A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF ÇANKIRI KARATEKİN UNIVERSITY

# COMPLEX INTERVAL-VALUED T-SPHERICAL FUZZY SETS AND THEIR APPLICATIONS IN MULTI-CRITERIA DECISION MAKING

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

BY

HUSSEIN HAMMOODI MAHMOOD AL-JAFAR

ÇANKIRI

2023

COMPLEX INTERVAL-VALUED T-SPHERICAL FUZZY SETS ABD THEIR APPLICATIONS IN MULTI-CRITERIA DECISION MAKING

By Hussein Hammodi Mahmood AL-JAFAR

May 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

**Advisor** : Assoc. Prof. Dr. Faruk KARAASLAN

**Examining Committee Members:**

**Chairman** : Assoc. Prof. Dr. İrfan DELİ
Mathematics Education
Kilis 7 Aralık University

**Member** : Asst. Prof. Dr Celalettin KAYA
Mathematics
Çankırı Karatekin University

**Member** : Assoc. Prof. Dr. Faruk KARAASLAN
Mathematics
Çankırı Karatekin University

**Approved for the Graduate School of Natural and Applied Sciences**

**Prof. Dr. Hamit ALYAR**
**Director of Graduate School**

**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

**Hussein Hammodi Mahmood AL-JAFAR**

## ABSTRACT

# COMPLEX INTERVAL-VALUED T-SPHERICAL FUZZY SETS AND THEIR APPLICATIONS IN MULTI-CRITERIA DECISION MAKING

Hussein Hammodi Mahmood AL-JAFAR
Master of Science in Mathematics
Advisor: Assoc. Prof. Dr. Faruk KARAASLAN
May 2023

This thesis consists of six chapters. In section 1, literature review, motivation of the thesis and contributions of the thesis in literature are presented. In section 2, the basic definitions and theorems that will be used throughout the thesis are given. In section 3, the concept of complex interval-valued T-spherical fuzzy (CIVT-SF) sets is recalled, and their set-theoretical operations such as union, intersection and complement and score and accuracy functions of CIVT-SF numbers are introduced. Also, arithmetic operations between two CIVT-SF numbers are defined and some results related to these operations are obtained. In section 4, some novel aggregation operators called complex interval-valued T-spherical fuzzy weighted arithmetic aggregation (CIVTSFWAA) operator and complex interval-valued T-spherical fuzzy weighted geometric aggregation (CIVTSFWGA) operator are introduced and some of their basic properties are investigated. In section 5, a decision-making method is developed based on defined averaging operators. Also, to show the progrees of the proposed method an application is given. In section 6, some explanations related to the results of the thesis are presented.

**2023, 45 pages**

**Keywords:** Fuzzy set, Complex fuzzy set, T-spherical fuzzy set, Interval-valued fuzzy sets, Complex T-spherical fuzzy set, Aggregation operator, Decision-making

**ÖZET**

# KOMPLEKS ARALIK DEĞERLİ T-KÜRESEL BULANIK KÜMELER VE ÇOK KRİTERLİ KARAR VERMEDEKİ UYGULAMALARI

Hussein Hammodi Mahmood AL-JAFAR
Matematik, Yüksek Lisans
Danışman: Doç. Dr. Faruk KARAASLAN
Mayıs 2023

Bu tez altı bölümden oluşmaktadır. 1. Bölümde, literatür incelemesi, tezin motivasyonu ve tezin literatüre katkısı sunulmuştur. 2. Bölümde, tez boyunca kullanılacak temel tanım ve teoremler verilir. 3. bölümde, kompleks aralık-değerli T-küresel bulanık küme (KADT-KBK) kavramı hatırlatıldı ve birleşim, kesişim, tümleyen gibi küme teorik işlemleri ve KADT-KB sayıların skor ve kesinlik fonksiyonları tanımlanmıştır. Ayrıca iki KADT-KB sayı arasındaki aritmetik işlemler tanımlanmış ve bu işlemlerle ilgili bazı sonuçlar elde edilmiştir. 4. Bölümde, karmaşık aralık değerli T-küresel bulanık ağırlıklı aritmetik birleştirme (KADTKBAAB) operatörü ve karmaşık aralık değerli T-küresel bulanık ağırlıklı geometrik birleştirme (KADTKBAGB) operatörü olarak adlandırılan bazı yeni birleştirme operatörleri tanıtılmakta ve bazı temel özellikleri araştırılmaktadır. Bölüm 5'te tanımlanan birleştirme operatörlerine dayalı olarak bir karar verme yöntemi geliştirilmiştir. Ayrıca önerilen yöntemin işleyişini göstermek için bir uygulama verilmiştir. Bölüm 6 da, tezin sonuçlarına ilişkin bazı açıklamalara yer verilmiştir.

**2023, 45 sayfa**

**Anahtar Kelimeler:** Bulanık küme, komplex bulanık küme, T-küresel bulanık küme, Aralık-değerli bulanık küme, Kompleks T-küresel bulanık küme, Birleştime operatörü, Karar verme.

## PREFACE AND ACKNOWLEDGEMENTS

To the soul of my father, may Allah have mercy on him. To my mother, may Allah prolong her life. To my brothers, may Allah protect them. To my wife, may Allah perpetuate her as a helper for me, and to my children, may Allah grant them success. I dedicate this thesis. I extend my sincere thanks to Assoc. Prof. Dr. Faruk KARAASLAN, who supervised my thesis. He enriched it scientifically and did not skimp on any information. To the owner of the greatest credit for completing the master's degree, my friend and colleague Mohammed Allaw DAWOOD. To Mr. Abdulrasool AL-HUSSEINAWI and to everyone who has favored me or wished me well.

**Hussein Hammodi Mahmood AL-JAFAR**
**Çankırı-2023**

**CONTENTS**

## LIST OF SYMBOLS

| | |
|---|---|
| $\oplus$ | Addition of q-ROFNs |
| $\mathfrak{x}$ | Element in the universal set |
| $\mathscr{F}$ | Fuzzy set |
| $\pi_{\mathfrak{g}}(\alpha)$ | Hesitancy degree of $\alpha \in \mathfrak{X}$ |
| $\mu$ | Membership function |
| $\otimes$ | Multiplication of q-ROFNs |
| $\nu$ | Non-membership function |
| $\mathbb{P}$ | Pythagorean fuzzy set |
| $\mathfrak{g}$ | q-rung orthopair fuzzy set |
| $\mathfrak{X}$ | Universal set |
| $\omega$ | Weight vector |

## LIST OF ABBREVIATIONS

| | |
|---|---|
| AO | Aggregation operator |
| CFS | Complex fuzzy set |
| CIVPFS | Complex interval-valued picture fuzzy set |
| CIVSFS | Complex interval-valued spherical fuzzy set |
| CIVTSFS | Complex interval-valued T-spherical fuzzy set |
| CIVTSFWAA | Complex interval-valued T-SF weighted arithmetic aggregation |
| CIVTSFWGA | Complex interval-valued T-SF weighted geometric aggregation |
| CIFS | Complex intuitionistic fuzzy set |
| CPFS | Complex picture fuzzy set |
| CSFS | Complex spherical fuzzy set |
| CTSFS | Complex T-spherical fuzzy set |
| FS | Fuzzy set |
| IFS | Intuitionistic fuzzy set |
| IVPFS | Interval-valued picture fuzzy set |
| IVSFS | Interval-valued spherical fuzzy set |
| IVTSFS | Interval-valued T-spherical fuzzy set |
| PFS | Picture fuzzy set |
| PFN | Picture fuzzy number |
| SFS | Spherical fuzzy set |
| TSFS | T-spherical fuzzy set |

## LIST OF TABLES

# 1. INTRODUCTION

Humans have struggled with dealing with uncertainty for a very long time. A proposition is either true or false according to traditional reasoning. However, there may be instances in daily life where an objective method cannot be assessed. In other words, a statement may not necessarily be wholly truthful or wholly falsity. These circumstances resemble the grey areas between black and white. To model difficulties involving situations where it is unclear whether something is true or false, numerous theories have been proposed. The most well-known of them is the fuzzy set theory, which, as a generalization of established sets, was put up by Zadeh (1965). In reality, a fuzzy set is a function from a non-empty universe to a closed interval [0,1]. The membership function is a function that expresses how much a given element in the universe belongs to a particular collection. A component's degree of fuzzy set membership depends on how large its value is under the membership function. An element, for instance, is said to have a high degree of belonging to the set if its membership degree is 0.9, and a very low degree of belonging to the set if it is 0.1. Additionally, a member of the set with a membership degree of 0.6 does not belong, with the degree of non-belonging being $1-0.6=0.4$. If you pay attention, you can determine the degree of non-membership by deducting the degree of membership from 1. However, this might not always be the case in real life. In other words, there are some circumstances where people might hesitate. Fuzzy set theory is currently insufficient to model such circumstances. A new set theory was able to evolve as a result of this. In order to generalize fuzzy sets, Atanassov (1986) developed the concept of intuitionistic fuzzy sets. From a universe $X$ to the bounding interval $[0,1]$, two functions were established: the membership ($mu$) and the non-membership ($nu$) functions, are used to describe an intuitionistic fuzzy set. An element $x$, for instance, can belong to a set to a degree of 0.5 but it doesn't belong to a degree of 0.3. It can be seen that there is 0.2 degree of hesitation in this situation with reference to the element. In an intuitionistic fuzzy set, an element's combined degree of membership and degree of non-membership cannot exceed 1. The total of an element's membership and non-membership degrees may, in some circumstances, exceed 1. Fuzzy intuitionistic sets cannot adequately represent problems involving such circumstances. Yager (2013) offered the Pythagorean fuzzy set as a solution to this problem. In Pythagorean fuzzy sets, the squares of an element's membership degree and non-membership degree must not exceed one. The membership degree of an element is 0.6 while the non-membership degree is 0.7. for instance, it cannot be stated as a number since $0.6+0.7=1.3$. However, it can be written as a Pythagorean fuzzy value because $0.6^2+0.7^2 \leq 1$. Here, you can wonder: "How should this situation be treated if the square of the sum of the membership degree and non-membership degree corresponding to an element is more than 1?". Yager (2016) and Yager *et al.* (2018) also answered this query by presenting the idea of a q-rung orthopair fuzzy set. We cannot model such elements using Pythagorean fuzzy values, for instance, if an element has a non-membership degree of 0.9 and a membership degree of 0.8. But $0.8^5+0.9^5 \leq 1$. The q-rung orthopair fuzzy set is an extension of the fuzzy set, intuitionistic fuzzy set, and Pythagorean fuzzy sets. It is a set that is defined from an initial universe into the range $[0,1]$ under the condition $\mu^q+\nu^q \leq 1$ $(q>2)$. It is distinguished by membership $(\mu)$ and non-membership $(\nu)$ functions.

Picture FS (PFS), described by Cuong (2013a, 2013b), is another extension of IFS. Because a PFS may simulate judgments about an object or an idea using degrees of yes, abstention, no, and rejection, it is a powerful tool for modeling human opinion. Using the standard $0 \leq \mu+\gamma+\nu \leq 1$, a PFS discovered three degrees of an element, referred to as MD $(\mu)$, abstinence degree (AD) or neutral degree $(\gamma)$, and NMD $(\nu)$. Despite the fact that PFS has numerous applications in areas like decision-making (DM), (Cao 2020, Garg 2017, Joshi 2020, Karaaslan and Dawood 2021, Peng and Dai 2017, Tian *et al.* 2020, Wei 2017a), similarity measure (Rafiq *et al.* 2019, Thao 2020, Wei and Gao 2018b, Wei 2018c), correlation coefficient (Ganie *et al.* 2020, Singh 2015), and clustering (Hao 2016, Son 2016). When $\mu+\gamma+\nu>1$, it is insufficient for modeling various issues. Because of this, Gungogdu and Kahraman (2019a, 2019b) introduced the design of the spherical fuzzy set (SFS), an extension of PFS that satisfies the requirement $0 \leq \mu^2+\gamma^2+\nu^2 \leq 1$. They also researched SFS operations and how to use this set in DM issues.

By combining the SFS and TOPSIS methods, Kahraman *et al.* (2019a) presented a DM strategy and gave an example of how to use the technique in selecting a hospital location. A definition of the T-spherical fuzzy set (T-SFS) as an extension of the SFS with the constraint may be found in Mahmood *et al.* (2019) $0 \leq \mu^q+\gamma^q+\nu^q \leq 1$ and several examples of T-SFS and SFS uses in medical diagnosis and decision-making challenges. The similarity measurements for T-SFSs were introduced by Ullah *et al.* (2018a) along with a pattern recognition application. In Garg *et al.* (2018), improved interactive aggregation operators for T-SFSs were presented, and their operational laws were examined. Ullah *et al.* (2018b) provided a numerical example involving a multi-attribute decision-making (MADM) problem and defined certain ordered weighted geometric (OWG) and hybrid geometric (HG) operators.

The complex FS (CFS) theory was investigated by Ramot *et al.* (2002, 2003). It contains the grade of complex-valued supporting relating to unit disc on a complex plane in the form of polar coordinates. Alkouri and Salleh (2012) also improved CFS to look into complex intuitionistic FS (CIFS), which has the supporting grade and the opposing grade in the form of a complex number in a unit disc with the restriction that the sum of the real part (also for the opposing grade's imaginary part) of the supporting grade and the grade of supporting against is not greater than a unit interval. pertaining to a multi-attribute decision-making (MADM) problem. The interval-valued complex intuitionistic fuzzy relation (IVCIFR) was proposed by Garg and Rani (2019). Ullah *et al.* (2019) extended CIFS to explore complex Pythagorean fuzzy sets (CPFS) under the condition that the complex PFS and the sum of the squares of the real parts be positive. Complex q-ROFs (Cq-ROFS) was defined by Liu *et al.* (2020). Ali *et al.* (2020) defined the concept of complex T-spherica fuzzy (CT-SF) sets (CT-SFSs) and studied on aggregation opertaions of them. Karaaslan and Dawood (2021) introduced Dombi aggregation operator under the CT-SF environment. They create a decision-making process by utilizing the defined operators. Interval-valued complex Spherical and T-spherical fuzzy sets were defined by Alothaim *et al.* (2022). They also developed interval-valued complex T-spherical partial order set and relation to use in cybersecurity application.

In this thesis, we present the accuracy and scoring functions of the CIVT-SF statistics, along with instances of their set theoretical and arithmetic operations. Both the complex interval-valued T-spherical-fuzzy weighted-arithmetic aggregation (CIVTSFWAA) operator and the complex interval-valued T-spherical-fuzzy weighted-geometric aggregation (CIVTSFWGA) operator are introduced, and some of their fundamental properties are examined. In addition, a mechanism for making decisions is created using predefined averaging operators. Additionally, an application is provided to demonstrate the progress of the suggested strategy.

Remainin part of the thesis is organized as follows: We went over the fundamental concepts of FS, IFS, IVIFS, CIFS, CIVIFS, PFS, IVPFS, SFS, TSFS, and IVTSFS in Section 2. In Section 3, the concept of CIVTSFS and how it functions are proposed. We constructed the aggregation operators in Section 4, and their well-known results are given. We created decision-making algorithms in Section 5 to address the issues and illustrate them with the use of numerical examples. In Section 6, a conclusion is provided.

## 2. PRELIMINARIES

This part introduces some fundamental concepts concerning FSs, CFSs, SFSs, T-SFSs, and CT-SFSs in order to make the paper self-contained. A finite universal set will also be represented by $\mathfrak{X}$. The marks "+" and "-" denote the upper and bottom halves, respectively, of the intervals for membership, neutrality, and non-membership.

**Definition 2.1.** (Zadeh 1965) Let $\mathfrak{X}$ be a non-empty universal set. A fuzzy set $\mathscr{F}$ on $\mathfrak{X}$ is described by a membership function:

$$s_{\mathscr{F}} : \mathfrak{X} \to [0,1].$$

A fuzzy set (FS) can be written as a set of pairs as follow:

$$\mathscr{F} = \{(\mathfrak{x}, s_{\mathscr{F}}(\mathfrak{x})) : \mathfrak{x} \in \mathfrak{X}\}$$

where the degree of membership of the element "$\mathfrak{x}$" in "$\mathfrak{X}$" is represented by $s_{\mathscr{F}}(\text{"}\mathfrak{x}\text{"})$.

**Definition 2.2.** (Ramot 2002, 2003) Let $\mathfrak{X}$ be a set that is not empty. The definition of a complex fuzzy set (CFS) is:

$$\mathscr{F} = \{(\mathfrak{x}, \tilde{r}_{\mathscr{F}}(\mathfrak{x})) : \mathfrak{x} \in \mathfrak{X}\},$$

where the membership functions of CFS $\mathscr{F}$ are denoted by $\tilde{r}_{\mathscr{F}}(\mathfrak{x})$ and receive any objects that are inside the complex plane's unit circle. As a result, it might be said to be $\tilde{r}_{\mathscr{F}}(\mathfrak{x}) = r_{\mathscr{F}}(\mathfrak{x})e^{i2\pi\varpi_{r_{\mathscr{F}}}(\mathfrak{x})}$, and it signifies a complex-valued grade of membership (CMF) of $\mathfrak{x} \in \mathfrak{X}$ to CFS $\mathscr{F}$. In this instance, $i = \sqrt{-1}$ and $0 \leq r_{\mathscr{F}}(\mathfrak{x}) \leq 1$ and $0 \leq \varpi_{r_{\mathscr{F}}}(\mathfrak{x}) \leq 1.$ , respectively.

**Definition 2.3.** (Atanassov 1986) Let $\mathfrak{X}$ be a set that is not empty. The following is the definition of an intuitionistic fuzzy set (IFS) $\kappa$ on $\mathfrak{X}$:

$$\kappa = \{(\mathfrak{x}, \mu_{\kappa}(\mathfrak{x}), \nu_{\kappa}(\mathfrak{x}) : \forall \mathfrak{x} \in \mathfrak{X}\}$$

Here $\mu_\kappa : \mathfrak{X} \to [0,1]$ and $\nu_\kappa : \mathfrak{X} \to [0,1]$ are called membership function and non-membership function of intuitionistic fuzzy set $\kappa$, respectively. Also,
$0 \le \mu_\kappa(\mathfrak{x}) + \nu_\kappa(\mathfrak{x}) \le 1$. $R(\mathfrak{x}) = 1 - \mu_\kappa(\mathfrak{x}) - \nu_\kappa(\mathfrak{x})$ denotes the degree of hesitancy of $\mathfrak{x} \in \mathfrak{X}$. The pair for the convenience $(\mu_\kappa, \nu_\kappa)$ is called intuitionistic fuzzy number.

**Definition 2.4.** (Alkouri and Salleh 2012) Let $\mathfrak{X}$ be a set that is not empty. A complex intuitionistic fuzzy-set (CIFS) $\mathscr{F}$ is described as:

$$\mathscr{F} = \{(\mathfrak{x}, \tilde{r}\mathscr{F}(\mathfrak{x}), \tilde{s}\mathscr{F}(\mathfrak{x})) : \mathfrak{x} \in \mathfrak{X}\},$$

where $\tilde{r}_{\mathscr{F}}(\mathfrak{x})$ and $\tilde{s}\mathscr{F}(\mathfrak{x})$ are referred to as membership and non-membership functions for CIFS ($\mathscr{F}$). They all get lying within the complex plane's unit circle. As a result, They fit the description of $\tilde{r}\mathscr{F}(\mathfrak{x}) = r_{\mathscr{F}}(\mathfrak{x})e^{i2\pi\varpi_{r_{\mathscr{F}}}(\mathfrak{x})}$, and $\tilde{s}\mathscr{F}(\mathfrak{x}) = s\mathscr{F}(\mathfrak{x})e^{i2\pi\varpi_{s_{\mathscr{F}}}(\mathfrak{x})}$, showing, the complex-valued membership and non-membership grades of the $\mathfrak{x} \in \mathfrak{X}$ to CIFS F. $\mathscr{F}$, respectively. Here $i = \sqrt{-1}$, and for all $\mathfrak{x} \in \mathfrak{X}$, $0 \le r_{\mathscr{F}}(\mathfrak{x}) + s_{\mathscr{F}}(\mathfrak{x}) \le 1$, and $0 \le \varpi_{r_{\mathscr{F}}}(\mathfrak{x}) + \varpi_{s_{\mathscr{F}}}(\mathfrak{x}) \le 1$. Furthermore,
$W_{\mathfrak{R}}(\mathfrak{x}) = (1 - r_{\mathscr{F}}(\mathfrak{x}) - s_{\mathscr{F}}(\mathfrak{x}))e^{(1-\varpi_{r_{\mathscr{F}}(\mathfrak{x})} - \varpi_{s_{\mathscr{F}}(\mathfrak{x})})}$ exemplifies the complex hesitation grade of $\mathfrak{x}$.

**Definition 2.5.** (Cuong 2013a, 2013b) A picture fuzzy set (PFS) $\mathscr{F}$ on a universe $\mathfrak{X}$ is an object in the form

$$\mathscr{F} = \{(\mathfrak{x}, \mu_{\mathscr{F}}(\mathfrak{x}), \eta_{\mathscr{F}}(\mathfrak{x}), \nu_{\mathscr{F}}(\mathfrak{x})) | \mathfrak{x} \in \mathfrak{X}\}.$$

Here $\mu_{\mathscr{F}}(\mathfrak{x}) \in [0,1]$ is called the degree of positive membership of $\mathfrak{x}$ in $\mathscr{F}$, $\eta_{\mathscr{F}}(\mathfrak{x}) \in [0,1]$ is called the degree of neutral membership of $\mathfrak{x}$ in $\mathscr{F}$, and $\nu_{\mathscr{F}}(\mathfrak{x}) \in [0,1]$ is called the degree of negative membership of $\mathfrak{x}$ in $\mathscr{F}$ such that
$0 \le \mu_{\mathscr{F}}(\mathfrak{x}) + \eta_{\mathscr{F}}(\mathfrak{x}) + \nu_{\mathscr{F}}(\mathfrak{x}) \le 1$, $(\forall \mathfrak{x} \in \mathfrak{X})$. Here
$R(\mathfrak{x}) = (1 - (\mu_{\mathscr{F}}(\mathfrak{x}) + \eta_{\mathscr{F}}(\mathfrak{x}) + \nu_{\mathscr{F}}(\mathfrak{x})))$ is referred to $\mathfrak{x}$ in $\mathscr{F}$. The membership degree of refusal membership degree of $\mathfrak{x}$ in $\mathscr{F}$. $\rho = (\mu_{\mathscr{F}}, \eta_{\mathscr{F}}, \nu_{\mathscr{F}})$ is called a Picture fuzzy number (PFN).

**Definition 2.6.** (Cuong 2014) An interval-valued picture fuzzy set (IVPFS) $F$ on a universe $\mathfrak{X}$ is an object in the form

$$F=\{\mathfrak{x},(\acute{\mu}F(\mathfrak{x}),\acute{\eta}F(\mathfrak{x}),\acute{\nu}_F(\mathfrak{x}))|\mathfrak{x}\in\mathfrak{X}\}$$

where $\acute{\mu}F=[\mu F^-,\mu_F^+]$, $\acute{\eta}F=[\eta F^-,\eta_F^+]$ and $\acute{\nu}F=[\nu F^-,\nu_F^+]$ with $0\le(\mu_F^+)+(\eta_F^+)+(\nu_F^+)\le 1$. The term $H_F(\mathfrak{x})=[1-(\mu_F^++\eta_F^++\nu_F^+), 1-(\mu_F^-+\eta_F^-+\nu_F^-)]$ is seen as the level of hesitation of $\mathfrak{x}$ in $\mathfrak{X}$. The IVPF number is stated by:$f=([\mu_f^-,\mu_f^+],[\eta_f^-,\eta_f^+],[\nu_f^-,\nu_f^+])$.

