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CURVATURE PROPERTIES OF METALLIC RIEMANNIAN MANIFOLDS

Ibrahim ISAH

Master's Thesis

DEPARTMENT OF MATHEMATICS

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This thesis, which was prepared according to the thesis writing rules of the Graduate School of Natural and Applied Sciences, Fırat University, was evaluated by the committee members who have signed the following signatures and was unanimously approved after the defense exam made open to the academic audience.

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DECLARATION

I hereby declare that I wrote this Master's Thesis titled "Curvature Properties of Metallic Riemannian Manifolds" in consistent with the thesis writing guide of the Graduate School of Natural and Applied Sciences, Fırat University. I also declare that all information in it is correct, that I acted according to scientific ethics in producing and presenting the findings, cited all the references I used, express all institutions or organizations or persons who supported the thesis financially. I have never used the data and information I provide here in order to get a degree in any way.

2 April 2020

Ibrahim ISAH

PREFACE

The basis for this work initially came from my passion for mathematical geometry. While the world moves forward, geometric studies with dated geometric methods are proving useful for studying new structutres. It is my passion both to discover out, and to create tools to break down gaps in geometry for future use.

First of all, I will like to thank Allah Almighty for giving me abundant of health to achieve this in his kind mercy. It gives me great pleasure in acknowledging the support and help of my able supervisor in person of Prof. Dr. Alper Osman ÖĞRENMİŞ who has the attitude and the substance of a genius in his excitement in teaching and guiding. Without his guidance and persistence help this thesis wouldn't have happened.

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At the same time, nobody was more important to me than my family members in the creation of this project. I would like to thank my parents; their support and encouragement in whatever I seek is with me. They are the true role models and their timely assistance, without which this study wouldn't have been successfully completed. Many thanks goes to my friends and all well-wishers who made a direct or indirect contributions to the success of this work.

However, I still thank Almighty Allah for which without his blessings this thesis wouldn't have been possible.

Ibrahim ISAH

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In this thesis, we study the basics of metallic structure, which is polynomial structure with structure polynomial on manifolds using metallic ratio, which is the generalization of Golden proportion or golden ratio. We study some mathematical terminologies, study some theorems and proof where possible of the metallic structures. We also study some examples of metallic structure. Also, we study some properties of the metallic Riemannian metric. Hence, study the J-sectional and J- Bisectional curvature of metallic Riemannian manifold.

Keywords: Metallic Structure, Golden Ratio, Curvature.

ÖZET

Metalik Riemann Manifoldların Eğrilik Özellikleri

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Bu tez çalışmasında, metalik oram kavramını kullanarak manifoldlar üzerinde polinomial metalik yapıların temel özelliklerini inceleyeceğiz. Bu metalik yapılar altın oran veya altın oranın genel hali olarak ele alınacaktır. Ayrıca metalik yapıların bazı teorem ve ispatlarını içeren matematiksel ifadelerden de bahsedeceğiz. Ardından metalik yapıların bazı özelliklerini inceleyeceğiz. Daha sonra, metalik riemann metriğinin bazı özelliklerini de vereceğiz. Son olarak, metalik riemann manifoldların bazı eğriliklerini inceleyeceğiz.

Anahtar Kelimeler: Metalik Yapi, Altin Oran, Riemann Manifold.

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Symbols and Abbreviations

Symbols

$d(,)$	: Euclidean distance
d & l	: Projectors
$\boldsymbol{g}$	: Riemannian Metric
J	: Metallic Structure
$\mathcal{L}(X, \mathbb{F})$	: Linear Functional
M	: Topological Manifold
$\mathbf{N_F}$	: Nijenhuis Tensor
R^3	: Euclidean 3-space
$\boldsymbol{T_x(R^3)}$	: Tangent space
$\boldsymbol{T_l^k M}$	: Tensors
U	: Open Set
$\boldsymbol{V}$	: Vectors
$\boldsymbol{V_p}$	: Tangent vector
$\boldsymbol{X}$ & $\boldsymbol{Y}$	: Vector Fields
$\mathbf{X}^*$	: Dual Space
$\|\ \ \|$	: Norm
∅	: 1-Form
π	: Projection Map
∐	: Disjoint Union of Set
λ	: Eigenvalue
∇	: Connection
ƛ$_{a,b}$	: Metallic Means Family

1. INTRODUCTION

Everyone knows the Golden Ration and believe it has a rich application area illustrated in architecture, painting, music, Optimization, design, Nature, and other continuous studies. In truth, though, it is probably fair to say however, that the golden ratio has inspired mathematicians from all fields like none else.

However, Crasmareanu and Hretcanu [1] researched the geometry of the golden structure on a manifold using a corresponding almost product structure, that is a polynomial structure with structure polynomial of the form $Q(J) = J^2 - aJ - bI = 0$ on manifolds using metallic ratio, which is the generalization of Golden proportion or in other word golden ratio, the golden proportion played a significant role. Özkan in [2], the complete and horizontal lift of a golden structure was studied, and the geometry of a golden structure on a tangent bundle was also investigated with regards to the almost product structure. Özkan, çıtlak, Taylan and Fatma again [3] and [4] studied the prolongations of golden structure on a tangent bundle of order 2, and order r respectively. Şahin and Akyol in [5] introduced the golden maps between the golden Riemannian manifolds, and showed such maps were harmonic, they also investigated the constancy of some certain maps from golden Riemannian manifolds to various manifold by adopting the holomorphic-like map condition. Özkan and Fatma [6], investigated the integrability and parallelism of metallic structures on differentiable manifolds and also provide some Riemannian metric properties.

Again, Crasmareanu and Hretcanu in [7], they examined the polynomial structure and defined it on Riemannian manifold, they checked for properties of the induced structure on submanifolds by metallic Riemannian structure, research for the necessary and sufficient condition for a submanifold to be a metallic Riemannian manifold in terms of invariance as well. Hretcanu and Adara [8] studied submanifolds in metallic Riemannian manifolds in which they focused on the structure induced on submanifolds named it $\sum-$ metallic Riemannian structures. They further extend to [9] focused on slant and semi-slant submanifolds, and gave some characterization for which submanifolds to be a slant or semi-slant submanifolds in metallic or golden Riemannian manifolds. Also, in [10] they studied the properties of invariant, anti-invariant and slant isometrically immersed submanifolds of metallic Riemannian manifolds which were given with a special view towards the induced Σ-structure. Gezer and Karaman in [11, 12], on metallic Riemannian structures where the integrability condition, curvature properties of the structure, and the golden hessian structure were studied respectively. In [13], Adara and Nannicini. A studied some properties of curvature tensors on Norden and metallic pseudo – Riemannian manifolds in which they introduce the notion of J- Sectional and J-Bisectional Curvature of metallic pseudo-Riemannian manifolds $(\bar{M}_n, g, J)$.

However, the metallic ratio, which is a generalization of the golden proportion. This generalization was implemented in 1999 by De spinadel [14] and called it as the metallic structures by [1,6,7]. As such, in our own case in this thesis we want to study the metallic structures and their curvature properties extensively.

2. BASIC PROPERTIES AND DEFINITIONS

We provide here some basic definitions and properties within the scope of this research work.

Definition 2.1

Euclidean space R^3 is the set of all reserved triples of real numbers, so that the triples $q = (q_1, q_2, q_3)$ is termed as a point of R^3 [15].

Definition 2.2

In consideration given to two points $q = (q_1, q_2, q_3)$ and $k = (k_1, k_2, k_3)$ in R^3 their dot product is given by [15].

$$q.k = q_1k_1 + q_2k_2 + q_3k_3 \tag{2.1}$$

[15].

The norm of a point $q = (q_1, q_2, q_3)$is the number

$$\|q\| = \sqrt{q.q} = \sqrt{q_1^2 + q_2^2 + q_3^2} \tag{2.2}$$

Therefore, the norm is real-valued function on R^3 [15].

Definition 2.3

If $q = (q_1, q_2, q_3)$ and $k = (k_1, k_2, k_3)$ are points in R^3, the Euclidean distance from q to k is the number

$$d(q,k) = \|q - k\| \tag{2.3}$$

$$q - k = (q_1 - k_1, q_2 - k_2, q_3 - k_3) \tag{2.4}$$

$$d(q,k) = \sqrt{(q_1 - k_1)^2 + (q_2 - k_2)^2 + (q_3 - k_3)^2} \tag{2.5}$$

[15]

Definition 2.4

A vector at a point $q \in R^{n+1}$ is a pair $V = (q, l)$ where $l \in R^{n+1}$. The vectors at a point $q \in R^{n+1}$ form what is called a vector space R_q^{n+1} of dimension $n + 1$. Given two vectors l and w at q, their dot and cross product are defined same using the standard Dot and Cross product on R^{n+1} respectively.

The length of v using dot product,

$$\|l\| = (l.l)^{\frac{1}{2}} \tag{2.6}$$

The angle between two vectors v and w is defined by

$$\cos\theta = \frac{l.w}{\|l\|\|w\|} \tag{2.7}$$

[16].

