Crossing the singularity of a gravitational wave collision

by

Kıvanç İbrahim Ünlütürk

A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Doctor of Philosophy

in

Physics

KOÇ ÜNİVERSİTESİ

February 5, 2024

Crossing the singularity of a gravitational wave collision

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a doctoral dissertation by

Kıvanç İbrahim Ünlütürk

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Prof. Tekin Dereli

Prof. Özay Gürtuğ

Assoc. Prof. Fethi M. Ramazanoğlu

Prof. O. Teoman Turgut

Prof. Özgür E. Müstecaplıoğlu

Date:

ABSTRACT

Crossing the singularity of a gravitational wave collision
Kıvanç İbrahim Ünlütürk
Doctor of Philosophy in Physics
February 5, 2024

A reformulation of general relativity inspired by the Belinski–Khalatnikov–Lifshitz conjecture had been introduced by Ashtekar, Henderson and Sloan which is based on variables closely related to the basic variables of loop quantum gravity, thereby providing a way of classically analyzing singularities that may be carried over to the quantum theory. It is reasonable to expect that these variables are regular at generic spacelike singularities. This has been shown on various examples—particularly, cosmological spacetimes. In this thesis we extend this analysis to the singularities of gravitational wave collision spacetimes, which are the result of the mutual focusing of the two waves. We focus on two specific examples and explicitly confirm that the said variables are regular at the singularity and can be smoothly continued beyond it.

ÖZETÇE

Bir kütleçekim dalgası tekilliğini geçmek
Kıvanç İbrahim Ünlütürk
Fizik, Doktora
5 Şubat 2024

Belinski–Halatnikov–Lifşits sanısından esinlenmiş, genel göreliliği yeniden formüle eden bir yaklaşım Ashtekar, Henderson ve Sloan tarafından oluşturulmuştur. Bu yaklaşım, döngü kuantum kütleçekimi temel değişkenleri ile yakından ilişkili değişkenlere dayanmaktadır ve bu sayede tekilliklerin incelenmesi için kuantum kurama da taşınabilecek bir yol sağlamaktadır. Bu değişkenlerin uzayımsı tekilliklerde genel olarak düzenli olduğunu beklemek akla yatkındır. Bu, çeşitli örneklerde—özellikle kozmolojik uzayzamanlarda—gösterilmiştir. Bu tezde bu analiz kütleçekimsel dalga çarpışma uzayzamanlarının tekilliklerine uygulanmıştır. Bu tekillikler, iki dalganın karşılıklı odaklanmasının sonucu olarak ortaya çıkmaktadır. Bu tezde iki somut örneğe odaklanılmıştır ve söz konusu değişkenlerin tekillikte düzenli olduğu ve tekilliğin ötesine sorunsuzca devam edebildiği doğrudan gösterilmiştir.

ACKNOWLEDGMENTS

The author was supported by TÜBİTAK BİDEB 2211-A General Domestic Doctoral Scholarship Program and TÜBİTAK Projects No. 117F111 and 122F097.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

Spacetime singularities in general relativity need to be understood better for the deterministic nature of the theory. Classical tools for understanding and resolving the issue of singularities are insufficient due to the scales at which singularities are formed. Thus, quantum tools are required to cope with the analysis of singularities. For instance, test quantum wave packets obeying Klein-Gordon, Dirac and Maxwell equations may be used to probe timelike singularities in static spacetimes [Horowitz and Marolf, 1995, Helliwell et al., 2003, Gurtug et al., 2014]. In this formalism, the quantum singularity is understood as a non-unique evolution of test quantum wave packets. In dynamical spacetimes such as cosmological ones, the interior of the Schwarzschild black hole or colliding gravitational wave spacetimes, our example of interest here, a new perspective is essential for analyzing spacelike singularities.

Classically, the Belinski–Khalatnikov–Lifshitz (BKL) conjecture [Belinskii et al., 1970] states that close to a spacelike singularity, the spatial derivatives of geometric quantities become negligible compared to the time derivatives, meaning that the field equations may be well approximated by ordinary differential equations. The geometry then exhibits an oscillatory and chaotic behaviour, where the metric is well described by the Bianchi I metric for long epochs of time, interrupted by short Bianchi II transitions causing the parameters p_i characterizing the Bianchi I metric to settle to new values. Furthermore, for most types of matter, the effect of matter sources on this behaviour should be negligible, with the exception of a massless scalar field or a stiff fluid.

Although the conjecture seems at first coordinate dependent and surprising, it has been shown that it can be made precise and there is now a body of numerical and analytical evidence in its favor [Weaver et al., 1998, Andersson and Rendall, 2001, Berger, 2002, Garfinkle, 2004, Garfinkle, 2007, Saotome et al., 2010]. Thus, it is intriguing to think about its consequences for quantum gravity. As pointed out by [Ashtekar et al., 2011], there are a number of indications that quantum effects become significant only when curvature or matter density are about a percent of the Planck scale. Therefore, it is possible that the BKL behaviour sets in already when the spacetime is sufficiently classical. This means that a quantization of the effective theory with ordinary differential equations could provide a qualitative understanding of quantum gravity effects close to spacelike singularities in general.

It is natural to expect that the singularities in classical general relativity are resolved in a fully quantum theory of gravity. Indeed, there is already an accumulated body of evidence in this direction, such as the ones coming from loop quantum cosmology (LQC). LQC predicts the Big Bang singularity to be replaced by a *Big Bounce*, where the discretization of spacetime introduces a maximum cutoff for the physical quantities such as curvature and matter density, at which point the geometry "bounces back" [Bojowald, 2001]. It is also known that in LQC, the singularities of the Bianchi I, II and IX metrics are resolved [Ashtekar and Wilson-Ewing, 2009a, Ashtekar and Wilson-Ewing, 2009b, Wilson-Ewing, 2010].

Considering the BKL conjecture, these results suggest that generic spacelike singularities may be all resolved in LQC. However, the standard formulations of the BKL conjecture are in terms of variables not particularly suited for quantization. For instance, in the formalism introduced by [Uggla et al., 2003], one normalizes relevant physical quantities by an appropriate power of the inverse of the trace of the extrinsic curvature K, which is expected to diverge at the singularity, thus giving finite quantities.

Loop quantum gravity is based on a density weighted triad formulation of general relativity, i.e., the basic configuration variables are the orthonormal triads on spacelike surfaces, scaled by the the square root of the determinant of the spatial metric; $E^a_i = \sqrt{q}e^a_i$. The determinant q is expected to vanish at spacelike singularities, just like K^{-1}. Thus, another strategy to formulate the BKL conjecture would be to multiply divergent quantities by appropriate powers of q[1].

Based on these observations Ashtekar, Henderson and Sloan (AHS) introduced a formulation of the BKL conjecture [Ashtekar et al., 2009, Ashtekar et al., 2011] based on variables closely related to the basic variables of loop quantum gravity, and therefore better suited for quantization. These variables are tensor densities involving positive powers of q, and therefore expected to be finite at the singularity. So far, the AHS variables have been used to study the singularities of the FLRW spacetime with a massless scalar field, Bianchi I and IX models and the part of the Schwarzschild spacetime inside the horizon [Valdés-Meller, 2021]. In all of these cases, the AHS variables are regular at the singularity, and can be continued beyond the singularity.

In this study, we extend this analysis to the singularities of colliding gravitational wave spacetimes. Colliding gravitational waves have been known to produce singularities as a result of the mutual focusing of the two waves. Though the development of singularities seem to be a generic feature of colliding gravitational waves, there also exist exceptional solutions where, instead of a singularity, a Cauchy horizon may develop in the region of interaction of the two waves [Bell and Szekeres, 1974, Chandrasekhar and Xanthopoulos, 1986, Ferrari and Ibanez, 1988]. However, it has been shown that these horizons are unstable in the full nonlinear theory against small but generic plane-symmetric perturbations of the incoming waves, and thus, spacelike curvature singularities are developed in the interaction region [Yurtsever, 1987].

Colliding gravitational wave spacetimes therefore provide interesting models for investigating singularities, which are relatively unexplored. Here, we calculate the AHS variables for two basic examples of colliding wave solutions and confirm explicitly that the AHS variables are regular at the singularity and can be continued smoothly through the singularity. The results of this thesis can also be found in [Dereli et al., 2024]

This thesis is organized as follows. In Chapter 2 we briefly review constrained Hamiltonian systems. In Chapter 3 we give an overview of the ADM formulation of

[1] We note that such a strategy was also considered by [Einstein and Rosen, 1935].

general relativity. The AHS variables are introduced in Chapter 4. In Chapter 5 we provide an overview of colliding gravitational waves. Next, in Chapter 6, we apply the AHS variables to the colliding gravitational wave metrics that we consider. Finally, in Chapter 7 we discuss the implications of our results and the further questions that need to be addressed. Throughout the thesis we use the mostly plus metric signature $(-, +, +, +)$ and $c = 1$.

Chapter 2

CONSTRAINED HAMILTONIAN SYSTEMS

The Hamiltonian formulation of general relativity results in a system with constraints. Treatment of constrained systems requires additional techniques, therefore, we start with the basic theory of constrained Hamiltonian systems. Here, we shall closely follow [Henneaux and Teitelboim, 1992].

2.1 Primary Constraints

Given an action

$$S[q] = \int \mathrm{d}t\, L(q, \dot{q}), \tag{2.1}$$

the Euler-Lagrange equations read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^n} - \frac{\partial L}{\partial q^n} = 0, \quad n = 1, \ldots, N. \tag{2.2}$$

This can be rewritten as

$$\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}} \ddot{q}^{n'} = \frac{\partial L}{\partial q^n} - \frac{\partial^2 L}{\partial \dot{q}^n \partial q^{n'}} \dot{q}^{n'}. \tag{2.3}$$

Therefore, the accelerations $\ddot{q}$ can be uniquely written as a function of q and $\dot{q}$ if and only if the determinant

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}}\right) \tag{2.4}$$

does not vanish. If the determinant vanishes, then the solution (if exists) will contain arbitrary functions.

The canonical momenta are defined by

$$p_n = \frac{\partial L}{\partial \dot{q}^n}. \tag{2.5}$$

We want to pass to the Hamiltonian formalism by writing $\dot{q}$ in terms of q and p. To be able to do this, the transformation $(q, \dot{q}) \to (q, p)$ defined by (2.5) must be non-singular, i.e. the Jacobian

$$J = \begin{pmatrix} I_n & 0 \\ \frac{\partial p_n}{\partial q^{n'}} & \frac{\partial p_n}{\partial \dot{q}^{n'}} \end{pmatrix} \tag{2.6}$$

must have a nonzero determinant:

$$\det J = \det \left(\frac{\partial p_n}{\partial \dot{q}^{n'}} \right) = \det \left(\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}} \right). \tag{2.7}$$

We therefore see that this is exactly the condition that $\ddot{q}$ is written uniquely in terms of q and $\dot{q}$. If this determinant is zero, then the transformation $(q, \dot{q}) \to (q, p)$ is not invertible, i.e. it maps the $2N$ dimensional space $(q, \dot{q})$ to a $2N - M$ dimensional subspace of (q, p), where

$$\text{rank} J = 2N - M \quad \text{or} \quad \text{rank} \left(\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}} \right) = N - M, \tag{2.8}$$

see Fig. 2.1. This means that the p's we get from (2.5) are not all independent, but there are M constraints

$$\phi_m(q, p) = 0, \quad m = 1, \ldots, M. \tag{2.9}$$

These are called *primary constraints* and we shall call the surface of the phase space given by Eq. (2.9) the *primary constraint surface.*

We want to choose the constraint functions ϕ_m to satisfy a regularity condition, namely:

$$\text{rank} \begin{pmatrix} \frac{\partial \phi_m}{\partial q^n} & \frac{\partial \phi_m}{\partial p_n} \end{pmatrix} = M, \tag{2.10}$$

in the vicinity of the primary constraint surface $\phi_m = 0$. This means that the ϕ_m can be chosen as the first M coordinates of a new coordinate system (ϕ, ψ) on the (q, p) space. E.g. the same surface can be represented as $p = 0$ or as $p^2 = 0$. In the first case the constraint function is regular; rank$(0\ 1) = 1$, whereas in the second case the matrix $(0\ 2p)$ vanishes on the constraint surface. Hence the former one is admissible, but the latter is not.

We now prove two important claims.