**Definition 2.7.** (Akram *et al.* 2020) Let $\mathfrak{X}$ be a set that isn't empty. A complex picture fuzzy set (CPFS) is described mathematically as:

$$\mathscr{F}=\{(\mathfrak{x},\tilde{r}\mathscr{F}(\mathfrak{x}),\tilde{s}\mathscr{F}(\mathfrak{x}),\tilde{t}_{\mathscr{F}}(\mathfrak{x})):\mathfrak{x}\in\mathfrak{X}\},$$

where $\mathscr{F}$ membership, neutral membership, and non-membership functions are denoted by $\tilde{r}_{\mathscr{F}}(\mathfrak{x}),\tilde{s}_{\mathscr{F}}(\mathfrak{x})$ and $\tilde{t}\mathscr{F}(\mathfrak{x})$ to the CPFS ($\mathscr{F}$). They are all positioned inside the complex plane's unit circle. As a result, they can be described as $\tilde{r}\mathscr{F}(\mathfrak{x})=r_{\mathscr{F}}(\mathfrak{x})e^{i2\pi\varpi_{r_{\mathscr{F}}}(\mathfrak{x})}$, $\tilde{s}\mathscr{F}(\mathfrak{x})=s\mathscr{F}(\mathfrak{x})e^{i2\pi\varpi_{s_{\mathscr{F}}}(\mathfrak{x})}$, and $\tilde{t}\mathscr{F}(\mathfrak{x})=t\mathscr{F}(\mathfrak{x})e^{i2\pi\varpi_{t_{\mathscr{F}}}(\mathfrak{x})}$, where they represent, respectively, the complex-valued categories of membership, non-membership, and neutral membership in CPFS $\mathscr{F}$. In this instance, $i=\sqrt{-1}$, and $\forall\mathfrak{x}\in\mathfrak{X}$, $0\le r_{\mathscr{F}}(\mathfrak{x})+s_{\mathscr{F}}(\mathfrak{x})+t_{\mathscr{F}}(\mathfrak{x})\le 1$, and $0\le\varpi_{r_{\mathscr{F}}}(\mathfrak{x})+\varpi_{s_{\mathscr{F}}}(\mathfrak{x})+\varpi_{t_{\mathscr{F}}}(\mathfrak{x})\le 1$. Furthermore, $W_{\mathfrak{R}}(\mathfrak{x})=(1-r_{\mathscr{F}}(\mathfrak{x})-s_{\mathscr{F}}(\mathfrak{x})-t_{\mathscr{F}}(\mathfrak{x}))e^{(1-\varpi_{r_{\mathscr{F}}(\mathfrak{x})}-\varpi_{s_{\mathscr{F}}(\mathfrak{x})}-\varpi_{t_{\mathscr{F}}(\mathfrak{x})})}$ expresses the complex hesitancy grade of $\mathfrak{x}$.

**Definition 2.8.** A complex interval-valued picture fuzzy set (CIVPFS) $C$ is defined as:

$$C=(\mathfrak{x},[r_c^-(\mathfrak{x}),r_c^+(\mathfrak{x})],[s_c^-(\mathfrak{x}),s_c^+(\mathfrak{x})],[t_c^-(\mathfrak{x}),t_c^+(\mathfrak{x})]|\mathfrak{x}\in\mathfrak{X})$$

where $r_c^-(\mathfrak{x})=\zeta_c^-(\mathfrak{x})e^{i2\pi\omega_{\zeta_c^-}(\mathfrak{x})},r_c^+(\mathfrak{x})=\zeta_c^+(\mathfrak{x})e^{i2\pi\omega_{\zeta_c^+}(\mathfrak{x})},s_c^-(\mathfrak{x})=\xi_c^-(\mathfrak{x})e^{i2\pi\omega_{\xi_c^-}(\mathfrak{x})}$, $s_c^+(\mathfrak{x})=\xi_c^+(\mathfrak{x})e^{i2\pi\omega_{\xi_c^+}(\mathfrak{x})}$, and $t_c^-(\mathfrak{x})=\varsigma_c^-(\mathfrak{x})e^{i2\pi\omega_{\varsigma_c^-}(\mathfrak{x})},t_c^+(\mathfrak{x})=\varsigma_c^+(\mathfrak{x})e^{i2\pi\omega_{\varsigma_c^+}(\mathfrak{x})}$ with conditions: $0\le\zeta_c^+(\mathfrak{x})+\xi_c^+(\mathfrak{x})+\varsigma_c^+(\mathfrak{x})\le 1$ and $0\le\omega_{\zeta_c^+}(\mathfrak{x})+\omega_{\xi_c^+}(\mathfrak{x})+\omega_{\varsigma_c^+}(\mathfrak{x})\le 1$.
The CIVPF number is stated by:

$$c=([\zeta_c^-,\zeta_c^+]e^{i2\pi[\omega_{\zeta_c^-},\omega_{\zeta_c^+}]},[\xi_c^-,\xi_c^+]e^{i2\pi[\omega_{\xi_c^-},\omega_{\xi_c^+}]},[\varsigma_c^-,\varsigma_c^+]e^{i2\pi[\omega_{\varsigma_c^-},\omega_{\varsigma_c^+}]}).$$

**Example 2.9.** Let $\mathfrak{X}$ be a set that isn't empty and

$$C = (\mathfrak{x}, [r_c^-(\mathfrak{x}), r_c^+(\mathfrak{x})], [s_c^-(\mathfrak{x}), s_c^+(\mathfrak{x})], [t_c^-(\mathfrak{x}), t_c^+(\mathfrak{x})] | \mathfrak{x} \in \mathfrak{X})$$

where $r_c^-(\mathfrak{x}) = 0.1e^{i2\pi 0.1}, r_c^+(\mathfrak{x}) = 0.2e^{i2\pi 0.2}, s_c^-(\mathfrak{x}) = 0.2e^{i2\pi 0.1}, s_c^+(\mathfrak{x}) = 0.3e^{i2\pi 0.3}$, and $t_c^-(\mathfrak{x}) = 0.1e^{i2\pi 0.3}, t_c^+(\mathfrak{x}) = 0.3e^{i2\pi 0.4}$ with conditions: $0 \leq 0.2 + 0.3 + 0.3 \leq 1$ and $0 \leq 0.2 + 0.3 + 0.4 \leq 1$. According to Definition 2.8, set $C$ is a CIVPFS.

**Definition 2.10.** (Gundogdu and Kahraman 2019a) Let $\mathfrak{X}$ be a non-empty set. The following is the definition of a spherical fuzzy set (SFS) $\mathscr{F}$ over $\mathfrak{X}$:

$$\mathscr{F} = \{(\mathfrak{x}, r_{\mathscr{F}}(\mathfrak{x}), s_{\mathscr{F}}(\mathfrak{x}), t_{\mathscr{F}}(\mathfrak{x})) : \mathfrak{x} \in \mathfrak{X}\},$$

where $r_{\mathscr{F}} : \mathfrak{X} \to [0,1], s_{\mathscr{F}} : \mathfrak{X} \to [0,1], t_{\mathscr{F}} : \mathfrak{X} \to [0,1]$ and
$0 \leq r_{\mathscr{F}}^2(x) + s_{\mathscr{F}}^2(x) + t_{\mathscr{F}}^2(x) \leq 1, \forall \mathfrak{x} \in \mathfrak{X}$. For $\mathfrak{x} \in \mathfrak{X}$, $r_{\mathscr{F}}(\mathfrak{x})$, $s_{\mathscr{F}}(\mathfrak{x})$ and $t_{\mathscr{F}}(\mathfrak{x})$ are the degree of membership, neutral and non-membership, respectively. When we consider $(\mathfrak{x}, r_{\mathscr{F}}, s_{\mathscr{F}}, t_{\mathscr{F}})$ is called a spherical fuzzy number (SFN).

**Definition 2.11.** (Duleba *et al.* 2021) An interval-valued spherical fuzzy set (IVSFS) $C$ is characterized as:

$$C = \{(\mathfrak{x}, \acute{r}_c(\mathfrak{x}), \acute{s}_c(\mathfrak{x}), \acute{t}_c(\mathfrak{x})) | \mathfrak{x} \in X\}$$

where $\acute{r}_c = [r_c^-, r_c^+]$, $\acute{s}_c = [s_c^-, s_c^+]$ and $\acute{t}_c = [t_c^-, t_c^+]$ such that $0 \leq (r_c^+)^2 + (s_c^+)^2 + (t_c^+)^2 \leq 1$. An IVSF number is stated by: $C = ([r_C^-, r_C^+], [s_C^-, s_C^+], [t_C^-, t_C^+])$.

**Definition 2.12.** (Akram *et al.* 2021) A starting universe that is distinct from the empty set is $\mathfrak{X}$. The definition of a complex spherical fuzzy set (CSFS) is as follows:

$$\mathscr{F} = \{(\mathfrak{x}, \tilde{r}_{\mathscr{F}}(\mathfrak{x}), \tilde{s}_{\mathscr{F}}(\mathfrak{x}), \tilde{t}_{\mathscr{F}}(\mathfrak{x})) : \mathfrak{x} \in \mathfrak{X}\},$$

Here $\tilde{r}_{\mathscr{F}}(\mathfrak{x}) = r_{\mathscr{F}}(\mathfrak{x})e^{i2\pi\varpi_{r_{\mathscr{F}}}(\mathfrak{x})}$, $\tilde{s}_{\mathscr{F}}(\mathfrak{x}) = s_{\mathscr{F}}(\mathfrak{x})e^{i2\pi\varpi_{s_{\mathscr{F}}}(\mathfrak{x})}$ and $\tilde{t}_{\mathscr{F}}(\mathfrak{x}) = t_{\mathscr{F}}(\mathfrak{x})e^{i2\pi\varpi_{t_{\mathscr{F}}}(\mathfrak{x})}$ indicate the degrees of abstinence, truth, and falsehood so that $0 \leq r^2_{\mathscr{F}}(\mathfrak{x}) + s^2_{\mathscr{F}}(\mathfrak{x}) + t^2_{\mathscr{F}}(\mathfrak{x}) \leq 1$ and $0 \leq \varpi^2_{r_{\mathscr{F}}}(\mathfrak{x}) + \varpi^2_{s_{\mathscr{F}}}(\mathfrak{x}) + \varpi^2_{t_{\mathscr{F}}}(\mathfrak{x}) \leq 1$. Furthermore, $W_{\mathfrak{R}}(\mathfrak{x}) = \sqrt[2]{1 - r_{\mathscr{F}}(\mathfrak{x})^2 - s_{\mathscr{F}}(\mathfrak{x})^2 - t_{\mathscr{F}}(\mathfrak{x})^2} e^{\sqrt[2]{(1-\varpi^2_{r_{\mathscr{F}}}(\mathfrak{x}) - \varpi^2_{s_{\mathscr{F}}}(\mathfrak{x}) - \varpi^2_{t_{\mathscr{F}}}(\mathfrak{x}))}}$ expresses the complex hesitancy grade of $\mathfrak{x}$.

**Definition 2.13.** (Alothaim *et al.* 2022) As defined by the interval-valued complex spherical fuzzy set (CIVSFS) $C$ :

$$C = (\mathfrak{x}, [r_c^-(\mathfrak{x}), r_c^+(\mathfrak{x})], [s_c^-(\mathfrak{x}), s_c^+(\mathfrak{x})], [t_c^-(\mathfrak{x}), t_c^+(\mathfrak{x})] | \mathfrak{x} \in \mathfrak{X})$$

where $r_c^-(\mathfrak{x}) = \zeta_c^-(\mathfrak{x})e^{i2\pi\omega_{\zeta_c^-}(\mathfrak{x})}, r_c^+(\mathfrak{x}) = \zeta_c^+(\mathfrak{x})e^{i2\pi\omega_{\zeta_c^+}(\mathfrak{x})}, s_c^-(\mathfrak{x}) = \xi_c^-(\mathfrak{x})e^{i2\pi\omega_{\xi_c^-}(\mathfrak{x})}$, $s_c^+(\mathfrak{x}) = \xi_c^+(\mathfrak{x})e^{i2\pi\omega_{\xi_c^+}(\mathfrak{x})}$, and $t_c^-(\mathfrak{x}) = \varsigma_c^-(\mathfrak{x})e^{i2\pi\omega_{\varsigma_c^-}(\mathfrak{x})}, t_c^+(\mathfrak{x}) = \varsigma_c^+(\mathfrak{x})e^{i2\pi\omega_{\varsigma_c^+}(\mathfrak{x})}$ with conditions: $0 \leq (\zeta_c^+(\mathfrak{x}))^2 + (\xi_c^+(\mathfrak{x}))^2 + (\varsigma_c^+(\mathfrak{x}))^2 \leq 1$ and
$0 \leq (\omega_{\zeta_c^+}(\mathfrak{x}))^2 + (\omega_{\xi_c^+}(\mathfrak{x}))^2 + (\omega_{\varsigma_c^+}(\mathfrak{x}))^2 \leq 1$.
The IVCSF number is stated by:

$$c = ([\zeta_c^-, \zeta_c^+]e^{i2\pi[\omega_{\zeta_c^-}, \omega_{\zeta_c^+}]}, [\xi_c^-, \xi_c^+]e^{i2\pi[\omega_{\xi_c^-}, \omega_{\xi_c^+}]}, [\varsigma_c^-, \varsigma_c^+]e^{i2\pi[\omega_{\varsigma_c^-}, \omega_{\varsigma_c^+}]}).$$

**Example 2.14.** Make $\mathfrak{X}$ a set that is not empty.

$$C = (\mathfrak{x}, [r_c^-(\mathfrak{x}), r_c^+(\mathfrak{x})], [s_c^-(\mathfrak{x}), s_c^+(\mathfrak{x})], [t_c^-(\mathfrak{x}), t_c^+(\mathfrak{x})] | \mathfrak{x} \in \mathfrak{X})$$

where $r_c^-(\mathfrak{x}) = 0.2e^{i2\pi 0.1}, r_c^+(\mathfrak{x}) = 0.3e^{i2\pi 0.3}, s_c^-(\mathfrak{x}) = 0.3e^{i2\pi 0.1}, s_c^+(\mathfrak{x}) = 0.4e^{i2\pi 0.5}$
and $t_c^-(\mathfrak{x}) = 0.2e^{i2\pi 0.3}, t_c^+(\mathfrak{x}) = 0.5e^{i2\pi 0.6}$ with conditions: $0 \leq 0.3^2 + 0.4^2 + 0.5^2 \leq 1$ and $0 \leq 0.3^2 + 0.5^2 + 0.6^2 \leq 1$.
According to Definition 2.13, set $C$ is an IVCSFS.

**Definition 2.15.** (Mahmood *et al.* 2019) Make $\mathfrak{X}$ a set that is not empty. The definition of a T-spherical fuzzy set (TSFS) over $\mathfrak{X}$ is as follows:

$$\mathscr{F} = \left\{ \left( \mathfrak{x}, \mu_{\mathscr{F}}(\mathfrak{x}), \eta_{\mathscr{F}}(\mathfrak{x}), \nu_{\mathscr{F}}(\mathfrak{x}) \right) : 0 \leq \mu^q_{\mathscr{F}}(\mathfrak{x}) + \eta^q_{\mathscr{F}}(\mathfrak{x}) + \nu^q_{\mathscr{F}}(\mathfrak{x}) \leq 1, q \in Z^+, \mathfrak{x} \in \mathfrak{X} \right\}.$$

$\rho = (\mu_{\mathscr{F}}, \eta_{\mathscr{F}}, \nu_{\mathscr{F}})$ is called a T-spherical fuzzy number (TSFN).

**Definition 2.16.** Let $\mathfrak{X}$ be a nonempty set. An interval-valued T-spherical fuzzy set (IVT-SFS) is defined over $\mathfrak{X}$ as follows: $\mathscr{F} = \{\mathfrak{x}, (\acute{r}_c(\mathfrak{x}), \acute{s}_c(\mathfrak{x}), \acute{t}_c(\mathfrak{x})) | \mathfrak{x} \in X\}$ where $\acute{r}_c = [r_c^-, r_c^+]$ , $\acute{s}_c = [s_c^-, s_c^+]$ and $\acute{t}_c = [t_c^-, t_c^+]$ with $0 \leq (r_c^+)^q + (s_c^+)^q + (t_c^+)^q \leq 1, q \in Z^+$ The IVT-SF number is stated by: $f = ([r_f^-, r_f^+], [s_f^-, s_f^+], [t_f^-, t_f^+])$.

**Definition 2.17.** (Ali *et al.* 2020) Assume that $\mathfrak{X}$ is an alternate starting universe to the empty set. The following characteristics (CT-SFS) are said to be present in a complex T-spherical fuzzy (CT-SF) set:

$$\mathscr{F} = \{(\mathfrak{x}, \tilde{r}_{\mathscr{F}}(\mathfrak{x}), \tilde{s}_{\mathscr{F}}(\mathfrak{x}), \tilde{t}_{\mathscr{F}}(\mathfrak{x})) : \mathfrak{x} \in \mathfrak{X}\}.$$

Here $\tilde{r}_{\mathscr{F}}(\mathfrak{x}) = r_{\mathscr{F}}(\mathfrak{x}) e^{i2\pi\varpi_{r_{\mathscr{F}}(\mathfrak{x})}}$, $\tilde{s}_{\mathscr{F}}(\mathfrak{x}) = s_{\mathscr{F}}(\mathfrak{x}) e^{i2\pi\varpi_{s_{\mathscr{F}}(\mathfrak{x})}}$, and $\tilde{t}_{\mathscr{F}}(\mathfrak{x}) = t_{\mathscr{F}}(\mathfrak{x}) e^{i2\pi\varpi_{t_{\mathscr{F}}(\mathfrak{x})}}$ indicate the degrees of truth, absence, and falsehood so that $0 \leq r^q_{\mathscr{F}}(\mathfrak{x}) + s^q_{\mathscr{F}}(\mathfrak{x}) + t^q_{\mathscr{F}}(\mathfrak{x}) \leq 1$ and $0 \leq \varpi^q_{r_{\mathscr{F}}(\mathfrak{x})} + \varpi^q_{s_{\mathscr{F}}(\mathfrak{x})} + \varpi^q_{t_{\mathscr{F}}(\mathfrak{x})} \leq 1, q \in Z^+$. Furthermore,
$W_{\mathfrak{R}}(\mathfrak{x}) = \sqrt[q]{1 - r_{\mathscr{F}}(\mathfrak{x})^q - s_{\mathscr{F}}(\mathfrak{x})^q - r_{\mathscr{F}}(\mathfrak{x})^q} e^{\sqrt[q]{(1 - \varpi^q_{r_{\mathscr{F}}(\mathfrak{x})} - \varpi^q_{s_{\mathscr{F}}(\mathfrak{x})} - \varpi^q_{t_{\mathscr{F}}(\mathfrak{x})})}}$ demonstrates the complex hesitation grade of $\mathfrak{x}$.

## 3. COMPLEX INTERVAL-VALUED T-SPHERICAL FUZZY SETS

The concept of interval-valued complex T-spherical fuzzy sets was defined by Alothaim *et al.* (2022). In this section, we define the score and accuracy functions of CIVT-SF number, set theoretical and arithmetical operations of CIVT-SFSs. We also obtain some results related to arithmetical operations of CIVT-SFSs

**Definition 3.1.** (Alothaim *et al.* 2022) An complex interval-valued T-spherical fuzzy set (CIVT-SFS) $C$ is defined as:

$$C = (\mathfrak{x}, [r_c^-(\mathfrak{x}), r_c^+(\mathfrak{x})], [s_c^-(\mathfrak{x}), s_c^+(\mathfrak{x})], [t_c^-(\mathfrak{x}), t_c^+(\mathfrak{x})] | \mathfrak{x} \in X)$$

where $r_c^-(\mathfrak{x}) = \alpha_c^-(\mathfrak{x})e^{i2\pi\omega_{\alpha_c^-}(\mathfrak{x})}, r_c^+(\mathfrak{x}) = \alpha_c^+(\mathfrak{x})e^{i2\pi\omega_{\alpha_c^+}(\mathfrak{x})}, s_c^-(\mathfrak{x}) = \beta_c^-(\mathfrak{x})e^{i2\pi\omega_{\beta_c^-}(\mathfrak{x})}$, $s_c^+(\mathfrak{x}) = \beta_c^+(\mathfrak{x})e^{i2\pi\omega_{\beta_c^+}(\mathfrak{x})}$ and $t_c^-(\mathfrak{x}) = \gamma_c^-(\mathfrak{x})e^{i2\pi\omega_{\gamma_c^-}(\mathfrak{x})}, t_c^+(\mathfrak{x}) = \gamma_c^+(\mathfrak{x})e^{i2\pi\omega_{\gamma_c^+}(\mathfrak{x})}$ with conditions: $0 \leq (\alpha_c^+(\mathfrak{x}))^q + (\beta_c^+(\mathfrak{x}))^q + (\gamma_c^+(\mathfrak{x}))^q \leq 1$ and
$0 \leq (\omega_{\alpha_c^+}(\mathfrak{x}))^q + (\omega_{\beta_c^+}(\mathfrak{x}))^q + (\omega_{\gamma_c^+}(\mathfrak{x}))^q \leq 1, q \in Z^+$.

The CIVT-SF number is stated by:

$$c = ([\alpha_c^-, \alpha_c^+]e^{i2\pi[\omega_{\alpha_c^-}, \omega_{\alpha_c^+}]}, [\beta_c^-, \beta_c^+]e^{i2\pi[\omega_{\beta_c^-}, \omega_{\beta_c^+}]}, [\gamma_c^-, \gamma_c^+]e^{i2\pi[\omega_{\gamma_c^-}, \omega_{\gamma_c^+}]}).$$

**Example 3.2.** Let $\mathfrak{X}$ be a non-empty set.
$C = (\mathfrak{x}, [r_c^-(\mathfrak{x}), r_c^+(\mathfrak{x})], [s_c^-(\mathfrak{x}), s_c^+(\mathfrak{x})], [t_c^-(\mathfrak{x}), t_c^+(\mathfrak{x})] | \mathfrak{x} \in X)$ where $r_c^-(\mathfrak{x}) = 0.3e^{i2\pi 0.4}$ $, r_c^+(\mathfrak{x}) = 0.4e^{i2\pi 0.5}, s_c^-(\mathfrak{x}) = 0.4e^{i2\pi 0.5}, s_c^+(\mathfrak{x}) = 0.5e^{i2\pi 0.7}$, and $t_c^-(\mathfrak{x}) = 0.7e^{i2\pi 0.7}$, $t_c^+(\mathfrak{x}) = 0.8e^{i2\pi 0.9}$ with conditions: $0 \leq 0.4^4 + 0.5^4 + 0.8^4 \leq 1$ and $0 \leq 0.5^4 + 0.7^4 + 0.9^4 \leq 1$. Note that, here we take $q = 4$.
According to Definition 3.1, set $C$ is a CIVT-SFS.

