COMPLEX CURVES IN R^4

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

In this study, the concepts complex elements and complex curves are extended to the real space R^4 which are defined for the real space R^3.

In the introduction it is summarized necessary knowledge about the concepts complex elements and complex curves in R^3 and this study is based on these concepts.

In the chapter two complex elements and complex curves are defined in the real space R^4 and cyclic four-faced is established.

In the chapter three minimal elements and minimal curves are defined. Necessary and sufficient conditions are presented for a complex flat to be a minimal flat. Frenet frame is established for a minimal curve in the real space R^4, and also it is shown that the tangent vectors and the binormal vectors of the curve are minimal vectors.

Also, the cyclic four-faced which is found in the chapter two is expressed with the position vectors of the minimal curves.

ÖZET

Bu çalışmada, R^3 gerçel uzayında tanımlanan sanal elemanlar ve sanal eğriler kavramları R^4 gerçel uzayına genişletilmiştir.

Giriş bölümünde R^3 gerçel uzayındaki sanal elemanlar ve sanal eğriler kavramları hakkında gerekli bilgiler özetlenmiştir ve çalışmamız bu kavramlar üzerine kurulmuştur.

İkinci bölümde R^4 gerçel uzayında sanal elemanlar ve sanal eğriler tanımlanmış ve siklik dörtyüzlü kurulmuştur.

Üçüncü bölümde R^4 gerçel uzayında minimal elemanlar ve minimal eğriler tanımlanmıştır. Bir sanal flatin minimal flat olması için gerek ve yeter şartlar ortaya konulmuştur. R^4 gerçel uzayında bir minimal eğrinin Frenet çatısı kurulmuş ve eğrinin teğet birim vektörlerinin ve binormal birim vektörlerinin minimal vektörler olduğu gösterilmiştir.

Ayrıca ikinci bölümde bulunan siklik dörtyüzlü minimal eğrilerin yer vektörlerine bağlı olarak ifade edilmiştir.

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CHAPTER ONE

INTRODUCTION

The concepts complex elements and complex curves are defined and the main theorems about these concepts are proved by Şemin. Then minimal elements are defined and the properties of the minimal elements are given by Şemin. Also, the minimal elements are defined by Woods in the analytic representation.

In this chapter, it is summarized necessary knowledge about the concepts complex elements and complex curves are defined, and also the main theorems are given without proof. Minimal elements and minimal curves are defined and the natural arc parameter of a minimal curve is given. Frenet frame of a minimal curve is established and cyclic three-faced is given by using Frenet frame. Finally the properties of a minimal curve are examined by using the Frenet frame and the cyclic three-faced of the minimal curve. This study is based on these concepts.

1.1. COMPLEX ELEMENTS IN R^3

Definition 1.1. Let Ox_p with the unit vectors $\vec{k}_p$ for $p = 1,2,3$ be an orthonormal axis system of the three dimensional real space R^3. A point $N(x_1, x_2, x_3)$ is called a complex point in R^3 if any component of the point is a complex number.

Definition 1.2. Any two complex points of which components are respectively conjugate to each other are said to be conjugate complex points.

Definition 1.3. Let $N(x_1, x_2, x_3)$ be a complex point in R^3. The complex numbers $X_1; X_2; X_3; X_4$ are said to be the homogeneous coordinates of the complex point if there is a relation $x_p = \frac{X_p}{X_4}$ for $p = 1,2,3$. Thus a complex point in R^3 is also denoted by $N(X_1, X_2, X_3, X_4)$.

Definition 1.4. The following equation

$$D = \sum_{p=1}^{3} A_p X_p = 0 \tag{1.1}$$

defines a complex line on the real plane Ox_1x_2 if any coefficient A_p is a complex number and there is not a relation $A_p = ca_p$ such that a_p are real numbers for $p = 1,2,3$ in the homogeneous coordinates.

Definition 1.5. Any two complex lines which are given by the equations

$$D = \sum_{p=1}^{3} A_p X_p = 0 \text{ and } \overline{D} = \sum_{p=1}^{3} \overline{A}_p X_p = 0 \tag{1.2}$$

where the numbers A_p and $\overline{A}_p$ for $p = 1,2,3$ are respectively conjugate to each other are said to be conjugate lines.

Theorem 1.1. There is only one real point on a complex line and the conjugate line of the complex line also passes through the same point.

Theorem 1.2. There is only one real line which passes through a complex point.

Theorem 1.3. A line which passes through any two conjugate complex points is a real line.

Definition 1.6. The following equation

$$\Delta = \sum_{p=1}^{4} A_p X_p = 0 \tag{1.3}$$

defines a complex plane in R^3 if any coefficient A_p is a complex number and there is not a relation $A_p = ca_p$ such that a_p are real numbers for $p = 1,2,3,4$ in the homogenous coordinates.

Definition 1.7. Any two complex planes which are given by two equations

$$\Delta = \sum_{p=1}^{4} A_p X_p = 0 \text{ and } \overline{\Delta} = \sum_{p=1}^{4} \overline{A}_p X_p = 0 \tag{1.4}$$

where the numbers A_p and $\overline{A}_p$ are respectively conjugate to each other for $p = 1,2,3,4$ are said to be conjugate planes.

Theorem 1.4. A complex plane has only one real line which is the intersection of the real planes which belong to the complex plane.

Theorem 1.5. A real line on a complex plane belongs to the conjugate plane of the complex plane. In other words, the intersection of a complex plane and conjugate of it is a real line.

Definition 1.8. The intersection of any two non-parallel complex plane is said to be a complex line in R^3.

Definition 1.9. Any two complex lines which are respectively determined by two complex planes and the conjugate planes of them are said be conjugate lines in R^3.

Theorem 1.6. A complex line and the conjugate line of it have only one real point.

Theorem 1.7. If a real line passes through a complex point then it passes through the conjugate point of the complex point.

Definition 1.10. Let θ be an angle between two axis on the real plane Ox_1x_2. A complex line

$$x_2 - a_2 = m(x_1 - a_1) \tag{1.5}$$

passing through a point $A(a_1, a_2)$ which is a real point or a complex point is called an izotrop complex line or an izotrop line or a minimal line where

$$m = -\cos\theta \pm \sqrt{-\sin^2\theta} = e^{-\varepsilon i\theta}, \ \varepsilon = \pm 1. \tag{1.6}$$

Theorem1.8. There is only one real point on a minimal line and any two minimal lines which pass through the same real point are conjugate to each other.

Theorem 1.9. A complex line is a minimal line iff the complex line is perpendicular to itself.

Theorem 1.10. A complex line is a minimal line iff the square of the distance between any two points of it is zero.

Definition 1.11. The complex points of which the square of the distance from the origin is zero define a cone which is called an izotrop cone or a minimal cone with the apex origin and is given by the equation

$$\sum_{p=1}^{3} x_p^2 = 0. \tag{1.7}$$

A minimal cone with the apex $A(a_1, a_2, a_3)$ which is a real point or a complex point is given by the equation

$$\sum_{p=1}^{3} (x_p - a_p)^2 = 0. \tag{1.8}$$

The intersections of a real plane passing through the origin and a minimal cone with the apex origin are generally two minimal lines passing through the origin. These minimal lines are two directrixes of the minimal cone so a minimal cone in the real space R^3 is also defined as below.

Definition 1.12. A locus of a minimal lines passing through the origin is a minimal cone with the apex origin.

Definition 1.13. A complex plane is called an izotrop plane or a minimal plane if the normal line of the complex plane is a minimal line.

The minimal cone given by the equation (1.8) is also given by the following equation in the homogeneous coordinates

$$\sum_{p=1}^{3}\left(X_p - a_p X_4\right)^2 = 0. \tag{1.9}$$

The intersection of a minimal cone and a plane at infinity is a quadratic curve and given by the following equation

$$\sum_{p=1}^{3} X_p^2 = 0, \; X_4^2 = 0 \tag{1.10}$$

The equation (1.10) defines all minimal cones.

Definition 1.14. This curve which is given by the equation (1.10) is said to be an absolute of the real plane at infinity. Now the third definition of a minimal cone can be given.

Definition 1.15. A cone is called a minimal cone in R^3 if the directrix of it is the absolute.

Definition 1.16. A direction is a minimal direction in R^3 if the point of the direction at infinity is on the absolute.

Theorem 1.11. A line which has the direction (b_1, b_2, b_3) is a minimal line iff

$$\sum_{p=1}^{3} b_p^2 = 0. \tag{1.11}$$

Definition 1.17. A vector which has a minimal direction is called an izotrop vector or a minimal vector.

Theorem 1.12. A vector $\vec{v}$ is a minimal vector iff

$$\vec{v}^2 = 0. \tag{1.12}$$

Definition 1.18. The intersection of a complex plane and the absolute is called cyclic point of the complex plane.

Definition 1.19. A plane which is tangent to the absolute is called a minimal plane.

Theorem 1.13. A plane given by the equation

$$b_1x_1 + b_2x_2 + b_3x_3 + b_4 = 0 \tag{1.13}$$

is a minimal plane iff

$$\sum_{p=1}^{3} b_p^2 = 0. \tag{1.14}$$

A normal line of a minimal plane at any point A is on the minimal plane and the minimal line is tangent to a minimal cone with the apex A along the minimal line. In other words, the normal line is a directrix of the minimal cone.