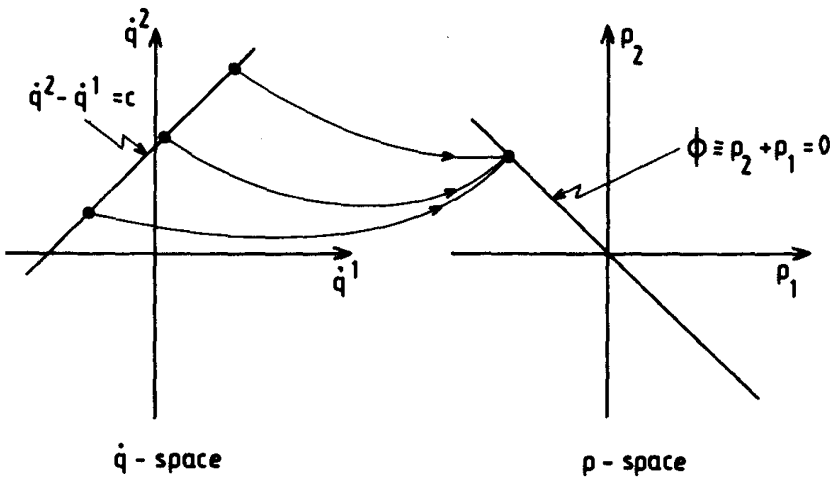

Figure 2.1: "The figure shows the example of a system with two q's and Lagrangian $\frac{1}{2}(\dot{q}^1 - \dot{q}^2)^2$. The momenta are $p_1 = \dot{q}^1 - \dot{q}^2$ and $p_2 = \dot{q}^2 - \dot{q}^1$. There is one primary constraint $\phi = p_1 + p_2 = 0$. All of $\dot{q}$-space is mapped on the straight line $p_1 + p_2 = 0$ of p-space. Moreover, all the $\dot{q}$'s on the straight line $\dot{q}^2 - \dot{q}^1 = c$ are mapped on the same point $p_1 = -c = -p_2$ belonging to the constraint surface $\phi = 0$. The transformation $\dot{q} \to p$ is thus neither one to-one nor onto. To render the transformation invertible, one needs to adjoin extra parameters to the p's (see below)." Image and description taken from from [Henneaux and Teitelboim, 1992].

Theorem 2.1. *If a phase space function G vanishes on the primary constraint surface, then $G = g^m \phi_m$ for some functions g^m.*

Proof. Choose a coordinate system (ϕ, ψ) on the phase space. Then, since $G(0, \psi) = 0$, we get

$$\begin{aligned} G(\phi, \psi) &= \int_0^1 \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} G(t\phi, \psi) \\ &= \phi_m \int_0^1 \mathrm{d}t \, \frac{\partial G}{\partial \phi_m}(t\phi, \psi). \end{aligned} \tag{2.11}$$

□

Theorem 2.2. *If*

$$\lambda_n \delta q^n + \mu^n \delta p_n = 0 \tag{2.12}$$

for arbitrary variations δq, δp tangent to the primary constraint surface, then

$$\lambda_n = u^m \frac{\partial \phi_m}{\partial q^n} \tag{2.13a}$$

$$\mu^n = u^m \frac{\partial \phi_m}{\partial p_n} \tag{2.13b}$$

for some functions u^m. The equalities here are on the constraint surface.

Proof. Since the primary constraint surface is $2N - M$ dimensional, at each point the tangent variations $(\delta q \ \delta p)$ form a $2N - M$ dimensional vector space. The vector $(\lambda \ \mu)$ belongs to the M dimensional orthogonal space of this under the standard euclidean inner product on $\mathbb{R}^{2N}$. The M gradients

$$\begin{pmatrix} \frac{\partial \phi_m}{\partial q^n} & \frac{\partial \phi_m}{\partial p_n} \end{pmatrix} \tag{2.14}$$

are clearly in this space, and they are linearly independent by the regularity assumption. They therefore form a basis of the orthogonal space and the claim holds. □

2.2 The Hamiltonian

The Hamiltonian is

$$H = p_n \dot{q}^n - L. \tag{2.15}$$

Defined in this way, the Hamiltonian is a function of q and $\dot{q}$, and $\dot{q}$ cannot be written as a function of q and p. Hence it seems at first that H too cannot be written as a function q and p. However, observe that for arbitrary variations of the coordinates and velocities:

$$\begin{aligned} \delta H &= \dot{q}^n \delta p_n + p_n \delta \dot{q}^n - \frac{\partial L}{\partial q^n} \delta q^n - \frac{\partial L}{\partial \dot{q}^n} \delta \dot{q}^n \\ &= \dot{q}^n \delta p_n - \frac{\partial L}{\partial q^n} \delta q^n. \end{aligned} \tag{2.16}$$

This means that H does not change as we move from a point $(q, \dot{q})$ to a nearby point $(q, \dot{q}')$ provided that $p(q, \dot{q}) = p(q, \dot{q}')$. Therefore H can be written as a function of q and p, at least locally around a point $(q, \dot{q})$. It should be noted of course that Eq.

(2.15) defines H only on the primary constraint surface; H can be extended off the constraint surface to the rest of the phase space in an arbitrary way.

Since we can write $H = H(q,p)$, Eq. (2.16) can be rewritten as

$$\left(\frac{\partial H}{\partial p_n} - \dot{q}^n\right)\delta p_n + \left(\frac{\partial H}{\partial q^n} + \frac{\partial L}{\partial q^n}\right)\delta q^n = 0. \tag{2.17}$$

The variations δq, δp here are tangent to the primary constraint surface, but otherwise arbitrary. We can therefore use Theorem 2.2 to conclude that, on the constraint surface,

$$\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n} \tag{2.18a}$$

$$\frac{\partial L}{\partial q^n} = -\frac{\partial H}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n}. \tag{2.18b}$$

The first of these tells us that we can recover the $\dot{q}$'s if we know the u's. Furthermore, the M vectors $\partial\phi_m/\partial p_n$ are linearly independent, hence no two different sets of u's can give the same $\dot{q}$'s. We prove this as follows. First,

$$\phi_m(q, p(q,\dot{q})) = 0 \tag{2.19}$$

for all q and $\dot{q}$. Differentiating this with respect to q^n yields

$$\frac{\partial \phi_m}{\partial q^n} + \frac{\partial \phi_m}{\partial p_{n'}}\frac{\partial p_{n'}}{\partial q^n} = 0. \tag{2.20}$$

For a fixed n, this means that the M-component vector $\partial\phi_m/\partial q^n$ can be written as a linear combination of the N vectors $\partial\phi_m/\partial p_{n'}$. Therefore the left half of the matrix

$$\operatorname{rank}\begin{pmatrix} \frac{\partial \phi_m}{\partial q^n} & \frac{\partial \phi_m}{\partial p_n} \end{pmatrix} \tag{2.21}$$

does not contribute to its rank. By the regularity condition, then,

$$\operatorname{rank}\begin{pmatrix} \frac{\partial \phi_m}{\partial p_n} \end{pmatrix} = M. \tag{2.22}$$

Hence the M vectors $\partial\phi_m/\partial p_n$ are linearly independent, which was to be shown.

We can indeed define an invertible mapping between the two $2N$-dimensional spaces, namely, the $(q,\dot{q})$ space and the $\phi_m = 0$ surface of the (q,p,u) space. This is given by

$$q^n = q^n \tag{2.23a}$$

$$p_n = \frac{\partial L}{\partial \dot{q}^n}(q, \dot{q}) \tag{2.23b}$$

$$u^m = u^m(q, \dot{q}), \tag{2.23c}$$

where $u^m(q, \dot{q})$ is given by the unique solution of

$$\dot{q}^n = \frac{\partial H}{\partial p_n}(q, p(q, \dot{q})) + u^m \frac{\partial \phi_m}{\partial p_n}(q, p(q, \dot{q})). \tag{2.24}$$

The inverse mapping is given by

$$q^n = q^n \tag{2.25a}$$

$$\dot{q}^n = \frac{\partial H}{\partial p_n}(q, p) + u^m \frac{\partial \phi_m}{\partial p_n}(q, p), \tag{2.25b}$$

where only those values of q and p are used that satisfy $\phi_m(q, p) = 0$. The only nontrivial part of showing that these two mappings are inverses of each other is using the fact that

$$\frac{\partial \phi_m}{\partial p_{n'}} \frac{\partial p_{n'}}{\partial \dot{q}^n} = 0, \tag{2.26}$$

which can be shown by differentiating $\phi_m(q, p) = 0$ with respect to $\dot{q}$ while keeping q fixed. Note that we do not need to know the functional form of $p(q, \dot{q})$ for this.

These allow us to conclude that the Euler-Lagrange equations are equivalent to Hamilton's equations of motion

$$\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n} \tag{2.27a}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n} \tag{2.27b}$$

$$\phi_m(q, p) = 0. \tag{2.27c}$$

These can also be derived from the Hamiltonian action

$$S[q, p, u] = \int \mathrm{d}t \, [p_n \dot{q}^n - H(q, p) - u^m \phi_m(q, p)] \tag{2.28}$$

for arbitrary variations, subject only to the condition $\delta q(t_1) = \delta q(t_2) = 0$.

The time evolution of an arbitrary phase space function $F(q,p)$ is then given by

$$\dot{F} = \{F, H\} + u^m \{F, \phi_m\}, \tag{2.29}$$

where the Poisson bracket is defined as usual:

$$\{A, B\} = \frac{\partial A}{\partial q^n}\frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial q^n}. \tag{2.30}$$

2.3 Secondary Constraints

For consistency, the constraints should be preserved in time, i.e.,

$$\dot{\phi}_m = \{\phi_m, H\} + u^{m'} \{\phi_m, \phi_{m'}\} = 0. \tag{2.31}$$

These equations may restrict the u's, or they may be independent of them, in which case they are constraint equations between the phase space variables q and p. In the latter case, if the constraints are independent of the primary consraints ϕ_m, they are called secondary constraints. If there is a secondary constraint, say $X(q,p)$, this must be preserved in time as well. Hence we have

$$\{X, H\} + u^m \{X, \phi_m\} = 0. \tag{2.32}$$

This may again restrict the u's, or yield new secondary constraints, and so on. Once the process in finished, we end up with K secondary constraints

$$\phi_k = 0, \quad k = M+1, \ldots, M+K. \tag{2.33}$$

We denote primary and secondary constraints using the same notation:

$$\phi_j = 0, \quad j = 1, \ldots, J, \tag{2.34}$$

where $J = M + K$. The reason for this is that the distinction between the primary and secondary constraints is not particularly important for the final form of the theory. We require the same regularity condition on the secondary constraints as well.

At this stage, it is useful to introduce the *weak equality* symbol "$\approx$". Two functions on the phase space are said to be weakly equal,

$$F \approx G, \tag{2.35}$$

if they are equal on the secondary constraint surface defined by the equations $\phi_j = 0$. Then, by Theorem 2.1 with ϕ_m replaced by ϕ_j, we have

$$F - G = g^j(q,p)\phi_j(q,p). \tag{2.36}$$

2.4 First Class and Second Class Functions

A function $F(q,p)$ is said to be first class, if its Poisson bracket with every constraint vanishes weakly:

$$\{F, \phi_j\} \approx 0. \tag{2.37}$$

A function which is not first class is called second class. An important feature of the first class property is that the Poisson bracket of two first class functions is also first class. To see this, note that if F and G are first class, then

$$\{F, \phi_j\} = f_j^{j'}\phi_{j'}, \quad \{G, \phi_j\} = g_j^{j'}\phi_{j'}. \tag{2.38}$$

Therefore, using the Jacobi identity we get

$$\begin{aligned}
\{\{F,G\},\phi_j\} &= \{\{F,\phi_j\},G\} + \{F,\{G,\phi_j\}\} \\
&= \left\{f_j^{j'}\phi_{j'}, G\right\} + \left\{F, g_j^{j'}\phi_{j'}\right\} \\
&= \left\{f_j^{j'}, G\right\}\phi_{j'} - f_j^{j'} g_{j'}^{j''}\phi_{j''} + \left\{F, g_j^{j'}\right\}\phi_{j'} + g_j^{j'} f_{j'}^{j''}\phi_{j''} \\
&\approx 0.
\end{aligned} \tag{2.39}$$

Let G be a function on the phase space. We say that G generates the infinitesimal transformation $F \to F + \delta F$ where

$$\delta F = \epsilon\{F, G\} \tag{2.40}$$

for any phase space function F and where ϵ is an infinitesimal parameter.

The Dirac conjecture states that all first class constraints generate gauge transformations. This is not strictly true, as one can construct counterexamples. However, one postulates that this is the case, i.e., one postulates that states that differ by a transformation generated by a first class constraint are physically the same state, since otherwise one runs into other conceptual problems, see [Henneaux and Teitelboim, 1992].

The second class constraints, on the other hand, do not generate gauge transformations. Their treatment classically and quantum mechanically is much more straightforward (though may be tedious in practice), since in this case one can effectively work with a lower dimensional phase space with no constraints. Note that, second class constraints should always come in even numbers.

In general, one can count the number of physical degrees of freedom as

$$\begin{aligned} 2\times &\begin{pmatrix}\text{Number of physical}\\ \text{degrees of freedom}\end{pmatrix} \\ &= \begin{pmatrix}\text{Total number of}\\ \text{canonical variables}\end{pmatrix} - \begin{pmatrix}\text{Number of second}\\ \text{class constraints}\end{pmatrix} \\ &\qquad - 2\times\begin{pmatrix}\text{Number of first}\\ \text{class constraints}\end{pmatrix}. \end{aligned} \tag{2.41}$$

We conclude this section with two examples that illustrate first and second class constraints.