**Corollary 3.3.** Every CIVPFS can be considered as CIVT-SFS but not conversely.

**Corollary 3.4.** Every CIVSFS can be considered as CIVT-SFS but not conversely.

**Definition 3.5.** Let

$A_1 = \{[\alpha^-_{A_1}, \alpha^+_{A_1}]e^{i2\pi[w^-_{\alpha A_1}, w^+_{\alpha A_1}]}, [\beta^-_{A_1}, \beta^+_{A_1}]e^{i2\pi[w^-_{\beta A_1}, w^+_{\beta A 1}]}, [\gamma^-_{A_1}, \gamma^+_{A_1}]e^{i2\pi[w^-_{\gamma A_1}, w^+_{\gamma A_1}]}\}$ and

$A_2 = \{[\alpha^-_{A_2}, \alpha^+_{A_2}]e^{i2\pi[w^-_{\alpha A_2}, w^+_{\alpha A_2}]}, [\beta^-_{A_2}, \beta^+_{A_2}]e^{i2\pi[w^-_{\beta A_2}, w^+_{\beta A_2}]}, [\gamma^-_{A_2}, \gamma^+_{A_2}]e^{i2\pi[w^-_{\gamma A_2}, w^+_{\gamma A_2}]}\}$

be two CIVT-SFNs and $q \in Z^+$. Then,

1) $A_1 \leq A_2$ if and only if $\alpha^-_{A_1} \leq \alpha^-_{A_2}, \alpha^+_{A_1} \leq \alpha^+_{A_2}$, $\beta^-_{A_1} \leq \beta^-_{A_2}, \beta^+_{A_1} \leq \beta^+_{A_2}$ and $\gamma^-_{A_1} \geq \gamma^-_{A_2}, \gamma^+_{A_1} \geq \gamma^+_{A_2}$, for amplitude terms and $w^-_{\alpha A_1} \leq w^-_{\alpha A_2}, w^+_{\alpha A_1} \leq w^+_{\alpha A_2}$, $w^-_{\beta A_1} \leq w^-_{\beta A_2}, w^+_{\beta A_1} \leq w^+_{\beta A_2}$, and $w^-_{\gamma A_1} \geq w^-_{\gamma A_2}, w^+_{\gamma A_1} \geq w^+_{\gamma A_2}$ for phase terms.

2) $A^c_1 = \{[\gamma^-_{A_1}, \gamma^+_{A_1}]e^{i2\pi[w^-_{\gamma A_1}, w^+_{\gamma A_1}]}, [\beta^-_{A_1}, \beta^+_{A_1}]e^{i2\pi[w^-_{\beta A_1}, w^+_{\beta A_1}]}, [\alpha^-_{A_1}, \alpha^+_{A_1}]e^{i2\pi[w^-_{\alpha A_1}, w^+_{\alpha A_1}]}\})$

3)

$$\begin{aligned} A_1 \vee A_2 \quad = \quad & \Big\{[max(\alpha^-_{A_1}, \alpha^-_{A_2}), max(\alpha^+_{A_1}, \alpha^+_{A_2})]e^{i2\pi[max(w^-_{\alpha A_1}, w^-_{\alpha A_2}), max(w^+_{\alpha A_1}, w^+_{\alpha A_2})]}, \\ & [min(\beta^-_{A_1}, \beta^-_{A_2}), min(\beta^+_{A_1}, \beta^+_{A_2})]e^{i2\pi[min(w^-_{\beta A_1}, w^-_{\beta A_2}), min(w^+_{\beta A_1}, w^+_{\beta A_2})]}, \\ & [min(\gamma^-_{A_1}, \gamma^-_{A_2}), min(\gamma^+_{A_1}, \gamma^+_{A_1})]e^{i2\pi[min(w^-_{\gamma A_1}, w^-_{\gamma A_2}), min(w^+_{\gamma A_1}, w^+_{\gamma A_2})]}\Big\} \end{aligned}$$

4)

$$\begin{aligned} A_1 \wedge A_2 \quad = \quad & \Big\{[min(\alpha^-_{A_1}, \alpha^-_{A_2}), min(\alpha^+_{A_1}, \alpha^+_{A_2})]e^{i2\pi[min(w^-_{\alpha A_1}, w^-_{\alpha A_2}), min(w^+_{\alpha A_1}, w^+_{\alpha A_2})]}, \\ & [min(\beta^-_{A_1}, \beta^-_{A_2}), min(\beta^+_{A_1}, \beta^+_{A_2})]e^{i2\pi[min(w^-_{\beta A_1}, w^-_{\beta B}), min(w^+_{\beta A_1}, w^+_{\beta A_2})]}, \\ & [max(\gamma^-_{A_1}, \gamma^-_{A_2}), max(\gamma^+_{A_1}, \gamma^+_{A_2})]e^{i2\pi[max(w^-_{\gamma A_1}, w^-_{\gamma A_2}), max(w^+_{\gamma A_1}, w^+_{\gamma A_2})]}\Big\} \end{aligned}$$

**Example 3.6.** Let $A_1 = \{[0.3,0.4]e^{i2\pi[0.4,0.5]},[0.4,0.5]e^{i2\pi[0.5,0.7]},[0.7,0.8]e^{i2\pi[0.7,0.9]}\}$ and $A_2 = \{[0.4,0.6]e^{i2\pi[0.5,0.6]},[0.6,0.7]e^{i2\pi[0.7,0.9]},[0.4,0.5]e^{i2\pi[0.2,0.5]}\}$ be two CIVT-SFNs for $q=4$. Then,

1) $A_1 \leq A_2$ if and only if $\alpha^-_{A_1} \leq \alpha^-_{A_2}$ for $0.3 \leq 0.4$, $\alpha^+_{A_1} \leq \alpha^+_{A_2}$ for $0.4 \leq 0.6$,
$\beta^-_{A_1} \leq \beta^-_{A_2}$ for $0.4 \leq 0.6$, $\beta^+_{A_1} \leq \beta^+_{A_2}$ for $0.5 \leq 0.7$ and
$\gamma^-_{A_1} \geq \gamma^-_{A_2}$ for $0.7 \geq 0.4$, $\gamma^+_{A_1} \geq \gamma^+_{A_2}$ for $0.8 \geq 0.5$,
for amplitude terms and $w^-_{\alpha A_1} \leq w^-_{\alpha A_2}$ for $0.4 \leq 0.5$, $w^+_{\alpha A_1} \leq w^+_{\alpha A_2}$ for $0.5 \leq 0.6$,
$w^-_{\beta A_1} \leq w^-_{\beta A_2}$ for $0.5 \leq 0.7$ ,$w^+_{\beta A_1} \leq w^+_{\beta A_2}$ for $0.7 \leq 0.9$ and
$w^-_{\gamma A_1} \geq w^-_{\gamma A_2}$ for $0.7 \geq 0.2$, $w^+_{\gamma A_1} \geq w^+_{\gamma A_2}$ for $0.9 \geq 0.5$ for phase terms.

2)

$$\begin{aligned}
A_1^c &= \{[\gamma^-_{A_1},\gamma^+_{A_1}]e^{i2\pi[w^-_{\gamma A_1},w^+_{\gamma A_1}]},[\beta^-_{A_1},\beta^+_{A_1}]e^{i2\pi[w^-_{\beta A_1},w^+_{\beta A_1}]},[\alpha^-_{A_1},\alpha^+_{A_1}]e^{i2\pi[w^-_{\alpha A_1},w^+_{\alpha A_1}]}\} \\
&= \{[0.7,0.8]e^{i2\pi[0.7,0.9]},[0.4,0.5]e^{i2\pi[0.5,0.7]},[0.3,0.4]e^{i2\pi[0.4,0.5]}\}
\end{aligned}$$

3)

$$\begin{aligned}
A_1 \vee A_2 &= \Big\{[max(\alpha^-_{A_1},\alpha^-_{A_2}),max(\alpha^+_{A_1},\alpha^+_{A_2})]e^{i2\pi[max(w^-_{\alpha A_1},w^-_{\alpha A_2}),max(w^+_{\alpha A_1},w^+_{\alpha A_2})]}, \\
&\quad [min(\beta^-_{A_1},\beta^-_{A_2}),min(\beta^+_{A_1},\beta^+_{A_2})]e^{i2\pi[min(w^-_{\beta A_1},w^-_{\beta A_2}),min(w^+_{\beta A_1},w^+_{\beta A_2})]}, \\
&\quad [min(\gamma^-_{A_1},\gamma^-_{A_2}),min(\gamma^+_{A_1},\gamma^+_{A_1})]e^{i2\pi[min(w^-_{\gamma A_1},w^-_{\gamma A_2}),min(w^+_{\gamma A_1},w^+_{\gamma A_2})]}\Big\} \\
&= \Big\{[max(0.3,0.4),max(0.4,0.6)]e^{i2\pi[max(0.4,0.5),max(0.5,0.6)]}, \\
&\quad [min(0.4,0.6),min(0.5,0.7)]e^{i2\pi[min(0.5,0.7),min(0.7,0.9)]}, \\
&\quad [min(0.7,0.4),min(0.8,0.5)]e^{i2\pi[min(0.7,0.2),min(0.9,0.5)]}\Big\} \\
&= \Big\{[0.4,0.6]e^{i2\pi[0.5,0.6]},[0.4,0.5]e^{i2\pi[0.5,0.7]},[0.4,0.5)]e^{i2\pi[0.2,0.5)]}\Big\}
\end{aligned}$$

4)

$$
\begin{aligned}
A_1 \wedge A_2 &= \Big\{[min(\alpha^-_{A_1},\alpha^-_{A_2}),min(\alpha^+_{A_1},\alpha^+_{A_2})]e^{i2\pi[min(w^-_{\alpha A_1},w^-_{\alpha A_2}),min(w^+_{\alpha A_1},w^+_{\alpha A_2})]},\\
&\quad [min(\beta^-_{A_1},\beta^-_{A_2}),min(\beta^+_{A_1},\beta^+_{A_2})]e^{i2\pi[min(w^-_{\beta A_1},w^-_{\beta B}),min(w^+_{\beta A_1},w^+_{\beta A_2})]},\\
&\quad [max(\gamma^-_{A_1},\gamma^-_{A_2}),max(\gamma^+_{A_1},\gamma^+_{A_2})]e^{i2\pi[max(w^-_{\gamma A_1},w^-_{\gamma A_2}),max(w^+_{\gamma A_1},w^+_{\gamma A_2})]}\Big\}\\
&= \Big\{[min(0.3,0.4),min(0.4,0.6)]e^{i2\pi[min(0.4,0.5),min(0.5,0.6)]},\\
&\quad [min(0.4,0.6),min(0.5,0.7)]e^{i2\pi[min(0.5,0.7),min(0.7,0.9)]},\\
&\quad [max(0.7,0.4),max(0.8,0.5)]e^{i2\pi[max(0.7,0.2),max(0.9,0.5)]}\Big\}\\
&= \Big\{[0.3,0.4]e^{i2\pi[0.4,0.5]},[0.4,0.5]e^{i2\pi[0.5,0.7]},[0.7,0.8)]e^{i2\pi[0.7,0.9)]}\Big\}
\end{aligned}
$$

**Definition 3.7.** Let us consider the following CIVT-SF number $A = \{[\alpha^-_A,\alpha^+_A]e^{i.2\pi[w^-_{\alpha A},w^+_{\alpha A}]},[\beta^-_A,\beta^+_A]e^{i.2\pi[w^-_{\beta A},w^+_{\beta A}]},[\gamma^-_A,\gamma^+_A]e^{i.2\pi[w^-_{\gamma A},w^+_{\gamma A}]}\}$ Score function denoted by $S(A)$ and accuracy function denoted by $H(A)$ of CIVT-SF number $A$ are defined as follows:

$$
S(A) = \frac{((\alpha^{-q}_A+\alpha^{+q}_A)+(\beta^{-q}_A+\beta^{+q}_A)-(\gamma^{-q}_A+\gamma^{+q}_A))+((w^{-q}_{\alpha A}+w^{+q}_{\alpha A})+(w^{-q}_{\beta A}+w^{+q}_{\beta A})-(w^{-q}_{\gamma A}+w^{+q}_{\gamma A}))}{4}
$$

$$
H(A) = \frac{((\alpha^{-q}_A+\alpha^{+q}_A)+(\beta^{-q}_A+\beta^{+q}_A)+(\gamma^{-q}_A+\gamma^{+q}_A))+((w^{-q}_{\alpha A}+w^{+q}_{\alpha A})+(w^{-q}_{\beta A}+w^{+q}_{\beta A})+(w^{-q}_{\gamma A}+w^{+q}_{\gamma A}))}{4}
$$

respectively. Here $S(A),H(A) \in [-1,1]$.

**Definition 3.8.** Two CIVT-SFNs A and B have an order relation that takes the form

1. If $S(A) > S(B)$ then $A > B$.
2. If $S(A) = S(B)$, then
   (a) If $H(A) > H(B)$, then $A > B$
   (b) If $H(A) < H(B)$, then $A < B$
   (c) If $H(A) = H(B)$, then $A = B$.

Using the above definitions, to obtain the following results.

**Example 3.9.** Let $A=\{[0.3,0.4]e^{i2\pi[0.4,0.5]},[0.4,0.5]e^{i2\pi[0.5,0.7]},[0.7,0.8]e^{i2\pi[0.7,0.9]}\}$ and $B=\{[0.2,0.6]e^{i2\pi[0.3,0.5]},[0.3,0.7]e^{i2\pi[0.4,0.7]},[0.6,0.8]e^{i2\pi[0.7,0.8]}\}$ be two CIVT-SFNs when $q=4$. Then,

$$
\begin{aligned}
S(A)=&\frac{1}{4}(((0.3^4+0.4^4)+(0.4^4+0.5^4)-(0.7^4+0.8^4))+(0.4^4+0.5^4)+(0.5^4+0.7^4)\\
&-(0.7^4+0.9^4))\\
=&\frac{1}{4}((0.0337+0.0881-0.6497)+(0.0881+0.3026-0.8962))\\
=&\frac{1}{4}(-0.5279-0.5055)\\
=&\frac{1}{4}(-1.0334)\\
=&-0.2584
\end{aligned}
$$

$$
\begin{aligned}
S(B)=&\frac{1}{4}(((0.2^4+0.6^4)+(0.3^4+0.7^4)-(0.6^4+0.8^4))+(0.3^4+0.5^4)+(0.4^4+0.7^4)\\
&-(0.7^4+0.8^4))\\
=&\frac{1}{4}((0.1312+0.2482-0.5392)+(0.0706+0.2657-0.6497))\\
=&\frac{1}{4}(-0.1598-0.3134)\\
=&\frac{1}{4}(-0.4732)\\
=&-0.1183
\end{aligned}
$$

Since $S(B)>S(A)$ then, $B>A$.

**Theorem 3.10. (Monotonicity of score function)**
Let $A_1=\{[\alpha^-_{A_1},\alpha^+_{A_1}]e^{i2\pi[w^-_{\alpha A_1},w^+_{\alpha A_1}]},[\beta^-_{A_1},\beta^+_{A_1}]e^{i2\pi[w^-_{\beta A_1},w^+_{\beta A^1}]},[\gamma^-_{A_1},\gamma^+_{A_1}]e^{i2\pi[w^-_{\gamma A_1},w^+_{\gamma A_1}]}\}$ and $A_2=\{[\alpha^-_{A_2},\alpha^+_{A_2}]e^{i2\pi[w^-_{\alpha A_2},w^+_{\alpha A_2}]},[\beta^-_{A_2},\beta^+_{A_2}]e^{i2\pi[w^-_{\beta A_2},w^+_{\beta A_2}]},[\gamma^-_{A_2},\gamma^+_{A_2}]e^{i2\pi[w^-_{\gamma A_2},w^+_{\gamma A_2}]}\}$ be two CIVT-SFNs for $q\in Z^+$. If $A_1\leq A_2$, then $S(A_1)\leq S(A_2)$.

*Proof.* Assume that $A_1\leq A_2$. Then,

$$
\begin{aligned}
\alpha^{-q}_{A_1}+\alpha^{+q}_{A_1} &\leq \alpha^{-q}_{A_2}+\alpha^{+q}_{A_2}\\
(\alpha^{-q}_{A_1}+\alpha^{+q}_{A_1})+(\beta^{-q}_{A_1}+\beta^{+q}_{A_1}) &\leq (\alpha^{-q}_{A_2}+\alpha^{+q}_{A_2})+(\beta^{-q}_{A_2}+\beta^{+q}_{A_2})\\
(\alpha^{-q}_{A_1}+\alpha^{+q}_{A_1})+(\beta^{-q}_{A_1}+\beta^{+q}_{A_1})-(\gamma^{-q}_{A_1}+\gamma^{+q}_{A_1}) &\leq (\alpha^{-q}_{A_2}+\alpha^{+q}_{A_2})+(\beta^{-q}_{A_2}+\beta^{+q}_{A_2})-(\gamma^{-q}_{A_2}+\gamma^{+q}_{A_2})
\end{aligned}
$$

and

$w_{\alpha A_1}^{-q}+w_{\alpha A_1}^{+q}\leq w_{\alpha A_2}^{-q}+w_{\alpha A_2}^{+q}$ and $(w_{\alpha A_1}^{-q}+w_{\alpha A_1}^{+q})+(w_{\beta A_1}^{-q}+w_{\beta A_1}^{+q})\leq (w_{\alpha A_2}^{-q}+w_{\alpha A_2}^{+q})+(w_{\beta A_2}^{-q}+w_{\beta A_2}^{+q})$, $(w_{\alpha A_1}^{-q}+w_{\alpha A_1}^{+q})+(w_{\beta A_1}^{-q}+w_{\beta A_1}^{+q})-(w_{\gamma A_1}^{-q}+w_{\gamma A_1}^{+q})\leq (w_{\alpha A_2}^{-q}+w_{\alpha A_2}^{+q})+(w_{\beta A_2}^{-q}+w_{\beta A_2}^{+q})-(w_{\gamma A_2}^{-q}+w_{\gamma A_2}^{+q})$
Then,

$$\begin{aligned}
S(A_1) &= \frac{\begin{pmatrix} (\alpha_{A_1}^{-q}+\alpha_{A_1}^{+q})+(\beta_{A_1}^{-q}+\beta_{A_1}^{+q})-(\gamma_{A_1}^{-q}+\gamma_{A_1}^{+q}) \\ +(w_{\alpha A_1}^{-q}+w_{\alpha A_1}^{+q})+(w_{\beta A_1}^{-q}+w_{\beta A_1}^{+q})-(w_{\gamma A_1}^{-q}+w_{\gamma A_1}^{+q}) \end{pmatrix}}{4} \\
&\leq \frac{\begin{pmatrix} (\alpha_{A_2}^{-q}+\alpha_{A_2}^{+q})+(\beta_{A_2}^{-q}+\beta_{A_2}^{+q})-(\gamma_{A_2}^{-q}+\gamma_{A_2}^{+q}) \\ +(w_{\alpha A_2}^{-q}+w_{\alpha A_2}^{+q})+(w_{\beta A_2}^{-q}+w_{\beta A_2}^{+q})-(w_{\gamma A_2}^{-q}+w_{\gamma A_2}^{+q}) \end{pmatrix}}{4} \\
&= S(A_2).
\end{aligned}$$

□

**Theorem 3.11.** (**Symmetry of Score Function**) Assume $A_j=\{[\alpha_j^-,\alpha_j^+]e^{i2\pi[w_{\alpha j}^-,w_{\alpha j}^+]},[\beta_j^-,\beta_j^+]e^{i2\pi[w_{\beta j}^-,w_{\beta j}^+]},[\gamma_j^-,\gamma_j^+]e^{i2\pi[w_{\gamma j}^-,w_{\gamma j}^+]}\}$, $j=1,2$ be two CIVT-SFNs and $q\in Z^+$. Then, $A_j^c=\{[\gamma_j^-,\gamma_j^+]e^{i2\pi[w_{\gamma j}^-,w_{\gamma j}^+]},[\beta_j^-,\beta_j^+]e^{i2\pi[w_{\beta j}^-,w_{\beta j}^+]},[\alpha_j^-,\alpha_j^+]\ e^{i2\pi[w_{\alpha j}^-,w_{\alpha j}^+]}\}$, $j=1,2$ be the complement functions of each. The following is our conclusion: $S(A_1)\leq S(A_2)$ if and only if $S(A_1^c)\leq S(A_2^c)$.