Definition 1.20. Let $\vec{k}_1, \vec{k}_2, \vec{k}_3$ be the unit vectors of a frame of R^3. The set A of the vectors given by

$$\begin{aligned}
\vec{e}_1 &= \frac{1}{\sqrt{2}}\left(\vec{k}_1 - i\vec{k}_3\right) \\
\vec{e}_2 &= \vec{k}_2 \\
\vec{e}_3 &= \frac{1}{\sqrt{2}}\left(\vec{k}_1 + i\vec{k}_3\right)
\end{aligned} \tag{1.15}$$

is called cyclic three-faced of Cartan.

The inner products of the vectors are given as below

$$\vec{e}_i.\vec{e}_j = \begin{cases} 0, & i+j \neq 4 \\ 1, & i+j=4. \end{cases} \tag{1.16}$$

The inner products show that the vectors $\vec{e}_1$ and $\vec{e}_3$ are minimal vectors, but the vector $\vec{e}_2$ is a complex vector. The vector products on the set A are

$$\begin{aligned}
\vec{e}_1 \wedge \vec{e}_2 &= i\vec{e}_1 \\
\vec{e}_2 \wedge \vec{e}_3 &= i\vec{e}_3 \\
\vec{e}_3 \wedge \vec{e}_1 &= i\vec{e}_2 .
\end{aligned} \tag{1.17}$$

Also, the following products occur on the set A

$$\left(\vec{e}_1, \vec{e}_2, \vec{e}_3\right) = \left(\vec{e}_2, \vec{e}_3, \vec{e}_1\right) = \left(\vec{e}_3, \vec{e}_1, \vec{e}_2\right) = i . \tag{1.18}$$

The unit vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ are also given in terms of the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\begin{aligned}
\vec{k}_1 &= \frac{1}{\sqrt{2}}\left(\vec{e}_1 + \vec{e}_3\right) \\
\vec{k}_2 &= \vec{e}_2 \\
\vec{k}_3 &= \frac{i}{\sqrt{2}}\left(\vec{e}_1 - \vec{e}_3\right).
\end{aligned} \tag{1.19}$$

Let the vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ be the unit vectors of a moving frames with respect to a variable t. Then

$$\vec{k}_p{}'(t) = a_{p1}\vec{k}_1(t) + a_{p2}\vec{k}_2(t) + a_{p3}\vec{k}_3(t) \tag{1.20}$$

where a_{p1}, a_{p2}, a_{p3} are real coefficients for $p = 1,2,3$. There are the following relations for $p = 1,2,3$ and $m = 1,2,3$

if $m = p$ then $a_{pm} = 0$,

if $m \neq p$ then $a_{pm} + a_{mp} = 0$. (1.21)

Using the relations (1.21) and the formulae (1.19) in the formulae (1.20) for $p = 1,2,3$ and $m = 1,2,3$ the following formulae are obtained

$$\begin{aligned}
\vec{e}_1{}'(t) &= \frac{i}{\sqrt{2}}\left[a_{13}\vec{e}_1(t) + \left(a_{12} + ia_{23}\right)\vec{e}_2(t)\right] \\
\vec{e}_2{}'(t) &= \frac{1}{\sqrt{2}}\left[\left(-a_{12} + ia_{23}\right)\vec{e}_1(t) - \left(a_{12} + ia_{23}\right)\vec{e}_3(t)\right] \\
\vec{e}_3{}'(t) &= \frac{1}{\sqrt{2}}\left[\left(a_{12} - ia_{23}\right)\vec{e}_2(t) - i\sqrt{2}a_{13}\vec{e}_3(t)\right].
\end{aligned} \tag{1.22}$$

If the vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ are respectively the vectors $\vec{T}, \vec{N}, \vec{B}$ which are Frenet Vectors of a curve with the arc parameter t then the coefficients are obtained as

$$a_{12} = \kappa,\ a_{13} = 0,\ a_{23} = \tau \tag{1.23}$$

such that κ and τ are respectively the first curvature and the second curvature of given curve. If these coefficients are applied to the formulae given by (1.22) then

$$\begin{aligned}
\dot{\vec{e}}_1(t) &= \frac{1}{\sqrt{2}}(\kappa + i\tau)\vec{e}_2(t) \\
\dot{\vec{e}}_2(t) &= \frac{1}{\sqrt{2}}\left[(-\kappa + i\tau)\vec{e}_1(t) - (\kappa + i\tau)\vec{e}_3(t)\right] \\
\dot{\vec{e}}_3(t) &= \frac{1}{\sqrt{2}}(\kappa - i\tau)\vec{e}_2(t)
\end{aligned} \tag{1.24}$$

where the notation $\bullet$ shows the first order derivative with respect to the arc parameter t of given curve.

1.2. MINIMAL CURVES IN R^3

Definition 1.21. Let the components $x_p(t)$ of a vector function $\vec{x}(t) = (x_1(t), x_2(t), x_3(t))$ for $p = 1,2,3$ be analytic functions with respect to the complex variable t. The vector function is also written as

$$\vec{x}(t) = \sum_{p=1}^{3} x_p(t)\vec{k}_p \tag{1.25}$$

where the vectors $\vec{k}_p$ for $p = 1,2,3$ are the unit vectors of a frame of R^3. The end points of the position vectors $\vec{x}(t)$ form a curve which consists of ∞^2 number of complex points. This curve is called a complex curve in R^3.

Definition 1.22. A complex curve is called a minimal curve or an izotrop curve if the square of the distance of any two points of it is zero.

Theorem 1.14. A curve is a minimal curve iff $ds^2 = 0$ such that s is the arc parameter of the curve.

Theorem 1.16. A curve is a minimal curve iff all tangent vectors of the curve are minimal vectors at all points of the curve where the components of the vector function of the curve are analytic.

Let a minimal curve be given as below

$$(x): \vec{x} = \vec{x}(t). \tag{1.26}$$

By the theorem 1.14

$$ds^2 = [d\vec{x}(t)]^2 \tag{1.27}$$

and the following relation is obtained

$$[\vec{x}'(t)]^2 = 0. \tag{1.28}$$

From this equation it is obvious that the tangent vector of the curve is a minimal vector and the normal plane of the curve is a minimal plane at all points. If both sides of the equation (1.28) is derived with respect to t then

$$\vec{x}'(t)\vec{x}''(t)=0. \tag{1.29}$$

From this equation it is obvious that the vector $\vec{x}''(t)$ is perpendicular to the vector $\vec{x}'(t)$ and is parallel to the principal normal of the curve at all points. The principal normal is a complex vector and the rectifying plane is a complex plane because in general

$$[\vec{x}''(t)]^2 \neq 0. \tag{1.30}$$

In addition to these, the binormal of the curve can be determined by the product of the vectors $\vec{x}'(t)$ and $\vec{x}''(t)$ at all points. The scalar square of the binormal vector of the curve is written as below

$$[\vec{x}'(t)\wedge\vec{x}''(t)]^2 = [\vec{x}'(t)]^2.[\vec{x}''(t)]^2 - [\vec{x}'(t)\vec{x}''(t)]^2. \tag{1.31}$$

Using the equations (1.28) and (1.29)

$$[\vec{x}'(t)\wedge\vec{x}''(t)]^2 = 0 \tag{1.32}$$

so the binormal vector is a minimal vector and the osculating plane is a minimal plane at all points.

Theorem 1.16. A curve is a minimal curve or on a minimal plane if the osculating planes of the curve are minimal planes.

It was given that the binormal vectors of the minimal curve at all points are minimal vectors so the binormal vector of the minimal curve is perpendicular to itself and lies on the osculating plane. On the other hand, there is only one minimal vector $\vec{x}'(t)$ on the osculating plane. Thus the minimal vector $\vec{x}'(t)\wedge\vec{x}''(t)$ is parallel to the minimal vector $\vec{x}'(t)$ and it can be written as

$$\lambda\vec{x}'(t) = \vec{x}'(t)\wedge\vec{x}''(t) \tag{1.33}$$

where λ is not zero. When λ is calculated from the relation (1.33)

$$\lambda = \sqrt{-[\vec{x}''(t)]^2}. \tag{1.34}$$

Thus

$$\vec{x}'(t)=\frac{\vec{x}'(t)\wedge\vec{x}''(t)}{\sqrt{-[\vec{x}''(t)]^2}}dt\,. \tag{1.35}$$

Using

$$\vec{x}'(t)=\frac{d\vec{x}(t)}{dt} \tag{1.36}$$

the following relation is obtained

$$d\vec{x}(t)=\frac{\vec{x}'(t)\wedge\vec{x}''(t)}{\sqrt{-[\vec{x}''(t)]^2}}dt\,. \tag{1.37}$$

Theorem 1.17. The relation (1.37) does not depend on a chosen variable t. In other words, it is a differential invariant.

Proof: Let t be a parametric function of s such that

$$t=f(s) \text{ and } \frac{df}{ds}=f^{\bullet}\neq 0\,. \tag{1.38}$$

If the first derivative of $\vec{x}$ with respect to s is denoted by $\vec{x}^{\bullet}$

$$\vec{x}^{\bullet}=\vec{x}'f^{\bullet} \tag{1.39}$$

$$\vec{x}^{\bullet\bullet}=\vec{x}''f^{\bullet 2}+\vec{x}'f^{\bullet\bullet}\,. \tag{1.40}$$

Using the relations (1.39) and (1.40)

$$\frac{\vec{x}^{\bullet}\wedge\vec{x}^{\bullet\bullet}}{\sqrt{-(\vec{x}^{\bullet\bullet})^2}}=\frac{\vec{x}'\wedge\vec{x}''}{\sqrt{-(\vec{x}'')^2}}f^{\bullet}\,. \tag{1.41}$$

Then

$$dx=\frac{\vec{x}^{\bullet}\wedge\vec{x}^{\bullet\bullet}}{\sqrt{-(\vec{x}^{\bullet\bullet})^2}}ds=\frac{\vec{x}'\wedge\vec{x}''}{\sqrt{-(\vec{x}'')^2}}dt\,. \tag{1.42}$$

This is the required result.