Example 2.3. *Particle in two dimensions restricted to a circle of radius* r_0. Start with the Lagrangian

$$L(r,\varphi,\lambda,\dot r,\dot\varphi,\dot\lambda) = \frac{1}{2}\left(\dot r^2 + r^2\dot\varphi^2\right) - \lambda\left(r - r_0\right). \tag{2.42}$$

We have a single primary constraint $\phi_1 = p_\lambda = 0$. The Hamiltonian is

$$H(r,\varphi,\lambda,p_r,p_\varphi,p_\lambda) = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + \lambda\left(r - r_0\right). \tag{2.43}$$

We require that the primary constraint is preserved in time:

$$\dot\phi_1 = 0 = \{\phi_1, H\} + u\{\phi_1,\phi_1\} = r_0 - r. \tag{2.44}$$

Thus, we get a secondary consraint, $\phi_2 = r_0 - r$. Proceeding in the same way, we get

$$\dot{\phi}_2 = 0 = -p_r \equiv \phi_3 \tag{2.45}$$

and

$$\dot{\phi}_3 = 0 = \lambda - \frac{p_\varphi^2}{r^3} \equiv \phi_4. \tag{2.46}$$

If we do this once more, we do not get a new constraint, but a restriction on u:

$$\dot{\phi}_4 = 0 = u + \frac{3p_r p_\varphi^2}{r^4} \approx u. \tag{2.47}$$

The equations of motion are

$$\begin{aligned} \dot{r} &= p_r, & \dot{\varphi} &= \frac{p_\varphi}{r^2}, & \dot{\lambda} &= 0, \\ \dot{p}_r &= \frac{p_\varphi^2}{r^3} - \lambda, & \dot{p}_\varphi &= 0, & \dot{p}_\lambda &= r_0 - r = 0, \end{aligned} \tag{2.48}$$

which are solved by

$$\begin{aligned} r &= r_0, & \varphi &= \frac{p_\varphi}{r_0^2} t + \varphi_0, & \lambda &= \frac{p_\varphi^2}{r_0^3}, \\ p_r &= 0, & p_\varphi &= \text{const}, & p_\lambda &= 0, \end{aligned} \tag{2.49}$$

Note that, the constraints are all second class, since we have $\{\phi_1, \phi_4\} = -1$ and $\{\phi_2, \phi_3\} = 1$. This is consistent with the fact that there is no gauge freedom, since there is no arbitrary function u in the solution. In this case, one can simply set the constraints equal to zero from the beginning, and work with the Hamiltonian

$$H = \frac{p_\varphi^2}{2r^2} \tag{2.50}$$

without constraints.

Example 2.4. *Particle in one dimension with gauge redundancy.* We take the Lagrangian

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}\left(\dot{x}_1 + \dot{x}_2\right)^2. \tag{2.51}$$

There is a single primary constraint: $\phi \equiv p_1 - p_2 = 0$. The Hamiltonian is

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2}p_1^2. \tag{2.52}$$

Here, we get neither a secondary constraint nor a restriction on u, since

$$\{\phi, H\} + u\,\{\phi, \phi\} = 0 \tag{2.53}$$

identically. Since there is only one constraint, it is first class. Hence, we should have gauge freedom.

The equations of motion are

$$\dot{x}_1 = p_1 + u(t), \quad \dot{x}_2 = -u(t), \quad \dot{p}_1 = 0, \quad \dot{p}_2 = 0, \quad p_1 = p_2. \tag{2.54}$$

The initial conditions must satisfy the constraint $p_1(0) = p_2(0)$. Then, the solution is given by

$$q_1(t) = q_1(0) + p_1(0)t + \int_0^t u(t')\mathrm{d}t', \tag{2.55a}$$

$$q_2(t) = q_2(0) - \int_0^t u(t')\mathrm{d}t', \tag{2.55b}$$

$$p_1(t) = p_2(t) = p_1(0). \tag{2.55c}$$

We see that the solutions for q_1 and q_2 indeed include an arbitrary function. This is a gauge redundancy. A gauge independent combination is

$$q_1(t) + q_2(t). \tag{2.56}$$

Chapter 3

ADM FORMALISM

The standard Hamiltonian formulation of general relativity is the Arnowitt-Deser-Misner (ADM) formulation. The triad formulation and the Ashtekar formulation are also closely related to the ADM formalism, hence, we shall review here the basics of the ADM formalism. We shall closely follow [Blau, 2023].

3.1 Hypersurfaces

A hypersurface Σ of a manifold M is a co-dimension one submanifold. If we have a metric g, we can define a vector field n normal to Σ, which is unique up to normalization. If Σ is spacelike, timelike or null, n will be timelike, spacelike or null, respectively. In the following, we consider spacelike and timelike hypersurfaces only, and assume that n is normalized with

$$n \cdot n = \sigma = \pm 1. \tag{3.1}$$

We can define the *projection tensor* as

$$q^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \sigma n^{\mu} n_{\nu}. \tag{3.2}$$

q is called such, since, given any vector V, q projects it tangent to Σ:

$$\left(q^{\mu}{}_{\nu} V^{\nu}\right) n_{\mu} = 0. \tag{3.3}$$

Furthermore, acting on tangent vectors V and W, q is equivalent to the metric g:

$$q_{\mu\nu} V^{\mu} W^{\nu} = g_{\mu\nu} V^{\mu} W^{\nu}. \tag{3.4}$$

Thus, if restricted to tangent vectors, q is the induced metric on Σ. One also says that q is the *first fundamental form* of Σ.

The *extrinsic curvature* of a hypersurface is defined by taking the covariant derivative of the normal vector and projecting it to the surface:

$$K_{\mu\nu} = q^{\alpha}{}_{\mu} q^{\beta}{}_{\nu} \nabla_\alpha n_\beta, \tag{3.5}$$

which is also called the *second fundamental form* of Σ. Although it may seem at first that $K_{\mu\nu}$ depends on how we extend n off the hypersurface, due to the tangential projections it does not, which we shall prove at the end of this section.

Let us remark that, as long as we keep n normalized off the surface, we have $n^\beta \nabla_\alpha n_\beta = 0$, hence, one of the projections in the definition Eq. (3.5) is unnecessary and we can write $K_{\mu\nu} = q^{\alpha}{}_{\mu} \nabla_\alpha n_\nu$.

We shall now that $K_{\mu\nu}$ is a symmetric tensor. To this end, note that since n is hypersurface orthogonal, it can be written as $n_\mu = g \nabla_\mu f$ for some functions g and f. Since ∇ is torsion-free, we have

$$\begin{aligned} \nabla_{[\mu} n_{\nu]} &= \nabla_{[\mu} g \nabla_{\nu]} f \\ &\sim (\nabla_\mu g)\, n_\nu - (\nabla_\nu g)\, n_\mu . \end{aligned} \tag{3.6}$$

This means that the antisymmetric part of $\nabla_\mu n_\nu$ does not contribute when projected. Hence,

$$\begin{aligned} K_{\mu\nu} &= q^{\alpha}{}_{\mu} q^{\beta}{}_{\nu} \nabla_{(\alpha} n_{\beta)} \\ &= q^{\alpha}{}_{(\mu} q^{\beta}{}_{\nu)} \nabla_\alpha n_\beta . \end{aligned} \tag{3.7}$$

Given a hypersurface Σ with the induced metric q, we have the Levi-Civita connection on Σ defined by q. We can also show that the projected covariant derivative,

$$D_\mu V^\nu = q^{\alpha}{}_{\mu} q^{\nu}{}_{\beta} \nabla_\alpha V^\beta \tag{3.8}$$

defines a connection on Σ. It turns out that D is the same as the Levi-Civita connection of q. This can be proven by showing that D is compatible with q and that it is torsion-free. Compatibility with q is easily shown. To see that D is torsion-free, let f be a function on Σ. Then,

$$\begin{aligned} D_\alpha D_\beta f &= q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \left(q^\gamma{}_\nu \nabla_\gamma f\right) \\ &= -\sigma q^\mu{}_\alpha q^\nu{}_\beta n^\gamma \left(\nabla_\mu n_\nu\right) \nabla_\gamma f + q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \nabla_\nu f. \end{aligned} \tag{3.9}$$

Note that, since ∇ is torsion-free, we have $q^\mu{}_{[\alpha} q^\nu{}_{\beta]} \nabla_\mu \nabla_\nu f = 0$. Therefore,

$$\begin{aligned} D_{[\alpha} D_{\beta]} f &= -\sigma q^\mu{}_{[\alpha} q^\nu{}_{\beta]} n^\gamma \left(\nabla_\mu n_\nu\right) \nabla_\gamma f \\ &= -\sigma K_{[\alpha\beta]} n^\gamma \nabla_\gamma f \\ &= 0. \end{aligned} \tag{3.10}$$

Furthermore, the four dimensional Ricci scalar R and the Ricci scalar ${}^3R$ of the three dimensional surface Σ are related through [Wald, 2009]

$$R = {}^3R + \sigma \left(K^2 - K^{\alpha\beta} K_{\alpha\beta}\right) + 2\sigma \nabla_\alpha \left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta\right) \tag{3.11}$$

where $K = g^{\mu\nu} K_{\mu\nu}$.

Before concluding this section, we want to prove that the extrinsic curvature tensor $K_{\mu\nu}$ does not depend on how we extend the normal vector n off a hypersurface Σ. For any hypersurface Σ, we can find the so-called Gaussian normal coordinates where the lapse is unity and the shift vanishes, so that the metric takes the form

$$g = \sigma \mathrm{d}z^2 + \gamma_{ab} \mathrm{d}y^a \mathrm{d}y^b. \tag{3.12}$$

Here, $\sigma = \pm 1$ and Σ is given by the condition $z = \text{const}$. In the following, we shall work with the Gaussian normal coordinates for simplicity. The normal vector is given by $n_0 = \partial_z|_\Sigma$. Let us extend n_0 arbitrarily off Σ:

$$n = N \left(\partial_z + f^a \partial_a\right), \tag{3.13}$$

where N and f^a are arbitrary except for $f^a|_\Sigma = 0$ and $N|_\Sigma = 1$.

On Σ we have

$$K_{\mu\nu} = \nabla_\mu n_\nu - \delta^z_\mu \nabla_z n_\nu, \tag{3.14}$$

from which we readily see that $K_{z\mu} = K_{\mu z} = 0$. Thus we only need to consider $K_{ab} = \nabla_a n_b$ where $a, b \neq z$.

Again, on Σ we get

$$\nabla_a n_b = \gamma_{bc}\partial_a f^c - \sigma\Gamma^z_{ab}. \tag{3.15}$$

Note, however, that

$$\partial_a f^c|_\Sigma = \partial_a \left(f^c|_\Sigma\right) = 0. \tag{3.16}$$

We therefore get $K_{ab} = -\sigma\Gamma^z_{ab}$, which shows that K_{ab} does not depend on N or f^a, i.e., on how we extend n_0 off the hypersurface.

3.2 ADM decomposition

We shall assume that the spacetime is topologically of the form $M = \mathbb{R}\times S$ for some three dimensional manifold S (space). We can then foliate spacetime with spacelike hypersurfaces $\Sigma_t \stackrel{\sim}{=} S$. On each Σ_t, define coordinates x^a where $a = 1, 2, 3$. Then, (t, x^a) can be used as coordinates on M.

Let n be a vector field normal to Σ_t at each point. Normalize n such that $n \cdot n = -1$. There remains a freedom in overall orientation ($\pm n$). Choose n to be future directed, which completely fixes n.

We do not require ∂_t to be normal to Σ_t. Hence, in general, ∂_t can be decomposed into a tangential and a normal component as

$$\partial_t = Nn + N^a\partial_a. \tag{3.17}$$

N and N^a are called the lapse function and the shift vector, respectively.

Note that we have $\partial_a \cdot \partial_b = q_{ab}$, $\partial_t \cdot \partial_a = N_a$ where $N_a = q_{ab}N^b$ (we raise/lower the indices of N^a with the spatial metric q) and finally $\partial_t \cdot \partial_t = -N^2 + N^aN_a$. These imply that the metric is given by

$$g = -N^2\mathrm{d}t^2 + q_{ab}\left(\mathrm{d}x^a + N^a\mathrm{d}t\right)\left(\mathrm{d}x^b + N^b\mathrm{d}t\right). \tag{3.18}$$

Let us now relate the determinants of the two metrics g and q. We have

$$\tilde{n} = -N\mathrm{d}t, \tag{3.19}$$

where $\tilde{n} = g(n)$ is the associated 1-form. Now, the volume element of spacetime M has to be equal to the product of the hypersurface element of Σ_t and the unit normal:

$$\sqrt{-g}\mathrm{d}t \wedge \mathrm{d}^3x = -\tilde{n} \wedge \left(\sqrt{q}\mathrm{d}^3x\right), \tag{3.20}$$

where the minus sign occurs because $\tilde{n}$ is timelike. From the last two equations, we get the relation between the determinants:

$$\sqrt{-g} = N\sqrt{q}. \tag{3.21}$$

Before writing down the ADM action, we finally want to show that the extrinsic curvature K_{ab} can be expressed nicely in terms of the time derivative of the spatial metric $\dot{q}_{ab}$, the lapse and the spatial derivatives of the shift vector as

$$K_{ab} = \frac{1}{2N}\left(\dot{q}_{ab} - D_a N_b - D_b N_a\right). \tag{3.22}$$

To see this, note first that in terms of the Christoffel symbols of g we have $K_{ab} = N\Gamma^t_{ab}$. Then, observe that

$$\Gamma^t_{ab} = -\frac{1}{2N^2}\left(\partial_a N_b + \partial_b N_a - \dot{q}_{ab}\right) + \frac{1}{N^2}\Gamma^c_{ab}N_c, \tag{3.23}$$

from which follows Eq. (3.22).

