then

$$J_D^2(x^2, \hat{y}^2(x^2)) = \begin{cases} -k_A \left(\frac{w_2 x_2}{k_A x_2 + c_2} + \frac{w_1 x_1}{k_A x_1 + k_D B + c_1} \right) & \text{if } \tilde{y}_2 \leq 0 \\ -k_A \left(\frac{(\sqrt{w_1} + \sqrt{w_2})(\sqrt{w_1 x_1} + \sqrt{w_2 x_2})}{k_A(x_1 + x_2) + k_D B + c_1 + c_2} \right) & \text{if } 0 < \tilde{y}_2 < B \\ -k_A \left(\frac{w_2 x_2}{k_A x_2 + k_D B + c_2} + \frac{w_1 x_1}{k_A x_1 + c_1} \right) & \text{if } \tilde{y}_2 \geq B. \end{cases} \quad (5.19)$$

Proof. For $x^2 = 0$, $x_1 = 0$, so the best response of the defender at target 1 given the budget b consists of the set $[0, b]$, with $U_D^1(b, x^1) = 0$, and (5.13) reduces to $u_2(x_2, y_2) = 0$, since $x_2 = 0$, so any $y_2 \in [0, B]$ is a best response at target 2. Hence any (y_1, y_2) that is feasible is a best response, and has expected utility zero.

Now suppose $x^2 \neq 0$. By Lemma 2, $V_D^2(x^2, y_2)$ is concave in $0 \leq y_2 \leq B$. Its unique stationary point $\tilde{y}_2$ is found by equating (5.14) to zero and solving for y_2 . This solution is given in (5.18). If $\tilde{y}_2 \leq 0$, it means that $V_D^2(x^2, y_2)$ is decreasing in $0 \leq y_2 \leq B$, so the best response at target 2 is zero; if $\tilde{y}_2 \in (0, B)$ is the maximizer of $V_D^2(x^2, y_2)$ on $[0, B]$, so it is the defender's best response at target 2; otherwise, $\tilde{y}_2 \geq B$, meaning that $V_D^2(x^2, y_2)$ is increasing in $0 \leq y_2 \leq B$, so the best response at target 2 is B . The amount left for target 1 is B in the first case, $B - \tilde{y}_2$ in the

second, and zero in the third one. If $x_1 > 0$, then the whole amount left is clearly the unique best response at target 1; if $x_1 = 0$, then

$$\tilde{y}_2 = \frac{\sqrt{w_2 x_2}(k_D B + c_1)}{k_D \sqrt{w_2 x_2}} > B,$$

so $\hat{y}_2(x^2) = B$ and the only feasible investment at target 1 is $B - \hat{y}_2(x^2) = 0$. Corresponding utilities are calculated by plugging in these best-response values. $\square$

We note that the defender's investments at the two targets are related, as the defender has to strategically divide its budget between them in order to minimize the overall expected damage.

5.3 Nash Equilibrium

A Nash equilibrium of the two-target attacker-defender game is a strategy profile (x^*, y^*) such that x^* is a best response of the attacker to the defender's strategy y^* , and y^* is a best response of the defender to the attacker's strategy x^* . Accordingly, x^* and y^* must satisfy the conditions specified in Propositions 3 and 4.

The existence of a Nash equilibrium in multi-player concave games, where the payoff function of each player is concave in its own strategy for fixed strategies of the other players and each player has a convex, closed, and bounded strategy set is shown by [Rosen, 1965]. While the strategy set of the defender is convex, closed, and bounded by definition, the attacker is allowed to allocate any arbitrarily large amount for the attack of the targets; however, in effect, investing any amount greater than the value of the target will not make up for the disutility of the investment and cause the attacker's expected utility to be negative. For this reason, without any loss of generality, the attacker's strategy set can be restricted to the bounded, closed, and convex set of x such that $x_n \in [0, v_n]$ for every $n = 1, \dots, N$. Consequently, the existence result of Rosen applies. In this section, we reach the same conclusion by explicitly finding the Nash equilibrium.

Theorem 3. *For the two-target attacker-defender game, a Nash equilibrium (x^*, y^*) exists and is as follows.*

(i) If $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$ and $v_1 k_A \leq k_D B + c_1$, then the set of Nash equilibria consists of any (x^*, y^*) such that $x^* = 0$, $y_1^* \geq (v_1 k_A - c_1)^+/k_D$, $y_2^* \geq (v_2 k_A - c_2)^+/k_D$, and $y_1^* + y_2^* \leq B$.

(ii) If both $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$ and $v_1 k_A > k_D B + c_1$ hold or both $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A \leq c_2$ hold, then the unique Nash equilibrium (x^*, y^*) has $x_2^* = y_2^* = 0$, $y_1^* = B$, and

$$x_1^* = \frac{\sqrt{v_1 k_A (k_D B + c_1)} - (k_D B + c_1)}{k_A}.$$

(iii) If $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A > c_2$, then there exists a unique Nash equilibrium (x^*, y^*) , which has $x_1^* > 0$, $x_2^* > 0$, and $y_1^* + y_2^* = B$. There are three possible cases:

Case 1: If $\frac{w_2}{v_2}(\sqrt{\frac{v_2 k_A}{c_2}} - 1) \leq \frac{w_1}{v_1}(\sqrt{\frac{v_1 k_A}{k_D B + c_1}} - 1)$, then $y_1^* = B$, $y_2^* = 0$,

$$\begin{aligned} x_1^* &= \frac{\sqrt{v_1 k_A (k_D B + c_1)} - (k_D B + c_1)}{k_A}, \\ x_2^* &= \frac{\sqrt{v_2 k_A c_2} - c_2}{k_A}. \end{aligned}$$

Case 2: If $\frac{w_1}{v_1}(\sqrt{\frac{v_1 k_A}{c_1}} - 1) \leq \frac{w_2}{v_2}(\sqrt{\frac{v_2 k_A}{k_D B + c_2}} - 1)$, then $y_1^* = 0$, $y_2^* = B$,

$$\begin{aligned} x_1^* &= \frac{\sqrt{v_1 k_A c_1} - c_1}{k_A}, \\ x_2^* &= \frac{\sqrt{v_2 k_A (k_D B + c_2)} - (k_D B + c_2)}{k_A}. \end{aligned}$$

Case 3: If $\frac{w_2}{v_2}(\sqrt{\frac{v_2 k_A}{c_2}} - 1) > \frac{w_1}{v_1}(\sqrt{\frac{v_1 k_A}{k_D B + c_1}} - 1)$ and $\frac{w_1}{v_1}(\sqrt{\frac{v_1 k_A}{c_1}} - 1) > \frac{w_2}{v_2}(\sqrt{\frac{v_2 k_A}{k_D B + c_2}} - 1)$, then $y_1^* = B - y_2^*$, $x_1^* > 0$, $x_2^* > 0$, and $y_2^* \in (0, B)$ uniquely satisfy:

$$\begin{aligned} v_1 k_A [k_D (B - y_2) + c_1] &= [k_A x_1 + k_D (B - y_2) + c_1]^2, \\ v_2 k_A (k_D y_2 + c_2) &= (k_A x_2 + k_D y_2 + c_2)^2, \\ (w_2 x_2) [k_A x_1 + k_D (B - y_2) + c_1]^2 &= (w_1 x_1) (k_A x_2 + k_D y_2 + c_2)^2. \end{aligned}$$

Proof. Starting with (i), by Proposition 4, the defender's best response to $x^* = 0$ is any $y^* \geq 0$ such that $y_1^* + y_2^* \leq B$, and by Proposition 3, the attacker's best

response to y^* such that $y_1^* \geq (v_1 k_A - c_1)^+ / k_D$, $y_2^* \geq (v_2 k_A - c_2)^+ / k_D$ is $x^* = 0$. Hence the strategy profiles described in (i) form a Nash equilibrium. The conditions $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$ and $v_2 k_A - c_2 \leq v_1 k_A - c_1 \leq k_D B + c_1$ guarantee that such strategy profiles exist, since adding the two inequalities and rearranging gives $k_D(y_1^* + y_2^*) \geq (v_1 k_A - c_1)^+ + (v_2 k_A - c_2)^+ \geq (v_1 + v_2)k_A - (c_1 + c_2)$. Conversely, the best response y^* to any $x^* \neq 0$ satisfies $y_1^* + y_2^*$ as shown in Proposition 4. For $x_1^*, x_2^* \neq 0$, the conditions $v_1 k_A > k_D y_1^* + c_1$, $v_2 k_A > k_D y_2^* + c_2$, and $y_1^* + y_2^*$ contradict $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$. For $x_1^* > 0 = x_2^*$ ($x_1^* = 0 < x_2^*$), $y_1^* = B$ and $v_1 k_A > k_D y_1^* + c_1$ ($y_2^* = B$ and $v_2 k_A > k_D y_2^* + c_2$) contradict $v_1 k_A \leq k_D B + c_1$. Hence under these conditions $x^* = 0$ must hold.

If $v_1 k_A > k_D B + c_1$, then $x_1^* = 0$ is not a best response to $y_1 \leq B$ so $x_1^* > 0$. This condition paired with $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$ implies $v_2 k_A \leq c_2$, so $v_2 k_A \leq k_D y_2 + c_2$ for any $y_2 \geq 0$, so the attacker's best response at target 2 must be $x_2^* = 0$. For $x_1^* > 0$ and $x_2^* = 0$, the best response of the defender is to allocate all to target 1 because (5.18) is negative, so (5.16) equals zero, and (5.17) gives $y_1^* = B$. Similarly, when $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A \leq c_2$, the inequality $v_1 k_A - c_1 > k_D B + c_1$ must hold, leading to the same Nash equilibrium.

Next suppose $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A > c_2$. The first condition eliminates the possibility of having $x^* = 0$ in any Nash equilibrium. Accordingly, there remain three possibilities. The attacker may invest positively only at target 1, so $x_1^* > 0 = x_2^*$, which makes (5.18) negative, so by (5.16)-(5.17), $y_1^* = B$ and $y_2^* = 0$. On the other hand, the attacker's best response at target 2 given $y_2^* = 0$ must be positive because of the condition $v_2 k_A > c_2$, so we eliminate this possibility. The second possibility is that the attacker invests a positive amount only at target 2, so $x_2^* > 0 = x_1^*$, which makes (5.18)

$$\tilde{y}_2 = \frac{\sqrt{w_2 x_2}(k_D B + c_1)}{k_D \sqrt{w_2 x_2}} > B,$$

so by (5.16)-(5.17), $y_1^* = 0$ and $y_2^* = B$. However, $v_2 k_A > c_2$ and the ordering of the targets imply $v_1 k_A > c_1$, contradicting $x_1^* = 0$ is best response to $y_1^* = 0$. Hence under the conditions of (iii), if there exists a Nash equilibrium, it must have $x_1^* > 0$ and $x_2^* > 0$. According to Proposition 4, there are three possible cases.

Case 1: $y^* = (B, 0)$. The positivity of the attacker's investments requires $v_1 k_A > k_D B + c_1$ and $v_2 k_A > c_2$. The corresponding x^* is found by using (5.6). In turn, $y_2^* = 0$ is a best response to x^* if and only if $\tilde{y}_2 \leq 0$, or equivalently,

$$\frac{\sqrt{w_2 x_2}}{(k_A x_2 + c_2)^2} \leq \frac{\sqrt{w_1 x_1}}{(k_A x_1 + k_D B + c_1)^2}.$$

Substituting $y^* = (B, 0)$ and (5.6) in this condition reduces it to the condition in Case 1 of (iii). This condition also implies that $v_1 k_A > k_D B + c_1$, so the given x_1^* is the attacker's best response to $y_1^* = B$.

Case 2: $y^* = (0, B)$. The positivity of the attacker's investments requires $v_1 k_A > c_1$ and $v_2 k_A > k_D B + c_2$. The corresponding x^* is found by using (5.6). In turn, $y_2^* = B$ is a best response to x^* if and only if $\tilde{y}_2 \geq B$, or equivalently,

$$\frac{\sqrt{w_1 x_1}}{(k_A x_1 + c_1)^2} \leq \frac{\sqrt{w_2 x_2}}{(k_A x_2 + k_D B + c_2)^2}.$$

Substituting $y^* = (0, B)$ and (5.6) in this condition reduces it to the condition in Case 2 of (iii). The condition $v_2 k_A > c_2$ of (iii) and the ordering of the targets assures $v_1 k_A > c_1$, which together with the condition of Case 3 implies $v_2 k_A > k_D B + c_2$, justifying the given x_2^* as the attacker's best response to $y_2^* = B$.

Case 3: $y^* = (B - \tilde{y}_2, \tilde{y}_2)$. The positivity of the attacker's investments requires $v_1 k_A > k_D(B - \tilde{y}_2) + c_1$ and $v_2 k_A > k_D \tilde{y}_2 + c_2$. The corresponding x^* is found by using (5.6). The best response to x^* is $\tilde{y}_2 \in (0, B)$ if and only if $0 < \tilde{y}_2 < B$, or equivalently, the conditions in Cases 1 and 2 fail. The first two equations guarantee that x^* is the best response of the attacker to y^* and the last one guarantees that the defender chooses the best defense-budget allocation given the attacker's strategy x^* . The conditions of Case 3 along with the first two equations imply that the solution y_2^* of the third equation is in $(0, B)$. By replacing x_1 and x_2 from the first two equations, the third equation can be reorganized as follows:

$$f(y_2) \equiv \frac{w_2}{v_2} \left[\sqrt{\frac{v_2 k_A}{k_D y_2 + c_2}} - 1 \right] - \frac{w_1}{v_1} \left[\sqrt{\frac{v_1 k_A}{k_D(B - y_2) + c_1}} - 1 \right] = 0. \quad (5.20)$$

The first condition in Case 3 guarantees $f(0) > 0$ while the second condition guarantees $f(B) < 0$. Since $f(y_2)$ is a continuous function of y_2 , by the Intermediate Value Theorem, there must be a solution $y_2 \in (0, B)$ that satisfies (5.20). The solution is

unique because $f(y_2)$ is a strictly decreasing function of $y_2 \in (0, B)$. The strategy profile $y_2^* = y_2$, $y_1^* = B - y_2^*$ combined with the corresponding x_n^* values gives the Nash equilibrium. $\square$

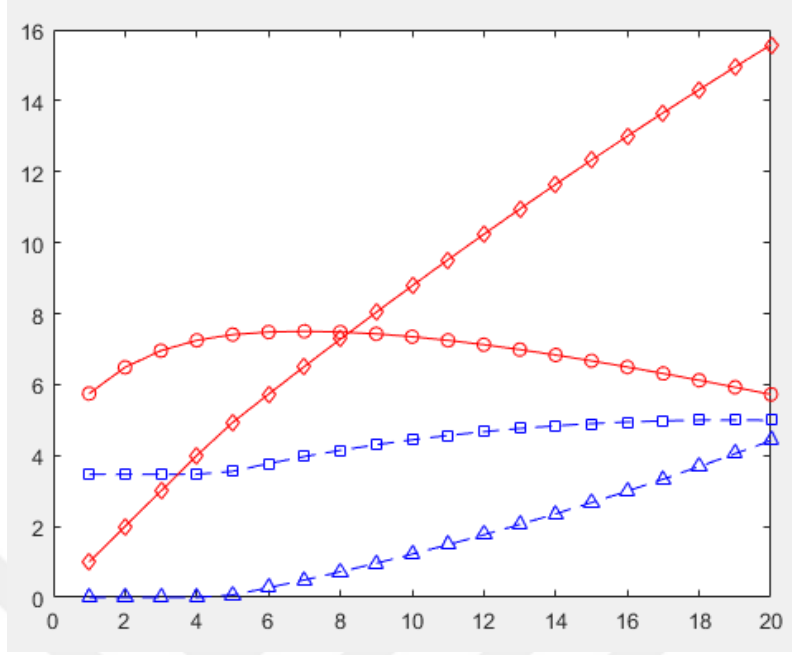
5.4 Numerical Experiments - Simultaneous-Move Game

This section illustrates the results of the two-target simultaneous-move game with numerical examples. The baseline values for our experiment are $v_1 = 30$, $v_2 = 20$, $w_1 = 80$, $w_2 = 20$, and $c_1 = c_2 = k_A = k_D = 2$. The inequality $v_1 > v_2$ combined with $k_A = k_D$ $c_1 = c_2$ ensures that the ordering of the targets is consistent with the analysis. Table 5.1 lists the equilibrium investments and expected utilities for the budget values between 1 and 20. Figure 5.1 and Figure 5.2 show the changes in the Nash equilibrium investments and expected utilities as functions of the budget varying on the interval $[1, 20]$.

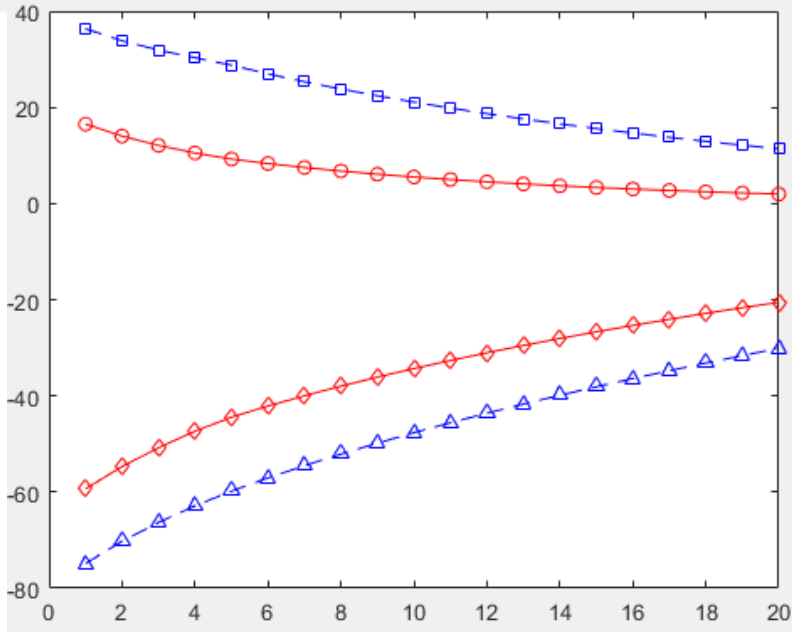
In Figure 5.1, we observe that the defender always invests more at target 1 as compared to target 2, whose value for the defender is smaller, at all budget values. When the defense budget is relatively small, the defender allocates all of it to target 1. Increases in the budget allows the defender to spare some of it for target 2. The defender's investment at each target increases as the budget increases. The attacker's investments are also increasing in the defense budget in a neighborhood of the origin, but after a threshold, they start to decrease. This trend is most clear for target 1, where the attacker invests most because it is more valuable to the attacker too. Figure 5.2 depicts the monotonic effect of budget changes on the equilibrium utilities, the defender's utility always improves and the attacker's utility decreases with an increase in the defense budget.

In our next experiment, we reverse the defender's valuations for the two targets, so we let $w_1 = 20$ and $w_2 = 80$, leaving all other parameters as before. Now target 1 is more valuable for the attacker, but less valuable for the defender. Figure 5.3 and Figure 5.4 show the equilibrium investments and utilities for this experiment. The results are listed in Table 5.2.

