# COMPRESSIBLE MODULES AND THEIR GENERALIZATION OF COMPRESSIBLE MODULES

by

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# ABSTRACT

COMPRESSIBLE MODULES AND THEIR GENERALIZATION OF COMPRESSIBLE MODULES

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This study consists of three chapters, firstly we present some definitions and theorems which are used in the subsequent chapters.

The characterization of a critically compressible module is given in terms of torsion free and uniform modules. A sufficient condition for a module to be critically compressible module is given. It has been shown that a critically compressible module is indecomposable. A sufficient condition for an injective hull of critically compressible module to be critically compressible module. Finally, it has been proved that noetherian critically compressible modules and noetherian compressible modules are equivalent.

we have studied essentially slightly compressible modules and rings, which are natural generalizations of essentially compressible modules and rings. It has also been given an example of an essentially slightly compressible module which is not essentially compressible module. Furthermore, ıt has been shown that direct sums of essentially slightly compressible modules are essentially slightly compressible modules.

Keywords: modules, compressible modules, slightly compressible modules, critically compressible modules and essentially compressible rings

# ÖZET

COMPRESSiBLE MODÜLLER ve COMPRESSİBLE MODÜLLERİN GENELLEMELERİ

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Bu çalışma üç bölümden oluşmaktadır, başlarken diğer bölümlerde kullanılan temel tanım ve teoremler verilmiştir.

Critically compressible modüllerin, uniform ve torsion free modüller cinsinden bir karakterizasyonu verilmiştir. Bir modülün critically compressible modül olması için sağlanması gereken koşul verilmiştir. Bir critically compressible modülün ayrıştırılamaz (indecomposable) bir modül olduğu verilmiştir. Bir critically compressible modülün injective kabuğunun (hull) critically compressible module olması için modülün sağlaması gereken koşul verilmiştir. Ayrıca, kanıtlanmıştır ki; noetherian critically compressible modüllerle,noetherian compressible modüller denktir.

Essential compressible modül ve halkaların bir doğal genellemesi olan essentially slightly compressible modül ve halkalar çalışılmıştır. Essentially slightly compressible modül olan ancak essentially compressible modül olmayan bir örnek verilmiştir. Ayrıca, essentially slightly compressible modüllerin dik toplamının da yine essentially slightly compressible modül olduğu kanıtlanmıştır.

Anahtar Kelimeler: modules, compressible modüller, slightly compressible modüller, critically compressible modüller ve essentially compressible halkalar

*ToMyFamily*

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# CHAPTER 1

# PRELIMINARIES

This chapter is mainly devoted to the collection of definitions and basic theorems, which are used in the subsequent chapters of the thesis.

**Definition 1.1.** Let $R$ be any ring, not necessarily containing an identity element and if $(M, +)$ is additive abelian group, then $M$ is left $R$-module in case there exists a mapping : $R \times M \to M$ defined by $(a, x) \to ax$ , $a \epsilon R$, $x \epsilon M$ satisfying the following conditions:

(i) $a(x + y) = ax + ay$

(ii) $(a + b)x = ax + bx$

(iii) $(ab)x = a(bx)$; for every $x, y \epsilon M$ and $a, b \epsilon R$.

Left $R$-module $M$ is *unital module* in case $R$ contains an identity element 1 and $1.x = x$ for every $x \epsilon M$.

**Definition 1.2.** Let $M$ and $N$ be left $R$-modules.

(a) Function $f : M \to N$ is called *a module homomorphism* if

$f(m_1 + m_2) = f(m_1) + f(m_2)$ and $f(rm) = rf(m)$ for all $r \in R$ and all $m, m_1, m_2 \in M$. The set of all module homomorphism from $M$ into $N$ denoted by $Hom_R(M, N)$.

(b) For a module homomorphism $f \in Hom_R(M, N\ )$ we define its kernel as $Ker(f) = \{m \in M \mid f(m) = 0\}$ .

(c) We say that $f$ is an isomorphism if it is both one-to-one and onto.

(d) Elements of $Hom_R(M, M)$ are called *endomorphisms*, and *isomorphisms* in $Hom_R(M, M\ )$ are called *automorphisms.* The set of endomorphisms of $M$ will be denoted by $End_R(M)$.

More generally, we may use $Hom(M, N)$, $End(M)$ rather than $Hom_R(M, N), End_R(M)$ respectively.$_n N$

**Theorem 1.3.** *(Fundamental homomorphism theorem) Let $M$ and $N$ be left $R$-modules. If $f : M \to N$ is any module homomorphism, then $f(M) \cong M/Ker(f)$.*

## 1.1 Homomorphism decomposition theorem

**Theorem 1.4.** *Let $f : A \to B$ be a module homomorphism and $g : A \to C$ be a module epimorphism. Then $Ker(g) \subseteq Ker(f)$ if and only if there exists a unique module homomorphism $h : C \to B$ such that $f = h \circ g$.*

**Proof.** Suppose $Ker(g) \subseteq Ker(f)$. Since $g$ is onto implies every element of $C$ is of the form $g(x)$ for some $x \in A$. Define $h : C \to B$ by $h(g(x)) = f(x)$. Now,

$$\begin{aligned} g(x) &= g(y) \Longrightarrow g(x - y) = 0 \\ &\Longrightarrow x - y \in Ker(g) \subseteq Ker(f) \\ &\Longrightarrow f(x - y) = 0 \\ &\Longrightarrow h(g(x)) = f(x) = f(y) = h(g(y)) \end{aligned}$$

Therefore $h$ is well defined mapping.

Take $g(x), g(y) \in C$ and $\alpha, \beta \in R$.

Then

$h(\alpha(g(x)) + \beta(g(y))) = h(g(\alpha x) + g(\beta y)) = h(g(\alpha x + \beta y)) = f(\alpha x + \beta y)$ $= f(\alpha x) + f(\beta y) = \alpha f(x) + \beta f(y) = \alpha h(g(x)) + \beta h(g(y))$.

**H**ence $h$ is homomorphism. Suppose $h' : C \to B$ is a homomorphism with $f = h' \circ g$.

Then $f = h \circ g = h' \circ g \implies h = h'$since $g$ is onto. Therefore $h$ is unique. Conversely, suppose that there exists a unique module homomorphism $g : C \to B$ such that $f = h \circ g$. Take $0 \neq x \in Ker(g)$ and suppose $x \notin Ker(f)$, then $g(x) = 0 \implies h \circ g(x) = 0 \implies f(x) = 0$ a contradiction, hence $x \in Ker(f)$. Therefore $Ker(g) \subseteq Ker(f)$ .

**Corollary 1.5.** *Let $f : A \to B$ be a module homomorphism and $g : A \to C$ be a module epimorphism with $Ker(g) = Ker(f)$. Then $h$ is a monomorphism.*

**Corollary 1.6.** *Let $f : A \to B$ and $g : A \to C$ be module epimorphisms with $Ker(g) \subseteq Ker(f)$. Then $h$ is onto .*

**Corollary 1.7.** *If $f$ and $g$ both are module epimorphisms with $Ker(g) = Ker(f)$ , then $h$ is isomorphism.*

## 1.2 Direct sum and Direct product

**Definition 1.8.** A family $(M_i)_{i\in I}$ of submodules of $M$ is called *a free family* if $M_i \cap \sum_{j\in I} M_j = 0$ for all $i \neq j \in I$.

**Definition 1.9.** A module $M$ is said to be it a direct sum of a family $(M_i)_{i\in I}$ of submodules of $M$ if ,

(i) $M = \sum_{i\in I} M_i$, and

(ii) Every element of $M$ can be expressed uniquely as a sum $m = m_{i_1} + m_{i_2} +$ ...........

$m_{i_j}, m_{i_j} \in M_{i_j}, i_j \in I$. We shall express this fact by $M = \bigoplus \sum_{i\in I} M_i$

**Definition 1.10.** A submodule $N$ of $M$ is called a *direct summand* of $M$ if there exists a submodule $L$ of $M$ such that $N \cap L = 0$ and $M = N \oplus L$. It is denoted by $N \leq_d M$ .

**Definition 1.11.** Let $(M_i)_{i\in I}$ be a family of $R$-modules. The totality $\prod M_i$ of all the functions $f : I \to \cup M_i$ defined by $\prod M_i = \{f : I \to \cup M_i : f(i) \in M_i$, for each $i \in I$ with respect to addition and scalar multiplication defined by:

(1) $(f+g)(i) = f(i) + g(i)$ and

(2) $(af)(i) = af(i)$, for all $a \in R$ and $i \in I$ , is a module. We call $\prod M_i$ the direct product of the family $(M_i)_{i \in I}$.

The submodule of $\prod M_i$ consisting of all $f \in \prod M_i$ such that $f(i) = 0$ for all but finitely many $i \in I$ , is called the external sum of modules $M_i$ and is denoted by $\bigoplus_{i \in I} M_i$. It can be shown that if $(M_i)_{i \in I}$ is free family of submodules of a module $M$, then $\sum\limits_{i \in I} M_i \cong \bigoplus\limits_{i \in I} M_i$

## 1.3 Small submodules, Minimal epimorphism and Exact sequence

**Definition 1.12.** A submodule $N$ of $M$ is called *small* (or *superflues*) in $M$, abbreviated as $N << M$ in case $N + L = M \implies L = M$ for any submodule $L \subseteq M$ .

**Proposition 1.13.** *If $A$ and $B$ are submodules of $M$ such that $B \subseteq A$ and $A$ is small in $M$ then $B$ is small in $M$.*

**Proposition 1.14.** *If $A$ and $B$ are small submodules of $M$ then $A + B$ is also small.*

**Remark 1.15.** *A finite sum of small submodules of $M$ is small in $M$*

**Definition 1.16.** An epimorphism $f : A \to B$ is called *minimal* if $Ker(f)$ is small in $A$.

**Proposition 1.17.** *The following conditions are equivalent:*

*(i) $N$ is small in $M$.*

*(ii) The natural epimorphism $\alpha : M \to M/N$ is minimal.*

*(iii) For every $g \in Hom_R(M, M)$, $\mathrm{Im}(g) + N = M \implies \mathrm{Im}(g) = M$.*

**Definition 1.18.** A chain $\rightarrow M_{n+1} \overrightarrow{\alpha_{n+1}} \ M_n \ \overrightarrow{\alpha_n} \ M_{n\text{-}1} \ \rightarrow$ where $M_n$ is an $R$ -module $\forall\ n$ and $\alpha_n$ is a homomorphism from $M_n \rightarrow M_{n-1} \forall n$ is called an *exact sequence* at $M_n$ if $Ker(\alpha_n) = \mathrm{Im}(\alpha_{n+1})$.

It is called *exact sequence* if it is exact at $M_n\ \forall n$ .

An exact sequence $0 \longrightarrow A \ \overrightarrow{f} \ B \ \ \overrightarrow{g} \ C \longrightarrow 0$

where 0 is the trivial module , is called a *short exact sequence.* Thus the chain

$0 \longrightarrow A \ \overrightarrow{f} \ B \ \ \overrightarrow{g} \ C \longrightarrow 0$

is exact sequence if and only if

(i) $Ker(f) = \{0\}$,

(ii) $Ker(g) = \mathrm{Im}(f)$,

(iii) $\mathrm{Im}(g) = C$.

**Proposition 1.19.** *The following conditions are equivalent:*

*(i)* $0 \rightarrow A \ \overrightarrow{f} \ B \ \overrightarrow{g} \ C \rightarrow 0$ *splits.*

*(ii) There exists a homomorphism* $j : B \rightarrow A$ *such that* $j \circ f = I_A$.

*(iii) There exists a homomorphism* $h : C \rightarrow B$ *such that* $g \circ h = I_C$.

**Theorem 1.20.** *Let* $M, N$ *be left* $R$*-modules.*

*(a) Let* $f : M \rightarrow N$ *and* $g : N \rightarrow M$ *be homomorphism such that* $f \circ g = I_N$. *Then* $M = Ker(f) \oplus \mathrm{Im}(g)$.

*(b) A one to one homomorphism* $g : N \rightarrow M$ *splits if and only if* $\mathrm{Im}(g)$ *is direct summand of* $M$.

*(c) An onto homomorphism* $f : M \rightarrow N$ *splits if and only if* $Ker(f)$ *is direct summand of M.*

## 1.4 Finitely generated, Free, Cyclic, Supplemented and Quotient modules

**Definition 1.21.** A left $R$-module $M$ is said to be finitely generated if there exist elements $m_1, m_2, ........., m_n \in M$ such that $M = \sum_{j=1} Rm_j$. In this case, we say that $\{m_1, m_2, ......., m_n\}$ is a set of *generators* of $M$.

**Definition 1.22.** A left $R$-module $M$ is called *free module* if there exists a subset $X \subseteq M$ such that each element $m \in M$ can be expressed uniquely as a finite sum $m = \sum_{i=1}^{n} r_i x_i$ with $r_1, .........., r_n \in R$ and $x_1, ..........., x_n \in X$.

**Proposition 1.23.** *Let $M$ be a left $R$-module.*

*(a) The module $M$ is free if and only if it is isomorphic to a direct sum $R^{(I)}$, for some index set $I$.*

*(b) Every module $M$ is homomorphic image of a free module.*

**Definition 1.24.** An $R$-module $M$ is said to be *cyclic* if there is an element $m_0 \in M$ such that every $m \in M$ is of the form $m = m_0 r$, where $r \in R$. Also $m_0$ is called *generator* of $M$ and we write $M = (m_0)R$.

**Definition 1.25.** A module $M$ is *supplemented* if $M = U + V$ implies the existence of a submodule $V'$ of $V$ minimal with respect to $M = U + V'$. This $V'$is called *supplement* of $U$ in $M$.

