STABILITY OF PLANAR PIECEWISE LINEAR SYSTEMS: A GEOMETRIC APPROACH

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By

Adamu Abdullahi

September, 2015

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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This thesis focuses on the stability analysis of piecewise linear systems. Such systems consist of linear subsystems, each of which is active in a particular region of the state-space. Many practical and theoretical systems can be modelled as piecewise linear systems. Despite their simple structure, analysis of piecewise linear systems can be rather complex. For instance, most of the results for stability can be based on a Lyapunov approach. However, a major drawback of applying this method is that, it usually only provides sufficient conditions for stability.

A geometric approach will be used to derive new stability criteria for planar piecewise linear systems. Any planar piecewise linear (multi-modal) system is shown to be globally asymptotically stable just in case each linear mode satisfies certain conditions that solely depend on how its eigenvectors stand relative to the cone on which it is defined. The stability conditions are in terms of the eigenvalues, eigenvectors, and the cone. The improvements on the known stability conditions are the following: i) The condition is directly in terms of the "givens" of the problem. ii) Non-transitive modes are identified. iii) Initial states and their trajectories are classified (basins of attraction and repulsion are indicated). iv) The known condition for bimodal systems is obtained as an easy corollary of the main result. Additionally, using our result on stability, we design a hybrid controller for a class of second order LTI systems that do not admit a static output feedback controller. The effectiveness of the proposed controller is illustrated on a magnetic levitation system.

Keywords: Piecewise linear systems, Stability analysis, Well-posedness, Basins of attraction and repulsion, Magnetic levitation.

ÖZET

DÜZLEMSEL PARÇALI DOĞRUSAL SİSTEMLERİN KARARLILIĞI: BİR GEOMETRİK YAKLAŞIM

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Bu tezde parçalı doğrusal sistemlerin kararlılığı incelenmektedir. Bu tür sistemler, durum uzayının belirli bölgesinde etkin olan doğrusal sistemlerden oluşmaktadır. Bir çok doğrusal olmayan sistem veya melez sistem, parçalı doğrusal bir sistem olarak modellenebilir. Basit bir yapıya sahip olan bu tür sistemlerin analizi çok zor olabilmektedir. Örneğin , kararlılık için elde edilen sonuçların çoğu Lyapunov yaklaşımına dayanır. Ancak, Lyapunov metodu genellikle yalnızca yeter şartlar üretebilmektedir.

Burada düzlemsel parçalı doğrusal sistemlerin kararlılığı için yeni geometrik kararlılık koşulları türetilecektir. En genel bir düzlemsel parçalı doğrusal sistemin kararlılığı için gerek ve yeter koşulların alt sistemlerin tanımlandıkları konisel düzlemlerin, sistem özvektörleriyle olan ilişkine bağlı olduğu gösterilecektir. Yani, verilen gerek ve yeter koşullar tamamen sistemlerin özdeğerleri, özvektörleri ve konisel düzlemleri tanımlayan vektörler cinsindendir . Literatürde bilenen gerek ve yeter koşullarla kıyaslandığında burada türetilen koşulların artıları şunlardır: i) Yalnızca problemin verileri cinsindendirler ii) Geçişli olmayan kipler tespit edilirmiştir iii) Başlangıç koşularına bağlı olarak yörüngelerin sınıflandırılması yapılmaktadır iv) Bilinen iki modlu sistemlerin kararlılık koşulları ana sonuçtan kolaylıkla çıkarabilmektedir. Elde edilen ana sonucun önemli bir uygulaması olarak, sabit geribesleme ile kararlı yapılmayan ikinci mertebeden sistemler için melez bir denetleyici tasarlanmiştir. Önerilen denetleyicinin performansı bir manyetik kaldırma sistemi üzerinde örneklendirilmiştir.

Anahtar sözcükler: Parçalı doğrusal sistemler, Kararlılık analizi, Doğru tanımlılık, Ön koşul havzaları, Manyetik kaldırma.

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In memory of my sister, Fatima (1993-2014).