Definition 2.5

A real vector space is a non-empty set X of items we define as vectors and the following two functions:

i. Vector Addition: $X \times X \to X: (x, y) \to (x + y)$
ii. Scalar Multiplication: $X \times R \to X: (x, a) \to xa$

Where it is commutative, associative, has an inverse and has 0 as identity element under addition and, it is distributive, and has an identity element 1 under Multiplication. [16].

Definition 2.6

A vector field X on $U \subset R^{n+1}$ is a function which allocates to each point of U a vector at that point. Thus, for some function $X: U \to R^{n+1}$.

$$X(q) = (q, X(q)) \tag{2.8}$$

[17].

Definition 2.7

A tangent vector V_q to R^3 consist of two points of R^3, its vector part V and its point of application q [15].

Definition 2.8

Let q be a point of R^3. The set $T_q(R^3)$ consisting of all tangent vectors that have a point q as point of application is called the tangent space of R^3 at q [15].

Definition 2.9

A coordinate patch $X: U \to R^3$ is a one-to-one regular mapping of an open set U of R^2 into R^3 [15].

Definition 2.10

A surface in R^3 is a subset D of R^3 such that for each point q of D there exist a proper patch in D whose image contains a neighborhood of q in D [15].

Remark 2.11

A 1-surface in R^2 is called a plane curve. A 2-surface in R^3 is called a Surface.
An n-surface in R^{n+1} is called a hypersurface [16].

Definition 2.12

A 1-form ρ on R^3 is a real-valued function on the set of all tangent vectors of R^3 such that ρ is linear at each point, that is

$$\rho(lv + kw) = l\rho(v) + k\rho(w) \tag{2.9}$$

For any $l, k \in R$ and tangent vectors v, w at the same point of R^3.

Remark 2.13

We emphasize that for every tangent vector v, a 1-form ρdefines a real number $\rho(v)$ for each point $l \in R^3$,the resulting function $\rho_l: T_l(R^3) \to R$ is linear. Thus, at each point l, ρ_l is an element of the dual space of $T_l(R^3)$. In this notion 1-form is dual to that of vector fields [15].

Definition 2.14

Let X be a vector space over a field $\mathbb{F}$, a linear functionals is the linear map $X \to \mathbb{F}$. In other words, a linear functional is $\mathcal{L}(X, \mathbb{F})$ [17].

Definition 2.15

Let X be a vector space over a field $\mathbb{F}$, then the dual space of X, denoted by X^* is the space of all linear functional of X. That is $X^* := \mathcal{L}(X, \mathbb{F})$ [17].

Definition 2.16

A C^k manifold M modelled on Euclidean space is a hausdorff topological space along with a collection of open sets $\{U_a | a \in A\}$ in M and the corresponding map $\phi_a: U_a \to X$, such that

i- $\bigcup_{a \in A} U_a = M$

ii- Each ϕ_a defines a homeomorphism $U_a \to \phi_a(U_a)$.

iii- If for any a, b on $U_a \cap U_b \neq \emptyset$, then the composite maps $\phi_a \circ \phi_b^{-1}$, $\phi_b \circ \phi_a^{-1}$, on the set $\phi_b(U_a)$, $\phi_a(U_b)$ on which they are define are C^k. In which from (ii) are homeomorphisms and we want them to be differentiable k times.

The pairs (U_a, ϕ_a) are called charts on M, and the set $\{(U_a, \phi_a)\}$ of all of them is an Atlas [17].

Definition 2.17

A map $f: M \to N$ between smooth manifolds is differentiable, (or C^k) at a point $q \in M$ if for some charts (U, ϕ) on M, (V, ψ) on N with $q \in U$ and $f(q) \in V$, the map $\psi \circ f \circ \phi^{-1}$ is differentiable at $\phi(q)$ [17].

Definition 2.18

A (*smooth*) k-dimenssional vector bundle is a pair of smooth manifolds E (the *total space*) and M (the *base*), together with a surjective map $\pi: E \to M$ (the *projection*), satisfying the following conditions:

i- Each set $Eq: = \pi^{-1}(q)$ (termed as the *fiber* of E over q) is furnished with the structure of a vector space.

ii- For each $q \in M$, there exists a neighborhood U of q and a diffeomorphism $\phi: \pi^{-1}(U) \to U \times R^k$.

iii- The restriction of ϕ to each fiber, $\phi: Eq \to (q) \times R^k$, is a linear isomorphism [18].

Definition 2.19

Let D be a finite-dimensional real vector space. As usual, D^* denotes the dual space of D (the space of convectors), or real-valued linear functionals on D, and we denote the natural pairing $D^* \times D \to$ R by either of the notations;

$$(w, y) \to \langle w, y \rangle \; or \; (w, y) \to w(y) \tag{2.9}$$

for $w \in D^*, y \in D$.

A covariant k-tensor on V is a multilinear map

$$F: \underbrace{D \times D \times \cdots \times D}_{k-copies} \to R \tag{2.11}$$

closely, a contravariant l-tensor on D^* is a multilinear map

$$F: \underbrace{D^* \times D^* \times \cdots \times D^*}_{l-copies} \to R \tag{2.12}$$

A tensor of type $\binom{t}{l}$, also t-covariant and l-contravariant tensor is a multilinear map

$$F: \underbrace{D^* \times D^* \times \cdots \times D^*}_{l-copies} \times \underbrace{D \times D \times \cdots \times D}_{t-copies} \to R \tag{2.13}$$

[18].

Definition 2.20

A $\binom{k}{l}$- tensor bundle at a point $q$on M is define as

$$T_l^k M = \coprod_{q\in M} T_l^k(T_q M) \tag{2.14}$$

Where $\coprod$ denotes the disjoint union $\binom{k}{l}$- tensor on the tangent space $T_q M$. Similarly, we can define the k-forms bundle as

$$\wedge^k M = \coprod_{q\in M} \wedge^k (T_q M) \tag{2.15}$$

[18].

Definition 2.21

A tensor field F at a point $q$on M is a smooth section of some tensor bundle $F_q \in T_l^k(T_qM)$ [18].

Definition 2.22

A Riemannian metric on a smooth manifold M is a 2-tensor field $g \in \mathcal{T}^2(M)$ that is symmetric $g(X,Y) = g(Y,X)$ and positive definite $g(X,X) > o$ if $X \neq 0$. A Riemannian metric thus established an inner product on each tangent space T_qM, which is typically written as

$$\langle X, Y\rangle = g(X,Y) \tag{2.16}$$

for $X, Y \in T_qM$ [18].

Definition 2.22

A manifold M equipped with a Riemannian metric g then, the manifold (M, g) is called a Riemannian Manifold [18].

Definition 2.23

A pseudo Riemannian metric often also called a semi- Riemannian metric on a smooth manifold M is a symmetric 2-tensor field g that is nondegenerate at each point $q \in M$. This imply that the only vector orthogonal to everything is the zero vector. More formally, $g(X,Y) = 0$, for all $Y \in T_qM$ if and only if $X = 0$ [18].

Definition 2.24

A vector $x \neq 0$ is an eigenvector of $A: X \to X$ if $Ax = x\lambda$ for some scalars λ. Then λ is an eigenvalue of A, and x is an eigenvector belonging to λ. Eigenvectors are sometimes called characteristics vectors, and correspondingly sometimes eigenvalues are called characteristics roots [18].

Definition 2.25

A connection on a manifold M is a function ∇ which assigns to every tangent vector t and s a vector field w on M. A vector $\nabla_t w$ in T_qM where t is at point q, such that,

i- $\nabla_{s+t} w = \nabla_s w + \nabla_t w$ for any s, t in the tangent space and vector field w.

ii- $\nabla_t(u + w) = \nabla_t u + \nabla_t w$ for any t $\in TM$ and u, w in the vector field on M.

iii-$\nabla_{\lambda t} w = \lambda\nabla_t w$ for any t $\in TM$ and w in the vector field on M.

iv-$\nabla_t(fw) = (\mathrm{t(f)w(q)} + \mathrm{f(q)}\nabla_t w$ for any t $\in TM$ and w in the vector field on M, and C^∞ $f: M \to$ R.

v- If t, w are C^∞ vector fields, so is $\nabla_t w: q \to \nabla_{t(q)} w$ [17].

Definition 2.26

A lie group G is a manifold modelled on a finite dimensional space on which there is a group structure such that

$$(g, h) \in G \times G \to gh^{-1} \in G \tag{2.17}$$

is differentiable. The neutral element of G will be denoted by e. Also, for every element of G we define the left and right translation, L_g , R_g respectively as

$L_g: G \to G; h \to gh.$

$R_g: G \to G; h \to hg$ [19].