Theorem 1.18. The relations, which are given by (1.28), (1.29), (1.32) do not depend on the variable t.

Proof: Let t be a parametric function of s. From the relations (1.39) and (1.40)

$$(\vec{x}^{\bullet})^2 = (\vec{x}')^2 f^{\bullet 2} = 0 \tag{1.43}$$

$$\vec{x}^{\bullet}\vec{x}^{\bullet\bullet} = \vec{x}'\vec{x}'' f^{\bullet 3} + (\vec{x}')^2 f^{\bullet} f^{\bullet\bullet} = 0 \tag{1.44}$$

and using these relations

$$(\vec{x}^{\bullet} \wedge \vec{x}^{\bullet\bullet})^2 = (\vec{x}^{\bullet})^2(\vec{x}^{\bullet\bullet})^2 - (\vec{x}^{\bullet}\vec{x}^{\bullet\bullet})^2 = 0\,. \tag{1.45}$$

This is the required result.

On the other hand, from the relation (1.40)

$$(\vec{x}^{\bullet\bullet})^2 = (\vec{x}^{\bullet})^2 f^{\bullet 4} \tag{1.46}$$

or

$$\left[-(\vec{x}^{\bullet\bullet})^2\right]^{\frac{1}{4}} ds = \left[-(\vec{x}^{\bullet})^2\right]^{\frac{1}{4}} dt \tag{1.47}$$

and taking

$$(\vec{x}^{\bullet\bullet})^2 = -1 \tag{1.48}$$

the following relation is obtained

$$s = \int_{t_0}^{t} \left[-(\vec{x}''(t))^2\right]^{\frac{1}{4}} dt\,. \tag{1.49}$$

Definition 1.23. The relation (1.49) given by Study and Vessiot is called the psedo-arc parameter or the natural arc parameter of the minimal curve (x): $\vec{x} = \vec{x}(t)$.

When the minimal curves are exanined the concept psedo-arc will be used spontaneously. The following relations occur among the various order derivatives of the position vector in the minimal curves.

$$\vec{x}^{\bullet} = \vec{x}^{\bullet} \wedge \vec{x}^{\bullet\bullet}$$
$$\vec{x}^{\bullet}\vec{x}^{\bullet\bullet\bullet} = 1$$
$$\vec{x}^{\bullet\bullet}\vec{x}^{\bullet\bullet\bullet} = 0$$
$$\vec{x}^{\bullet}\vec{x}^{\bullet\bullet\bullet\bullet} = 0$$
$$\left(\vec{x}^{\bullet}, \vec{x}^{\bullet\bullet}, \vec{x}^{\bullet\bullet\bullet}\right) = 1$$
$$\left(\vec{x}^{\bullet}, \vec{x}^{\bullet\bullet}, \vec{x}^{\bullet\bullet\bullet\bullet}\right) = 0$$
$$\vec{x}^{\bullet\bullet}\vec{x}^{\bullet\bullet\bullet\bullet} + \left(\vec{x}^{\bullet\bullet\bullet}\right)^2 = 0 \tag{1.50}$$
$$\vec{x}^{\bullet\bullet\bullet}\vec{x}^{\bullet\bullet\bullet\bullet} - \frac{1}{2}\left[\left(\vec{x}^{\bullet\bullet\bullet}\right)^2\right]^{\bullet} = 0$$
$$\vec{x}^{\bullet\bullet} = \vec{x}^{\bullet} \wedge \vec{x}^{\bullet\bullet\bullet}$$
$$\vec{x}^{\bullet\bullet} \wedge \vec{x}^{\bullet\bullet\bullet} = \vec{x}^{\bullet\bullet\bullet} - \left(\vec{x}^{\bullet\bullet\bullet}\right)^2 \vec{x}^{\bullet}$$
$$\vec{x}^{\bullet\bullet} \wedge \vec{x}^{\bullet\bullet\bullet} = \vec{x}^{\bullet\bullet\bullet} - \vec{x}^{\bullet}\vec{x}^{\bullet\bullet\bullet\bullet}$$
$$\vec{x}^{\bullet\bullet\bullet\bullet} = \frac{1}{2}\beta^{\bullet}\vec{x}^{\bullet} + \beta\vec{x}^{\bullet\bullet}, \quad \beta = \left(\vec{x}^{\bullet\bullet\bullet}\right)^2$$
$$\left(\vec{x}^{\bullet\bullet\bullet\bullet}\right)^2 = -\beta^2 = -\left(\vec{x}^{\bullet\bullet\bullet}\right)^4$$

Definition 1.24. Let a three-faced of the following vectors be connected to every point P of a minimal curve (x): $\vec{x} = \vec{x}(t)$ which is in class of C^n such that $n \geq 3$.

$$\begin{aligned} \vec{e}_1 &= \vec{x}^{\bullet} \\ \vec{e}_2 &= i\vec{x}^{\bullet\bullet} \\ \vec{e}_3 &= -\frac{\beta}{2}\vec{x}^{\bullet} + \vec{x}^{\bullet\bullet\bullet} \end{aligned} \tag{1.51}$$

where $\beta = \left(\vec{x}^{\bullet\bullet\bullet}\right)^2$.

This three-faced is called cyclic three-faced of Cartan at the point P of the curve.

It is easily seen that the vectors given by (1.51) satisfy the relations (1.16) and (1.17) so the vectors $\vec{e}_1$ and $\vec{e}_3$ are minimal vectors. The vector $\vec{e}_1$ determines the tangent vector of the curve at the point P. The vector $\vec{e}_2$, which is perpendicular to the vector $\vec{e}_1$ and on the osculating plane at the point P, determines the principal normal because it is parallel to the vector $\vec{x}^{\bullet\bullet}$. The binormal vector of the curve at the point P is the minimal vector $\vec{e}_1$, which is normal to the osculating plane and on the osculating plane so the osculating plane is a minimal plane. The minimal vectors $\vec{e}_1$ and $\vec{e}_3$, which are perpendicular to the vector $\vec{e}_2$, determine the rectifying plane at the point P. The minimal vector $\vec{e}_3$ is perpendicular to both of the vector $\vec{e}_2$ and the normal plane, which is determined by the vectors $\vec{e}_2$ and $\vec{e}_3$, so the normal plane is also a minimal plane. The rectifying plane is not a minimal plane. A minimal cone with the apex P is tangent to the osculating plane along the vector $\vec{e}_3$.

The first order derivatives of the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ with respect to the psedo-arc parameter s occur as below

$$\begin{aligned} \vec{e}_1^{\bullet} &= -i\vec{e}_2 \\ \vec{e}_2^{\bullet} &= i\left(k\vec{e}_1 + \vec{e}_3\right) \\ \vec{e}_3^{\bullet} &= -ik\vec{e}_2 \end{aligned} \tag{1.52}$$

where $k = \frac{1}{2}\beta = \frac{1}{2}\left(\vec{x}^{\bullet\bullet\bullet}\right)^2$ is the psedo curvature of the minimal curve (x): $\vec{x} = \vec{x}(t)$ in the class of C^n such that $n \geq 4$. (Şemin, 1983)

1.3. REAL ELEMENTS IN $\mathbf{R}^4$

Definition 1.25. Let Ox_p for $p = 1,2,3,4$ be an orthogonal axis system of the three dimensional real space R^4. An equation

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + A_5 = 0, \tag{1.53}$$

is said to state a flat in R^4 for the real coefficients A_p. The normal vector of the flat is the vector $\vec{V}\left(A_1, A_2, A_3, A_4\right)$.

Definition 1.26. The intersection of the following system

$$\begin{aligned} A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + A_5 = 0 \\ B_1x_1 + B_2x_2 + B_3x_3 + B_4x_4 + B_5 = 0 \end{aligned} \tag{1.54}$$

defines a plane in R^4 if the flats are non-parallel to each other. A plane in R^4 has two normal vectors that belong to the flats which determine the plane.

Definition 1.27. The intersection of three flats defines a line in R^4 if any two of them are non-parallel to each other. (Forsyth, 1930)

CHAPTER TWO

COMPLEX ELEMENTS IN R^4

In this chapter, the concepts complex points and complex flats in the real space R^4 are defined and by using these definitions complex elements and complex lines are defined for R^4. The theorems about complex elements are proved for the real space R^4. By using these theorems the properties of the complex elements are determined.

2.1. COORDINATES OF A COMPLEX POINT IN R^4

Definition 2.1. Let Ox_p with the unit vectors k_p for $p=1,2,3,4$ be an orthogonal axis system of the four dimensional real space R^4. A point $N(x_1,x_2,x_3,x_4)$ is called a complex point in R^4 if any component of the point is a complex number.

Definition 2.2. Any two complex points of which the components are respectively conjugate to each other are said to be conjugate complex points.