**Theorem 1.26.** *If $M$ is a module and $M = U + V$ , then a necessary and sufficient condition for $V$ to be supplement of $U$ in $M$ is that $U \cap V$ be small in $V$.*

**Definition 1.27.** If $A$ is any submodule of an $R$- module $M$. Then the set $M/A$ of all cosets $m + A$, where $m$ is any arbitrary element of $M$, is a module over $R$ for the addition and scalar multiplication compositions defined as follows;

(i) $(m_1 + A) + (m_2 + A) = (m_1 + m_2) + A$, and

(ii) $r(m+A) = rm + A$ for all $r \in R, m \in M$. $R$-module $M/A$ is called the *quotient module* of $M$ relative to the submodule $A$.

## 1.5 Essential, Closed, Complemented, Stable and Uniform submodules

**Definition 1.28.** A submodule $N$ of an $R$-module $M$ is called an *essential submodule* or a *large submodule* if $N \cap B \neq 0$ for each nonzero submodule $B$ of $M$. In this case $M$ is called an *essential extension* of $N$. It is denoted by $N \leq_e M$ .

**Definition 1.29.** A submodule $C$ of an $R$-module $M$ is called *closed submodule* if it has no proper essential extension in $M$. It is denoted by $C \leq_c M$.

**Definition 1.30.** Submodules $A$ and $B$ of $M$ are said to be *complemented* to each other if each of these is maximal with respect to intersecting the other is zero. Furthermore, if $A \oplus B = M$, then $A$ and $B$ are said to be *co-maximal.*

**Definition 1.31.** A submodule $N$ of an $R$-module $M$ is called *stable* (or *fully invariant*) if and only if $\alpha(N) \subset N$ for every endomorphism $\alpha \in End(M)$ , $End(M)$ is called endomorphism ring of $M$.

**Definition 1.32.** An $R$-module $M$ is *uniform* if every nonzero submodule of $M$ is essential in $M$.

**Remark 1.33.** *Any nonzero submodule of a uniform module is uniform, and any essential extension of a uniform module is uniform.*

## 1.6 Order ideal, Singular submodule, Torsion and Torsion free modules

**Definition 1.34.** Let $M$ be an $R$-module. For any nonzero $x \in M$ define $o(x) = \{\alpha \epsilon R; \alpha x = 0\}$. Then $o(x)$ is left ideal of $R$ and is called the *order ideal* (*annihilator ideal*) of $x$ . It follows that $Rx \cong R/o(x)$

**Definition 1.35.** Let $M$ be an $R$- module. The set of all nonzero elements $x$ of $M$ such that $o(x)$ is an *essential ideal* of $R$, is a submodule and is called the *singular submodule* of $M$. It is denoted by $Z(M)$. i.e. $Z(M) = \{m \in M \mid o(m) \leq_e R\}$. If $Z(M) = 0$ then $M$ is called a *nonsingular module* and if $Z(M) = M$ then $M$ is called *singular module.*

**Definition 1.36.** An $R$-module $M$ is said to be a *torsion module* if $o(x)$ is nonzero for every nonzero $x \in M$ .

**Definition 1.37.** An $R$-module $M$ is called *torsion free* module if $o(x)$ is zero for every nonzero $x \in M$ or the torsion submodule of $M$ is zero.

## 1.7 Local modules, Local rings and Socle of a module

**Definition 1.38.** An $R$-module $M$ is called *local* if it has a unique maximal submodule, which contains every proper submodule of $M$ and a ring $R$ is called *local* if it is local as an $R$-module.

**Definition 1.39.** A ring which has only a finite number of distinct maximal left ideals is called a *semi-local* ring.

## 1.8 Simple, Semisimple modules and Finite Goldie Dimension

**Definition 1.40.** A nonzero module $M$ is called a *simple module* if it contains no nontrivial proper submodule.

**Definition 1.41.** A module $M$ is called *indecomposable* if it is nonzero and has no proper direct summand.

**Definition 1.42.** A nonzero module $M$ is said to be *semisimple* if it is expressible as sum of simple submodules.

**Definition 1.43.** The socle of any module $M$ is the sum of all simple submodules of $M$ . It is denoted by $Soc(M)$. If $M$ has no simple submodules, then $Soc(M) = 0$.

**Theorem 1.44.** *For a module $M$ the following conditions are equivalent :*

*(i) $M$ is semisimple.*

*(ii) $M$ is direct sum of simple submodules.*

*(iii) Every submodule of $M$ is a direct summand.*

**Definition 1.45.** An $R$-module $M$ is said to have *Finite Goldie Dimension (FGD)* if every free family of its nonzero submodules is finite.

## 1.9 Division ring, Endomorphism ring, Quotient ring , V-Ring,Prime ideal and Schur's lemma

**Definition 1.46.** A ring with unit element in which every nonzero element has an inverse with respect to multiplication is called a *division ring* or *skew field*.

**Definition 1.47.** Let $G$ be an abelian group and $R$ be a set of all homomorphisms of $G$ in $G$ forms a ring for $(+)$ and $(.)$ defined by

(i) $(f + g)(a) = f(a) + g(a)$, and

(ii) $(fg)(a) = f(g(a))$; where $f, g \in R, a \in G$. Then $R$ is called endomorphism ring of $G$.

**Definition 1.48.** Let $R$ be any ring and $N$ be any ideal of $R$ , then the system $(R/N, +, \cdot)$ where $R/N = \{r + n; r \in R\}$ and $(+)$ and $(.)$ are binary compositions on $R/N$ defined by

(i) $(r + N) + (s + N) = (r + s) + N$

(ii) $(r + N)(s + N) = rs + N$ for all $r$,$s \in R$, is a ring called quotient ring with respect to the ideal $N$.

**Definition 1.49.** A ring $R$ is called a *left $V$-ring* if every simple left $R$-module is injective.

**Definition 1.50.** A left ideal $P$ of a ring $R$ is called *prime ideal* if whenever a product of two elements of $R$ is in $P$ , then at least one of the factors in $P$.

**Definition 1.51.** ***(Schur's lemma)*** If $A$ is a simple $R$-module, then $End(A)$ is a division ring.

## 1.10 Ascending chain condition, Descending chain condition, Left Noetherian ring, Left Artinian ring, Jacobson radical and Semisimple artinian ring

**Definition 1.52.** A module $M$ is said to satisfy ascending chain condition (ACC) if there is no infinite strictly ascending chain

$M_1 \subset M_2 \subset M_3 \subset ........,$

of submodules of $M$ . Similarly if $M$ does not contain any infinite strictly descending chain of submodules , it is to satisfy descending chain condition (DCC).

**Definition 1.53.** A ring $R$ is called a *left Noetherian ring* if any one of the following equivalent statements holds:

(i) $R$ satisfies ACC for its left ideals.

(ii) Every non empty set of left ideals of $R$ contains a maximal element.

(iii) Every left ideal in $R$ is finitely generated.

**Definition 1.54.** A ring $R$ is called a *left Artinian ring* if $R$ satisfies DCC for its left ideals.

**Definition 1.55.** The intersection of all maximal left ideals of a ring $R$ is called the *radical* of $R$ and is denoted by $Rad(R)$ or $J(R)$. It is also called the *Jacobson radical* of $R$.

**Note:** A left ideal $M$ of a ring $R$ is called a maximal left ideal if $M \neq R$ and $M$ is not properly contained in any proper left ideal of $R$.

**Definition 1.56.** The intersection of all prime left ideals of $R$ is called the *prime radical* of $R$.

**Remark 1.57.** *The prime radical of $R$ is always contained in the Jacobson radical of $R$.*

**Theorem 1.58.** *Let $R$ be a ring then its Jacobson radical $J(R)$ is the sum of all small ideals of $R$.*

**Remark 1.59.** *It is well known that $J(M) = \{m \in M \mid Rm \text{ is small in } M\}$.*

**Definition 1.60.** A ring $R$ is called *semisimple* if $R$ satisfies DCC and $J(R) = 0$.

**Theorem 1.61.** *The following conditions are equivalent for a ring $R$:*

*(i) $R$ is semisimple.*

*(ii) All $R$-modules are semisimple.*

*(iii) All $R$-modules are projective.*

*(iv) All $R$-modules are injective.*

*(v) Every ideal of $R$ is a direct summand.*

## 1.11 Noetherian module, Artinian module and Jacobson radical of a module

**Definition 1.62.** An $R$-module $M$ is called *Noetherian* if it satisfies ACC on its submodules.

**Theorem 1.63.** *For a module $M$ the following conditions are equivalent :*

*(i) $M$ is Noetherian.*

*(ii) Every submodule of $M$ is finitely generated.*

*(iii) Every nonempty set $S$ of submodules of $M$ has a maximal element.*

**Definition 1.64.** An $R$-module $M$ is called *Artinian* if it satisfies DCC on its submodules.

**Theorem 1.65.** *For any module $M$ the following are equivalent :*

*(i) $M$ is Artinian.*

*(ii) Every nonempty set $S$ of submodules of $M$ has minimal element.*

**Definition 1.66.** The Jacobson radical of a module $M$ is defined as the intersection of all maximal submodules of $M$ and is denoted by $J(M)$. If $M$ has no maximal submodule, then $J(M) = M$.

## 1.12 Regular element, Regular ideal,Prime ring,semi-primary ring and Goldie ring

**Definition 1.67.** A nonzero element $\alpha \in R$ is called a *proper zero divisor* $R$ if there exists a nonzero element $\beta \in R$ such that $\alpha\beta = 0$. If an element $\alpha \in R$ is not a proper zero divisor in $R$ then it is called a *regular element* of $R$.

**Definition 1.68.** A left ideal $A$ of ring $R$ is said to be *regular ideal* if it contains a regular element.

**Definition 1.69.** A ring $R$ is called a *prime ring* if 0 is a prime ideal of $R$ and it is called *semiprimitive ring* if its Jacobson radical is zero and it is also called *semiprime ring* if its prime radical is zero.

**Definition 1.70.** An ideal $N$ of a ring $R$ is *nilpotent* if some power of it is zero. A ring with radical $N$ is semi primary if $N$ is nilpotent and $R/N$ satisfies DCC.

**Definition 1.71.** A ring $R$ is said to be a *Goldie ring* if it has the following properties.

(i) $R$ satisfies ACC on annihilator ideals, and

(ii) $R$ does not contain any infinite direct sum of left ideals.

**Theorem 1.72.** *Any left ideal of semi prime Goldie ring is essential if and only if it contains a regular element.*

## 1.13 Injective,Quasi-injective modules and Injective hull of a modules,CS,Quasi-Continuous,Continuous module.

**Definition 1.73.** An $R$-module $M$ is called *injective* if given any $R$-modules $A$ and $B$, a monomorphism $f : A \to B$ and a homomorphism $g : A \to M$ , if there exists a homomorphism $h : B \to M$ such that $g = h \circ f$

**Definition 1.74.** An $R$-module $M$ is said to be *quasi-injective* in case each homomorphism of any submodule $N$ of $M$ can be extended to a homomorphism of $M$ into $M$.

**Definition 1.75.** A module $N$ is called an *injective hull* of $M$ if $N$ is an injective module containing $M$ and there is no injective module $P$ satisfying $M \subset P \subset N$ and it is denoted by $E(M)$ .

Consider the following conditions for a module $M$;

($C_1$) Every submodule of $M$ is essential in a direct summand of $M$ .

($C_2$) If a submodule of $M$ is isomorphic to a direct summand of $M$, then it is itself a direct summand of $M$.

($C_3$) If $A$ and $B$ are direct summand of $M$ with $A \cap B = 0$ then $A + B$ is also a direct summand of $M$.

**Definition 1.76.** $M$ is called $CS$ module (*extending module*) if it satisfies $C_1$.

**Definition 1.77.** $M$ is called *continuous module* if it satisfies $C_1$ and $C_2$.

**Definition 1.78.** $M$ is called *quasi continuous* if it satisfies $C_1$ and $C_3$.

## 1.14 Projective module, Quasi-projective module

**Definition 1.79.** An $R$-module $M$ is called *projective* if given any $R$- modules $A$ and $B$, an epimorphism $g : B \to A$ and a homomorphism $f : M \to A$, if there exists a homomorphism $h : M \to B$ such that f=g ∘ h.

**Theorem 1.80.** *Every free module is projective.*

**Corollary 1.81.** $R_R$ *is projective.*

**Theorem 1.82.** *Every R-module is projective if and only if R is semisimple.*

**Note:** An epimorphism $M \to N$ is called direct if its Kernel is a direct summand of $M$.

**Theorem 1.83.** *An R-module $M$ is projective if and only if every epimorphism $\pi : B \to M$ is direct.*

**Definition 1.84.** An $R$-module $M$ is said to be *Quasi-projective* if for every epimorphism $f : M \to N$ and every homomorphism $g : M \to N$, $N$ being an $R$-module there exists $h \in End(M)$ such that, $f \circ h = g$ .

**Theorem 1.85.** *For a ring R , the following conditions are equivalent :*

*(i) R is semisimple artinian.*

*(ii) Every finitely generated module is quasi-projective.*

*(iii) Direct sum of two quasi-projective module is always quasi-projective.*

## 1.15 Left perfect ring, Semi perfect ring, Quasi-perfect ring, Regular ring, Hereditary ring and Almost perfect ring

**Definition 1.86.** A ring $R$ is said to be *left perfect ring* if every $R$-module has a projective cover.

**Definition 1.87.** A ring $R$ is said to be *semi perfect* ring if every finitely generated $R$-module has a projective cover.

**Theorem 1.88.** *Every module has quasi-projective cover if and only if $R$ is left perfect.*

**Theorem 1.89.** *A ring $R$ is semi perfect if and only if every finitely generated module has a quasi-projective cover.*

**Definition 1.90.** A ring $R$ with identity is called *left (right) quasi-perfect* if each left(right) artinian $R$-module has projective cover.

**Definition 1.91.** A ring $R$ is said to be *regular* (in the sense of Von-Neumann) if for each $r \in R$ there exists $x \in R$ such that $r = rxr$

**Definition 1.92.** A ring $R$ is said to be *hereditary(semi-hereditary)* if every ideal (finitely generated ideal) of $R$ is projective as an $R$-module.