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Chapter 1

Introduction

In recent years, stability analysis of switched systems and switched controller synthesis has received considerable attention in the control community [2–11]. Among the many subclasses of switched systems, one that has received even greater attention is piecewise linear system [12], [13]. A piecewise linear system consists of linear subsystems, each of which is active in a particular region of the state-space. The concern with such systems has mainly been application oriented, because there are so many naturally hybrid, multi-modal plants all around [14], [15], [16]. Examples can be found in electronic systems. In [14], an adaptive controller was implemented using a microcontroller, for the control of a bimodal piecewise linear electric circuit. In [15], a DC-DC series resonant converter was modelled as a piecewise linear system. Examples are not limited to physical systems and, sophisticated switched systems come up in the domain of social sciences [17], [18]. Many nonlinear dynamical systems can be modelled accurately using a piecewise linear approach [19], [20]. A piecewise linear system framework was used to model and control an automotive all-wheel drive (AWD) clutch system in [19] .

There are many systems that can not be stabilized by a continuous state feedback control law, but can be stabilized by a switched state feedback control law. Such systems are also piecewise linear in the closed-loop [21], [22]. Switched stabilization of linear subsystems can result in a piecewise linear system [23–29]. In addition, other forms of switched system can also be represented as a piecewise linear system: linear complementarity systems [30]; fuzzy control systems [31]; mixed logical dynamical (MLD) systems [32]; relay systems [33]; linear systems with actuator saturation [34] are example of this.

Despite their simple structure, analysis of piecewise linear systems is complex. Extending the results on the much celebrated notions of controllability, observability, and stability for linear systems to this class of systems is not trivial. This is mainly because, such properties may hold for all subsystems while not holding for the piecewise linear system itself. For example, there are many examples of piecewise linear systems where switching between stable subsystems results in an unstable trajectory [9]. Nevertheless, many results have been obtained on controllability and observability [35], [36], [37]; well-posedness [38–41]; stability analysis [42–59]; controller synthesis [60–64]. See [6], for recent results on the stability analysis of piecewise linear systems.

Determining necessary and sufficient condition on stability of a piecewise linear system is very desirable. Although Lyapunov approach has provided many sufficient conditions for stability of more general cases, the method becomes quickly stagnant by the requirement to concoct Lyapunov functions for a set of systems [45], [48], [53], [57]. One can claim that, the most complete result so far on, for instance, stability of piecewise linear systems has turned out to be on the very special case of two-state systems [50], [52] (also see [56]).

The result on stability of Iwatani and Hara [50] on planar, multi-modal systems is complete from a test of stability point of view but it is still worth a second look for many reasons. The condition offered for stability is in terms of the zeros of a subsidiary system, the relation of which to the original plant is indirect. This prevents an insightful interpretation of the condition. They devise a very useful notion of 'transitive modes' but also use a not so natural notion of 'weakly-transitive' modes, which are also transitive, so a clear distinction between the two notions is not possible. What are the 'non-transitive modes'? The main result of [52], obtained independently of [50], is based on a clean characterization of trajectories escaping convex cones (i.e., transitive modes) and brings the eigenvectors of each mode into the picture via the notions of "visible eigenvectors" and "stable eigenspaces". However, the result in [52], similar to that of [50], does not provide any intuition concerning "non-transitive cones." In [51], a hybrid automaton approach is followed to study the stability of planar systems and a decision algorithm that is based on "contractive cycles" that are, in essence, stable transitive trajectories are given. Also in [58] and [46], integral expressions are derived to characterize the "expansion factors" when trajectories go through transitive modes.

Here we take a different approach to the same problem and obtain a new set of necessary and sufficient conditions. Any planar piecewise linear system is shown to be globally asymptotically stable just in case each linear mode satisfies certain conditions that only depend on how its eigenvectors stand relative to the cone on which it is defined. The conditions are in terms of the eigenvalues, eigenvectors, and the cones. The improvements on both [50] and [52] are the following: *i)* The condition is directly in terms of the "givens" of the problem. *ii)* Non-transitive modes are identified. *iii)* Initial states and their trajectories are classified (basins of attraction and repulsion are indicated). *iv)* The known condition for bimodal systems is obtained as an easy corollary of the main result.

Using the results on stability, we construct a hybrid controller for a class of planar LTI systems. A hybrid controller is designed to stabilize a planar LTI system. The effectiveness of the controller is illustrated on a magnetic levitation system.