Definition 2.27

A vector space a (over a field $\mathbb{F}$) is called a Lie algebra if it is equipped with a bilinear map $a \times a \to a$ (A multiplication) denoted $(c, w) \to [c, w]$ such that

$$[c, w] = -[c, w] \tag{2.18}$$

and such that we have the Jacobi identity

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0. \tag{2.19}$$

[20].

Definition 2.28

Let F, M, and E be a smooth manifolds and let $\pi: E \to M$ be a smooth map. The quadruple (F, π, M, E) is called a (locally trivial) fiber bundle, if for each point $q \in M$ the open set Z containing q and a C^r diffeomorphism $\varphi: \pi^{-1}(Z) \to Z \times F$ in Figure 2.1 commutes.

$$\begin{tikzcd}
\pi^{-1}(Z) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Z \times F \arrow[dl, "q\pi_1"] \\
& Z &
\end{tikzcd}$$

Figure 2.1. Fiber Bundle [20]

If (F, π, M, E) is a smooth fiber bundle like in the Figure 2.1 above, then E is termed the total space, π is termed the bundle projection, M is termed the base space and F is termed the typical fiber. For each $q \in M$, the set $E_q = \pi^{-1}(q)$ s termed the fiber over q [20].

Definition 2.29

A (global) smooth section of a fiber bundle $\xi = (F, \pi, M, E)$ is a smooth map $\sigma: M \to E$ such that $\pi \circ \sigma = id_M$ (i.e. $\sigma(q) \in E_q$). A local smooth section over an open set U is a smooth map $\sigma: M \to E$ such that $\pi \circ \sigma = id_U$. The set of smooth sections of ξ is named by $\Gamma(\xi)$ or sometimes by $\Gamma(E)$ or $\Gamma(\pi)$ [20].

Definition 2.30

Let $\pi: P \to M$ be a smooth fiber bundle with typical fiber a Lie group G. The bundle (P, π, M, G) is called a principal G-bundle if the right action of G is smooth and free on P such that;

i- The action keeps fibers; $\pi(ug) = \pi(u)$ for all $u \in P$ and $g \in G$.

ii- For each $q \in M$, there is a bundle chart (U, ϕ) with $q \in U$ and such that if $\phi = (\pi, \Phi)$, then $\Phi(\mathrm{ug}) = \Phi(\mathrm{u})\mathrm{g}$ for all $u \in \pi^{-1}(U)$ and $g \in G$.

If the group G is understood, then we may refer to (P, π, M, G) simply as a principal bundle.

Remark 2.31

Define a right action on $U \times G$ by

$$(q, g_1)g = (q, g_1 g) \tag{2.20}$$

then $U \times G \rightarrow U$ is a trivial principal bundle [20].

Definition 2.32

Clifford Algebra CL_2 is a $4-$dimensional real linear space with basis elements $1, e_1, e_2, e_{12}$ which have the following multiplication in Table 2.1 below.

Table 2.1. Clifford Algebra [21]

1	e_1	e_2	e_{12}
e_1	1	e_{12}	e_2
e_2	$-e_{21}$	1	$-e_1$
e_{12}	$-e_2$	e_1	-1

We define the Clifford product of two vectors $A = a_1 e_1 + a_2 e_2$ and $B = b_1 e_1 + b_2 e_2$ to be $AB = a_1 b_1 + a_2 b_2 + (a_1 b_2 - a_2 b_1) e_{12}$ [21].

Definition 2.33

A quaternion algebra H is an associative, non-commutative division ring with four basic elements $\{1, e_1, e_2, e_3\}$ satisfying the equalities

$$e_1^2 = e_2^2 = e_3^2 = -1$$
$$e_1 e_2 = -e_2 e_1 = e_3$$
$$e_2 e_3 = -e_3 e_2 = e_1$$
$$e_3 e_1 = -e_1 e_3 = e_2$$

we write any sort of quaternion as

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \tag{2.21}$$

or some time $q = S_q + \overrightarrow{V_q}$ where the symbol $S_q = q_0$ and $\overrightarrow{V_q} = q_1 e_1 + q_2 e_2 + q_3 e_3$ represent the scalar and vector parts of the quaternion q [22].

Remark 2.34

If $S_q = 0$, q is named a pure quaternion. The conjugate of the quaternion is termed by $\bar{q}$ and fixed by $\bar{q} = S_q - \overrightarrow{V_q}$. The norm of quaternion is established by;

$\sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ and denoted by N_q. The unit quaternion fixed by $q_0 = \frac{q}{N_q}$ where $q \neq 0$. As such every unit quaternion could be in the form $q_0 = \cos\theta + \overrightarrow{s_0}\sin\theta$ where $\overrightarrow{s_0}$ is a unit vector satisfying $\overrightarrow{s_0}^2 = \overrightarrow{s_0}\overrightarrow{s_0} = -1$ and is named as the axis of the quaternion.

The quaternion product $p = p_0 + p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is defined as

$$pq = S_pS_q - \langle\overrightarrow{V_p}, \overrightarrow{V_q}\rangle + S_p\overrightarrow{V_q} + S_q\overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q} \tag{2.22}$$

where $\langle , \rangle$ and $\wedge$ are Euclidean inner product and vector product, respectively [22].

Definition 2.35

Let $1, e_1, e_2, e_3$ be the four basis quaternion elements and $z_0, z_1, z_2, z_3 \in \mathbb{C}$ then

$$q = z_0 + z_1e_1 + z_2e_2 + z_3e_3 \tag{2.23}$$

is said to be a bi-quaternion (complexified quaternion). As a consequence of this definition, a bi-quaternion q can also be given as

$$q = q' + iq''$$

where q' is scalar and iq'' is the imaginary part of q [22].

Definition 2.36

A split quaternion algebra is an associative, non-commutative non-division ring with four basic elements {1, e1, e2, e3} satisfying the equalities

$$e_1^2 = -1, e_2^2 = e_3^2 = 1$$
$$e_1e_2 = -e_2e_1 = e_3$$
$$e_2e_3 = -e_3e_2 = -e_1$$
$$e_3e_1 = -e_1e_3 = e_2$$

we write any sort of quaternion as

$$q = q_0 + q_1e_1 + q_2e_2 + q_3e_3 \tag{2.24}$$

or some time $q = S_q + \overrightarrow{V_q}$ where the symbol $S_q = q_0$ and $\overrightarrow{V_q} = q_1e_1 + q_2e_2 + q_3e_3$ represent the scalar and vector parts of the quaternion q [22].

Remark 2.37

The split quaternion product $p = p_0 + p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is defined as

$$pq = S_pS_q + g(\overrightarrow{V_p}, \overrightarrow{V_q}) + S_p\overrightarrow{V_q} + S_q\overrightarrow{V_p} + \overrightarrow{V_p} \wedge_L \overrightarrow{V_q} \tag{2.25}$$

where g is an indefinite metric [22].

Definition 2.38

A displaced map f from a space X to a space Y and another map g from a space Z to a space Y, a lift is a map h from X to Z such that $gh = f$. In other words, a lift of f is a map h, so that the diagram commutes like in Figure 2.2 shown below.

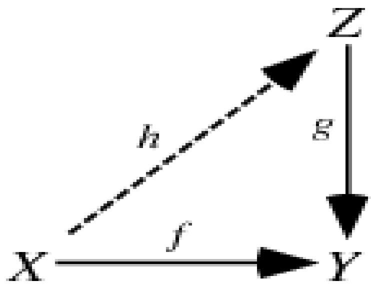

Figure 2.2. Lift [23]

Remark 2.39

If f is the identity from Y to Y as a base manifold as shown in Figure 2.2, and if g is the bundle projection from the tangent bundle to Y, the lifts are quite vector fields. If g is a bundle projection from any fiber bundle to Y, then lifts are quite sections [23].

Definition 2.40

Let N be a $n \times n$ matrix, then N is an orthogonal matrix if

$$NN^T = I \tag{2.26}$$

where N^T is the transpose of A and I the identity matrix. Specifically, an orthogonal matrix has an inverse always.

$$N^{-1} = N^T \tag{2.27}$$

for every dimension $n > 0$, the orthogonal group $O(n)$ is the group of $n \times n$ orthogonal matrices. These matrices form a group because they are close under multiplication and taking inverses [23].

Definition 2.41

Let X, Y and Z be any vector field, and $\langle , \rangle$ denote the metric. The notation $[X, Y]$ is the commutator of vector fields, $[XY - YX]$

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{2.28}$$

and is compatible with the metric

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{2.29}$$

This is a canonical relation called the Levi-Civita, also known sometimes as the Riemannian connection or covariant derivative. It is a connection on the tangent bundle, which is the solitary torsion -free connection ∇ on the tangent bundle compatible with the metric [23].

Definition 2.42

Let V be any vector space then the linear map $f: V \to V$ is named an endomorphism. In algebra, an endomorphism of a group, module, ring or vector space, etc. is a homeomorphism from one object to itself [23].