**Theorem 1.93.** *The following statements are equivalent for a ring $R$;*

*(i) $R$ is left hereditary.*

*(ii) Every submodule of a projective left $R$-module is projective.*

*(iii) Every quotient module of an injective module is injective.*

# CHAPTER 2

# COMPRESSIBLE MODULES

In this chapter we have studied the critically compressible modules defined by J.M. Zelmanowitz in 1976 and provided their characterization in terms of uniform modules and torsion free modules . A sufficient condition for a compressible module to be critically compressible is also given. Further a sufficient condition has been for a critically compressible module to be continuous. It has been shown that noetherian compressible and noetherian critically compressible modules are equivalent by A.K.Singh in 2007. Finally, we have studied the essentially slightly compressible modules and rings which are generalizations of essentially compressible modules and rings defined by P.F.Smith in 2006. It has been shown that for uniform modules over a noetherian ring, essentially slightly compressible modules and nonzero homomorphism from $M$ in to $U$ for nonzero uniform submodules $U$ of $M$ are equivalent by A.K.Singh in 2007. Unless stated otherwise, throughout this chapter by a ring $R$ always mean an associative ring with identity and all modules are unitary left $R$-modules.

## 2.1 Slightly Compressible Modules And Rings

**Definition 2.1.** An $R$-module $M$ is called *compressible (Slightly compressible)* if for every nonzero submodule $N$ of $M$, there exists a monomorphism (nonzero homomorphism) from $M$ to $N$.

**Definition 2.2.** A homomorphism from a submodule of $R$-module $M$ into $M$ is called a *partial endomorphism* on $M$.

**Definition 2.3.** An $R$-module $M$ is called *critically compressible* if it is compressible and it can not be embedded in any of its proper factor modules.

**Definition 2.4.** A nonzero $R$-module $M$ is called *self similar* if every nonzero submodule of $M$ is isomorphic to $M$.

**Definition 2.5.** A ring with unit 1 is said to be an *SP ring* if $Z(M)$ is a direct summand of $M$ for every $R$-module $M$.

**Example 2.6.** *Every semisimple $R$-module $M$ is slightly compressible module. However, every semisimple $R$-module $M$ is compressible if and only if $M$ is simple.*

**Proof.** It is clear that if $M$ is semisimple $R$-module, then every submodule of $M$ is a direct summand of $M$. Suppose that $M$ be a semisimple compressible module . Let $N$ be a simple submodule of $M$. Then there exist a monomorphism $f : M \to N$. Since $N$ is simple, $f$ is an isomorphism and so $M$ is simple $R$-module.

**Lemma 2.7.** *Let $A$ be a nonzero idempotent right ideal of a ring $R$ such that $A$ contains a nonzero right ideal $B$ of $R$ such that $BA = 0$. Then the right $R$-module $A$ is not slightly compressible.*

**Proof.** Suppose that $f : A \to B$ is a homomorphism. Then

$f(A) = f(A^2) = f(A)A \subseteq BA = 0.$

It follows that $A$ is not slightly compressible.

**Example 2.8.** *Let $S$ be any (nonzero) ring and let $R$ denote the ring of $2 \times 2$ upper triangular matrices over $S$. Let $A$ be the right ideal of $R$ consisting of all matrices in $R$ with bottom row zero. Then $A$ is a cyclic projective right $R$-module which is not slightly compressible.*

**Proof.** Note first that $A = eR$ where $e$ is the idempotent element of R with 1 as its (1,1) entry and all other entries 0. Thus A is cyclic, idempotent and projective . Let B denote the right ideal of R consisting of all matrices with all entries 0 except for possibly the (1,2) entry. Then B is contained in A and BA=0. By Lemma 2.7, A is not slightly compressible.

**Proposition 2.9.** *Let $R$ be a semi prime ring. Then every right ideal $A$ of $R$ is a slightly compressible $R$-module.*

**Proof.** Let $B$ any nonzero right ideal of $R$ such that $B$ is contained in $A$. Then $B^2$ is nonzero so that $BA$ is nonzero . There exists an element $b$ of $B$ such that $bA$ is nonzero. Define a nonzero homomorphism $f : A \to B$ by $f(a) = ba$ for all $a$ in $A$ .

Let $R$ be a ring. In general the class of slightly compressible $R$-modules is not closed under taking submodules, factor modules or extensions. In Example 2.8 the right $R$-module $R$ is slightly compressible but the submodule $A$ is not. Moreover, $A$ is also a homomorphic image of $R$ and is not slightly compressible. In Example 2.8 let $S$ denote a field. Then the right ideal B in Example 2.8. is a simple R-module and the factor module $A/B$ is also simple. Clearly $B$ and $A/B$ are both (slightly) compressible but $A$ is not slightly compressible. The situation for direct sums is much better, however, as the next result shows.

**Proposition 2.10.** *Let $R$ be a ring . Then any direct sum of slightly compressible $R$-modules is slightly compressible module.*

**Proof.** Let an $R$-module $M = \bigoplus_{i \in I} M_i$ be a direct sum of slightly compressible submodules $M_i (i \in I)$. Let $N$ be any nonzero submodule of $M$. Let $m$ be any nonzero element of $N$. There exists a finite subset $J$ of $I$ such that $m$ belongs to $\bigoplus_{i \in J} M_i$. If there exists a nonzero homomorphism $f : \bigoplus_{i \in J} M_i \to mR$ and $p : M \to \bigoplus_{i \in J} M_i$ is the canonical projection then $fp : M \to mR$ is a nonzero homomorphism and hence $Hom_R(M, N)$ is nonzero.

Thus without loss of generality we can suppose that $I$ is a finite set. By induction it is sufficient to prove the result when $I = \{1, 2\}$. Let $M = M_1 \oplus M_2$ be a direct sum of slightly compressible submodules $M_1$ and $M_2$. Let $L$ be a nonzero submodule of $M$. If $L \cap M_1 \neq 0$, then there exists a nonzero homomorphism from $M_1$ to $L \cap M_1$ and hence then there exists a nonzero homomorphism from

$M$ to $L$. If $L \cap M_1$ is zero then $L$ is isomorphic to a submodule $K$ of $M_2$. There exists a nonzero homomorphism from $M_2$ to $K$ and hence there exists a nonzero homomorphism from $M$ to $L$.

**Definition 2.11.** A ring $R$ is called a *right $V$-ring* if every simple right $R$-module is injective.

**Theorem 2.12.** *Let $R$ be a right $V$-ring . Then every projective right $R$-module is slightly compressible.*

**Proof.** Let $X$ be any nonzero projective $R$-module. Let $x$ be any nonzero element of $X$. Let $Y$ be any maximal submodule of $xR$. Then $xR/Y$ is injective and hence is a direct summand of $X/Y$. It follows that $X$ has a submodule $Z$ such that $X/Z$ is isomorphic to $xR/Y$. Hence there exists a nonzero homomorphism $f : X \to xR/Y$. Because $X$ is a projective module, $f$ can be lifted to a nonzero homomorphism $g : X \to xR$. It follows that $X$ is slightly compressible.

**Lemma 2.13.** *Let $R$ be any ring and let $M$ be a right $R$-module such that every nonzero submodule contains a nonzero direct summand of $M$. Then $M$ is slightly compressible.*

**Proof.** Let $N \leq M$. By our assumption there exists a direct summand $L$ of $M$, such that $L \leq N$.

Since $L \leq_d M$. $M = L \oplus K$ for some $K \leq M$. By modularity

$N = N \cap M = N \cap (L \oplus K) = L \oplus (N \cap K)$.

Since $L$ is direct summand of $M$, there exists a projection map:

$\pi : M \to L$. ($\pi$ is nonzero)

**Corollary 2.14.** *Let $R$ be any ring and let $M$ be a nonsingular injective right $R$-module . Then $M$ is slightly compressible if and only if every nonzero submodule contains a nonzero direct summand of $M$.*

**Proof.** The sufficiency follows by Lemma 2.13. Conversely, suppose that $M$ is slightly compressible. Let $N$ be any nonzero submodule of $M$. By hypothesis, there exists a nonzero homomorphism $f : M \to N$. Because $M \diagup \ker f$ is nonsingular, being isomorphic to a submodule of $N$, $\ker f$ is injective and hence is a direct summand of $M$. It follows that $f(M)$ is injective. Thus $f(M)$ is a nonzero direct summand of $M$ and $f(M) \subseteq N$.

**Definition 2.15.** A ring $R$ is called *right hereditary* if every right ideal is projective, equivalently if every submodule of every projective right $R$-module is projective.

**Corollary 2.16.** *Let $R$ be a right hereditary ring and let $M$ be an injective right $R$-module. Then $M$ is slightly compressible if and only if every nonzero submodule contains a nonzero direct summand of $M$.*

**Proof.** Since $R$ is a hereditary ring, every homomorphic image of $M$ is injective by [**3, p.215, Example**]. The result follows by the proof of Corollary 2.14.

**Corollary 2.17.** *Let $R$ be any ring and let $M$ be a slightly compressible injective right $R$-module. Then any nonsingular uniform submodule of $M$ is simple and injective.*

**Proof.** Let $U$ be a nonsingular uniform submodule of $M$. Let $V$ be any nonzero submodule of $U$. By adapting the proof of Corollary 2.16, there exists a nonzero injective submodule $W$ of $V$. Then $W = V = U$. Thus $U$ is simple injective.

**Corollary 2.18.** *Let $R$ be a right hereditary ring and let $M$ be a slightly compressible injective right $R$-module. Then any uniform submodule of $M$ is simple and injective.*

**Proof.** By Corollary 2.16 and the proof of Corollary 2.17.

Let $R$ be any ring and let $M$ be any $R$-module. For an element $m$ of $M$ we set $r(m) = \{r \in R : mr = 0\}$. Note that $r(m)$ is a right ideal of $R$ and $R \diagup r(m) \cong mR$ for all $m \in M$.

**Theorem 2.19.** *Let $R$ be any ring and let $M$ be a nonsingular injective right $R$-module such that $mR$ has finite rank for every element $m$ of $M$. Then $M$ is slightly compressible if and only if $M$ is semisimple.*

**Proof.** The sufficiency is clear. Conversely suppose that $M$ is slightly compressible. Let $m \in M$. Then $R/r(m)$ is a nonsingular $R$-module which has finite rank. By [ **9, section5.10**] $R$ satisfies the ascending chain condition on right ideals of the form $r(m), m \in M$. Now [**9, section 8.2**] gives that $M$ is a direct sum of uniform submodules. Finally, by Corollary 2.17 $M$ is semisimple.

A ring $R$ is called right nonsingular if the right $R$-module $R$ which we shall denote by $R_R$ is nonsingular. Theorem 2.19 has the following consequence.

**Corollary 2.20.** *Let $R$ be a right nonsingular right Goldie ring. Then $E(R_R)$ is a slightly compressible right $R$-module if and only if $R$ is a semiprime Artinian ring. In this case, every right $R$-module is slightly compressible.*

**Proof.** The sufficiency is clear. Conversely suppose that $E(R_R)$ is a slightly compressible $R$-module. Note that $E(R_R)$ is nonsingular and has finite rank. By Theorem 2.19 the $R$-module $E(R_R)$ is semisimple and hence the $R$-module $R$ is semisimple, as required. The last part is clear.

**Corollary 2.21.** *Let $R$ be a right Noetherian ring such that every nonzero right $R$-module is slightly compressible. Then $R$ is right Artinian.*

**Proof.** Let $P$ be any prime ideal of $R$. Let $X$ denote the $R/P$ -module which is the injective hull of the $(R/P)$ -module $R/P$. Then $X$ is an $R$-module. By hypothesis, $X$ is slightly compressible as an $R$-module and hence also as an $(R/P)$-module. Now Corollary 2.20 gives that $R/P$ is a right Artinian ring. Thus $R/P$ is a right Artinian ring for every prime ideal $P$ of $R$. Because $R$ is right Noetherian, there exists a positive integer $n$ and prime ideals $P_i (1 \leq i \leq n)$ of $R$ such that $P_1 .......P_n = 0$. By considering the chain

$R \supseteq P_1 \supseteq P_1P_2 \supseteq ......... \supseteq P_1.....P_n = 0,$

we conclude that the ring $R$ is right Artinian.

**Proposition 2.22.** *Let $R$ be a right and left Noetherian ring such that there exists a nonzero $R$-homomorphism $f : E(R_R) \to R$. Then $R$ has nonzero right socle.*

**Proof.** By [ **30**, **Corollary2.4**].

**Proposition 2.23.** *Let $R$ be any prime ring. Then the right $R$-module $E(R_R)$ is slightly compressible if and only if $Hom_R(E(R_R), R) \neq 0$ .*

**Proof.** ($\Rightarrow$:) Suppose that $E(R_R)$ is slightly compressible module. Then there exists a nonzero homomorphism $f : E(R_R) \to R$, and also $Hom_R(E(R_R), R) \neq 0$ (:$\Leftarrow$) Suppose that $Hom_R(E(R_R), R) \neq 0$. Then there exists a nonzero homomorphism $f : E(R_R) \to R$. Let $N$ be a submodule of $E(R_R)$. Then $N \cap R \neq 0$ because $R \leq_e E(R_R)$. Then $f(E)$ and $N \cap R$ are two nonzero ideals of $R$ ($E = E(R_R)$). Because $R$ is a prime ring then $(N \cap R)f(E) \neq 0$ and hence $af(E) \neq 0$ for some $a \in (N \cap R)$. Define a mapping $g : E(R_R) \to N$ by $g(e) = af(e)$ for all $e \in E(R_R)$. Thus $g \neq 0$ and so $E(R_R)$ is slightly compressible.