*Proof.* Assume that $S(A_1)\leq S(A_2)$. Then,

$$\frac{\begin{pmatrix} (\alpha_{A_1}^{-q}+\alpha_{A_1}^{+q})+(\beta_{A_1}^{-q}+\beta_{A_1}^{+q})-(\gamma_{A_1}^{-q}+\gamma_{A_1}^{+q}) \\ +(w_{\alpha A_1}^{-q}+w_{\alpha A_1}^{+q})+(w_{\beta A_1}^{-q}+w_{\beta A_1}^{+q})-(w_{\gamma A_1}^{-q}+w_{\gamma A_1}^{+q}) \end{pmatrix}}{4} \leq \frac{\begin{pmatrix} (\alpha_{A_2}^{-q}+\alpha_{A_2}^{+q})+(\beta_{A_2}^{-q}+\beta_{A_2}^{+q})-(\gamma_{A_2}^{-q}+\gamma_{A_2}^{+q}) \\ +(w_{\alpha A_2}^{-q}+w_{\alpha A_2}^{+q})+(w_{\beta A_2}^{-q}+w_{\beta A_2}^{+q})-(w_{\gamma A_2}^{-q}+w_{\gamma A_2}^{+q}) \end{pmatrix}}{4}$$

if and only if

$(\alpha_{A_1}^{-q} + \alpha_{A_1}^{+q}) + (\beta_{A_1}^{-q} + \beta_{A_1}^{+q}) - (\gamma_{A_1}^{-q} + \gamma_{A_1}^{+q}) \leq (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) + (\beta_{A_2}^{-q} + \beta_{A_2}^{+q}) - (\gamma_{A_2}^{-q} + \gamma_{A_2}^{+q})$
$(\alpha_{A_1}^{-q} + \alpha_{A_1}^{+q}) + (\beta_{A_1}^{-q} + \beta_{A_1}^{+q}) - (\gamma_{A_1}^{-q} + \gamma_{A_1}^{+q}) - 2(\alpha_{A_1}^{-q} + \alpha_{A_1}^{+q}) + 2(\gamma_{A_1}^{-q} + \gamma_{A_1}^{+q}) \leq (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) + (\beta_{A_2}^{-q} + \beta_{A_2}^{+q}) - (\gamma_{A_2}^{-q} + \gamma_{A_2}^{+q}) - 2(\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) + 2(\gamma_{A_2}^{-q} + \gamma_{A_2}^{+q})(\gamma_{A_1}^{-q} + \gamma_{A_1}^{+q}) + (\beta_{A_1}^{-q} + \beta_{A_1}^{+q}) - (\gamma_{A_1}^{-q} + \gamma_{A_1}^{+q}) \leq (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) + (\beta_{A_2}^{-q} + \beta_{A_2}^{+q}) - (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q})$

and

$(w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) + (w_{\beta A_1}^{-q} + w_{\beta A_1}^{+q}) - (w_{\gamma A_1}^{-q} + w_{\gamma A_1}^{+q}) \leq (w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+q}) + (w_{\beta A_2}^{-q} + w_{\beta A_2}^{+q}) - (w_{\gamma A_2}^{-q} + w_{\gamma A_2}^{+q})(w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) + (w_{\beta A_1}^{-q} + w_{\beta A_1}^{+q}) - (w_{\gamma A_1}^{-q} + w_{\gamma A_1}^{+q}) - 2(w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) + 2(w_{\gamma A_1}^{-q} + w_{\gamma A_1}^{+q}) \leq (w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+q}) + (w_{\beta A_2}^{-q} + w_{\beta A_2}^{+q}) - (w_{\gamma A_2}^{-q} + w_{\gamma A_2}^{+q}) - 2(w_{\alpha A_2}^{-} + w_{\alpha A_2}^{+}) + 2(w_{\gamma A_2}^{-q} + w_{\gamma A_2}^{+q})(w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) + (w_{\beta A_1}^{-q} + w_{\beta A_1}^{+q}) - (w_{\gamma A_1}^{-q} + w_{\gamma A_1}^{+q}) \leq (w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+}) + (w_{\beta A_2}^{-} + w_{\beta A_2}^{+q}) - (w_{\gamma A_2}^{-q} + w_{\gamma A_2}^{+q})$

if and only if

$$\frac{\begin{pmatrix} (\gamma_{A_1}^{-q} + \gamma_{A_1}^{+q}) + (\beta_{A_1}^{-q} + \beta_{A_1}^{+q}) - (\alpha_{A_1}^{-q} + \zeta_{A_1}^{+q}) \\ +(w_{\gamma A_1}^{-q} + w_{\gamma A_1}^{+q}) + (w_{\beta A_1}^{-q} + w_{\beta A_1}^{+q}) - (w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) \end{pmatrix}}{4} \leq \frac{\begin{pmatrix} (\gamma_{A_2}^{-q} + \gamma_{A_2}^{+q}) + (\beta_{A_2}^{-q} + \beta_{A_2}^{+q}) - (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) \\ +(w_{\gamma A_2}^{-q} + w_{\gamma A_2}^{+q}) + (w_{\beta A_2}^{-q} + w_{\beta A_2}^{+q}) - (w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+q}) \end{pmatrix}}{4},$$

if and only if $\quad S(A_1^c) \leq S(A_2^c)$

□

**Theorem 3.12.** (**Monotonicity of Accuracy Function**). Let $A_j = \{[\alpha_{A_j}^-, \alpha_{A_j}^+]e^{i2\pi[w_{\alpha A_j}^-, w_{\alpha A_j}^+]}, [\beta_{A_j}^-, \beta_{A_j}^+]e^{i2\pi[w_{\beta A_j}^-, w_{\beta A_j}^+]}, [\gamma_{A_j}^-, \gamma_{A_j}^+]e^{i2\pi[w_{\gamma A_j}^-, w_{\gamma A_j}^+]}\}$, $j = 1,2$ be two CIVT-SFNs and $q \in Z^+$. Then, the accuracy-function

$$H(A) = \frac{1}{4}|((\alpha_A^{-q} + \alpha_A^{+q}) + (\beta_A^{-q} + \beta_A^{+q}) + (\gamma_A^{-q} + \gamma_A^{+q})) + ((w_{\alpha A}^{-q} + w_{\alpha A}^{+q}) + (w_{\beta A}^{-q} + w_{\beta A}^{+q}) + (w_{\gamma A}^{-q} + w_{\gamma A}^{+q}))|$$

is a monotonic increasing function.

*Proof.* Assume that $A_1 \leq A_2$. Then,

$$\begin{aligned}
\alpha_{A_1}^{-q} + \alpha_{A_1}^{+q} &\leq \alpha_{A_2}^{-q} + \alpha_{A_2}^{+q} \\
(\alpha_{A_1}^{-q} + \alpha_{A_1}^{+q}) + (\beta_{A_1}^{-q} + \beta_{A_1}^{+q}) &\leq (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) + (\beta_{A_2}^{-q} + \beta_{A_2}^{+q}) \\
(\alpha_{A_1}^{-q} + \alpha_{A_1}^{+q}) + (\beta_{A_1}^{-q} + \beta_{A_1}^{+q}) + (\gamma_{A_1}^{-q} + \gamma_{A_1}^{+q}) &\leq (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) + (\beta_{A_2}^{-q} + \beta_{A_2}^{+q}) + (\gamma_{A_2}^{-q} + \gamma_{A_2}^{+q})
\end{aligned}$$

and

$w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q} \leq w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+q}$, $(w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) + (w_{\beta A_1}^{-q} + w_{\beta A_1}^{+q}) \leq (w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+q}) + (w_{\beta A_2}^{-q} + w_{\beta A_2}^{+q})$, $(w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) + (w_{\beta A_1}^{-q} + w_{\beta A_1}^{+q}) + (w_{\gamma A_1}^{-q} + w_{\gamma A_1}^{+q}) \leq (w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+q}) + (w_{\beta A_2}^{-q} + w_{\xi A_2}^{+q}) + (w_{\gamma A_2}^{-q} + w_{\gamma A_2}^{+q})$.

Then,

$$\begin{aligned}
H(A_1) &= \frac{\begin{pmatrix} (\alpha_{A_1}^{-q} + \alpha_{A_1}^{+q}) + (\beta_{A_1}^{-q} + \beta_{A_1}^{+q}) + (\gamma_{A_1}^{-q} + \varsigma_{A_1}^{+q}) \\ +(w_{\alpha A_1}^{-q} + w_{\alpha A_1}^{+q}) + (w_{\beta A_1}^{-q} + w_{\beta A_1}^{+q}) + (w_{\gamma A_1}^{-q} + w_{\gamma A_1}^{+q}) \end{pmatrix}}{4} \\
&\leq \frac{\begin{pmatrix} (\alpha_{A_2}^{-q} + \alpha_{A_2}^{+q}) + (\beta_{A_2}^{-q} + \beta_{A_2}^{+q}) + (\gamma_{A_2}^{-q} + \gamma_{A_2}^{+q}) \\ +(w_{\alpha A_2}^{-q} + w_{\alpha A_2}^{+q}) + (w_{\beta A_2}^{-q} + w_{\beta A_2}^{+q}) + (w_{\gamma A_2}^{-q} + w_{\gamma A_2}^{+q}) \end{pmatrix}}{4} \\
&= H(A_2).
\end{aligned}$$

□

**Theorem 3.13.** (**Symmetry of Accuracy Function**) Let $A_j = \{[\alpha_{A_j}^-, \alpha_{A_j}^+]e^{i2\pi[w_{\alpha A_j}^-, w_{\alpha A_j}^+]}, [\beta_{A_j}^-, \beta_{A_j}^+]e^{i2\pi[w_{\beta A_j}^-, w_{\beta A_j}^+]}, [\gamma_{A_j}^-, \gamma_{A_j}^+]e^{i2\pi[w_{\gamma A_j}^-, w_{\gamma A_j}^+]}\}$, $j = 1,2$ be two CIVT-SFNs and $q \in Z^+$. Then, $A_j^c = \{[\gamma_{A_j}^-, \gamma_{A_j}^+]e^{i2\pi[w_{\gamma A_j}^-, w_{\gamma A_j}^+]}, [\beta_{A_j}^-, \beta_{A_j}^+]e^{i2\pi[w_{\beta A_j}^-, w_{\xi A_j}^+]}, [\alpha_{A_j}^-, \alpha_{A_j}^+]e^{i2\pi[w_{\alpha A_j}^-, w_{\alpha A_j}^+]}\}$, $j = 1,2$ be the complement functions of each. We have the following conclusion: $H(A_1) \leq H(A_2)$ if and only if $H(A_1^c) \leq H(A_2^c)$.

*Proof.* Assume that $H(A_1) \leq H(A_2)$. Then,

$$\frac{\left(\begin{array}{c}(\alpha_{A_1}^{-q}+\alpha_{A_1}^{+q})+(\beta_{A_1}^{-q}+\beta_{A_1}^{+q})+(\gamma_{A_1}^{-q}+\gamma_{A_1}^{+q})\\+(w_{\alpha A_1}^{-q}+w_{\beta A_1}^{+q})+(w_{\beta A_1}^{-q}+w_{\beta A_1}^{+q})+(w_{\gamma A_1}^{-q}+w_{\gamma A_1}^{+q})\end{array}\right)}{4}$$

$$\leq\frac{\left(\begin{array}{c}(\alpha_{A_2}^{-q}+\alpha_{A_2}^{+q})+(\beta_{A_2}^{-q}+\beta_{A_2}^{+q})+(\gamma_{A_2}^{-q}+\gamma_{A_2}^{+q})\\+(w_{\alpha A_2}^{-q}+w_{\alpha A_2}^{+q})+(w_{\beta A_2}^{-q}+w_{\beta A_2}^{+q})+(w_{\gamma A_2}^{-q}+w_{\gamma A_2}^{+q})\end{array}\right)}{4}$$

if and only if

$$=\frac{\left(\begin{array}{c}(\gamma_{A_1}^{-q}+\gamma_{A_1}^{+q})+(\beta_{A_1}^{-q}+\beta_{A_1}^{+q})+(\alpha_{A_1}^{-q}+\zeta_{A_1}^{+q})\\+(w_{\gamma A_1}^{-q}+w_{\gamma A_1}^{+q})+(w_{\beta A_1}^{-q}+w_{\beta A_1}^{+q})+(w_{\alpha A_1}^{-q}+w_{\alpha A_1}^{+q})\end{array}\right)}{4}$$

$$\leq\frac{\left(\begin{array}{c}(\gamma_{A_2}^{-q}+\gamma_{A_2}^{+q})+(\beta_{A_2}^{-q}+\beta_{A_2}^{+q})+(\alpha_{A_2}^{-q}+\alpha_{A_2}^{+q})\\+(w_{\gamma A_2}^{-q}+w_{\gamma A_2}^{+q})+(w_{\beta A_2}^{-q}+w_{\beta A_2}^{+q})+(w_{\alpha A_2}^{-q}+w_{\alpha A_2}^{+q})\end{array}\right)}{4}$$

if and only if $\quad H(A_1^c) \leq H(A_2^c)$,

$\square$

### 3.1 Operational Laws of CIVT-SFSs

The operating rules for CIVT-SFNs that we suggest the following based on Archimedean t-norm procedures in this paragraph.

**Definition 3.14.** Let
$A_j = \{[\alpha_j^-, \alpha_j^+]e^{i.2\pi[w_{\alpha_j}^-, w_{\alpha_j}^+]}, [\beta_j^-, \beta_j^+]e^{i.2\pi[w_{\beta_j}^-, w_{\beta_j}^+]}, [\gamma_j^-, \gamma_j^+]e^{i.2\pi[w_{\gamma_j}^-, w_{\gamma_j}^+]}\},\ j = 1,2$ be any two CIVT-SFNs, and choose any real integer $\lambda > 0$ and $q \in Z^+$. Then,

1. $$A_1 \oplus A_2 = \begin{pmatrix} [(\alpha_1^{-q}+\alpha_2^{-q}-\alpha_1^{-q}\alpha_2^{-q})^{1/q},(\alpha_1^{+q}+\alpha_2^{+q}-\alpha_1^{+q}\alpha_2^{+q})^{1/q}]e^{i2\pi[(w_{\alpha_1}^{-q}+w_{\alpha_2}^{-q}-w_{\alpha_1}^{-q}w_{\alpha_2}^{-q})^{1/q},(w_{\alpha_1}^{+q}+w_{\alpha_2}^{+q}-w_{\alpha_1}^{+q}w_{\alpha_2}^{+q})^{1/q}]}, \\ [\beta_1^-\beta_2^-,\beta_1^+\beta_2^+]e^{i2\pi[w_{\beta_1}^-w_{\beta_2}^-,w_{\beta_1}^+w_{\beta_2}^+]}, \\ [\gamma_1^-\gamma_2^-,\gamma_1^+\gamma_2^+]e^{i2\pi[w_{\gamma_1}^-w_{\gamma_2}^-,w_{\gamma_1}^+w_{\gamma_2}^+]} \end{pmatrix}.$$

2.

$$A_1 \otimes A_2 = \begin{pmatrix} [\alpha_1^-\alpha_2^-,\alpha_1^+\alpha_2^+]e^{i2\pi[w_{\alpha_1}^-w_{\alpha_2}^-,w_{\alpha_1}^+w_{\alpha_2}^+]}, \\ [\beta_1^-\beta_2^-,\beta_1^+\beta_2^+]e^{i2\pi[w_{\beta_1}^-w_{\beta_2}^-,w_{\beta_1}^+w_{\beta_2}^+]}, \\ [(\gamma_1^{-q}+\gamma_2^{-q}-\gamma_1^{-q}\gamma_2^{-q})^{1/q},(\gamma_1^{+q}+\gamma_2^{+q}-\gamma_1^{+q}\gamma_2^{+q})^{1/q}]e^{i2\pi[(w_{\gamma_1}^{-q}+w_{\gamma_2}^{-q}-w_{\gamma_1}^{-q}w_{\gamma_2}^{-q})^{1/q},(w_{\gamma_1}^{+q}+w_{\gamma_2}^{+q}-w_{\gamma_1}^{+q}w_{\gamma_2}^{+q})^{1/q}]} \end{pmatrix}.$$

3. $$\lambda A_1 = \begin{pmatrix} [(1-(1-\alpha_1^{-q})^\lambda)^{1/q},(1-(1-\alpha_1^{+q})^\lambda)^{1/q}]e^{i2\pi[(1-(1-w_{\alpha_1}^{-q})^\lambda)^{1/q},(1-(1-w_{\alpha_1}^{+q})^\lambda)^{1/q}]}, \\ [\beta_1^{-\lambda},\beta_1^{+\lambda}]e^{i2\pi[w_{\beta_1}^{-\lambda},w_{\beta_1}^{+\lambda}]}, \\ [\gamma_1^{-\lambda},\gamma_1^{+\lambda}]e^{i2\pi[w_{\gamma_1}^{-\lambda},w_{\gamma_1}^{+\lambda}]} \end{pmatrix}$$

4. $$A_1^\lambda = \begin{pmatrix} [\alpha_1^{-\lambda},\alpha_1^{+\lambda}]e^{i2\pi[w_{\alpha_1}^{-\lambda},w_{\alpha_1}^{+\lambda}]}, \\ [\beta_1^{-\lambda},\beta_1^{+\lambda}]e^{i2\pi[w_{\beta_1}^{-\lambda},w_{\beta_1}^{+\lambda}]}, \\ [(1-(1-\gamma_1^{-q})^\lambda)^{1/q},(1-(1-\gamma_1^{+q})^\lambda)^{1/q}]e^{i2\pi[(1-(1-w_{\gamma_1}^{-q})^\lambda)^{1/q},(1-(1-w_{\gamma_1}^{+q})^\lambda)^{1/q}]} \end{pmatrix}$$

**Example 3.15.** Let $A_1 = \{[0.3,0.4]e^{i2\pi[0.4,0.5]},[0.4,0.5]e^{i2\pi[0.5,0.7]},\ [0.7,0.8]e^{i2\pi[0.7,0.9]}\}$ and $A_2 = \{[0.2,0.6]e^{i2\pi[0.3,0.5]},[0.3,0.7]e^{i2\pi[0.4,0.7]},[0.6,0.8]e^{i2\pi[0.7,0.8]}\}$ be any two CIVT-SFNs and let $\lambda = 3$ and $q = 4$. Then,

1. $$A_1 \oplus A_2 = \begin{pmatrix} [(\alpha_1^{-q}+\alpha_2^{-q}-\alpha_1^{-q}\alpha_2^{-q})^{1/q},(\alpha_1^{+q}+\alpha_2^{+q}-\alpha_1^{+q}\alpha_2^{+q})^{1/q}]e^{i2\pi[(w_{\alpha_1}^{-q}+w_{\alpha_2}^{-q}-w_{\alpha_1}^{-q}w_{\alpha_2}^{-q})^{1/q},(w_{\alpha_1}^{+q}+w_{\alpha_2}^{+q}-w_{\alpha_1}^{+q}w_{\alpha_2}^{+q})^{1/q}]}, \\ [\beta_1^-\beta_2^-,\beta_1^+\beta_2^+]e^{i2\pi[w_{\beta_1}^-w_{\beta_2}^-,w_{\beta_1}^+w_{\beta_2}^+]}, \\ [\gamma_1^-\gamma_2^-,\gamma_1^+\gamma_2^+]e^{i2\pi[w_{\gamma_1}^-w_{\gamma_2}^-,w_{\gamma_1}^+w_{\gamma_2}^+]} \end{pmatrix}.$$

$$\begin{aligned} = \ & \{[0.314,0.624]e^{i2\pi[0.428,0.59]},[0.12,0.35]e^{i2\pi[0.2,0.49]}, \\ & [0.42,0.64]e^{i2\pi[0.49,0.72]}\} \end{aligned}$$

$$2.\ A_1 \bigotimes A_2 = \begin{pmatrix} [\alpha_1^- \alpha_2^-, \alpha_1^+ \alpha_2^+] e^{i2\pi[w_{\alpha_1}^- w_{\alpha_2}^-, w_{\alpha_1}^+ w_{\alpha_2}^+]} \quad , \\ [\beta_1^- \beta_2^-, \beta_1^+ \beta_2^+] e^{i2\pi[w_{\beta_1}^- w_{\beta_2}^-, w_{\beta_1}^+ w_{\beta_2}^+]} \quad , \\ [(\gamma_1^{-q} + \gamma_2^{-q} - \gamma_1^{-q}\gamma_2^{-q})^{1/q}, (\gamma_1^{+q} + \gamma_2^{+q} - \gamma_1^{+q}\gamma_2^{+q})^{1/q}] e^{i2\pi[(w_{\gamma_1}^{-q} + w_{\gamma_2}^{-q} - w_{\gamma_1}^{-q} w_{\gamma_2}^{-q})^{1/q}, (w_{\gamma_1}^{+q} + w_{\gamma_2}^{+q} - w_{\gamma_1}^{+q} w_{\gamma_2}^{+q})^{1/q}]} \end{pmatrix}$$

$$\begin{aligned} = \ & \{[0.06, 0.24] e^{i2\pi[0.12, 0.25]}, [0.12, 0.35] e^{i2\pi[0.2, 0.49]}, \\ & [0.763, 0.898] e^{i2\pi[0.806, 0.945]}\} \end{aligned}$$

$$3.\ \lambda A_1 = \begin{pmatrix} [(1-(1-\alpha_1^{-q})^\lambda)^{1/q}, (1-(1-\alpha_1^{+q})^\lambda)^{1/q}] e^{i2\pi[(1-(1-w_{\alpha_1}^{-q})^\lambda)^{1/q}, (1-(1-w_{\alpha_1}^{+q})^\lambda)^{1/q}]} \quad , \\ [\beta_1^{-\lambda}, \beta_1^{+\lambda}] e^{i2\pi[w_{\beta_1}^{-\lambda}, w_{\beta_1}^{+\lambda}]} \quad , \\ [\gamma_1^{-\lambda}, \gamma_1^{+\lambda}] e^{i2\pi[w_{\gamma_1}^{-\lambda}, w_{\gamma_1}^{+\lambda}]} \end{pmatrix}$$

$$= \{[0.394, 0.523] e^{i2\pi[0.523, 0.648]}, [0.064, 0.125] e^{i2\pi[0.125, 0.343]}, [0.343, 0.512] e^{i2\pi[0.343, 0.729]}\}$$

$$4.\ A_1^\lambda = \begin{pmatrix} [\alpha_1^{-\lambda}, \alpha_1^{+\lambda}] e^{i2\pi[w_{\alpha_1}^{-\lambda}, w_{\alpha_1}^{+\lambda}]} \quad , \\ [\beta_1^{-\lambda}, \beta_1^{+\lambda}] e^{i2\pi[w_{\beta_1}^{-\lambda}, w_{\beta_1}^{+\lambda}]} \quad , \\ [(1-(1-\gamma_1^{-q})^\lambda)^{1/q}, (1-(1-\gamma_1^{+q})^\lambda)^{1/q}] e^{i2\pi[(1-(1-w_{\gamma_1}^{-q})^\lambda)^{1/q}, (1-(1-w_{\gamma_1}^{+q})^\lambda)^{1/q}]} \end{pmatrix}$$

$$= \{[0.027, 0.064] e^{i2\pi[0.064, 0.125]}, [0.064, 0.125] e^{i2\pi[0.125, 0.343]}, [0.866, 0.944] e^{i2\pi[0.866, 0.99]}\}$$

**Theorem 3.16.** Consider $A_j = \{[\alpha_j^-, \alpha_j^+]e^{i.2\pi[w_{\alpha_j}^-, w_{\alpha_j}^+]}, [\beta_j^-, \beta_j^+]e^{i.2\pi[w_{\beta_j}^-, w_{\beta_j}^+]}, [\gamma_j^-, \gamma_j^+]e^{i.2\pi[w_{\gamma_j}^-, w_{\gamma_j}^+]}\},\ j = 1,2$ be any two CIVT-SFNs and let $n_1, n_2 > 0$ be any real numbers and $q \in Z^+$. We have

1. $A_1 \oplus A_2 = A_2 \oplus A_1$
2. $A_1 \otimes A_2 = A_2 \otimes A_1$
3. $n_1(A_1 \oplus A_2) = n_1 A_1 \oplus n_1 A_2$
4. $n_1 A_1 \oplus n_2 A_1 = (n_1 \oplus n_2) A_1$
5. $A_1^{n_1} \otimes A_1^{n_2} = A_1^{n_1+n_2}$
6. $A_1^{n_1} \otimes A_2^{n_1} = (A_1 \otimes A_2)^{n_1}$.