The thesis is organized as follows. Chapter 2 gives a mathematical model of the classes of piecewise linear system considered in this thesis. Chapter 3 is devoted to the structural analysis of a linear time-invariant system inside a cone. Necessary and sufficient conditions for well-posedness of conewise linear systems are obtained in Chapter 4. Chapter 5 is dedicated to the stability analysis of conewise linear systems, computationally tractable stability criteria are given. In Chapter 6, results obtained in Chapter 3 and 5 are used to design a hybrid output feedback controller for a planar linear time-invariant (LTI) system. Concluding remarks are given in Chapter 7. The codes used in this thesis are included in the Appendix.

The following notations will be used throughout this thesis. We denote the real numbers, n-dimensional real vector space, and the set of real $n \times m$ matrices by $\mathbf{R}$, $\mathbf{R}^n$, and $\mathbf{R}^{n\times m}$, respectively. The norm of a vector $\mathbf{v} \in \mathbf{R}^n$ will be denoted by $|\mathbf{v}|$. The natural basis vectors in $\mathbf{R}^n$ will be denoted by $\mathbf{e}_i, i = 1, ..., n$. In particular, when $n = 3$, we will use $\mathbf{k} := \mathbf{e}_3$. If $\mathbf{v}, \mathbf{w} \in \mathbf{R}^3$, then $\mathbf{v} \times \mathbf{w}$ will denote the cross product of the vectors and $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$, their dot product, where 'T' denotes 'transpose.' If $\mathbf{v}, \mathbf{w} \in \mathbf{R}^2$, then by $\mathbf{v} \times \mathbf{w}$, we mean $\det[\mathbf{v} \;\; \mathbf{w}]\mathbf{k}$, where 'det' means 'determinant,' i.e., cross product of vectors in the plane will be computed by imbedding them in the space. The set of complex n-vectors will be $\mathbf{C}^n$ and $j \in \mathbf{C}$ will be the imaginary number. For convenience, we will use the cross product of $\mathbf{v}, \mathbf{w} \in \mathbf{C}^2$ as well and define $\mathbf{v} \times \mathbf{w} := \det[\mathbf{v} \quad \mathbf{w}]\mathbf{k}$. By $\log z$, $z \in \mathbf{C}$, we denote the complex principal logarithm $\log z = \ln|z| + j\angle z$ with $-\pi < \angle z \leq \pi$.

Chapter 2

Piecewise Linear Systems

In this chapter, we define the classes of piecewise linear systems (Fig. 2.1) studied in this thesis. We will give a mathematical model that describes these systems. Three types of piecewise linear systems: Conewise Linear, Bimodal Piecewise Linear and Bimodal Quadratic Piecewise Linear, will be studied. The latter two types of systems can actually be posed in the category of a conewise linear system as we will show below.

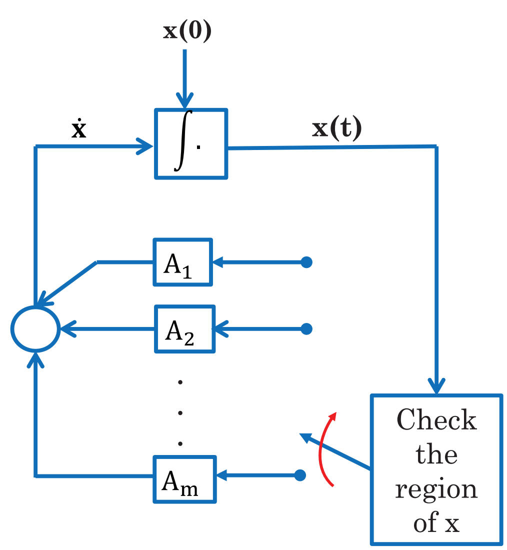


Figure 2.1: Piecewise linear systems

2.1 Conewise Linear Systems

A planar conewise linear system [35] is a special type of piecewise linear system (PLS) consisting of linear time-invariant (LTI) subsystems, each of which is active in a convex cone (Fig. 2.2).