In the instance where the attacker and the defender value the two target in the

Figure 5.1: Equilibrium Investments vs. B

$\circ x_1^*, \diamond y_1^*, \square x_2^*, \triangle y_2^*,$

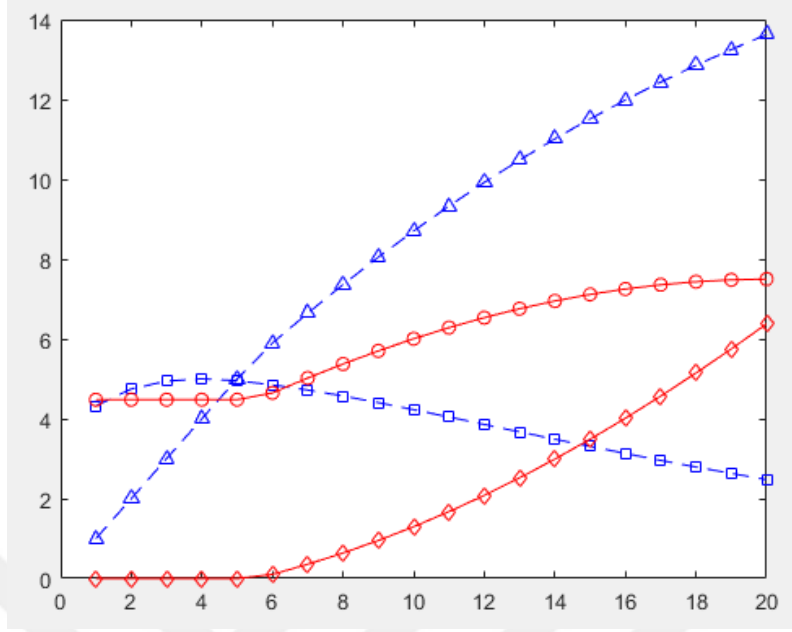
Figure 5.2: Equilibrium Utilities vs. B

$\circ U_A^1(B), \diamond U_D^1(B), \square U_A^2(B), \triangle U_D^2(B)$

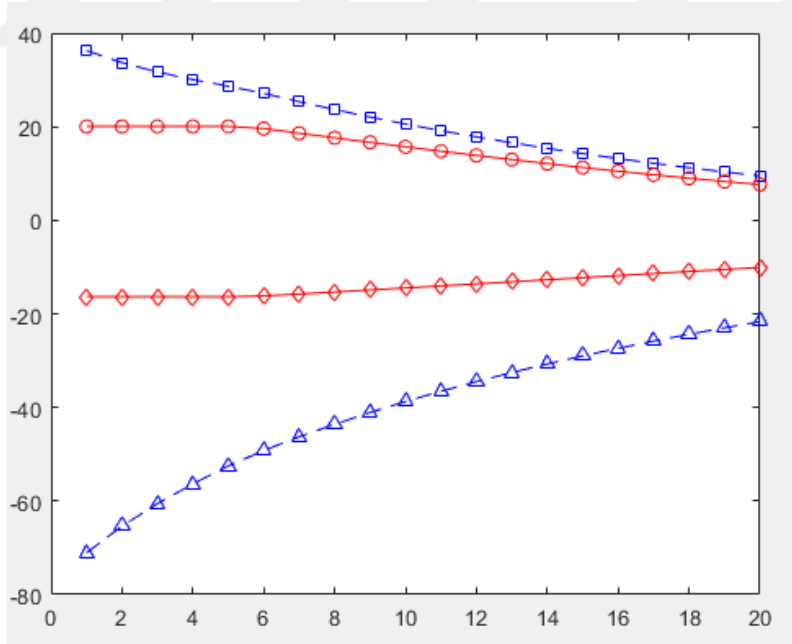
Table 5.1: Equilibrium Investments and Utilities in the Simultaneous-Move Game
 $[v_1 = 30, v_2 = 20, w_1 = 80, w_2 = 20, c_1 = c_2 = k_A = k_D = 2]$

B	y_2^*	x_2^*	y_1^*	x_1^*	U_D^2	U_A^2	U_D^1	U_A^1
1	0	3.4721	1.0000	5.7460	-74.8720	36.3277	-59.3441	16.5081
2	0	3.4721	2.0000	6.4868	-70.2296	33.8460	-54.7018	14.0263
3	0	3.4721	3.0000	6.9545	-66.3160	31.9108	-50.7881	12.0911
4	0	3.4721	4.0000	7.2474	-62.8680	30.3248	-47.3401	10.5051
5	0.0669	3.5524	4.9331	7.4083	-59.8036	28.7690	-44.4229	9.2503
6	0.2725	3.7723	5.7275	7.4790	-57.0711	26.9751	-42.1159	8.3144
7	0.4904	3.9693	6.5096	7.5000	-54.5147	25.3317	-39.9744	7.4904
8	0.7206	4.1455	7.2794	7.4807	-52.1068	23.8144	-37.9729	6.7591
9	0.9629	4.3027	8.0371	7.4284	-49.8264	22.4048	-36.0921	6.1061
10	1.2175	4.4421	8.7825	7.3486	-47.6574	21.0888	-34.3170	5.5203
11	1.4842	4.5645	9.5158	7.2458	-45.5870	19.8552	-32.6357	4.9926
12	1.7629	4.6707	10.2371	7.1235	-43.6047	18.6948	-31.0384	4.5158
13	2.0537	4.7613	10.9463	6.9849	-41.7019	17.6002	-29.5170	4.0840
14	2.3565	4.8368	11.6435	6.8323	-39.8714	16.5652	-28.0647	3.6920
15	2.6713	4.8976	12.3287	6.6678	-38.1070	15.5847	-26.6759	3.3356
16	2.9979	4.9440	13.0021	6.4933	-36.4036	14.6543	-25.3456	3.0112
17	3.3364	4.9764	13.6636	6.3104	-34.7566	13.7701	-24.0694	2.7156
18	3.6866	4.9949	14.3134	6.1203	-33.1621	12.9288	-22.8436	2.4461
19	4.0485	4.9999	14.9515	5.9242	-31.6165	12.1277	-21.6649	2.2002
20	4.4220	4.9915	15.5780	5.7231	-30.1169	11.3641	-20.5303	1.9758

opposite order, the order of their equilibrium investments is also reversed for budget values above a threshold. The defender now invests more in the protection of the more valuable target 2 whereas the attacker concentrates the attack efforts on target 1, which is the more valuable target for the attacker. We observe that when budget is scarce, the attacker's investment can be greater at target 2 than target 1. This

Figure 5.3: Equilibrium Investments vs. B

$\circ x_1^*$, $\diamond y_1^*$, $\square x_2^*$, $\triangle y_2^*$,

Figure 5.4: Equilibrium Utilities vs. B

$\circ U_A^1(B)$, $\diamond U_D^1(B)$, $\square U_A^2(B)$, $\triangle U_D^2(B)$

Table 5.2: Equilibrium Investments and Utilities in the Simultaneous-Move Game
 $[v_1 = 30, v_2 = 20, w_1 = 20, w_2 = 80, c_1 = c_2 = k_A = k_D = 2]$

B	y_2^*	x_2^*	y_1^*	x_1^*	U_D^2	U_A^2	U_D^1	U_A^1
1	1.0000	4.3246	0	4.4772	-71.0503	36.2342	-16.3485	20.0455
2	2.0000	4.7460	0	4.4772	-65.3646	33.6806	-16.3485	20.0455
3	3.0000	4.9443	0	4.4772	-60.5714	31.6849	-16.3485	20.0455
4	4.0000	5.0000	0	4.4772	-56.3485	30.0455	-16.3485	20.0455
5	5.0000	4.9545	0	4.4772	-52.5307	28.6594	-16.3485	20.0455
6	5.8977	4.8477	0.1023	4.6483	-49.1848	27.1353	-16.1662	19.6010
7	6.6447	4.7203	0.3553	5.0212	-46.2889	25.3345	-15.7490	18.6024
8	7.3618	4.5702	0.6382	5.3722	-43.5985	23.6492	-15.3264	17.6173
9	8.0481	4.4041	0.9519	5.7003	-41.0896	22.0650	-14.8985	16.6475
10	8.7032	4.2275	1.2968	6.0041	-38.7434	20.5716	-14.4661	15.6951
11	9.3267	4.0446	1.6733	6.2821	-36.5445	19.1611	-14.0298	14.7626
12	9.9189	3.8587	2.0811	6.5331	-34.4800	17.8275	-13.5905	13.8527
13	10.4801	3.6725	2.5199	6.7561	-32.5388	16.5664	-13.1493	12.9678
14	11.0110	3.4880	2.9890	6.9504	-30.7109	15.3737	-12.7071	12.1103
15	11.5127	3.3067	3.4873	7.1153	-28.9874	14.2463	-12.2649	11.2821
16	11.9861	3.1298	4.0139	7.2506	-27.3602	13.1813	-11.8237	10.4849
17	12.4325	2.9580	4.5675	7.3563	-25.8219	12.1760	-11.384	9.7199
18	12.8533	2.7920	5.1467	7.4327	-24.3658	11.2279	-10.9471	8.9879
19	13.2500	2.6319	5.7500	7.4802	-22.9854	10.3346	-10.5132	8.2895
20	13.6239	2.4781	6.3761	7.4995	-21.6750	9.4939	-10.0829	7.6249

can be explained by the fact that for these budget values, the defender allocates a minimal amount to protect target 1, so the vulnerability at this target is already high and the attacker may enjoy a sufficiently large probability of damage without investing a lot. However, increasing the defense budget leads the attacker to invest more at target 1 rather than target 2. As in the previous experiment, the defender's

equilibrium investments are increasing in the available budget while the attacker's investments are not monotonic.

We next perform some experiments to observe the effects of changes in parameters other than the budget. Our baseline values in these experiments are $B = 10$, $v_1 = 30$, $v_2 = 20$, $w_1 = w_2 = 40$, and $c_1 = c_2 = k_A = k_D = 2$. We change one model parameter at a time and keep the others constant.

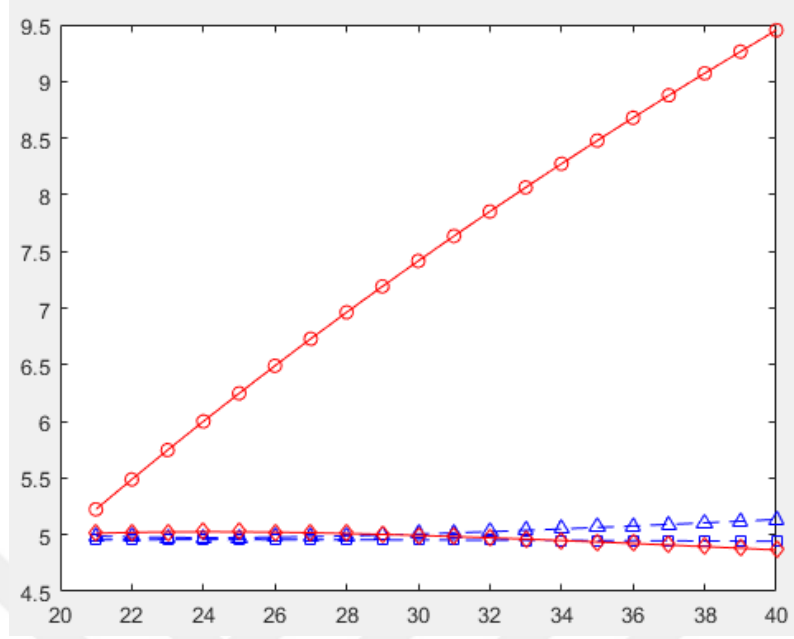
Figure 5.5 and Figure 5.6 plot the changes in equilibrium strategies and utilities respectively varying the attacker's valuation v_1 of target 1. As one would expect, the attacker's investment at target 1 increases as v_1 increases. However, the defender's investment at target 1 decreases slowly whereas the investment to target 2 increases slowly. This result tells us that the attacker's target valuation of target 1 does not affect the defender's investment strategy much.

Figure 5.7 and Figure 5.8 demonstrate respectively how the equilibrium strategies and utilities change varying the attacker's valuation of target 2. The attacker's investment at target 2 increases as v_2 increases as expected. The defender's investment to target 2 also increases for small values of v_2 while the investment to target 1 decreases. However, after a certain level, the increase in the attacker's valuation of target 2 barely effects the defender's investment strategy.

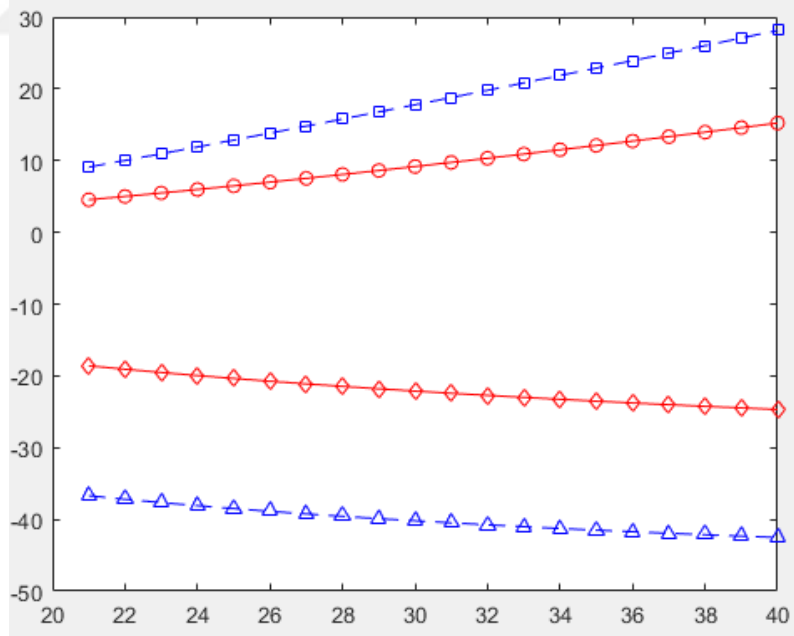
Figure 5.9 and Figure 5.10 represent the changes in equilibrium strategies and utilities respectively as functions of the defender's valuation w_1 of target 1. When w_1 is small, the defender invests a very small amount if not zero at target 1. As w_1 increases, the defender's investment at target 1 also increases.

Figure 5.11 and Figure 5.12 show the sensitivity of equilibrium strategies and utilities to changes in the defender's valuation w_2 of target 2. Similarly to an increase in w_1 , an increase in w_2 increases the defender's incentive to invest at this target. Figures 5.9-5.12 show that the defender's target valuation has a small impact on the attacker's strategy.

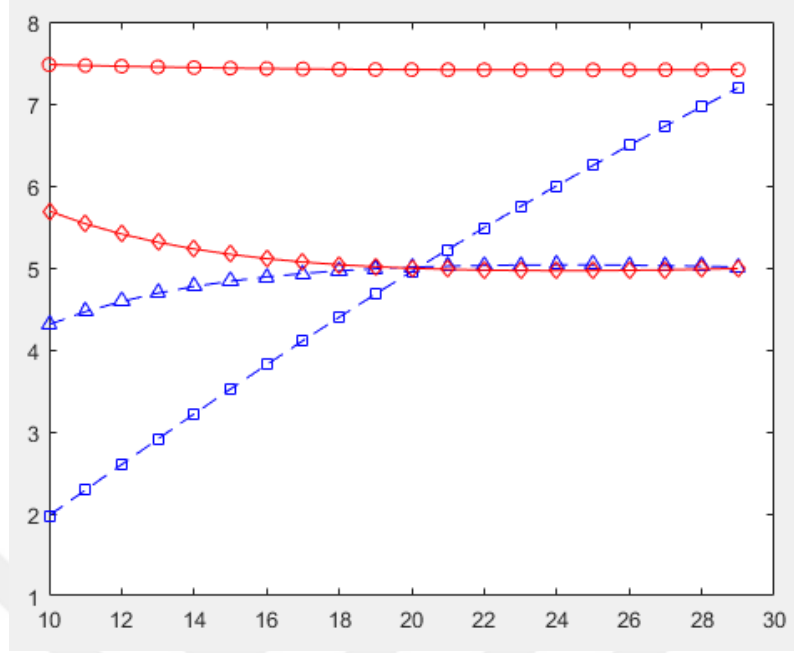
Figure 5.13 and Figure 5.14 show the changes in equilibrium strategies and utilities respectively as functions of the inherent defense level c_1 at target 1. An increase in c_1 allows the defender to spare more resources for target 2. Figure 5.14 suggests that the defender enjoys a substantial utility increase while the attacker suffers.

Figure 5.5: Equilibrium Investments vs. v_1

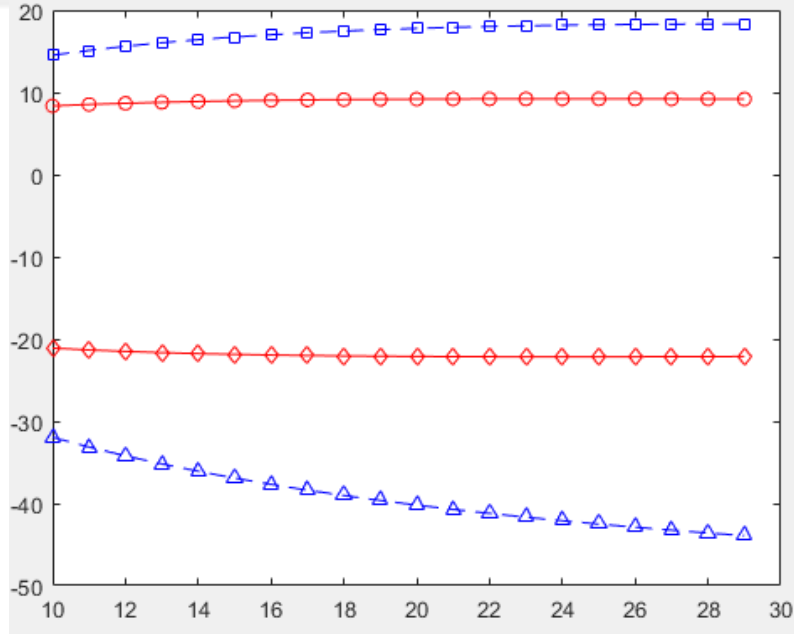
$\circ x_1^*$, $\diamond y_1^*$, $\square x_2^*$, $\triangle y_2^*$

Figure 5.6: Equilibrium Utilities vs. v_1

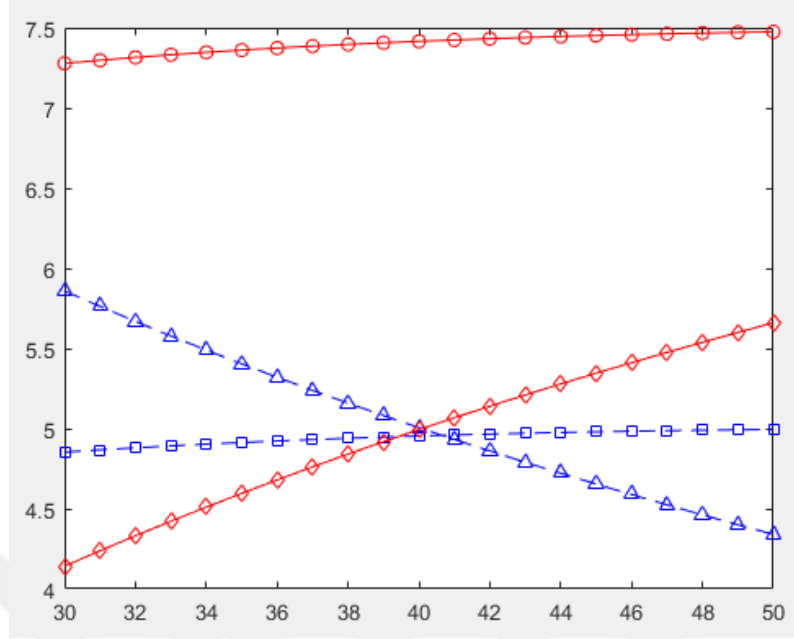
$\circ U_A^1(B)$, $\diamond U_D^1(B)$, $\square U_A^2(B)$, $\triangle U_D^2(B)$,

Figure 5.7: Equilibrium Investments vs. v_2

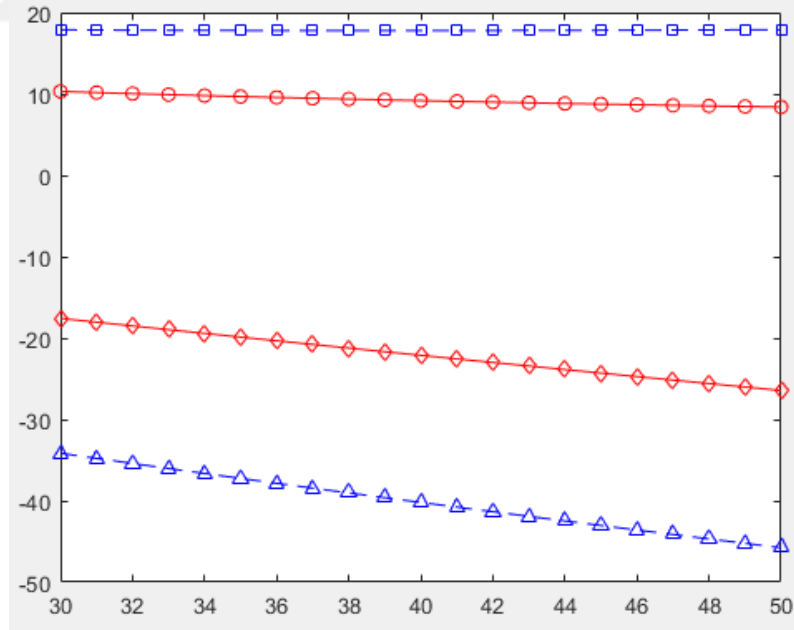
$\circ x_1^*, \diamond y_1^*, \square x_2^*, \triangle y_2^*$

Figure 5.8: Equilibrium Utilities vs. v_2

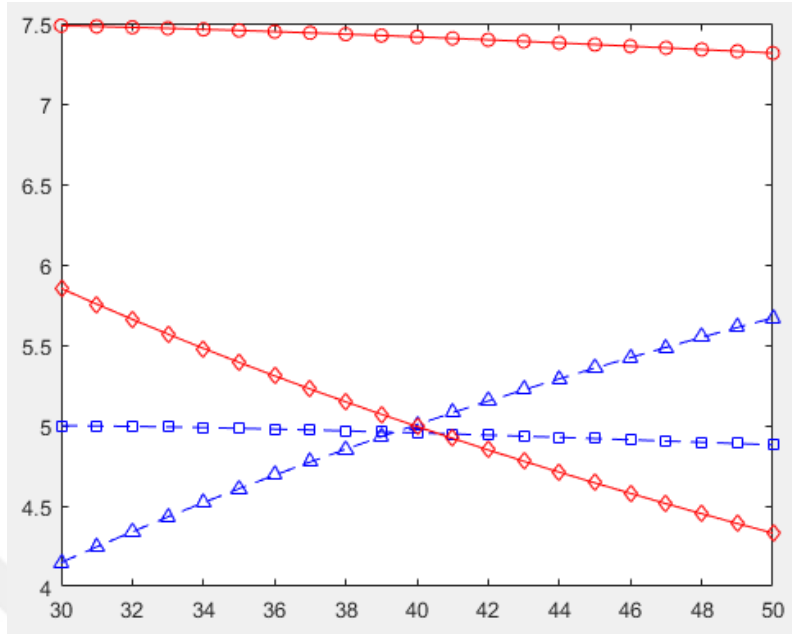
$\circ U_A^1(B), \diamond U_D^1(B), \square U_A^2(B), \triangle U_D^2(B)$