3.3 ADM Hamiltonian and action

We want to write the Einstein-Hilbert action in terms of the three metric q and its time derivatives. This can be done using Eq. (3.11), which yields

$$\begin{aligned} S &= \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R \\ &= \frac{1}{2\kappa}\int \mathrm{d}t\ \mathrm{d}^3x\sqrt{q}N\left({}^3R - K^2 + K^{ab}K_{ab}\right), \end{aligned} \tag{3.24}$$

where $\kappa = 8\pi G$ and we have discarded the boundary term and used the fact that $K_{\alpha\beta}$ is a spatial tensor, i.e., $K^{\alpha\beta}K_{\alpha\beta} = K^{ab}K_{ab}$. It is understood that in this action, the variables N, N^a and q_{ab} must be varied independently.

The Lagrangian reads as

$$\mathcal{L} = \frac{1}{2\kappa}\sqrt{q}N\left({}^3R - K^2 + K^{ab}K_{ab}\right). \tag{3.25}$$

The momenta conjugate to q_{ab} are defined as

$$P^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}}. \tag{3.26}$$

Note that the Lagrangian depends on $\dot{q}_{ab}$ through the K_{ab} as $K_{ab} = (2N)^{-1}\dot{q}_{ab} + \cdots$. We therefore get

$$P^{ab} = \frac{\sqrt{q}}{2\kappa}\left(K^{ab} - Kq^{ab}\right). \tag{3.27}$$

This can also be inverted as

$$\frac{\sqrt{q}}{2\kappa}K^{ab} = P^{ab} - \frac{1}{2}Pq^{ab} \tag{3.28}$$

with $P = q_{ab}P^{ab}$, which, using Eq. (3.22) in turn allows the "velocities" $\dot{q}_{ab}$ to be expressed in terms of the momenta as

$$\dot{q}_{ab} = 2NK_{ab} + 2D_{(a}N_{b)}. \tag{3.29}$$

Final thing to note before applying the Legendre transformation is that the time derivatives of N and N^a do not show up in the action, thus we have the primary constraints

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0, \quad \Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{N}^a} = 0. \tag{3.30}$$

We can now apply the Legendre transformation to pass over to the Hamiltonian. We get, up to constraint terms

$$\mathcal{H}_{\mathrm{ADM}} = P^{ab}\dot{q}_{ab} - \mathcal{L}. \tag{3.31}$$

Using the fact that

$$K^{ab}K_{ab} - K^2 = \frac{(2\kappa)^2}{q}\left(P^{ab}P_{ab} - \frac{1}{2}P^2\right) \tag{3.32}$$

and discarding boundary terms, we can write Eq. (3.31) as

$$\mathcal{H}_{\mathrm{ADM}} = N\mathcal{H} + N_a\mathcal{H}^a, \tag{3.33}$$

where

$$\mathcal{H} = \frac{2\kappa}{\sqrt{q}}\left(P^{ab}P_{ab} - \frac{1}{2}P^2\right) - \frac{\sqrt{q}}{2\kappa}\ {}^3R$$
$$= \frac{\sqrt{q}}{2\kappa}\left(K^{ab}K_{ab} - K^2 - {}^3R\right) \tag{3.34}$$

and

$$\mathcal{H}^a = -2D_bP^{ab}. \tag{3.35}$$

We can thus see from Eq. (3.33) that N and N^a act as Lagrange multipliers only, and the Hamiltonian is a sum of constraints. The Hamiltonian action can be written as

$$S[q_{ab}, P^{ab}, N, N_a] = \int \mathrm{d}t\ \mathrm{d}^3x \left(P^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a\right). \tag{3.36}$$

The classical Poisson bracket between two functionals on the phase space A and B is given by

$$\begin{aligned}\{A,B\} = &\int \mathrm{d}^3x \left(\frac{\delta A}{\delta q_{ab}(x)}\frac{\delta B}{\delta P^{ab}(x)} - \frac{\delta A}{\delta P^{ab}(x)}\frac{\delta B}{\delta q_{ab}(x)}\right)\\ &+ \int \mathrm{d}^3x \left(\frac{\delta A}{\delta N(x)}\frac{\delta B}{\delta \Pi(x)} - \frac{\delta A}{\delta \Pi(x)}\frac{\delta B}{\delta N(x)}\right)\\ &+ \int \mathrm{d}^3x \left(\frac{\delta A}{\delta N^a(x)}\frac{\delta B}{\delta \Pi_a(x)} - \frac{\delta A}{\delta \Pi_a(x)}\frac{\delta B}{\delta N^a(x)}\right),\end{aligned} \tag{3.37}$$

where δ denotes the functional derivative. The time evolution of an observable A is then given by

$$\dot{A} = \{A, H_{\mathrm{ADM}}\}, \tag{3.38}$$

where

$$H_{\mathrm{ADM}} = \int \mathrm{d}^3x\ \mathcal{H}_{\mathrm{ADM}} \tag{3.39}$$

is the total Hamiltonian.

3.4 Constraints

We have seen that we have four primary constraints Eq. (3.30). For consistency, these should be conserved in time. We therefore impose the conditions $\{\Pi, H_{\mathrm{ADM}}\} = 0$ and $\{\Pi_a, H_{\mathrm{ADM}}\} = 0$, which yield the four secondary constraints

$$\mathcal{H} = 0, \quad \mathcal{H}_a = 0. \tag{3.40}$$

These are called the scalar and vector constraints, or also Hamiltonian and momentum constraints, respectively, and they are equivalent to the normal components of the Einstein field equations,

$$G_{nn} = G_{\mu\nu} n^\mu n^\nu = 0, \quad G_{na} = G_{\mu a} n^\mu = 0. \tag{3.41}$$

As the next step, one should check whether these lead to other (tertiary) constraints. At this stage, it proves convenient to work with the so-called *smeared* constraints. These are functionals that we get by integrating the constraints multiplied by test functions. Namely, we define

$$H[V] = \int \mathrm{d}^3x \; V\mathcal{H}, \quad P[V_a] = \int \mathrm{d}^3x \; V_a \mathcal{H}^a. \tag{3.42}$$

Then, Eq. (3.40) is equivalent to condition that H and P vanish for all functions V and vectors V_a, respectively. Thus, we can equivalently check the time evolution of H and P to see whether they give new constraints.

Since the Hamiltonian is a sum of H and P, we need the Poisson brackets between them. One can show that [Blau, 2023]

$$\{H[V_1], H[V_2]\} = P[V_1 D_a V_2 - V_2 D_a V_1] \tag{3.43a}$$

$$\{P[V_a], H[V]\} = H[V^a D_a V] \tag{3.43b}$$

$$\{P[V_1^a], P[V_2^a]\} = P[[V_1, V_2]^a], \tag{3.43c}$$

where on the right hand side of Eq. (3.43c), $[\cdot, \cdot]$ denotes the Lie bracket between vector fields:

$$[V_1, V_2]^a = V_1^b D_b V_2^a - V_2^b D_b V_1^a. \tag{3.44}$$

From Eq. (3.43) we see that the secondary constraints do not lead to new constraints, thus the procedure stops at this point. We have 8 constraints in total, all of whose Poisson brackets vanish weakly. They are therefore first class constraints, and generate gauge transformations. Since the phase space has 20 dimensions and there are 8 first class constraints per spatial point, we have $(20 - 2 \times 8)/2 = 2$ propagating degrees of freedom, as expected.

Chapter 4

ASHTEKAR–HENDERSON–SLOAN VARIABLES

Motivated by the BKL conjecture, a set of variables was introduced by [Ashtekar et al., 2009, Ashtekar et al., 2011], which provides a way of analyzing spacelike singularities in a purely classical setting. These variables are based on the formulation of general relativity in terms of a density weighted triad, which is also the starting point of loop quantum gravity. In this chapter, we shall review the triad formulation of general relativity and the Ashtekar-Henderson-Sloan (AHS) variables. We shall then summarize the application of these variables to the Schwarzschild singularity as an illustration of the idea, which will also help us introduce some of the ideas we are going to use in our calculations.

4.1 *Triad formulation of general relativity*

Consider spacetimes of the form $\mathcal{M} = \mathbb{R} \times M$, where M is a three-dimensional manifold. We foliate $\mathcal{M}$ into spacelike slices Σ_t and denote the spatial metric induced on these slices by q. An orthonormal co-triad e_a^i on Σ_t is defined by

$$q_{ab} = \delta_{ij} e_a^i e_b^j. \tag{4.1}$$

Here, the letters $a, b, c, \ldots$ denote the spatial indices and $i, j, k, \ldots$ denote internal SO(3) indices. The internal indices can be raised/lowered freely using δ_{ij}, while the spatial indices are raised/lowered using the spatial metric q_{ab}. We denote the orthonormal triad dual to e_a^i by the same letter, e_i^a; they may be distinguished by the placement of the indices. The triad and the co-triad satisfy $e_a^i e_j^a = \delta_j^i$ and $e_a^i e_i^b = \delta_a^b$.

We shall work with the density weighted triad $\tilde{E}_i^a = \widetilde{\sqrt{q}} e_i^a$, or

$$\tilde{E}_i^a \tilde{E}^{bi} = \tilde{\tilde{q}} q^{ab}, \tag{4.2}$$

where $\tilde{\tilde{q}}$ denotes the determinant of q_{ab} and in this section we want to make the density weights explicit with tildes; a tilde above a variable means a density weight of $+1$, and a tilde below means a density weight of -1.

Note that, an orthonormal co-triad $e^i = e^i_a \mathrm{d}x^a$ determines a unique connection $\omega^i{}_j = \omega^i_{aj}\mathrm{d}x^a$ such that $\omega_{ij} + \omega_{ji} = 0$ and $\mathrm{d}e^i + \omega^i{}_j \wedge e^j = 0$, or, equivalently,

$$D_a e^b_i - \omega^j_{ai} e^b_j = 0 \tag{4.3}$$

in terms of the triad, where D_a is the Levi-Civita connection of q_{ab}. We should emphasize that D_a only acts on spacetime indices whereas ω acts only on the internal indices. Expanding ω in the SO(3) basis as $\omega^i{}_j = \epsilon^i{}_{jk}\Gamma^k$ and multiplying by $\widetilde{q^{1/2}}$, Eq. (4.3) can also be written as

$$D_a \tilde{E}^b_i - \epsilon^j{}_{ik}\Gamma^k_a \tilde{E}^b_j = 0. \tag{4.4}$$

The variable canonical to $\tilde{E}^a_i$ is K^i_a, which, on solutions, is related to the extrinsic curvature K_{ab} of Σ_t by

$$K_{ab} = K^i_{(a} e_{b)i}. \tag{4.5}$$

The standard ADM variables $(q_{ab}, \tilde{P}^{ab})$, where

$$\tilde{P}^{ab} = \frac{\widetilde{\sqrt{q}}}{2\kappa}\left(K^{ab} - K^c{}_c q^{ab}\right) \tag{4.6}$$

can be found using Eqs. (4.2) and (4.5). Note, however, that the variables $\tilde{E}^a_i$ have more degrees of freedom than q_{ab}, which is obvious from Eq. (4.2); two triads that differ by an SO(3) rotation yield the same metric q_{ab}. Therefore, in going over to the pair $(\tilde{E}^a_i, K^i_a)$ from the ADM variables we have introduced further gauge redundancy and extended the phase space. Thus, in order to recover the same physics, one needs to introduce a set of three constraints which generates the SO(3) rotations. This additional vector constraint is called the Gauss constraint $\tilde{G}^i$.

In terms of these new variables, the scalar and vector constraints of general relativity and the Gauss constraint are given, in units with $2\kappa = 1$, by [Ashtekar et al., 2011]

$$\tilde{\tilde{S}} = -\tilde{q}R - 2\tilde{E}^a_{[i}\tilde{E}^b_{j]}K^i_aK^j_b, \tag{4.7a}$$

$$\tilde{V}_a = 4D_{[a}\left(K^i_{b]}\tilde{E}^b_i\right), \tag{4.7b}$$

$$\tilde{G}^k = \epsilon_i{}^{jk}\tilde{E}^a_jK^i_a, \tag{4.7c}$$

where R and D_a are the scalar curvature and the Levi-Civita connection of the spatial metric q_{ab}, respectively. The Hamiltonian is a sum of the constraints:

$$H = \int \mathrm{d}^3x \left(\underset{\sim}{N}\tilde{\tilde{S}} + N^a\tilde{V}_a + \Lambda_i\tilde{G}^i\right), \tag{4.8}$$

and the equations of motion can be found by taking the Poisson bracket with H;

$$\dot{\tilde{E}}^a_i = \left\{\tilde{E}^a_i, H\right\}, \quad \dot{K}^i_a = \left\{K^i_a, H\right\}. \tag{4.9}$$

4.2 The variables

The AHS variables [Ashtekar et al., 2009, Ashtekar et al., 2011] are defined as[1]

$$\tilde{P}_i{}^j = \tilde{E}^a_iK^j_a - \delta^j_i\tilde{E}^a_kK^k_a, \tag{4.10a}$$

$$\tilde{C}_i{}^j = \tilde{E}^a_i\Gamma^j_a - \delta^j_i\tilde{E}^a_k\Gamma^k_a. \tag{4.10b}$$

Furthermore, a derivative operator $\tilde{D}_i$ is defined as

$$\tilde{D}_i = \tilde{E}^a_iD_a. \tag{4.11}$$

Note that these carry internal indices only; they are spacetime scalars (with density weight +1). Physically, the $\tilde{P}^{ij}$ are related directly to the canonical ADM momenta $\tilde{P}^{ab}$, whereas the $\tilde{C}^{ij}$ encode information about the $\tilde{D}_i$ derivatives of the triad $\tilde{E}^a_i$ [Ashtekar et al., 2011]. As mentioned previously, the determinant q is expected to vanish at the singularity and therefore the AHS variables, being densities, are expected to be finite there. In what follows, we shall omit the tildes for simplicity. Thus, each of E^a_i, $C_i{}^j$, $P_i{}^j$ and D_i carries a density weight of +1, while the lapse

[1] Note that, due to the placement of the indices, our convention for Γ^i_a and consequently for $\tilde{C}_i{}^j$ differs from that of [Ashtekar et al., 2009, Ashtekar et al., 2011] by a sign.