Definition 2.3. Let $N(x_1,x_2,x_3,x_4)$ be a complex point in R^4. The complex numbers $X_1;X_2;X_3;X_4;X_5$ are said to be homogeneous coordinates of the complex point if there is a relation $x_p=\dfrac{X_p}{X_5}$ for $p=1,2,3,4$. Thus a complex point in R^4 is also denoted by $N(X_1,X_2,X_3,X_4,X_5)$ in the homogeneous coordinates. Also, a complex point at infinity is denoted by $N(X_1,X_2,X_3,X_4,0)$.

2.2. COMPLEX FLATS IN R^4

Definition 2.4. The following equation

$$\Delta = \sum_{p=1}^{5} A_p X_p = 0 \tag{2.1}$$

defines a complex flat in R^4 if any coefficient A_p is a complex number and there is not a relation $A_p = ca_p$ such that a_p are real numbers for $p = 1,2,3,4,5$ in the homogeneous coordinates. The flat given by the equation (2.1) is also given as below

$$\Delta = \sum_{p=1}^{4} \left(A_p x_p\right) + A_5 = 0 . \tag{2.2}$$

Theorem 2.1. The real points satisfying a complex flat equation are on a real plane which has a finite or an infinite distance from the origin.

Proof: Let a complex flat be given by the equation

$$\Delta = \sum_{p=1}^{5} A_p X_p = 0 \tag{2.3}$$

where A_p for $p = 1,2,3,4,5$ are complex numbers. By using $A_p = a_p + ia_p^*$ where a_p and a_p^* are real numbers for $p = 1,2,3,4,5$

$$\Delta = \sum_{p=1}^{5} \left(a_p + ia_p^*\right) X_p = 0 , \tag{2.4}$$

$$\Delta = \sum_{p=1}^{5} a_p X_p + i \sum_{p=1}^{5} a_p^* X_p = 0 . \tag{2.5}$$

Taking

$$\sum_{p=1}^{5} a_p X_p = \Delta_1 \text{ and } \sum_{p=1}^{5} a_p^* X_p = \Delta_2 \tag{2.6}$$

the equation (2.3) is in the following form

$$\Delta = \Delta_1 + i\Delta_2 = 0 . \tag{2.7}$$

Then the homogeneous equations $\Delta_1 = 0$ and $\Delta_2 = 0$ state real flats. If the real flats $\Delta_1 = 0$ and $\Delta_2 = 0$ are non-parallel to each other the intersection of them is a real plane which has a finite distance from the origin. If the real flats are parallel to each other then the plane is at infinity.

Theorem 2.2. There is no complex flat passing through any two real planes which intersect each other in R^4.

Proof: There is only one real plane on a complex flat so a complex flat passing through two intersecting real planes can not be found.

Definition 2.5. Any two complex flats which are given by the equations

$$\Delta = \sum_{p=1}^{5} A_p X_p = 0 \text{ and } \overline{\Delta} = \sum_{p=1}^{5} \overline{A}_p X_p = 0 \tag{2.8}$$

where the numbers A_p and $\overline{A}_p$ for $p = 1,2,3,4,5$ are respectively conjugate to each other are said to be conjugate flats.

Theorem 2.3. The complex flat $\overline{\Delta} = 0$ passes through the real plane which is in the conjugate complex flat $\Delta = 0$ of $\overline{\Delta} = 0$.

Proof: Let a complex flat be given by an equation

$$\Delta = \Delta_1 + i\Delta_2 = 0. \tag{2.9}$$

Then the conjugate flat of it is given by an equation

$$\overline{\Delta} = \Delta_1 - i\Delta_2 = 0. \tag{2.10}$$

This complex flat also satisfies the equations $\Delta_1 = 0$ and $\Delta_2 = 0$ so the complex flat $\overline{\Delta} = 0$ passes through the real plane which is in the complex flat $\Delta = 0$.

2.3. COMPLEX PLANES IN R^4

Definition 2.6. An intersection of any two non-parallel planes are said to state a complex plane in R^4. In other words, the equation system

$$\Delta = 0 \text{ and } \Delta' = 0 \tag{2.11}$$

state a complex plane in R^4 such that the flats $\Delta = 0$ and $\Delta' = 0$ are non-parallel to each other.

Theorem 2.4. There is only one real line on a complex plane in R^4.

Proof: Let a complex plane be given by the following equation system in R^4

$$\Delta = \Delta_1 + i\Delta_2 = 0 \text{ and } \Delta = \Delta_3 + i\Delta_4 = 0. \tag{2.12}$$

It was given that there is only one real plane on a complex flat so the intersection of the planes, which are determined by the complex flats $\Delta = 0$ and $\Delta' = 0$, determines only one line if the planes are non-parallel to each other. This is the required result.

2.4. COMPLEX LINES IN R^4

Definition 2.8. The intersection of three complex flats defines a complex line in R^4 if any two of them are non-parallel to each other. In other words, a complex line is given by the equation system

$$\begin{aligned} \Delta &= \sum_{p=1}^{5} A_p X_p = 0 \\ \Delta' &= \sum_{p=1}^{5} B_p X_p = 0 \\ \Delta'' &= \sum_{p=1}^{5} C_p X_p = 0 \end{aligned} \tag{2.13}$$

where $\Delta = 0$, $\Delta' = 0$ and $\Delta'' = 0$ are any three non-parallel complex flats.

Definition 2.9. A complex line is called a conjugate line of the complex line given by (2.13) if the complex line is the intersection of the conjugate flats of the flats $\Delta = 0$, $\Delta' = 0$ and $\Delta'' = 0$.

Theorem 2.5. There is only one real point on a complex line.

Proof: It was given that an intersection of any three complex flats which any two of them are non-parallel to each other determines a complex line and there is only one real plane on a complex flat. Let a complex line be determined by the complex flats Δ, Δ' and Δ''. The real planes, which are on the complex flats, meet in a real point because the complex flats are non-parallel to each other. This is the required result.

2.5. REAL ELEMENTS PASSING THROUGH A COMPLEX POINT

Theorem 2.6. A real flat passing through a complex point $N(a+ia^*,b+ib^*,c+ic^*,d+id^*)$ has the normal vector $\vec{V}(b^*,-a^*,d^*,-c^*)$ and a real point which is the point $M(a,b,c,d)$.

Proof: Let a real flat be given by the equation

$$K_1x_1+K_2x_2+K_3x_3+K_4x_4+K_5=0. \tag{2.14}$$

The components of the point N satisfy the equation (2.14) if the real flat passes through the complex point N.

$$K_1(a+ia^*)+K_2(b+ib^*)+K_3(c+ic^*)+K_4(d+id^*)+K_5=0. \tag{2.15}$$

Then

$$(K_1a+K_2b+K_3c+K_4d+K_5)+i(K_1a^*+K_2b^*+K_3c^*+K_4d^*)=0. \tag{2.16}$$

By using the equality of the complex numbers

$$K_1a+K_2b+K_3c+K_4d+K_5=0,$$

$$K_1a^*+K_2b^*+K_3c^*+K_4d^*=0. \tag{2.17}$$

A particular solution of the equation system (2.17) is

$$K_1=b^*,\ K_2=-a^*,\ K_3=d^*,\ K_4=-c^*,\ K_5=-ab^*+a^*b-cd^*+c^*d\ . \tag{2.18}$$

By using the particular solution in the equation (2.14)

$$b^*(x_1-a)-a^*(x_2-b)+d^*(x_3-c)-c^*(x_4-d)=0. \tag{2.19}$$

The equation (2.19) is a flat equation, so a real point of the flat is the point $M(a,b,c,d)$ and the normal vector of the flat is $\vec{V}(b^*,-a^*,d^*,-c^*)$.

Theorem 2.7. In the real flat $x_4 = 0$ a real plane passing through a complex point $N(a+ia^*,b+ib^*,c+ic^*)$ has the normal vector $\vec{V}(-b^*c^*,2a^*c^*,a^*b^*)$ and a real point which is the point $M(a,b,c)$.

Proof: Let a real plane in $x_4 = 0$ be given by the equation

$$K_1x_1 + K_2x_2 + K_3x_3 + K_4 = 0. \tag{2.20}$$

The components of the point N satisfy the equation (2.20) if the real flat passes through the complex point N.

$$K_1(a+ia^*) + K_2(b+ib^*) + K_3(c+ic^*) + K_4 = 0. \tag{2.21}$$

Then

$$(K_1a + K_2b + K_3c + K_4) + i(K_1a^* + K_2b^* + K_3c^*) = 0. \tag{2.22}$$

By using the equality of the complex numbers

$$K_1a + K_2b + K_3c + K_4 = 0,$$

$$K_1a^* + K_2b^* + K_3c^* = 0. \tag{2.23}$$

A particular solution of the equation system (2.23) is

$$K_1 = b^*c^*,\ K_2 = -2a^*c^*,\ K_3 = a^*b^*,\ K_4 = ab^*c^* + 2a^*bc^* - a^*b^*c. \tag{2.24}$$

By using the particular solution in the equation (2.20)

$$b^*c^*(x_1 - a) - 2a^*c^*(x_2 - b) + a^*b^*(x_3 - c) = 0. \tag{2.25}$$

The equation (2.25) states a plane so a real point of the plane is the point $M(a,b,c)$ and the normal vector of the plane is $\vec{V}(b^*c^*,-2a^*c^*,a^*b^*)$.