Figure 5.9: Equilibrium Investments vs. w_1

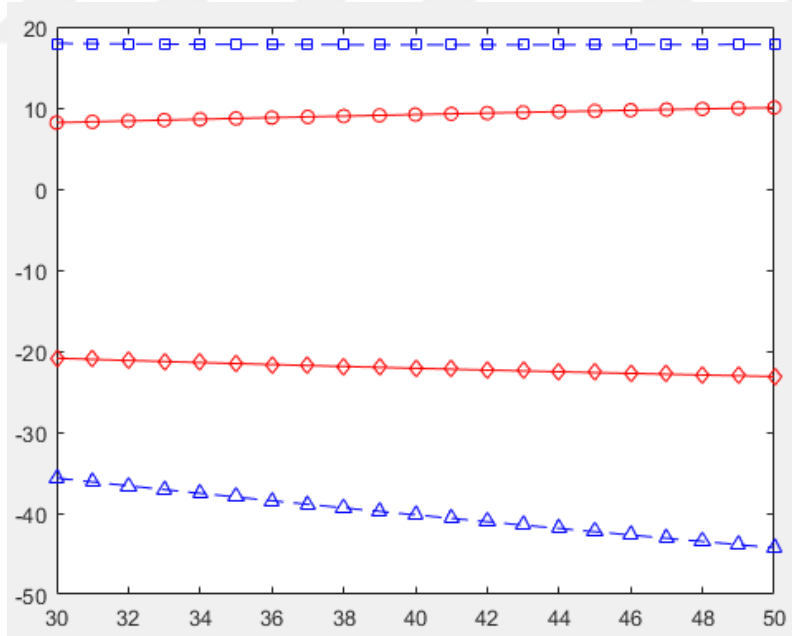
○ x_1^* , ◇ y_1^* , □ x_2^* , △ y_2^*

Figure 5.10: Equilibrium Utilities vs. w_1

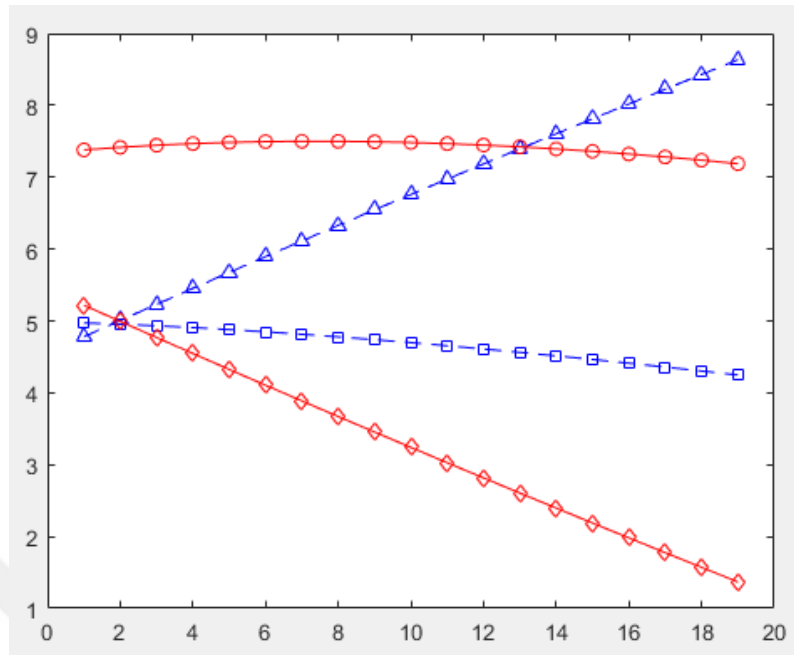
○ $U_A^1(B)$, ◇ $U_D^1(B)$, □ $U_A^2(B)$, △ $U_D^2(B)$

Figure 5.11: Equilibrium Investments vs. w_2

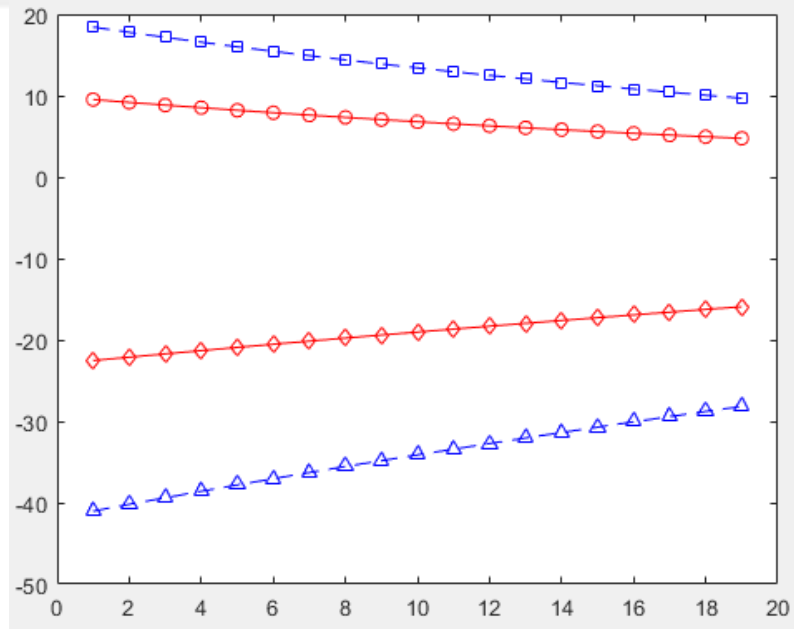
$\circ x_1^*$, $\diamond y_1^*$, $\square x_2^*$, $\triangle y_2^*$

Figure 5.12: Equilibrium Utilities vs. w_2

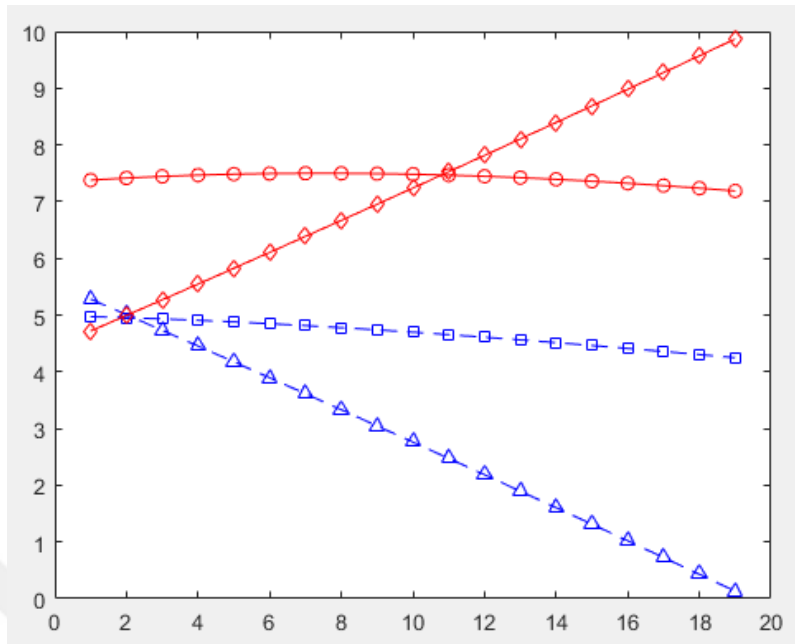
$\circ U_A^1(B)$, $\diamond U_D^1(B)$, $\square U_A^2(B)$, $\triangle U_D^2(B)$

Figure 5.13: Equilibrium Investments vs. c_1

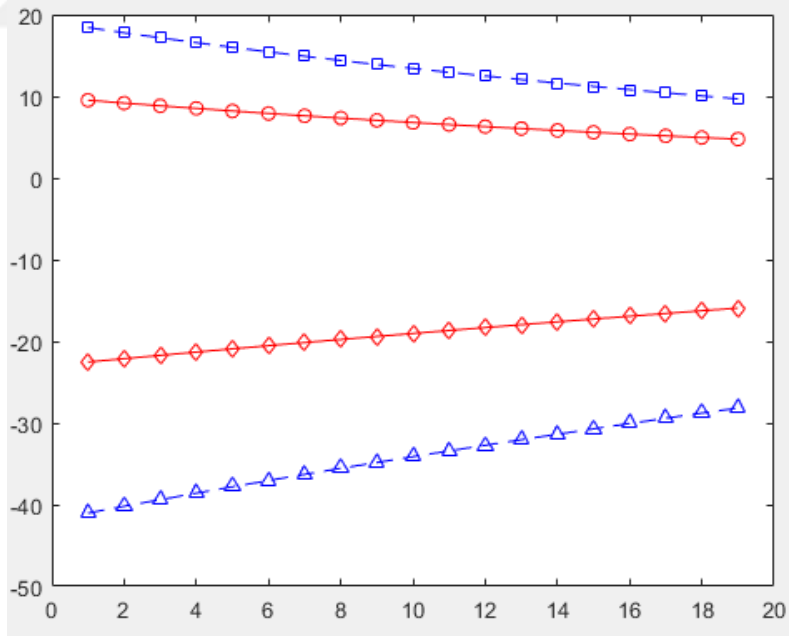
$\circ x_1^*, \diamond y_1^*, \square x_2^*, \triangle y_2^*$

Figure 5.14: Equilibrium Utilities vs. c_1

$\circ U_A^1(B), \diamond U_D^1(B), \square U_A^2(B), \triangle U_D^2(B)$

Figure 5.15: Equilibrium Investments vs. c_2

$\circ x_1^*, \diamond y_1^*, \square x_2^*, \triangle y_2^*$

Figure 5.16: Equilibrium Utilities vs. c_2

$\circ U_A^1(B), \diamond U_D^1(B), \square U_A^2(B), \triangle U_D^2(B)$

In Figure 5.15 and Figure 5.16, we observe that the effects of changes in c_2 are parallel to those of changes in c_1 . As c_2 increases, the defender allocates the majority of its resources to target 1, and the attacker invests less at target 2. The defender's expected utility continuously increases while the attacker's expected utility decreases.

We conclude this subsection with a few observations. Our numerical experiments suggest that the attacker's strategy is affected most by the target valuations v_1 and v_2 , and the defense budget B , the effects of other changes on the attacker's strategy seem to be minimal. In general, the defender's strategy appears to be more sensitive to changes than the attacker's strategy. This may be due to the somewhat myopic nature of the attacker's decisions. Both players tend to concentrate their efforts at the targets they value higher, but this may not hold depending on the particular values model parameters, as demonstrated by the following figure, where $w_1 = 20$, and $w_2 = 80$.

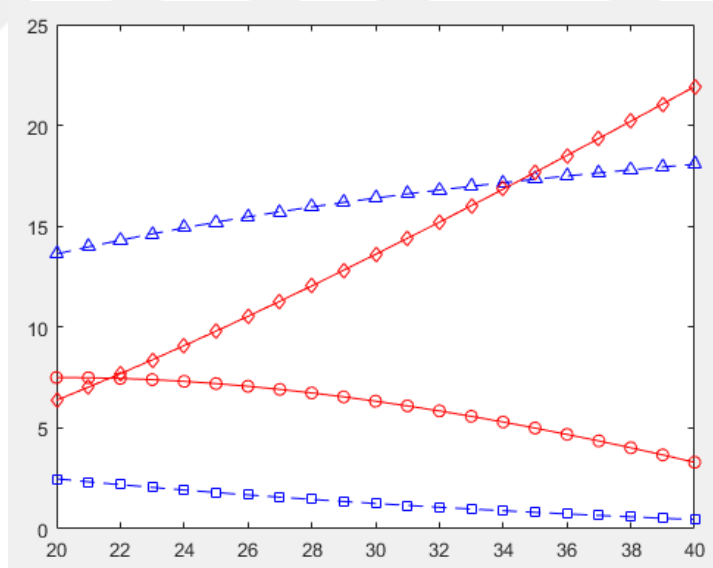


Figure 5.17: Equilibrium Investments vs. B

$\circ x_1^*, \diamond y_1^*, \square x_2^*, \triangle y_2^*$

5.5 Subgame-Perfect Nash Equilibrium in the Sequential-Move Game

In the sequential-move version of the two-target attacker-defender game, we assume that the defender moves first and allocated the limited defense budget among the two targets. After observing the defender's strategy, the attacker decides on the attack strategy. The resulting subgame-perfect Nash equilibrium can be found with a backward argument. For any given defense strategy $y^2 = (y_1, y_2)$, the attacker will allocate

$$\hat{x}_n(y_n) = \begin{cases} \frac{\sqrt{v_n k_A (k_D y_n + c_n)} - (k_D y_n + c_n)}{k_A} & \text{if } v_n k_A > k_D y_n + c_n \\ 0 & \text{if } v_n k_A \leq k_D y_n + c_n \end{cases} \quad (5.21)$$

to target $n = 1, 2$. Knowing that the attacker is rational and consequently will form the attack strategy according to (5.21), the defender tries to optimize the expected utility through the following optimization problem:

$$\max_{y_1, y_2} J_D^2(\hat{x}^2(y^2), y^2) = u_1^D(\hat{x}_1(y_1), y_1) + u_2^D(\hat{x}_2(y_2), y_2) \quad (5.22)$$

$$\text{subject to } 0 \leq y_1 + y_2 \leq B, \quad (5.23)$$

$$y_1, y_2 \geq 0. \quad (5.24)$$

Theorem 4. *The subgame-perfect Nash equilibrium of the two-target attacker-defender game where the attacker decides after observing the investment of the defender is as follows.*

- (i) *If $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$ and $v_1 k_A \leq k_D B + c_1$, then in any subgame-perfect Nash equilibrium, the defender selects a strategy satisfying $\bar{y}_1 \geq (v_1 k_A - c_1)^+ / k_D$, $\bar{y}_2 \geq (v_2 k_A - c_2)^+ / k_D$, and $\bar{y}_1 + \bar{y}_2 \leq B$, while the attacker chooses not to invest at any target, i.e., $\bar{x} = 0$.*
- (ii) *If both $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$ and $v_1 k_A > k_D B + c_1$ hold or both $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A \leq c_2$ hold, then in any subgame-perfect Nash equilibrium, the attacker invests a positive amount only in target 1 and therefore the defender allocates all the budget to the defense of target 1. The attacker invests $\bar{x}_1 = [\sqrt{v_1 k_A (k_D B + c_1)} - (k_D B + c_1)] / k_A$ and the defender sets $\bar{y}_1 = B$.*

(iii) If $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A > c_2$, then in the unique subgame-perfect Nash equilibrium, $\bar{y}_1 = B - \bar{y}_2$,

$$\bar{y}_2 = \begin{cases} [B - (v_1 k_A - c_1)/k_D]^+ & \text{if } \tilde{y}_2 \leq [B - (v_1 k_A - c_1)/k_D]^+ \\ \min\{B, (v_2 k_A - c_2)/k_D\} & \text{if } \tilde{y}_2 \geq \min\{B, (v_2 k_A - c_2)/k_D\} \\ \tilde{y}_2 & \text{otherwise,} \end{cases}$$

where

$$\tilde{y}_2 \equiv \frac{w_2^2 v_1 (k_D B + c_1) - w_1^2 v_2 c_2}{k_D (w_1^2 v_2 + w_2^2 v_1)}, \quad (5.25)$$

and

$$\begin{aligned} \bar{x}_1 &= \frac{\sqrt{v_1 k_A [k_D (B - \bar{y}_2) + c_1]} - (k_D (B - \bar{y}_2) + c_1)}{k_A}, \\ \bar{x}_2 &= \frac{\sqrt{v_2 k_A (k_D \bar{y}_2 + c_2)} - (k_D \bar{y}_2 + c_2)}{k_A}. \end{aligned}$$

Proof. If $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$ and $v_1 k_A \leq k_D B + c_1$, the defender can obtain the maximum possible expected utility by making the attacker invest zero, $\bar{x}(y) = 0$, that is by choosing a defense strategy such that $\bar{y}_1 \geq (v_1 k_A - c_1)^+/k_D$, $\bar{y}_2 \geq (v_2 k_A - c_2)^+/k_D$, and $\bar{y}_1 + \bar{y}_2 \leq B$.

If $v_1 k_A > k_D B + c_1$, then $\bar{x}_1(y_1) > 0$ for any $y_1 \leq B$. Also if $(v_1 + v_2)k_A \leq k_D B + c_1 + c_2$, then $v_2 k_A \leq c_2$, so $v_2 k_A \leq k_D y_2 + c_2$ for any $y_2 \geq 0$, so the attacker's best response at target 2 must be $\bar{x}_2(y_2) = 0$. Knowing the attacker will allocate $\bar{x}_1(y_1) > 0$ to target 1 and $\bar{x}_2(y_2) = 0$ to target 2, the problem reduces to a single-target problem with $v_1 k_A > k_D B + c_1$, which has the unique subgame-perfect equilibrium given in (ii). When $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A \leq c_2$, $v_1 k_A > k_D B + c_1$, so the unique subgame-perfect equilibrium is the same.

Now suppose $(v_1 + v_2)k_A > k_D B + c_1 + c_2$ and $v_2 k_A > c_2$. Due to the first condition, the defender cannot make the attacker invest zero at both targets. If $v_2 k_A > k_D B + c_2$, then $v_1 k_A > k_D B + c_1$, so for any feasible y , $\bar{x}_1(y_1) > 0$ and $\bar{x}_2(y_2) > 0$, and the problem of the defender is to maximize

$$V_D^2(y_2) \equiv w_1 \left(\sqrt{\frac{k_D (B - y_2) + c_1}{v_1 k_A}} - 1 \right) + w_2 \left(\sqrt{\frac{k_D y_2 + c_2}{v_2 k_A}} - 1 \right), \quad (5.26)$$

over $0 \leq y_2 \leq B$.

If $v_2 k_A - c_2 \leq k_D B < v_1 k_A - c_1$, then for any feasible y , $\bar{x}_1(y_1) > 0$, but $\bar{x}_2(y_2) = 0$ for $y_2 \geq (v_2 k_A - c_2)/k_D$. To allocate most to target 1 (for which the probability of damage is positive) without creating an incentive for the attacker to invest positively at target 2, the defender would choose $y_2 = (v_2 k_A - c_2)/k_D$ among $y_2 \geq (v_2 k_A - c_2)/k_D$. This reduces the defender's problem to maximizing (5.26) over $0 \leq y_2 \leq (v_2 k_A - c_2)/k_D$. By the argument above, the maximizer of (5.26) is zero if $\tilde{y}_2 \leq 0$ (equivalently, $w_2^2 v_1 (k_D B + c_1) \leq w_1^2 v_2 c_2$), $y_2 = (v_2 k_A - c_2)/k_D$ if $\tilde{y}_2 \geq (v_2 k_A - c_2)/k_D$, and $\bar{y}_2 = \tilde{y}_2 \in (0, (v_2 k_A - c_2)/k_D)$ otherwise.

If $v_1 k_A \leq k_D B + c_1$, then for $y_2 \geq (v_2 k_A - c_2)/k_D$, the attacker invests $\bar{x}_2(y_2) = 0$ at target 2. Because $(v_1 + v_2)k_A > k_D B + c_1 + c_2$, $v_1 k_A > k_D(B - y_2) + c_1$, so $\bar{x}_1(y_1) > 0$. By the argument in the previous paragraph, the defender's problem becomes maximizing (5.26) over $0 \leq y_2 \leq (v_2 k_A - c_2)/k_D$. Alternatively, the defender can choose $B - y_2 \geq (v_1 k_A - c_1)/k_D$, so that the attacker invests $\bar{x}_1(y_1) = 0$ at target 1. Because $(v_1 + v_2)k_A > k_D B + c_1 + c_2$, $v_2 k_A > k_D y_2 + c_2$, so $\bar{x}_2(y_2) > 0$. To allocate most to target 2 (for which the probability of damage is positive) without creating an incentive for the attacker to invest positively at target 1, the defender would choose $y_2 = B - (v_1 k_A - c_1)/k_D$ among $y_2 \leq B - (v_1 k_A - c_1)/k_D$. Then the defender's problem becomes maximizing (5.26) over $B - (v_1 k_A - c_1)/k_D \leq y_2 \leq B$.

Consequently, the defender's problem is to maximize (5.26) over $y_2 \in [0, B] \cap [B - (v_1 k_A - c_1)/k_D, (v_2 k_A - c_2)/k_D]$. The first and second derivatives of the function $V_D^2(y_2)$ are

$$\frac{dV_D^2(y_2)}{dy_2} = \frac{-w_1 k_D}{2\sqrt{v_1 k_A [k_D(B - y_2) + c_1]}} + \frac{w_2 k_D}{2\sqrt{v_2 k_A (k_D y_2 + c_2)}}, \quad (5.27)$$

$$\frac{d^2 V_D^2(y_2)}{dy_2^2} = \frac{-w_1 k_D^2}{4\sqrt{v_1 k_A [k_D(B - y_2) + c_1]^3}} - \frac{w_2 k_D^2}{4\sqrt{v_2 k_A (k_D y_2 + c_2)^3}}. \quad (5.28)$$

Since (5.28) is negative, $V_D^2(y_2)$ is concave. Its maximum value is attained either at the stationary point where (5.27) equals zero or at a boundary. The unique solution of setting (5.27) equal to zero is (5.25). The maximizer of $V_D^2(y_2)$ equals $[B - (v_1 k_A - c_1)/k_D]^+$ if $\tilde{y}_2 \leq [B - (v_1 k_A - c_1)/k_D]^+$, $\min\{B, (v_2 k_A - c_2)/k_D\}$ if $\tilde{y}_2 \geq \min\{B, (v_2 k_A - c_2)/k_D\}$, and $\tilde{y}_2 \in ([B - (v_1 k_A - c_1)/k_D]^+, \min\{B, (v_2 k_A - c_2)/k_D\})$

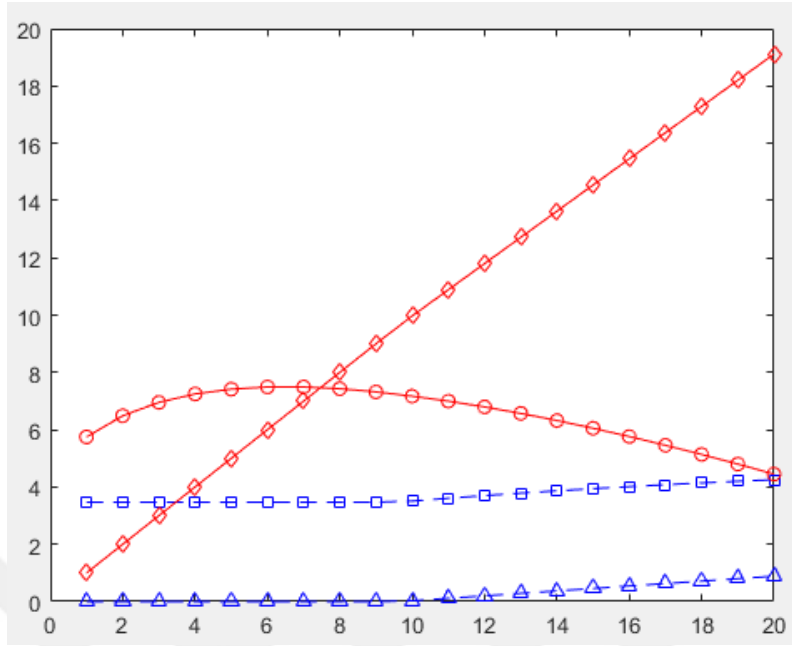
otherwise. These three situations correspond to the three cases in (iii). $\square$

5.6 Numerical Experiments - Sequential-Move Game

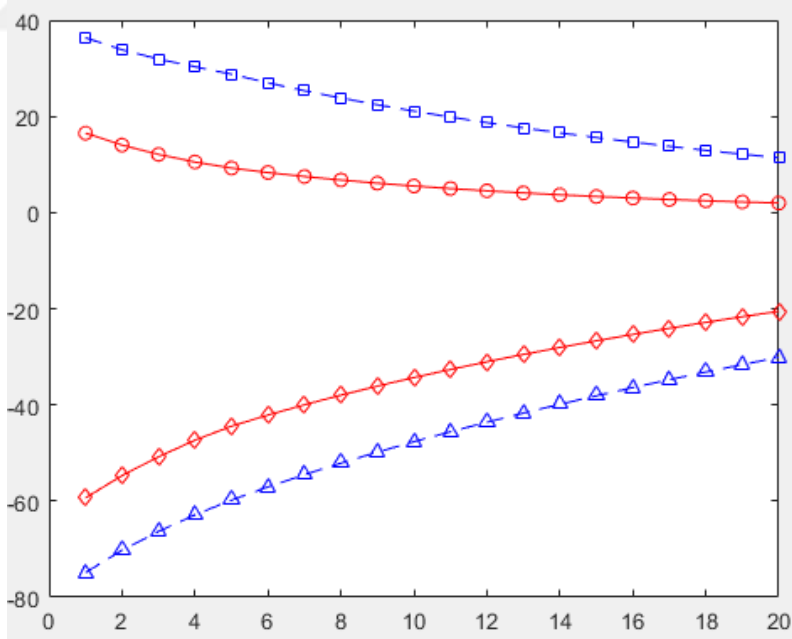
In this section, we report our findings through numerical experiments with the sequential-move game where the defender moves first. First, we consider the baseline values $v_1 = 30$, $v_2 = 20$, $w_1 = 80$, $w_2 = 20$, and $c_1 = c_2 = k_A = k_D = 2$.