**Theorem 2.24.** *Let $R$ be a right hereditary ring and let $M$ be an injective right $R$-module such that $R$ satisfies the ascending chain condition on right ideals of the form $r(m)$ where $m \in M$. Then $M$ is slightly compressible if and only if $M$ is semisimple.*

**Proof.** By corollary 2.18 and the proof of Theorem 2.19

**Proposition 2.25.** *If $R$ is self injective ring then every finitely generated right $R$-module $M$ (not equal $R_R$) is semisimple.*

**Proof.** Since $R$ is right self-injective so it is right simple injective.

**Step 1:** Since $R$ is right simple injective ring then for any ideal $I$ of $R$, $cI$ is simple, for some $c \in R$. Write $cI$ to be of the form $a_iR$.

The epimorphism $\alpha_i : R \to a_iR$; $\alpha_i(r) = a_ir$ will induce the epimorphism $\alpha : \bigoplus_n R \to \bigoplus_n a_iR$, $\alpha(a_i) = (a_ir_i)$ and $Ker\alpha = \{(a_i) : (a_ir_i) = 0\}$.

Being $a_iR$ simple implies that $\bigoplus_n a_iR$ is semi simple.

**Step 2:** Let $M$ be a finitely generated module with generators $\{m_i : i = 1, 2, ........n\}$ so there is an epimorphism $\beta$ and a free module $F$ such that;

$\beta : F \to M$. Since each finitely generated module is a factor of a free module with finite rank so $\beta : F \cong \underset{n}{\oplus} R \to M \cong \frac{\underset{n}{\oplus} R}{K}$ is an epimorphism given by $\beta(x_i) = \sum m_i x_i$, where $K = Ker\beta$. On the other hand, $m_i = (r_i) + K$ so

$$\beta(x_i) = \sum[(r_i) + K](x_i)$$

$$Ker\beta = \{(x_i) \in \underset{n}{\oplus} R : \sum m_i x_i = 0\} = \{(x_i) \in \underset{n}{\oplus} R : (r_i)(x_i) \in K\}$$

If $(y_i) \in Ker\alpha$; $\alpha$ is as in step 1, then $\alpha((y_i)) = (a_i)(y_i) = 0$ for some $(a_i) \in \underset{n}{\oplus} R$. By which we get

$0 = \beta(0) = \beta(a_i y_i) = \sum[(r_i) + K](a_i y_i) = \sum[(r_i a_i) + K](y_i) \Rightarrow (y_i) \in Ker\beta$.

**H**ence $Ker\alpha \subseteq Ker\beta$ and $\beta$ induce an epimorphism.

$$\bar{\beta} : \underset{n}{\oplus} a_iR \cong \underset{n}{\oplus} R/Ker\alpha \to M.$$

Hence $M$ is epimorphic image of a semisimple module by which $M$ is semisimple and finitely generated.

**Proposition 2.26.** *Let $R$ be any right self-injective ring. Then any nonsingular right $R$-module is slightly compressible.*

**Proof.** Let $M$ be any nonzero nonsingular $R$-module. Let $0 \neq m \in M$. Then $R/r(m) \cong mR$ and hence $R/r(m)$ is nonsingular. Since $R$ is an injective $R$-module it follows that $r(m)$ is an injective $R$-module and hence so to is $mR$. Thus every cyclic submodule of $M$ is injective. By Lemma 2.13 $M$ is slightly compressible.

A ring $R$ is called right semiartinian provided every nonzero right $R$-module has nonzero socle. Clearly right Artinian rings are right semiartinian. Many examples of right semiartinian right $V$-rings are given in **[9].** Note the following result which should be compared with Theorem 2.12 .

**Proposition 2.27.** *Let $R$ be a right semiartinian right $V-$ring. Then every right $R-$module is slightly compressible.*

**Proof** Let $M$ be any nonzero right $R$-module. Let $N$ be any nonzero submodule of $M$. Let $U$ be a simple submodule of $N$. Then $U$ is injective so that $U$ is a direct summand of $M$. Clearly there exists a nonzero homomorphism $f : M \to U$ and hence $Hom_R(M, N) \neq O$.

## 2.2 Modules over Noetherian rings

**Lemma 2.28.** *Let $I$ be an ideal of an arbitrary ring $R$ and let $M$ be a right $R$-module. Then $Hom_R(M, R/I) \neq 0$ if and only if $Hom_{R/I}(M/MI, R/I) \neq O$.*

**Proof.** Suppose first that there exists a nonzero homomorphism $f : M \to R/I$. Then $f(MI) = f(M)I \subseteq (R/I)I = 0$, so that $f$ induces a nonzero $(R/I)$-homomorphism $f_1 : M/MI \to R/I$ defined by $f_1(m + MI) = f(m)$ for all $m \in M$. Conversely, suppose that there exists a nonzero $(R/I)$-homomorphism $g : M/MI \to R/I$. Then $g$ is an $R$-homomorphism. Let $p : M \to M/MI$ $p(m) = m + MI$ denote the canonical epimorphism. Then $gp : M \to R/I$ is a nonzero $R$-homomorphism.

**Lemma 2.29.** *Let $R$ be a right Noetherian ring. Then the following statements are equivalent for a right $R$-module $M$.*

*i) $M$ is slightly compressible .*

*(ii) $Hom_R(M, U)$ is nonzero for every uniform submodule $U$ of $M$.*

*(iii) $Hom_R(M, U)$ is nonzero for every cyclic uniform submodule $U$ of $M$.*

**Proof.**

(i) $\to$ (ii) $\to$ (iii) Clear.

(iii) $\to$ (i) Let $N$ be any nonzero submodule of $M$. Let $m$ be any nonzero element of $N$. By hypothesis, the module $mR$, is Noetherian and hence $mR$

contains a uniform submodule $U$. Let $u$ be any nonzero element of $U$. Then $uR$ is a cyclic uniform submodule of $M$. By (iii) $Hom_R(M, uR)$ is nonzero and hence $Hom_R(M, N)$ is nonzero.

**Definition 2.30.** An annihilator prime for a module $M$ over a ring $R$ is any prime ideal $P$ of $R$ such that $P = ann_R(N)$ where $N$ is a nonzero submodule of $M$.

**Definition 2.31.** An associated prime of $M$ is any annihilator prime $P = ann_R(N)$ where $N$ is a nonzero submodule of a $R$-module $M$ such that for every nonzero submodule $L$ of $N$, $P = ann_R(L)$. The set of associated prime ideals of $M$ will be denoted by $Ass(M)$. Nonzero right $R$-modules $M$ and $M'$will be said to be related if $Ass(M) \cap Ass(M')$ is nonempty.

**Lemma 2.32.** *If $R$ is a ring which satisfies the ACC on ideals and $P$ is maximal in, $\{ann_R(N) : 0 \neq N \leq M\}$, then $P$ is associated prime ideal of $M$.*

**Proof.** By assumption, there exists a nonzero submodule $N$ of $M$ such that $P = ann_R(N)$. Let $L$ be a nonzero submodule of $N$. If $P \neq ann_R(L)$, then $ann_R(L) \supsetneq P$. A contradiction by maximality of $P$ in $\{ann_R(N) : 0 \neq N \leq M\}$. Thus $P$ is an associated prime ideal of $M$.

**Lemma 2.33.** *Let $U$ be a uniform right module over a right Noetherian ring $R$. Then there is a unique prime ideal $P$ in $R$ such that $P$ equals the annihilator of some nonzero submodule of $U$ and $P$ contains the annihilators of all nonzero submodule of $U$. Moreover, $P$ is the unique associated prime of $U$.*

**Proof.** Let $0 \neq A \leq U$ such that the ideal $P = ann_R(A)$ is maximal among annihilators of nonzero submodule of $U$. If $B$ is nonzero submodule of $U$, then $A \cap B \neq 0$ and $P \leq ann_R(A \cap B)$, and so $ann_R(A \cap B) = P$ by the maximality of $P$, whence $ann_R(B) \leq P$. Thus $P$ contains the annihilators of all nonzero submodule of $U$. Let $Q$ be an associated prime of $U$. Then there is a nonzero submodule $C$ of $U$ such that $ann_R(C) = Q$. Since $0 \neq A \cap C$, $P = ann_R(A \cap C)$, then $P = Q$.

**Lemma 2.34.** *Let $R$ be a right Noetherian ring and let $N$ be a nonzero submodule of a nonzero right $R$-module $M$. Then $Ass(N) \subseteq Ass(M)$. In particular $M$ and $N$ are related.*

**Proof.** Clear.

**Theorem 2.35.** *Let $R$ be a right Noetherian ring and let $M$ be a nonzero right $R$-module such that $Hom_R(M, R/P) \neq 0$ for every associated prime ideal $P$ of $M$. Then $Hom_R(M, M') \neq 0$ for every right $R$-module $M'$ which is related to $M$. In particular, $M$ is slightly compressible.*

**Proof** Let $M'$ be any nonzero right $R$-module which is related to $M$. Let $P$ be any prime ideal of $R$ which belongs to $Ass(M) \cap Ass(M')$. There exists a nonzero submodule $N'$ of $M'$such that $P = ann_R(L')$ for every nonzero submodule $L'$of $N'$. By hypothesis, there exists a nonzero homomorphism $f : M \rightarrow R/P$. Then $f(M) = E/P$ for some right ideal $E$ of $R$ properly containing $P$. Clearly $N'E \neq 0$ and hence there exists $0 \neq x' \in N'$ such that $x'E \neq 0$. Define a mapping $g : E/P \rightarrow M'$ by $g(e + P) = x'e$ $(e \in E)$. Since $x'P = 0$ it follows that $g$ is well defined. Clearly $g$ is an $R$-homomorphism. Finally $gf : M \rightarrow M'$ is a nonzero homomorphism because $gf(m) = x'E \neq 0$. The last part follows by Lemma 2.34.

Note that the converse of Theorem 2.35 is false in general because there exist many right ideal Noetherian rings $R$ and nonzero slightly compressible right $R$-modules $M$ such that $Hom_R(M, R/P) = 0$ for some associated prime ideal $P$ of $M$. For example, let $R$ be a simple right Noetherian ring which is not Artinian. Let $M$ be any nonzero semisimple $R$-module. Clearly 0 is the only associated prime ideal of $M$. Thus $M$ is slightly compressible but $Hom_R(M, R) = 0$.

**Definition 2.36.** A ring $R$ is called right bounded if every essential right ideal an ideal which is essential as a right ideal. Next a ring $R$ is called *fully right bounded* provided every prime factor ring of $R$ is right bounded. By a right $FBN$ ring we shall mean a fully right bounded right Noetherian ring. For any ring $R$, a right

$R$-module $M$ will be called *bounded* if $R/P$ is a right bounded ring for every associated prime ideal $P$ of $M$. There exist right and left Noetherian rings which are not fully right (nor left) bounded such that every unfaithful right $R$-module is bounded. [ **31,Exercise 8B**]. An element $c$ of a general ring $R$ is called *regular* if, given $r \in R, rc = 0$ or $cr = 0$ implies $r = 0$. An $R$-module $M$ will be called *torsion free* provided, for any given $m \in M$, $mc = 0$ for some regular element $c$ in $R$ implies that $m = 0$.

**Definition 2.37.** A module $M$ over a ring $R$ is a faithful $R$-module if $ann_R(M) = 0$. A module $M$ over a ring $R$ is fully faithful provided $M$ and all nonzero submodules of $M$ are faithful $R$-modules. On the other hand, $M$ is called *torsion* if for each $x$ in $M$ there exists a regular element $d$ of $R$ such that $xd = 0$. Note that if $R$ is a semiprime right Goldie ring then a right $R$-module $M$ is *torsion free* if and only if $M$ is nonsingular [ **see for example 31, Proposition 6.9**]

**Lemma 2.38.** *Let $R$ be a semiprime right Goldie ring and let $M$ be a right $R$-module . If $M$ is not torsion, then $M$ has a uniform submodule isomorphic to a right ideal of $R$.*

**Proof.** Let $x \in M$ be not torsion, then $ann_R(x)$ is not essential right ideal of $R$, and hence there is a nonzero right ideal $I$ of $R$ such that $I \cap ann_R(x) = 0$. Since $R_R$ has finite rank, $I$ must contain a uniform right ideal $J$. $J \cap ann_R(x) = 0$ implies that $J \cong xJ$. Therefore $xJ$ is uniform submodule of $M$.

**Lemma 2.39.** *Let $P$ be a prime ideal of a ring $R$ such that the ring $R/P$ is a prime right bounded right Goldie ring. Let $U$ be a uniform right $R$-module with assassinator $P$. Then there exist a nonzero submodule $V$ of $U$ and an $R$-monomorphism $f : V \to R/P$.*

**Proof.** There exists a nonzero submodule $N$ of $U$ such that $P = ann_R(L)$ for every nonzero submodule $L$ of $N$. Consider the right $(R/P)$-module $N$. Let $u \in N$ such that $u(c + P) = 0$ for some $c \in R$ with the property that $c + P$ is a regular element of the ring $R/P$. By [**31, Lemma 5.5**] $(cR + P)/P$ is an essential right ideal of $R/P$ and hence there exists an ideal $A$ of $R$ properly containing $P$ such that $A \subseteq cR + P$. It follows that $u.A = 0$ and hence $A \subseteq ann_R(uR)$. But $P = ass(U)$, so that $u = 0$. Thus $N$ is not a torsion right $(R/P)$-module. By [ **31, Lemma 2.38**] there exist a nonzero submodule $V$ of $N$ and an $(R/P)$-monomorphism. $f : V \to R/P$. Finally note that $f$ is an $R$-monomorphism.