Definition 2.43

A hyper complex structure on a smooth manifold M is a triple of integrable almost complex structures I, J, K convincing (2.30)

$$IJ = -JI = K \tag{2.30}$$

Let $(M; I, J, K)$be a hyper-complex manifold. Then M agrees a unique torsion-free connection ∇ that maintain the hyper-complex structure by

$$\nabla \mathrm{I} = \nabla \mathrm{J} = \nabla \mathrm{k} = 0 \tag{2.31}$$

this relation is termed the Obata connection [24].

3. METALLIC STRUCTURES ON DIFFERENTIABLE MANIFOLD

In this section, we study metallic structure on a differentiable manifold, at end we also provide some study examples.

3.1. Fibonacci Sequence

In [14], the Fibonacci sequence is a list beginning with numbers 1 or 0, going along by a 1 and continues according to the number (called the Fibonacci number) analogous to the sum of the two prior numbers, the following relations were then formed:

$$F(m+1) = F(m) + F(m-1).$$

the above relation can be given generally using:

$$G(m+1) = aG(m) + bG(m-1), \qquad m \geq 1$$

where $a, b, G(0) = c$ and $G(1) = d$ are real numbers.

If m=1 $\quad G(2) = aG(1) + bG(0) = ad + bc$

If m=2

$$G(3) = a\ G(2) + bG(1) = a(ad+bc) + bd$$

$$c, d, ad+bc, a(ad+bc)+bd, \cdots$$

are the numbers of the generalized Fibonacci Sequence. Letting the numbers a and b to be positive integers. The favorable root of the equation below;

$$x^2 - ax - b = 0$$

is called a family of metallic means because there is no physical importance for the negative root [14]. This root is indicated by

$$\lambda_{a,b} = \frac{a + \sqrt{a^2 + 4b}}{2} \tag{3.1}$$

and called it metallic ratio.

In Equation 3.1, the following ratios are obtained for different values of a and b:

- ✓ If $a = b = 1$ $\lambda_{1,1} = \frac{1+\sqrt{1^2+4\times 1}}{2} = \frac{1+\sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}$, we obtained the Golden mean, which is the proportion of two successive classical Fibonacci numbers [7].
- ✓ If $a = 2$ and $b = 1$, $\lambda_{2,1} = \frac{2+\sqrt{2^2+4\times 1}}{2} = \frac{2+\sqrt{4+4}}{2} = \frac{2+\sqrt{8}}{2} = \frac{2+\sqrt{4\times 2}}{2} = 1 + \sqrt{2}$, we obtained the Silver mean, which is the proportion of two successive Pell numbers [7, 25, 26, 27].
- ✓ If $a = 3$ and $b = 1$, $\lambda_{3,1} = \frac{3+\sqrt{3^2+4\times 1}}{2} = \frac{3+\sqrt{9+4\times 1}}{2} = \frac{3+\sqrt{13}}{2}$, we obtained the Bronze Mean [6, 12].

- ✓ If $a = 4$ and $b = 1$, $\lambda_{4,1} = \frac{4+\sqrt{4^2+4\times 1}}{2} = \frac{4+\sqrt{16+4\times 1}}{2} = \frac{4+\sqrt{20}}{2} = \frac{4+\sqrt{4\times 5}}{2} = 2 + \sqrt{5}$, we obtained the Subtle mean [7, 28].
- ✓ If $a = 1$ and $b = 2$, $\lambda_{1,2} = \frac{1+\sqrt{1^2+4\times 2}}{2} = \frac{1+\sqrt{1+8}}{2} = \frac{1+\sqrt{9}}{2}$ =2, we obtained the Copper mean [7]
- ✓ If $a = 1$ and $b = 3$, $\lambda_{1,2} = \frac{1+\sqrt{1^2+4\times 3}}{2} = \frac{1+\sqrt{1+12}}{2} = \frac{1+\sqrt{13}}{2}$,we obtained the Nickel mean [7].

3.2. Polynomial Structure on $\bar{M}$

Let $\bar{M}$ be defined as differentiable n-dimensional manifold. If F satisfies the algebraic equation where F is a C^{∞}- tensor field of type (1,1) on a manifold $\bar{M}$, F is said to describe a polynomial framework.

$$Q(x) = x^n + a_n x^{n-1} + \cdots + a_2 x + a_1 I = 0$$

where I is the identity map and $F^{n-1}(p), F^{n-2}(p), \cdots, F^2(p), F(p), I$ are linearly independent for each $p \in \bar{M}$. Obviously, $a_1 \neq 0$ if and only if F is non-singular. As such $Q(x)$ is said to be polynomial structure [29].

Theorem 3.1

A polynomial structure on a differentiate manifold $\bar{M}$ identified by a C^{∞} tensor field F of type (1,1), produces an almost product structure on $\bar{M}$. The number of structural distributions equal to the number of distinct irreducible factors over *R* of the structure polynomial, and the projectors are demonstrated as polynomials in F [29].

Proof: Let $E(x)$ be the structure polynomial and $E(x) = E_1(x).E_2(x).E_3(x) \cdots E_k(x)$ be the distinct monic irreducible polynomials over *R*. Since $E_1(x).E_2(x).E_3(x) \cdots E_k(x)$ are relatively prime in the ring *F[x]* of polynomials over *R*, applying the Euclidean algorithm, we obtain polynomials $k_1(x), k_2(x), k_3(x), \cdots, k_k(x)$ such that

$$E_1(x).k_1(x) + E_2(x).k_2(x) + E_3(x).k_3(x) + \cdots + E_k(x)k_k(x) = 1$$

If we define $h_i = E_i(F).k_i(F)$ for $i = 1,2,3,...$ k,

$$h_1 + h_2 + h_3 + \cdots h_k = I$$

The $h_i, i = 1,2,3,...$ k are thus C^{∞} projectors defining an Almost product structure with distributions $T_{i,} i = 1,2,3,...$ k.

Remark 3.2 For X a (1,1) tensor field and I identity map we have:

For $Q(X) = X^2 - I$ we get an Almost product structure.

For $Q(X) = X^2 + I$ we get an Almost complex structure.

For $Q(X) = X^2$ we get an Almost tangent structure [1].

3.3. Metallic Structure on Manifold

Let $\bar{M}$ described as n-dimensional manifold. A polynomial structure on a manifold $\bar{M}$ is said to be a metallic structure if so defined with a (1,1)- tensor field J that meets the equation,

$$J^2 = aJ + bI \tag{3.2}$$

For a, b are positive integers, I the identity operator on the lie algebra $X(\bar{M})$ of vector fields on $\bar{M}$ [6].

The ensuing proposition provide us with the primary characteristics of metallic structure

Proposition 3.3

i. The eigenvalues of J are the metallic ratios $\lambda_{a,b}$ and $a - \lambda_{a,b}$.
ii. A metallic structure J is an isomorphism on the tangent space of the manifold $T_p(\bar{M})$ for every point $p \in \bar{M}_n$.
iii. J is invertible and its inverse $J^{-1} = \hat{J}$ yields

$$b\hat{J}^2 + a\hat{J} = I = 0 \tag{3.3}$$

Proof:

Let $J^2 = aJ + bI$ be a polynomial determined by an (1,1)- tensor field J with identity operator I that gives the endomorphism. $J: T\bar{M} \to T\bar{M}$

$$J^2 - aJ - bI = 0$$

applying the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.4}$$

where $a = 1, b = -a, c = -b$

we obtained

$$J = \frac{-(-a) \pm \sqrt{a^2 - 4(1)(-b)}}{2(1)}$$

$$J_1 = \frac{a+\sqrt{a^2+4b}}{2} \text{ and } J_2 = \frac{a-\sqrt{a^2+4b}}{2}$$

From Equation 3.1, $\lambda_{a,b} = \frac{a+\sqrt{a^2+4b}}{2}$

implies

$$\lambda_{a,b} = \frac{a + \sqrt{a^2 + 4b}}{2} = J_1$$

if J_1 is a metallic structure then $a - J_1$

$$a - J_1 = a - \left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)$$

$$= \frac{2a - a + \sqrt{a^2 + 4b}}{2}$$

$$= \frac{a + \sqrt{a^2 + 4b}}{2}$$

$$J_1 = a - J_1 = a - ƛ_{a,b}$$

which are the eigenvalues and hence the metallic ratios.

From Theorem 3.1, in [1, 29] A polynomial structure on a differentiable manifold $\bar{M}$ described by a C^∞ tensor field F of type (1,1), produces an almost product structure on $\bar{M}$.

Let $F_p = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ be a (1,1) tensor field on $\bar{M}$ with I as an identity operator then,

$$F_p^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

imply F is an almost product structure on $\bar{M}$. Let $F = J$ then, J is invertible if

$$JJ^{-1} = J\hat{J} = I \tag{3.5}$$

$$J^{-1} = \frac{1}{\det(J)}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= -1\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

from Equation 3.5

$$JJ^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

An almost product structure F, an almost tangent structure T and an almost complex structure C are shown in pairs. That is, $-F, -T\ and - C$ are also an almost product structure, an almost tangent structure and an almost complex structure, respectively [1].