Figure 5.20 and Figure 5.21 show the behavior of the equilibrium strategies and expected utilities as functions of the budget. The results are also provided in Table 5.3. With a small budget, the defender allocates all resources to target 1. When budget is sufficient enough to protect target 1, the defender can invest some of the resources at target 2. The attacker's investment at target 1 is higher as compared to target 2; however, as the defense budget increases, the attacker tends to invest more to target 2. Moreover, increasing the budget reduces the attacker's expected utility and increases the defender's expected utility.

Next we consider the case where the defender's valuation of target 2 is higher than target 1. We let $w_1 = 20$ and $w_2 = 80$. Equilibrium investments and utilities are given in Figures 5.22-5.23 and Table 5.4. When the defender values target 2 more than target 1, the defender allocates all the budget to target 2 almost at every value of the budget on the interval $[1, 20]$. When target 2 is protected enough, the defender allocates the rest of the budget to protect target 1. For a small defense budget, the attacker's investments at the two targets are not so different, but as the defender's investment at target 2 increases, the attacker starts to invest more to target 1 which is the relatively more vulnerable target. Comparing Table 5.1 with Table 5.3 and Table 5.2 with Table 5.4 shows that the equilibrium outcomes in the simultaneous- and sequential-move games are quite different. In particular, the attacker's utility is higher in the simultaneous-move game whereas the defender's utility is higher in the sequential-move game, so the defender experiences first-mover advantage in the sequential-move game.

Figure 5.18: Equilibrium Investments vs. B

○ $\bar{x}_1$, ◇ $\bar{y}_1$, □ $\bar{x}_2$, △ $\bar{y}_2$

Figure 5.19: Equilibrium Utilities vs. B

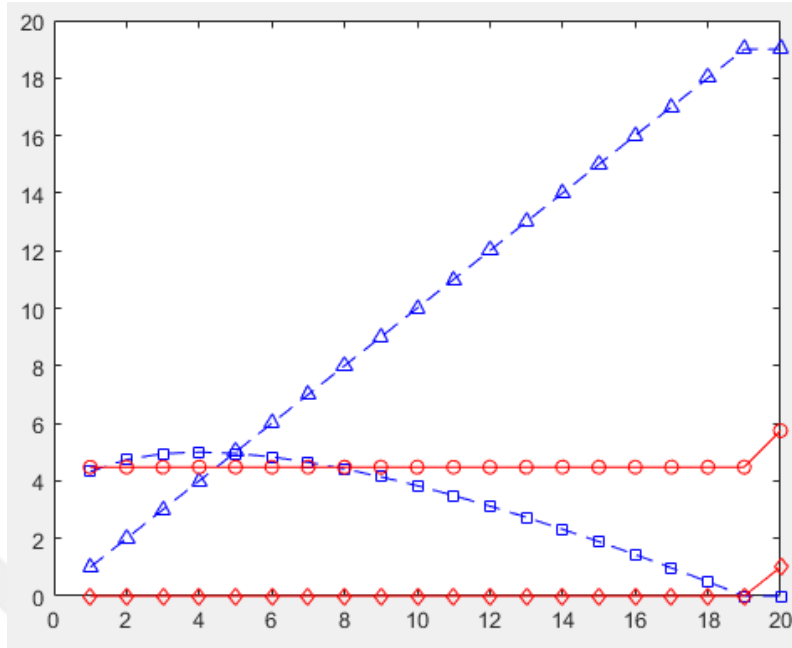
○ $U_A^1(B)$, ◇ $U_D^1(B)$, □ $U_A^2(B)$, △ $U_D^2(B)$

Table 5.3: Equilibrium Investments and Utilities in the Sequential-Move Game [$v_1 = 30, v_2 = 20, w_1 = 80, w_2 = 20, c_1 = c_2 = k_A = k_D = 2$]

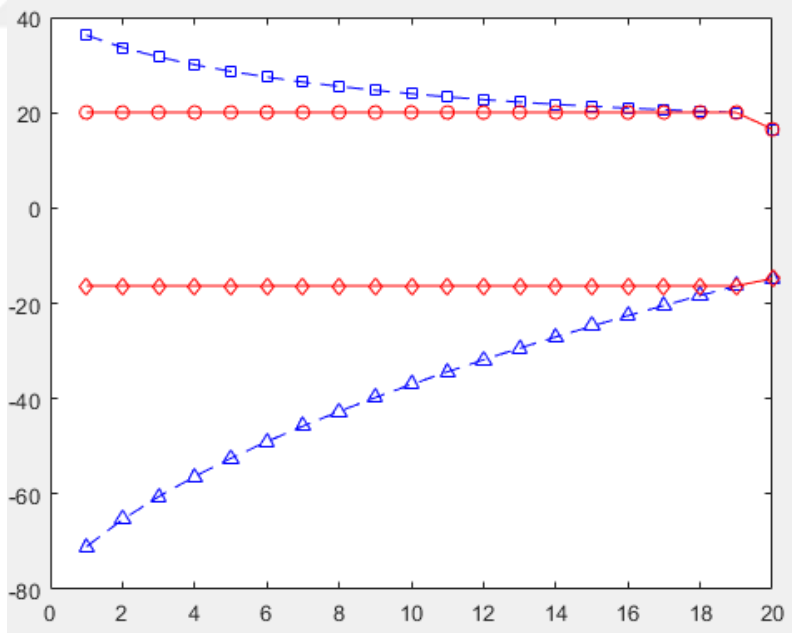
B	$\bar{y}_2$	$\bar{x}_2$	$\bar{y}_1$	$\bar{x}_1$	U_D^2	U_A^2	U_D^1	U_A^1
1	0	3.4721	1.0000	5.7460	-74.8720	36.3277	-59.3441	16.5081
2	0	3.4721	2.0000	6.4868	-70.2296	33.8460	-54.7018	14.0263
3	0	3.4721	3.0000	6.9545	-66.3160	31.9108	-50.7881	12.0911
4	0	3.4721	4.0000	7.2474	-62.8680	30.3248	-47.3401	10.5051
5	0	3.4721	5.0000	7.4164	-59.7508	28.9868	-44.2229	9.1672
6	0	3.4721	6.0000	7.4914	-56.8842	27.8369	-41.3563	8.0172
7	0	3.4721	7.0000	7.4919	-54.2160	26.8358	-38.6882	7.0161
8	0	3.4721	8.0000	7.4317	-51.7101	25.9563	-36.1822	6.1366
9	0	3.4721	9.0000	7.3205	-49.3398	25.1786	-33.8120	5.3590
10	0.0286	3.5070	9.9714	7.1709	-47.0850	24.3765	-31.6205	4.6868
11	0.1143	3.6065	10.8857	6.9974	-44.9243	23.4319	-29.6451	4.1195
12	0.2000	3.6990	11.8000	6.7959	-42.8452	22.5607	-27.7442	3.6082
13	0.2857	3.7852	12.7143	6.5694	-40.8392	21.7553	-25.9101	3.1469
14	0.3714	3.8658	13.6286	6.3203	-38.8990	21.0091	-24.1362	2.7307
15	0.4571	3.9413	14.5429	6.0508	-37.0185	20.3167	-22.4169	2.3556
16	0.5429	4.0121	15.4571	5.7625	-35.1926	19.6733	-20.7475	2.0178
17	0.6286	4.0786	16.3714	5.4571	-33.4167	19.0750	-19.1239	1.7143
18	0.7143	4.1411	17.2857	5.1359	-31.6870	18.5183	-17.5424	1.4425
19	0.8000	4.2000	18.2000	4.8000	-30.0000	18.0000	-16.0000	1.2000
20	0.8857	4.2555	19.1143	4.4505	-28.3527	17.5174	-14.4939	0.9847

We next set $B = 10$ and conduct sensitivity analysis with respect to other parameters, with base values $v_1 = 30, v_2 = 20, w_1 = w_2 = 40$, and $c_1 = c_2 = k_A = k_D = 2$.

Figures 5.22 and 5.23 show the sensitivity of equilibrium outcomes to the attacker's target valuation v_1 . As v_1 increases, the attacker invests more at target 1,

Figure 5.20: Equilibrium Investments vs. B

○ $\bar{x}_1$, ◇ $\bar{y}_1$, □ $\bar{x}_2$, △ $\bar{y}_2$

Figure 5.21: Equilibrium Utilities vs. B

○ $U_A^1(B)$, ◇ $U_D^1(B)$, □ $U_A^2(B)$, △ $U_D^2(B)$

Table 5.4: Equilibrium Investments and Utilities in the Sequential-Move Game [$v_1 = 30, v_2 = 20, w_1 = 20, w_2 = 80, c_1 = c_2 = k_A = k_D = 2$]

B	$\bar{y}_2$	$\bar{x}_2$	$\bar{y}_1$	$\bar{x}_1$	U_D^2	U_A^2	U_D^1	U_A^1
1	1.0000	4.3246	0	4.4772	-71.0503	36.2342	-16.3485	20.0455
2	2.0000	4.7460	0	4.4772	-65.3646	33.6806	-16.3485	20.0455
3	3.0000	4.9443	0	4.4772	-60.5714	31.6849	-16.3485	20.0455
4	4.0000	5.0000	0	4.4772	-56.3485	30.0455	-16.3485	20.0455
5	5.0000	4.9545	0	4.4772	-52.5307	28.6594	-16.3485	20.0455
6	6.0000	4.8322	0	4.4772	-49.0199	27.4651	-16.3485	20.0455
7	7.0000	4.6491	0	4.4772	-45.7521	26.4228	-16.3485	20.0455
8	8.0000	4.4164	0	4.4772	-42.6829	25.5045	-16.3485	20.0455
9	9.0000	4.1421	0	4.4772	-39.7800	24.6902	-16.3485	20.0455
10	10.0000	3.8324	0	4.4772	-37.0189	23.9646	-16.3485	20.0455
11	11.0000	3.4919	0	4.4772	-34.3808	23.3157	-16.3485	20.0455
12	12.0000	3.1245	0	4.4772	-31.8505	22.7343	-16.3485	20.0455
13	13.0000	2.7332	0	4.4772	-29.4157	22.2125	-16.3485	20.0455
14	14.0000	2.3205	0	4.4772	-27.0665	21.7443	-16.3485	20.0455
15	15.0000	1.8885	0	4.4772	-24.7943	21.3242	-16.3485	20.0455
16	16.0000	1.4391	0	4.4772	-22.5922	20.9478	-16.3485	20.0455
17	17.0000	0.9737	0	4.4772	-20.4539	20.6114	-16.3485	20.0455
18	18.0000	0.4936	0	4.4772	-18.3742	20.3116	-16.3485	20.0455
19	19.0000	0	0	4.4772	-16.3485	20.0455	-16.3485	20.0455
20	19.0000	0	1.0000	5.7460	-14.8360	16.5081	-14.8360	16.5081

which is the more valuable target. Large investments of the attacker at target 1 leads the defender to invest more to target 2. In this case, the defender finds it more beneficial to concentrate the efforts at target 2 rather than wasting resources at a target that is more valuable to the attacker. Still, the defender does not completely risk target 1 and allocates small amount of resources to target 1.

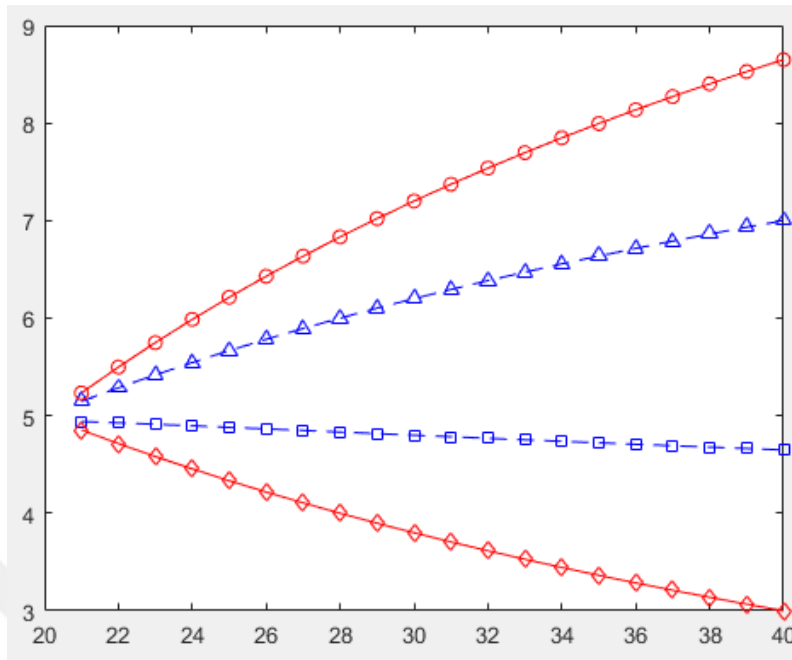
Figures 5.24 and 5.25 show the impact of the attacker's target valuation v_2 on the equilibrium strategies and utilities respectively. The attacker's investment at target 2 increases as v_2 increases, and the defender tends to allocate resources mostly at target 1. However, the defender's investment at target 1 does not affect much the attacker's investment at target 1. We note that as v_2 increases, the defender's investment at target 2 increases in the simultaneous-move game, but decreases in the sequential-move game.

Figures 5.26 and 5.27 illustrate the changes in equilibrium investments and utilities as the defender's target valuation w_1 increases. As target 1 becomes more valuable to the defender, the defense investment at this target exceeds the investment at target 2. The attacker's investment at each target also increases slowly as a function of w_1 .

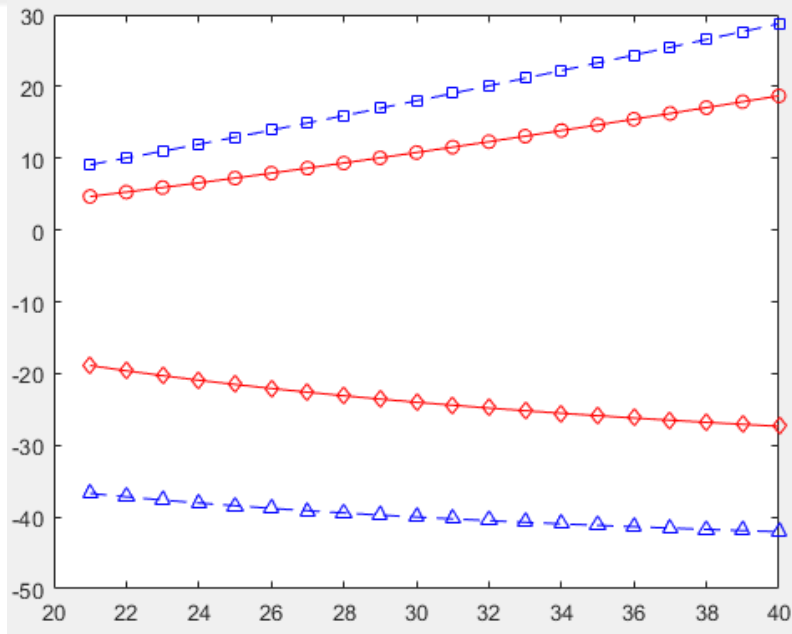
The effects of changes in the defender's target valuation w_2 , shown in Figures 5.28 and 5.29, are similar to the effects of changes in w_1 . As w_2 increases, the defender invests more at target 2. Differently from the changes in w_1 , the attacker's investment at each target decreases as w_2 increases, but still the changes in the attacker's investments are slow relative to the changes in the defender's investments.

Figures 5.30 and 5.32 suggest the importance of existing defense levels at the targets. Figure 5.30 shows the changes in equilibrium strategies as functions of c_1 . As c_1 increases, the defender's investment decreases at target 1 and increases at target 2. A strong enough defense level that is already present at target 1 allows the defender to concentrate efforts at the other target.

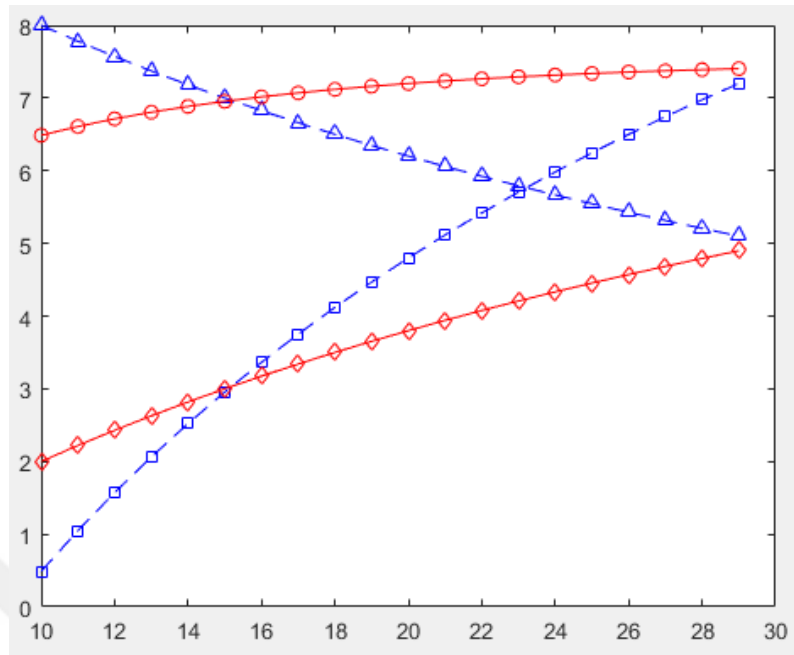
Similar observations hold when the inherent defense level c_2 at target 2 increases, as shown in Figure 5.32. Increases in the inherent defense level c_2 motivates the defender to invest exclusively at target 1. Figures 5.31 and 5.33 demonstrate that as a result of an increase in the inherent defense at any target, the attacker's utility decreases whereas the defender's utility increases.