**Theorem 2.40.** *Let $R$ be a right Noetherian ring. Then the following statements are equivalent for a nonzero bounded right $R$-module $M$.*

*(i) $M$ is slightly compressible.*

*(ii) $Hom_R(M, R/P) \neq 0$ for every associated prime ideal $P$ of $M$.*

*(iii) $Hom_{R/P}(M/MP, R/P) \neq 0$ for every associated prime ideal $P$ of $M$.*

*(iv) $Hom_R(M, M') \neq 0$ for every nonzero right $R$-module $M'$ which is related to $M$.*

**Proof.**

(ii) $\Longrightarrow$ (iv) By Theorem 2.35
(ii) $\Longrightarrow$ (iii) By Lemma 2.28
(iv) $\Longrightarrow$ (i) By Theorem 2.35
(i) $\Longrightarrow$ (ii) Suppose that $M$ is slightly compressible. Let $P$ be any associated prime ideal of $M$. There exists a nonzero submodule $N$ of $M$ such that $P = ann_R(L)$ for every nonzero submodule $L$ of $N$. Let $U$ be any uniform submodule of $N$. Clearly $P = ass(U)$. By Lemma 2.39 there exists a nonzero submodule $V$ of $U$ and a monomorphism $f : V \to R/P$. By hypothesis there exists a nonzero homomorphism $g : M \to V$. Then $fg : M \to R/P$ is a nonzero homomorphism.

**Corollary 2.41.** *Let $R$ be a right $FBN$ ring. Then the following statements are equivalent for a nonzero right $R-$module $M$.*

*(i)* $M$ *is slightly compressible.*

*(ii)* $Hom_R(M, R/P) \neq O$ *for every associated prime ideal* $P$ *of* $M$*.*

*(iii)* $Hom_{R/P}(M, R/P) \neq 0$ *for every associated prime ideal* $P$ *of* $M$*.*

*(iv)* $Hom_R(M, M') \neq 0$ *for every nonzero right* $R$*-module* $M'$ *which is related to* $M$*.*

**Proof.** By Theorem 2.40

**Corollary 2.42.** *Let* $R$ *be right Noetherian ring and let* $M$ *be a nonzero slightly compressible bounded right* $R$*-module. Then* $M \neq MP$ *for every associated prime ideal* $P$ *of* $M$*.*

**Proof.** Clearly by Theorem 2.40

**Corollary 2.43.** *Let* $R$ *be a right Noetherian ring. Then any nonzero bounded projective right* $R$*-module* $X$ *is slightly compressible if and only if* $X \neq XP$ *for every associated prime ideal* $P$ *of* $X$*.*

**Proof.** The necessity is clear by Corollary 2.42. Conversely, suppose that $X \neq XP$ for any associated prime ideal $P$ of $X$. Let $Q$ be any associated prime ideal of $X$. By hypothesis, $X/XQ$ is a nonzero projective $(R/Q)$-module and hence $Hom_{R/Q}(X/XQ, R/Q)$ is nonzero. By Theorem 2.40 it follows that $X$ is slightly compressible.

## 2.3 Critically Compressible Modules

**Proposition 2.44.** *Let* $M$ *be compressible module. Then the following conditions are equivalent:*

*(1)* $M$ *is a critically compressible module.*

*(2) Every nonzero partial endomorphism of* $M$ *is a monomorphism.*

**Proof.**

(1)⇒(2) Let $N$ be a nonzero submodule of $M$ and let $f : N \to M$ be a nonzero partial endomorphism. Then we have an isomorphism $\bar{f} : N/\ker f \to f(N)$, and $f(N)$ is a nonzero submodule of $M$. Since $M$ is compressible, therefore there exists a monomorphism $h : M \to f(N)$. But the composition $M \to f(N) \to N/(\ker f) \subseteq M/(\ker f)$ is a monomorphism. Since $M$ is critically compressible, therefore $\ker f = 0$

(2)⇒(1) Assume that $M$ is not a critically compressible module. Then there is a nonzero proper submodule $N$ of $M$ such that $M$ is embedded in $M/N$, say $g : M \to M/N$ is the monomorphism. Let $T$ be a submodule such that $T/N = g(M)$. Then we have the composition, $T \xrightarrow{\pi} T/N \to M$ is a nonzero partial endomorphism which must be a monomorphism by assumption, where $\pi$ is canonical projection, and the kernel of composition is nonzero, which is a contradiction. Hence, $M$ is critically compressible.

**Proposition 2.45.** *Let $M$ be a compressible module over a ring $R$. Then following conditions are equivalent :*

*(1) $M$ is uniform and torsion free.*

*(2) Every nonzero partial endomorphism on $M$ is monomorphism. Moreover, if $R$ is assumed to be commutative, then one more equivalent condition (3) is obtained*

*(3) $M$ is critically compressible.*

**Proof.**

(1)⇒(2) Let $N$ be a nonzero submodule of $M$. Let $f : N \to M$ be a nonzero partial endomorphism, then $f(n) \neq 0$ for some $0 \neq n \in N$. Suppose $\ker f \neq 0$, then due to uniformity of $M$, $\ker f \cap Rn \neq 0$. Hence there exists a nonzero $\alpha n \in \ker f (\alpha \in R)$, such that $f(\alpha n) = 0 = \alpha f(n)$, which is a contradiction because $M$ is torsion free and $\alpha$ is nonzero. Hence $\ker f = 0$, thus every nonzero partial endomorphism on $M$ is a monomorphism.

(2)$\Rightarrow$(3) It follows from Proposition 2.44

(3)$\Rightarrow$(1) It is easy to prove that $M$ is uniform. Since $M$ is critically compressible, therefore every nonzero partial endomorphism on $M$ is a monomorphism. Suppose $M$ is not torsion free and take a torsion element $x \in M$. Then there exists a nonzero element $\alpha \in R$ such that $\alpha x = 0$.
Now define a nonzero partial endomorphism $f_\alpha : M \to M$ $f_\alpha(y) = \alpha y$ where $y \in M$. Then $f_\alpha(x) = \alpha x = 0$ and so $x \in \ker f_\alpha$, but due to critically compressibility of $M$, $f_\alpha$ will be a monomorphism, then $x = 0$ which is a contradiction. Therefore, $M$ is torsion free.

**Proposition 2.46.** *Let $M$ be a compressible module. If every nonzero submodule of $M$ is cyclic and torsion free, then $M$ will be critically compressible.*

**Proof.** Let $N$ be a nonzero cyclic submodule of $M$ and $f$ be a nonzero partial endomorphism from $N$ to $M$, then $f(n) \neq 0$ for some $0 \neq n \in N$. Since otherwise $f(N) = f(Rn) = 0$, it is a contradiction. Now let $\alpha n \in \ker f \Rightarrow f(\alpha n) = 0 \Rightarrow \alpha f(n) = 0$.

Since $M$ is torsion free and $f(n) \neq 0 \Rightarrow \ker f = 0$, every nonzero partial endomorphism on $M$ is a monomorphism. Then by the Proposition 2.45, $M$ will be critically compressible.

**Proposition 2.47.** *If $R$ be a torsion free ring with unity then $R_R$ is a compressible module if and only if $R_R$ is critically compressible module.*

**Proof.** Let $R_R$ be a compressible module. Now $R = R.1_R$, then $R$ is cyclic. Since $R$ is also torsion free, then from the Proposition 2.46, $R_R$ is critically compressible. The converse is obvious.

**Proposition 2.48.** *Every critically compressible module $M$ is indecomposable.*

**Proof.** Let $M$ be a critically compressible module and $M = M_1 \oplus M_2$ with $0 \neq M_1, M_2 \neq M$. Let $p_i$ be the projection map from $M$ to $M_i$ for $i = 1, 2$. Due to compressibility of $M$, $p_i$ is a monomorphism, then $Kerp_i = M_j = 0$ for $i \neq j$. Hence, $M$ is indecomposable.

**Proposition 2.49.** *An injective indecomposable compressible module $M$ whose nonzero partial endomorphisms has uniform nonsingular kernels, then $M$ will be critically compressible.*

**Proof.** Let $N$ be a nonzero submodule of $M$ and let $f : N \to M$ be a nonzero partial endomorphism and suppose $\ker f \neq 0$. Since $\ker f$ is uniform, therefore $M = E(\ker f)$ [**18, Proposition 2.2**], then $M$ is nonsingular. By [**11, Proposition 5, p.59**] $M$ is a rational extension of $\ker f$. Hence $f(N) = 0 \Rightarrow f = 0$ which is a contradiction. Therefore, $\ker f = 0$ implies that every nonzero partial endomorphism on $M$ is monomorphism. Therefore, by Proposition 2.44, $M$ is critically compressible.

**Proposition 2.50.** *Every critically compressible module is a $CS$ (extending module) and it is continuous if and only if every partial monomorphism is an isomorphism.*

**Proof.** Let $M$ be a critically compressible module, then by Proposition 2.45, $M$ will be uniform. Hence, every critically compressible module is a $CS$ module(extending module). Let $f : M \to M$ be a nonzero partial endomorphism on $M$. Due to critically compressibility of $M$, $f$ will be a monomorphism. Suppose $f$ is not surjective, then $f(M) \subset M$ and $f(M)$ will be a proper essential submodule of $M$. Now define $g : M \to f(M)$ by $g(x) = f(x)$, then $g$ is an isomorphism. Since $M$ is continuous, $f(M)$ being an isomorphic copy of $M$, can not be a proper essential submodule of $M$, which is a contradiction. Hence, $f$ is surjective. The converse is obvious.

**Proposition 2.51.** *Let $M$ and $E(M)$ be compressible modules. If $E(M)$ is critically compressible, then $M$ is also critically compressible and the converse holds if:*

*(1) $M$ is stable under partial endomorphism of $E(M)$.*

*(2) $E(M)$ is a rational extension of $M$.*

**Proof.** Let $E(M)$ be a critically compressible module. Then every nonzero partial endomorphism on $E(M)$ is a monomorphism. Let $N$ be a nonzero submodule of $M$ and $f : N \to M$ be a nonzero partial endomorphism on $M$ with $\ker f \neq 0$. Let $g : M \to E(M)$ be any nonzero partial endomorphism on $E(M)$, then due to critically compressibility of $E(M)$, $g$ will be a monomorphism. Now $\ker g \supseteq \ker f = 0$. Hence $f$ is a monomorphism. Therefore, by the Proposition 2.44, $M$ will be critically compressible. Conversely, let $M$ be a critically compressible module and suppose $M_1$ be a nonzero submodule of $E(M)$ and $h : M_1 \to E(M)$ be a nonzero partial endomorphism on $E(M)$. Suppose $\ker h \neq 0$. Since $M$ is essentially submodule of $E(M)$, then $\ker h \cap M \neq 0$ and also $M \cap M_1 = N \neq 0$. Now $f = h_{|N} : N \to M$ is a partial endomorphism on $M$ since $M$ is stable under partial endomorphism on $E(M)$. Since $M$ is critically compressible, $f$ is a monomorphism. On the other hand $\ker f = \ker h \cap N = (M_1 \cap M) \cap \ker h = M \cap \ker h \neq 0$ which is a contradiction. Hence $E(M)$ is critically compressible.

**Corollary 2.52.** *Let $M$ and $E(M)$ be compressible modules and $M$ be a nonsingular module stable under partial endomorphism of $E(M)$. Then $E(M)$ is a critically compressible module if and only if $M$ is a compressible module.*

**Proof.** By Proposition 2.51.

**Proposition 2.53.** *Let $R$ be a S.P. ring. Then the critically compressible module $R_R$ is either singular or nonsingular.*

**Proof.** Let $R_R$ be neither singular nor nonsingular, then $Z(R_R)$ is a nonzero direct summand and proper submodule of $R_R$. Now by the Proposition 2.48, $R_R$ is indecomposable. Hence $Z(R_R) = R_R$, which is a contradiction. Hence $R_R$ is either singular or nonsingular.

**Proposition 2.54.** *A ring $R$ with unit $1_R \neq 0$ and $R_R$ is a left $R$-module in which every nonzero endomorphism is a monomorphism is self similar if and only if it is a principal ideal ring without zero divisors.*

**Proof.** Assume $R_R$ is self similar in which every nonzero endomorphism is a monomorphism. Then $R$ has no zero divisors. Now to prove that $R$ is a principal ideal ring, let $S$ be any nonzero ideal of $R$ and $r \neq 0$ be an arbitrary element in $R$. Let $f : R \to rR$ and $g : rR \to S$ are modules isomorphisms.

Suppose $f(1_R) = rx$ and $g(rx) = b$ for some $x \in R$ and $b \in S$.

Hence for any $y \in S$ then there exists an $a \in R$ such that $y = (g \circ f)(a)$ for some $a \in R$, $y = (g \circ f)(a \cdot 1) = g(af(1)) = ag(rx) = ab \in R$. Hence, $S = Rb$ is a principal ideal. The converse is obvious.

**Corollary 2.55.** *If $R$ is a principal ideal ring without zero divisors, then $R_R$ is critically compressible.*

**Proof.** If $R$ is a Principal ideal ring without zero divisors, then by Proposition 2.54, $R_R$ is self-similar in which every nonzero endomorphism is a monomorphism. Then from [**Theorem 4.1, 17**], $R_R$ is critically compressible.

**Proposition 2.56.** *Let $M$ be a compressible module and every nonzero $f \in Hom(M, M')$ is a monomorphism, where $M'$ is a quasi-injective hull of $M$. Then $M$ is critically compressible module.*

**Proof.** For a nonzero submodule $N$ of $M$. Let $g : N \to M$ be a nonzero partial endomorphism on $M$, such that $f_{|N} = g$. By assumption, $f$ is a monomorphism and so $g$ is monomorphism. Hence, every nonzero partial endomorphism on $M$ is a monomorphism. Therefore, $M$ is critically compressible by the Proposition 2.44.

**Proposition 2.57.** *Let $R$ be a noetherian ring, then following conditions are equivalent:*

*(i)* *$R_R$ is a compressible module.*

*(ii)* *$R_R$ is a critically compressible module.*

*(iii)* *Monomorphism $f : R \to U$ is nonzero for every uniform submodule $U$ of $R$.*

*(iv) For every cyclic uniform submodule $U$ of $R$ there exists a monomorphism $f : R \to U$*

**Proof.**

(i)$\Rightarrow$ *(ii)* It follows from [**17, Theorem 3.2**]

(ii)$\Rightarrow$(iii)$\Rightarrow$ *(iv)*. It is obvious.