N carries a density weight of -1. The scalar and vector constraints S and V_i (introduced below) carry a density weight of $+2$, and the Gauss constraint G^i carries a density weight of $+1$.

The remarkable thing about these variables is that the constraints (4.7a)-(4.7c) can be expressed entirely in terms of ${P_i}^j$ and ${C_i}^j$ (and their D_i derivatives) [Ashtekar et al., 2011]:

$$S = -2\epsilon^{ijk} D_i C_{jk} + 4C_{[ij]}C^{[ij]} + C_{ij}C^{ji} - \frac{1}{2}C^2 + P_{ij}P^{ji} - \frac{1}{2}P^2, \tag{4.12a}$$

$$V_i = -2D_j {P_i}^j + 2\epsilon_{jkl}{P_i}^j C^{kl} + \epsilon_{ijk} C P^{jk} - 2\epsilon_{ijk}P^{jl}{C_l}^k, \tag{4.12b}$$

$$G^k = \epsilon^{ijk} P_{ji}, \tag{4.12c}$$

where P and C are the traces of ${P_i}^j$ and ${C_i}^j$, respectively.

One can write down the equations of motion for the variables ${P_i}^j$, ${C_i}^j$ and the operator D_i by taking the Poisson brackets with the Hamiltonian. We shall not use the equations of motion directly, and therefore do not repeat them here (see [Ashtekar et al., 2011] for the full equations of motion). The important thing to note, however, is that the time derivative of the triplet $({P_i}^j, {C_i}^j, D_i)$ can be written entirely in terms of the triplet itself (and the Lagrange multipliers N, N^i and Λ^i), i.e., without reference to (E^a_i, K^i_a).

These variables can therefore be used as follows. On an initial time slice, one can construct the triplet $({P_i}^j, {C_i}^j, D_i)$ from the pair (E^a_i, K^i_a) and then evolve the triplet $({P_i}^j, {C_i}^j, D_i)$ without referring to (E^a_i, K^i_a). As we have mentioned, these variables are expected to be finite at the singularity, where the metric formulation fails, hence one can continue time evolution past the singularity. Once we cross the singularity, we can, in principle, reconstruct the triad using the evolution equation

$$\dot{E}^a_i = -N {P_i}^j E^a_j, \tag{4.13}$$

which is an ordinary differential equation for E^a_i.

4.3 Application to Schwarzschild spacetime

In order to demonstrate the use of AHS variables with a simple, analytical example, we shall first summarize their application to the Schwarzschild singularity, which has been done by [Valdés-Meller, 2021].

Let us first look at the metric

$$g = -T(\tau)\mathrm{d}\tau^2 + X(\tau)\mathrm{d}x^2 + A(\tau)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{4.14}$$

where

$$T = \frac{4\tau^4}{2M-\tau^2}, \quad X = \frac{2M-\tau^2}{\tau^2}, \quad A = \tau^4. \tag{4.15}$$

The portion of this spacetime given by $\tau \in (-\sqrt{2M}, 0)$ is precisely the interior of a Schwarzschild black hole, as can be seen by the coordinate transformation

$$t = x, \quad r = \tau^2. \tag{4.16}$$

However, the spacetime in Eq. (4.14) also includes the part $\tau \in (0, \sqrt{2M})$, which is a time–reversed copy of the black hole interior, i.e., a white hole interior (see Fig. 4.1), since the transformation from r to τ is a two-to-one mapping. As discussed by [D'Ambrosio and Rovelli, 2018], such a spacetime with two copies of the Schwarzschild metric may actually provide an effective description of an underlying quantum geometry, where there is a high–curvature cutoff that causes the geometry to "bounce back" instead of becoming singular, similarly to the Big Bounce scenario from LQC in the case of FLRW spacetimes.

The metric (4.14) has singularities at $\tau = \pm\sqrt{2M}$, which, of course, correspond to the horizon of the Schwarzschild black hole, and are merely coordinate singularities.

It also has a singularity at $\tau = 0$, which is the central singularity of the Schwarzschild black hole. Note however, that at both sides of the singularity, i.e. at $\tau \in (-\sqrt{2M}, 0) \cup (0, \sqrt{2M})$, we have a well defined metric tensor. However, in the standard metric formulation, one cannot evolve the metric from one side to the other, past the singularity. This brings up the question, can there be a formulation of the gravitational field equations which does not face the same problem at the singularity?

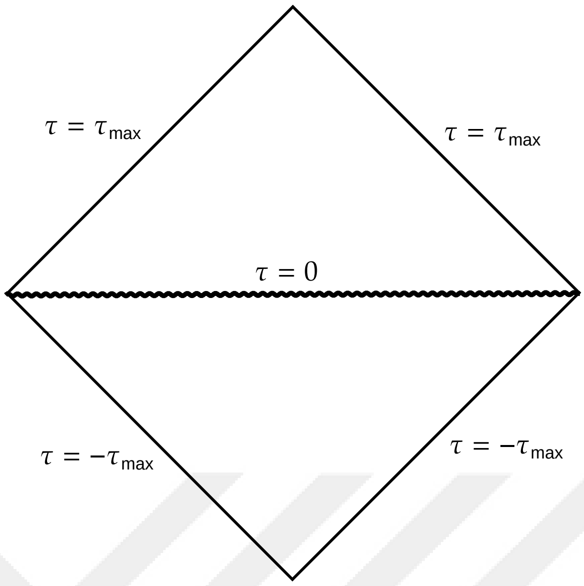

Figure 4.1: Spacetime diagram of the metric (4.14). We have a well-defined metric tensor in the two regions $\tau < 0$ and $\tau > 0$, which are separated by the singularity, $\tau = 0$. Here, $\tau_{\max} = \sqrt{2M}$.

It is always possible in principle that a differential equation is stated in terms of "wrong" variables, and the same equation can be stated in terms of other variables which has well-behaved solutions. To quote the simple example from [D'Ambrosio and Rovelli, 2018], the differential equation $y\ddot{y} - 2\dot{y}^2 - y^2 = 0$ has the solution $y = 1/\sin t$, which is singular at $t = 0$. Formulated in terms of $x = y^{-1}$, however, the equation is simply the harmonic oscillator equation $\ddot{x} + x = 0$ and the corresponding solution $x = \sin t$ is regular. The idea of the AHS variables is that, although the metric formulation fails at the singularity, the AHS variables can still remain finite and provide a well-behaved description of the gravitational field.

We can now calculate the AHS variables Eq. (4.10) for the metric in Eq. (4.14). The spatial metric on the $\tau = \text{const}$ slices reads as

$$q = X(\tau)\mathrm{d}x^2 + A(\tau)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{4.17}$$

Working with the triad

$$E_1^a = A\sin\theta\delta_x^a, \quad E_2^a = \sqrt{XA}\sin\theta\delta_\theta^a, \quad E_3^a = \sqrt{XA}\delta_\phi^a, \tag{4.18}$$

the only non–vanishing AHS variables are [Valdés-Meller, 2021]

$$P^{11} = -2\sin\theta\left(2M - \tau^2\right), \tag{4.19a}$$

$$P^{22} = P^{33} = -\sin\theta\left(2M - \tau^2\right), \tag{4.19b}$$

$$C^{31} = -\cos\theta\ \tau\sqrt{2M - \tau^2}. \tag{4.19c}$$

Hence we see that the AHS variables are indeed finite at $\tau = 0$, and can be continued from one side of the singularity to the other.

Chapter 5

GRAVITATIONAL WAVE COLLISION SPACETIMES

Before introducing the colliding gravitational wave solutions we are going to work with, it would be appropriate here to briefly review the fundamental concepts of the subject. When studying such collision solutions, it is convenient to divide spacetime into four regions, as shown in Fig. 5.1, with two dimensions suppressed. In a typical collision scenario, one has two approaching waves coming from infinity, so that regions II and III contain the incoming waves, whereas region I contains no waves and is therefore described by the—typically flat—background metric. After the point $u = v = 0$, the two waves interact, and region IV is described by the interaction metric, which is to be determined.

In Rosen coordinates, the interaction of arbitrarily polarized gravitational waves possibly coupled with matter fields is described [Griffiths, 1991] by a metric of the form

$$g = -2e^{-M}\mathrm{d}u\mathrm{d}v + e^{-U}\left[\cosh W\left(e^{V}\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right) - 2\sinh W\mathrm{d}x\mathrm{d}y\right], \qquad (5.1)$$

where the metric functions $M = M(u,v)$, $U = U(u,v)$, $V = V(u,v)$ and $W = W(u,v)$ are functions of the null coordinates u and v. These functions will assume different forms in each region, which must satisfy certain junction conditions at each boundary separating two regions.

The function W measures the second or cross polarization of the waves, and vanishes in the case of linear polarization. In this work, we shall only consider the case $W = 0$, thus the metric (5.1) simplifies to

$$g = -2e^{-M}\mathrm{d}u\mathrm{d}v + e^{-U}\left(e^{V}\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right). \qquad (5.2)$$

We will be interested in the singularities of the examples we work with. A convenient way to examine the singularity of a colliding gravitational wave spacetime

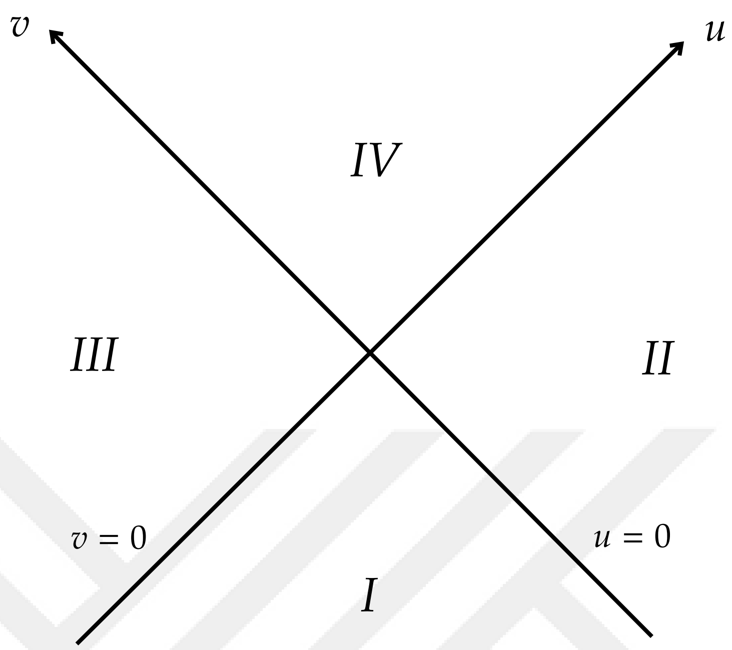

Figure 5.1: General structure of a colliding wave spacetime. Regions II and III contain the incoming waves, possibly including impulsive waves on the boundaries $u = 0$ and $v = 0$. There are no waves in region I, which is therefore described by the background metric. The waves start interacting after the point $u = v = 0$, so that the metric in region IV is determined by the interaction of the waves. Generically, the interaction results in a curvature singularity in region IV.

is by looking at the Weyl-NP scalars Ψ_i (see Appendix A). For instance, if $\Psi_1 = \Psi_3 = 0$, as will be the case in the examples we look at, the two polynomial curvature invariants I and J [Penrose and Rindler, 2003] can be written as

$$I = 2\Psi_0\Psi_4 + 6\Psi_2^2, \quad J = 6\left(\Psi_0\Psi_4 - \Psi_2^2\right)\Psi_2. \tag{5.3}$$

Note in particular that if Ψ_2 becomes unbounded, either I or J must also be unbounded, hence, showing that Ψ_2 diverges is a sufficient proof of a curvature singularity in this case.

The metric in Eq. (5.2) suggests a natural choice for the null tetrad:

$$l_\alpha = e^{-M/2}\delta^u_\alpha, \tag{5.4a}$$

$$n_\alpha = e^{-M/2}\delta^v_\alpha, \tag{5.4b}$$

$$m_\alpha = \frac{e^{-U/2}}{\sqrt{2}}\left(e^{V/2}\delta^x_\alpha + ie^{-V/2}\delta^y_\alpha\right), \tag{5.4c}$$

$$\overline{m}_\alpha = \frac{e^{-U/2}}{\sqrt{2}}\left(e^{V/2}\delta^x_\alpha - ie^{-V/2}\delta^y_\alpha\right). \tag{5.4d}$$

Working with this, it is straightforward to see that for the metric (5.2), we have $\Psi_1 = \Psi_3 = 0$ and

$$\Psi_0 = -\frac{1}{2}e^M\left[\left(M_v - U_v\right)V_v + V_{vv}\right], \tag{5.5a}$$

$$\Psi_2 = \frac{1}{6}e^M\left(M_{uv} - U_{uv} + V_uV_v\right), \tag{5.5b}$$

$$\Psi_4 = -\frac{1}{2}e^M\left[\left(M_u - U_u\right)V_u + V_{uu}\right], \tag{5.5c}$$

where the subscript denotes a partial derivative with respect to the coordinate.