Figure 5.22: Equilibrium Investments vs. v_1

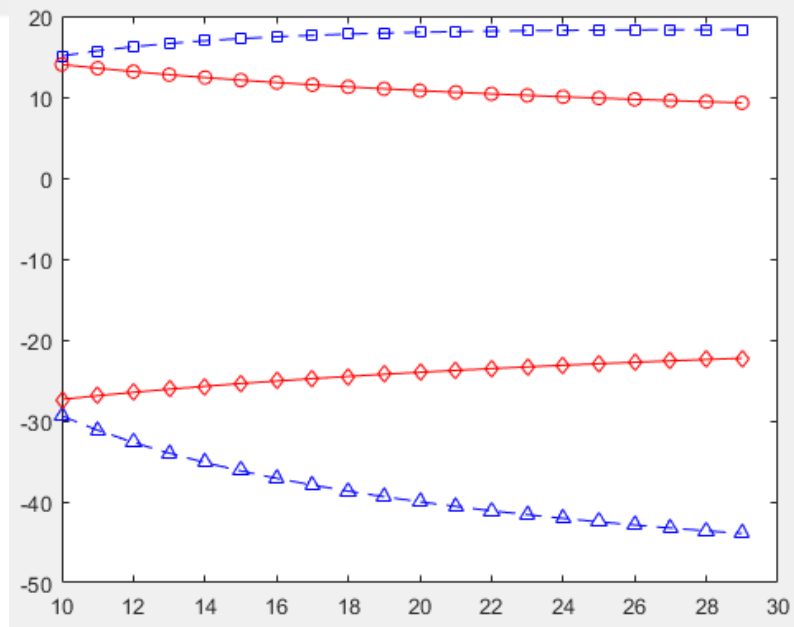
○ $\bar{x}_1$, ◇ $\bar{y}_1$, □ $\bar{x}_2$, △ $\bar{y}_2$

Figure 5.23: Equilibrium Utilities vs. v_1

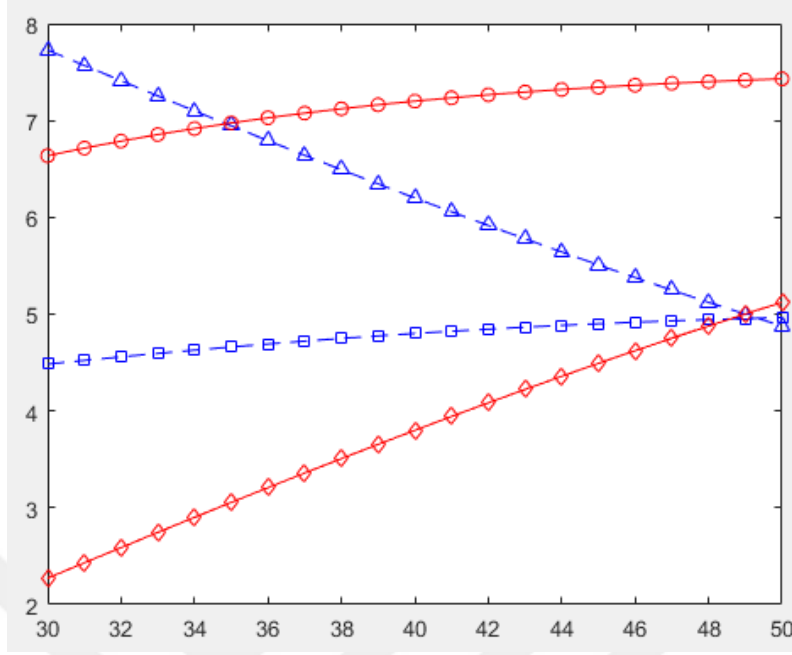
○ $U_A^1(B)$, ◇ $U_D^1(B)$, □ $U_A^2(B)$, △ $U_D^2(B)$,

Figure 5.24: Equilibrium Investments vs. v_2

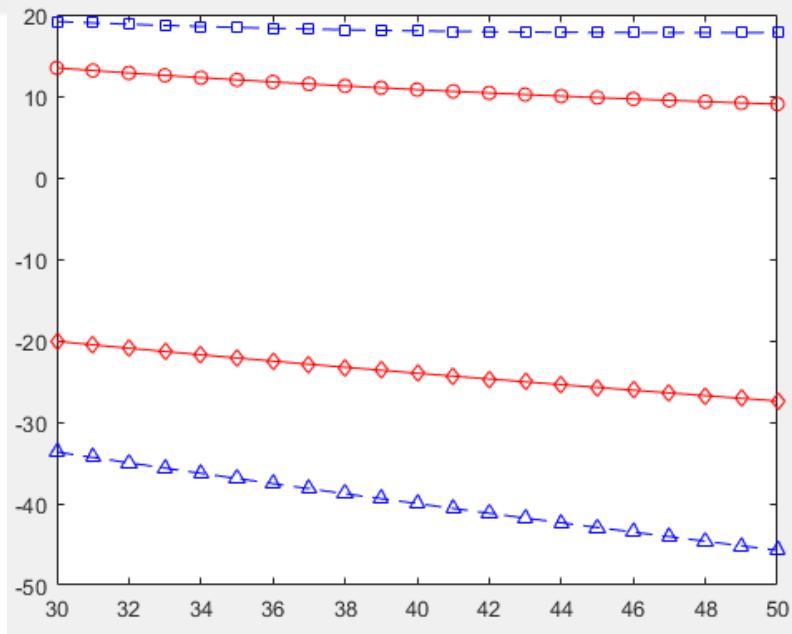
$\circ \bar{x}_1$, $\diamond \bar{y}_1$, $\square \bar{x}_2$, $\triangle \bar{y}_2$

Figure 5.25: Equilibrium Utilities vs. v_2

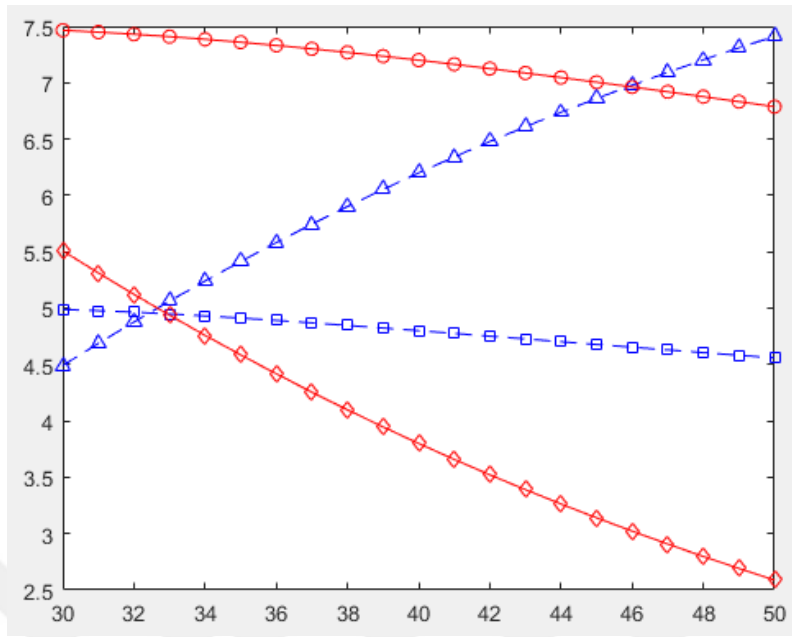
$\circ U_A^1(B)$, $\diamond U_D^1(B)$, $\square U_A^2(B)$, $\triangle U_D^2(B)$

Figure 5.26: Equilibrium Investments vs. w_1

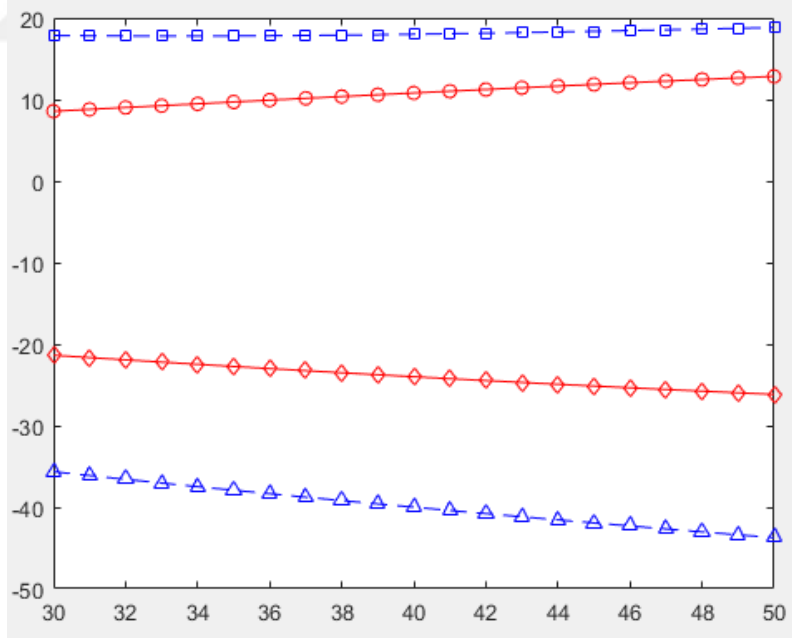
$\circ \bar{x}_1, \diamond \bar{y}_1, \square \bar{x}_2, \triangle \bar{y}_2$

Figure 5.27: Equilibrium Utilities vs. w_1

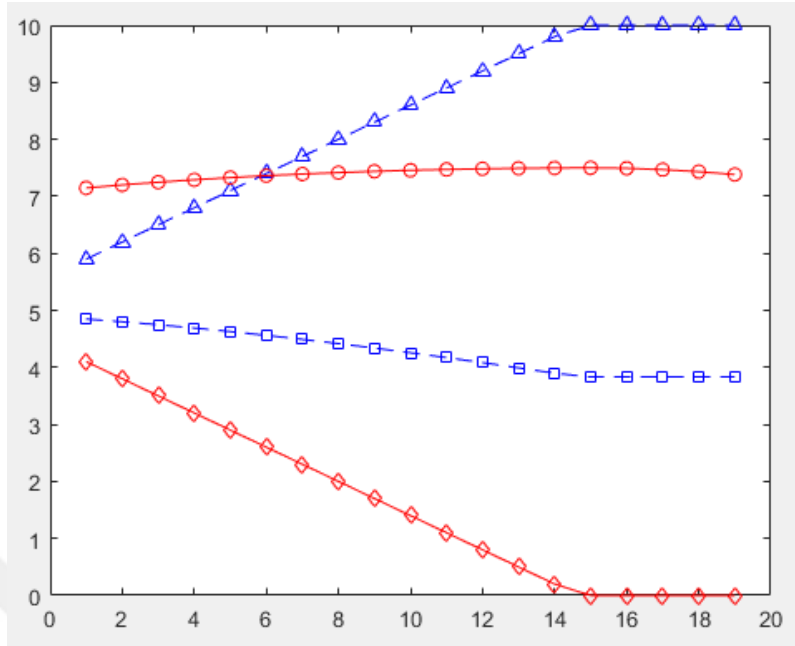
$\circ U_A^1(B), \diamond U_D^1(B), \square U_A^2(B), \triangle U_D^2(B)$

Figure 5.28: Equilibrium Investments vs. w_2

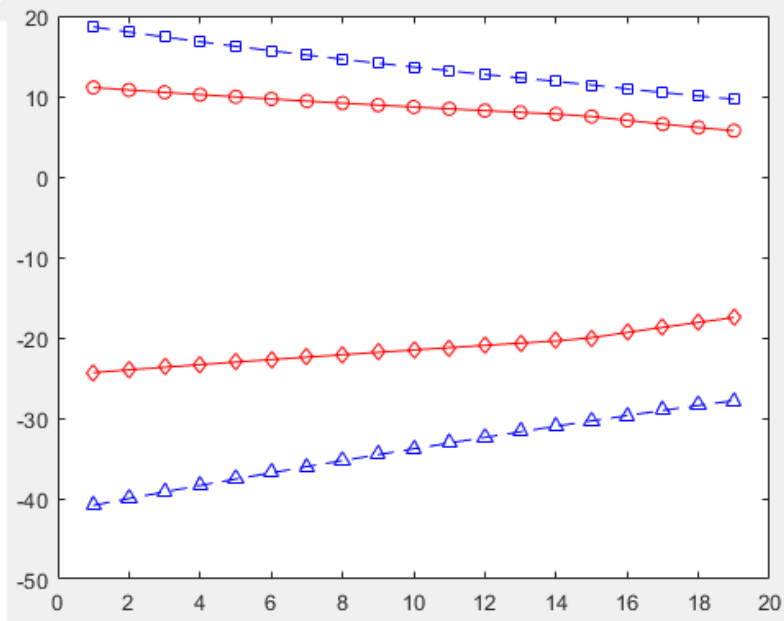
$\circ \bar{x}_1$, $\diamond \bar{y}_1$, $\square \bar{x}_2$, $\triangle \bar{y}_2$

Figure 5.29: Equilibrium Utilities vs. w_2

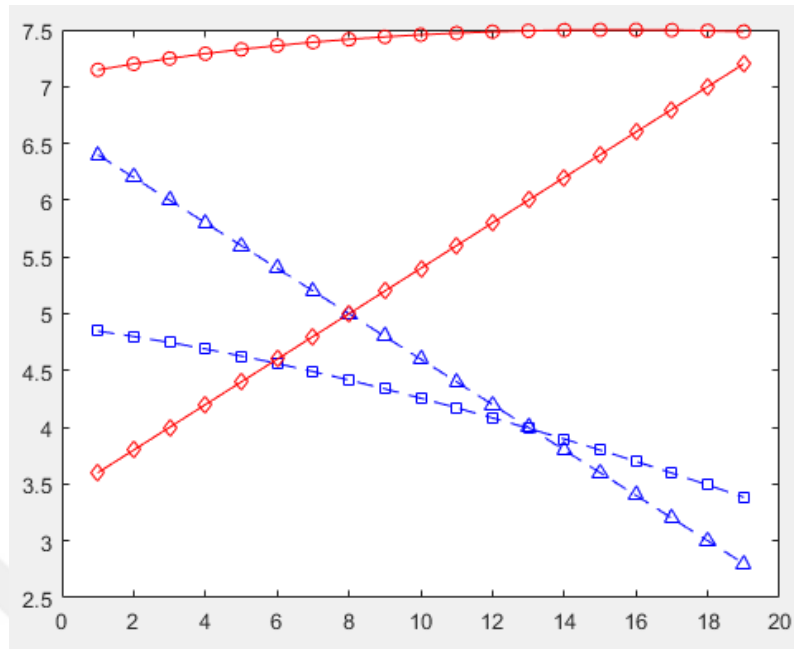
$\circ U_A^1(B)$, $\diamond U_B^1(B)$, $\square U_A^2(B)$, $\triangle U_B^2(B)$

Figure 5.30: Equilibrium Investments vs. c_1

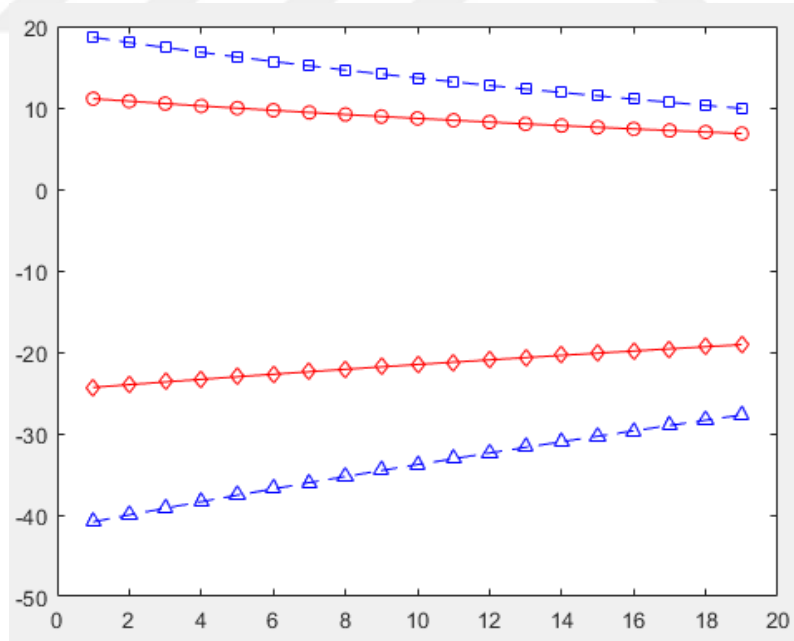
$\circ \bar{x}_1, \diamond \bar{y}_1, \square \bar{x}_2, \triangle \bar{y}_2$

Figure 5.31: Equilibrium Utilities vs. c_1

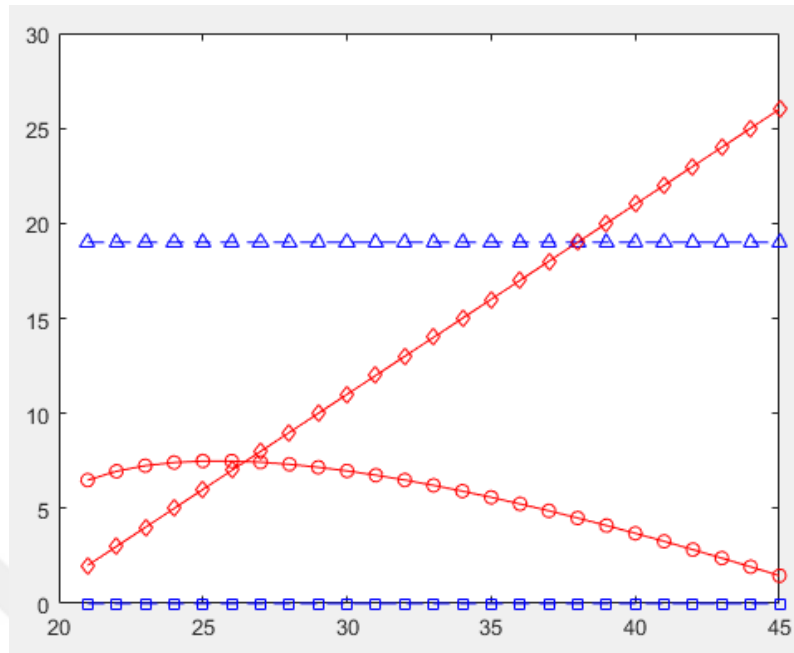
$\circ U_A^1(B), \diamond U_D^1(B), \square U_A^2(B), \triangle U_D^2(B)$

Figure 5.32: Equilibrium Investments vs. c_2

○ $\bar{x}_1$, ◇ $\bar{y}_1$, □ $\bar{x}_2$, △ $\bar{y}_2$

Figure 5.33: Equilibrium Utilities vs. c_2

○ $U_A^1(B)$, ◇ $U_D^1(B)$, □ $U_A^2(B)$, △ $U_D^2(B)$

Figure 5.34: Equilibrium Investments vs. B

$\circ \bar{x}_1$, $\diamond \bar{y}_1$, $\square \bar{x}_2$, $\triangle \bar{y}_2$

Figure 5.34, analogously to Figure 5.17, shows that in the sequential-move game, too, the defender can invest more at the target with a smaller valuation, depending on the parameters of the problem.

Chapter 6

MULTI-TARGET ATTACKER-DEFENDER GAME

In this chapter, we formulate the attacker-defender game for an arbitrary number of targets and show that this game has a Nash equilibrium. We let N be the number of targets. The attacker's valuations for the targets are denoted by $v_1, \dots, v_n$, and those of the defender by $w_1, \dots, w_n$. The defender is subject to a budget constraint. The available defense budget is B , and the inherent defense levels at the targets are $c_1, \dots, c_n$. The efficiency factors for the attacker and the defender are k_A and k_D , respectively.

We recall from Chapter 3 that in any Nash equilibrium of the simultaneous-move game, given the defender's strategy y^N , the attacker's optimization problem is as follows:

$$\max_{x^N} \quad J_A^N(x^N, y^N) = \sum_{n=1}^N \frac{v_n k_A x_n}{k_A x_n + k_D y_n + c_n} - \sum_{n=1}^N x_n \quad (6.1)$$

$$\text{subject to} \quad x_n \geq 0 \quad \text{for all } n = 1, 2, \dots, N, \quad (6.2)$$

and given the attacker's strategy x^N , the defender's optimization problem is as follows:

$$\max_{y^N} \quad J_D^N(x^N, y^N) = - \sum_{n=1}^N \frac{w_n k_A x_n}{k_A x_n + k_D y_n + c_n} \quad (6.3)$$

$$\text{subject to} \quad \sum_{n=1}^N y_n \leq B, \quad (6.4)$$

$$y_n \geq 0 \quad \text{for all } n = 1, 2, \dots, N. \quad (6.5)$$

6.1 Attacker's Best Response

As with the two-target problem, for any given defense strategy $y^N = (y_1, \dots, y_N)$ of the defender, the attacker's optimization problem (6.1)-(6.2) can be decomposed

into N single-target problems, the solution of each of which can be obtained from Proposition 1 of Chapter 4. Proposition 5 states this result.

Proposition 5. *The best response of the attacker to the defense strategy $y^N = (y_1, \dots, y_N) \geq 0$ is $\hat{x}^N(y^N) = (\hat{x}_1(y_1), \dots, \hat{x}_N(y_N))$ with*

$$\hat{x}_n(y_n) = \begin{cases} \frac{\sqrt{v_n k_A (k_D y_n + c_n)} - (k_D y_n + c_n)}{k_A} & \text{if } v_n k_A > k_D y_n + c_n \\ 0 & \text{if } v_n k_A \leq k_D y_n + c_n, \end{cases} \quad (6.6)$$

for $n = 1, \dots, N$. The attacker's expected utility is then

$$J_A^N(\hat{x}^N(y^N), y^N) = \sum_{n=1}^N \left[\left(\sqrt{v_n} - \sqrt{\frac{k_D y_n + c_n}{k_A}} \right)^+ \right]^2. \quad (6.7)$$

Proof. The proposition follows from decomposing the attacker's problem into N subproblems, one for each target, and applying Proposition 1. $\square$

6.2 Defender's Best Response

For any given strategy $x^N = (x_1, \dots, x_N)$ of the attacker, the defender's optimization problem can be solved using dynamic programming. The dynamic-programming recursion (3.13) of the defender developed in Section 3.2 of Chapter 3 is as follows:

$$U_D^n(b, x^n) = \max_{0 \leq y_n \leq b} \{u_n^D(x_n, y_n) + U_D^{n-1}(b - y_n, x^{n-1})\}, \quad (6.8)$$

for $0 \leq b \leq B$ and $n = 1, \dots, N$, with $U_D^0(b, x^0) \equiv 0$ for $0 \leq b \leq B$ and x^0 vacuous. To argue that the existence of a solution for each n , we show that the expression in braces in (6.8) is concave in $y_n \in [0, b]$. The expected utility at target n is

$$u_n^D(x_n, y_n) = \frac{w_n k_A x_n}{k_A x_n + k_D y_n + c_n},$$

which is concave decreasing in $y_n \geq 0$ if $x_n \neq 0$ and equals zero for all $y_n \geq 0$ if $x_n = 0$. To show that the expression in braces is concave, it is enough to show that $U_D^{n-1}(b - y_n, x^{n-1})$ is concave in $y_n \geq 0$, so to show that each $U_D^n(b, x^n)$ is concave in $y_n \in [0, b]$.