Theorem 2.8. In the real plane Ox_1x_2 a real line passing through a complex point $N(a+ia^*,b+ib^*)$ has the direction (a^*,b^*) and a real point which is the point $M(a,b)$.

Proof: Let a real line in the real plane Ox_1x_2 be given by the equation

$$K_1x_1 + K_2x_2 + K_3 = 0. \quad (2.26)$$

The coordinates of the point N satisfy the equation (2.26) if the real flat passes through the complex point N.

$$K_1(a + ia^*) + K_2(b + ib^*) + K_3 = 0. \quad (2.27)$$

Then

$$(K_1a + K_2b + K_3) + i(K_1a^* + K_2b^*) = 0. \quad (2.28)$$

By using the equality of the complex numbers

$$K_1a + K_2b + K_3 = 0,$$

$$K_1a^* + K_2b^* = 0. \quad (2.29)$$

A particular solution of the equation system (2.29) is

$$K_1 = b^*,\ K_2 = -a^*,\ K_3 = -ab^* + a^*b. \quad (2.30)$$

By using the particular solution in the equation

$$\frac{x_1 - a}{a^*} = \frac{x_2 - b}{b^*}. \quad (2.31)$$

The equation (2.31) is a line equation so a real point of the line is the point $M(a,b)$ and the direction of the line is (a^*,b^*).

CHAPTER THREE

MINIMAL ELEMENTS AND CURVES IN R^4

In this chapter, minimal elements and the minimal curves are defined for the real space R^4. It is presented that necessary and sufficient condition for a complex flat to be a minimal flat. By using minimal flats minimal planes and minimal lines are defined for R^4. Minimal curves are defined and the properties of the minimal curves are determined by using the Frenet frame and cyclic four-faced of the minimal curves.

3.1. MINIMAL HYPERCONES AND MINIMAL ELEMENTS

Definition 3.1. The complex points having the square of the distances from the origin is zero define a hypercone which is called an izotrop hypercone or a minimal hypercone with the apex origin and is given by the equation

$$\sum_{p=1}^{4} x_p^2 = 0. \tag{3.1}$$

A minimal hypercone with the apex $A(a_1, a_2, a_3, a_4)$ which is a real point or a complex point is given by the equation

$$\sum_{p=1}^{4} \left(x_p - a_p\right)^2 = 0. \tag{3.2}$$

"A hypercone through the vertex intersects the hypercone in general in an ordinary cone of minimal lines, and a plane through the vertex intersects the hypercone in general in two minimal lines" (Woods, p. 379, 1961). The minimal lines are two directrixes of the minimal hypercone so a minimal hypercone in the real space R^4 is also defined as below.

Definition 3.2. A locus of minimal lines which are belong to minimal planes passing through the origin is called a minimal hypercone with the apex origin.

The minimal hypercone given by the equation (3.2) is also given by the following equation

$$\sum_{p=1}^{4}\left(X_p - a_p X_5\right)^2 = 0 \tag{3.3}$$

in the homogeneous coordinates. The intersection of the minimal hypercone and a flat at infinity is a quadratic curve given by the equation

$$\sum_{p=1}^{4} X_p^2 = 0,\ X_5 = 0. \tag{3.4}$$

The equation (3.4) defines all minimal hypercones.

Definition 3.3. The curve which is given by the equation (3.4) is said to be an absolute of the real flat at infinity. (Woods, 1961)

Definition 3.4. A hypercone is called a minimal hypercone in R^4 if the directrix of the hypercone is the curve given by the equation (3.4).

Definition3.5. A direction is a minimal direction in R^4 if the point of the direction at infinity is on the absolute.

Theorem 3.1. A line which has the direction (b_1, b_2, b_3, b_4) is a minimal line iff

$$\sum_{p=1}^{4} b_p^2 = 0. \tag{3.5}$$

Proof: Let a line which has the direction (b_1,b_2,b_3,b_4) be a minimal line. From the definition of the minimal direction the point of the direction at infinity is on the absolute the components of (b_1,b_2,b_3,b_4) satisfy the equation (3.4). Then it is clear that $\sum_{p=1}^{4} b_p^2 = 0$. Conversely if the components of the direction satisfy the equation (3.5) then the direction has a point which is on the absolute. It is clear that the line is a minimal line from the definition 3.5.

Definition 3.6. A vector which has a minimal direction is called a minimal vector or an izotrop vector.

Theorem 3.6. A vector $\vec{v}$ is a minimal vector iff $\vec{v}^2 = 0$.

Proof: From the definition 3.6 if $\vec{v}$ is a minimal vector then $\vec{v}$ has a minimal direction. By the theorem 3.1 it is obvious that $\vec{v}^2 = 0$. Conversely if $\vec{v}^2 = 0$ then $\vec{v}$ has a minimal direction so the vector is a minimal vector.

Definition 3.7. The intersection of a complex plane and the absolute is called siklik point of the plane.

Definition 3.8. A flat which is tangent to the absolute is called a minimal flat.

Theorem 3.3. A flat is a minimal flat iff the normal vector of the flat is a minimal vector.

Proof: Let a flat be tangent to a minimal hypercone with the apex origin and be given by the equation

$$\bar{x}_1 x_1 + \bar{x}_2 x_2 + \bar{x}_3 x_3 + \bar{x}_4 x_4 = 0 . \tag{3.6}$$

Another flat which is parallel to the tangent flat of the minimal hypercone and passes through the origin

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0 \tag{3.7}$$

is the tangent flat iff

$$\frac{A_p}{x_p} = \lambda \tag{3.8}$$

for $p = 1,2,3,4$ and the real constant $\lambda \neq 0$. By using the relation (3.8) in the relation (3.6)

$$\sum_{p=1}^{4} \frac{A_p}{\lambda} x_p = 0. \tag{3.9}$$

From the definition of the absolute

$$\sum_{p=1}^{4} \left(\frac{A_p}{\lambda}\right)^2 = 0. \tag{3.10}$$

The relation (3.10) is also written as below

$$\frac{1}{\lambda^2} \sum_{p=1}^{4} A_p^2 = 0. \tag{3.11}$$

Then we have

$$\sum_{p=1}^{4} A_p^2 = 0. \tag{3.12}$$

This means that the normal vector of the flat is a minimal vector. Conversely if the flat given by the equation (3.8) satisfies the condition (3.13) then the normal vector of the flat is a minimal vector. There exists a minimal hypercone that the flat is tangent along the directrix of the minimal hypercone so the flat is a minimal flat.

Definition 3.9. The intersection of two minimal flats defines a minimal plane or an izotrop plane in R^4 if the minimal flats are non-parallel to each other

Theorem 3.4. If a plane is a minimal plane in R^4 then the normal vectors of the plane are minimal vectors.

Proof: Let a plane be a minimal plane. From the definition 3.9 the minimal plane is generated by two fixed minimal flats. The normal vectors of the plane are minimal vectors because the normal vectors of the minimal plane are the normal vectors of the flats.

A normal line of a minimal flat at a point A is in the minimal flat and the minimal flat is tangent to a minimal hypercone with the apex A along the minimal line. In other words, the normal line is a directrix of the minimal hypercone. Also, there is a minimal hypercone that the normal line of the plane are tangent to it for every minimal plane.

3.2. CYCLIC FOURFACED

Definition 3.10. Let $\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4$ be unit vectors of a frame of four dimensional real space R^4. The set of the vectors given by

$$\begin{aligned}
\vec{e}_1 &= \frac{1}{\sqrt{2}}\left(\vec{k}_1 - i\vec{k}_3\right) \\
\vec{e}_2 &= \vec{k}_2 \\
\vec{e}_3 &= \frac{1}{\sqrt{2}}\left(\vec{k}_1 + i\vec{k}_3\right) \\
\vec{e}_4 &= \mu\left(\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3\right) = i\vec{k}_4, \ \mu = \mp 1
\end{aligned} \tag{3.13}$$

is called cyclic four-faced.

The inner products of the vectors are given as below

$$\vec{e}_i \vec{e}_j = \begin{cases} 0, & i+j \equiv 1,2,3 \pmod 4 \\ 1, & i+j=4 \\ -1, & i+j=8 \end{cases} \tag{3.14}$$

Above inner products show that the vectors $\vec{e}_1$ and $\vec{e}_3$ are minimal vectors but $\vec{e}_2$ is a real vector and $\vec{e}_4$ is a complex vector. The vector products on the set A are given as below

$$\begin{aligned}
\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 &= \vec{e}_4 \\
\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_4 &= -\vec{e}_1 \\
\vec{e}_1 \wedge \vec{e}_3 \wedge \vec{e}_4 &= \vec{e}_2 \\
\vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 &= -\vec{e}_3
\end{aligned} \tag{3.15}$$

and the following product occurs on the set A

$$\left(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\right) = -1. \tag{3.16}$$

The unit vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4$ are given as below in the terms of $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$

$$\begin{aligned}
\vec{k}_1 &= \frac{1}{\sqrt{2}}\left(\vec{e}_1 + \vec{e}_3\right) \\
\vec{k}_2 &= \vec{e}_2 \\
\vec{k}_3 &= \frac{i}{\sqrt{2}}\left(\vec{e}_1 - \vec{e}_3\right) \\
\vec{k}_4 &= -i\vec{e}_4.
\end{aligned} \tag{3.17}$$