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if & \mathbf{x} \in \mathbf{S}_1, \\ A_2\mathbf{x} & if & \mathbf{x} \in \mathbf{S}_2, \\ \vdots & \vdots & \vdots \\ A_m\mathbf{x} & if & \mathbf{x} \in \mathbf{S}_m, \end{cases} \tag{2.1}$$

where $x \in \mathbf{R}^2$, $A_i \in \mathbf{R}^{2\times 2}$, $C_i \in \mathbf{R}^{2\times 2}$ and $\mathbf{S}_i$ is a convex cone defined by,

$$\mathbf{S}_i := \{\mathbf{x} \in \mathbf{R}^2 : C_i\mathbf{x} \geq 0\},$$

for $i = 1, 2, ..., m$, where i represents the mode of the system. Without loss of generality, we assume that C_i is nonsingular with $\det C_i > 0$. The nonsingularity assumption implies that the conewise linear system is truly multi-modal ($m \geq 2$) and that $int(\mathbf{S}_i) \neq \emptyset$. In addition, we assume that the interior of each pairwise intersection is empty i.e., $int(\mathbf{S}_i \cap \mathbf{S}_k) = \emptyset$, for all $i \neq k$ and $\mathbf{S}_1 \cup ... \cup \mathbf{S}_m = \mathbf{R}^2$. These assumptions will ensure that the system (2.1) is memoryless i.e., the switching rule depends only on the present state. Note that the assumption $\det C_i > 0$ does not impose any constraint, and only requires a permutation of rows of C_i if necessary.

Each mode of (2.1) is defined by $(A_i, \mathbf{S}_i)$, where A_i denotes the system matrix of mode i and,

$$S_i = \begin{bmatrix} \mathbf{s}_{i1} & \mathbf{s}_{i2} \end{bmatrix} := C_i^{-1} = \begin{bmatrix} \mathbf{c}_{i1}^T \\ \mathbf{c}_{i2}^T \end{bmatrix}^{-1}.$$

Since $\det S_i = \frac{1}{\det C_i} > 0$, $\mathbf{s}_{i1} \times \mathbf{s}_{i2}$ points upward using the right-hand rule, i.e., is positively oriented.

It is easy to see that, if each $\mathbf{S}_i$, $i = 1, ..., m$ is strictly contained in a half-plane,

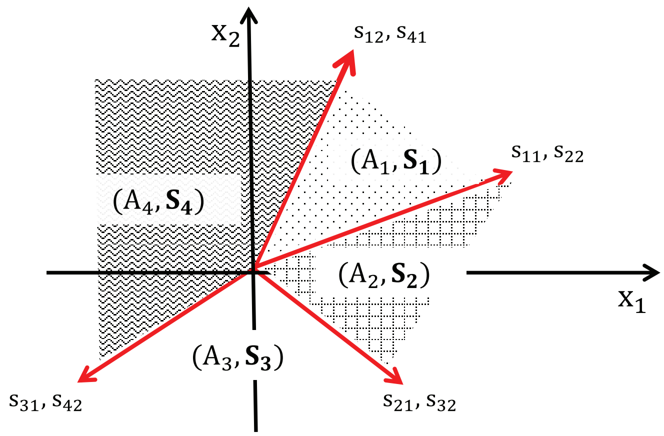


Figure 2.2: Planar conewise linear system, m=4

then $\mathbf{S}_i$ is a convex cone,

$$\mathbf{S}_i = \{\alpha \mathbf{s}_{i1} + \beta \mathbf{s}_{i2} : \alpha, \beta \geq 0\};$$

and the boundary of $\mathbf{S}_i$ is the union two rays,

$$\mathsf{B}_{ik} = \{\alpha \mathbf{s}_{ik} : \alpha \geq 0\},\ k = 1, 2,$$

where B_{i1} and B_{i2} represents the left and right border, respectively. If a mode $(A_i, \mathbf{S}_i)$ is defined on a half-plane or a sector larger than a half-plane, then it can be split into two modes having the same dynamics (the same A_i matrix), so that each is still defined on a cone. The splitting must be done with care, as we will clarify later.

The eigenvalues of a mode i will be denoted by $\lambda_{i1}, \lambda_{i2} \in \mathbf{C}$ and, when the eigenvalues are real and distinct, they will be indexed by $\lambda_{i1} > \lambda_{i2}$. If the eigenvalues of a mode i are real and distinct, $\mathbf{v}_{i1}$ and $\mathbf{v}_{i2}$ will denote the eigenvectors associated with λ_{i1} and λ_{i2}, respectively. If they are repeated ($\lambda_{i1} = \lambda_{i2}$), $\mathbf{v}_{i2}$ is an eigenvector and $\mathbf{v}_{i1}$ is the generalized eigenvector. For non-real eigenvalues, $\mathbf{v}_{i1} + j\mathbf{v}_{i2}$ will denote the eigenvector associated with $\sigma_i - j\omega_i$, where, $\lambda_{i1} = \bar{\lambda}_{i2} = \sigma_i + j\omega_i$ with $\omega_i > 0$.