Lemma 3. *For any $n = 1, \dots, N$, given $x^n = (x_1, \dots, x_n) \geq 0$, $U_D^n(b, x^n)$ is concave increasing in $b \geq 0$ if $x^n \neq 0$ and zero if $x^n = 0$.*

Proof. We prove the lemma by induction. In Chapter 4, Proposition 2 showed

$$U_D^1(b, x^1) = -\frac{w_1 k_A x_1}{k_A x_1 + k_D b + c_1}, \quad (6.9)$$

which is concave increasing in $b \geq 0$ if $x_1 \neq 0$ and zero otherwise. Suppose the claim is true for $n - 1 \geq 1$. For $x^n = 0$, $U_D^n(b, x^n) = 0$. For $x^n \neq 0$ such that $x^{n-1} = 0$, $x_n > 0$ and

$$U_D^n(b, x^n) = \max_{0 \leq y_n \leq b} \{u_n^D(x_n, y_n)\} = -\frac{w_n k_A x_n}{k_A x_n + k_D b + c_n},$$

where the last equality is an application of Proposition 2. As shown above, this expression is concave increasing in $b \geq 0$. For $x^n \neq 0$ such that $x^{n-1} \neq 0$,

$$U_D^n(b, x^n) = \max_{0 \leq y_n \leq b} \{u_n^D(x_n, y_n) + U_D^{n-1}(b - y_n, x^{n-1})\}$$

, is increasing in $b \geq 0$ because $U_D^{n-1}(b - y_n, x^{n-1})$ is so and the feasible interval is larger for larger b . The Hessian matrix of the function inside the maximization with respect to y_n and b is

$$\begin{bmatrix} \frac{\partial^2 u_n^D(x_n, y_n)}{\partial y_n^2} + \frac{\partial^2 U_D^{n-1}(b - y_n, x^{n-1})}{\partial b^2} & -\frac{\partial^2 U_D^{n-1}(b - y_n, x^{n-1})}{\partial b^2} \\ -\frac{\partial^2 U_D^{n-1}(b - y_n, x^{n-1})}{\partial b^2} & \frac{\partial^2 U_D^{n-1}(b - y_n, x^{n-1})}{\partial b^2} \end{bmatrix}, \quad (6.10)$$

which has negative diagonals because $u_n^D(x_n, y_n)$ has nonpositive second derivative in $y_n \geq 0$ and $U_D^{n-1}(b - y_n, x^{n-1})$ has negative second derivative at all $b - y_n \geq 0$ by the induction hypothesis. Additionally, its determinant equals

$$\frac{\partial^2 u_n^D(x_n, y_n)}{\partial y_n^2} \frac{\partial^2 U_D^{n-1}(b - y_n, x^{n-1})}{\partial b^2} \geq 0,$$

so $u_n^D(x_n, y_n) + U_D^{n-1}(b - y_n, x^{n-1})$ is concave in (y_n, b) . Since the projection of concave functions is concave (a proof of this result can be found in [Veinott, 2005]), we conclude that $U_D^n(b, x^n)$ is concave increasing in $b \geq 0$. $\square$

Proposition 6. *The best response of the defender $\hat{y}^N(x^N)$ to the attack strategy $x^N = (x_1, \dots, x_N) \geq 0$ is as follows.*

- (i) *If $x^N = 0$, then any $(y_1, \dots, y_N) \geq 0$ such that $\sum_n y_n \leq B$ is a best response. The defender's expected utility is then $J_D^N(x^N, \hat{y}^N(x^N)) = 0$.*

- (ii) If $x^N \neq 0$, the defender's best response at target n given budget b to be allocated over targets $1, \dots, n$ is the unique maximizer of the concave function $u_n^D(x_n, y_n) + U_D^{n-1}(b - y_n, x^{n-1})$ over $y_n \in [0, b]$, with $U_D^0(b, x^0) \equiv 0$.

Proof. For $x^N = 0$, any feasible $(y_1, \dots, y_N)$ gives zero expected utility, so is optimal. If $x^{n-1} = 0$ and $x_n > 0$, then the objective function in (6.8) reduces to $u_n(x_n, y_n)$, which is concave, so has a unique maximizer. If $x^{n-1} \neq 0$, then $U_D^{n-1}(b - y_2, x^{n-1})$ must be concave in $y_2 \leq b$, since $u_n(x_n, y_n)$ is also either concave or equal to zero, the objective function in (6.8) is concave in $0 \leq y_2 \leq b$, so has a unique maximizer. $\square$

6.3 Nash Equilibrium

Theorem 5 characterizes the Nash equilibrium for the multi-target problem.

Theorem 5. *For the multi-target attacker-defender game, a Nash equilibrium (x^*, y^*) exists and is as follows:*

- (i) If $\sum_n v_n k_A \leq k_D B + \sum_n c_n$ and $v_n k_A \leq k_D B + c_n$ for all $n = 1, \dots, N$, then any (x^*, y^*) such that $x^* = 0$, $y_n^* \geq (v_n k_A - c_n)^+ / k_D$ for all $n = 1, \dots, N$, and $\sum_n y_n^* \leq B$ is a Nash equilibrium.
- (ii) If $\sum_n v_n k_A \leq k_D B + \sum_n c_n$ and $v_n k_A > k_D B + c_n$ for some $n = 1, \dots, N$, or $\sum_n v_n k_A > k_D B + \sum_n c_n$, then $x^* \neq 0$, and for each $n = 1, \dots, N$,

$$x_n^* = \frac{\left[\sqrt{v_n k_A (k_D y_n^* + c_n)} - (k_D y_n + c_n) \right]^+}{k_A}, \quad (6.11)$$

and y_n^* is the unique maximizer of the concave function $u_n^D(x_n^*, y_n^*) + U_D^{n-1}(B - \sum_{i=n+1}^N y_i^*, x_1^*, \dots, x_{n-1}^*)$ over $y_n \in [0, b]$, with $U_D^0(b, x^0) \equiv 0$.

Proof. By Proposition 6, any feasible y , including one given in (i) is a best response to $x^* = 0$, and by Proposition 5, $x^* = 0$ is the unique best response to y^* with $y_n^* \geq (v_n k_A - c_n)^+ / k_D$, so the strategy profile described in (i) is a Nash equilibrium. The conditions $\sum_n v_n k_A \leq k_D B + \sum_n c_n$ and $v_n k_A \leq k_D B + c_n$ for all $n = 1, \dots, N$ guarantee the existence of such profiles.

To show (ii), we note that (6.11) is the unique best response of the attacker to y^* by Proposition 5. The investment at target n such that $v_n k_A > k_D B + c_n$ must be positive, so $x^* \neq 0$. If $\sum_n v_n k_A > k_D B + \sum_n c_n$, then no feasible y^* can make the attacker invest $x^* = 0$. In both cases, the unique best response of the defender to $x^* \neq 0$ is described by Proposition 6.

The existence of a Nash equilibrium follows from [Rosen, 1965], the concavity of $J_A^N(x^N, y^N)$ in x^N and of $J_D^N(x^N, y^N)$ in y^N , the convexity and compactness of the effective strategy sets. For the defender, the strategy set is convex and compact by definition, and for the attacker, one can restrict attention to $0 \leq x_n \leq v_n$ without loss of generality. The concavity of $J_A^N(x^N, y^N)$ in x^N follows from the concavity of each $u_n^A(x_n, y_n)$ in x_n as stated by Lemma 1. The concavity of $J_D^N(x^N, y^N)$ in y^N follows from the concavity of each $u_n^D(x_n, y_n)$ in y_n . $\square$

Theorem 5 provides a characterization for the Nash equilibrium of the multi-target attacker-defender game and a method to find a Nash equilibrium. However, as the number of targets increases, the computational effort required increases. Preprocessing of the targets may help shrinking the set of targets where positive investments are viable. More specifically, one can first calculate $v_n k_A - c_n$, label targets in decreasing order of this index, and eliminate all targets with $v_n k_A - c_n \leq 0$, as the attacker will never invest a positive amount at such targets.

6.4 Subgame-Perfect Nash Equilibrium in the Sequential-Move Game

In the sequential-move game, the attacker decides after observing the defense level at every target. As the leader, the defender adopts a strategy that maximizes the expected utility by taking into account the attacker's future response. Section 6.1 showed that given defense level y_n at target n , the attacker's best response is

$$\hat{x}_n(y_n) = \begin{cases} \frac{\sqrt{v_n k_A (k_D y_n + c_n)} - (k_D y_n + c_n)}{k_A} & \text{if } v_n k_A > k_D y_n + c_n \\ 0 & \text{if } v_n k_A \leq k_D y_n + c_n. \end{cases} \quad (6.12)$$

Rolling backwards, the defender's optimization problem is then:

$$\max_{y^N} J_D^N(\hat{x}^N(y^N), y^N) = \sum_{n=1}^N u_n^D(\hat{x}_n(y_n), y_n) \quad (6.13)$$

$$\text{subject to} \quad \sum_{n=1}^N y_n \leq B, \quad (6.14)$$

$$y_n \geq 0. \quad (6.15)$$

Theorem 6. *For the multi-target attacker-defender game where the attacker decides after observing the investment of the defender, there exists a subgame-perfect Nash equilibrium $(\bar{x}, \bar{y})$ and satisfies the following:*

- (i) *If $\sum_n v_n k_A \leq k_D B + \sum_n c_n$ and $v_n k_A \leq k_D B + c_n$ for all $n = 1, \dots, N$, then in any subgame-perfect Nash equilibrium, the defender selects a strategy satisfying $\bar{y}_n \geq (v_n k_A - c_n)^+ / k_D$ for all $n = 1, \dots, N$ and $\sum_n \bar{y}_n \leq B$, while the attacker chooses not to invest at any target, i.e., $\bar{x} = 0$.*
- (ii) *If $\sum_n v_n k_A \leq k_D B + \sum_n c_n$ and $v_n k_A > k_D B + c_n$ for some $n = 1, \dots, N$, or $\sum_n v_n k_A > k_D B + \sum_n c_n$, the subgame-perfect Nash equilibrium is such that $\bar{y}$ maximizes*

$$V_D^N(y) \equiv \sum_{n=1}^N w_n \left(\sqrt{\frac{k_D y_n + c_n}{v_n k_A}} - 1 \right)^+$$

, over $y \geq 0$ such that $\sum_n y_n \leq B$, and the attacker invests

$$\bar{x}_n = \frac{[\sqrt{v_n k_A (k_D y_n + c_n)} - (k_D y_n + c_n)]^+}{k_A}.$$

Proof. If $\sum_n v_n k_A \leq k_D B + \sum_n c_n$ and $v_n k_A \leq k_D B + c_n$ for all $n = 1, \dots, N$, the defender can obtain the maximum possible expected utility, which is zero, by making the attacker invest zero at all targets, $\bar{x}(y) = 0$, that is by choosing a defense strategy such that $\bar{y}_n \geq (v_n k_A - c_n)^+ / k_D$ for all $n = 1, \dots, N$ and $\sum_n \bar{y}_n \leq B$. Such a strategy exists under the given conditions.

On the other hand, if $\sum_n v_n k_A \leq k_D B + \sum_n c_n$ and $v_n k_A > k_D B + c_n$ for some $n = 1, \dots, N$, $\bar{x}_n > 0$. If $\sum_n v_n k_A > k_D B + \sum_n c_n$, then $\bar{x} = 0$ is not a best response to any feasible y . For this reason, under the conditions of (ii), $\bar{x} \neq 0$ and

$V_D^N(y) \neq 0$, where $V_D^N(y)$ is obtained by substituting (6.12) in (6.13). Since $V_D^N(y)$ is a real-valued continuous function and the set of $y \geq 0$ such that $\sum_n y_n \leq B$ is compact, Extreme-Value Theorem applies, so there exists a maximizer $\bar{y}$. $\square$

6.5 Numerical Experiments - Sequential-Move Game

To illustrate the results of Section 6.4, we set two experiments.

Experiment 1. We consider the competition of an attacker and a defender over ten targets, for the defense budget $B = 40$, efficiency factors $k_A = k_D = 2$, and equal inherent defense levels $c_n = 0.5$ at all targets. Table 6.1 provides the attacker's and the defender's target valuations along with their investments in the subgame-perfect Nash equilibrium.

Table 6.1: Attacker's and Defender's Valuations and Investments for 10 Targets

Target	Target Valuations		Equilibrium Solutions	
	v_n	w_n	$\bar{x}_n$	$\bar{y}_n$
1	21	63	2.36	0.1
2	21	190	4.99	2.94
3	3	75	0	2.74
4	21	27	2.04	0
5	27	180	5.53	1.98
6	30	120	4.28	0.64
7	75	1000	18.33	24.52
8	180	75	6.46	0
9	6	120	0.72	4.2
10	30	225	6.56	2.88
Total			51.27	40
Attacker's equilibrium utility		268.9		
Defender's equilibrium utility		-1089		

We observe that the defender allocates a large portion of the budget to the most valuable target 7. Even though target 7 is the second most valuable target to the attacker, the attacker also makes the largest investment there. This can be explained with the structure of the game. Even though the attacker associates a higher value to target 8, upon observing that the defender does not allocate anything to target 8 and allocates a large amount to target 7, the attacker may assume the valuable target 8 almost as granted and concentrate the investment at target 7 instead to raise the chances of destroying this one. The low inherent defense level at targets also plays a role in this choice, since c_8 being very small in the absence of any additional defense budget allocated to target 8, the probability of damage at this target is almost 1, in other words, even without making a large investment, the attacker enjoys a large expected utility here. The defender does not invest in the protection of target 4 eit-

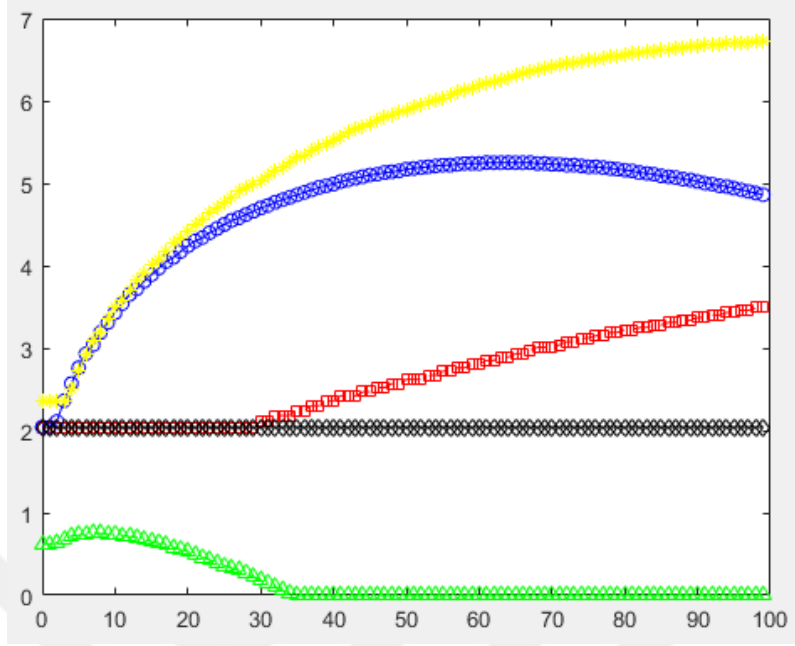
Table 6.2: Attacker's and Defender's Valuations and Investments for 10 Targets

Target	Target Valuations		Equilibrium Investments	
	v_n	w_n	$\bar{x}_n$	$\bar{y}_n$
1	63	21	5.59	0.36
2	190	21	6.64	0
3	75	3	4.08	0
4	27	21	4.78	1.18
5	180	27	7.59	0.10
6	120	30	8.18	0.4
7	1000	75	21.65	0.24
8	75	180	15.42	37.62
9	120	6	5.23	0
10	225	30	8.52	0.1
Total			87.68	40
Attacker's equilibrium utility		1857		
Defender's equilibrium utility		-271.7		

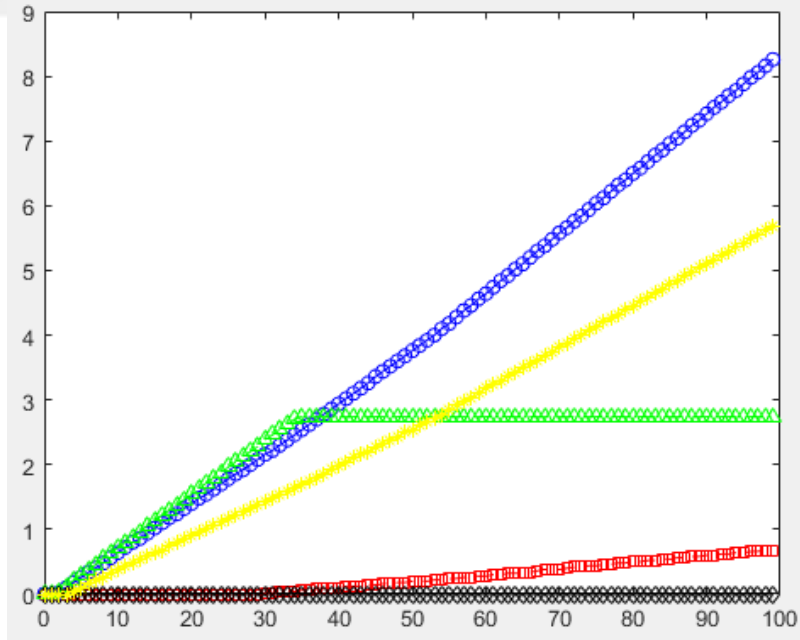
her because this is the least valuable target for the defender. Similarly, the attacker does not allocate anything to target 3, which is the least valuable target for the attacker. Table 6.1 also shows that a player can allocate a different amount to equally valuable targets.

We solved the same problem after interchanging the target valuations for the attacker and the defender. Table 6.2 displays the valuations and the equilibrium investments. In this variation, we observe that both the attacker and the defender invest most in the target that is most valuable to them, the attacker at target 7 and the defender at target 8. Surprisingly, the defender now makes the second largest investment at target 4 even though its value is much lower than some others. A plausible explanation for this is that the defender wishes to guarantee the protection of a relatively low-valued target that is not of high value for the attacker either instead of wasting efforts at more valuable targets, where the attacker may be stronger.

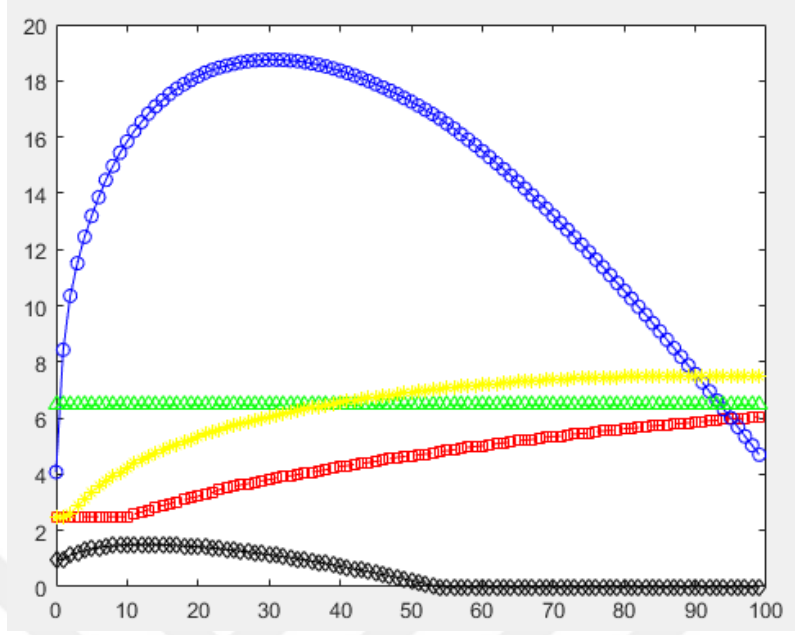
In order to gain insight into the effects of the changes in the defense budget and the inherent defense levels, we conduct sensitivity analysis with baseline values $B = 40$, $c_n = 0.5$ for all $n = 1, \dots, 10$, $k_A = k_D = 2$ and target valuations listed in Table 6.1. Figures 6.1-6.4 plot the changes in the equilibrium investments as functions of B . The defender's investments mostly increase as B increases. For low budget values, the defender is not able to deter any attack, i.e., to make the attacker invest zero; however, as the budget gets larger and larger, the defender starts investing positively at every target, and after a certain point is able to deter any potential attack. The defender's investment at target 7 is the highest for every budget value. Comparing the defense investments at targets 2, 3, and 5, we observe that when the budget is scarce, the defender invests more at target 3, but after a certain threshold, the investments at targets 2 and 5 go beyond the investment at target 3. The reason for this is that since the attacker's valuation of target 3 is low, the defender is able to deter a potential attack at this target by choosing a relatively low investment. As the budget increases, the amount allocated to target 3 does not change much while the investments made at more valuable targets increase. Increases in the defender's investments at the targets have a non-monotonic effect on the attacker's investments as shown in Chapter 4. Specifically, as the defender's investment at

Figure 6.1: Equilibrium Investments of the Attacker vs. B

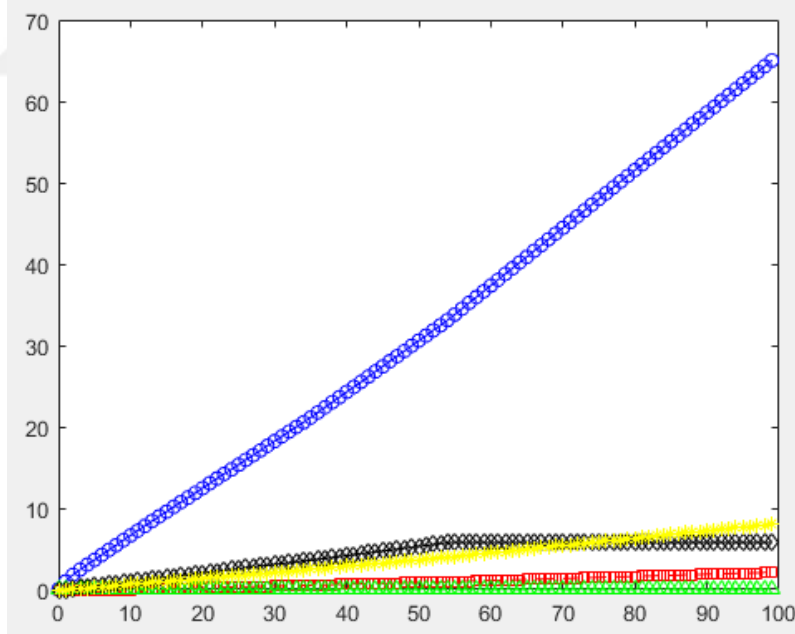
$\square \bar{x}_1$, $\circ \bar{x}_2$, $\triangle \bar{x}_3$, $\diamond \bar{x}_4$, $\ast \bar{x}_5$

Figure 6.2: Equilibrium Investments of the Defender vs. B

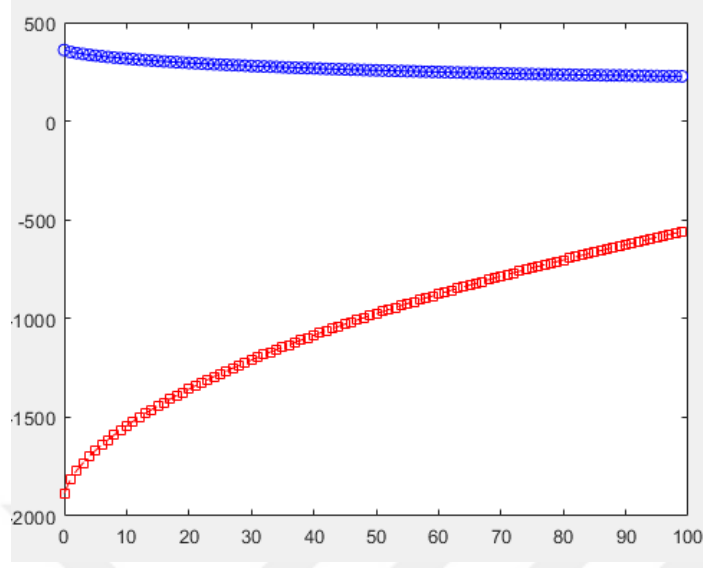
$\square \bar{y}_1$, $\circ \bar{y}_2$, $\triangle \bar{y}_3$, $\diamond \bar{y}_4$, $\ast \bar{y}_5$

Figure 6.3: Equilibrium Investments of the Attacker vs. B

$\square \bar{x}_6$, $\circ \bar{x}_7$, $\triangle \bar{x}_8$, $\diamond \bar{x}_9$, $\ast \bar{x}_{10}$

Figure 6.4: Equilibrium Investments of the Defender vs. B

$\square \bar{y}_6$, $\circ \bar{y}_7$, $\triangle \bar{y}_8$, $\diamond \bar{y}_9$, $\ast \bar{y}_{10}$

Figure 6.5: Equilibrium Utilities vs. B

○ $U_A^{10}(B)$, □ $U_D^{10}(B)$

a target increases, the attacker's response first increases, then decreases, which is consistent with Figures 6.1 and 6.3. Figure 6.5 shows that as the defense budget increases, the attacker's utility decreases while the defender is better off.