We can proceed from now on to say that metallic structures appeared in pairs:

Proposition 3.4

If J is a metallic structure, then $\tilde{J} = aI - J$ is also a metallic structure.

proof. Let

$$J = ƛ_{a,b} = \frac{aI + \sqrt{a^2 + 4b}}{2}$$

$$aI - J = aI - \left(\frac{aI + \sqrt{a^2 + 4b}}{2}\right)$$

$$= \frac{2aI - aI + \sqrt{a^2 + 4b}}{2}$$

$$= \frac{aI + \sqrt{a^2 + 4b}}{2}$$

$$aI - J = aI - ƛ_{a,b} = \tilde{J}$$

A polynomial structure on a manifold $\bar{M}$ is known to induce a generalized almost product structure F. We connect the metallic structure with the almost product structure on the manifold M [1] .

Proposition 3.5

Every almost product structure F induces two metallic structures on a manifold $\bar{M}$ given as follows:

$J_1 = \frac{aI}{2} + \left(\frac{2\lambda_{a,b}-a}{2}\right)F$ and $J_2 = \frac{aI}{2} - \left(\frac{2\lambda_{a,b}-a}{2}\right)F$

The other way around, every metallic structure J on $\bar{M}$ produces two almost product structures on the manifold $\bar{M}$ as:

$$F = \pm\left(\frac{2}{2\lambda_{a,b}-a}J - \frac{a}{2\lambda_{a,b}-a}I\right) \tag{3.6}$$

[7].

Proof:

Let $J_1 = \frac{aI}{2} + \left(\frac{2\lambda_{a,b}-a}{2}\right)F$ and $J_2 = \frac{aI}{2} - \left(\frac{2\lambda_{a,b}-a}{2}\right)F$ be the metallic structures induced by the almost product structure F then,

$$J_1 = \frac{aI}{2} + \left(\frac{2\lambda_{a,b}-a}{2}\right)F$$

$$J_1 = \frac{aI + \left(2\lambda_{a,b}-a\right)F}{2}$$

$$2J_1 = aI + \left(2\lambda_{a,b}-a\right)F$$

$$2J_1 - aI = \left(2\lambda_{a,b}-a\right)F$$

$$\left(\frac{2}{2\lambda_{a,b}-a}J_1 - \frac{a}{2\lambda_{a,b}-a}I\right) = F$$

but F induces two metallic structures therefor:

$$F = \pm\left(\frac{2}{2\lambda_{a,b}-a}J - \frac{a}{2\lambda_{a,b}-a}I\right).$$

using Proposition 3.3, the relation allying the almost product structure and the metallic structure, we may state the following structures:

3.4. Tangent Metallic Structure

In [6], Let T be an almost tangent structure on $\bar{M}_n$

$$J_t = \frac{aI}{2} + \left(\frac{2\lambda_{a,b} - a}{2}\right)T \tag{3.7}$$

Equation 3.7, is termed as tangent metallic structure on $(\bar{M}_n, T)$ and the following equation is verified:

$$J_t^2 - aJ_t + \frac{a^2}{4}I = 0. \tag{3.8}$$

considering Equation 3.8 in the real field R, we have

$$y^2 - ay + \frac{a^2}{4}I = 0$$

from Equation 3.4, the quadratic formula where $a = 1, b = -a, c = \frac{a^2}{4}$, we obtain

$$J_t = \frac{-(-a) \pm \sqrt{a^2 - 4(1)\left(\frac{a^2}{4}\right)}}{2(1)}$$

$$= \frac{a \pm \sqrt{a^2 - a^2}}{2}$$

$$= \frac{a}{2}$$

we have the tangent metallic ratio:

$$\lambda_t = \frac{a}{2} \tag{3.9}$$

3.5. Complex Metallic Structure

Let C be an almost complex structure on $\bar{M}_n$. Then in [6]

$$J_c = \frac{aI}{2} + \left(\frac{2\lambda_{a,b} - a}{2}\right)C \tag{3.10}$$

Equation 3.10 is termed as complex metallic structure on $(\bar{M}_n, C)$. The polynomial appeased to J_c is:

$$J_c^2 - aJ_c + \frac{a^2+2b}{4}I = 0 \tag{3.11}$$

Similarly, for $\overline{M}_n = R$, we have the equation

$$y^2 - ay + \frac{a^2+2b}{2}I = 0$$

from Equation 3.4, the quadratic formula where $a = 1, b = -a, c = \frac{a^2+2b}{2}$, we get

$$J_c = ƛ_c = \frac{-(-a) \pm \sqrt{a^2 - 4(1)\left(\frac{a^2+2b}{2}\right)}}{2(1)}$$

$$= \frac{a \pm \sqrt{a^2 - 2(a^2+2b)}}{2}$$

$$= \frac{a \pm \sqrt{a^2 - 2a^2 + 4b}}{2}$$

$$= \frac{a \pm \sqrt{-a^2 + 4b}}{2}$$

$$= \frac{a \pm i\sqrt{a^2 + 4b}}{2}$$

The positive root cause of the above solution:

$$ƛ_c = \frac{a + \sqrt{a^2+4b}}{2}i \tag{3.12}$$

is called a complex metallic ratio.

3.6. Examples of Metallic structures

In this section we give examples of the metallic structures.

3.6.1. Clifford Algebras

Let $CL(n)$ be the Clifford Algebra of the positive definite form $\sum_{i=1}^{n}(x^i)^2$ of R^n, from the discerption in Equation 2.32 and Table 2.1, in [21], the Clifford product in the standard base of R^n satisfied the followings:

$$\begin{cases} e_i^2 = 1, & i = j \\ e_ie_j + e_je_i = 0, & i \neq j \end{cases} \tag{3.13}$$

using the metallic structure from Equation 3.1 we get $J_i = \frac{a+\sqrt{a^2+4be_i}}{2}$ and from Equation 3.13, we can drive new representation of the Clifford Algebra as follows:

for $i = j$ we have,

$$J_i = \frac{a + \sqrt{a^2 + 4be_i}}{2}$$

for $i \neq j$ we have,

$$J_i . J_j = \left(\frac{a + \sqrt{a^2 + 4be_i}}{2}\right)\left(\frac{a + \sqrt{a^2 + 4be_j}}{2}\right)$$

$$= \frac{1}{4}\left(a + \sqrt{a^2 + 4be_i}\right)\left(a + \sqrt{a^2 + 4be_j}\right)$$

$$= \frac{1}{4}\left(a^2 + a\sqrt{a^2 + 4be_j} + a\sqrt{a^2 + 4be_i} + (a^2 + 4b)e_i e_j\right)$$

$$= \frac{1}{4}\left(a^2 + a\sqrt{a^2 + 4be_j} + a\sqrt{a^2 + 4be_i} + 0\right)$$

$$= \frac{1}{4}\left(a^2 + a\sqrt{a^2 + 4be_j} + a\sqrt{a^2 + 4be_i}\right)$$

similarly,

$$J_j . J_i = \frac{1}{4}\left(a^2 + a\sqrt{a^2 + 4be_i} + a\sqrt{a^2 + 4be_j}\right)$$

$$J_i . J_j + J_j . J_i = \frac{1}{4}\left(2a^2 + 2a\sqrt{a^2 + 4be_i} + 2a\sqrt{a^2 + 4be_j}\right) + 0$$

$$= \left(\frac{a^2}{2} + \frac{a\sqrt{a^2 + 4be_i}}{2} + \frac{a\sqrt{a^2 + 4be_j}}{2}\right) + \frac{a^2}{2} - \frac{a^2}{2}$$

$$= \left(\frac{a^2}{2} + \frac{a\sqrt{a^2 + 4be_i}}{2} + \frac{a\sqrt{a^2 + 4be_j}}{2} + \frac{a^2}{2}\right) - \frac{a^2}{2}$$

$$= a\left(\frac{a}{2} + \frac{\sqrt{a^2 + 4be_i}}{2} + \frac{\sqrt{a^2 + 4be_j}}{2} + \frac{a}{2}\right) - \frac{a^2}{2}$$

$$= a\left(J_i + J_j\right) - \frac{a^2}{2}$$

new representation is:

$$\begin{cases} J_i, \quad metallic\ structre & i = j \\ J_i J_j + J_j J_i = a\left(J_i + J_j\right) - \dfrac{a^2}{2}, & i \neq j \end{cases} \qquad (3.14)$$