Let the vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4$ be the unit vectors of a moving frame with respect to a variable t. Then

$$\vec{k}_p{}'(t) = a_{p1}\vec{k}_1(t) + a_{p2}\vec{k}_2(t) + a_{p3}\vec{k}_3(t) + a_{p4}\vec{k}_4(t) \tag{3.18}$$

where $a_{p1}, a_{p2}, a_{p3}, a_{p4}$ are real coefficients $p = 1,2,3,4$. There are the following relations $p = 1,2,3,4$ and $m = 1,2,3,4$

$$a_{pm} = \begin{cases} 0, & m = p \\ -a_{mp}, & m \neq p. \end{cases} \tag{3.19}$$

If the relations given in (3.19) is substituted in the formula (3.18) then

$$\begin{aligned}
\vec{k}_1{}'(t) &= a_{12}\vec{k}_2(t) + a_{13}\vec{k}_3(t) + a_{14}\vec{k}_4(t) \\
\vec{k}_2{}'(t) &= -a_{12}\vec{k}_1(t) + a_{23}\vec{k}_3(t) + a_{24}\vec{k}_4(t) \\
\vec{k}_3{}'(t) &= -a_{13}\vec{k}_1(t) - a_{23}\vec{k}_2(t) + a_{34}\vec{k}_4(t) \\
\vec{k}_4{}'(t) &= -a_{14}\vec{k}_1(t) - a_{24}\vec{k}_2(t) - a_{34}\vec{k}_3(t).
\end{aligned} \tag{3.20}$$

When the formulae given by (3.17) are substituted in (3.20)

$$\vec{k}_1{}'(t) = -\frac{a_{13}}{i\sqrt{2}}\vec{e}_1(t) + a_{12}\vec{e}_2(t) + \frac{a_{13}}{i\sqrt{2}}\vec{e}_3(t) - ia_{14}\vec{e}_4(t)$$
$$\vec{k}_2{}'(t) = -\frac{ia_{12} + a_{23}}{i\sqrt{2}}\vec{e}_1(t) + \frac{-ia_{12} + a_{23}}{i\sqrt{2}}\vec{e}_3(t) - ia_{24}\vec{e}_4(t)$$
$$\vec{k}_3{}'(t) = -\frac{a_{13}}{\sqrt{2}}\vec{e}_1(t) - a_{23}\vec{e}_2(t) - \frac{a_{13}}{\sqrt{2}}\vec{e}_3(t) - ia_{24}\vec{e}_4(t) \qquad (3.21)$$
$$\vec{k}_4{}'(t) = \frac{-ia_{14} + a_{34}}{i\sqrt{2}}\vec{e}_1(t) - a_{24}\vec{e}_2(t) - \frac{ia_{14} + a_{34}}{i\sqrt{2}}\vec{e}_3(t).$$

By using the derivative of the formulae in (3.14), the vectors $\vec{e}_1{}'(t), \vec{e}_2{}'(t), \vec{e}_3{}'(t), \vec{e}_4{}'(t)$ are obtained in the terms of the vectors $\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t), \vec{e}_4(t)$ as below

$$\vec{e}_1{}'(t) = ia_{13}\vec{e}_1(t) + \frac{a_{12} + ia_{23}}{\sqrt{2}}\vec{e}_2(t) - \frac{a_{34} + ia_{14}}{\sqrt{2}}\vec{e}_4(t)$$
$$\vec{e}_2{}'(t) = \frac{-a_{12} + ia_{23}}{\sqrt{2}}\vec{e}_2(t) - ia_{13}\vec{e}_3(t) - ia_{24}\vec{e}_4(t)$$
$$\vec{e}_3{}'(t) = \frac{a_{12} - ia_{23}}{\sqrt{2}}\vec{e}_2(t) - ia_{13}\vec{e}_3(t) + \frac{a_{34} - ia_{14}}{\sqrt{2}}\vec{e}_4(t) \qquad (3.22)$$
$$\vec{e}_4{}'(t) = \frac{a_{34} - ia_{14}}{\sqrt{2}}\vec{e}_1(t) - ia_{24}\vec{e}_2(t) - \frac{a_{34} + ia_{14}}{\sqrt{2}}\vec{e}_3(t).$$

If the unit vectors $\vec{k}_1(t), \vec{k}_2(t), \vec{k}_3(t), \vec{k}_4(t)$ are respectively the vectors $\vec{T}, \vec{N}, \vec{B}, \vec{E}$ which are Frenet Vectors of a curve with the arc parameter t then the coefficients are obtained as

$$a_{12} = \kappa, \;\; a_{23} = \tau, \;\; a_{34} = \sigma, \;\; a_{13} = a_{14} = a_{24} = 0 \qquad (3.23)$$

such that κ, τ and σ are respectively the first curvature, the second curvature and the third curvature of given curve. If the coefficients are applied to the formulae given by (1.22) then

$$\vec{e}_1{}^{\bullet}(t) = \frac{1}{\sqrt{2}}\left[(\kappa + i\tau)\vec{e}_2(t) - \sigma\vec{e}_4(t)\right]$$
$$\vec{e}_2{}^{\bullet}(t) = \frac{1}{\sqrt{2}}\left[(-\kappa + i\tau)\vec{e}_1(t) - (\kappa + i\tau)\vec{e}_3(t)\right]$$
$$\vec{e}_3{}^{\bullet}(t) = \frac{1}{\sqrt{2}}\left[(\kappa - i\tau)\vec{e}_2(t) + \sigma\vec{e}_4(t)\right] \qquad (3.24)$$
$$\vec{e}_4{}^{\bullet}(t) = \frac{1}{\sqrt{2}}\left[\sigma\vec{e}_1(t) - \sigma\vec{e}_3(t)\right].$$

where the notation $\bullet$ shows the first order derivatives of the vectors with respect to the complex arc parameter t of the curve.

It should be noted that the vectors of cyclic four-faced satisfy the inner products given by (3.15) so the vectors $\vec{e}_1(t)$ and $\vec{e}_3(t)$ are minimal vectors and the vectors $\vec{e}_2(t)$ and $\vec{e}_4(t)$ are complex vectors.

3.3. MINIMAL CURVES

Definition 3.11. Let the components of a vector function

$$\vec{x}(t) = \left(x_1(t), x_2(t), x_3(t), x_4(t)\right) \tag{3.25}$$

be analytic functions with respect to a complex variable t. The vector function is also written as

$$\vec{x}(t) = \sum_{p=1}^{4} x_p(t)\vec{k}_p \tag{3.26}$$

where the vectors $\vec{k}_p$ for $p = 1,2,3,4$ are the unit vectors of a frame of the real space R^4. The end points of the position vectors of the vectors $\vec{x}(t)$ form a curve which consists of ∞^2 number of complex points. This curve is called a complex curve in R^4.

Definition 3.12. A complex curve is called a minimal curve or an izotrop curve if the square of the distance of any two points of it is zero.

Theorem 3.5. A complex curve is a minimal curve if and only if $ds^2 = 0$ such that s is the arc parameter of the curve.

Proof: Let (x): $\vec{x} = \vec{x}(t)$ be a minimal curve and any two points on the curve are $P(x_1, x_2, x_3, x_4)$ and $Q(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3, x_4 + \Delta x_4)$ such that $\Delta x_i > 0$ for

$i = 1,2,3,4$. The square of the distance between the points P and Q is written as below

$$|PQ|^2 = \Delta x_1{}^2 + \Delta x_2{}^2 + \Delta x_3{}^2 + \Delta x_4{}^2 = 0\,. \tag{3.27}$$

When $\Delta x_i \to 0$ for $i = 1,2,3,4$ it is obvious that

$$dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + dx_4{}^2 = ds^2 = 0\,. \tag{3.28}$$

Conversely if $ds^2 = 0$ then it is clear that the square of the distance of any two points of the curve is zero so the curve is a minimal curve.

Theorem 3.6. All tangent lines of a minimal curve at all regular points are minimal lines.

Proof: Let (x): $\vec{x} = \vec{x}(t)$ be a minimal curve with respect to arc parameter s. By the theorem 3.5

$$ds^2 = d\vec{x}^2 = 0\,. \tag{3.29}$$

Also,

$$\frac{d\vec{x}}{ds} = \vec{x}^{\bullet}(s) \neq 0 \tag{3.30}$$

at all regular points of the curve. Consequently

$$(\vec{x}^{\bullet}(s))^2 = 0\,. \tag{3.31}$$

By the theorem 3.1 all tangent lines of a minimal curve at all regular points are minimal lines.

The following definition can be given as a result of the theorem 3.6.

Definition 3.13. A curve is called a minimal curve if all tangent lines of the curve at all regular points are minimal lines.