*Proof.* We consider Definition 3.14

1.

$$A_1 \bigoplus A_2 = \begin{pmatrix} \left( [(\alpha_1^{-q} + \alpha_2^{-q} - \alpha_1^{-q}\alpha_2^{-q})^{1/q}, (\alpha_1^{+q} + \alpha_2^{+q} - \alpha_1^{+q}\alpha_2^{+q})^{1/q}]e^{i2\pi[(w_{\alpha_1}^{-q} + w_{\alpha_2}^{-q} - w_{\alpha_1}^{-q}w_{\alpha_2}^{-q})^{1/q}, (w_{\alpha_1}^{+q} + w_{\alpha_2}^{+q} - w_{\alpha_1}^{+q}w_{\alpha_2}^{+q})^{1/q}]} \right), \\ \left( [\beta_1^- \beta_2^-, \beta_1^+ \beta_2^+]e^{i2\pi[w_{\beta_1}^- w_{\beta_2}^-, w_{\beta_1}^+ w_{\beta_2}^+]} \right), \\ \left( [\gamma_1^- \gamma_2^-, \gamma_1^+ \gamma_2^+]e^{i2\pi[w_{\gamma_1}^- w_{\gamma_2}^-, w_{\gamma_1}^+ w_{\gamma_2}^+]} \right) \end{pmatrix}.$$

$$= \begin{pmatrix} \left( [(\alpha_2^{-q} + \alpha_1^{-q} - \alpha_2^{-q}\alpha_1^{-q})^{1/q}, (\alpha_2^{+q} + \alpha_1^{+q} - \alpha_2^{+q}\alpha_1^{+q})^{1/q}]e^{i2\pi[(w_{\alpha_2}^{-q} + w_{\alpha_1}^{-q} - w_{\alpha_2}^{-q}w_{\alpha_1}^{-q})^{1/q}, (w_{\alpha_2}^{+q} + w_{\alpha_1}^{+q} - w_{\alpha_2}^{+q}w_{\alpha_1}^{+q})^{1/q}]} \right), \\ \left( [\beta_2^- \beta_1^-, \beta_2^+ \beta_1^+]e^{i2\pi[w_{\beta_2}^- w_{\beta_1}^-, w_{\beta_2}^+ w_{\beta_1}^+]} \right), \\ \left( [\gamma_2^- \gamma_1^-, \gamma_2^+ \gamma_1^+]e^{i2\pi[w_{\gamma_2}^- w_{\gamma_1}^-, w_{\gamma_2}^+ w_{\gamma_1}^+]} \right) \end{pmatrix}.$$

$=A_2 \bigoplus A_1$

2. $$A_1 \bigotimes A_2 = \begin{pmatrix} \left( [\alpha_1^- \alpha_2^-, \alpha_1^+ \alpha_2^+]e^{i2\pi[w_{\alpha_1}^- w_{\alpha_2}^-, w_{\alpha_1}^+ w_{\alpha_2}^+]} \right), \\ \left( [\beta_1^- \beta_2^-, \beta_1^+ \beta_2^+]e^{i2\pi[w_{\beta_1}^- w_{\beta_2}^-, w_{\beta_1}^+ w_{\beta_2}^+]} \right), \\ \left( [(\gamma_1^{-q} + \gamma_2^{-q} - \gamma_1^{-q}\gamma_2^{-q})^{1/q}, (\gamma_1^{+q} + \gamma_2^{+q} - \gamma_1^{+q}\gamma_2^{+q})^{1/q}]e^{i2\pi[(w_{\gamma_1}^{-q} + w_{\gamma_2}^{-q} - w_{\gamma_1}^{-q}w_{\gamma_2}^{-q})^{1/q}, (w_{\gamma_1}^{+q} + w_{\gamma_2}^{+q} - w_{\gamma_1}^{+q}w_{\gamma_2}^{+q})^{1/q}]} \right) \end{pmatrix}.$$

$$= \begin{pmatrix} \left( [\alpha_2^- \alpha_1^-, \alpha_2^+ \alpha_1^+]e^{i2\pi[w_{\alpha_2}^- w_{\alpha_1}^-, w_{\alpha_2}^+ w_{\alpha_1}^+]} \right), \\ \left( [\beta_2^- \beta_1^-, \beta_2^+ \beta_1^+]e^{i2\pi[w_{\beta_2}^- w_{\beta_1}^-, w_{\beta_2}^+ w_{\beta_1}^+]} \right), \\ \left( [(\gamma_2^{-q} + \gamma_1^{-q} - \gamma_2^{-q}\gamma_1^{-q})^{1/q}, (\gamma_2^{+q} + \gamma_1^{+q} - \gamma_2^{+q}\gamma_1^{+q})^{1/q}]e^{i2\pi[(w_{\gamma_2}^{-q} + w_{\gamma_1}^{-q} - w_{\gamma_2}^{-q}w_{\gamma_1}^{-q})^{1/q}, (w_{\gamma_2}^{+q} + w_{\gamma_1}^{+q} - w_{\gamma_2}^{+q}w_{\gamma_1}^{+q})^{1/q}]} \right) \end{pmatrix}.$$

$=A_2 \bigotimes A_1$

3.

$$n_1(A_1 \bigoplus A_2) = n_1 \begin{pmatrix} \left[ \begin{array}{c} (\alpha_1^{-q} + \alpha_2^{-q} - \alpha_1^{-q}\alpha_2^{-q})^{1/q}, \\ (\alpha_1^{+q} + \alpha_2^{+q} - \alpha_1^{+q}\alpha_2^{+q})^{1/q} \end{array} \right] e^{i2\pi[(w_{\alpha_1}^{-q} + w_{\alpha_2}^{-q} - w_{\alpha_1}^{-q} w_{\alpha_2}^{-q})^{1/q}, (w_{\alpha_1}^{+q} + w_{\alpha_2}^{+q} - w_{\alpha_1}^{+q} w_{\alpha_2}^{+q})^{1/q}]} , \\ [\beta_1^- \beta_2^-, \beta_1^+ \beta_2^+] e^{i2\pi[w_{\beta_1}^- w_{\beta_2}^-, w_{\beta_1}^+ w_{\beta_2}^+]} , \\ [\gamma_1^- \gamma_2^-, \gamma_1^+ \gamma_2^+] e^{i2\pi[w_{\gamma_1}^{-q} w_{\gamma_2}^-, w_{\gamma_1}^+ w_{\gamma_2}^+]} \end{pmatrix}.$$

$$= \begin{pmatrix} \left[ \begin{array}{c} (1-(1-\begin{pmatrix} \alpha_1^{-q} + \alpha_2^{-q} \\ -\alpha_1^{-q}\alpha_2^{-q} \end{pmatrix})^{n_1})^{1/q}, \\ (1-(1-\begin{pmatrix} \alpha_1^{+q} + \alpha_2^{+q} \\ -\alpha_1^{+q}\alpha_2^{+q} \end{pmatrix})^{n_1})^{1/q} \end{array} \right] e^{i2\pi\left[(1-(1-\begin{pmatrix} w_{\alpha_1}^{-q} + W_{\alpha_2}^{-q} \\ -w_{\alpha_1}^{-q} W_{\alpha_2}^{-q} \end{pmatrix})^{n_1})^{1/q} (1-(1-\begin{pmatrix} w_{\alpha_1}^{+q} + W_{\alpha_2}^{+q} \\ -w_{\alpha_1}^{+q} W_{\alpha_2}^{+q} \end{pmatrix})^{n_1})^{1/q}\right]}, \\ [(\beta_1^- \beta_2^-)^{n_1}, (\beta_1^+ \beta_2^+)^{n_1}] e^{i2\pi[(w_{\beta_1}^- w_{\beta_2}^-)^{n_1}, (w_{\beta_1}^+ w_{\beta_2}^+)^{n_1}]} , \\ [(\gamma_1^- \gamma_2^-)^{n_1}, (\gamma_1^+ \gamma_2^+)^{n_1}] e^{i2\pi[(w_{\gamma_1}^- w_{\gamma_2}^-)^{n_1}, (w_{\gamma_1}^+ w_{\gamma_2}^+)^{n_1}]} \end{pmatrix}.$$

Let

$$n_1 A_1 = \begin{pmatrix} [(1-(1-\alpha_1^{-q})^{n_1})^{1/q}, (1-(1-\alpha_1^{+q})^{n_1})^{1/q}] e^{i2\pi[(1-(1-w_{\alpha_1}^{-q})^{n_1})^{1/q}, (1-(1-w_{\alpha_1}^{+q})^{n_1})^{1/q}]} , \\ [\beta_1^{-n_1}, \beta_1^{+n_1}] e^{i2\pi[w_{\beta_1}^{-n_1}, w_{\beta_1}^{+n_1}]} , \\ [\gamma_1^{-n_1}, \gamma_1^{+n_1}] e^{i2\pi[w_{\gamma_1}^{-n_1}, w_{\gamma_1}^{+n_1}]} \end{pmatrix}$$

and

$$n_1 A_2 = \begin{pmatrix} [(1-(1-\alpha_2^{-q})^{n_1})^{1/q}, (1-(1-\alpha_2^{+q})^{n_1})^{1/q}] e^{i2\pi[(1-(1-w_{\alpha_2}^{-q})^{n_1})^{1/q}, (1-(1-w_{\alpha_2}^{+q})^{n_1})^{1/q}]} , \\ [\beta_2^{-n_1}, \beta_2^{+n_1}] e^{i2\pi[w_{\beta_2}^{-n_1}, w_{\beta_2}^{+n_1}]} , \\ [\gamma_2^{-n_1}, \gamma_2^{+n_1}] e^{i2\pi[w_{\gamma_2}^{-n_1}, w_{\gamma_2}^{+n_1}]} \end{pmatrix}.$$

Then, we have

$$n_1A_1 \bigoplus n_1A_2 = \left( \begin{array}{c} \begin{bmatrix} (1-(1-\alpha_1^{-q})^{n_1})^{1/q}, \\ (1-(1-\alpha_1^{+q})^{n_1})^{1/q} \end{bmatrix} e^{i2\pi \begin{bmatrix} (1-(1-w_{\alpha_1}^{-q})^{n_1})^{1/q}, \\ (1-(1-w_{\alpha_1}^{+q})^{n_1})^{1/q} \end{bmatrix}}, \\ \\ [\beta_1^{-n_1}, \beta_1^{+n_1}] e^{i2\pi [w_{\beta_1}^{-n_1}, w_{\beta_1}^{+n_1}]}, \\ \\ [\gamma_1^{-n_1}, \gamma_1^{+n_1}] e^{i2\pi [w_{\gamma_1}^{-n_1}, w_{\gamma_1}^{+n_1}]} \end{array} \right)$$

$$\bigoplus \left( \begin{array}{c} \begin{bmatrix} (1-(1-\alpha_2^{-q})^{n_1})^{1/q}, \\ (1-(1-\alpha_2^{+q})^{n_1})^{1/q} \end{bmatrix} e^{i2\pi \begin{bmatrix} (1-(1-w_{\alpha_2}^{-q})^{n_1})^{1/q}, \\ (1-(1-w_{\alpha_2}^{+q})^{n_1})^{1/q} \end{bmatrix}}, \\ \\ [\beta_2^{-n_1}, \beta_2^{+n_1}] e^{i2\pi [w_{\beta_2}^{-n_1}, w_{\beta_2}^{+n_1}]}, \\ \\ [\gamma_2^{-n_1}, \gamma_2^{+n_1}] e^{i2\pi [w_{\gamma_2}^{-n_1}, w_{\gamma_2}^{+n_1}]} \end{array} \right)$$

$$= \left( \begin{array}{c} \begin{bmatrix} \left( \begin{array}{c} ((1-(1-\alpha_1^{-q})^{n_1})^{1/q})^q + \\ ((1-(1-\alpha_2^{-q})^{n_1})^{1/q})^q - \\ ((1-(1-\alpha_1^{-q})^{n_1})^{1/q})^q ((1-(1-\alpha_2^{-q})^{n_1})^{1/q})^q \end{array} \right)^{1/q}, \\ \\ \left( \begin{array}{c} ((1-(1-\alpha_1^{+q})^{n_1})^{1/q})^q + \\ ((1-(1-\alpha_2^{+q})^{n_1})^{1/q})^q - \\ ((1-(1-\alpha_1^{+q})^{n_1})^{1/q})^q ((1-(1-\alpha_2^{+q})^{n_1})^{1/q})^q \end{array} \right)^{1/q} \end{bmatrix} e^{i2\pi \begin{bmatrix} \left( \begin{array}{c} ((1-(1-w_{\alpha_1}^{-q})^{n_1})^{1/q})^q + \\ ((1-(1-w_{\alpha_2}^{-q})^{n_1})^{1/q})^q - \\ ((1-(1-w_{\alpha_1}^{-q})^{n_1})^{1/q})^q ((1-(1-w_{\alpha_2}^{-q})^{n_1})^{1/q})^q \end{array} \right)^{1/q}, \\ \\ \left( \begin{array}{c} ((1-(1-w_{\alpha_1}^{+q})^{n_1})^{1/q})^q + \\ ((1-(1-w_{\alpha_2}^{+q})^{n_1})^{1/q})^q - \\ ((1-(1-w_{\alpha_1}^{+q})^{n_1})^{1/q})^q ((1-(1-w_{\alpha_2}^{+q})^{n_1})^{1/q})^q \end{array} \right)^{1/q} \end{bmatrix}}, \\ \\ \left[ (\beta_1^- \beta_2^-)^{n_1}, (\beta_1^+ \beta_2^+)^{n_1} \right] e^{i2\pi \left[ (w_{\beta_1}^- w_{\beta_2}^-)^{n_1} (w_{\beta_1}^+ w_{\beta_2}^+)^{n_1} \right]}, \\ \\ \left[ (\gamma_1^- \gamma_2^-)^{n_1}, (\gamma_1^+ \gamma_2^+)^{n_1} \right] e^{i2\pi \left[ (w_{\gamma_1}^- w_{\gamma_2}^-)^{n_1} (w_{\gamma_1}^+ w_{\gamma_2}^+)^{n_1} \right]} \end{array} \right)$$

$$= n_1(A_1 \bigoplus A_2).$$

Hence the part(3) have provided.

(4) Obviously

Next we are taking part (5), we have

$$A_1^{n_1} = \begin{pmatrix} [\alpha_1^{-n_1}, \alpha_1^{+n_1}] e^{i2\pi[w_{\alpha_1}^{-n_1}, w_{\alpha_1}^{+n_1}]} \ , \\ \\ [\beta_1^{-n_1}, \beta_1^{+n_1}] e^{i2\pi[w_{\beta_1}^{-n_1}, w_{\beta_1}^{+n_1}]} \ , \\ \\ [(1-(1-\gamma_1^{-q})^{n_1})^{1/q}, (1-(1-\gamma_1^{+q})^{n_1})^{1/q}] e^{i2\pi[(1-(1-w_{\gamma_1}^{-q})^{n_1})^{1/q}, (1-(1-w_{\gamma_1}^{+q})^{n_1})^{1/q}]} \end{pmatrix}$$

and

$$A_1^{n_2} = \begin{pmatrix} [\alpha_1^{-n_2}, \alpha_1^{+n_2}] e^{i2\pi[w_{\alpha_1}^{-n_2}, w_{\alpha_1}^{+n_2}]} \ , \\ \\ [\beta_1^{-n_2}, \beta_1^{+n_2}] e^{i2\pi[w_{\beta_1}^{-n_2}, w_{\beta_1}^{+n_2}]} \ , \\ \\ [(1-(1-\gamma_1^{-q})^{n_2})^{1/q}, (1-(1-\gamma_1^{+q})^{n_2})^{1/q}] e^{i2\pi[(1-(1-w_{\gamma_1}^{-q})^{n_2})^{1/q}, (1-(1-w_{\gamma_1}^{+q})^{n_2})^{1/q}]} \end{pmatrix}$$

Then, we have

$$A_1^{n_1} \otimes A_1^{n_2} = \begin{pmatrix} [\alpha_1^{-n_1}, \alpha_1^{+n_1}] e^{i2\pi[w_{\alpha_1}^{-n_1}, w_{\alpha_1}^{+n_1}]} \ , \\ \\ [\beta_1^{-n_1}, \beta_1^{+n_1}] e^{i2\pi[w_{\beta_1}^{-n_1}, w_{\beta_1}^{+n_1}]} \ , \\ \\ \begin{bmatrix} (1-(1-\gamma_1^{-q})^{n_1})^{1/q}, \\ (1-(1-\gamma_1^{+q})^{n_1})^{1/q} \end{bmatrix} e^{i2\pi \begin{bmatrix} (1-(1-w_{\gamma_1}^{-q})^{n_1})^{1/q}, \\ (1-(1-w_{\gamma_1}^{+q})^{n_1})^{1/q} \end{bmatrix}} \end{pmatrix}$$

$$\otimes \begin{pmatrix} [\alpha_1^{-n_2}, \alpha_1^{+n_2}] e^{i2\pi[w_{\alpha_1}^{-n_2}, w_{\alpha_1}^{+n_2}]} \ , \\ \\ [\beta_1^{-n_2}, \beta_1^{+n_2}] e^{i2\pi[w_{\beta_1}^{-n_2}, w_{\beta_1}^{+n_2}]} \ , \\ \\ \begin{bmatrix} (1-(1-\gamma_1^{-q})^{n_2})^{1/q}, \\ (1-(1-\gamma_1^{+q})^{n_2})^{1/q} \end{bmatrix} e^{i2\pi \begin{bmatrix} (1-(1-w_{\gamma_1}^{-q})^{n_2})^{1/q}, \\ (1-(1-w_{\gamma_1}^{+q})^{n_2})^{1/q} \end{bmatrix}} \end{pmatrix}$$

$$
= \begin{pmatrix}
\left[\alpha_1^{-n_1}\alpha_1^{-n_2}, \alpha_1^{+n_1}\alpha_1^{+n_2}\right] e^{i2\pi\left[(w_{\alpha_1}^{-n_1} w_{\alpha_1}^{-n_2}, (w_{\alpha_1}^{+n_1} w_{\alpha_1}^{+n_1}\right]}, \\
\left[\beta_1^{-n_1}\beta_1^{-n_2}, \beta_1^{+n_1}\beta_1^{+n_2}\right] e^{i2\pi\left[(w_{\beta_1}^{-n_1} w_{\beta_1}^{-n_2}, (w_{\beta_1}^{+n_1} w_{\beta_1}^{+n_1}\right]}, \\
\left[\begin{matrix} \left(\begin{matrix} ((1-(1-\gamma_1^{-q})^{n_1})^{1/q})^q + \\ ((1-(1-\gamma_1^{-q})^{n_2})^{1/q})^q - \\ ((1-(1-\beta_1^{-q})^{n_1})^{1/q})^q((1-(1-\gamma_1^{-q})^{n_2})^{1/q})^q \end{matrix}\right)^{1/q}, \\ \left(\begin{matrix} ((1-(1-\gamma_1^{+q})^{n_1})^{1/q})^q + \\ ((1-(1-\gamma_1^{+q})^{n_2})^{1/q})^q - \\ ((1-(1-\gamma_1^{+q})^{n_1})^{1/q})^q((1-(1-\gamma_1^{+q})^{n_2})^{1/q})^q \end{matrix}\right)^{1/q} \end{matrix}\right] e^{i2\pi\left[\begin{matrix} \left(\begin{matrix} ((1-(1-w_{\gamma_1}^{-q})^{n_1})^{1/q})^q + \\ ((1-(1-w_{\gamma_1}^{-q})^{n_2})^{1/q})^q - \\ ((1-(1-w_{\gamma_1}^{-q})^{n_1})^{1/q})^q((1-(1-w_{\gamma_1}^{-q})^{n_2})^{1/q})^q \end{matrix}\right)^{1/q}, \\ \left(\begin{matrix} ((1-(1-w_{\gamma_1}^{+q})^{n_1})^{1/q})^q + \\ ((1-(1-w_{\gamma_1}^{+q})^{n_2})^{1/q})^q - \\ ((1-(1-w_{\gamma_1}^{+q})^{n_1})^{1/q})^q((1-(1-w_{\gamma_1}^{+q})^{n_2})^{1/q})^q \end{matrix}\right)^{1/q} \end{matrix}\right]}
\end{pmatrix}
$$

$$= (A_1)^{n_1+n_2}$$

Hence, $A_1^{n_1} \bigotimes A_1^{n_2} = (A_1)^{n_1+n_2}$

□

**Example 3.17.** Let
$A_1 = \{[0.3, 0.4]e^{i2\pi[0.4,0.5]}, [0.4, 0.5]e^{i2\pi[0.5,0.7]}, [0.7, 0.8]e^{i2\pi[0.7,0.9]}\}$ and
$A_2 = \{[0.2, 0.6]e^{i2\pi[0.3,0.5]}, [0.3, 0.7]e^{i2\pi[0.4,0.7]}, [0.6, 0.8]e^{i2\pi[0.7,0.8]}\}$
be any two CIVT-SFNs and $q = 4$. Then,

$$
1. A_1 \oplus A_2 = \begin{pmatrix}
[(\alpha_1^{-q} + \alpha_2^{-q} - \alpha_1^{-q}\alpha_2^{-q})^{1/q}, (\alpha_1^{+q} + \alpha_2^{+q} - \alpha_1^{+q}\alpha_2^{+q})^{1/q}] e^{i2\pi[(w_{\alpha_1}^{-q} + w_{\alpha_2}^{-q} - w_{\alpha_1}^{-q} w_{\alpha_2}^{-q})^{1/q}, (w_{\alpha_1}^{+q} + w_{\alpha_2}^{+q} - w_{\alpha_1}^{+q} w_{\alpha_2}^{+q})^{1/q}]} , \\
[\beta_1^{-}\beta_2^{-}, \beta_1^{+}\beta_2^{+}] e^{i2\pi[w_{\beta_1}^{-} w_{\beta_2}^{-}, w_{\beta_1}^{+} w_{\beta_2}^{+}]} , \\
[\gamma_1^{-}\gamma_2^{-}, \gamma_1^{+}\gamma_2^{+}] e^{i2\pi[w_{\gamma_1}^{-} w_{\gamma_2}^{-}, w_{\gamma_1}^{+} w_{\gamma_2}^{+}]}
\end{pmatrix}
$$

$$
\begin{aligned}
= \quad & \{[0.314, 0.624]e^{i2\pi[0.428,0.59]}, [0.12, 0.35]e^{i2\pi[0.2,0.49]}, \\
& [0.42, 0.64]e^{i2\pi[0.49,0.72]}\}
\end{aligned}
$$

and

$$A_2 \bigoplus A_1 = \left( \begin{array}{c} [(\alpha_2^{-q}+\alpha_1^{-q}-\alpha_2^{-q}\alpha_1^{-q})^{1/q},(\alpha_2^{+q}+\alpha_1^{+q}-\alpha_2^{+q}\alpha_1^{+q})^{1/q}]e^{i2\pi[(w_{\alpha_2}^{-q}+w_{\alpha_1}^{-q}-w_{\alpha_2}^{-q}w_{\alpha_1}^{-q})^{1/q},(w_{\alpha_2}^{+q}+w_{\alpha_1}^{+q}-w_{\alpha_2}^{+q}w_{\alpha_1}^{+q})^{1/q}]} \quad , \\ [\beta_2^-\beta_1^-,\beta_2^+\beta_1^+]e^{i2\pi[w_{\beta_2}^- w_{\beta_1}^-,w_{\beta_2}^+ w_{\beta_1}^+]} \quad , \\ [\gamma_2^-\gamma_1^-,\gamma_2^+\gamma_1^+]e^{i2\pi[w_{\gamma_2}^- w_{\gamma_1}^-,w_{\gamma_2}^+ w_{\gamma_1}^+]} \end{array} \right)$$

$$\begin{aligned} = \quad & \{[0.314,0.624]e^{i2\pi[0.428,0.59]},[0.12,0.35]e^{i2\pi[0.2,0.49]}, \\ & [0.42,0.64]e^{i2\pi[0.49,0.72]}\} \end{aligned}$$

Then, $A_1 \bigoplus A_2 = A_2 \bigoplus A_1$

2. $$A_1 \bigotimes A_2 = \left( \begin{array}{c} [\alpha_1^-\alpha_2^-,\alpha_1^+\alpha_2^+]e^{i2\pi[w_{\alpha_1}^- w_{\alpha_2}^-,w_{\alpha_1}^+ w_{\alpha_2}^+]} \quad , \\ [\beta_1^-\beta_2^-,\beta_1^+\beta_2^+]e^{i2\pi[w_{\beta_1}^- w_{\beta_2}^-,w_{\beta_1}^+ w_{\beta_2}^+]} \quad , \\ [(\gamma_1^{-q}+\gamma_2^{-q}-\gamma_1^{-q}\gamma_2^{-q})^{1/q},(\gamma_1^{+q}+\gamma_2^{+q}-\gamma_1^{+q}\gamma_2^{+q})^{1/q}]e^{i2\pi[(w_{\gamma_1}^{-q}+w_{\gamma_2}^{-q}-w_{\gamma_1}^{-q}w_{\gamma_2}^{-q})^{1/q},(w_{\gamma_1}^{+q}+w_{\gamma_2}^{+q}-w_{\gamma_1}^{+q}w_{\gamma_2}^{+q})^{1/q}]} \end{array} \right)$$

$$\begin{aligned} = \quad & \{[0.06,0.24]e^{i2\pi[0.12,0.25]},[0.12,0.35]e^{i2\pi[0.2,0.49]}, \\ & [0.763,0.898]e^{i2\pi[0.806,0.945]}\} \end{aligned}$$

and

$$A_2 \bigotimes A_1 = \left( \begin{array}{c} [\alpha_2^-\alpha_1^-,\alpha_2^+\alpha_1^+]e^{i2\pi[w_{\alpha_2}^- w_{\alpha_1}^-,w_{\alpha_2}^+ w_{\alpha_1}^+]} \quad , \\ [\beta_2^-\beta_1^-,\beta_2^+\beta_1^+]e^{i2\pi[w_{\beta_2}^- w_{\beta_1}^-,w_{\beta_2}^+ w_{\beta_1}^+]} \quad , \\ [(\gamma_2^{-q}+\gamma_1^{-q}-\gamma_2^{-q}\gamma_1^{-q})^{1/q},(\gamma_2^{+q}+\gamma_1^{+q}-\gamma_2^{+q}\gamma_1^{+q})^{1/q}]e^{i2\pi[(w_{\gamma_2}^{-q}+w_{\gamma_1}^{-q}-w_{\gamma_2}^{-q}w_{\gamma_1}^{-q})^{1/q},(w_{\gamma_2}^{+q}+w_{\gamma_1}^{+q}-w_{\gamma_2}^{+q}w_{\gamma_1}^{+q})^{1/q}]} \end{array} \right)$$

$$\begin{aligned} = \quad & \{[0.06,0.24]e^{i2\pi[0.12,0.25]},[0.12,0.35]e^{i2\pi[0.2,0.49]}, \\ & [0.763,0.898]e^{i2\pi[0.806,0.945]}\} \end{aligned}$$

Then, $A_1 \bigotimes A_2 = A_2 \bigotimes A_1$

## 4. COMPLEX INTERVAL-VALUED T-SPHERICAL FUZZY AOs

Here $\varpi = (\varpi_1, \varpi_2, ..., \varpi_k)^T$ shows the weight vectors in a manner that $\sum_{j=1}^{k} \varpi = 1$, where $j = 1, 2, ..., k$ Moreover

$$A_j = \{[\alpha_j^-, \alpha_j^+]e^{i2\pi[\mathfrak{W}_{\alpha_j}^-, \mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-, \beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-, \mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-, \gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-, \mathfrak{W}_{\gamma_j}^+]}\},$$

$j = 1, 2, ..., k.$ have acted as the collective CIVT-SFNs.