Table 6.3 shows how the equilibrium changes as the inherent defense level increases at all targets equally. As a result of an increase in c_n , the defender's utility increases while the attacker's decreases. A raise from 0.5 to 5 enables the defender to deter a potential attack at one more target.

Experiment 2. In this experiment, we apply our results to the real dataset provided by [Willis et al., 2005]. The dataset was also used by [Bier et al., 2008], [Hao et al., 2009], [Nikoofal and Zhuang, 2012], [Shan and Zhuang, 2013b], [Shan and Zhuang, 2013a], [Nikoofal and Zhuang, 2015], [Guan et al., 2017], [Zhang et al., 2018], [Shan and Zhuang, 2018], and [Yolmeh and Baykal-Gürsoy, 2019]. We restrict our attention to the ten most valuable areas listed in Table 6.4. For the defense budget, we use the total amount allocated to the ten most valuable areas from Fiscal Year 2004 Urban Areas Security Initiative (FY 2004 UASI). We assume that the attacker's and the defender's target valuations are the same and given by the expected property losses.

Table 6.3: Effects of Changing Inherent Defense Level on Equilibrium

Target	Target Valuations		Equilibrium Investments					
			$c_n = 0.5$		$c_n = 5$		$c_n = 10$	
	v_n	w_n	$\bar{x}_n$	$\bar{y}_n$	$\bar{x}_n$	$\bar{y}_n$	$\bar{x}_n$	$\bar{y}_n$
1	21	63	2.36	0.1	4.75	0	5.25	0
2	21	190	4.99	2.94	5.21	1.9	5.25	0.54
3	3	75	0	2.74	0	0.5	0	0
4	21	27	2.04	0	4.75	0	5.25	0
5	27	180	5.53	1.98	6.04	0.58	6.62	0
6	30	120	4.28	0.64	6.16	0	7.25	0
7	75	1000	18.33	24.52	16.45	31.7	13.78	38.02
8	180	75	6.46	0	18.71	0	25	0
9	6	120	0.72	4.2	0	3.5	0	1
10	30	225	6.56	2.88	7.06	1.82	7.33	0.44
Total			51.27	40	69.13	40	75.73	40
Attacker's equilibrium utility			268.9		210.75		174.55	
Defender's equilibrium utility			-1089		-888.04		-733.63	

Table 6.4 shows the equilibrium investments for $B = \$268$ million, $c_n = 0.1$ for all $n = 1, \dots, 10$, and $k_A = k_D = 1$.

We observe that both players concentrate their efforts to highly valued areas. As compared to the allocation of FY 2004 UASI, the defender's equilibrium investments appear to be higher at areas with large expected property losses, and lower at areas with small expected property losses. According to the equilibrium found, the defender should allocate significantly more to New York, at least three times higher than the FY 2004 UASI allocation, and the investment to Seattle should be approximately seven times lower than the FY 2004 UASI allocation.

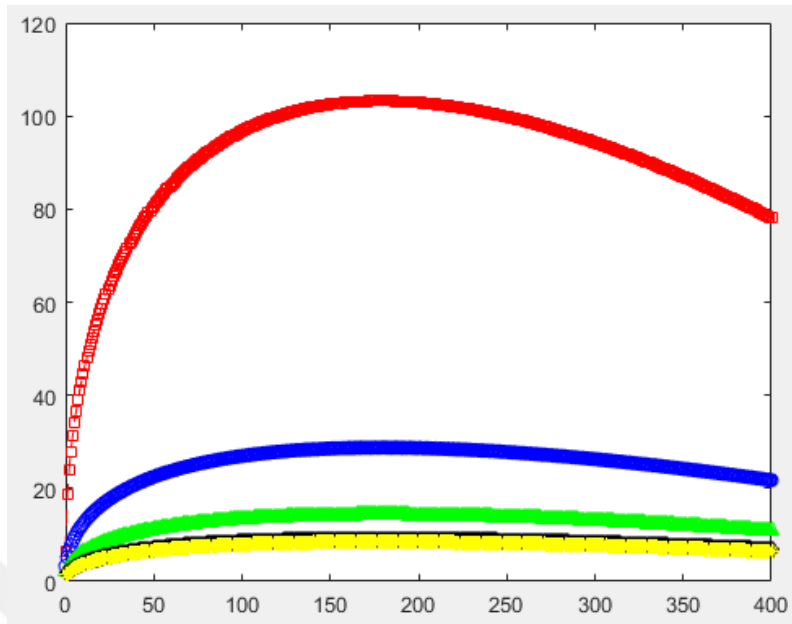
Figures 6.6-6.9 represent the changes in equilibrium strategies of the attacker and the defender for ten urban areas. As budget increases, defender's investment to every urban area increases linearly, and the attacker's investment first increases

Table 6.4: Expected Property Losses, FY 2004 UASI Resource Allocations and Equilibrium Investments for 10 Urban Areas (\$ million)

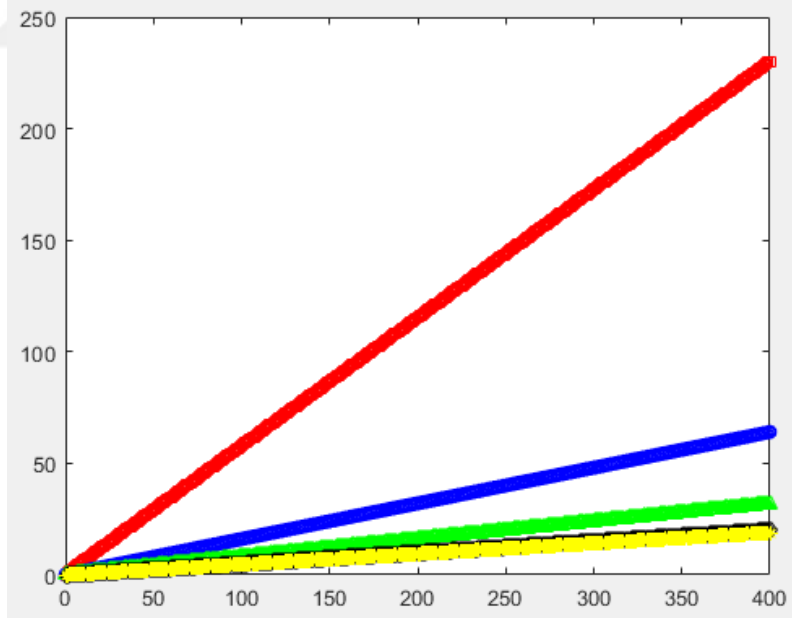
Target	Expected property losses	Equilibrium Investments		FY 2004 UASI Allocation
		$\bar{x}_n$	$\bar{y}_n$	
New York	413	98.1	154.42	47
Chicago	115	27.32	42.92	34
San Francisco	57	13.54	21.22	26
Washington	36	8.55	13.36	29
Los Angeles	34	8.08	12.62	40
Philadelphia	21	4.99	7.76	23
Boston	18	4.27	6.64	19
Houston	11	2.61	4.02	19
Newark	7.3	1.73	2.64	15
Seattle	6.7	1.59	2.4	16
Total		170.8	268	268
Attacker's equilibrium utility	108.43			
Defender's equilibrium utility	-279.21			

then decreases.

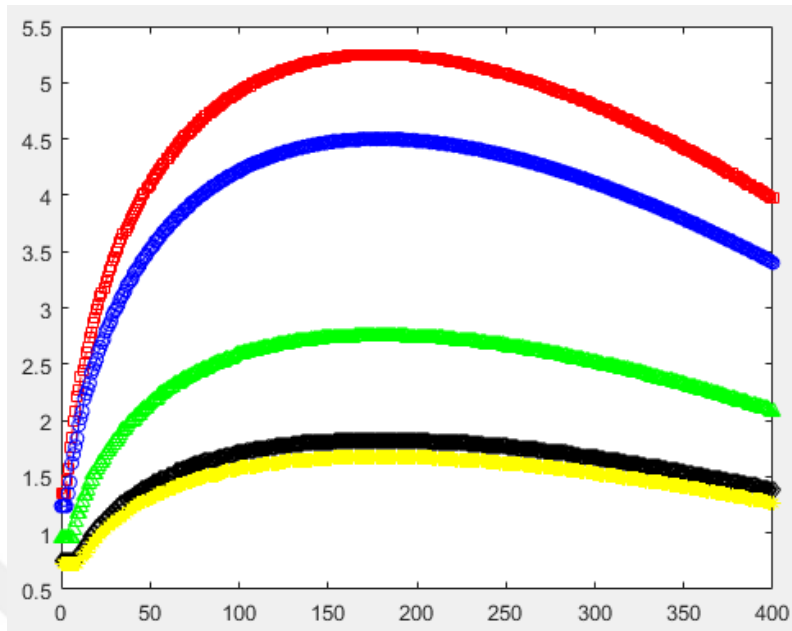
As shown in Table 6.5, an increase in the inherent defense level c_n increases the defender's capacity to deter an attack. Raising it to 50 allows the defender to deter attacks at six areas. As this existing defense increases, the defender can allocate its budget more to the areas that are highly valued, as the inherent defense level becomes sufficient to deter attacks at low-valued areas. Furthermore, Figure 6.10 and Table 6.5 indicate that an increase in the defense budget or the inherent defense level increases the defender's utility while decreasing the attacker's.

Figure 6.6: Equilibrium Investments of the Attacker vs. B

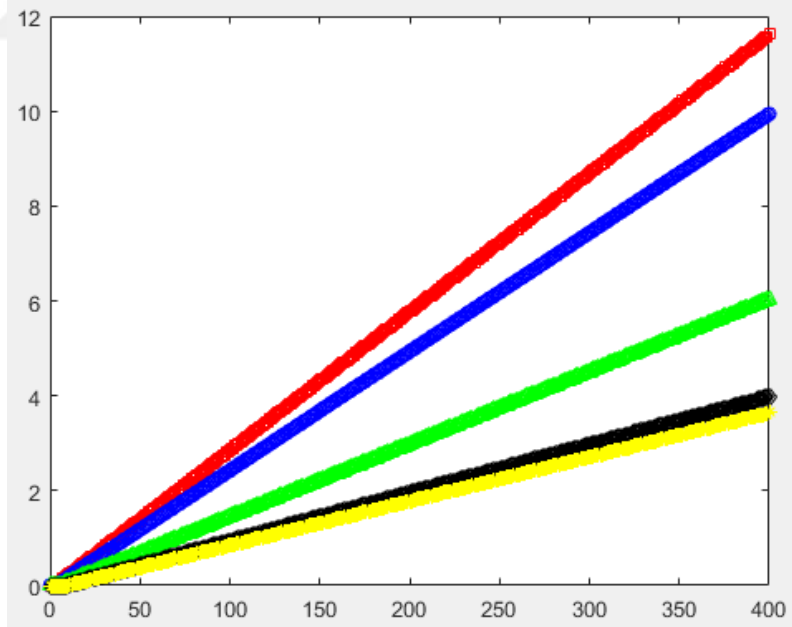
□ New York, ○ Chicago, △ San Francisco, ◇ Washington, * Los Angeles

Figure 6.7: Equilibrium Investments of the Defender vs. B

□ New York, ○ Chicago, △ San Francisco, ◇ Washington, * Los Angeles

Figure 6.8: Equilibrium Investments of the Attacker vs. B

□ Philadelphia, ○ Boston, △ Houston, ◇ Newark, * Seattle

Figure 6.9: Equilibrium Investments of the Defender vs. B

□ Philadelphia, ○ Boston, △ Houston, ◇ Newark, * Seattle

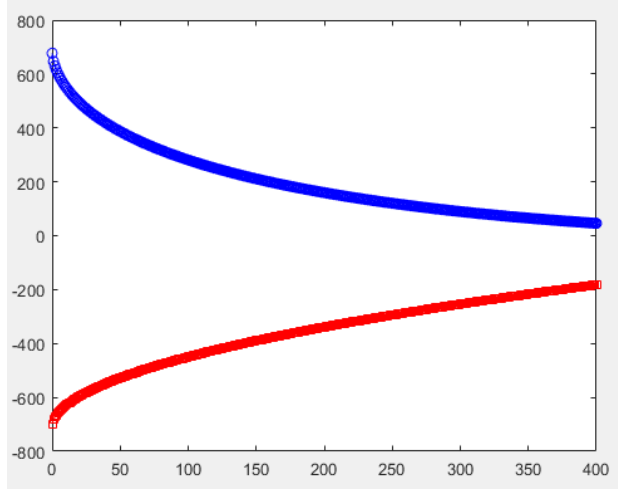


Figure 6.10: Equilibrium Utilities for the Urban Areas

○ $U_A^{10}(B)$, □ $U_D^{10}(B)$

Table 6.5: Effects of Changing Inherent Defense Level on Equilibrium

Urban Areas	Equilibrium Investments					
	$c_n = 0.1$		$c_n = 10$		$c_n = 50$	
	$\bar{x}_n$	$\bar{y}_n$	$\bar{x}_n$	$\bar{y}_n$	$\bar{x}_n$	$\bar{y}_n$
New York	98.1	154.42	87.29	190.4	56.95	237.84
Chicago	27.32	42.92	24.31	17.66	15.85	30.16
San Francisco	13.54	21.22	12.05	45.8	3.39	0
Washington	8.55	13.36	7.61	7.46	7.61	0
Los Angeles	8.08	12.62	7.19	6.5	0	0
Philadelphia	4.99	7.76	4.44	0.18	0	0
Boston	4.27	6.64	3.42	0	0	0
Houston	2.61	4.02	0.49	0	0	0
Newark	1.73	2.64	0	0	0	0
Seattle	1.59	2.4	0	0	0	0
Total	170.8	268	146.8	268	76.19	268
Attacker's EU	108.43		63.43		14.63	
Defender's EU	-279.21		-210.21		-90.81	

Chapter 7

CONCLUSION

In this thesis, we investigated how a single attacker and a single defender allocate their resources among several potential targets. We developed a set of dynamic-programming recursion for calculating the equilibrium investments and expected utilities of the players for the simultaneous-move game and a sequential-move version where the defender decides first. We assumed that while the attacker tries to maximize the expected damage over all targets in a cost-efficient manner, the defender aims to allocate its limited budget for the protection of these targets so as to minimize the expected total damage. One of our main observations was that the defense budget has a significant effect on the equilibrium outcome. A defender that has a sufficiently large budget can deter attacks at all targets by selecting the right allocation regardless of whether it is a simultaneous- or a sequential-move game; however, as the budget gets scarce, efficiently protecting becomes more challenging and indeed this is when a nontrivial resource-allocation game between the attacker and the defender arises.

In addition to the budget available to the defender, we also analyzed the impact of the changes in defender's and attacker's target valuations, already existing defense levels at the targets, and the efficiency factors of the defender and the attacker in converting investment to defense or attack efforts, on the defender's and the attacker's equilibrium investments and utility values.

In our model, cost efficiency is incorporated directly by the utility function of the attacker instead of being modeled with budgetary constraints. For this reason, the attacker's problem can be decomposed into independent problem for a given allocation of the defender whereas this is not possible for the defender due to the budget constraint that involves investments at all targets.

Analytical results suggest that the attacker's best-response investment is a con-

cave function of the defender's investment at each target. Consequently, as the defender increases the defense level at a target, the attacker responds to this by increasing the attack effort provided that the value of the target for the attacker is large enough as compared to the defense level. However, after a certain defense level, an increase in the defense effort leads the attacker to reduce the attack investment at this target, because the value of the target no longer makes up for the disutility of the investment.

While our numerical experiments suggest that both the attacker and the defender concentrate their efforts at the most valuable targets, the inherent defense levels at the targets, i.e., the measures that are already in place at the beginning of the competition, are also critical in determining the effective values of the targets. For example, a target that is very valuable for the attacker, but also already very well protected by the defender may not be worth the effort for the attacker, so a sufficiently large existing defense level at a target can help the defender deter a possible attack without even making any additional investments.

Our comparison of the Nash equilibrium in the simultaneous-move game and the subgame-perfect Nash equilibrium in the sequential-move version over numerical experiments suggests that the defender enjoys the advantage of being the first mover by obtaining a larger expected utility when there are two or more targets.

Potential directions for future research include the extension of the model to include multiple defenders and multiple attackers, who compete over several targets and may or may not be allowed to cooperate among themselves, the relaxation of the complete or perfect information assumptions, and the dynamic game where the attacker and the defender compete over a time horizon, making decision at several time points.

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Appendix A

MATLAB CODES

A.1 Single-Target Game

```

function onetargetgame
%Initialize parameters
B=2;
kA=2;
kD=2;
c1=2;
v1=10;
w1=15;
i=18;
eq=[];

%Find the equilibrium investments and utilities vs. w1
for w1=14:0.2:i
    x1=(sqrt(v1*kA*(kD*B+c1))-(kD*B+c1))/kA;
    if x1<0
        x1=0;
    end
    y1=B;
    UA1=(sqrt(v1)-sqrt((kD*B+c1)/kA))^2;
    UD1=w1*((sqrt((kD*B+c1)/(v1*kA)))-1);
    eq=[eq; w1, x1, y1, UA1, UD1];
end

figure
plot(eq(:,1),eq(:,2:5))
plot(eq(:,1),eq(:,2),'b—s',eq(:,1),eq(:,3),'b^—',eq(:,1),eq(:,4),'ro
    —',eq(:,1),eq(:,5),'rd—')
legend('x1','y1','UA1','UD1')

```

```

%Find the equilibrium investments and utilities vs. v1
w1=15;
i=13;
eq=[];
for v1=7:0.2:i
    x1=(sqrt(v1*kA*(kD*B+c1))-(kD*B+c1))/kA;
    if x1<0
        x1=0;
    end
    y1=B;
    UA1=(sqrt(v1)-sqrt((kD*B+c1)/kA))^2;
    UD1=w1*((sqrt((kD*B+c1)/(v1*kA)))-1);
    eq=[eq; v1, x1, y1, UA1, UD1];
end
figure
plot(eq(:,1),eq(:,2:5))
plot(eq(:,1),eq(:,2),'b—s',eq(:,1),eq(:,3),'b^—',eq(:,1),eq(:,4),'ro
—',eq(:,1),eq(:,5),'rd—')
legend('x1','y1','UA1','UD1')

%Find the equilibrium investments and utilities vs. kA
v1=10;
i=6;
eq=[];
for kA=1:0.2:i
    x1=(sqrt(v1*kA*(kD*B+c1))-(kD*B+c1))/kA;
    if x1<0
        x1=0;
    end
    y1=B;
    UA1=(sqrt(v1)-sqrt((kD*B+c1)/kA))^2;
    UD1=w1*((sqrt((kD*B+c1)/(v1*kA)))-1);
    eq=[eq; kA, x1, y1, UA1, UD1];
end
figure
plot(eq(:,1),eq(:,2:5))
plot(eq(:,1),eq(:,2),'b—s',eq(:,1),eq(:,3),'b^—',eq(:,1),eq(:,4),'ro

```

```

    -',eq(:,1),eq(:,5),'rd-')
legend('x1','y1','UA1','UD1')

%Find the equilibrium investments and utilities vs. kD
kA=2;
i=6;
eq=[];
for kD=1:0.2:i
    x1=(sqrt(v1*kA*(kD*B+c1))-(kD*B+c1))/kA;
    if x1<0
        x1=0;
    end
    y1=B;
    UA1=(sqrt(v1)-sqrt((kD*B+c1)/kA))^2;
    UD1=w1*((sqrt((kD*B+c1)/(v1*kA)))-1);
    eq=[eq; kD, x1, y1, UA1, UD1];
end
figure
plot(eq(:,1),eq(:,2:5))
plot(eq(:,1),eq(:,2),'b—s',eq(:,1),eq(:,3),'b^—',eq(:,1),eq(:,4),'ro
    -',eq(:,1),eq(:,5),'rd-')
legend('x1','y1','UA1','UD1')

%Find the equilibrium investments and utilities vs. c1
kD=2;
i=7;
eq=[];
for c1=0.2:0.2:i
    x1=(sqrt(v1*kA*(kD*B+c1))-(kD*B+c1))/kA;
    if x1<0
        x1=0;
    end
    y1=B;
    UA1=(sqrt(v1)-sqrt((kD*B+c1)/kA))^2;
    UD1=w1*((sqrt((kD*B+c1)/(v1*kA)))-1);
    eq=[eq; c1, x1, y1, UA1, UD1];
end