It is necessary that Frenet Vectors of the minimal curves are determined when the properties of the minimal curves are examined. Let (x): $\vec{x} = \vec{x}(t)$ be a minimal curve. By the theorem 3.6.

$$\left(\vec{x}'(t)\right)^2 = 0. \tag{3.32}$$

When the relation (3.32) is derived with respect to t

$$\vec{x}'(t)\vec{x}''(t) = 0. \tag{3.33}$$

From the last expression it is clear that the vector $\vec{x}''(t)$ is perpendicular to the vector $\vec{x}'(t)$ and parallel to the principal normal $\vec{N}$ of the curve. If the expression (3.33) is derived with respect to t then

$$\left(\vec{x}''(t)\right)^2 = -\vec{x}'(t)\vec{x}'''(t). \tag{3.34}$$

Since in general $\left(\vec{x}''(t)\right)^2 \neq 0$ the vector $\vec{x}''(t)$ is not a minimal vector. Besides, the vector

$$\vec{u}(t) = \vec{x}'(t) \wedge \vec{x}''(t) \wedge \vec{x}'''(t) \tag{3.35}$$

is parallel to the trinormal vector $\vec{E}$ of the curve. Since

$$\left(\vec{u}(t)\right)^2 = -\left(\vec{x}''(t)\right)^6 \neq 0 \tag{3.36}$$

the vector $\vec{E}$ is not a minimal vector. The vector

$$\vec{v}(t) = \vec{u}(t) \wedge \vec{x}'(t) \wedge \vec{x}''(t) \tag{3.37}$$

is parallel to the binormal vector $\vec{B}$ of the curve. Also, the vector $\vec{v}(t)$ is a minimal vector because

$$\left(\vec{v}(t)\right)^2 = 0. \tag{3.38}$$

Finally Frenet Vectors of the minimal curve (x): $\vec{x} = \vec{x}(t)$ are written as below

$$\begin{aligned}
\vec{T} &= \vec{x}'(t) \\
\vec{N} &= \frac{\vec{x}''(t)}{\left\|\vec{x}''(t)\right\|} \\
\vec{B} &= \left(\vec{x}'(t) \wedge \vec{x}''(t) \wedge \vec{x}'''(t)\right) \wedge \vec{x}'(t) \wedge \vec{x}''(t) \\
\vec{E} &= \frac{\vec{x}'(t) \wedge \vec{x}''(t) \wedge \vec{x}'''(t)}{\left\|\vec{x}'(t) \wedge \vec{x}''(t) \wedge \vec{x}'''(t)\right\|}
\end{aligned} \tag{3.39}$$

Since the tangent vector and the binormal vector of the curve (x): $\vec{x} = \vec{x}(t)$ are minimal vectors the $\left(\vec{N} - \vec{B} - \vec{E}\right)$ flat and the $\left(\vec{E} - \vec{T} - \vec{N}\right)$ flat are minimal flats and the intersection of these minimal flats called $\left(\vec{E} - \vec{N}\right)$ plane is a minimal plane.

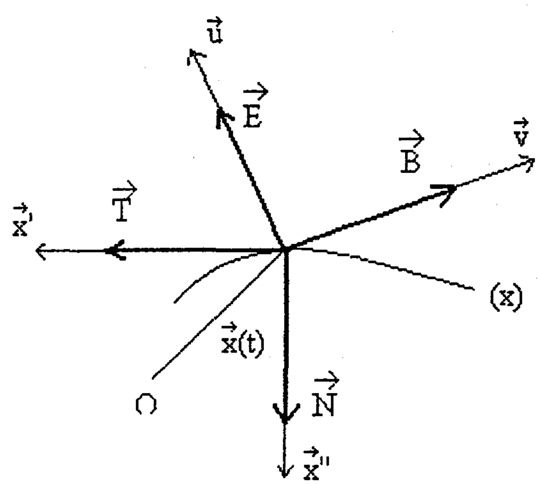

Figure 3.1.

It is known that a binormal vector of a minimal curve at a regular point is a minimal vector and a minimal vector is perpendicular to itself so the binormal vector lies in the $(\vec{E}-\vec{T}-\vec{N})$ flat in which $\vec{T}$ is a unique minimal vector. Then the binormal vector $\vec{B}$ is parallel to the tangent vector $\vec{T}$. Thus

$$\lambda\vec{x}'(t)=\vec{u}(t)\wedge\vec{x}'(t)\wedge\vec{x}''(t). \tag{3.40}$$

Then

$$\vec{u}(t)\wedge\lambda\vec{x}'(t)\wedge\vec{x}''(t)=\vec{u}(t)\wedge\left[\vec{u}(t)\wedge\vec{x}'(t)\wedge\vec{x}''(t)\right]\wedge\vec{x}''(t). \tag{3.41}$$

By using the relation (3.40)

$$\lambda^2\vec{x}'(t)=\vec{u}(t)\wedge\left[\vec{u}(t)\wedge\vec{x}'(t)\wedge\vec{x}''(t)\right]\wedge\vec{x}''(t). \tag{3.42}$$

By using the following relation in (3.42)

$$\vec{u}(t)\wedge\vec{x}'(t)\wedge\vec{x}''(t)=\|\vec{u}(t)\|.\|\vec{x}''(t)\|.\vec{x}'(t) \tag{3.43}$$

the following relation is obtained

$$\lambda^2\vec{x}'(t)=\|\vec{u}(t)\|^2.\|\vec{x}''(t)\|^2.\vec{x}'(t) \tag{3.44}$$

or

$$\lambda^2\vec{x}'(t)=\left[\vec{u}(t)\right]^2.\left[\vec{x}''(t)\right]^2.\vec{x}'(t). \tag{3.45}$$

From the last equation

$$\lambda=\sqrt{\left[\vec{u}(t)\right]^2.\left[\vec{x}''(t)\right]^2}. \tag{3.46}$$

By using the relation (3.36)

$$\lambda = \sqrt{-[\vec{x}''(t)]^8} \tag{3.47}$$

so the vector $\vec{x}'(t)$ can be written as

$$\vec{x}'(t) = \frac{\vec{u}(t) \wedge \vec{x}'(t) \wedge \vec{x}''(t)}{\sqrt{-[\vec{x}''(t)]^8}}. \tag{3.48}$$

Using $\dfrac{d\vec{x}(t)}{dt} = \vec{x}'(t)$

$$d\vec{x}(t) = \frac{\vec{u}(t) \wedge \vec{x}'(t) \wedge \vec{x}''(t)}{\sqrt{-[\vec{x}''(t)]^8}} dt. \tag{3.49}$$

Theorem 3.7. The relations which are given by (3.32),(3.33),(3.38) do not depend on a chosen variable t.

Proof: Let t be a parametric function of s such that

$$t = f(s) \text{ and } \frac{df}{ds} = f^{\bullet} \neq 0. \tag{3.50}$$

If the first order derivative of $\vec{x}(s)$ with respect to s is denoted by $\vec{x}^{\bullet}$

$$\vec{x}^{\bullet} = \vec{x}' f^{\bullet}. \tag{3.51}$$

Then

$$(\vec{x}^{\bullet})^2 = (\vec{x}')^2 f^{\bullet 2}. \tag{3.52}$$

By using the relation (3.32)

$$(\vec{x}^{\bullet})^2 = 0. \tag{3.53}$$

If the first order derivative of the vector $\vec{x}^{\bullet}$ with respect to s is taken

$$\vec{x}^{\bullet\bullet} = \vec{x}'' f^{\bullet 2} + \vec{x}' f^{\bullet\bullet}. \tag{3.54}$$

Then

$$\vec{x}^{\bullet}\vec{x}^{\bullet\bullet} = \vec{x}'\vec{x}'' f^{\bullet 3} + \vec{x}'^2 f^{\bullet} f^{\bullet\bullet}. \tag{3.55}$$

By using the relations (3.32) and (3.33)

$$\vec{x}^{\bullet}\vec{x}^{\bullet\bullet} = 0. \tag{3.56}$$

If the first order derivative of the vector $\vec{x}^{\bullet\bullet}$ with respect to s is taken

$$\vec{x}^{\bullet\bullet\bullet} = \vec{x}''' f^{\bullet 3} + 3\vec{x}'' f^{\bullet} f^{\bullet\bullet} + \vec{x}' f^{\bullet\bullet\bullet}. \tag{3.57}$$

Also, the vector $\vec{u}(s)$ can be written as below

$$\vec{u}(s) = \vec{x}^{\bullet}(s) \wedge \vec{x}^{\bullet\bullet}(s) \wedge \vec{x}^{\bullet\bullet\bullet}(s). \tag{3.58}$$

By using the relation (3.51), (3.52) and (3.56)

$$\vec{u}(s) = (\vec{x}' \wedge \vec{x}'' \wedge \vec{x}''')f^{\bullet 6}. \tag{3.59}$$

Then

$$\vec{u}(s) = \vec{u}(t)f^{\bullet 6}. \tag{3.60}$$

Also, the vector $\vec{v}(s)$ can be written as below

$$\vec{v}(s) = \vec{u}(s) \wedge \vec{x}^{\bullet}(s) \wedge \vec{x}^{\bullet\bullet}(s). \tag{3.61}$$

Then

$$\vec{v}(s) = (\vec{u}(t) \wedge \vec{x}' \wedge \vec{x}'')f^{\bullet 9}, \tag{3.62}$$

$$\vec{v}(s) = \vec{v}(t)f^{\bullet 9}. \tag{3.63}$$

Since $[\vec{v}(t)]^2 = 0$

$$[\vec{v}(s)]^2 = 0. \tag{3.64}$$

This is the required result.

Theorem 3.8. The relation (3.49) does not depend on a chosen variable t. In other words, it is a differential invariant.