In [21], e_1, e_2 are orthogonal basis vectors of R_2^2, determined by

$$1 = I_2, \qquad e_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

using Equation 3.14 and the orthogonal basis vectors, we get the metallic structures as:

$$J_1 = \frac{a + \sqrt{a^2 + 4be_1}}{2}$$
$$J_1 = \frac{1}{2}\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \sqrt{a^2+4b}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$
$$= \begin{pmatrix} \frac{a}{2} & 0 \\ 0 & \frac{a}{2} \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{a^2+4b}}{2} & 0 \\ 0 & \frac{-\sqrt{a^2+4b}}{2} \end{pmatrix}$$
$$= \begin{bmatrix} \frac{a+\sqrt{a^2+4b}}{2} & 0 \\ 0 & \frac{a-\sqrt{a^2+4b}}{2} \end{bmatrix}$$
$$= \begin{pmatrix} \lambda_{a,b} & 0 \\ 0 & a - \lambda_{a,b} \end{pmatrix} \tag{3.15}$$

$$J_2 = \frac{a + \sqrt{a^2 + 4be_2}}{2}$$
$$= \frac{1}{2}\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \sqrt{a^2+4b}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$
$$= \frac{1}{2}\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{a^2+4b} \\ \sqrt{a^2+4b} & 0 \end{pmatrix}\right)$$
$$= \frac{1}{2}\begin{pmatrix} a & \sqrt{a^2+4b} \\ \sqrt{a^2+4b} & a \end{pmatrix}$$

3.6.2. 2D Metallic Matrices

For $J \in R_n^n$, if J provides Equation 3.1,

$$J^2 = aJ + bI_n$$

then this matrix is termed a metallic matrix where I_n is the identity matrix on R_n^n. In R_2^2, we get a two family of the metallic structures by solving Equation 3.1. In [1]

i-For $r, t \in R$ and $s \in R - \{0\}$,

ii-

$$J_{r,s} = \begin{pmatrix} r & -\frac{1}{s}(r^2 - ar - b) \\ s & a - r \end{pmatrix}$$

or (3.16)

$$J_{s,t} = \begin{pmatrix} a-t & -\frac{1}{s}(t^2-at-b) \\ s & t \end{pmatrix}$$

iii- For $r = \lambda_{a,b}, s \in R$

$$J_{\lambda_{a,b},s} = \begin{pmatrix} \lambda_{a,b} & 0 \\ s & a-\lambda_{a,b} \end{pmatrix} \text{ or } J_{a-\lambda_{a,b},s} = \begin{pmatrix} a-\lambda_{a,b} & 0 \\ s & \lambda_{a,b} \end{pmatrix}$$

$$J_{\lambda_{a,b},s} = \begin{pmatrix} \lambda_{a,b} & s \\ 0 & a-\lambda_{a,b} \end{pmatrix} \text{ or } J_{a-\lambda_{a,b},s} = \begin{pmatrix} a-\lambda_{a,b} & s \\ 0 & \lambda_{a,b} \end{pmatrix}$$

iv- For $r = \lambda_{a,b}, s = 0$

$$J_{\lambda_{a,b},s} = \begin{pmatrix} \lambda_{a,b} & 0 \\ 0 & a-\lambda_{a,b} \end{pmatrix} \text{ or } J_{a-\lambda_{a,b},s} = \begin{pmatrix} a-\lambda_{a,b} & 0 \\ 0 & \lambda_{a,b} \end{pmatrix}$$

then from Equation 3.15 and using Equation 3.16 we get

$$J_1 = \lim_{s\to 0} J_{\lambda_{a,b},s} \text{ and } J_2 = J_{\frac{a}{2},\frac{\sqrt{a^2+4b}}{2}}.$$

3.6.3. Metallic Reflection

In a Euclidean space E, the reflection with respect to a hyperplane H with the normal $v \in E\setminus\{0\}$ got the principles:

$$r_v(x) = x - \frac{2\langle x,v\rangle}{\langle v,v\rangle}, x \in E. \tag{3.17}$$

for $r_v{}^2 = I_E$, where I_E is the identity on E. In [31]
we can interpret a metallic reflection with respect to v as:

$$J_v = \frac{1}{2}\left(aI_E + \sqrt{a^2+4b}r_v\right) \tag{3.18}$$

where v is an eigenvector of J_v with the corresponding eigenvalues $a - \lambda_{a,b}$. In [31, page 314], subject to the condition that X is an orthogonal group of E,

$$XJ_vX^{-1} = J_v(x)$$

r transformation can be stated explicitly as:

$$J_v(x) = \lambda_{a,b}x - \frac{\sqrt{a^2+4b}\langle x, v\rangle}{\langle v, v\rangle}, x \in E. \tag{3.19}$$

3.6.4. Triple structure in terms of metallic structures

Let F and T denote two (1, 1)-tensor fields on a manifold M_n so that the endomorphisms defined by these are: both almost product, or one almost product and other almost complex, or both almost complex, which commute or anti-commute. With the triplet $(F, K, T = K \circ F)$ we can form the following four structures:

1) $F^2 = T^2 = I$, and $T \circ F - F \circ T = 0$; then $K^2 = I$
2) $F^2 = T^2 = I$, and $T \circ F + F \circ T = 0$; then $K^2 = -I$
3) $F^2 = T^2 = -I$, and $T \circ F - F \circ T = 0$; then $K^2 = I$
4) $F^2 = T^2 = -I$, and $T \circ F + F \circ T = 0$; then $K^2 = -I$

These structures are called (1) almost hyper product (ahp), (2) almost bi-product complex (abpc), (3) almost product bi-complex (apbc), (4) almost hyper complex (ahc), In [1].

Taking into consideration Proposition 3.3, we get

$$J_F = \frac{aI}{2} + \left(\frac{2\lambda_{a,b} - a}{2}\right)F,\ J_T = \frac{aI}{2} + \left(\frac{2\lambda_{a,b} - a}{2}\right)T,\ J_K = \frac{aI}{2} + \left(\frac{2\lambda_{a,b} - a}{2}\right)K \tag{3.20}$$

Then, the relation defines between J_F, J_T, J_K is given as,

$$\sqrt{a^2+4b}J_K = 2J_TJ_F - aJ_T - aJ_F + {\lambda_{a,b}}^2 I - bI \tag{3.21}$$

hence, the triple (J_F, J_T, J_K) can be described as follows:

i- An almost hyper product structure (ahp) if and only if J_F, J_T are metallic structures and that $J_FJ_T = J_TJ_F$, then J_K is a metallic structure.

ii- An almost bi-product complex structure (abpc) if and only if J_F, J_T are metallic structures and that $J_TJ_F + J_FJ_T = a(J_T + J_F) - \frac{a^2}{2}I$ then J_K is a complex metallic structure.

iii- An almost product bi-complex (apbc) if and only if J_F, J_T are complex metallic structures and that $J_FJ_T = J_TJ_F$ then J_K is a metallic structure.

An almost hyper complex structure (ahc) if and only if J_F, J_T are metallic structures and that $J_TJ_F + J_FJ_T = a(J_T + J_F) - \frac{a^2}{2}I$ then J_K is a complex metallic structure.

3.6.5. Quaternion Algebras

A quaternion algebra H is an associative, non-commutative division ring with four basic elements $\{1, e_1, e_2, e_3\}$ satisfying the equalities

$$e_1^2 = e_2^2 = e_3^2 = -1$$

$$e_1 e_2 = -e_2 e_1 = e_3$$

$$e_2 e_3 = -e_3 e_2 = e_1$$

$$e_3 e_1 = -e_1 e_3 = e_2$$

We write in this form to any quaternion

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$$

or some time $q = S_q + \overrightarrow{V_q}$ where the symbol $S_q = q_0$ and $\overrightarrow{V_q} = q_1 e_1 + q_2 e_2 + q_3 e_3$ denote the scalar and vector parts of the quaternion q. [22], the norm of quaternion is defined by $\sqrt{\mathrm{q}\bar{q}} = \sqrt{\bar{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ and denoted by N_q. The unit quaternion defined by $q_0 = \frac{q}{N_q}$ where $q \neq 0$. As such every unitary quaternion can be written as $q_0 = \cos\theta + \overrightarrow{s_0}\sin\theta$ where $\overrightarrow{s_0}$ is a unit vector satisfying $\overrightarrow{s_0}^2 = \overrightarrow{s_0}\overrightarrow{s_0} = -1$ and is called the axis of the quaternion.