(iv)$\Rightarrow$(i) Let $N$ be a nonzero submodule of $R$. Let $m$ be a nonzero element of $N$. By hypothesis, $mR$ is Noetherian and hence $mR$ contains a uniform submodule $U$ of $R$. Let $u$ be a nonzero element of $U$. Then $uR$ is a cyclic uniform submodule of $R$. By (iv) monomorphism from $R$ into $uR$ is nonzero. Hence the monomorphism from $R$ into $N$ is nonzero. Hence, $R_R$ is a compressible module.

# CHAPTER 3

# ESSENTIALLY COMPRESSIBLE MODULES AND RINGS

In this chapter, we have introduced the concept of essentially compressible modules and rings which are generalizations of essentially compressible modules and rings defined by P.F.Smith [**27**] in 2006. Throughout this chapter, all rings are associative with identity and all modules are unital right $R$-modules.

## 3.1 Essentially Compressible Module

**Definition 3.1.** An $R$-module $M$ is called *essentially compressible* if for every essential submodule $N$ of $M$, there exists a nonzero monomorphism from $M$ to $N$.

**Definition 3.2.** An $R$ module $M$ is said to be *co-Hopfian* if every injective endomorphism of $M_R$ is an isomorphism.

**Definition 3.3.** We shall call an $R$-module $M$ is *essentially slightly compressible* if for each nonzero essential submodule $N$ of $M$, there exists a nonzero homomorphism from $M$ to $N$.

**Definition 3.4.** We shall say that an $R$-module $M$ is *subisomorphic* to an $R$-module $M^1$ if there exist $R$-monomorphisms $f : M \to M^1$ and $g : M^1 \to M$, and in this case we call the modules $M$ and $M^1$ subisomorphic.

**Example 3.5.** *Let $R = (Z_4, +, \cdot)$ (the ring of integer modulo 4). Then $R_R$ is an essentially slightly compressible module but not an essentially compressible module.*

**Proposition 3.6.** *The following statements are equivalent for a module $M$;*

*(a) $M$ is essentially compressible module.*

*(b) $M$ is subisomorphic to an essentially compressible module.*

*(c) $M$ contains an essentially compressible submodule $N$ such that there exists a monomorphism $\varphi : M \to N$.*

*(d) There is an essentially monomorphism $\psi : M \to M'$ for some essentially compressible module $M'$.*

**Proof.**

(a) $\Rightarrow$(b) and

(a) $\Rightarrow$(d) are clear.

(b) $\Rightarrow$(c) Suppose that there exist an essentially compressible module $M'$and monomorphisms $\alpha : M \to M'$, $\beta : M' \to M$. Let $N = \beta(M')$. Then $N$ is an essentially compressible submodule of $M$ and $\beta\alpha : M \to N$ is a monomorphism.

(c) $\Rightarrow$(a) Let $L$ be essential submodule of $M$. Then $L \cap N$ is an essential submodule of $N$ and so there is a monomorphism $\theta : N \to L \cap N$. If $i : L \cap N \to L$ is the inclusion mapping then $i\theta\varphi : M \to L$ is a monomorphism. It follows that $M$ is essentially compressible.

(d) $\Rightarrow$(b) Let $N = \psi(M)$. By our assumption. $M'$ can be embedded in $N$ and hence in $M$. Thus $M$ and $M'$are subisomorphic.

**Proposition 3.7.** *Every direct sum of essentially compressible modules is essentially compressible.*

**Proof.** Let a module $M = \bigoplus_{i \in I} M_i$be a direct sum of essentially compressible modules $M_i (i \in I)$, for some (nonempty) index set $I$. Let $L$ be an essential submodule of $M$. Then, for each $i \in I$ , $L \cap M_i$ is an essential submodule of $M_i$

and hence there exists a monomorphism $\theta_i : M_i \to L \cap M_i$. Clearly the mapping $\theta = \sum_{i \in I} \theta_i : M \to L$ is a monomorphism.

**Proposition 3.8.** *Let $M$ be a nonzero essentially compressible right $R$-module.*

*(a) If $N$ is either an essential submodule of $M_R$ or a submodule which is invariant under injective endomorphisms of $M_R$ then $N_R$ is also an essentially compressible module.*

*(b) If $N$ is a submodule of $M_R$ such that $\theta(N) + \theta^{-1}(N) \subseteq N$ for every monomorphism $\theta : M \to M$, then $M/N$ is an essentially compressible module.*

*(c) $M_R$ is semisimple co-Hopfian if and only if $M_R$ has a co-Hopfian essentially compressible submodule.*

*(d) $ann_R(M)$ is a semiprime ideal of $R$.*

*(e) $\hat{M}_R$ has no fully invariant essential submodule.(where $\hat{M}_R$ is an injective hull of $M$ )*

*(f) If an addition $M_R$ is finitely generated, then $M_R$ does not contain an infinite direct sum of nonzero fully invariant submodules.*

**Proof**

(a) Let $N$ be a submodule of $M$. If $N$ is essential then it is easy to check that $N_R$ is essentially compressible. Let $N$ be invariant under injective endomorphisms of $M_R$ and let $K$ be any essential submodule of $N_R$. There exists a submodule $N'$ of $M_R$ such that $K \cap N' = 0$ and $N \oplus N'$ is an essential submodule of $M_R$. Then $K \oplus N'$ is essential in $N \oplus N'$, and hence $K \oplus N'$ is essential in $M_R$. So by our assumption there is a monomorphism $f : M \to K \oplus N'$. Now $f(N) \subseteq N$ by our assumption, so

$f(N) \cap N' = 0$. It follows that $f(N)$, and hence $N$, is embedded in

$(K \oplus N')/N' \simeq K$. Therefore $N_R$ is essentially compressible.

(b) Let $L$ be any submodule of $M_R$, containing $N$, such that $L/N$ is an essential submodule of $M/N$. It is easy to check that $L$ is an essential submodule of $M_R$. By hypothesis, there exists a monomorphism $\varphi : M \to L$.

Because $\varphi(N) + \varphi^{-1}(N) \subseteq N$, it follows that the induced mapping $\bar{\varphi} : M/N \to L/N$, defined by $\bar{\varphi}(m + N) = \varphi(m) + N$ for each $m \in M$, is a monomorphism. The result follows.

(c) The necessity is clear. Conversely, clearly any co-Hopfian essentially compressible module is semisimple. Thus by (a), $M_R$ has an essential socle. It follows that $M_R$ is semisimple.

(d) Let $A = ann_R(M)$. Note that $A$ is a proper ideal of $R$. Let $B$ be any ideal of $R$ such that $B^2 \subseteq A$. Let $L = \{m \in M : mB = 0\}$. If $0 \neq m \in M$ then $mB^2 = 0$ so that there exists $n \in \{0, 1\}$ such that $mB^n \neq 0$ and $mB^{n+1} = 0($ say $B^0 = R)$ and in this case $0 \neq mB^n \subseteq mR \cap L$. It follows that $L$ is an essential submodule of $M_R$. By hypothesis, there exists a monomorphism $\theta : M \to L$.

Then $\theta(MB) = \theta(M)B \subseteq LB = 0$, so that $MB = 0$ and hence $B \subseteq A$. Thus $A$ is a semiprime ideal.

(e) Let $N$ be a fully invariant essential submodule of $\hat{M}_R$. By hypothesis, there exists a submodule $L$ of $N$ and an isomorphism $\theta : L \to M$. Now $\theta$ can be extended to $\bar{\theta} \in End_R(\hat{M})$. Thus we have $M = \theta(L) = \bar{\theta}(L) \subseteq \bar{\theta}(N) \subseteq N$. It follows that $\hat{M} = End_R(\hat{M})M \subseteq End_R(\hat{M})N = N$.

(f) Let $N = N_1 \oplus N_2 \oplus ......$ be any direct sum of fully invariant submodules of $M_R$. It is well known that there exists a submodule $K$ of $M_R$ such that $N \cap K = 0$ and $N \oplus K$ is an essential submodule of $M_R$. By our assumption, there is a monomorphism $\varphi : M \to N \oplus K$. Since $M_R$ is finitely generated , we can assume that $\varphi(M) \subseteq N_1 \oplus ...... \oplus N_t \oplus K$ for some positive integer $t$. It follows, by hypothesis, that

$\varphi(N_{t+1} \oplus N_{t+2}.....) \subseteq \varphi(M) \cap (N_{t+1} \oplus N_{t+2}.....) = 0.$

Thus $N_{t+1} \oplus N_{t+2}....$ must be zero.

**Corollary 3.9.** *Let $M$ be an essentially compressible module over a commutative ring $R$. Then the uniform dimension of ${}_sM$ is finite where $S = End_R(M)$.*

**Proof.**

Let $N$ be an $S$-submodule of $M$ and $r \in R$. Define $\theta_r : M \to M$ by $\theta_r(m) = mr$. Because $R$ is commutative, $\theta_r \in S$. Thus $\theta_r(N) \subseteq N$. It follows that $N$ is a fully invariant submodule of $M_R$. The result is now clear by proposition 3.8(f).

Let $R$ be a ring. A nonzero right $R$-module $M$ is said to be prime if $ann_R(M) = ann_R(N)$ for all nonzero submodules $N$ of $M_R$ ( similarly, prime left $R$-modules are defined).

**Proposition 3.10.** *Let $M_R$ be a finitely generated essentially compressible module with $S = End_R(M)$. If ${}_sM$ is a prime module then for each nonzero submodule $U$ of $M_R$, there exist a positive integer $n$ and $f_i \in Hom_R(U, M)(1 \leq i \leq n)$ such that $M$ can be embedded into $\sum_i^n f_i(U)$. If furthermore, $M_R$ is nonsingular then $M_R$ has finite uniform dimension if and only if $M_R$ has a uniform submodule.*

**Proof.** Let $U$ be any nonzero submodule of $M_R$.

Let $N = \sum\{f(U); f : U_R \to M_R\}$. It is easy to check that $N$ is a nonzero fully invariant submodule of $M_R$. There exists a submodule $K$ of $M_R$ such that $N \oplus K$ is essential in $M_R$. By hypothesis, there is monomorphism $\theta : M \to N \oplus K$. Let $\varphi = \pi\theta$ where $\pi : N \oplus K \to K$ is the canonical projection.

Now $\varphi(N) \subseteq N \cap K = 0$, and so by the prime condition on ${}_sM$, $\varphi$ is zero element of $S$.

It follows that $\theta(M) \subseteq N$. Since $M_R$ is finitely generated, there exist a positive integer $n$ and $f_i \in Hom_R(U, M)(1 \leq i \leq n)$ such that;

$\theta(M) \subseteq f_1(U) + ..... + f_n(U)$. Now the first statement is clear. For the second part, let furthermore, $M_R$ be nonsingular. If $U$ is uniform, define a mapping

$\Phi : U^{(n)} \rightarrow f_1(U) + ...... + f_n(U)$ by $\Phi(u_1, .....u_n) = f_1(u_1) + ..... + f_n(u_n)$ for all $u_i \in U (1 \leq i \leq n)$. Clearly $\Phi$ is a homomorphism. Note that the module $U^{(n)}/\ker\Phi$, is nonsingular. By [**9, 1.10 and 30,5. 10**], $\sum_{i=1}^{n} f_i(U) \simeq U^{(n)}/\ker\Phi$ has finite uniform dimension and hence so too does $M_R$. It is well known that every module with finite uniform dimension has a uniform submodule. Thus $M_R$ has finite uniform dimension if and only if $M_R$ has a uniform submodule, as desired.

The following lemma is also needed.

**Lemma 3.11.** *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform submodules $M_i (i \in I)$. Let $N$ be any nonzero submodule of $M$. Then there exists a subset $I'$ of $I$ and an essential monomorphism $\theta : N \rightarrow \bigoplus_{i \in I'} M_i$.*

**Proof.** If $N \cap M_i \neq 0$ for all $i \in I$ then $N$ is an essential submodule of $M$ and there is nothing to prove. Suppose that $N \cap M_j = 0$ for some $j \in I$.

By Zorn's Lemma there exists a maximal subset $I''$ of $I$ such that; $N \cap (\bigoplus_{i \in I''} M_i) = 0$. Note that $I''$ is a proper subset of $I$ and hence $I' = I \backslash I''$ is a nonempty set. Let $\pi : M \rightarrow \bigoplus_{i \in I'} M_i$ denote the canonical projection. Then $\pi_{|_N} : N \rightarrow \bigoplus_{i \in I'} M_i$ is a monomorphism because $\ker(\pi_{|_N}) = N \cap (\bigoplus_{i \in I''} M_i) = 0$. Let $k \in I'$. By the choice of $I''$, $N \cap \{M_k \oplus (\bigoplus_{i \in I''} M_i)\} \neq 0$ and hence $\pi(N) \cap M_k \neq 0$. It follows that $\pi(N) \cap M_i \neq 0$ for all $i \in I'$ so that $\pi(N)$ is an essential submodule of $\bigoplus_{i \in I'} M_i$.

**Proposition 3.12.** *Let $M$ be a direct sum of uniform compressible modules. Then any nonzero submodule of $M$ is an essentially compressible module.*

**Proof.**

By Lemma 3.11 and Proposition 3.7 and 3.8(a).