2.2 Bimodal Piecewise Linear Systems

A bimodal piecewise linear system is given by

$$\dot{\mathbf{x}} = \begin{cases} B_1\mathbf{x} & if \quad \mathbf{c}^T\mathbf{x} \geq 0, \\ B_2\mathbf{x} & if \quad \mathbf{c}^T\mathbf{x} \leq 0, \end{cases} \tag{2.2}$$

where $B_1, B_2 \in \mathbf{R}^{2\times 2}$, $\mathbf{c} \in \mathbf{R}^2$, and $\mathbf{c} \neq \mathbf{0}$.

The system in (2.2) can be viewed as a conewise linear system. This can be done by selecting a vector $\mathbf{c}_0$ such that $d := \det[\mathbf{c}\ \mathbf{c}_0]$ is positive. Then, an equivalent description of (2.2) is

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad C_1\mathbf{x} \geq 0, \\ A_2\mathbf{x} & if \quad C_2\mathbf{x} \geq 0, \\ A_3\mathbf{x} & if \quad C_3\mathbf{x} \geq 0, \\ A_4\mathbf{x} & if \quad C_4\mathbf{x} \geq 0, \end{cases} \tag{2.3}$$

with $A_1 = A_2 = B_1$, $A_3 = A_4 = B_2$ and; $C_1^T := [\mathbf{c}\ \mathbf{c}_0]$, $C_2^T := [-\mathbf{c}_0\ \ \mathbf{c}]$, $C_3 = -C_1$, $C_4 = -C_2$. Note that, our initial assumption of nonsingularity is satisfied (i.e., $S_i := C_i^{-1}$ satisfies $\det S_i > 0$ for i =1,2,3,4). In order for (2.3) to be well-posed, $\mathbf{c}_0$ needs to satisfy a further condition. This will be elaborated further in Chapter 4 below.

2.3 Bimodal Quadratic Piecewise Linear Systems

A bimodal quadratic piecewise linear system is given by

$$\dot{\mathbf{x}} = \begin{cases} B_1\mathbf{x} & if \quad \mathbf{x}_1\mathbf{x}_2 \geq 0, \\ B_2\mathbf{x} & if \quad \mathbf{x}_1\mathbf{x}_2 \leq 0, \end{cases} \tag{2.4}$$

where $B_1, B_2 \in \mathbf{R}^{2\times 2}$.

The description in (2.4) simply means that, B_1 is active in the first and third quadrant, while B_2 is active in the second and fourth quadrant. Therefore, such a system can be viewed as a conewise linear system (Fig. 2.3):

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad C_1\mathbf{x} \geq 0, \\ A_2\mathbf{x} & if \quad C_2\mathbf{x} \geq 0, \\ A_3\mathbf{x} & if \quad C_3\mathbf{x} \geq 0, \\ A_4\mathbf{x} & if \quad C_4\mathbf{x} \geq 0, \end{cases} \tag{2.5}$$