```

figure

plot(eq(:,1),eq(:,2:5))

plot(eq(:,1),eq(:,2),'b—s',eq(:,1),eq(:,3),'b^—',eq(:,1),eq(:,4),'ro—',eq(:,1),eq(:,5),'rd—')

legend('x1','y1','UA1','UD1')

%Find the equilibrium investments and utilities vs. B

c1=2;

i=7;

eq=[];

for B=0.2:0.2:i

 x1=(**sqrt**(v1*kA*(kD*B+c1))-(kD*B+c1))/kA;

if x1<0

 x1=0;

end

 y1=B;

 UA1=(**sqrt**(v1)-**sqrt**((kD*B+c1)/kA))^2;

 UD1=w1*((**sqrt**((kD*B+c1)/(v1*kA)))-1);

 eq=[eq; B, x1, y1, UA1, UD1];

end

figure

plot(eq(:,1),eq(:,2:5))

plot(eq(:,1),eq(:,2),'b—s',eq(:,1),eq(:,3),'b^—',eq(:,1),eq(:,4),'ro—',eq(:,1),eq(:,5),'rd—')

legend('x1','y1','UA1','UD1')

A.2 Two-Target Game

function twotargetgame

c1=2;

c2=2;

v1=30;

v2=20;

w1=80; %20 %40

w2=20; %80 %40

kA=2;

kD=2;

```

B=10;
j=1;

nashx1=[];
nashy1=[];
nashx2=[];
nashy2=[];
utilityD2=[];
utilityA2=[];
utilityD1=[];
utilityA1=[];
SolutionMatrix=[];

%Set the parameter to conduct sensitivity analysis
answer = inputdlg({'Choose the game type','Choose the parameter'},...
    'Sensitivity Analysis', [1 50; 1 50]);

%Find the equilibrium investments and utilities
if answer{2,1}=='B'
    for B=j:20
        if answer{1,1}=='sim'
            [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
                nashy1 , nashx1]=myfunsimultaneous(v1 , v2 , w1 , w2 , c1 , c2 , kA ,
                kD , B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 ,
                nashx2 , nashy1 , nashx1);
        elseif answer{1,1}=='seq'
            [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
                nashy1 , nashx1]=myfunsequential(v1 , v2 , w1 , w2 , c1 , c2 , kA , kD ,
                B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2
                , nashy1 , nashx1);
        end
        y2=nashy2(B-(j-1));
        x2=nashx2(B-(j-1));
        y1=nashy1(B-(j-1));
        x1=nashx1(B-(j-1));
        D2=utilityD2(B-(j-1));
        A2=utilityA2(B-(j-1));
    end
end

```

```

D1=utilityD1(B-(j-1));
A1=utilityA1(B-(j-1));
SolutionMatrix=[SolutionMatrix; B, y2, x2, y1, x1, D2, A2, D1,
    A1];
end
disp('      B          y_2          x_2          y_1          x_1          UD2
      UA2          UD1          UA1      ')
disp(SolutionMatrix)
elseif answer{2,1}=='v1'
    for v1=j:40
        if answer{1,1}=='sim'
            [utilityD2, utilityA2, utilityD1, utilityA1, nashy2, nashx2,
                nashy1, nashx1]=myfunsimultaneous(v1, v2, w1, w2, c1, c2, kA,
                kD, B, utilityD2, utilityA2, utilityD1, utilityA1, nashy2,
                nashx2, nashy1, nashx1);
        elseif answer{1,1}=='seq'
            [utilityD2, utilityA2, utilityD1, utilityA1, nashy2, nashx2,
                nashy1, nashx1]=myfunsequential(v1, v2, w1, w2, c1, c2, kA, kD,
                B, utilityD2, utilityA2, utilityD1, utilityA1, nashy2, nashx2
                , nashy1, nashx1);
        end
        y2=nashy2((v1-(j-1)));
        x2=nashx2((v1-(j-1)));
        y1=nashy1((v1-(j-1)));
        x1=nashx1((v1-(j-1)));
        D2=utilityD2((v1-(j-1)));
        A2=utilityA2((v1-(j-1)));
        D1=utilityD1((v1-(j-1)));
        A1=utilityA1((v1-(j-1)));
        SolutionMatrix=[SolutionMatrix; v1, y2, x2, y1, x1, D2, A2, D1
            , A1];
    end
    disp('      v1          y_2          x_2          y_1          x_1          UD2
      UA2          UD1          UA1      ')
    disp(SolutionMatrix)
elseif answer{2,1}=='v2'
    for v2=j:29

```

```

if answer{1,1}=='sim'
    [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
     nashy1 , nashx1]=myfunsimultaneous(v1 , v2 , w1 , w2 , c1 , c2 , kA ,
     kD , B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 ,
     nashx2 , nashy1 , nashx1);
elseif answer{1,1}=='seq'
    [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
     nashy1 , nashx1]=myfunsequential(v1 , v2 , w1 , w2 , c1 , c2 , kA , kD ,
     B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2
     , nashy1 , nashx1);

end
y2=nashy2((v2-(j-1)));
x2=nashx2((v2-(j-1)));
y1=nashy1((v2-(j-1)));
x1=nashx1((v2-(j-1)));
D2=utilityD2((v2-(j-1)));
A2=utilityA2((v2-(j-1)));
D1=utilityD1((v2-(j-1)));
A1=utilityA1((v2-(j-1)));
SolutionMatrix=[SolutionMatrix; v2 , y2 , x2 , y1 , x1 , D2 , A2 , D1
    , A1];

end
disp('      v2          y_2          x_2          y_1          x_1          UD2
      UA2          UD1          UA1      ')

disp(SolutionMatrix)
elseif answer{2,1}=='c1'
    for c1=j:19
        if answer{1,1}=='sim'
            [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
             nashy1 , nashx1]=myfunsimultaneous(v1 , v2 , w1 , w2 , c1 , c2 , kA ,
             kD , B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 ,
             nashx2 , nashy1 , nashx1);
        elseif answer{1,1}=='seq'
            [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
             nashy1 , nashx1]=myfunsequential(v1 , v2 , w1 , w2 , c1 , c2 , kA , kD ,
             B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2
             , nashy1 , nashx1);
        
```

```

end
y2=nashy2(c1);
x2=nashx2(c1);
y1=nashy1(c1);
x1=nashx1(c1);
D2=utilityD2(c1);
A2=utilityA2(c1);
D1=utilityD1(c1);
A1=utilityA1(c1);
SolutionMatrix=[SolutionMatrix; c1, y2, x2, y1, x1, D2, A2, D1
, A1];
end
disp('      c1      y_2      x_2      y_1      x_1      UD2
      UA2      UD1      UA1  ')
disp(SolutionMatrix)
elseif answer{2,1}=='c2'
for c2=j:19
if answer{1,1}=='sim'
[utilityD2, utilityA2, utilityD1, utilityA1, nashy2, nashx2,
nashy1, nashx1]=myfunsimultaneous(v1, v2, w1, w2, c1, c2, kA,
kD, B, utilityD2, utilityA2, utilityD1, utilityA1, nashy2,
nashx2, nashy1, nashx1);
elseif answer{1,1}=='seq'
[utilityD2, utilityA2, utilityD1, utilityA1, nashy2, nashx2,
nashy1, nashx1]=myfunsequential(v1, v2, w1, w2, c1, c2, kA, kD,
B, utilityD2, utilityA2, utilityD1, utilityA1, nashy2, nashx2
, nashy1, nashx1);
end
y2=nashy2(c2);
x2=nashx2(c2);
y1=nashy1(c2);
x1=nashx1(c2);
D2=utilityD2(c2);
A2=utilityA2(c2);
D1=utilityD1(c2);
A1=utilityA1(c2);
SolutionMatrix=[SolutionMatrix; c2, y2, x2, y1, x1, D2, A2, D1

```

```

        , A1];

    end

    disp('      c2      y_2      x_2      y_1      x_1      UD2
          UA2      UD1      UA1      ')

    disp(SolutionMatrix)
elseif answer{2,1}=='w1'
    for w1=j:50
        if answer{1,1}=='sim'
            [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
              nashy1 , nashx1]=myfunsimultaneous(v1 , v2 , w1 , w2 , c1 , c2 , kA ,
              kD , B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 ,
              nashx2 , nashy1 , nashx1);
        elseif answer{1,1}=='seq'
            [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
              nashy1 , nashx1]=myfunsequential(v1 , v2 , w1 , w2 , c1 , c2 , kA , kD ,
              B , utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2
              , nashy1 , nashx1);
        end
        y2=nashy2((w1-(j-1)));
        x2=nashx2((w1-(j-1)));
        y1=nashy1((w1-(j-1)));
        x1=nashx1((w1-(j-1)));
        D2=utilityD2((w1-(j-1)));
        A2=utilityA2((w1-(j-1)));
        D1=utilityD1((w1-(j-1)));
        A1=utilityA1((w1-(j-1)));
        SolutionMatrix=[SolutionMatrix; w1, y2, x2, y1, x1, D2, A2, D1
            , A1];
    end

    disp('      w1      y_2      x_2      y_1      x_1      UD2
          UA2      UD1      UA1      ')

    disp(SolutionMatrix)
elseif answer{2,1}=='w2'
    for w2=j:50
        if answer{1,1}=='sim'
            [utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
              nashy1 , nashx1]=myfunsimultaneous(v1 , v2 , w1 , w2 , c1 , c2 , kA ,

```

```

        kD,B, utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 ,
        nashx2 , nashy1 , nashx1 );
    elseif answer{1,1}=='seq'
        [ utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 ,
        nashy1 , nashx1]=myfunsequential(v1,v2,w1,w2,c1,c2,kA,kD,
        B, utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2
        , nashy1 , nashx1 );
    end
    y2=nashy2((w2-(j-1)));
    x2=nashx2((w2-(j-1)));
    y1=nashy1((w2-(j-1)));
    x1=nashx1((w2-(j-1)));
    D2=utilityD2((w2-(j-1)));
    A2=utilityA2((w2-(j-1)));
    D1=utilityD1((w2-(j-1)));
    A1=utilityA1((w2-(j-1)));
    SolutionMatrix=[SolutionMatrix; w2, y2, x2, y1, x1, D2, A2, D1
    , A1];
end
disp('      w2          y_2      x_2          y_1          x_1      UD2
      UA2          UD1      UA1      ')
disp(SolutionMatrix)
end
figure
plot(SolutionMatrix(:,1),SolutionMatrix(:,3),'b—s',SolutionMatrix
(:,1),SolutionMatrix(:,2),'b^—',SolutionMatrix(:,1),SolutionMatrix
(:,5),'ro—',SolutionMatrix(:,1),SolutionMatrix(:,4),'rd—')
legend('x2','y2','x1','y1')
figure
plot(SolutionMatrix(:,1),SolutionMatrix(:,7),'b—s',SolutionMatrix
(:,1),SolutionMatrix(:,6),'b^—',SolutionMatrix(:,1),SolutionMatrix
(:,9),'ro—',SolutionMatrix(:,1),SolutionMatrix(:,8),'rd—')
legend('UA2','UD2','UA1','UD1')

```

A.2.1 Two-Target Simultaneous-Move Game

```

function [ utilityD2 , utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 , nashy1
, nashx1]=myfunsimultaneous(v1,v2,w1,w2,c1,c2,kA,kD,B, utilityD2 ,

```

```

    utilityA2 , utilityD1 , utilityA1 , nashy2 , nashx2 , nashy1 , nashx1)
clear x1; clear y1; clear x2; clear y2; clear UD1; clear UD2; clear
    UA1; clear UA2;
syms a1; syms a2; syms y;
format short
sol = [];
if ((v1+v2)*kA>kD*B+c1+c2 & v2*kA<=c2) || ((v1+v2)*kA<=kD*B+c1+c2 & v1
    *kA>kD*B+c1)
    y1=B;
    y2=0;
    x1=(sqrt(v1*kA*(kD*B+c1))-kD*B-c1)/kA;
    x2=0;
    UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
    UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
elseif v2*kA>c2 & (v1+v2)*kA>kD*B+c1+c2
    if (w2/v2)*(sqrt(v2*kA/c2)-1)<=(w1/v1)*(sqrt(v1*kA/(kD*B+c1))-1)
        y2=0;
        x2=(sqrt(v2*kA*c2)-c2)/kA;
        y1=B;
        x1=(sqrt(v1*kA*(kD*B+c1))-kD*B-c1)/kA;
        UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+c1);
        UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
        UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
        UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
    elseif (w1/v1)*(sqrt(v1*kA/c1)-1)<=(w2/v2)*(sqrt(v2*kA/(kD*B+c2))-
        -1)
        y1=0;
        y2=B;
        x1=(sqrt(v1*kA*c1)-c1)/kA;
        x2=(sqrt(v2*kA*(kD*B+c2))-kD*B-c2)/kA;
        UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
        UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
        UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+c1);
        UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
    elseif (w2/v2)*(sqrt(v2*kA/c2)-1)>(w1/v1)*(sqrt(v1*kA/(kD*B+c1))

```

```

-1) & (w1/v1)*(sqrt(v1*kA/c1)-1)>(w2/v2)*(sqrt(v2*kA/(kD*B+c2))
-1)
foc1=(kA*a1+kD*(B-y)+c1)^2-v1*kA*(kD*(B-y)+c1);
foc2=(kA*a2+kD*y+c2)^2-v2*kA*(kD*y+c2);
foc3=(w1*a1)*(kA*a2+kD*y+c2)^2-(w2*a2)*(kA*a1+kD*(B-y)+c1)^2;
sol1 = feval(symengine, 'numeric::solve', {foc1, foc2, foc3});
for i=1:length(sol1)
    s=sol1(i);
    s1=double(solve(s(1)));
    s2=double(solve(s(2)));
    s3=double(solve(s(3)));
    if s1>=0 & s2>=0 & s3>=0 & s3<=B
        y2=s3;
        x1=s1;
        x2=s2;
        y1=B-y2;
        UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
        UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
        UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+
        c1);
        UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1
        +c1)-x1;
    end
end
end
elseif (v1+v2)*kA<=kD*B+c1+c2 & v1*kA<=kD*B+c1
    x1=0;
    x2=0;
    y1=max(0, (v1*kA-c1)/kD);
    y2=max(0, (v2*kA-c2)/kD);
    UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
    UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
end
nashy2=[nashy2 y2];
nashx2=[nashx2 x2];

```

```

nashy1=[nashy1 y1];
nashx1=[nashx1 x1];
utilityD2=[utilityD2 UD2];
utilityA2=[utilityA2 UA2];
utilityD1=[utilityD1 UD1];
utilityA1=[utilityA1 UA1];

```

A.2.2 Two-Target Sequential-Move Game

```

function [utilityD2,utilityA2,utilityD1,utilityA1,nashy2,nashx2,nashy1,
    ,nashx1]=myfunsequential(v1,v2,w1,w2,c1,c2,kA,kD,B,utilityD2,
    ,utilityA2,utilityD1,utilityA1,nashy2,nashx2,nashy1,nashx1)
clear x1; clear y1; clear x2; clear y2; clear UD1; clear UD2; clear
    UA1; clear UA2;
syms a1; syms a2; syms y;
if ((v1+v2)*kA>kD*B+c1+c2 & v2*kA<=c2) || ((v1+v2)*kA<=kD*B+c1+c2 & v1
    *kA>kD*B+c1)
    y2=0;
    x2=0;
    y1=B;
    x1=(sqrt(v1*kA*(kD*B+c1))-kD*B-c1)/kA;
    UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
    UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
elseif v2*kA>c2 & (v1+v2)*kA>kD*B+c1+c2
    tildey=(w2^2*v1*(kD*B+c1)-w1^2*v2*c2)/(kD*(w1^2*v2+w2^2*v1));
    if tildey>=min(B,(v2*kA-c2)/kD)
        y2=min(B,(v2*kA-c2)/kD);
    elseif tildey<=max(0,B-(v1*kA-c1)/kD)
        y2=max(0,B-(v1*kA-c1)/kD);
    else
        y2=tildey;
    end
    x2=(sqrt(v2*kA*(kD*y2+c2))-kD*y2-c2)/kA;
    y1=B-y2;
    x1=(sqrt(v1*kA*(kD*(B-y2)+c1))-kD*(B-y2)-c1)/kA;
    UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+c1);

```

```

    UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
    UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
elseif (v1+v2)*kA<=kD*B+c1+c2 & v1*kA<=kD*B+c1
    y2=max(0,(v2*kA-c2)/kD);
    x2=0;
    y1=max(0,(v1*kA-c1)/kD);
    x1=0;
    UD2=-w2*x2*kA/(kA*x2+kD*y2+c2)-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA2=v1*x2*kA/(kA*x2+kD*y2+c2)-x2+v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
    UD1=-w1*x1*kA/(kA*x1+kD*y1+c1);
    UA1=v1*x1*kA/(kA*x1+kD*y1+c1)-x1;
end
nashy2=[nashy2 y2];
nashx2=[nashx2 x2];
nashy1=[nashy1 y1];
nashx1=[nashx1 x1];
utilityD2=[utilityD2 UD2];
utilityA2=[utilityA2 UA2];
utilityD1=[utilityD1 UD1];
utilityA1=[utilityA1 UA1];

```

A.3 Multi-Target Sequential-Move Game

```

function multitargetgame
%Initialize parameters
kA=2;
kD=2;
i=41;
r=0.02;
c=[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5];
v=[21 21 3 21 27 30 75 180 6 30];
w=[63 190 75 27 180 120 1000 75 120 225];
format short

%Find optimum defensive investment expecting the attacker's best
response

```

```

for n=1:10
    for B=1:r:i
        if n==1
            x1=(sqrt(v(1,1)*kA*(kD*(B-1)+c(1,1)))-kD*(B-1)-c(1,1))/kA;
            if x1<=0 || v(1,1)*kA<=kD*(B-1)+c(1,1)
                x1=0;
            end
            y(1,round(B/r))=B-1;
            JD(1,round(B/r))=-w(1,1)*kA*x1/(kA*x1+kD*(B-1)+c(1,1));
        else
            J2=[];
            for j=0:r:(B-1)
                x1=(sqrt(v(1,n)*kA*(kD*j+c(1,n)))-kD*j-c(1,n))/kA;
                if x1<=0 || v(1,n)*kA<=kD*j+c(1,n)
                    x1=0;
                end
                J2=[J2,-w(1,n)*kA*x1/(kA*x1+kD*j+c(1,n))+JD(n-1,round
                    ((B-j)/r))];
                [maxv, ind]=max(J2);
            end
            y(n,round(B/r))=(ind-1)*r;
            JD(n,round(B/r))=maxv;
        end
    end
end

```

%Identify defender's investments and utilities

```

for B=i:-r:1
    investmenty(round(B/r),n)=y(n,round(B/r));
    utilityD(round(B/r),n)=JD(n,round(B/r));
end
for p=n-1:-1:1
    for B=i:-r:1
        investmenty(round(B/r),p)=y(p,round((B-sum(investmenty(round(B
            /r),p+1:end))))/r));
        utilityD(round(B/r),p)=JD(p,round((B-sum(investmenty(round(B/r
            ),p+1:end))))/r));
    end
end

```

```

    end
end

%Identify attacker's investments and utilities
for B=1:r:i
    for p=1:n
        investmentx(round(B/r),p)=(sqrt(v(1,p)*kA*(kD*investmenty(
            round(B/r),p)+c(1,p)))-kD*investmenty(round(B/r),p)-c(1,p))
            /kA;
        if investmentx(round(B/r),p)<=0 || v(1,p)*kA<=kD*investmenty(
            round(B/r),p)+c(1,p)
            investmentx(round(B/r),p)=0;
        end
        if p==1
            utilityA(round(B/r),p)=v(1,p)*kA*investmentx(round(B/r),p)
                /(kA*investmentx(round(B/r),p)+kD*investmenty(round(B/r),
                p)+c(1,p))-investmentx(round(B/r),p);
        else
            utilityA(round(B/r),p)=v(1,p)*kA*investmentx(round(B/r),p)
                /(kA*investmentx(round(B/r),p)+kD*investmenty(round(B/r),
                p)+c(1,p))-investmentx(round(B/r),p)+utilityA(round(B
                /r),p-1);
        end
    end
end
end

utility=[];
defence=[];
offence=[];
for B=1:i
    utility=[utility; B-1, utilityD(round(B/r),n:-1:1), utilityA(round
        (B/r),n:-1:1)];
    defence=[defence; B-1, round(investmenty(round(B/r),1:n),2)];
    offence=[offence; B-1, round(investmentx(round(B/r),1:n),2)];
end
utility
defence

```

offence

figure

```
plot(defence(:,1),defence(:,2),'r—s',defence(:,1),defence(:,3),'bo—',
      defence(:,1),defence(:,4),'g^—',defence(:,1),defence(:,5),'kd—',
      defence(:,1),defence(:,6),'y*—')
```

```
legend('y1','y2','y3','y4','y5')
```

figure

```
plot(defence(:,1),defence(:,7),'r—s',defence(:,1),defence(:,8),'bo—',
      defence(:,1),defence(:,9),'g^—',defence(:,1),defence(:,10),'kd—',
      defence(:,1),defence(:,11),'y*—')
```

```
legend('y6','y7','y8','y9','y10')
```

figure

```
plot(offence(:,1),offence(:,2),'r—s',offence(:,1),offence(:,3),'bo—',
      offence(:,1),offence(:,4),'g^—',offence(:,1),offence(:,5),'kd—',
      offence(:,1),offence(:,6),'y*—')
```

```
legend('x1','x2','x3','x4','x5')
```

figure

```
plot(offence(:,1),offence(:,7),'r—s',offence(:,1),offence(:,8),'bo—',
      offence(:,1),offence(:,9),'g^—',offence(:,1),offence(:,10),'kd—',
      offence(:,1),offence(:,11),'y*—')
```

```
legend('x6','x7','x8','x9','x10')
```

figure

```
plot(utility(:,1),utility(:,2),'r—s',utility(:,1),utility(:,12),'bo—'
      )
```

```
legend('UD10','UA10')
```