**Proposition 3.13.** *Let $\hat{M}_R$ be an essentially compressible module. Then:*

*(a) Either $M_R$ is semisimple module or $M_R$ has an infinite descending chain $B_1 \supset A_1 \supset B_2 \supset A_2 \supset .....$ such that each $A_i$ is an essential submodule of $B_i$ and each $B_i$ is a submodule of $M_R$ isomorphic to $\hat{M}_R$;*
*(b) If $M_R$ has DCC on direct summands then $M_R$ is a semisimple module; and*
*(c) $\hat{M} = Z(\hat{M}) \oplus L$ for some essential compressible submodule $L$.*

**Proof.**

(a) Let $\hat{M}_R$ be essentially compressible and not semisimple, then $M_R$ has a submodule $B_1$ isomorphic to $\hat{M}_R$. Since $M_R$ is not semisimple, $B_1$ is not semisimple and so it has a proper essential submodule $A_1$. Again by the essentially compressible condition on $B_1$, $A_1$ has a submodule $B_2$ isomorphic to $B_1$. But since $A_1$ is not $M$-injective (other-wise $A_1 = B_1$), $B_2$ is a proper submodule of $A_1$. Proceed to obtain the infinite descending chain
$B_1 \supset A_1 \supset B_2 \supset A_2 \supset .....$, as desired.
(b) It follows from (a).
(c) Let $Z = Z(\hat{M}_R)$. Then there exists a submodule $K$ of $\hat{M}_R$ such that $N = Z \oplus K$ is essential in $\hat{M}_R$. By our assumption $N$ has a submodule $A$ isomorphic to $\hat{M}_R$.
Because $A$ is $M$-injective, there is a submodule $B$ of $N$ such that $N = A \oplus B$. Now, we have $Z = Z(N) = Z(A) \oplus Z(B)$. Consequently, we have $A \subseteq N = Z(A) \oplus C$ where $C = Z(B) \oplus K$. It follows that $A = Z(A) \oplus (C \cap A)$. Because $A \simeq \hat{M}_R$, $\hat{M} = Z \oplus L$ for some submodule $L$. Also $L \simeq \hat{M}/Z$ is an essentially compressible right $R$-module by proposition 3.8(b), as desired.

**Theorem 3.14.** *The following statements are equivalent for a module $M$.*

*(a) $M$ is essentially compressible.*

*(b) $M \simeq M_1 \oplus M_2$ where $M_1$ is a semisimple module and $M_2$ is an essentially compressible module with zero socle.*

*(c) $M$ is subisomorphic to $M_1 \oplus M_2$ where $M_1$ is a nonsingular essentially compressible module and $M_2$ is a singular essentially compressible module.*

**Proof.**

(b)$\Rightarrow$(a) and (c)$\Rightarrow$(a), by Proposition 3.6 and 3.7

(a)$\Rightarrow$(b) Let $M$ be an essentially compressible module and $S = Soc(M)$. Then there exists a submodule $K$ of $M$ such that $N = S \oplus K$ is an essential submodule. By hypothesis, there is a monomorphism $\theta : M \to N$. Let $U = Soc(\theta(M))$. Then $U \subseteq Soc(N) = S$. Because $S$ is semisimple, $S = U \oplus L$ for some submodule $L$ of $S$. Thus we have $U \subseteq \theta(M) \subseteq N = U \oplus L \oplus K$. It follows that $U$ is a direct summand of $\theta(M)$. Because $M$ is isomorphic to $\theta(M)$, there is submodule $N'$ of $M$ such that $M = S \oplus N'$. Now by Proposition 3.8(b) $N'$ is also essentially compressible, as desired.

(a)$\Rightarrow$(c) Let $Z = Z(M)$ and $Z \oplus L$ be essential in $M$ for some submodule $L$ of $M$. The module $Z \oplus L$ is essentially compressible by Proposition 3.8(a) and is subisomorphic to $M$ because there is a monomorphism $\varphi : M \to Z \oplus L$. By Proposition 3.8, $Z$ and $L \simeq (Z \oplus L)/Z$ are essentially compressible modules.

**Theorem 3.15.** *For any ring $R$, every nonsingular essentially compressible $R$-module is isomorphic to a submodule of a free $R-$module.*

**Proof.**

Let $M$ be any nonzero nonsingular essentially compressible right $R$-module. By Zorn's Lemma there exist an index set $I$ and nonzero elements $m_i \in M(i \in I)$ such that $\sum_{i\in I} m_i R$ is an essential submodule of $M_R$. Let $i \in I$. Let $A_i = r.ann_R(m_i)$. Then $A_i$ is a right ideal of $R$ and hence there is a right ideal $B_i$ of $R$ such that $A_i \oplus B_i$ is an essential right ideal of $R$. Let $r \in R$ such that $m_i r \neq 0$. Then $rE \subseteq A_i \oplus B_i$ for some essential right ideal $E$ of $R$. Because $M$ is nonsingular, we have $0 \neq m_i rE \subseteq m_i B_i$. Thus $m_i B_i$ is an essential submodule of $m_i R$. Note further that the mapping $\varphi_i : B_i \to m_i B_i$ defined by $\varphi_i(b) = m_i b (b \in B_i)$ is clearly an isomorphism.

Finally note that $\bigoplus_{i\in I} m_i B_i$ is an essential submodule of $M$ and is isomorphic to the submodule $\bigoplus_{i\in I} B_i$ of the free $R$-module $R^{(I)}$. Because $M$ is essentially compressible, there exists a monomorphism $\theta : M \to R^{(I)}$.

**Theorem 3.16.** *The following statements are equivalent for a nonzero module $M$ over an arbitrary ring $R$.*

*(a) $M_R$ is nonsingular, essentially compressible and has enough uniforms.*

*(b) $M_R$ is subisomorphic to a direct sum $\bigoplus_{i\in I} A_i$ of nonsingular uniform right ideals $A_i$ $(i \in I)$ of $R$ such that $A_i$ does not contain a nonzero nilpotent right ideal of $R$ for each $i \in I$.*

*(c) $M$ is nonsingular and $M$ embeds in a direct sum of uniform compressible right $R$-modules.*

**Proof.**

(a)⇒(b) By Zorn's Lemma there exists a maximal collection of uniform cyclic submodules $U_i(i \in I)$ of $M_R$ such that $\sum_{i\in I} U_i$ is direct. It is easy to check that $\bigoplus_{i\in I} U_i$ is an essential submodule of $M_R$. Let $i \in I$ and let $U = U_i$. There is $x \in U$ such that $U = xR$. Let $C = r.ann_R(x)$. Note that $U$ is a nonsingular $R$-module so that $C$ is not an essential right ideal of $R$. There exists a nonzero right ideal $A$ of $R$ such that $A \cap C = 0$. Note that $A \simeq xA$ and that $A$ is a nonsingular uniform right ideal of $R$.

Let $B$ a right ideal of $R$ such that $B^2 = 0$ and $B \subseteq A$. By Proposition 3.8(d), $MB = 0$ so that $xB = 0$ and we have $B \subseteq A \cap C = 0$. Thus $A$ does not contain a nonzero nilpotent right ideal of $R$. Consequently, for each $i \in I$ there is a nonsingular uniform right ideal $A_i$ of $R$ such that $A_i$ does not contain a nonzero nilpotent right ideal of $R$ and $A_i$ is isomorphic to a nonzero submodule of $U_i$. It follows that $\bigoplus_{i\in I} A_i$ is isomorphic to an essential submodule of $M_R$. Because $M_R$ is essentially compressible, there exists a monomorphism $\theta : M \to \bigoplus_{i\in I} A_i$. This proves (b).

(b)⇒(c) Suppose that (b) holds. Clearly $M_R$ is nonsingular. It is enough to show that $A_i$ is a compressible right $R$-module for each $i \in I$. So suppose that $i \in I$ and let $A = A_i$.

If $B$ is a nonzero submodule of $A$ then by our assumption, $BA \neq 0$. Then there exists $b \in B$ such that $bA \neq 0$.

It follows that the mapping $\theta_b : A \to B$ defined by $\theta_b(a) = ba$ for all $a \in A$, is a nonzero homomorphism. Note that any nonzero homomorphism from a uniform module to a nonsingular module is a monomorphism. This proves that $A$ is compressible, as desired.

(c)$\Rightarrow$(a) If $m$ is any nonzero element of $M$ then $mR$ can be embedded in a finite direct sum of uniform modules and hence $mR$ contains a uniform submodule. Thus $M$ has enough uniforms.

Finally, $M$ is essentially compressible by Proposition 3.12.

## 3.2 Essentially Slightly Compressible Modules

**Proposition 3.17.** *Let $R$ be a ring. Then every direct sum of essentially slightly compressible $R$-modules is essentially slightly compressible.*

**Proof.** Let an $R$-module $M = \bigoplus_{i \in I} M_i$ be a direct sum of essentially slightly compressible modules $M_i (i \in I)$, for some index set $I$ and $N$ be essential submodule of $M$, then for each $i$, $N \cap M_i$ is an essential submodule of $M_i$. Hence there exists a nonzero homomorphism $\theta_i : M_i \to N \cap M_i$. Let $\theta : M \to N$; $\theta(m) = \sum_{i \in I} \theta_i(m)$ is a nonzero homomorphism.

**Lemma 3.18.** *Let $R$ be a ring and $M$ be a right $R$-module such that every nonzero essential submodule contains a nonzero direct summand of $M$. Then $M$ is essentially slightly compressible.*

**Proof.** It is obvious.

**Corollary 3.19.** *Let $R$ be a right hereditary ring and $M$ an injective $R$-module. Then $M$ is essentially slightly compressible if and only if every nonzero essential submodule contains a nonzero direct summand of $M$.*

**Proof.** It follows from [**25, Corollary 1.8**]

**Corollary 3.20.** *Let $R$ be a ring and $M$ a nonsingular injective $R$-module. Then $M$ is essentially slightly compressible if and only if every nonzero essential submodule contains a nonzero direct summand of $M$.*

**Proof.** The sufficiency is obvious from Lemma 2.13. Since every slightly compressible module is essentially slightly compressible, then from the Corollary 2.14, if every nonzero essential submodule contains a nonzero direct summand of $M$, then $M$ is essentially slightly compressible.

**Corollary 3.21.** *Let $R$ be a ring and $M$ is an essentially slightly compressible injective $R$-module. Then any nonsingular uniform submodule of $M$ is simple and injective.*

**Proof.** Let $U$ be a nonsingular uniform submodule of $M$. Let $V$ be a nonzero submodule of $U$. From Corollary 3.20, there exists an injective submodule $W$ of $V$ such that $W = V = U$. Hence $U$ is simple and injective.

**Proposition 3.22.** *Let $R$ be any ring and $M$, a nonsingular injective $R$-module such that $mR$ has finite rank for every element $m \in M$ . Then the following conditions are equivalent:*

*(i)* $M$ *is essentially slightly compressible.*

*(ii)* $M$ *is semi simple.*

*(iii)* $M$ *is essentially compressible.*

**Proof.**

(i)$\Rightarrow$ $(ii)$ Suppose $M$ is essentially slightly compressible and let $m \in M$. Therefore $R/o(m)$ is a nonsingular $R$-module which has a finite rank, and by[**9, Lemma 5.10.**] $R$ satisfies the ascending chain condition on right ideal of the $o(m), m \in M$. Now according to [**9, Corollary 8.2**], $M$ is a direct sum of uniform submodules. Hence, by Corollary 3.19, $M$ is semisimple.
(ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(i) It is obvious.

**Proposition 3.23.** *Let $R$ be a right noetherian ring. Then the following statements are equivalent for a uniform $R$-module $M$:*

*(i)* *$M$ is essentially slightly compressible.*

*(ii)* *$M$ is slightly compressible.*

*(iii)* *$Hom_R(M, U)$ is nonzero for every submodule $U$ of $M$.*

*(iv)* *$Hom_R(M, U)$ is nonzero for every cyclic submodule of $M$.*

**Proof.**

(i) $\Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$ It is obvious.
(iv) $\Rightarrow (i)$. Let $N$ be any nonzero essential submodule of $M$ and $m$ is any nonzero element of $N$. Since $mR$ is Noetherian, $mR$ contains a uniform submodule $U$. Let $u$ be a nonzero element of $U$. Then $uR$ is a cyclic submodule of $M$ and by (iv), $Hom_R(M, uR)$ is nonzero and hence $Hom_R(M, N)$ is nonzero.

**Proposition 3.24.** *The following statements are equivalent for a module $M$:*

*(i)* *$M$ contains an essentially slightly compressible module $N$ such that there exists a nonzero homomorphism $f : M \to N$.*

*(ii)* *$M$ is essentially slightly compressible.*

*(iii)* *There is an essential homomorphism $g : M \to M^1$ for an essentially slightly compressible module $M^1$.*

**Proof.**

(i)$\Rightarrow$(ii) Let $K$ be an essential submodule of $M$, then $K \cap N$ is an essential submodule of $N$, so there is a homomorphism. $\Psi : N \to K \cap N$.
If $i : K \cap N \to K$ is the inclusion map then $i\Psi f : M \to K$ is a homomorphism.
Hence, $M$ is essentially slightly compressible.
(ii)$\Rightarrow$(iii) It is obvious.
(iii)$\Rightarrow$(i) It is obvious.

**Corollary 3.25.** *If $N$ is either an essential submodule of $M$ or a submodule which is invariant under an endomorphism of $M_R$. Therefore, if $M$ is essentially slightly compressible, then $N_R$ is also essentially slightly compressible.*

**Proof.** It follows from [**27, Proposition 1.4(a)**]. It is well known that every module with a finite uniform dimension has a uniform submodule. Thus, $M_R$ has a finite uniform dimension if and only if $M_R$ has a uniform submodule. In 1971, Tiwary and Pandey [**28**] defined an $R$-module $M$ as being quasi polysimple if each of its nonzero submodule contains a nonzero submodule with free family condition (FFC). In fact a module has enough uniforms [ **defined by 27**] is a quasi polysimple module.

**Proposition 3.26.** *The following statements are equivalent for a nonzero $R$-module $M$ over an arbitrary ring $R$:*

*(i)* $M_R$ *is nonsingular, essentially compressible and has enough uniform (quasi polysimple module).*

*(ii)* $M_R$ *is nonsingular, essentially compressible and* $M_R$ *is an essential extension of direct sums of absolutely indecomposable submodules.*

*(iii)* $M_R$ *is nonsingular, essentially compressible and* $M_R$ *is an essential extension of direct sums of indecomposable modules.*

*(iv)* $M_R$ *is subisomorphic to a direct sum* $\bigoplus_{i\in I} A_i$ *of nonsingular uniform right ideal* $A_i$ $(i \in I)$ *of* $R$ *such that* $A_i$ *does not contain a nonzero nilpotent right ideal of* $R$ *for each* $i \in I$.