with $A_1 = A_3 = B_1,\ A_2 = A_4 = B_2$
and

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ C_3 = -C_1,\ C_4 = -C_2.$$

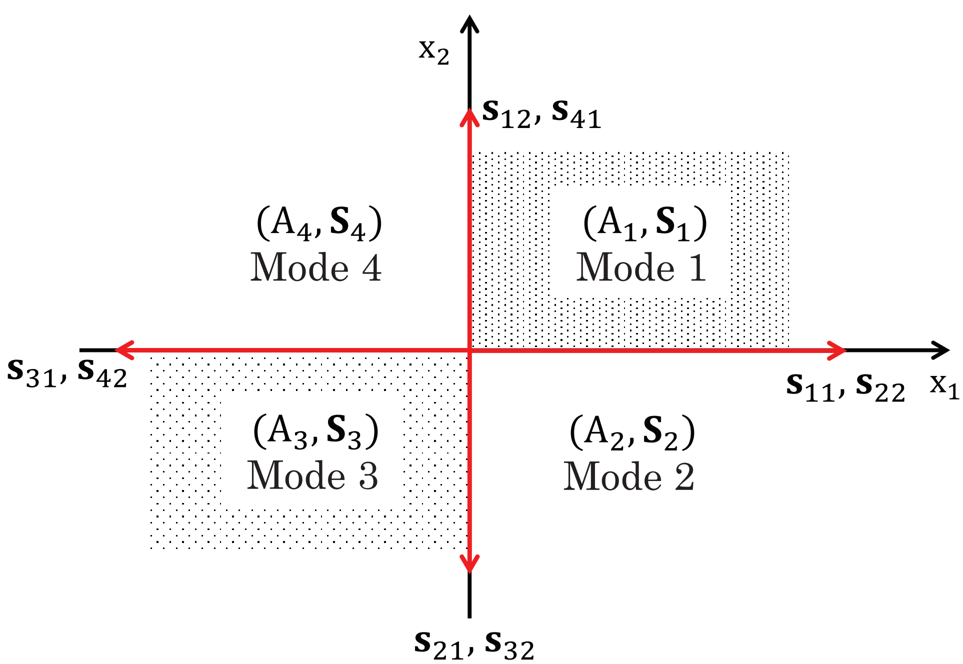


Figure 2.3: Bimodal quadratic piecewise linear systems

Chapter 3

Structural Analysis

In this chapter, in order to study the structural properties of a planar linear time-invariant system inside a cone, we focus on a single mode

$$\dot{\mathbf{x}} = A\mathbf{x},\ \mathbf{x} \in \mathbf{S} \subset \mathbf{R}^2, \mathbf{S} = \{\alpha\mathbf{s}_1 + \beta\mathbf{s}_2 : \alpha, \beta \geq 0\}, \tag{3.1}$$

where $S = [\mathbf{s}_1\ \ \mathbf{s}_2]$ with $\det S > 0$. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^2$ be such that

$$AV = V\Lambda,\ V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix},$$

where Λ is equal to

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \text{ and } \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}.$$

respectively, when the eigenvalues are real and distinct, real and repeated, and non-real. The trivial case $A = \lambda I$ of real and repeated eigenvalues is left out for reasons to be stated in Chapter 4, Remark 4.1.1. We also define

$$W = \begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \end{bmatrix} := V^{-1}.$$

Note that $\det V > 0$ if and only if $\mathbf{v}_1 \times \mathbf{v}_2$ is positively oriented. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors in case λ_1, λ_2 are real and distinct. If eigenvalues are repeated, then $\mathbf{v}_2$ is the eigenvector and $\mathbf{v}_1$ is a generalized eigenvector. In case of non-real eigenvalues, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the real and imaginary parts of the eigenvector associated with $\sigma - j\omega$, where $\lambda_1 = \bar{\lambda}_2 = \sigma + j\omega$ with $\omega > 0$.

3.1 Trajectory

The trajectory at $t \geq 0$ of (3.1) starting at $\mathbf{x}(0) = \mathbf{b} \in \mathbf{S}$ at time 0 can be written as

$$\mathbf{x}(t, \mathbf{b}) = e^{At}\mathbf{b}, \tag{3.2}$$

where

$$\mathbf{x}(t, \mathbf{b}) = \begin{cases} e^{\lambda_1 t}\mathbf{w}_1^T\mathbf{b}\,\mathbf{v}_1 + e^{\lambda_2 t}\,\mathbf{w}_2^T\mathbf{b}\,\mathbf{v}_2, \\ e^{\lambda t}[\mathbf{w}_1^T\mathbf{b}\,\mathbf{v}_1 + (t\,\mathbf{w}_1^T\mathbf{b} + \mathbf{w}_2^T\mathbf{b})\mathbf{v}_2], \\ e^{\sigma t}\{[\mathbf{w}_1^T\mathbf{b}\cos(\omega t) - \mathbf{w}_2^T\mathbf{b}\sin(\omega t)]\,\mathbf{v}_1 \\ \quad +[\mathbf{w}_1^T\mathbf{b}\sin(\omega t) + \mathbf{w}_2^T\mathbf{b}\cos(\omega t)]\,\mathbf{v}_2\}. \end{cases} \tag{3.3}$$

respectively, when the eigenvalues are real and distinct, real and repeated, and non-real.