Typically, the Weyl-NP scalars $\Psi_4 = \Psi_4(u)$ in region II and $\Psi_0 = \Psi_0(v)$ in region III represent the incoming gravitational waves that participate in the collision. In the interaction region these become functions of both of the null coordinates; $\Psi_0 = \Psi_0(u, v)$ and $\Psi_4 = \Psi_4(u, v)$. In addition, a finite Coulomb component $\Psi_2 = \Psi_2(u, v)$ also develops as a result of the nonlinear interaction.

On the other hand, the Ricci-NP scalars Φ_{ij} represent the electromagnetic or other matter fields participating in the collision, e.g., $\Phi_{00}(v)$ and $\Phi_{22}(u)$ represent electromagnetic wave components in the incoming regions. In the interaction region, $\Phi_{00}(u, v)$ and $\Phi_{22}(u, v)$ are necessarily nonzero and $\Phi_{02}(u, v)$ develops as a result of the nonlinear interaction.

In this study, our focus will be on two specific colliding gravitational wave solutions. The first is the Khan-Penrose solution, which was the first solution discovered in this context and which describes the collision of two impulsive plane symmetric gravitational waves. The second is the collision of a plane electromagnetic wave with plane impulsive gravitational waves accompanied by shock gravitational waves. The common point in each solution is the existence of a spacelike curvature singularity in the interaction region.

Since our focus in this study is the singularity structures of these spacetimes, which are located in the interaction region, we shall not be interested in the solutions in regions I, II and III, and only consider region IV.

5.1 Khan-Penrose solution

The first exact solution that considers the collision of two gravitational waves was discovered by [Khan and Penrose, 1971], where the colliding waves are linearly polarized, impulsive gravitational waves described by $\Psi_0 = \delta(v)$ and $\Psi_4 = \delta(u)$.

The mutual focusing of the two waves eventually produces a spacelike singularity. The interaction region of the Khan-Penrose spacetime is described by the metric (5.2), where the metric functions are given by

$$e^{-U} = 1 - u^2 - v^2, \tag{5.6a}$$

$$e^{-V} = \frac{1 + u\sqrt{1-v^2} + v\sqrt{1-u^2}}{1 - u\sqrt{1-v^2} - v\sqrt{1-u^2}}, \tag{5.6b}$$

$$e^{-M} = \frac{\left(1 - u^2 - v^2\right)^{3/2}}{w^2\sqrt{1-u^2}\sqrt{1-v^2}}, \tag{5.6c}$$

with $w = uv + \sqrt{1-u^2}\sqrt{1-v^2}$.

In this region, the nonzero Weyl-NP scalars read as

$$\Psi_0 = \frac{\delta(v)}{\sqrt{1-u^2}} + 3e^{7U/2}\frac{w^3u\left(1-u^2\right)}{\sqrt{1-v^2}}, \tag{5.7a}$$

$$\Psi_2 = e^{7U/2}w^2\left(w^2 - uv\sqrt{1-u^2}\sqrt{1-v^2}\right), \tag{5.7b}$$

$$\Psi_4 = \frac{\delta(u)}{\sqrt{1-v^2}} + 3e^{7U/2}\frac{w^3v\left(1-v^2\right)}{\sqrt{1-u^2}}. \tag{5.7c}$$

Looking at these components, we see that on the spacelike surface given by the equation

$$u^2 + v^2 = 1, \tag{5.8}$$

i.e. $e^{-U} = 0$, the solution becomes singular. The singularity structure of the Khan-Penrose spacetime is illustrated in Fig. 5.2.

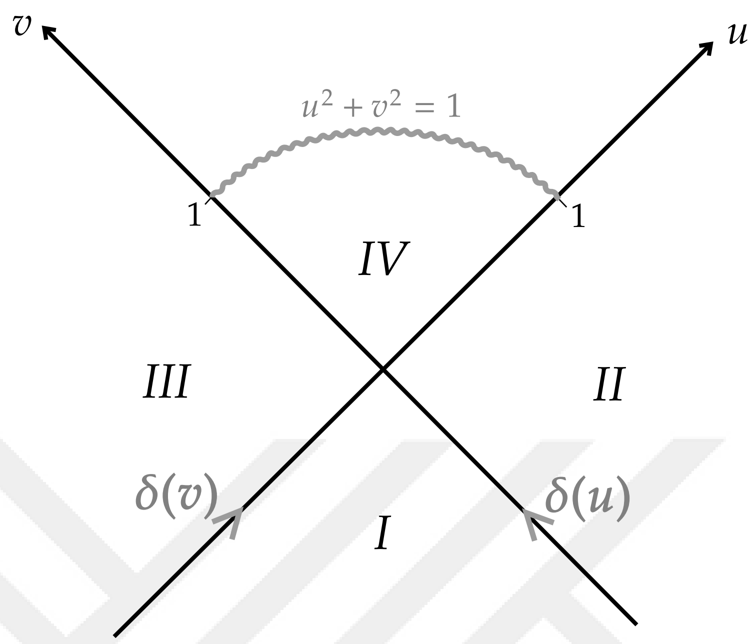

Figure 5.2: Structure of the Khan-Penrose spacetime. The incoming Dirac-δ waves start interacting after the point $u = v = 0$, and eventually create a singularity on the surface $u^2 + v^2 = 1$. The part of region IV below the singularity is described by the metric (5.2) with the metric functions given by Eq. (5.6).

5.2 *Colliding electromagnetic and gravitational waves*

The second example we shall look at is the collision of electromagnetic waves with plane symmetric impulsive gravitational waves accompanied by shock gravitational waves. This solution was obtained by [Gurtug et al., 2006] together with an additional scalar field. Here, we shall take the vanishing scalar field limit of this solution. The incoming gravitational wave is described by $\Psi_4 = \delta(u) - 3(1 + \sin u)^{-3}$ in region II and the incoming electromagnetic wave may be described by the Ricci-NP scalar $\Phi_{00} = 1$ in region III.

The interaction region is described again by Eq. (5.2), this time with the metric

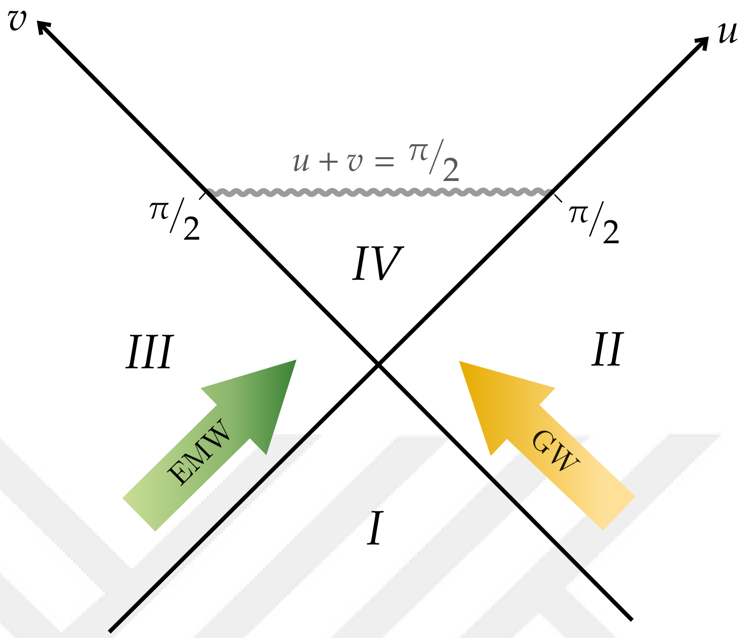

Figure 5.3: Structure of the gravitational-electromagnetic wave collision spacetime. The gravitational waves (GW) coming from region II and the electromagnetic waves (EMW) coming from region III start interacting after the point $u = v = 0$, and the interaction results in a singularity on the surface $u + v = \pi/2$. The part of region IV below the singularity is described by the metric (5.2) with the metric functions given by Eq. (5.9).

functions

$$e^{-U} = \cos^2 u + \cos^2 v - 1, \tag{5.9a}$$

$$e^{-V} = (1 + \sin u)^2 \,, \tag{5.9b}$$

$$e^{-M} = \cos u \cos v \, e^{-V+U/2}, \tag{5.9c}$$

and the electromagnetic potential $A_\mu = A\delta^y_\mu$, where

$$A = \sqrt{2} \sin v \, (1 + \sin u) \,. \tag{5.10}$$

The Ricci and Weyl-NP scalars are given by

$$\Phi_{00} = e^{M+U} \cos^2 v, \tag{5.11a}$$

$$\Phi_{02} = \frac{e^{V+U/2} \sin v}{1 + \sin u}, \tag{5.11b}$$

$$\Phi_{22} = e^{M+U+V} \sin^2 v \cos^2 u, \tag{5.11c}$$

and

$$\Psi_2 = -e^{3U/2} \frac{\sin u \sin v}{(1 + \sin u)^2}, \tag{5.12a}$$

$$\Psi_4 = \delta(u) - 3e^V \frac{1 - \sin u}{\cos u \cos v} \left(\frac{e^{-U/2}}{1 + \sin u} + e^{U/2} \sin u \right). \tag{5.12b}$$

Again, looking at the Weyl-NP scalars, we see that on the surface $e^{-U} = 0$, i.e.,

$$\cos^2 u + \cos^2 v = 1, \tag{5.13}$$

we have a spacelike singularity. Note that Eq. (5.13) is satisfied for

$$u + v = \frac{\pi}{2}. \tag{5.14}$$

This is illustrated in Fig. 5.3.

Chapter 6

SINGULARITIES OF GRAVITATIONAL WAVE COLLISIONS

In this chapter, we shall analyze the behaviour of the AHS variables at the singularities of the Khan-Penrose metric and the gravitational-electromagnetic wave collision solution introduced in Ch. 5. This requires a time-space split of the spacetime, thus, it is suitable to go from the null coordinates in Eq. (5.2) to a pair of timelike and spacelike coordinates. We therefore define $t = (u+v)/\sqrt{2}$ and $z = (u-v)/\sqrt{2}$. With these, Eq. (5.2) becomes

$$g = e^{-M}\left(-\mathrm{d}t^2 + \mathrm{d}z^2\right) + e^{-U}\left(e^{V}\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right), \tag{6.1}$$

which will be our starting point in what follows. Note that in these coordinates, the singularity of the Khan-Penrose metric is located at $e^{-U} = 1 - t^2 - z^2 = 0$, whereas the singularity of the the second metric is at a constant–t surface, $t = 2^{-3/2}\pi$.

6.1 *Khan-Penrose metric*

To calculate the AHS variables for the Khan-Penrose metric, we want to choose such a foliation that the singular surface $e^{-U} = 0$ is a limiting case of the spacelike slices. In other words, we want to introduce a timelike variable τ, such that the singularity corresponds to a level surface, conveniently chosen to be $\tau = 0$. In addition, just as was done in the Schwarzschild case, we let τ enter quadratically so that we get a spacetime with two copies of the interaction region connected by the singularity. We therefore define

$$\tau^2 = 1 - t^2 - z^2. \tag{6.2}$$

Then, our metric becomes

$$g = \frac{e^{-M}}{\xi + z^2}\left(-\tau^2 \mathrm{d}\tau^2 - 2\tau z \mathrm{d}\tau \mathrm{d}z + \xi \mathrm{d}z^2\right) + e^{-U}\left(e^V \mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right), \tag{6.3}$$

with the shorthand notation $\xi = 1 - \tau^2 - 2z^2$.

The spatial metric on the surfaces of constant τ reads as

$$q = e^{-M}\frac{\xi}{\xi + z^2}\mathrm{d}z^2 + e^{-U}\left(e^V \mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right). \tag{6.4}$$

We work with the obvious choice for the density-weighted triad, namely,

$$E_1 = e^{-U}\partial_z, \quad E_2 = e^{-(M+U+V)/2}\sqrt{\frac{\xi}{\xi + z^2}}\partial_x, \quad E_3 = e^{-(M+U-V)/2}\sqrt{\frac{\xi}{\xi + z^2}}\partial_y. \tag{6.5}$$

Given the triad, the connection 1-forms are uniquely determined as

$$\Gamma^1 = 0, \tag{6.6a}$$

$$\Gamma^2 = -\frac{1}{2}e^{(M-U-V)/2}\left(U_z + V_z\right)\sqrt{\frac{\xi + z^2}{\xi}}\mathrm{d}y, \tag{6.6b}$$

$$\Gamma^3 = \frac{1}{2}e^{(M-U+V)/2}\left(U_z - V_z\right)\sqrt{\frac{\xi + z^2}{\xi}}\mathrm{d}x, \tag{6.6c}$$

with $\omega^i{}_j = \epsilon^i{}_{jk}\Gamma^k$.