The quaternion product $p = p_0 + p_1 e_1 + p_2 e_2 + p_3 e_3$ and $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ is defined as

$$pq = S_p S_q - \langle \overrightarrow{V_p}, \overrightarrow{V_q} \rangle + S_p \overrightarrow{V_q} + S_q \overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q}$$

where $\langle , \rangle$ and $\wedge$ are Euclidean inner product and vector product, respectively [22]. Hence by the motivation of [22], we can describe the metallic bi-quaternion structure and the split quaternion as follows:

$$J_q = \frac{a}{2} + \frac{\sqrt{a^2 + 4b}i}{2} \vec{S}_0, \vec{S}_0^{\ 2} = -1 and\ i^2 = -1 \tag{3.22}$$

$$J_q = \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} \vec{S}_0, \vec{S}_0^{\ 2} = 1 \tag{3.23}$$

Proof:

Let J be a quaternion satisfying Equation 3.2,

$$J^2 = aJ + bI$$

$$= {S_J}^2 - \langle \vec{V}_J, \vec{V}_J \rangle + 2S_J\vec{V}_J - a(S_J + \vec{V}_J) = bI$$

$$= {S_J}^2 - \langle \vec{V}_J, \vec{V}_J \rangle + 2S_J\vec{V}_J - aS_J - a\vec{V}_J = bI$$

we can have that:

$$\begin{cases} {S_J}^2 - \langle \vec{V}_J, \vec{V}_J \rangle - aS_J = bI \\ 2S_J\vec{V}_J - a\vec{V}_J = 0 \end{cases} \tag{3.23}$$

from above Equation 3.23,

$$2S_J\vec{V}_J - a\vec{V}_J = (2S_J - a)\vec{V}_J = 0$$

$$\vec{V}_J = 0 \; or \; (2S_J - a) = 0$$

$$S_J = \frac{a}{2} \tag{3.24}$$

using Equation 3.24 in Equation 3.23 we get the following:

$${S_J}^2 - \langle \vec{V}_J, \vec{V}_J \rangle - aS_J = \left(\frac{a}{2}\right)^2 - \left(\frac{a^2}{2}\right) - bI = \langle \vec{V}_J, \vec{V}_J \rangle$$

$$\left(\frac{a}{2}\right)^2 - \left(\frac{a^2}{2}\right) - bI = \langle \vec{V}_J, \vec{V}_J \rangle$$

$$\frac{a^2}{4} - \frac{a^2}{2} - bI = \langle \vec{V}_J, \vec{V}_J \rangle$$

$$\frac{-(a^2 + 4bI)}{4} = \langle \vec{V}_J, \vec{V}_J \rangle$$

$$\vec{V}_J = \sqrt{\frac{-(a^2 + 4bI)}{4}} = i\frac{\sqrt{(a^2 + 4b)}}{2} \tag{3.25}$$

also, let $1, e_1, e_2, e_3$ be the four basis quaternion elements and $z_0, z_1, z_2, z_3 \in \mathbb{C}$ then

$$J = z_0 + z_1e_1 + z_2e_2 + z_3e_3$$

J is a biquaternion and from Equation 3.24 and Equation 3.25 we have the metallic biquaternion as follows:

given the unitary bi-quaternion $J_q = \cos\theta + \vec{S}_0 \sin\theta$ and unit $z_0 = \frac{J}{N_J} = S_J = \frac{a}{2} = \cos\theta$

and $\vec{V}_J = z_1e_1 + z_2e_2 + z_3e_3 = \frac{\sqrt{(a^2+4b)}i}{2}$, therefor

$J_q = \cos\theta + \vec{S}_0 \sin\theta = \frac{a}{2} + \frac{\sqrt{a^2+4b}i}{2}\vec{S}_0$. Hence proved

From Equation 2.25 split quaternion products is given by;

$$pq = S_pS_q + g\left(\overrightarrow{V_p}, \overrightarrow{V_q}\right) + S_p\overrightarrow{V_q} + S_q\overrightarrow{V_p} + \overrightarrow{V_p} \wedge_L \overrightarrow{V_q}$$

therefore, similar to Equation 3.23 we have that:

$$\begin{cases} {S_J}^2 + g\,\langle \vec{V}_J, \vec{V}_J \rangle - aS_J = bI \\ \quad 2S_J\vec{V}_J - a\vec{V}_J = 0 \end{cases} \tag{3.26}$$

$S_J = \frac{a}{2}$ and

$$\vec{V}_J = \sqrt{\frac{(a^2 + 4bI)}{4}} = \frac{\sqrt{(a^2 + 4b)}}{2}$$

J is a split quaternion and from Equation 3.24 and Equation 3.25 we have the metallic split quaternion as:

given the unitary split quaternion as $J_q = \cos\theta + \vec{S}_0 \sin\theta$ and unit $q_0 = \frac{J}{N_J} = S_J = \frac{a}{2} = \cos\theta$ and $\vec{V}_J = q_1e_1 + q_2e_2 + q_3e_3 = \frac{\sqrt{(a^2+4b)}}{2}$, therefore

$J_q = \cos\theta + \vec{S}_0 \sin\theta = \frac{a}{2} + \frac{\sqrt{a^2+4b}}{2}\vec{S}_0$. Hence proved

4. METALLIC RIEMANNIAN MANIFOLD AND ITS CURVATURE TENSOR

We will give in this section the properties of Riemann curvature tensor of metallic Riemannian manifold $(\bar{M}_n, g, J)$ and give some concept J −sectional curvature and J −bi-sectional curvature

4.1. Almost Product Riemannian Structure

In [30,31,32]. Let F be an almost product structure on $\bar{M}$ and let g be a Riemannian metric or (semi-Riemannian metric) defined by

$$g\big(F(X), F(Y)\big) = g(X, Y) \tag{4.1}$$

Or equally, F is an endomorphism of g-symmetric denoted as:

$$g\big(F(X), F(Y)\big) = g(X, Y), \qquad \forall\, X, Y \in x(\bar{M})$$

the pair (g, F) is therefore called an Almost product Riemannian structure or (an Almost product semi-Riemannian structure).

we have the following by the means of Equation 3.1 and Proposition 3.5.

Proposition 4.1

The almost product structure F is a g-symmetric endomorphism if and only if the metallic structure J is also a g- symmetric endomorphism.

4.2. Metallic Riemannian Structure

In [7]. Let $(\bar{M}_n, g, J)$ be a metallic Riemannian manifold where g a Riemannian metric on $\bar{M}$ and a metric tensor compatible to J such that:

$$g(J(X), Y) = g(X, J(Y)), \qquad \forall\ vector\ fields\ X, Y \in x(\bar{M}) \tag{4.2}$$

J is a (1,1) tensor field define on $\bar{M}$ by means of Equation 3.2

$J^2 = aJ + bI$, $a, b \in$ N and I an identity on the lie algebra.

$$\begin{aligned} g(JX, JY) &= g(J^2X, Y) \\ &= g(aJ + bIX, Y) \\ &= ga(JX, Y) + gb(X, Y) \end{aligned} \tag{4.3}$$

Therefore (g, J) is called a metallic Riemannian structure. Also, $(\bar{M}_n, g, J)$ is called a metallic Riemannian manifold.

Corollary 4.2.1 On a metallic Riemannian manifold,

i. The projectors l, d are g-symmetric, that is:

$$g(l(X), Y) = g\big(X, l(Y)\big)$$
$$g(d(X), Y) = g(X, d(Y))$$

ii. The distributions L, D are g-orthagonal, that is:

$$g(l(X), Y) = g(X, d(Y)) = 0$$

Example

Let consider the real space $R^N = R^d \times R^{N-d}$ equipped with the metric g given by

$$g = \begin{pmatrix} \delta_j^i & 0 \\ 0 & \delta_{\tilde{j}}^{\tilde{\imath}} \end{pmatrix}, i, j = 1, \dots, d, \tilde{\imath}, \tilde{j} = N+1, \dots., N.$$

The canonical product structure on R^N is given by, where N is dimension of R

$$F = \begin{pmatrix} \delta_j^i & 0 \\ 0 & \delta_{\tilde{j}}^{\tilde{\imath}} \end{pmatrix}, i, j = 1, \dots, d, \tilde{\imath}, \tilde{j} = N+1, \dots., N.$$

As such, the triple (R^N, F, g) is locally decomposable Euclidean space. The metallic Structures $J_\pm$ obtained from the product structure F are given by using Proposition 3.5

$$J_\pm = \begin{pmatrix} \frac{a}{2}\delta_j^i & \pm\left(\frac{2\lambda_{a,b} - a}{2}\right)\delta_j^i \\ \pm\left(\frac{2\lambda_{a,b} - a}{2}\right)\delta_{\tilde{j}}^{\tilde{\imath}} & \frac{a}{2}\delta_{\tilde{j}}^{\tilde{\imath}} \end{pmatrix}$$

Therefore, the triple structure $(R^N, J_\pm, g)$ is a locally decomposable metallic Euclidean space.