3.1.1 Direction of Trajectory

The direction the trajectory moves at time t can be determined by checking the sign of $\psi(t)$. In Fig. 3.1, if a trajectory beginning at $\mathbf{b}$ is positively oriented, then it may hit the border B_2 at time t_2. If it is negatively oriented, then it may hit the border B_1 at time t_1.

Fact 3.1.1. *Trajectory* $\mathbf{x}(t, \mathbf{b})$ *moves in positive direction at time* $t \geq 0$ *if for*

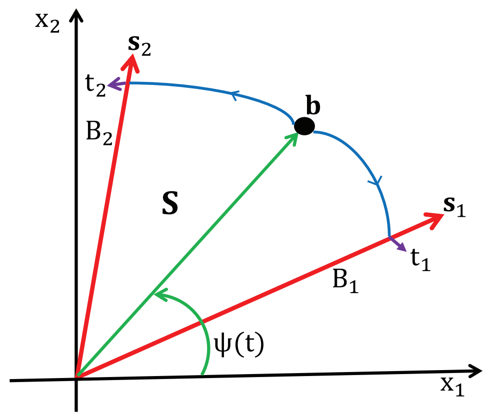


Figure 3.1: Direction of trajectory

every real eigenvector $\mathbf{v}_k \times \mathbf{b} \neq 0$ *for* $k = 1, 2$ *and*

$$\begin{cases} \det V \left(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}\right) > 0 & \textit{if eigenvalues are real and distinct,} \\ \det V > 0 & \textit{otherwise.} \end{cases} \tag{3.4}$$

Proof. Since $\mathbf{w}_l^T \mathbf{v}_k = 0$ for $l \neq k$, a trajectory moves radially along an eigendirection if and only if $\mathbf{v}_k \times \mathbf{b} = 0$, by (3.3). For any other $\mathbf{b}$, the angle of the vector $\mathbf{v}_k \times \mathbf{b}$, and hence, the direction of a trajectory is well-defined. Let $\mathbf{e}_i$ denote the i-th natural vector for $i = 1, 2$ and let $\mathbf{x}(t, \mathbf{b}) = \rho(t) \angle \psi(t)$ be in polar representation. Then,

$$\dot{\psi}(t) = \frac{(\mathbf{e}_2^T \dot{\mathbf{x}})(\mathbf{e}_1^T \mathbf{x}) - (\mathbf{e}_1^T \dot{\mathbf{x}})(\mathbf{e}_2^T \mathbf{x})}{\rho^2} \tag{3.5}$$

so that, by an easy computation, we get

$$\dot{\psi}(t) = \begin{cases} \dfrac{\det V (\lambda_1 - \lambda_2)(\mathbf{w}_1^T \mathbf{b})(\mathbf{w}_2^T \mathbf{b})}{\rho^2 e^{-(\lambda_1 + \lambda_2)t}} & \textit{if eigenvalues are real and distinct,} \\ \dfrac{\det V (\mathbf{w}_1^T \mathbf{b})^2}{\rho^2 e^{-2\lambda t}} & \textit{if eigenvalues are real and repeated,} \\ \dfrac{\det V\, \omega[(\mathbf{w}_1^T \mathbf{b})^2 + (\mathbf{w}_2^T \mathbf{b})^2]}{\rho^2 e^{-2\sigma t}} & \textit{if eigenvalues are non-real.} \end{cases} \tag{3.6}$$

If the eigenvalues are real and distinct, then by (3.6) $\dot{\psi}(t) > 0$ if and only if $\det V\,(\mathbf{w}_1^T\mathbf{b})(\mathbf{w}_2^T\mathbf{b}) < 0$. Since, $\mathbf{v}_1 \times \mathbf{b} = \det V(\mathbf{w}_2^T\mathbf{b})\mathbf{k}$ and $\mathbf{v}_2 \times \mathbf{b} = -\det V(\mathbf{w}_1^T\mathbf{b})\mathbf{k}$ (where $\mathbf{k}$ denotes the (positively oriented) cross product of the unit vectors in x_1 and x_2 directions), $\dot{\psi}(t) > 0$ if and only if (3.4) holds.

By (3.6), if the eigenvalues are real and repeated or non-real, $\dot{\psi}(t) > 0$ if and only if (3.4) holds. ■

That is, if the eigenvalues are real and repeated or non-real, then the direction of the trajectory is independent of the initial state $\mathbf{b}$ and is solely determined by $\det V$. In case of real and distinct eigenvalues, how the initial state is situated with respect to the two eigenvectors also matters.