**Definition 4.1.** The following mapping represents the complex interval-valued T-spherical fuzzy weighted arithmetic aggregation (CIVTSWAA) operator:
$CIVTSWAA : Z^k \to Z$, and it is defined as follows:

$$CIVTSWAA(A_1, A_2, ..., A_k) = \varpi_1 A_1 \bigoplus \varpi_2 A_2 \bigoplus ... \bigoplus \varpi_k A_k = \bigoplus_{j=1}^{k} \varpi_j A_j$$

where Z denotes the set of CIVT-SFNs.

The following theorem can be found by using the CIVT-SFNs' operating laws.

**Theorem 4.2.** Let's think about Definition 4.1, and then we get

$$CIVTSFWAA(A_1, A_2, ..., A_k) = \left( \begin{array}{c} \left[ \begin{array}{c} \left(1 - \Pi_{j=1}^{k} (1 - \alpha_j^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1 - \Pi_{j=1}^{k} (1 - \alpha_j^{+q})^{\varpi_j}\right)^{1/q} \end{array} \right] e^{i2\pi \left[ \begin{array}{c} \left(1 - \Pi_{j=1}^{k} (1 - \mathfrak{W}_{\alpha_j}^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1 - \Pi_{j=1}^{k} (1 - \mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j}\right)^{1/q} \end{array} \right]}, \\ \left[ \Pi_{j=1}^{k} \beta_j^{-\varpi_j}, \Pi_{j=1}^{k} \beta_j^{+\varpi_j} \right] e^{i2\pi \left[ \Pi_{j=1}^{k} \mathfrak{W}_{\beta_j}^{-\varpi_j}, \Pi_{j=1}^{k} \mathfrak{W}_{\beta_j}^{+\varpi_j} \right]}, \\ \left[ \Pi_{j=1}^{k} \gamma_j^{-\varpi_j}, \Pi_{j=1}^{k} \gamma_j^{+\varpi_j} \right] e^{i2\pi \left[ \Pi_{j=1}^{k} \mathfrak{W}_{\gamma_j}^{-\varpi_j}, \Pi_{j=1}^{k} \mathfrak{W}_{\gamma_j}^{+\varpi_j} \right]} \end{array} \right)$$

*Proof.* The proof is made by using mathematical Indication.
Case 1: For $k = 2$

$$\varpi_1 A_1 = \begin{pmatrix} [(1-(1-\alpha_1^{-q})^{\varpi_1})^{1/q},(1-(1-\alpha_1^{+q})^{\varpi_1})^{1/q}]e^{i2\pi[(1-(1-\mathfrak{W}_{\alpha_1}^{-q})^{\varpi_1})^{1/q},(1-(1-\mathfrak{W}_{\alpha_1}^{+q})^{\varpi_1})^{1/q}]}, \\ [\beta_1^{-\varpi_1},\beta_1^{+\varpi_1}]e^{i2\pi[\mathfrak{W}_{\beta_1}^{-\varpi_1},\mathfrak{W}_{\beta_1}^{+\varpi_1}]}, \\ [\gamma_1^{-\varpi_1},\gamma_1^{+\varpi_1}]e^{i2\pi[\mathfrak{W}_{\gamma_1}^{-\varpi_1},\mathfrak{W}_{\gamma_1}^{+\varpi_1}]} \end{pmatrix},$$

and

$$\varpi_2 A_2 = \begin{pmatrix} [(1-(1-\alpha_2^{-q})^{\varpi_2})^{1/q},(1-(1-\alpha_2^{+q})^{\varpi_2})^{1/q}]e^{i2\pi[(1-(1-w_{\alpha_2}^{-q})^{\varpi_2})^{1/q},(1-(1-w_{\alpha_2}^{+q})^{\varpi_2})^{1/q}]}, \\ [\beta_2^{-\varpi_2},\beta_2^{+\varpi_2}]e^{i2\pi[\mathfrak{W}_{\beta_2}^{-\varpi_2},\mathfrak{W}_{\beta_2}^{+\varpi_2}]}, \\ [\gamma_2^{-\varpi_2},\gamma_2^{+\varpi_2}]e^{i2\pi[\mathfrak{W}_{\gamma_2}^{-\varpi_2},\mathfrak{W}_{\gamma_2}^{+\varpi_2}]} \end{pmatrix}.$$

Then,

$CIVTSFWAA(A_1,A_2) = \varpi_1 A_1 \bigoplus \varpi_2 A_2 =$

$$\begin{pmatrix} [(1-(1-\alpha_1^{-q})^{\varpi_1})^{1/q},(1-(1-\alpha_1^{+q})^{\varpi_1})^{1/q}]e^{i2\pi[(1-(1-\mathfrak{W}_{\alpha_1}^{-q})^{\varpi_1})^{1/q},(1-(1-\mathfrak{W}_{\alpha_1}^{+q})^{\varpi_1})^{1/q}]}, \\ [\beta_1^{-\varpi_1},\beta_1^{+\varpi_1}]e^{i2\pi[\mathfrak{W}_{\beta_1}^{-\varpi_1},\mathfrak{W}_{\beta_1}^{+\varpi_1}]}, \\ [\gamma_1^{-\varpi_1},\gamma_1^{+\varpi_1}]e^{i2\pi[\mathfrak{W}_{\gamma_1}^{-\varpi_1},\mathfrak{W}_{\gamma_1}^{+\varpi_1}]} \end{pmatrix}$$

$$\oplus \begin{pmatrix} [(1-(1-\alpha_2^{-q})^{\varpi_2})^{1/q},(1-(1-\alpha_2^{+q})^{\varpi_2})^{1/q}]e^{i2\pi[(1-(1-\mathfrak{W}_{\alpha_2}^{-q})^{\varpi_2})^{1/q},(1-(1-\mathfrak{W}_{\alpha_2}^{+q})^{\varpi_2})^{1/q}]}, \\ [\beta_2^{-\varpi_2},\beta_2^{+\varpi_2}]e^{i2\pi[\mathfrak{W}_{\beta_2}^{-\varpi_2},\mathfrak{W}_{\beta_2}^{+\varpi_2}]}, \\ [\gamma_2^{-\varpi_2},\gamma_2^{+\varpi_2}]e^{i2\pi[\mathfrak{W}_{\gamma_2}^{-\varpi_2},\mathfrak{W}_{\gamma_2}^{+\varpi_2}]} \end{pmatrix}$$

$$
= \left( \begin{array}{c}
\left[ \begin{array}{c} \left( \begin{array}{c} ((1-(1-\alpha_1^{-q})^{\varpi_1})^{1/q})^q + \\ ((1-(1-\alpha_2^{-q})^{\varpi_2})^{1/q})^q - \\ ((1-(1-\alpha_1^{-q})^{\varpi_1})^{1/q})^q((1-(1-\alpha_2^{-q})^{\varpi_2})^{1/q})^q \end{array} \right)^{1/q}, \\ \left( \begin{array}{c} ((1-(1-\alpha_1^{+q})^{\varpi_1})^{1/q})^q + \\ ((1-(1-\alpha_2^{+q})^{\varpi_2})^{1/q})^q - \\ ((1-(1-\alpha_1^{+q})^{\varpi_1})^{1/q})^q((1-(1-\alpha_2^{+q})^{\varpi_2})^{1/q})^q \end{array} \right)^{1/q} \end{array} \right] e^{i2\pi \left[ \begin{array}{c} \left( \begin{array}{c} ((1-(1-\mathfrak{W}_{\alpha_1}^{-q})^{\varpi_1})^{1/q})^q + \\ ((1-(1-\mathfrak{W}_{\alpha_2}^{-q})^{\varpi_2})^{1/q})^q - \\ ((1-(1-\mathfrak{W}_{\alpha_1}^{-q})^{\varpi_1})^{1/q})^q((1-(1-w_{\alpha_2}^{-q})^{\varpi_2})^{1/q})^q \end{array} \right)^{1/q}, \\ \left( \begin{array}{c} ((1-(1-\mathfrak{W}_{\alpha_1}^{+q})^{\varpi_1})^{1/q})^q + \\ ((1-(1-\mathfrak{W}_{\alpha_2}^{+q})^{\varpi_2})^{1/q})^q - \\ ((1-(1-\mathfrak{W}_{\alpha_1}^{+q})^{\varpi_1})^{1/q})^q((1-(1-\mathfrak{W}_{\alpha_2}^{+q})^{\varpi_2})^{1/q})^q \end{array} \right)^{1/q} \end{array} \right]}, \\
\left[\beta_1^{-\varpi_1}\beta_2^{-\varpi_2}, \beta_1^{+\varpi_1}\beta_2^{+\varpi_2}\right] e^{i2\pi\left[\mathfrak{W}_{\beta_1}^{-\varpi_1}\mathfrak{W}_{\beta_2}^{-\varpi_2}, \mathfrak{W}_{\beta_1}^{+\varpi_1}\mathfrak{W}_{\beta_2}^{+\varpi_2}\right]}, \\
\left[\gamma_1^{-\varpi_1}\gamma_2^{-\varpi_2}, \gamma_1^{+\varpi_1}\gamma_2^{+\varpi_2}\right] e^{i2\pi\left[\mathfrak{W}_{\gamma_1}^{-\varpi_1}\mathfrak{W}_{\gamma_2}^{-\varpi_2}, \mathfrak{W}_{\gamma_1}^{+\varpi_1}\mathfrak{W}_{\gamma_2}^{+\varpi_2}\right]}
\end{array} \right)
$$

$$
= \left( \begin{array}{c}
\left[ \begin{array}{c} (1-(1-\alpha_1^{-q})^{\varpi_1}(1-\alpha_2^{-q})^{\varpi_2})^{1/q}, \\ (1-(1-\alpha_1^{+q})^{\varpi_1}(1-\alpha_2^{+q})^{\varpi_2})^{1/q}, \end{array} \right] e^{i2\pi\left[ \begin{array}{c} (1-(1-\mathfrak{W}_{\alpha_1}^{-q})^{\varpi_1}(1-\mathfrak{W}_{\alpha_2}^{-q})^{\varpi_2})^{1/q}, \\ (1-(1-\mathfrak{W}_{\alpha_1}^{+q})^{\varpi_1}(1-\mathfrak{W}_{\alpha_2}^{+q})^{\varpi_2})^{1/q}, \end{array} \right]}, \\
\left[\beta_1^{-\varpi_1}\beta_2^{-\varpi_2}, \beta_1^{+\varpi_1}\beta_2^{+\varpi_2}\right] e^{i2\pi\left[\mathfrak{W}_{\beta_1}^{-\varpi_1}\mathfrak{W}_{\beta_2}^{-\varpi_2}, \mathfrak{W}_{\beta_1}^{+\varpi_1}\mathfrak{W}_{\beta_2}^{+\varpi_2}\right]}, \\
\left[\gamma_1^{-\varpi_1}\gamma_2^{-\varpi_2}, \gamma_1^{+\varpi_1}\gamma_2^{+\varpi_2}\right] e^{i2\pi\left[w_{\gamma_1}^{-\varpi_1}w_{\gamma_2}^{-\varpi_2}, w_{\gamma_1}^{+\varpi_1}w_{\gamma_2}^{+\varpi_2}\right]}
\end{array} \right)
$$

$$
= \left( \begin{array}{c}
\left[ \begin{array}{c} (1-\prod_{j=1}^{2}(1-\alpha_j^{-q})^{\varpi_j})^{1/q}, \\ (1-\prod_{j=1}^{2}(1-\alpha_j^{+q})^{\varpi_j})^{1/q}, \end{array} \right] e^{i2\pi\left[ \begin{array}{c} (1-\prod_{j=1}^{2}(1-\mathfrak{W}_{\alpha_j}^{-q})^{\varpi_j})^{1/q}, \\ (1-\prod_{j=1}^{2}(1-\mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j})^{1/q}, \end{array} \right]}, \\
\left[\prod_{j=1}^{2}\beta_j^{-\varpi_j}, \prod_{j=1}^{2}\beta_j^{+\varpi_j}\right] e^{i2\pi\left[\prod_{j=1}^{2}w_{\beta_j}^{-\varpi_j}, \prod_{j=1}^{2}w_{\beta_j}^{+\varpi_j}\right]}, \\
\left[\prod_{j=1}^{2}\gamma_j^{-\varpi_j}, \prod_{j=1}^{2}\gamma_j^{+\varpi_j}\right] e^{i2\pi\left[\prod_{j=1}^{2}w_{\gamma_j}^{-\varpi_j}, \prod_{j=1}^{2}w_{\gamma_j}^{+\varpi_j}\right]}
\end{array} \right)
$$

The formula is valid.

Case 2: Assume the equation is true for $k = m$. Which is

$$CIVTSFWAA(A_1,A_2,...,A_m)=\left(\begin{array}{c}\left[\begin{array}{c}(1-\prod_{j=1}^{m}(1-\alpha_j^{-q})^{\varpi_j})^{1/q},\\(1-\prod_{j=1}^{m}(1-\alpha_j^{+q})^{\varpi_j})^{1/q},\end{array}\right]e^{i2\pi\left[\begin{array}{c}(1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j}^{-q})^{\varpi_j})^{1/q},\\(1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j})^{1/q},\end{array}\right]},\\ \left[\prod_{j=1}^{m}\beta_j^{-\varpi_j},\prod_{j=1}^{m}\beta_j^{+\varpi_j}\right]e^{i2\pi\left[\prod_{j=1}^{m}\mathfrak{W}_{\beta_j}^{-\varpi_j},\prod_{j=1}^{m}\mathfrak{W}_{\beta_j}^{+\omega_j}\right]},\\ \left[\prod_{j=1}^{m}\gamma_j^{-\varpi_j},\prod_{j=1}^{m}\gamma_j^{+\varpi_j}\right]e^{i2\pi\left[\prod_{j=1}^{m}\mathfrak{W}_{\gamma_j}^{-\varpi_j},\prod_{j=1}^{m}\mathfrak{W}_{\gamma_j}^{+\varpi_j}\right]}\end{array}\right)$$

We verify that $k=m+1$ so that

$CIVTSFWAA(A_1,A_2,...,A_{m+1})=$

$$\left(\begin{array}{c}\left[\begin{array}{c}(1-\prod_{j=1}^{m}(1-\alpha_j^{-q})^{\varpi_j})^{1/q},\\(1-\prod_{j=1}^{m}(1-\alpha_j^{+q})^{\varpi_j})^{1/q},\end{array}\right]e^{i2\pi\left[\begin{array}{c}(1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j}^{-q})^{\varpi_j})^{1/q},\\(1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j})^{1/q},\end{array}\right]},\\ \left[\prod_{j=1}^{m}\beta_j^{-\varpi_j},\prod_{j=1}^{m}\beta j^{+\varpi_j}\right]e^{i2\pi\left[\prod_{j=1}^{m}\mathfrak{W}_{\beta_j}^{-\varpi_j},\prod_{j=1}^{m}\mathfrak{W}_{\beta_j}^{+\varpi_j}\right]},\\ \left[\prod_{j=1}^{m}\gamma_j^{-\varpi_j},\prod_{j=1}^{m}\gamma_j^{+\varpi_j}\right]e^{i2\pi\left[\prod_{j=1}^{m}\mathfrak{W}_{\gamma_j}^{-\varpi_j},\prod_{j=1}^{m}\mathfrak{W}_{\gamma_j}^{+\varpi_j}\right]}\end{array}\right)\oplus\varpi_{m+1}A_{m+1}$$

$$=\left(\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}((1-\prod_{j=1}^{m}(1-\alpha_j^{-q})^{\varpi_j})^{1/q})^q+\\((1-(1-\alpha_{m+1}^{-q})^{\varpi_{m+1}})^{1/q})^q-\\((1-\prod_{j=1}^{m}(1-\alpha_j^{-q})^{\varpi_j})^{1/q})^q\times\\((1-(1-\alpha_{m+1}^{-q})^{\varpi_{m+1}})^{1/q})^q\end{array}\right)^{1/q},\\ \left(\begin{array}{c}((1-\prod_{j=1}^{m}(1-\alpha_j^{+q})^{\varpi_j})^{1/q})^q+\\((1-(1-\alpha_{m+1}^{+q})^{\varpi_{m+1}})^{1/q})^q-\\((1-\prod_{j=1}^{m}(1-\alpha_j^{+q})^{\varpi_j})^{1/q})^q\times\\((1-(1-\alpha_{m+1}^{+q})^{\varpi_{m+1}})^{1/q})^q\end{array}\right)^{1/q}\end{array}\right]e^{i2\pi\left[\begin{array}{c}\left(\begin{array}{c}((1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j}^{-q})^{\varpi_j})^{1/q})^q+\\((1-(1-\mathfrak{W}_{\alpha_{m+1}}^{-q})^{\varpi_{m+1}})^{1/q})^q-\\((1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j^{-q}})^{\varpi_j})^{1/q})^q\times\\((1-(1-\mathfrak{W}_{\alpha_{m+1}}^{-q})^{\varpi_{m+1}})^{1/q})^q\end{array}\right)^{1/q},\\ \left(\begin{array}{c}((1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j})^{1/q})^q+\\((1-(1-\mathfrak{W}_{\alpha_{m+1}}^{+q})^{\varpi_{m+1}})^{1/q})^q-\\((1-\prod_{j=1}^{m}(1-\mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j})^{1/q})^q\times\\((1-(1-\mathfrak{W}_{\alpha_{m+1}}^{+q})^{\varpi_{m+1}})^{1/q})^q\end{array}\right)^{1/q}\end{array}\right]},\\ \left[\prod_{j=1}^{m}\beta_j^{-\varpi_j}\beta_{m+1}^{-\varpi_{m+1}},\prod_{j=1}^{m}\beta_j^{+\varpi_j}\beta_{m+1}^{+\varpi_{m+1}}\right]e^{i2\pi\left[\prod_{j=1}^{m}\mathfrak{W}_{\beta_j}^{-\varpi_j}\mathfrak{W}_{\beta_{m+1}}^{-\varpi_{m+1}},\prod_{j=1}^{m}\mathfrak{W}_{\beta_j}^{+\varpi_j}\mathfrak{W}_{\beta_{m+1}}^{+\varpi_{m+1}}\right]},\\ \left[\prod_{j=1}^{m}\gamma_j^{-\varpi_j}\gamma_{m+1}^{-\varpi_{m+1}},\prod_{j=1}^{m}\gamma_j^{+\varpi_j}\gamma_{m+1}^{+\varpi_{m+1}}\right]e^{i2\pi\left[\prod_{j=1}^{m}\mathfrak{W}_{\gamma_j}^{-\varpi_j}\mathfrak{W}_{\gamma_{m+1}}^{-\varpi_{m+1}},\prod_{j=1}^{m}\mathfrak{W}_{\gamma_j}^{+\varpi_j}\mathfrak{W}_{\gamma_{m+1}}^{+\varpi_{m+1}}\right]}\end{array}\right)$$

$$
= \begin{pmatrix} \begin{bmatrix} (1-\prod_{j=1}^{m+1}(1-\alpha_j^{-q})^{\varpi_j})^{1/q}, \\ (1-\prod_{j=1}^{m+1}(1-\alpha_j^{+q})^{\varpi_j})^{1/q} \end{bmatrix} e^{i2\pi\begin{bmatrix} (1-\prod_{j=1}^{m+1}(1-\mathfrak{W}_{\alpha_j}^{-q})^{\varpi_j})^{1/q}, \\ (1-\prod_{j=1}^{m+1}(1-\mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j})^{1/q} \end{bmatrix}}, \\ \left[\prod_{j=1}^{m+1}\beta_j^{-\varpi_j}, \prod_{j=1}^{m+1}\beta_j^{+\varpi_j}\right] e^{i2\pi\left[\prod_{j=1}^{m+1}\mathfrak{W}_{\beta_j}^{-\varpi_j}, \prod_{j=1}^{m+1}\mathfrak{W}_{\beta_j}^{+\varpi_j}\right]}, \\ \left[\prod_{j=1}^{m+1}\gamma_j^{-\varpi_j}, \prod_{j=1}^{m+1}\gamma_j^{+\varpi_j}\right] e^{i2\pi\left[\prod_{j=1}^{m+1}\mathfrak{W}_{\gamma_j}^{-\varpi_j}, \prod_{j=1}^{m+1}\mathfrak{W}_{\gamma_j}^{+\varpi_j}\right]} \end{pmatrix}
$$

Hence, the proof is completed.