Proof: Let t be a parametric function of s such that

$$t = f(s) \text{ and } \frac{df}{ds} = f^{\bullet} \neq 0. \tag{3.65}$$

Now we consider the relation (3.49) with respect to s instead of t.

$$d\vec{x}(s) = \frac{\vec{u}(s) \wedge \vec{x}^{\bullet}(s) \wedge \vec{x}^{\bullet\bullet}(s)}{\sqrt{-[\vec{x}^{\bullet\bullet}(s)]^8}} ds. \tag{3.66}$$

Also, from the relation (3.52)

$$[\vec{x}^{\bullet\bullet}]^2 = [\vec{x}'']^2 f^{\bullet 4}. \tag{3.67}$$

By using the relations (3.62), (3.67) and (3.65) in (3.65)

$$d\vec{x}(s)=\frac{\vec{u}(t)\wedge\vec{x}'(t)\wedge\vec{x}''(t)}{\sqrt{-[\vec{x}''(t)]^8}}dt. \tag{3.68}$$

Using the relation (3.49) the last relation can be written as below

$$d\vec{x}(s)=d\vec{x}(t).dt. \tag{3.69}$$

This is the required result.

On the other hand, the relation (3.67) can be written as below

$$\sqrt[4]{-(\vec{x}^{\bullet\bullet})^2}.ds=\sqrt[4]{-(\vec{x}'')^2}.dt \tag{3.70}$$

and taking

$$(\vec{x}^{\bullet\bullet})^2=-1 \tag{3.71}$$

the following relation is obtained

$$s=\int_{t_0}^{t}\left[-(\vec{x}'')^2\right]^{\frac{1}{4}}dt. \tag{3.72}$$

Definition 3.14. The relation (3.72) is called the psedo-arc parameter or the natural arc parameter of the minimal curve (x): $\vec{x}=\vec{x}(t)$.

The relation (3.72) shows that the natural arc parameter of a minimal curve in R^4 is the same as the natural arc parameter of a minimal curve in R^3. Thus the position vector of a minimal curve in R^4 satisfies the relations given by (1.50) in the chapter one.

Definition 3.15. Let a four-faced of the following vectors be connected to every point P of a minimal curve (x): $\vec{x}=\vec{x}(s)$ which is in the class of C^n such that $n\geq 4$.

$$\begin{aligned}&\vec{e}_1=\vec{x}^{\bullet}\\&\vec{e}_2=i\vec{x}^{\bullet\bullet}\\&\vec{e}_3=-\frac{\beta}{2}\vec{x}^{\bullet}+\vec{x}^{\bullet\bullet\bullet},\ \beta=(\vec{x}^{\bullet\bullet\bullet})^2\\&\vec{e}_4=\mu(\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3),\ \mu=\mp1.\end{aligned} \tag{3.73}$$

This four-faced is called cyclic four-faced at the point P of the curve (x): $\vec{x}=\vec{x}(s)$.

The vector $\vec{e}_4$ will be determined in terms of the vectors $\vec{x}^{\bullet}, \vec{x}^{\bullet\bullet}, \vec{x}^{\bullet\bullet\bullet}, \vec{x}^{\bullet\bullet\bullet\bullet}$ by using the relations given by (3.14). Let

$$\vec{e}_4 = a\vec{x}^{\bullet} + b\vec{x}^{\bullet\bullet} + c\vec{x}^{\bullet\bullet\bullet} + d\vec{x}^{\bullet\bullet\bullet\bullet} \tag{3.74}$$

where a, b, c and d are complex coefficients.

$$\vec{e}_1\vec{e}_4 = a\left(\vec{x}^{\bullet}\right)^2 + b\vec{x}^{\bullet}\vec{x}^{\bullet\bullet} + c\vec{x}^{\bullet}\vec{x}^{\bullet\bullet\bullet} + d\vec{x}^{\bullet}\vec{x}^{\bullet\bullet\bullet\bullet}$$

and it is known that $\vec{e}_1\vec{e}_4 = 0$, $\left(\vec{x}^{\bullet}\right)^2 = 0$, $\vec{x}^{\bullet}\vec{x}^{\bullet\bullet} = 0$ and $\vec{x}^{\bullet}\vec{x}^{\bullet\bullet\bullet\bullet} = 0$ from the relations given by (3.14) and (1.50). Thus

$$c = 0. \tag{3.75}$$

$$\vec{e}_2\vec{e}_4 = ai\vec{x}^{\bullet\bullet}\vec{x}^{\bullet} + bi\left(\vec{x}^{\bullet\bullet}\right)^2 + ci\vec{x}^{\bullet\bullet}\vec{x}^{\bullet\bullet\bullet} + di\vec{x}^{\bullet\bullet}\vec{x}^{\bullet\bullet\bullet\bullet}$$

and by using the relations given by (3.14) and (1.50)

$$b = -d\beta. \tag{3.76}$$

$$\vec{e}_3\vec{e}_4 = \left(-\frac{\beta}{2}\vec{x}^{\bullet} + \vec{x}^{\bullet\bullet\bullet}\right)\left(a\vec{x}^{\bullet} + b\vec{x}^{\bullet\bullet} + c\vec{x}^{\bullet\bullet\bullet} + d\vec{x}^{\bullet\bullet\bullet\bullet}\right),$$

so

$$a = -\frac{1}{2}d\beta^{\bullet}. \tag{3.77}$$

$$\left(\vec{e}_4\right)^2 = \left(a\vec{x}^{\bullet} + b\vec{x}^{\bullet\bullet} + c\vec{x}^{\bullet\bullet\bullet} + d\vec{x}^{\bullet\bullet\bullet\bullet}\right)^2$$

and by using the relations given by (1.50)

$$d = \frac{\mp i}{\sqrt{\beta^2 + \gamma}} \tag{3.78}$$

where

$$\gamma = \left(\vec{x}^{\bullet\bullet\bullet\bullet}\right)^2. \tag{3.79}$$

When the relations (3.75), (3.76), (3.77), (3.78) are substituted in the relation given by (3.74)

$$\vec{e}_4 = d\left(-\frac{1}{2}\beta^{\bullet}\vec{x}^{\bullet} - \beta\vec{x}^{\bullet\bullet} + \vec{x}^{\bullet\bullet\bullet\bullet}\right) \tag{3.80}$$

where $d = \dfrac{\mp i}{\sqrt{\beta^2 + \gamma}}$, $\beta = \left(\vec{x}^{\bullet\bullet\bullet}\right)^2$ and $\gamma = \left(\vec{x}^{\bullet\bullet\bullet\bullet}\right)^2$.

Consequently a cyclic four-faced of a minimal curve (x): $\vec{x} = \vec{x}(s)$ is given as below

$$\begin{aligned} \vec{e}_1 &= \vec{x}^{\bullet} \\ \vec{e}_2 &= i\vec{x}^{\bullet\bullet} \\ \vec{e}_3 &= -\frac{\beta}{2}\vec{x}^{\bullet} + \vec{x}^{\bullet\bullet\bullet} \\ \vec{e}_4 &= d\left(-\frac{1}{2}\beta^{\bullet}\vec{x}^{\bullet} - \beta\vec{x}^{\bullet\bullet} + \vec{x}^{\bullet\bullet\bullet\bullet}\right). \end{aligned} \tag{3.81}$$

If the vector $\vec{e}_4$ is derived with respect to s which is the natural arc parameter of the minimal curve (x): $\vec{x} = \vec{x}(s)$ then

$$\vec{e}_4^{\bullet} = d^{\bullet}\left(-\frac{1}{2}\beta^{\bullet}\vec{x}^{\bullet} - \beta\vec{x}^{\bullet\bullet} + \vec{x}^{\bullet\bullet\bullet\bullet}\right) + d\left(-\frac{1}{2}\beta^{\bullet\bullet}\vec{x}^{\bullet} - \frac{1}{2}\beta^{\bullet}\vec{x}^{\bullet\bullet} - \beta^{\bullet}\vec{x}^{\bullet\bullet} - \beta\vec{x}^{\bullet\bullet\bullet} + \vec{x}^{\bullet\bullet\bullet\bullet}\right)$$

$$\vec{e}_4^{\bullet} = -\frac{1}{2}d\left(\beta^{\bullet\bullet} + \beta\right)\vec{e}_1 - d\beta\vec{e}_3 + \frac{d^{\bullet}}{d}\vec{e}_4. \tag{3.82}$$

Finally the vectors $\vec{e}_1^{\bullet}, \vec{e}_2^{\bullet}, \vec{e}_3^{\bullet}, \vec{e}_4^{\bullet}$ are given as below in terms of the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$

$$\begin{aligned} \vec{e}_1^{\bullet} &= -i\vec{e}_2 \\ \vec{e}_2^{\bullet} &= ik\vec{e}_1 + i\vec{e}_3 \\ \vec{e}_3^{\bullet} &= -ik\vec{e}_2 \\ \vec{e}_4^{\bullet} &= -d\left(k^{\bullet\bullet} + dk\right)\vec{e}_1 - dk\vec{e}_3 + \frac{d^{\bullet}}{d}\vec{e}_4 \end{aligned} \tag{3.83}$$

where $k = \dfrac{1}{2}\beta$ is the psedo curvature of the minimal curve (x): $\vec{x} = \vec{x}(s)$ in the class of C^n such that $n \geq 5$.

CONCLUSIONS

Complex elements and minimal elements are defined to examine minimal curves in the real space R^4. Then the properties of the complex elements and the minimal elements are given. It is proved that there is only one real plane on a complex flat in R^4.

Frenet frame of a minimal curve is established and natural arc parameter of a minimal curve is presented for R^4. Then it is shown that the tangent vectors and the binormal vectors of the minimal curves are minimal vectors.

Also, the cyclic four-faced which is found in the chapter two is expressed with the position vectors o the minimal curves.

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