Furthermore, the extrinsic curvature of these spatial hypersurfaces K_{ab} is diagonal in this chart such that the conjugate momentum 1-forms $K^i = K_a{}^b e^i_b \mathrm{d}x^a$ are given by

$$K^1 = \frac{1}{2\sqrt{\xi + z^2}}\left(\frac{\xi}{\tau}M_\tau + zM_z - 2\frac{1-\tau^2}{\xi}\right)\mathrm{d}z, \tag{6.7a}$$

$$K^2 = \frac{e^{(M-U+V)/2}}{2\tau\sqrt{\xi}}\left[\xi\left(U_\tau - V_\tau\right) + \tau z\left(U_z - V_z\right)\right]\mathrm{d}x, \tag{6.7b}$$

$$K^3 = \frac{e^{(M-U-V)/2}}{2\tau\sqrt{\xi}}\left[\xi\left(U_\tau + V_\tau\right) + \tau z\left(U_z + V_z\right)\right]\mathrm{d}y. \tag{6.7c}$$

Given the triad (6.5), connection (6.6) and extrinsic curvature (6.7), the AHS variables can be directly computed. The non-vanishing AHS variables read as

$$P^{11} = -\frac{e^{-U}}{\tau\sqrt{\xi+z^2}}\mathcal{D}U, \tag{6.8a}$$

$$P^{22} = -\frac{e^{-U}}{2\tau\sqrt{\xi+z^2}}\left[\mathcal{D}\left(M+U+V\right)-2\tau\frac{1-\tau^2}{\xi}\right], \tag{6.8b}$$

$$P^{33} = -\frac{e^{-U}}{2\tau\sqrt{\xi+z^2}}\left[\mathcal{D}\left(M+U-V\right)-2\tau\frac{1-\tau^2}{\xi}\right], \tag{6.8c}$$

$$C^{23} = \frac{1}{2}\tau^2\left(U_z-V_z\right), \tag{6.8d}$$

$$C^{32} = -\frac{1}{2}\tau^2\left(U_z+V_z\right), \tag{6.8e}$$

where we have defined the shorthand operator notation $\mathcal{D}=\xi\partial_\tau+\tau z\partial_z$. Note that for any function f we have

$$\mathcal{D}f = -\frac{\tau}{\sqrt{2}}\sqrt{\xi+z^2}\left(f_u+f_v\right)+\frac{\tau z}{\sqrt{2}}\left(f_u-f_v\right) \tag{6.9}$$

and

$$f_z = \frac{1}{\sqrt{2}}\left(f_u-f_v\right)-\frac{z}{\sqrt{2}}\frac{1}{\sqrt{\xi+z^2}}\left(f_u+f_v\right). \tag{6.10}$$

Therefore, the question whether the AHS variables P^{ij} and C^{ij} are regular at $\tau=0$ is reduced to whether $e^{-U}f_{u,v}$ are regular, where f is any of the functions U, V and M.

This is easily seen to be the case for $f=U$, for which we have $\mathcal{D}U=-2\xi\tau^{-1}$ and $U_z=0$.

For M, we get

$$M_u = 3ue^U-\frac{u}{1-u^2}-2\frac{u\sqrt{1-v^2}-v\sqrt{1-u^2}}{w\sqrt{1-u^2}}. \tag{6.11}$$

Note that the second and third terms on the right hand side of Eq. (6.11) are regular everywhere in the region $0<u,v<1$. Thus, as $\tau\to 0$ we get

$$e^{-U}M_u \approx 3u = \frac{3}{\sqrt{2}}\left(\sqrt{1-\tau^2-z^2}+z\right), \tag{6.12}$$

which is also regular. Since M and U are symmetric functions of u and v, the same holds for $e^{-U}M_v$.

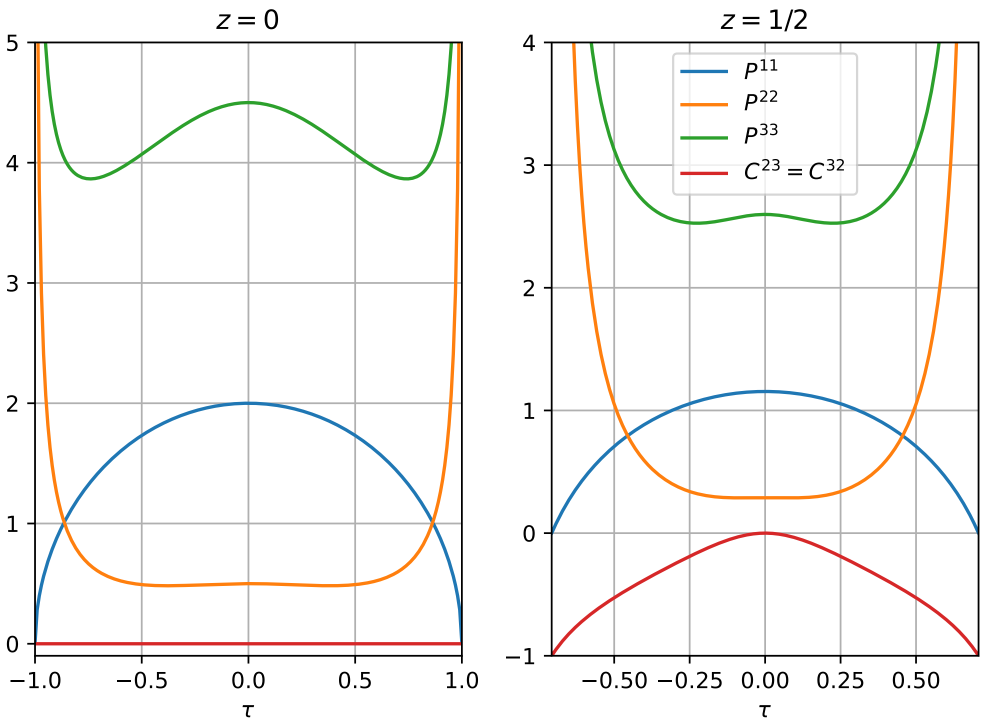

Figure 6.1: Non-vanishing AHS variables for the Khan-Penrose metric at two different points of space, $z = 0$ (left) and $z = 1/2$ (right). For this metric we always have $C^{23} = C^{32}$, which exactly vanish at $z = 0$. It can be seen that all of the variables are regular at the singularity. We note that P^{22} and P^{33} diverge at the boundaries of our region, $\tau = \pm\sqrt{1 - 2z^2}$. As explained in the text, however, this does not signal a physical breakdown since the foliation we have used becomes degenerate at the boundaries.

Lastly, for V we have

$$e^{-U} V_u = -\frac{2}{\sqrt{1-u^2}} \frac{\left(u + \sqrt{1-v^2}\right)\left(v + \sqrt{1-u^2}\right)}{1 + u\sqrt{1-v^2} + v\sqrt{1-u^2}}, \tag{6.13}$$

which is regular as well. Again, by symmetry, the same is true for $e^{-U} V_v$. Thus, all of the AHS variables in Eq. (6.8) are regular throughout the region in question, including the singularity $\tau = 0$, and they can be evolved smoothly from one side of the singularity to the other. Fig. 6.1 shows the AHS variables for the Khan-Penrose metric as functions of τ at two sample points.

We note that P^{22} and P^{33} diverge at the boundaries of our region, $\tau = \pm\sqrt{1-2z^2}$, due to the ξ^{-1} terms inside the square brackets in Eqs. (6.8b) and (6.8c). This does not imply a physical breakdown, however, because the boundaries are the null surfaces $u = 0$ and $v = 0$, or $t = |z|$, i.e., in the limit $\xi \to 0$ the foliation we have used becomes not spacelike, but null, which is also apparent from Eq. (6.4). If one wants to use the foliation we have used to calculate and evolve the AHS variables, one simply needs to start somewhere in the interior, where the $\tau = \text{const}$ surfaces are spacelike and evolve from there.

6.2 Gravitational wave-electromagnetic wave collision

We now turn to the gravitational-electromagnetic wave collision solution. Similarly to the previous cases, we define a timelike variable τ as

$$\tau^2 = \frac{\pi}{2\sqrt{2}} - t, \tag{6.14}$$

so that the metric becomes

$$g = e^{-M}\left(-4\tau^2 \mathrm{d}\tau^2 + \mathrm{d}z^2\right) + e^{-U}\left(e^{V}\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right). \tag{6.15}$$

The spatial metric on the surfaces of constant τ reads as

$$q = e^{-M}\mathrm{d}z^2 + e^{-U}\left(e^{V}\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right). \tag{6.16}$$

We work with the triad

$$E_1 = e^{-U}\partial_z, \quad E_2 = e^{-(M+U+V)/2}\partial_x, \quad E_3 = e^{-(M+U-V)/2}\partial_y. \tag{6.17}$$

Then, the connection is given by

$$\Gamma^1 = 0, \quad \Gamma^2 = -\frac{1}{2}e^{(M-U-V)/2}\left(U_z + V_z\right)\mathrm{d}y, \quad \Gamma^3 = \frac{1}{2}e^{(M-U+V)/2}\left(U_z - V_z\right)\mathrm{d}x. \tag{6.18}$$

The extrinsic curvature 1-forms can be computed as

$$K^1 = \frac{M_\tau}{4\tau}\mathrm{d}z, \quad K^2 = e^{(M-U+V)/2}\frac{U_\tau - V_\tau}{4\tau}\mathrm{d}x, \quad K^3 = e^{(M-U-V)/2}\frac{U_\tau + V_\tau}{4\tau}\mathrm{d}y. \tag{6.19}$$

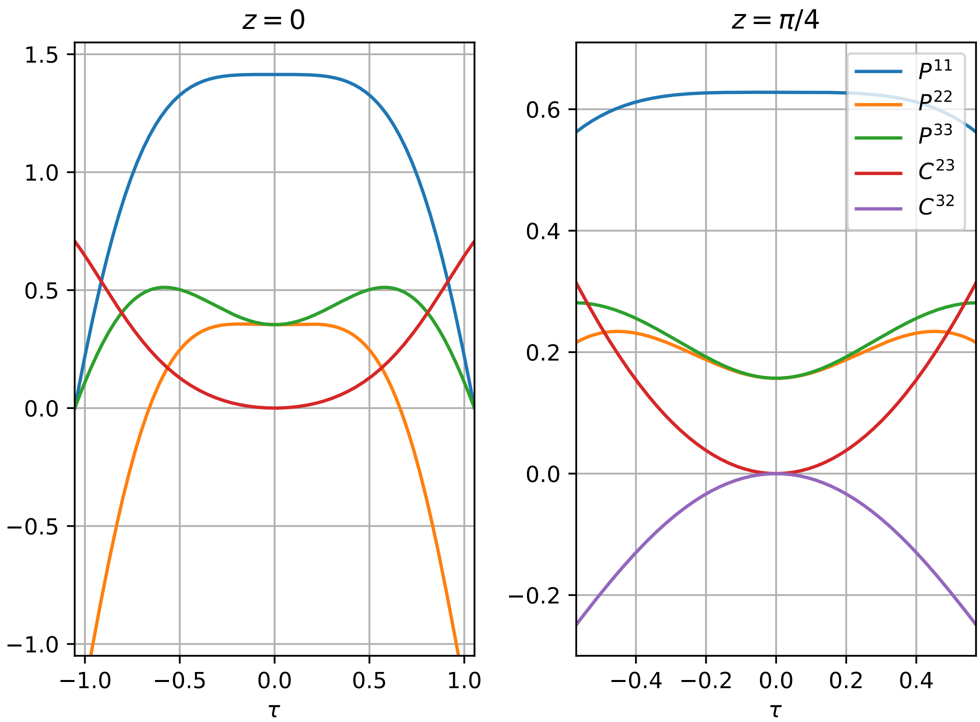

Figure 6.2: Non-vanishing AHS variables for the gravitational-electromagnetic wave collision at two different points of space, $z = 0$ (left) and $z = \pi/4$ (right). Note that C^{23} and C^{32} coincide at $z = 0$. It can be seen that all of the variables are regular at the singularity.

With the triad (6.17), connection (6.18) and extrinsic curvature (6.19), the non-vanishing AHS variables for this case read as

$$P^{11} = -\frac{e^{-U}}{2\tau} U_\tau, \tag{6.20a}$$

$$P^{22} = -\frac{e^{-U}}{4\tau} \left(M_\tau + U_\tau + V_\tau \right), \tag{6.20b}$$

$$P^{33} = -\frac{e^{-U}}{4\tau} \left(M_\tau + U_\tau - V_\tau \right), \tag{6.20c}$$

$$C^{23} = \frac{1}{2} e^{-U} \left(U_z - V_z \right), \tag{6.20d}$$

$$C^{32} = -\frac{1}{2} e^{-U} \left(U_z + V_z \right). \tag{6.20e}$$

Inserting our known solution and inspecting term by term, we see that

$$U_\tau = -\sqrt{2}\tau e^U \left(\sin 2u + \sin 2v\right) \tag{6.21a}$$

$$V_\tau = 2\sqrt{2}\tau \frac{\cos u}{1 + \sin u} \tag{6.21b}$$

$$M_\tau = -\sqrt{2}\tau \left(\tan u + \tan v\right) + V_\tau - \frac{1}{2}U_\tau \tag{6.21c}$$

$$U_z = \frac{e^U}{\sqrt{2}} \left(\sin 2u - \sin 2v\right) \tag{6.21d}$$

$$V_z = -\sqrt{2}\frac{\cos u}{1 + \sin u}, \tag{6.21e}$$

where u and v are understood to be functions of τ and z. We therefore observe that in the interior where $0 < u, v < \pi/2$, all of the variables P^{ij} and C^{ij} remain finite and differentiable, including the singularity $u + v = \pi/2$. Fig. 6.2 shows the AHS variables for the gravitational-electromagnetic wave collision as functions of τ at two sample points.