□

**Example 4.3.** Suppose

$$
\tau_1 = \begin{pmatrix} ([0.5, 0.85]e^{i2\pi[0.7,0.9]}), \\ ([0.6, 0.77]e^{i2\pi[0.6,0.8]}), \\ ([0.7, 0.88]e^{i2\pi[0.5,0.8]}) \end{pmatrix}, \tau_2 = \begin{pmatrix} ([0.6, 0.8]e^{i2\pi[0.5,0.65]}), \\ ([0.66, 0.88]e^{i2\pi[0.7,0.87]}), \\ ([0.65, 0.85]e^{i2\pi[0.55,0.90]}) \end{pmatrix},
$$

$$
\tau_3 = \begin{pmatrix} ([0.45, 0.75]e^{i2\pi[0.6,0.8]}), \\ ([0.7, 0.85]e^{i2\pi[0.65,0.88]}), \\ ([0.75, 0.88]e^{i2\pi[0.55,0.85]}) \end{pmatrix}, \tau_4 = \begin{pmatrix} ([0.35, 0.85]e^{i2\pi[0.75,0.8]}), \\ ([0.6, 0.77]e^{i2\pi[0.66,0.89]}), \\ ([0.67, 0.88]e^{i2\pi[0.65,0.8]}) \end{pmatrix}
$$

be four CIVTSFNs and the weight vectors ($\varpi = (0.15, 0.20, 0.25, 0.40)^T$). The CIVTSFWAA operator for $q = 7$ is then found in such a way that

$$
CIVTSFWAA(\tau_1, \tau_2, \tau_3, \tau_4) = \begin{pmatrix} \begin{bmatrix} \left(1-\Pi_{j=1}^{k}(1-\alpha_j^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1-\Pi_{j=1}^{k}(1-\alpha_j^{+q})^{\varpi_j}\right)^{1/q} \end{bmatrix} e^{i2\pi\begin{bmatrix} \left(1-\Pi_{j=1}^{k}(1-\mathfrak{W}_{\alpha_j}^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1-\Pi_{j=1}^{k}(1-\mathfrak{W}_{\alpha_j}^{+q})^{\varpi_j}\right)^{1/q} \end{bmatrix}}, \\ \left[\Pi_{j=1}^{k}\beta_j^{-\varpi_j}, \Pi_{j=1}^{k}\beta_j^{+\varpi_j}\right] e^{i2\pi\left[\Pi_{j=1}^{k}\mathfrak{W}_{\beta_j}^{-\varpi_j}, \Pi_{j=1}^{k}\mathfrak{W}_{\beta_j}^{+\varpi_j}\right]}, \\ \left[\Pi_{j=1}^{k}\gamma_j^{-\varpi_j}, \Pi_{j=1}^{k}\gamma_j^{+\varpi_j}\right] e^{i2\pi\left[\Pi_{j=1}^{k}\mathfrak{W}_{\gamma_j}^{-\varpi_j}, \Pi_{j=1}^{k}\mathfrak{W}_{\gamma_j}^{+\varpi_j}\right]} \end{pmatrix}
$$

$$CIVTSFWAA(\tau_1,\tau_2,\tau_3,\tau_4) = (\{[0.502,0.823]e^{i2\pi[0.692,0.810]}, [0.636,0.811]e^{i2\pi[0.648,0.871]}, [0.690,0.874]e^{i2\pi[0.832,0.668]}\}).$$

**Theorem 4.4.** (Idempotent) Let
$A = A_j = \{[\alpha_j^-,\alpha_j^+]e^{i2\pi[\mathfrak{W}_{\alpha_j}^-,\mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-,\beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-,\mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-,\gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-,\mathfrak{W}_{\gamma_j}^+]}\}$,
for $j = 1,2,...,k$. Then,

$$CIVTSFWAA(A_1,A_2,...,A_k) = A$$

**Theorem 4.5.** (Monotonicity) Let
$A_j = \{[\alpha_j^-,\alpha_j^+]e^{i2\pi[\mathfrak{W}_{\alpha_j}^-,\mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-,\beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-,\mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-,\gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-,\mathfrak{W}_{\gamma_j}^+]}\}$
and $\acute{A}_j = \{[\acute{\alpha}_j^-,\acute{\alpha}_j^+]e^{i2\pi[\acute{\mathfrak{W}}_{\alpha_j}^-,\acute{\mathfrak{W}}_{\alpha_j}^+]}, [\acute{\beta}_j^-,\acute{\beta}_j^+]e^{i2\pi[\acute{\mathfrak{W}}_{\beta_j}^-,\acute{\mathfrak{W}}_{\beta_j}^+]}, [\acute{\gamma}_j^-,\acute{\gamma}_j^+]e^{i2\pi[\acute{\mathfrak{W}}_{\gamma_j}^-,\acute{\mathfrak{W}}_{\gamma_j}^+]}\}$,
$j = 1,2,...,k.$ If $\alpha_j^+ \geq \acute{\alpha}_j^+, \beta_j^+ \leq \acute{\beta}_j^+, \gamma_j^+ \leq \acute{\gamma}_j^+, \mathfrak{W}_{\alpha_j}^+ \geq \acute{\mathfrak{W}}_{\alpha_j}^+, \mathfrak{W}_{\beta_j}^+ \leq \acute{\mathfrak{W}}_{\beta_j}^+$ and $\mathfrak{W}_{\gamma_j}^+ \leq \acute{\mathfrak{W}}_{\gamma_j}^+$. Then,

$$CIVTSFWAA(A_1,A_2,...,A_k) \geq CIVTSFWAA(\acute{A}_1,\acute{A}_2,...,\acute{A}_k)$$

**Theorem 4.6.** (Boundedness) Let consider the collection of CIVT-SFNs

$A_j = \{[\alpha_j^-,\alpha_j^+]e^{i2\pi[\mathfrak{W}_{\alpha_j}^-,\mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-,\beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-,\mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-,\gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-,\mathfrak{W}_{\gamma_j}^+]}\}, j = 1,2,...,k,$
if

$$A_j^+ = \begin{pmatrix} \left[\max_{1\leq j\leq k}\alpha_j^-, \max_{1\leq j\leq k}\alpha_j^+\right] e^{i2\pi\left[\max_{1\leq j\leq k}\mathfrak{W}_{\alpha_j}^-, \max_{1\leq j\leq k}\mathfrak{W}_{\alpha_j}^+\right]}, \\ \left[\min_{1\leq j\leq k}\beta_j^-, \min_{1\leq j\leq k}\beta_j^+\right] e^{i2\pi\left[\min_{1\leq j\leq k}\mathfrak{W}_{\beta_j}^-, \min_{1\leq j\leq k}\mathfrak{W}_{\beta_j}^+\right]}, \\ \left[\min_{1\leq j\leq k}\gamma_j^-, \min_{1\leq j\leq k}\gamma_j^+\right] e^{i2\pi\left[\min_{1\leq j\leq k}\mathfrak{W}_{\gamma_j}^-, \min_{1\leq j\leq k}\mathfrak{W}_{\gamma_j}^+\right]} \end{pmatrix}$$

$$A_j^- = \begin{pmatrix} \left[\min_{1\le j\le k}\alpha_j^-, \min_{1\le j\le k}\alpha_j^+\right] e^{i2\pi\left[\min_{1\le j\le k}\mathfrak{W}_{\alpha_j}^-, \min_{1\le j\le k}\mathfrak{W}_{\alpha_j}^+\right]}, \\ \left[\min_{1\le j\le k}\beta_j^-, \min_{1\le j\le k}\beta_j^+\right] e^{i2\pi\left[\min_{1\le j\le k}\mathfrak{W}_{\beta_j}^-, \min_{1\le j\le k}\mathfrak{W}_{\beta_j}^+\right]}, \\ \left[\max_{1\le j\le k}\gamma_j^-, \max_{1\le j\le k}\gamma_j^+\right] e^{i2\pi\left[\max_{1\le j\le k}\mathfrak{W}_{\gamma_j}^-, \max_{1\le j\le k}\mathfrak{W}_{\gamma_j}^+\right]} \end{pmatrix}$$

then, $A_j^- \le CIVTSFWAA(A_1, A_2, ..., A_k) \le A_j^+$.

**Definition 4.7.** The following mapping represents the complex interval-valued T-spherical fuzzy weighted geometric aggregation (CIVTSFWGA) operator: $CIVTSFWGA : Z^k \to Z$, and is defined as follows:

$$CIVTSFWGA(A_1, A_2, ..., A_k) = A_1^{\varpi_1} \bigotimes A_2^{\varpi_2} \bigotimes ... \bigotimes A_k^{\varpi_k} = \bigotimes_{j=1}^{k} A_j^{\varpi_j}$$

The following theorem is obtained using the operating laws of the CIVTSFNs.

**Theorem 4.8.** Let's think about Definition 4.7. Then we get

$$CIVTSFWGA(A_1, A_2, ..., A_k)$$
$$= \begin{pmatrix} \left[\Pi_{j=1}^k \alpha_j^{-\varpi_j}, \Pi_{j=1}^k \alpha_j^{+\varpi_j}\right] e^{i2\pi\left[\Pi_{j=1}^k \mathfrak{W}_{\alpha_j}^{-\varpi_j}, \Pi_{j=1}^k \mathfrak{W}_{\alpha_j}^{+\varpi_j}\right]}, \\ \left[\Pi_{j=1}^k \beta_j^{-\varpi_j}, \Pi_{j=1}^k \beta_j^{+\varpi_j}\right] e^{i2\pi\left[\Pi_{j=1}^k \mathfrak{W}_{\beta_j}^{-\varpi_j}, \Pi_{j=1}^k \mathfrak{W}_{\beta_j}^{+\varpi_j}\right]}, \\ \left[\begin{matrix} \left(1-\Pi_{j=1}^k (1-\gamma_j^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1-\Pi_{j=1}^k (1-\gamma_j^{+q})^{\varpi_j}\right)^{1/q} \end{matrix}\right] e^{i2\pi\left[\begin{matrix} \left(1-\Pi_{j=1}^k (1-\mathfrak{W}_{\gamma_j}^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1-\Pi_{j=1}^k (1-\mathfrak{W}_{\gamma_j}^{+q})^{\varpi_j}\right)^{1/q} \end{matrix}\right]} \end{pmatrix}$$

**Example 4.9.** Suppose

$$\tau_1 = \begin{pmatrix} ([0.5, 0.85]e^{i2\pi[0.7,0.9]}), \\ ([0.6, 0.77]e^{i2\pi[0.6,0.8]}), \\ ([0.7, 0.88]e^{i2\pi[0.5,0.8]}) \end{pmatrix}, \tau_2 = \begin{pmatrix} ([0.6, 0.8]e^{i2\pi[0.5,0.65]}), \\ ([0.66, 0.88]e^{i2\pi[0.7,0.87]}), \\ ([0.65, 0.85]e^{i2\pi[0.55,0.90]}) \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} ([0.45, 0.75]e^{i2\pi[0.6,0.8]}), \\ ([0.7, 0.85]e^{i2\pi[0.65,0.88]}), \\ ([0.75, 0.88]e^{i2\pi[0.55,0.85]}) \end{pmatrix}, \tau_4 = \begin{pmatrix} ([0.35, 0.85]e^{i2\pi[0.75,0.8]}), \\ ([0.6, 0.77]e^{i2\pi[0.66,0.89]}), \\ ([0.67, 0.88]e^{i2\pi[0.65,0.8]}) \end{pmatrix}$$

be four CIVT-SFNs and the weight vectors ($\varpi = (0.15, 0.20, 0.25, 0.40)^T$). The CIVTSFWGA operator for $q = 7$ is thus determined in such a way that

$$CIVTSFWGA(A_1, A_2, ..., A_k)$$

$$= \begin{pmatrix} \left[\Pi_{j=1}^{k}\alpha_j^{-\varpi_j}, \Pi_{j=1}^{k}\alpha_j^{+\varpi_j}\right] e^{i2\pi\left[\Pi_{j=1}^{k}\mathfrak{W}_{\alpha_j}^{-\varpi_j}, \Pi_{j=1}^{k}\mathfrak{W}_{\alpha_j}^{+\varpi_j}\right]}, \\ \left[\Pi_{j=1}^{k}\beta_j^{-\varpi_j}, \Pi_{j=1}^{k}\beta_j^{+\varpi_j}\right] e^{i2\pi\left[\Pi_{j=1}^{k}\mathfrak{W}_{\beta_j}^{-\varpi_j}, \Pi_{j=1}^{k}\mathfrak{W}_{\beta_j}^{+\varpi_j}\right]}, \\ \left[\begin{array}{c} \left(1 - \Pi_{j=1}^{k}(1 - \gamma_j^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1 - \Pi_{j=1}^{k}(1 - \gamma_j^{+q})^{\varpi_j}\right)^{1/q} \end{array}\right] e^{i2\pi\left[\begin{array}{c} \left(1 - \Pi_{j=1}^{k}(1 - \mathfrak{W}_{\gamma_j}^{-q})^{\varpi_j}\right)^{1/q}, \\ \left(1 - \Pi_{j=1}^{k}(1 - \mathfrak{W}_{\gamma_j}^{+q})^{\varpi_j}\right)^{1/q} \end{array}\right]} \end{pmatrix}$$

$$\begin{aligned} CIVTSFWGA(\tau_1, \tau_2, \tau_3, \tau_4) \quad = \quad & (\{[0.438, 0.814]e^{i2\pi[0.647,0.781]}, \\ & [0.636, 0.811]e^{i2\pi[0.648,0.871]}, \\ & [0.698, 0.875]e^{i2\pi[0.599,0.842]}\}). \end{aligned}$$

**Theorem 4.10.** (Idempotent) Let CIVT-SFNs $A = A_j = \{[\alpha_j^-, \alpha_j^+]e^{i2\pi[\mathfrak{W}_{\alpha_j}^-, \mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-, \beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-, \mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-, \gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-, \mathfrak{W}_{\gamma_j}^+]}\}$ , $j = 1, 2, ..., k$. Then,

$$CIVTSFWGA(A_1, A_2, ..., A_k) = A$$

**Theorem 4.11.** (Monotonicity) Let
$A_j = \{[\alpha_j^-, \alpha_j^+]e^{i2\pi[\mathfrak{W}_{\alpha_j}^-, \mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-, \beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-, \mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-, \gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-, \mathfrak{W}_{\gamma_j}^+]}\}$
and $\acute{A}_j = \{[\acute{\alpha}_j^-, \acute{\alpha}_j^+]e^{i2\pi[\acute{\mathfrak{W}}_{\alpha_j}^-, \acute{\mathfrak{W}}_{\alpha_j}^+]}, [\acute{\beta}_j^-, \acute{\beta}_j^+]e^{i2\pi[\acute{\mathfrak{W}}_{\beta_j}^-, \acute{\mathfrak{W}}_{\beta_j}^+]}, [\acute{\gamma}_j^-, \acute{\gamma}_j^+]e^{i2\pi[\acute{\mathfrak{W}}_{\gamma_j}^-, \acute{\mathfrak{W}}_{\gamma_j}^+]}\}$,
$j = 1, 2, ..., k$ be CIVT-SFNs. If $\alpha_j^+ \geq \acute{\alpha}_j^+, \beta_j^+ \leq \acute{\beta}_j^+, \gamma_j^+ \leq \acute{\gamma}_j^+, \mathfrak{W}_{\alpha_j}^+ \geq \acute{\mathfrak{W}}_{\alpha_j}^+, \mathfrak{W}_{\beta_j}^+ \leq \acute{\mathfrak{W}}_{\beta_j}^+$ and $\mathfrak{W}_{\gamma_j}^+ \leq \acute{\mathfrak{W}}_{\gamma_j}^+$, then

$$CIVTSFWGA(A_1, A_2, ..., A_k) \geq CIVTSFWGA(\acute{A}_1, \acute{A}_2, ..., \acute{A}_k)$$

**Theorem 4.12.** (Boundedness) Let us consider CIVT-SFNs

$A_j = \{[\alpha_j^-, \alpha_j^+]e^{i2\pi[\mathfrak{W}_{\alpha_j}^-, \mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-, \beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-, \mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-, \gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-, \mathfrak{W}_{\gamma_j}^+]}\}, j = 1, 2, ..., k$.
If

$$A_j^+ = \begin{pmatrix} \left[\max_{1\leq j\leq k} \alpha_j^-, \max_{1\leq j\leq k} \alpha_j^+\right] e^{i2\pi\left[\max_{1\leq j\leq k} \mathfrak{W}_{\alpha_j}^-, \max_{1\leq j\leq k} \mathfrak{W}_{\alpha_j}^+\right]}, \\ \left[\min_{1\leq j\leq k} \beta_j^-, \min_{1\leq j\leq k} \beta_j^+\right] e^{i2\pi\left[\min_{1\leq j\leq k} \mathfrak{W}_{\beta_j}^-, \min_{1\leq j\leq k} \mathfrak{W}_{\beta_j}^+\right]}, \\ \left[\min_{1\leq j\leq k} \gamma_j^-, \min_{1\leq j\leq k} \gamma_j^+\right] e^{i2\pi\left[\min_{1\leq j\leq k} \mathfrak{W}_{\gamma_j}^-, \min_{1\leq j\leq k} \mathfrak{W}_{\gamma_j}^+\right]} \end{pmatrix}$$

$$A_j^- = \begin{pmatrix} \left[\min_{1\leq j\leq k} \alpha_j^-, \min_{1\leq j\leq k} \alpha_j^+\right] e^{i2\pi\left[\min_{1\leq j\leq k} \mathfrak{W}_{\alpha_j}^-, \min_{1\leq j\leq k} \mathfrak{W}_{\alpha_j}^+\right]}, \\ \left[\min_{1\leq j\leq k} \beta_j^-, \min_{1\leq j\leq k} \beta_j^+\right] e^{i2\pi\left[\min_{1\leq j\leq k} \mathfrak{W}_{\beta_j}^-, \min_{1\leq j\leq k} \mathfrak{W}_{\beta_j}^+\right]}, \\ \left[\max_{1\leq j\leq k} \gamma_j^-, \max_{1\leq j\leq k} \gamma_j^+\right] e^{i2\pi\left[\max_{1\leq j\leq k} \mathfrak{W}_{\gamma_j}^-, \max_{1\leq j\leq k} \mathfrak{W}_{\gamma_j}^+\right]} \end{pmatrix}$$

then, $A_j^- \leq CIVTSFWGA(A_1, A_2, ..., A_k) \leq A_j^+$.

## 5. MULTI-ATTRIBUTE DECISION-MAKING UNDER CIVTSF VALUES

### 5.1 Method Based on Aggregation Operators

In this section we propose an MADM method based on the CIVTSFWAA operator and CIVTSFWGA operator. In the problem, $A = \{A_1, A_2, ..., A_j\}$ and $B = \{B_1, B_2, ..., B_k\}$ use weight vectors to represent the j alternative and k attributes $\varpi = (\varpi_1, \varpi_2, ..., \varpi_k)$ , $\sum_{i=1}^{k} \varpi_i = 1$, $\varpi_i \in [0,1]$. A decision-matrix $D = [d_{ij}]_{m\times n}$ whose each of entries is denoted by $d_{ij} = \{[\alpha_j^-, \alpha_j^+]e^{i2\pi[w_{\alpha_j}^-, \mathfrak{W}_{\alpha_j}^+]}, [\beta_j^-, \beta_j^+]e^{i2\pi[\mathfrak{W}_{\beta_j}^-, \mathfrak{W}_{\beta_j}^+]}, [\gamma_j^-, \gamma_j^+]e^{i2\pi[\mathfrak{W}_{\gamma_j}^-, \mathfrak{W}_{\gamma_j}^+]}\}$.
$d_{ij}$ indicates the value at which an alternative $(A_i)$ was evaluated for attributes $B_j$. With the aid of the CIVTSFWAA and CIVTSFWGA operators, we can order the choices; the stages are as follows:
**Step 1:** Normalize the decision matrix
Benefits and cost are two different sorts of qualities that are considered in this procedure. Using the following formula, we need to convert costs into benefits:

$$d_{ij} = \begin{cases} ([r_{ij}^-, r_{ij}^+], [s_{ij}^-, s_{ij}^+], [t_{ij}^-, t_{ij}^-]) & \text{for benefit attribute} \\ ([t_{ij}^-, t_{ij}^+], [s_{ij}^-, s_{ij}^+], [r_{ij}^-, r_{ij}^-]) & \text{for cost attribute} \end{cases}$$

**Step 2:** Find the aggregated values for all attribute values $d_{ij}$ by using the CIVTSFWAA operator

$$D_i = CIVTSFWAA(d_{i1}, d_{i2}, d_{i3}, ..., d_{ik})$$

Similarly for CIVTSFWGA such that

$$D_i = CIVTSFWGA(d_{i1}, d_{i2}, d_{i3}, ..., d_{ik})$$

**Step 3:** Rank $\mathscr{D}_i$ according to score function and accuracy function given in Definition 3.7

**Step 4:** The option with the highest score value, $\mathscr{D}_i$, is picked as the best one.

**Example 5.1.** A purchasing department of a hospital wants to buy medical tools. For this, there are five alternatives $A_1, A_2, A_3, A_4$ and $A_5$ are five attributes $B_1, B_2, B_3, B_4$ and $B_5$ to evaluate the alternatives. The attributes are follows as:

$B_1$: price.
$B_2$: Growth condition.
$B_3$: technical specifications.
$B_4$: warranty.
$B_5$: maintenance.

The weight vector of the attributes is $\varpi = (0.30, 0.24, 0.11, 0.17, 0.18)$ of attributes. Purchasing department evaluates the alternatives according the attributes and they obtain decision matrix as shown in Table 5.1.