4.3. Riemannian Curvature Tensor

In [13], if we let $(\bar{M}_n, g)$ be a metallic Riemannian manifold and $\chi\overline{(M)}$ be a vector space of smooth vector fields on $\bar{M}$. Then the map R denoted as the Riemannian curvature tensor of g

$$R: \chi\overline{(M)} \times \chi\overline{(M)} \times \chi\overline{(M)} \to \chi\overline{(M)}$$

Define by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \; for \; X, Y, Z \in \chi\overline{(M)} \tag{4.4}$$
$$R(X, Y)Z = -R(Y, X)Z$$

In [13], $R(X, Y, W, Z) := g(R(X, Y)W, Z) = -g(R(Y, X)W, Z)$ for all $X, Y, W, Z \in \chi\overline{(M)}$

Let J be a $(1,1)$ tensor field define on $\bar{M}$ by means of Equation 3.2 on Equation 4.4, and if

$$\nabla_{(JX)}Y = \nabla_{JX}Y - \mathrm{J}\nabla_X Y$$

If

$$\nabla_{(JX)}Y = 0$$
$$= \nabla_{JX}Y - \mathrm{J}\nabla_X Y$$
$$\nabla_{JX}Y = \mathrm{J}\nabla_X Y$$

Which implies ∇_{JX} is parallel to Y

we then obtained the following:

$$\begin{aligned} R(JX,Y)Z &= \nabla_{JX}(\nabla_Y Z) - \nabla_Y\big(\nabla_{JX}Z\big) - \nabla_{[JX,Y]}Z \\ &= \mathrm{J}\nabla_X(\nabla_Y Z) - (\nabla_Y(J\nabla_X Z) - (\nabla \mathrm{J}_{[X,Y]}Z - Y(J)XZ) \\ &= \mathrm{J}\nabla_X(\nabla_Y Z) - \nabla_Y J(\nabla_X Z) - \mathrm{J}\nabla_Y(\nabla_X Z) - \mathrm{J}\nabla_{[X,Y]}Z + Y(J)XZ \\ &= \mathrm{J}\nabla_X(\nabla_Y Z) - \mathrm{J}\nabla_Y(\nabla_X Z) - Y(J)XZ - \mathrm{J}\nabla_{[X,Y]}Z + Y(J)XZ \\ &= \mathrm{J}\nabla_X(\nabla_Y Z) - \mathrm{J}\nabla_Y(\nabla_X Z) - \mathrm{J}\nabla_{[X,Y]}Z \\ &= \mathrm{J}(\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z) \\ &= JR(X,Y)Z \end{aligned} \tag{4.5}$$

we get the above result by the use of Equation 4.4. We can also get the following using similar computations:

$$R(X,JY)Z = JR(X,Y)Z \tag{4.6}$$

$$R(JX,Y)Z = R(X,JY)Z = R(X,Y)JZ = JR(X,Y)Z$$

In [13], we obtain successively from direct calculations that:

$$\begin{aligned} &gR(JX,Y,W,Z) = g(R(JX,JY)W,Z) = g(R(W,Z)JX,Y) = \\ &g(J(R(W,Z)X),Y) = g((R(W,Z)X),JY) = gR(X,JY,W,Z) \end{aligned} \tag{4.7}$$

Similarly,

$$gR(X,Y,JW,Z) = gR(X,Y,W,JZ) \tag{4.8}$$

$$\begin{aligned} gR(X,Y,JW,JZ) &= ga(R(X,Y)W),JZ) + gb(R(X,Y)W),Z) \\ g(R(JX,JY,W,Z) &= ga(R(X,JY)W,Z) + gb(R(X,Y)W,Z) \end{aligned} \tag{4.9}$$

we get from Equation 4.6 and Equation 4.7 that:

$$R(X,JY,JW,Z) = R(JX,Y,JW,Z) = R(JX,Y,W,JZ) = R(X,JY,W,JZ) \tag{4.10}$$

from the above Equations 4.6, 4.7, 4.8, 4.9 and 4.10 we can get the following:

Proposition 4.3.1 If $(\bar{M}_n, g, J)$ is a metallic Riemannian manifold, then the Riemann curvature tensor R of g defined by $R(X,Y,W,Z) \coloneqq g(R(X,Y)W,Z)$ satisfied the relations:

1. $R(X,Y,JW,Z) = R(X,Y,W,JZ)$
2. $R(JX,Y,W,Z) = R(X,JY,W,Z)$
3. $R(X,Y,JW,JZ) = aR(X,Y,W,JZ) + bR(X,Y,W,Z)$
4. $R(JX,JY,W,Z) = aR(X,JY,W,Z) + bR(X,Y,W,Z)$
5. $R(X,JY,JW,Z) = R(JX,Y,JW,Z) = R(JX,Y,W,JZ) = R(X,JY,W,JZ)$
   $for\ any\ X,Y,W,Z \in \chi(\overline{M})$ [13].

4.4. Sectional Curvature of metallic Riemannian Manifold

The Similar holomorphic sectional curvature for the non-degenerate plane section of the Riemannian metallic manifold $(\bar{M}_n, g, J)$ will be describe as follows.

Definition 4.4.1 In [19], let ƛ be a $2-$ dimensional subspace of a tangent space $T_p\bar{M}$ of a Riemannian manifold. Let $\{e_1, e_2\}$ be the basis for the plane ƛ. Then:

$$K(e_1, e_2) = K(ƛ) = \frac{g < R(e_1, e_2)e_1, e_2) >}{\| e_1 \|^2 \| e_2 \|^2 - < e_1, e_2 >} \tag{4.11}$$

The above is termed as the Riemann or Sectional curvature of the plane ƛ.

Using Equation 4.11 we can now define the Sectional curvature of metallic Riemannian as follows:

Definition 4.4.2 In [13], let X be a non-zero tangent vector field and let $ƛ_X, JX$ be the plane generated by X and JX. We can define the sectional curvature provided that $g(X,X)g(JX,JX) - [g(X,JX)]^2 \neq 0$ which implies the plane $ƛ_X, JX$ is non degenerate, then:

$$K(X,JX) = K(ƛ_X) = \frac{g(R(X,JX)X,JX)}{g(X,X)g(JX,JX) - [g(X,JX)]^2} \tag{4.12}$$

If $K(ƛ_X)$ is constant, from Equation 4.10 we say that $\bar{M}$ has a constant sectional curvature defined by

$$K(ƛ_X) := R(X,JX,X,JX) = c \tag{4.13}$$

for any $X \in \chi(\overline{M})$.

Proposition 4.4.3 If $(\bar{M}_n, g, J)$ is a Metallic Riemannian manifold, then $K(ƛ_X) = 0$ for any $X \in \chi(\overline{M})$ and the sectional curvature of the non-degenerate plane sections vanishes.

Proof: from Equation 4.13 we have:

$$K(\lambda_X) := R(X, JX, X, JX)$$

In [13], let $K(\lambda_X) = 0$ Using Equation 4.10 and Proposition 4.3.1 we have:

$$K(\lambda_X) := R(X, JX, X, JX) = R(JX, X, X, JX) = -R(X, JX, X, JX) = -K(\lambda_X) \tag{4.14}$$

for any $X \in \chi(\overline{M})$.

Definition 4.4.4 In [13], let X and Y be two linearly non-zero tangent vector field and let λ_X, JX and λ_Y, JY be the plane generated by X and JX, and Y and JY, respectivly. We can define the bi-sectional curvature provided that $g(X,X)g(JX,JX) - [g(X,JX)]^2 \neq 0$ and $g(Y,Y)g(JY,JY) - [g(Y,JY)]^2 \neq 0$ which implies the plane λ_X, JX and λ_Y, JY are non-degenerate, then:

$$K(\lambda_{X,Y}) = \frac{g(R(X,JX)Y,JY)\sqrt{|g(X,X)g(JX,JX)-[g(X,JX)]^2|}\sqrt{|g(Y,Y)g(JY,JY)-[g(Y,JY)]^2|}}{(g(X,X)g(JX,JX)-[g(X,JX)]^2 g(Y,Y)g(JY,JY)-[g(Y,JY)]^2)} \tag{4.15}$$

If $K(\lambda_{X,Y})$ is constant, from Equation 4.10 we say that $\overline{M}$ has a constant bi-sectional curvature defined by

$$K(\lambda_{X,Y}) := R(X, JX, Y, JY) = c \tag{4.16}$$

for any $X, Y \in \chi(\overline{M})$.

Proposition 4.4.5 If $(\overline{M}_n, g, J)$ is a metallic Riemannian manifold, then $K(\lambda_{X,Y}) = 0$ for any $X, Y \in \chi(\overline{M})$ and the bi-sectional curvature of the non-degenerate plane sections vanishes.

Proof: from Equation 4.13 we have:

$$K(\lambda_{X,Y}) := R(X, JX, Y, JY)$$

In [13], let $K(\lambda_{X,Y}) = 0$ Using Equation 4.10 and Proposition 4.3.1 we have:

$$K(\lambda_{X,Y}) := R(X, JX, Y, JY) = R(X, JX, JY, Y) = -R(JX, X, Y, JY) = -K(\lambda_{X,Y}) \tag{4.14}$$

for any $X, Y \in \chi(\overline{M})$.

CONCLUSION

In this thesis, basics of the metallic structure was studied, which was a polynomial structure with structure polynomial of the form $Q(J) = J^2 - aJ - bI = 0$ on manifolds by the use of metallic ratio, which was the generalization of Golden proportion or golden ratio. Some of the metallic structure mathematical terminologies were studied, we stated some theorems and proved where possible of the metallic structures. We also studied some examples of the metallic structure. Also, in this thesis some properties of the metallic Riemannian metric were studied. The J-sectional and J- Bisectional curvatures of metallic Riemannian manifold were studied.

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