*(v)* $M$ *is nonsingular and* $M$ *imbeds in a direct sum of uniform compressible right* $R$*-modules.*

**Proof.**

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) It is obvious from [**21, Lemma 2.1** ]

(iv) $\Rightarrow$ (v) $\Rightarrow$ (i). It is obvious by [**27, Theorem 3.16**]

## 3.3 Essentially Compressible Rings

Let $R$ be a ring. We say that $R$ is a right essentially compressible ring if the right $R$-module $R$ is essentially compressible. It is well known that any semiprime right Goldie ring is right essentially compressible. Clearly any domain is a right essentially compressible ring but it is not necessarily right Goldie.

**Lemma 3.27.** *A ring $R$ is right essentially compressible if and only if every essential right ideal contains a right regular element of $R$.*

**Proof.**

This is routine

**Corollary 3.28.** *Let $R$ be a semiprime ring which satisfies ACC and DCC on right annihilators. Then every free right (or left) $R$-module is essentially compressible.*

**Proof.**

By [**30, Theorem 1.19**] and Lemma 3.27 the ring $R$ is right and left essentially compressible. The result follow by Proposition 3.7.

**Corollary 3.29.** *Let $R$ be a semiprime left Goldie ring. Then every free right $R$-module is essentially compressible.*

**Proof.** By Corollary 3.27 and [**31, Lemma 5.8**]

The following proposition is a consequence of parts (a) and (d) of Proposition 3.8 and Proposition 3.7.

**Proposition 3.30.** *Let $R$ be a right essentially compressible ring.*

*(a) Every ideal of $R$ is an essentially compressible right $R$-module.*

*(b) Every essential submodule of a free right $R$-module is an essential compressible $R$-module.*

*(c) If $A$ is any ideal of $R$ with $A^2 \neq A$ then $R/A^2$ is not an essentially compressible right $R$-module.*

**Proof.** By Proposition 3.7 and Proposition 3.8.

**Proposition 3.31.** *Every essentially compressible ring is a right nonsingular semiprime ring.*

**Proof.**

Let $R$ be essentially compressible ring. The semiprimeness of $R$ is a consequence of Proposition 3.8(d). By Theorem 3.14 there exists a monomorphism $\theta : R \to Z \oplus A$ for some singular right $R$-module $Z$ and nonsingular right $R$-module $A$.

Let $\theta(1) = c = a + z$ for some $a \in A$ and $z \in Z$.

Note that $0 = r.ann_R(c) = r.ann_R(a) \cap r.ann_R(z)$ and $r.ann_R(z)$ is an essential right ideal of $R$. Thus $r.ann_R(a) = 0$. But $R \simeq aR \subseteq A$, hence $Z(R_R) = 0$.

**Proposition 3.32.** *Every essentially compressible ring does not contain an infinite direct sum of nonzero ideals.*

**Proof.**

By Proposition 3.8(f).

**Corollary 3.33.** *A commutative ring $R$ is essentially compressible if and only if $R$ is a semiprime Goldie ring.*

**Proof.**

By Proposition 3.30 and 3.31.

**Theorem 3.34.** *Let $S$ denote the socle of a right essentially compressible ring $R$ . Then $S = eR$ for some central idempotent element $e$ of $R$. Consequently, the ring $R$ is a direct sum of a semiprime artinian ring and a right essentially compressible ring with zero right socle.*

**Proof.**

By Theorem 3.14, $S$ is a direct summand of $R$ and so there is an idempotent $e$ in $R$ such that $S = eR$.

Since $S$ is an ideal, $(1-e)\,\mathrm{Re} \subseteq (1-e)eR = 0$ and hence $[eR(1-e)R]^2 = 0$ , so that $eR(1-e) = 0$ by Proposition 3.8.

Thus $er = ere = re$ for all $r \in R$,i.e. $e$ is central.

For the last statement consider that $R/S$ is an essentially compressible right $R$-module by Proposition 3.8(b), and so it is essentially compressible as a right $R/S$-module. It follows that $(1-e)R$ is a right essentially compressible ring.

**Lemma 3.35.** *Let $R$ be a semiprime ring and $U$ be an ideal of $R$. Then*

$r.ann_R U = l.ann_R U.$

**Proof.**

This is clear.

**Theorem 3.36.** *A ring $R$ is right essentially compressible if and only if there exist a positive integer $n$ and prime ideals $P_i (1 \le i \le n)$ of $R$ such that;*

$P_1 \cap ....... \cap P_n = 0$ *and the ring $R/P_i$ is right essentially compressible for each* $1 \le i \le n$.

**Proof.**

Let $R$ be a right essentially compressible. Then by Proposition 3.30 and 3.32 and [**32, 11.41 and 11.43, p.336**], $R$ has a finite number of minimal prime ideals $P_1, ....., P_n$ such that $P_i = ann_R U_i$ for some uniform ideal $U_i$ of $R$ for each $1 \le i \le n$ and $P_1 \cap .... \cap P_n = 0$.

Let $i \in \{1, .....n\}$, let $P = P_i$ and let $U = U_i$. Let $E$ be a right ideal of $R$ such that $P \subseteq E$ and $E/P$ is an essential right ideal of the ring $R/P$. Then $E$ is an essential right ideal of $R$. By Lemma 3.17, $E$ contains a right regular element $c$ of $R$. Let $r \in R$ such that $cr \in P$. Then $crU = 0$ and hence $rU = 0$. This implies that $r \in P$. Thus $E/P$ contains the right regular element $c + P$ of $R/P$. By Lemma 3.27, the ring $R/P$ is a right essentially compressible.

Conversely, suppose that $P_1 \cap ..... \cap P_n = 0$ for some positive integer $n$ and prime ideals $P_i (1 \le i \le n)$ such that $R/P_i$ is a right essentially compressible ring for each $1 \le i \le n$. Without loss of generality we can suppose that $P_i \nsubseteq P_j$ for all $1 \leq i \neq j \leq n$. Let $F$ be any essential right ideal of $R$. Let $A$ be a right ideal of $R$ such that $(F \cap P_2 \cap ..... \cap P_n) \cap A \subseteq P_1$. Then $F \cap (P_2 \cap ..... \cap P_n \cap A) \subseteq P_1 \cap ..... \cap P_n = 0$ so that $P_2 \cap ..... \cap P_n \cap A = 0 \subseteq P_1$ and hence $A \subseteq P_1$. It follows that the right ideal $[(F \cap P_2 \cap ....... \cap P_n) + P_1]/P_1$ of the ring $R/P_1$ is essential. By Lemma 3.27, there exist $d_1 \in (F \cap P_2 \cap ... \cap P_n) \cap C'(P_1)$, where $C'(P_1)$=$\{r \in R; r + P_1$ is a regular element of $R/P_1\}$. Similarly, for each $2 \leq i \leq n$ there exist $d_i \in (F \cap P_1 \cap .... \cap P_{i-1} \cap P_{i+1} \cap .... \cap P_n) \cap C'(P_i)$. Let $d = d_1 + .....d_n \in F$. It is easy to check that if $ds = 0$ for some $s \in R$ then $d_i s \in P_i$ and hence $s \in P_i (1 \leq i \leq n)$, so that $s \in P_1 \cap ... \cap P_n$ and $s = 0$. Thus $d$ is a right regular element of $R$. By Lemma 3.27, $R$ is right essentially compressible.

**Lemma 3.37.** *Let $U$ be a uniform right ideal in the ring $R$ and let there be prime ideals $P_i (1 \leq i \leq n)$ of $R$ such that $P_i \nsubseteq P_j$ for all $1 \leq i \neq j \leq n$ and $P_1 \cap ...... \cap P_n = 0$. Then $(U + P_i)/P_i$ is a uniform right ideal of $R/P_i$ for each $1 \leq i \leq n$.*

**Proof.**

Assume that $P = P_1$ and $A, B$ are right ideals of $R$ such that $P \subseteq A \subseteq (U+P)$, $P \subseteq B \subseteq (U + P)$ and $A \cap B \subseteq P$. Let $A' = A \cap U$, $B' = B \cap U$. We have $A' \cap (P_2 \cap ..... \cap P_n \cap B') \subseteq P_1 \cap ..... \cap P_n = 0$. Hence by the uniform condition on $U$, $A' = 0$ or $(P_2 \cap .... \cap P_n \cap B') = 0$. Now if $A' = 0$ then $U \cap P \subseteq A' = 0$ and so $(U + P)/P \simeq U$ is a uniform right $R$-module as well as a uniform $R/P$-module, and if $(P_2 \cap ..... \cap P_n \cap B') = 0 \subseteq P$ then $B' \subseteq P$.

This in turn implies that $B \subseteq P$, because if $u + p = b \in B$ where $u \in U$ and $p \in P$, then $u = (b - p) \in B \cap U = B' \subseteq P$ and hence $b = (u + p) \in P$.

Consequently, $(U+P)/P$ is a uniform right ideal of $R/P$. Similarly, $(U+P_i)/P_i$ is an uniform right ideal of $R/P_i$ for each $2 \leq i \leq n$.

**Theorem 3.38.** *The following statements are equivalent for a ring $R$.*

*(a) $R$ is a semiprime right Goldie ring.*

*(b) $R$ is a right essentially compressible ring with at least one uniform right ideal.*

**Proof.**

(a)⇒(b) By [**27, Corollary 2.4(a)**]

(b)⇒(a) By Proposition 3.31, $R$ is a right nonsingular semiprime ring, so it is enough to show that the right uniform dimension of $R$ , is finite. For a moment assume that $R$ is prime, then by Proposition 3.10 $R$ has finite uniform dimension. Thus, by Theorem 3.36 and Lemma 3.37, we can conclude that the uniform dimension $R$ is finite.

## 3.4 Essentially Slightly Compressible Rings

$R$ is said to be essentially slightly compressible ring if $R_R$ is essentially slightly compressible module.

**Lemma 3.39.** *Let $A$ be a nonzero idempotent right ideal of a ring $R$ such that $A$ contains a nonzero essential right ideal $B$ with $BA = 0$.*

*Then the right $R$-module $A$ is not essentially slightly compressible.*

**Proof.** Suppose $f : A \to B$ is homomorphism then,

$f(A) = f(A^2) = f(A)A \subseteq BA = 0.$

It follows that $A$ is not essentially slightly compressible.

Using Lemma 3.39, we next give an example of a projective right ideal $A$ of a ring $R$ such that $A$ is not essentially slightly compressible module.

**Example 3.40.** *Let $S$ be any nonzero ring and $R$ denote the ring of $(2 \times 2)$ lower triangular matrices over $S$. Let $A$ be the right ideal of $R$ consisting of all matrices in $R$ with top row zero. Then $A$ is a cyclic projective $R$-module which is not essentially slightly compressible.*

**Proof.** Clearly $A = eR$ where $e$ is the idempotent element of $R$ with 1 as its (2,2) entry and all other entries zero. Thus $A$ is a direct summand and hence projective. Let $B$ denote the essential right ideal of $R$ consisting of all matrices with all entries zero except at (2,1) place.

Therefore $B \subseteq A$ and $BA = 0$. Hence by Lemma 3.39, $A$ is not essentially slightly compressible.

**Proposition 3.41.** *Let $R$ be a semi prime ring. Then every right ideal of $R$ is slightly compressible and hence essentially slightly compressible.*

**Proof.** It follows from [**27, Proposition 3.8**]

**Corollary 3.42.** *Let $R$ be a ring, which has no nilpotent essential ideal of index 2. Then every ideal $A$ of $R$ is essentially slightly compressible.*

**Proof.** Let $A \leq R$ and $B$ be a nonzero essential ideal of $R$ contained in $A$. Now, according to assumption $B^2 \neq 0 \Rightarrow BA \neq 0$, there exists an element $b$ of $B$ such that $bA$ is nonzero.

Let us define a nonzero homomorphism $f : A \to B$ by $f(a) = ba\ \forall a \in A$, then $A$ is essentially slightly compressible.

But the converse of the above corollary is not true in general. Let $R = (Z_4, +, \cdot)$ (ring of integer module 4) and $A = \{0, 2\}_{\bmod 4}$be an essential right ideal of $R$ such that $A^2 = (0)$. But $R$ is essentially slightly compressible.

**Proposition 3.43.** *Let $R$ be a prime ring. Then the right $R$-module $E(R_R)$ is essentially slightly compressible if and only if $Hom_R(E(R_R), R) \neq 0$.*

**Proof.** Since $R$ is essential in $E(R_R)$, hence the necessity is obvious. Conversely, suppose there exists a nonzero homomorphism $f : E \to R$ where $E(R_R) = E$. Let $N$ be a nonzero essential submodule of $E$. Therefore $N \cap R$ and $f(R)$ are both nonzero ideals of $R$. Since $R$ is prime, then we have $(N \cap R) \cap f(E) \neq 0$, hence $af(E) \neq 0$ for some $a \in N \cap R$. Let us define $g : E \to N$ by $g(e) = af(e)$ $\forall\ e \in E$. Clearly, $g$ is nonzero homomorphism.

**Lemma 3.44.** *If every essential right ideal of $R$ contains a regular element then $R$ is a right essentially slightly compressible ring.*

**Proof.** It is obvious.

But a right essentially slightly compressible ring is possible in which there exists an essential right ideal which does not contain a regular element.

Let $(Z_4, +, \cdot)$ (ring of integer module 4) and $I = \{0, 2\}_{\bmod 4}$ be essential ideal of $Z_4$. But $I$ does not contain any regular element.

**Proposition 3.45.** *For a right essentially slightly compressible ring, the following conditions are equivalent:*

*(i) Every ideal of $R$ is essentially slightly compressible.*

*(ii) Every essential submodule of a free $R$-module is an essential slightly compressible $R$-module.*

*(iii) If $A$ is an ideal of $R$ then $R/A$ is an essential slightly compressible right $R$ -module.*

**Proof.** It is obvious.

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