Chapter 7

CONCLUSION

The well-behavedness of the AHS variables at the singularities of the FLRW spacetimes, Bianchi I and IX models and the Schwarzschild metric had already been demonstrated. Here we have shown this to be the case for two specific colliding wave solutions. Although our results are for these two examples only, this further motivates the expectation that these variables are well behaved at generic spacelike singularities.

The first example we have looked at is the Khan-Penrose metric, which describes the collision of two impulsive gravitational waves. In order to calculate the AHS variables, we have chosen a foliation adapted to the singularity of the metric. Our results show explicitly that the AHS variables are finite at the singularity and can be continued through it without any problem.

The second example we have looked at is an exact solution which describes the collision of a gravitational wave with an electromagnetic wave. Here, too, are the AHS variables regular on the entire spacetime, including the singularity.

We should emphasize that since the singularities of general relativity occur at scales where quantum effects should be dominant and thus the classical theory is not necessarily a good approximation, investigating these singularities classically—as we have done—should be seen as a first step towards a fuller understanding of the problem. In fact, as we mentioned previously, there are a number of results indicating that quantum corrections become meaningful only when curvature or matter density are about a percent of the Planck scale. Therefore, it is possible that the classical results such as the BKL conjecture or the behaviour of the AHS variables at the singularity reliably describe what effectively happens near the singularity.

Of course, a full understanding of the problem requires a consistent theory of quantum gravity. However, semi-classical analyses such as looking at quantum fields on classical backgrounds [Ashtekar et al., 2021] may also shed some light on the issue. This is what is referred to as "level 2" in [Ashtekar et al., 2022] and has been done, e.g., for cosmological models, and extending these semi-classical methods to colliding gravitational waves seems to be a reasonable way to gain further insight into their singularity structures.

Appendix A

NEWMAN-PENROSE FORMALISM

The Newman–Penrose (NP) formalism [Newman and Penrose, 1962] is a special case of the tetrad formalism of general relativity, where the tensors of the theory are expressed in terms of their components with respect to a complete vector basis at each point. In the case of the NP formalism, the vector basis chosen is a complex null tetrad.

A complex null tetrad is a basis of vectors $(l, n, m, \bar{m})$ at each point of spacetime such that l and n are real null vectors, m is a complex null vector and $\bar{m}$ is the complex conjugate of m, i.e.,

$$l \cdot l = n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0. \tag{A.1}$$

In the mostly plus sign convention $(-, +, +, +)$ which we adopt here, the vectors are normalized such that

$$l \cdot n = -1, \quad m \cdot \bar{m} = 1, \tag{A.2}$$

while the two pairs are orthogonal to each other;

$$l \cdot m = l \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0. \tag{A.3}$$

The above equations imply that, in a local chart, the metric components can be written as

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu. \tag{A.4}$$

The main feature of the NP formalism is the introduction of spin coefficients, which are complex linear combinations of the Ricci rotation coefficients associated with the null tetrad. However, we do not use the spin coefficients directly in this thesis, hence do not repeat them here (see, e.g., [Griffiths, 1991]). Instead, we shall be interested only with the Weyl-NP, Ricci-NP and Maxwell-NP scalars, which we introduce below.

The gravitational field is completely described by the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$, which has 20 independent components in four dimensions. These 20 components can be divided into the 10 components of the Weyl tensor $C_{\alpha\beta\mu\nu}$ and the 10 components of the Ricci tensor $R_{\alpha\beta}$. The 10 components of the Weyl tensor can be conveniently expressed as 5 complex scalars, called the Weyl-NP scalars:

$$\Psi_0 = C_{\alpha\beta\mu\nu} l^\alpha m^\beta l^\mu m^\nu, \tag{A.5a}$$

$$\Psi_1 = C_{\alpha\beta\mu\nu} l^\alpha n^\beta l^\mu m^\nu, \tag{A.5b}$$

$$\Psi_2 = C_{\alpha\beta\mu\nu} l^\alpha m^\beta \bar{m}^\mu n^\nu, \tag{A.5c}$$

$$\Psi_3 = C_{\alpha\beta\mu\nu} l^\alpha n^\beta \bar{m}^\mu n^\nu, \tag{A.5d}$$

$$\Psi_4 = C_{\alpha\beta\mu\nu} n^\alpha \bar{m}^\beta n^\mu \bar{m}^\nu. \tag{A.5e}$$

The 10 components of the Ricci tensor, on the other hand, can be further divided into a trace part (scalar curvature R) and the remaining 9 components, which can be expressed as a 3×3 Hermitian matrix $\Phi_{AB} = \overline{\Phi_{BA}}$ with $A, B = 0, 1, 2$:

$$\Phi_{00} = \frac{1}{2} R_{\alpha\beta} l^\alpha l^\beta, \tag{A.6a}$$

$$\Phi_{01} = \frac{1}{2} R_{\alpha\beta} l^\alpha m^\beta, \tag{A.6b}$$

$$\Phi_{02} = \frac{1}{2} R_{\alpha\beta} m^\alpha m^\beta, \tag{A.6c}$$

$$\Phi_{11} = \frac{1}{4} R_{\alpha\beta} \left(l^\alpha n^\beta + m^\alpha \bar{m}^\beta \right), \tag{A.6d}$$

$$\Phi_{12} = \frac{1}{2} R_{\alpha\beta} m^\alpha n^\beta, \tag{A.6e}$$

$$\Phi_{22} = \frac{1}{2} R_{\alpha\beta} n^\alpha n^\beta, \tag{A.6f}$$

$$\Lambda = \frac{R}{24}. \tag{A.6g}$$

Together, these are called the Ricci-NP scalars.

Similarly, if there is an electromagnetic field $F_{\alpha\beta}$ present, one can encode the 6 independent degrees of freedom of the electromagnetic field in 3 complex scalars called the Maxwell-NP scalars:

$$\phi_0 = F_{\alpha\beta} l^\alpha m^\beta, \tag{A.7a}$$

$$\phi_1 = \frac{1}{2} F_{\alpha\beta} \left(l^\alpha n^\beta + \bar{m}^\alpha m^\beta \right), \tag{A.7b}$$

$$\phi_2 = F_{\alpha\beta} \bar{m}^\alpha n^\beta. \tag{A.7c}$$

With these, the Einstein-Maxwell field equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{A.8}$$

where

$$T_{\alpha\beta} = \frac{1}{\mu_0} \left(F_{\alpha\mu} F_\beta{}^\mu - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right), \tag{A.9}$$

can be written simply as

$$\Phi_{AB} = \frac{\kappa}{\mu_0} \phi_A \overline{\phi_B}. \tag{A.10}$$

BIBLIOGRAPHY

[Andersson and Rendall, 2001] Andersson, L. and Rendall, A. D. (2001). Quiescent cosmological singularities. *Communications in Mathematical Physics*, 218(3):479–511.

[Ashtekar et al., 2021] Ashtekar, A., De Lorenzo, T., and Schneider, M. (2021). Probing the big bang with quantum fields. *Advances in Theoretical and Mathematical Physics, 25, 7, 1651-1702 (2021)*, 25(7):1651–1702.

[Ashtekar et al., 2022] Ashtekar, A., del Río, A., and Schneider, M. (2022). Spacelike singularities of general relativity: A phantom menace? *General Relativity and Gravitation*, 54:45.

[Ashtekar et al., 2009] Ashtekar, A., Henderson, A., and Sloan, D. (2009). Hamiltonian general relativity and the Belinskii–Khalatnikov–Lifshitz conjecture. *Classical and Quantum Gravity*, 26(5):052001.

[Ashtekar et al., 2011] Ashtekar, A., Henderson, A., and Sloan, D. (2011). Hamiltonian formulation of the Belinskii-Khalatnikov-Lifshitz conjecture. *Physical Review D*, 83(8):084024.

[Ashtekar and Wilson-Ewing, 2009a] Ashtekar, A. and Wilson-Ewing, E. (2009a). Loop quantum cosmology of Bianchi type I models. *Physical Review D*, 79(8):083535.

[Ashtekar and Wilson-Ewing, 2009b] Ashtekar, A. and Wilson-Ewing, E. (2009b). Loop quantum cosmology of Bianchi type II models. *Physical Review D*, 80(12):123532.

[Belinskii et al., 1970] Belinskii, V. A., Khalatnikov, I. M., and Lifshitz, E. M. (1970). Oscillatory approach to a singular point in the relativistic cosmology. *Advances in Physics*, 19(80):525–573.

[Bell and Szekeres, 1974] Bell, P. and Szekeres, P. (1974). Interacting electromagnetic shock waves in general relativity. *General Relativity and Gravitation*, 5(3):275–286.

[Berger, 2002] Berger, B. K. (2002). Numerical approaches to spacetime singularities. *Living Reviews in Relativity*, 5(1):1.

[Blau, 2023] Blau, M. (2023). Lecture notes on general relativity. Lecture notes.

[Bojowald, 2001] Bojowald, M. (2001). Absence of a singularity in loop quantum cosmology. *Physical Review Letters*, 86(23):5227–5230.

[Chandrasekhar and Xanthopoulos, 1986] Chandrasekhar, S. and Xanthopoulos, B. C. (1986). A new type of singularity created by colliding gravitational waves. *Proceedings of the Royal Society of London A*, 408:175–208.

[Dereli et al., 2024] Dereli, T., Gurtug, O., and Ünlütürk, K. İ. (2024). Crossing the singularity of a gravitational wave collision. *arXiv:2401.11975 [gr-qc]*.

[D'Ambrosio and Rovelli, 2018] D'Ambrosio, F. and Rovelli, C. (2018). How information crosses Schwarzschild's central singularity. *Classical and Quantum Gravity*, 35(21):215010.

[Einstein and Rosen, 1935] Einstein, A. and Rosen, N. (1935). The particle problem in the general theory of relativity. *Physical Review*, 48(1):73–77.

[Ferrari and Ibanez, 1988] Ferrari, V. and Ibanez, J. (1988). Type D solutions describing the collisions of plane fronted gravitational waves. *Proceedings of the Royal Society of London A*, 417:417–431.

[Garfinkle, 2004] Garfinkle, D. (2004). Numerical simulations of generic singuarities. *Physical Review Letters*, 93(16):161101.

[Garfinkle, 2007] Garfinkle, D. (2007). Numerical simulations of general gravitational singularities. *Classical and Quantum Gravity*, 24(12):S295–S306.

[Griffiths, 1991] Griffiths, J. B. (1991). *Colliding Plane Waves in General Relativity*. Oxford University Press.

[Gurtug et al., 2014] Gurtug, O., Halilsoy, M., and Mazharimousavi, S. H. (2014). Quantum probes of timelike naked singularities in the weak field regime of f(r) global monopole spacetime. *Journal of High Energy Physics*, 2014:178.

[Gurtug et al., 2006] Gurtug, O., Halilsoy, M., and Unver, O. (2006). Reflection of electromagnetic waves from mixtures of plane gravitational and scalar waves. *Physical Review D*, 74:044020.

[Helliwell et al., 2003] Helliwell, T. M., Konkowski, D. A., and Arndt, V. (2003). Quantum singularity in quasiregular spacetimes, as indicated by Klein-Gordon, Maxwell and Dirac fields. *General Relativity and Gravitation*, 35(1):79–96.

[Henneaux and Teitelboim, 1992] Henneaux, M. and Teitelboim, C. (1992). *Quantization of Gauge Systems*. Physics. Princeton University Press, Princeton, New Jersey.

[Horowitz and Marolf, 1995] Horowitz, G. T. and Marolf, D. (1995). Quantum probes of spacetime singularities. *Physical Review D*, 52(10):5670–5675.

[Khan and Penrose, 1971] Khan, K. A. and Penrose, R. (1971). Scattering of two impulsive gravitational plane waves. *Nature*, 229(5281):185–186.

[Newman and Penrose, 1962] Newman, E. and Penrose, R. (1962). An approach to gravitational radiation by a method of spin coefficients. *Journal of Mathematical Physics*, 3(3):566–578.

[Penrose and Rindler, 2003] Penrose, R. and Rindler, W. (2003). *Spinors and Space-Time*, volume 2. Cambridge University Press.

[Saotome et al., 2010] Saotome, R., Akhoury, R., and Garfinkle, D. (2010). Examining gravitational collapse with test scalar fields. *Classical and Quantum Gravity*, 27(16):165019.

[Uggla et al., 2003] Uggla, C., van Elst, H., Wainwright, J., and Ellis, G. F. R. (2003). Past attractor in inhomogeneous cosmology. *Physical Review D*, 68(10):103502.

[Valdés-Meller, 2021] Valdés-Meller, N. (2021). Revisiting gravity and crossing the singularity. Master's thesis, Scholars International Program, Perimeter Institute.

[Wald, 2009] Wald, R. M. (2009). *General Relativity*. University of Chicago Press.

[Weaver et al., 1998] Weaver, M., Isenberg, J., and Berger, B. K. (1998). Mixmaster behavior in inhomogeneous cosmological spacetimes. *Physical Review Letters*, 80(14):2984–2987.

[Wilson-Ewing, 2010] Wilson-Ewing, E. (2010). Loop quantum cosmology of Bianchi type IX models. *Physical Review D*, 82(4):043508.

[Yurtsever, 1987] Yurtsever, U. (1987). Instability of Killing-Cauchy horizons in plane-symmetric spacetimes. *Physical Review D*, 36(6):1662–1672.