**Table 5.1** Input Decision-matrix

| | B1 | B2 | B3 | B4 | B5 |
|---|---|---|---|---|---|
| $A_1$ | $([0.5,0.85]e^{i2\pi[0.7,0.9]})$, $([0.6,0.77]e^{i2\pi[0.6,0.8]})$, $([0.7,0.88]e^{i2\pi[0.5,0.8]})$ | $([0.45,0.75]e^{i2\pi[0.6,0.8]})$, $([0.7,0.85]e^{i2\pi[0.65,0.88]})$, $([0.75,0.88]e^{i2\pi[0.55,0.85]})$ | $([0.6,0.8]e^{i2\pi[0.5,0.65]})$, $([0.66,0.88]e^{i2\pi[0.7,0.87]})$, $([0.65,0.85]e^{i2\pi[0.55,0.90]})$ | $([0.35,0.85]e^{i2\pi[0.75,0.8]})$, $([0.6,0.77]e^{i2\pi[0.66,0.89]})$, $([0.67,0.88]e^{i2\pi[0.65,0.8]})$ | $([0.55,0.77]e^{i2\pi[0.6,0.71]})$, $([0.61,0.8]e^{i2\pi[0.5,0.79]})$, $([0.67,0.88]e^{i2\pi[0.4,0.9]})$ |
| $A_2$ | $([0.6,0.75]e^{i2\pi[0.6,0.8]})$, $([0.6,0.8]e^{i2\pi[0.45,0.83]})$, $([0.8,0.9]e^{i2\pi[0.55,0.89]})$ | $([0.51,0.81]e^{i2\pi[0.61,0.69]})$, $([0.61,0.86]e^{i2\pi[0.72,0.73]})$, $([0.71,0.88]e^{i2\pi[0.52,0.96]})$ | $([0.53,0.68]e^{i2\pi[0.67,0.9]})$, $([0.62,0.82]e^{i2\pi[0.56,0.7]})$, $([0.71,0.94]e^{i2\pi[0.45,0.8]})$ | $([0.55,0.85]e^{i2\pi[0.47,0.93]})$, $([0.62,0.77]e^{i2\pi[0.46,0.8]})$, $([0.71,0.88]e^{i2\pi[0.45,0.69]})$ | $([0.45,0.88]e^{i2\pi[0.57,0.79]})$, $([0.46,0.7]e^{i2\pi[0.56,0.9]})$, $([0.57,0.88]e^{i2\pi[0.55,0.8]})$ |
| $A_3$ | $([0.54,0.83]e^{i2\pi[0.7,0.85]})$, $([0.64,0.82]e^{i2\pi[0.6,0.85]})$, $([0.7,0.85]e^{i2\pi[0.5,0.85]})$ | $([0.35,0.9]e^{i2\pi[0.77,0.84]})$, $([0.46,0.8]e^{i2\pi[0.66,0.87]})$, $([0.57,0.8]e^{i2\pi[0.55,0.8]})$ | $([0.35,0.78]e^{i2\pi[0.73,0.86]})$, $([0.36,0.92]e^{i2\pi[0.63,0.86]})$, $([0.37,0.81]e^{i2\pi[0.53,0.84]})$ | $([0.45,0.87]e^{i2\pi[0.57,0.81]})$, $([0.56,0.82]e^{i2\pi[0.56,0.76]})$, $([0.67,0.79]e^{i2\pi[0.55,0.91]})$ | $([0.45,0.96]e^{i2\pi[0.7,0.64]})$, $([0.64,0.6]e^{i2\pi[0.7,0.75]})$, $([0.77,0.8]e^{i2\pi[0.5,0.9]})$ |
| $A_4$ | $([0.6,0.7]e^{i2\pi[0.5,0.67]})$, $([0.6,0.79]e^{i2\pi[0.5,0.91]})$, $([0.7,0.88]e^{i2\pi[0.5,0.86]})$ | $([0.51,0.82]e^{i2\pi[0.72,0.74]})$, $([0.6,0.66]e^{i2\pi[0.6,0.9]})$, $([0.72,0.9]e^{i2\pi[0.51,0.68]})$ | $([0.44,0.72]e^{i2\pi[0.67,0.94]})$, $([0.55,0.96]e^{i2\pi[0.66,0.7]})$, $([0.66,0.68]e^{i2\pi[0.45,0.96]})$ | $([0.6,0.91]e^{i2\pi[0.47,0.84]})$, $([0.6,0.81]e^{i2\pi[0.6,0.84]})$, $([0.7,0.73]e^{i2\pi[0.45,0.85]})$ | $([0.35,0.7]e^{i2\pi[0.7,0.8]})$, $([0.63,0.7]e^{i2\pi[0.6,0.8]})$, $([0.5,0.7]e^{i2\pi[0.5,0.8]})$ |
| $A_5$ | $([0.45,0.79]e^{i2\pi[0.71,0.82]})$, $([0.64,0.89]e^{i2\pi[0.61,0.83]})$, $([0.7,0.8]e^{i2\pi[0.5,0.84]})$ | $([0.53,0.87]e^{i2\pi[0.72,0.8]})$, $([0.63,0.85]e^{i2\pi[0.62,0.7]})$, $([0.7,0.83]e^{i2\pi[0.5,0.9]})$ | $([0.5,0.69]e^{i2\pi[0.77,0.94]})$, $([0.66,0.79]e^{i2\pi[0.6,0.8]})$, $([0.7,0.89]e^{i2\pi[0.5,0.7]})$ | $([0.5,0.89]e^{i2\pi[0.7,0.77]})$, $([0.66,0.84]e^{i2\pi[0.6,0.81]})$, $([0.7,0.79]e^{i2\pi[0.55,0.85]})$ | $([0.5,0.8]e^{i2\pi[0.7,0.8]})$, $([0.6,0.7]e^{i2\pi[0.6,0.8]})$, $([0.7,0.88]e^{i2\pi[0.5,0.8]})$ |

**Step 1:** Given that $B_1$ is a cost type, we normalize Table's values 5.1 and the results are given in Table 5.2.

**Table 5.2** Normalized decision matrix

| | B1 | B2 | B3 | B4 | B5 |
|---|---|---|---|---|---|
| $A_1$ | $([0.7,0.88]e^{i2\pi[0.5,0.8]})$, $([0.6,0.77]e^{i2\pi[0.6,0.8]})$, $([0.5,0.85]e^{i2\pi[0.7,0.9]})$ | $([0.45,0.75]e^{i2\pi[0.6,0.8]})$, $([0.7,0.85]e^{i2\pi[0.65,0.88]})$, $([0.75,0.88]e^{i2\pi[0.55,0.85]})$ | $([0.6,0.8]e^{i2\pi[0.5,0.65]})$, $([0.66,0.88]e^{i2\pi[0.7,0.87]})$, $([0.65,0.85]e^{i2\pi[0.55,0.90]})$ | $([0.35,0.85]e^{i2\pi[0.75,0.8]})$, $([0.6,0.77]e^{i2\pi[0.66,0.89]})$, $([0.67,0.88]e^{i2\pi[0.65,0.8]})$ | $([0.55,0.77]e^{i2\pi[0.6,0.71]})$, $([0.61,0.8]e^{i2\pi[0.5,0.79]})$, $([0.67,0.88]e^{i2\pi[0.4,0.9]})$ |
| $A_2$ | $([0.8,0.75]e^{i2\pi[0.55,0.8]})$, $([0.6,0.8]e^{i2\pi[0.45,0.83]})$, $([0.6,0.9]e^{i2\pi[0.6,0.89]})$ | $([0.51,0.81]e^{i2\pi[0.61,0.69]})$, $([0.61,0.86]e^{i2\pi[0.72,0.73]})$, $([0.71,0.88]e^{i2\pi[0.52,0.96]})$ | $([0.53,0.68]e^{i2\pi[0.67,0.9]})$, $([0.62,0.82]e^{i2\pi[0.56,0.7]})$, $([0.71,0.94]e^{i2\pi[0.45,0.8]})$ | $([0.55,0.85]e^{i2\pi[0.47,0.93]})$, $([0.62,0.77]e^{i2\pi[0.46,0.8]})$, $([0.71,0.88]e^{i2\pi[0.45,0.69]})$ | $([0.45,0.88]e^{i2\pi[0.57,0.79]})$, $([0.46,0.7]e^{i2\pi[0.56,0.9]})$, $([0.57,0.88]e^{i2\pi[0.55,0.8]})$ |
| $A_3$ | $([0.7,0.85]e^{i2\pi[0.5,0.85]})$, $([0.64,0.82]e^{i2\pi[0.6,0.85]})$, $([0.54,0.83]e^{i2\pi[0.7,0.85]})$ | $([0.35,0.9]e^{i2\pi[0.77,0.84]})$, $([0.46,0.8]e^{i2\pi[0.66,0.87]})$, $([0.57,0.8]e^{i2\pi[0.55,0.8]})$ | $([0.35,0.78]e^{i2\pi[0.73,0.86]})$, $([0.36,0.92]e^{i2\pi[0.63,0.86]})$, $([0.37,0.81]e^{i2\pi[0.53,0.84]})$ | $([0.45,0.87]e^{i2\pi[0.57,0.81]})$, $([0.56,0.82]e^{i2\pi[0.56,0.76]})$, $([0.67,0.79]e^{i2\pi[0.55,0.91]})$ | $([0.45,0.96]e^{i2\pi[0.7,0.64]})$, $([0.64,0.6]e^{i2\pi[0.7,0.75]})$, $([0.77,0.8]e^{i2\pi[0.5,0.9]})$ |
| $A_4$ | $([0.7,0.88]e^{i2\pi[0.5,0.86]})$, $([0.6,0.79]e^{i2\pi[0.5,0.91]})$, $([0.6,0.7]e^{i2\pi[0.5,0.76]})$ | $([0.51,0.82]e^{i2\pi[0.72,0.74]})$, $([0.6,0.66]e^{i2\pi[0.6,0.9]})$, $([0.72,0.9]e^{i2\pi[0.51,0.68]})$ | $([0.44,0.72]e^{i2\pi[0.67,0.94]})$, $([0.55,0.96]e^{i2\pi[0.66,0.7]})$, $([0.66,0.68]e^{i2\pi[0.45,0.96]})$ | $([0.6,0.91]e^{i2\pi[0.47,0.84]})$, $([0.6,0.81]e^{i2\pi[0.6,0.84]})$, $([0.7,0.73]e^{i2\pi[0.45,0.85]})$ | $([0.35,0.7]e^{i2\pi[0.7,0.8]})$, $([0.63,0.7]e^{i2\pi[0.6,0.8]})$, $([0.5,0.7]e^{i2\pi[0.5,0.8]})$ |
| $A_5$ | $([0.7,0.8]e^{i2\pi[0.5,0.84]})$, $([0.64,0.89]e^{i2\pi[0.61,0.83]})$, $([0.45,0.79]e^{i2\pi[0.71,0.82]})$ | $([0.53,0.87]e^{i2\pi[0.72,0.8]})$, $([0.63,0.85]e^{i2\pi[0.62,0.7]})$, $([0.7,0.83]e^{i2\pi[0.5,0.9]})$ | $([0.5,0.69]e^{i2\pi[0.77,0.94]})$, $([0.66,0.79]e^{i2\pi[0.6,0.8]})$, $([0.7,0.89]e^{i2\pi[0.5,0.7]})$ | $([0.5,0.89]e^{i2\pi[0.7,0.77]})$, $([0.66,0.84]e^{i2\pi[0.6,0.81]})$, $([0.7,0.79]e^{i2\pi[0.55,0.85]})$ | $([0.5,0.8]e^{i2\pi[0.7,0.8]})$, $([0.6,0.7]e^{i2\pi[0.6,0.8]})$, $([0.7,0.88]e^{i2\pi[0.5,0.8]})$ |

**Step 2:** Aggregated values for all attribute values $d_{ik}$ by using CIVTSFWAA operator are obtained as follows: Here we take $q = 7$.

$$\mathscr{D}_1 = \left([0.612, 0.830]e^{i2\pi[0.628,0.778]}, [0.631, 0.806]e^{i2\pi[0.612,0.839]}, [0.628, 0.868]e^{i2\pi[0.574,0.870]}\right)$$
$$\mathscr{D}_2 = \left([0.690, 0.815]e^{i2\pi[0.584,0.842]}, [0.580, 0.792]e^{i2\pi[0.539,0.796]}, [0.649, 0.892]e^{i2\pi[0.527,0.842]}\right)$$
$$\mathscr{D}_3 = \left([0.597, 0.896]e^{i2\pi[0.688,0.824]}, [0.543, 0.780]e^{i2\pi[0.627,0.821]}, [0.580, 0.808]e^{i2\pi[0.579,0.855]}\right)$$
$$\mathscr{D}_4 = \left([0.613, 0.844]e^{2\pi[0.651,0.847]}, [0.600, 0.733]e^{i2\pi[0.574,0.850]}, [0.629, 0.746]e^{i2\pi[0.488,0.752]}\right)$$
$$\mathscr{D}_5 = \left([0.610, 0.836]e^{i2\pi[0.690,0.840]}, [0.636, 0.824]e^{i2\pi[0.608,0.785]}, [0.613, 0.826]e^{i2\pi[0.565,0.825]}\right)$$

**Step 3:** Score value $S(\mathscr{D}_i)$ of $\mathscr{D}_i$ $(i = 1, 2, 3, 4, 5)$ are obtained as follows:

$$S(\mathscr{D}_1) = 0.6989, S(\mathscr{D}_2) = 0.6821, S(\mathscr{D}_3) = 0.6912, S(\mathscr{D}_4) = 0.7741, S(\mathscr{D}_5) = 0.7497.$$

$$\mathscr{D}_2 \leq \mathscr{D}_3 \leq \mathscr{D}_1 \leq \mathscr{D}_5 \leq \mathscr{D}_4$$

**Step 4:** When all alternatives are ranked, we see that the best alternative is $A_4$.

Following is a summary of the computational stages if we apply the CIVTSFWGA operator to the previously mentioned approach.

**Step 1:** Table 5.2 contains the normalized decision matrix.

**Step 2:** Values for all attribute values $d_{ij}$ were aggregated using the CIVTSFWGA operator's formula for $q = 7$.

$$\mathscr{D}_1 = \left([0.526, 0.813]e^{i2\pi[0.578,0.765]}, [0.631, 0.806]e^{i2\pi[0.612,0.839]}, [0.672, 0.869]e^{i2\pi[0.630,0.878]}\right)$$
$$\mathscr{D}_2 = \left([0.581, 0.795]e^{i2\pi[0.565,0.801]}, [0.580, 0.792]e^{i2\pi[0.539,0.796]}, [0.669, 0.896]e^{i2\pi[0.547,0.891]}\right)$$
$$\mathscr{D}_3 = \left([0.471, 0.875]e^{i2\pi[0.628,0.800]}, [0.543, 0.780]e^{i2\pi[0.627,0.821]}, [0.651, 0.810]e^{i2\pi[0.617,0.865]}\right)$$
$$\mathscr{D}_4 = \left([0.530, 0.815]e^{i2\pi[0.593,0.842]}, [0.600, 0.733]e^{i2\pi[0.574,0.850]}, [0.660, 0.795]e^{i2\pi[0.492,0.823]}\right)$$
$$\mathscr{D}_5 = \left([0.561, 0.818]e^{i2\pi[0.644,0.821]}, [0.636, 0.824]e^{i2\pi[0.608,0.785]}, [0.668, 0.836]e^{i2\pi[0.619,0.843]}\right)$$

**Step 3:** The Definition of Score function can be utilized to score values of all alternatives are abstained as follows: .

$S(\mathscr{D}_1) = 0.6306, S(\mathscr{D}_2) = 0.6111, S(\mathscr{D}_3) = 0.6218, S(\mathscr{D}_4) = 0.6867, S(\mathscr{D}_5) = 0.6826.$

$$\mathscr{D}_2 \leq \mathscr{D}_3 \leq \mathscr{D}_1 \leq \mathscr{D}_5 \leq \mathscr{D}_4$$

**Step 4:** When all alternatives are ranked, we see that the best alternative is $A_4$.
Clearly, the outcomes from the two separate notions were the same.

# 6. CONCLUSIONS AND RECOMMENDATION

In realistic decision theory, MADM is an effective method for dealing with difficult and unreliable information. The benefits of complex T-spherical fuzzy environments, which allow degrees of truth, neutrality, and falsehood to take the shape of intervals, and interval-valued T-spherical fuzzy environments, which can express three dimensions of information in a single set, serve as the motivation for this study. In this thesis, we defined arithmetical operations of CIVTSFNs and score and accuracy functions of them. We also obtained some properties of CIVTSFNs. In addition, we presented and specified some of the properties of the arithmetic and geometric aggregation operators. We also created the MADM approach for the CIVTSFS environment using the suggested aggregation operators and score functions. An example is provided to illustrate the proposed MADM method's workflow. The application's results show that the same alternative is the best component for both of the two aggregation operators. Therefore our method is consistent each other. We think that this thesis is a reference for the researchers to study. Also, this thesis is original and it contributes the literature.

## REFERENCES


Akram, M., Bashir, A., Garg, H. 2020. Decision-making model under complex picture fuzzy Hamacher aggregation operators, Computational and Applied Mathematics, 39(3): 1-38.

Akram, M., Khan, A. and Karaaslan, F. 2021. Complex spherical dombi fuzzy aggregation operators. Journal of Multiple Valued Logic and Soft Computing., 37(5-6): 503-531.

Ali, Z., Mahmood, T., and Yang, M. S. 2020. Complex T-spherical fuzzy aggregation operators with application to multi-attribute decision making. Symmetry, 12(8): 1311.

Alkouri, A.M.J.S. and Salleh, A.R. 2012, Complex intuitionistic fuzzy sets. AIP conference proceedings, 1482(1): 464–470.

Alothaim, A., Hussain, S. and Al-Hadhrami, S. 2022, Analysis of cybersecurities within industrial control systems using interval-valued complex spherical fuzzy information, Computational Intelligence and Neuroscience., 2022: 3304333

Atanassov, K. T. 1986. Intuitionistic fuzzy sets. Fuzzy Sets Syst., 20: 87-96.

Cao, G. 2020. A Multi-criteria Picture Fuzzy Decision-making Model for Green Supplier Selection based on Fractional Programming, International Journal of Computers, Communications and Control, 15(1): 1-14.

Cuong, B. 2013a. Picture fuzzy sets-first results, Part 1 Seminar Neuro-Fuzzy Systems with Applications, Preprint 03/2013, Institute of Mathematics, Hanoi, May 2013.

Cuong, B. 2013b. Picture fuzzy sets-first results, Part 2 Seminar Neuro-Fuzzy Systems with Applications, Preprint 04/2013, Institute of Mathematics, Hanoi, June .

Coung, B.C. 2014. Picture fuzzy sets. J. Comput. Sci. Cybern, 30: 409–420

Duleba, S., Gundogdu, F.K and Moslem, S. 2021. Interval-valued spherical fuzzy analytic hierarchy process method to evaluate public transportation development. Informatica, 32(4): 661-686.

Ganie, A.H. Singh, S., Bhatia, P.K. 2020. Some new correlation coefficients of picture fuzzy sets with applications, Neural Computing and Applications, 32: 12609-12625.

Garg, H. 2017. Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. Arabian Journal for Science and Engineering, 42(12): 5275–5290.

Garg, H., Munir, M., Ullah K., Mahmood T. and Jan N. 2018. Algorithm for t-spherical fuzzy multi-attribute decision making based on improved interactive aggregation operators. Symmetry, 10(12): 670.

Garg, H., Rani, D. 2019. Complex interval-valued intuitionistic fuzzy sets and their aggregation operators. Fundamenta Informaticae, 164(1): 61-101.

Gundogdu, F. and Kahraman, C. 2019a. Spherical fuzzy sets and spherical fuzzy topsis method. Journal of intelligent and fuzzy systems, 36(1): 337–352.

Gundogdu, F.K. and Kahraman, C. 2019b. Spherical fuzzy sets and decision making applications. International Conference on Intelligent and Fuzzy Systems, 979–987.

Hao, N.D., Son, L.H., Thong, P.H. 2016. Some improvements of fuzzy clustering algorithms using picture fuzzy sets and applications for geographic data clustering. VNU Journal of Science: Computer Science and Communication Engineering, 32(3): 32-38.

Joshi, R. 2020. A novel decision-making method using r-norm concept and VIKOR approach under picture fuzzy environment, Expert Systems with Applications 147: 113228.

Karaaslan, F. and Dawood, M.A.D. 2021. Complex t-spherical fuzzy dombi aggregationoperators and their applications in multiple-criteria decision-making. Complex and Intelligent Systems, 7(5): 2711–2734.

Liu, P., Mahmood, T., Ali, Z. 2020. Complex Q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making. Information, 11(1): 5.

Mahmood, T., Ullah, K., Khan, Q. and Jan, N. 2019. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. Neural Computing and Applications, 31(11): 7041–7053.

Peng, X. and Dai, J. 2017. Algorithm for picture fuzzy multiple attribute decision-making based on new distance measure. International Journal for Uncertainty Quantification, 7(2): 177–187.

Rafi, M., Ashraf, S., Abdullah, S. Mahmood, T., Muhammad, S. 2019. The cosine similarity measures of spherical fuzzy sets and their applications in decision making, Journal of Intelligent and Fuzzy Systems, 36(6): 6059-6073.

Ramot, D. Milo, R. Friedman, M. and Kandel, A. 2002. Complex fuzzy sets. IEEE Transactions on Fuzzy Systems, 10(2): 171–186.

Ramot, D. Friedman, M., Langholz, G. and Kandel, A. 2003. Complex fuzzy logic. IEEE

Transactions on Fuzzy Systems, 11(4): 450–461.

Singh, P. 2015. Correlation coefficients for picture fuzzy sets, Journal of Intelligent and Fuzzy Systems, 28(2): 591-604.

Son, L.H. 2016. Generalized picture distance measure and applications to picture fuzzy clustering, Applied Soft Computing, 46(C): 284-295.

Thao, N.X. 2020. Similarity measures of picture fuzzy sets based on entropy and their application in MCDM, Pattern Analysis and Applications, 23(3): 1203-1213.

Tian, C., Peng, J., Zhang, W., Zhang, S. and Wang, J. 2020. Tourism environmental impact assessment based on improved AHP and picture fuzzy PROMETHEE II methods Technological and Economic Development of Economy, 26(2): 355-378.

Ullah, K., Mahmood, T. and Jan, N. 2018a. Similarity measures for t-spherical fuzzy sets with applications in pattern recognition. Symmetry, 10(6): 193.

Ullah, K., Mahmood, T., Jan, N. and Ali, Z. 2018b. A note on geometric aggregation operators in t-spherical fuzzy environment and their applications in multi-attribute

Ullah, K., Mahmood, T., Ali, Z., Jan, N. 2019. On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. Complex Intell. Syst., 6: 15-27.

Wei, G. 2017a. Picture fuzzy aggregation operators and theirapplication to multiple attribute decision making. Journal of Intelligent and Fuzzy Systems, 33(2): 713-724.

Wei, G. 2017b. Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making, Informatica, 28(3): 547-564.

Wei, G. and Gao, H. 2018. The generalized Dice similarity measures for picture fuzzy sets and their applications, Informatica, 29(1): 107-124.

Wei, G, 2018. Some similarity measures for picture fuzzy sets and their applications, Iranian Journal of Fuzzy Systems, 15(1): 77-89.

Yager, R. R. 2013. Pythagorean fuzzy subsets. In 2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS) (pp. 57-61). IEEE.

Yager, R. R. 2016. Generalized orthopair fuzzy sets. IEEE Transactions on Fuzzy Systems, 25(5): 1222-1230.

Yager, R. R., Alajlan, N. and Bazi, Y. 2018. Aspects of generalized orthopair fuzzy sets. Int. J Intell. Syst., 33(11): 2154-2174.

Zadeh, L. A. 1965. Fuzzy sets. Info. Comput., 8: 338-353.

## CURRICULUM VITAE

### Personal Information

Name Surname : Hussein Hammoodi Mahmood AL-JAFAR

### Education

| | | |
|---|---|---|
| MSc | Çankırı Karatekin University<br>Graduate School of Natural and Applied Sciences<br>Department of Mathematics | 2020-2023 |
| Undergraduate | Anbar University<br>College Eduacation | 1997-2001 |