Note. By the first equation in (3.6), if $A = \lambda I$, where $\lambda = \lambda_1 = \lambda_2$; $\dot{\psi}(t) = 0$ for all $\mathbf{b} \in \mathbf{S}$, which implies that the trajectory will always move radially along the direction of $\mathbf{b}$ for all $\mathbf{b} \in \mathbf{S}$.

3.1.2 Existence of t_1 and t_2

We will now classify the trajectories that hit the border ($\mathbf{s}_1$ or $\mathbf{s}_2$) of the cone.

Fact 3.1.2. *(i) There exists (a finite) $t_1 > 0$ such that $\mathbf{x}(t_1, \mathbf{b})$ intersects B_1 if and only if*

$$\begin{cases} \det V\,(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}) < 0 \;\;\&\;\; \mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_1 \times \mathbf{s}_1 > 0, \\ \det V < 0 \;\;\&\;\; \mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{s}_1 > 0, \\ \det V < 0 \end{cases} \tag{3.7}$$

respectively, when eigenvalues are such that $\lambda_1 > \lambda_2$ (real and distinct), $\lambda := \lambda_1 = \lambda_2$ (real and repeated), and $\lambda_1 = \bar{\lambda}_2 = \sigma + j\omega$ (non-real).

(ii) There exists (a finite) $t_2 > 0$ such that $\mathbf{x}(t_2, \mathbf{b})$ intersects B_2 if and only

if

$$\begin{cases} \det V \,(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}) > 0 \;\; \& \;\; \mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_1 \times \mathbf{s}_2 > 0, \\ \det V > 0 \;\; \& \;\; \mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{s}_2 > 0, \\ \det V > 0. \end{cases} \tag{3.8}$$

respectively, when eigenvalues are such that $\lambda_1 > \lambda_2$ *(real and distinct),* $\lambda := \lambda_1 = \lambda_2$ *(real and repeated), and* $\lambda_1 = \bar{\lambda_2} = \sigma + j\omega$ *(non-real).*

Proof. (i) **Real and distinct eigenvalues:** An intersection time $t_1 > 0$ exists if and only if

$$\mathbf{c}_2^T \mathbf{x}(t_1, \mathbf{b}) = e^{\lambda_1 t_1}(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + e^{\lambda_2 t_1}(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b}) = 0, \tag{3.9}$$

which gives

$$e^{(\lambda_1 - \lambda_2)t_1} = -\frac{(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})}{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b})} > 1, \tag{3.10}$$

where the inequality is by $\lambda_1 > \lambda_2$. Note that $\mathbf{c}_2^T \mathbf{v}_1 \neq 0$ and $\mathbf{w}_1^T \mathbf{b} \neq 0$ since otherwise either $\mathbf{v}_1 \times \mathbf{s}_1 = 0$ or $\mathbf{v}_2 \times \mathbf{b} = 0$, i.e., either there is a sliding mode (the trajectory stays on $\mathbf{s}_1$) or the initial condition is along an eigenvector.

Now, the condition (3.9) is equivalent to

$$1 + \frac{(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})}{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b})} = \frac{\det V \det S \, \mathbf{c}_2^T \mathbf{b}}{\mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_1 \times \mathbf{s}_1} < 0, \tag{3.11}$$

by the identities $\mathbf{c}_2^T(\mathbf{v}_1 \mathbf{w}_1^T + \mathbf{v}_2 \mathbf{w}_2^T)\mathbf{b} = \mathbf{c}_2^T \mathbf{b}$, $\det S \,(\mathbf{c}_2^T \mathbf{v}_1)\mathbf{k} = -\mathbf{v}_1 \times \mathbf{s}_1$ and, $\det V \,(\mathbf{w}_1^T \mathbf{b})\mathbf{k} = -\mathbf{v}_2 \times \mathbf{b}$.

In (3.11), $\mathbf{c}_2^T \mathbf{b} > 0$ because, as $\mathbf{s}_1 \times \mathbf{s}_2$ is positively oriented and $\mathbf{b}$ is in the interior of $\mathbf{S}$, $\mathbf{s}_1 \times \mathbf{b} = \det S \,(\mathbf{c}_2^T \mathbf{b})\mathbf{k}$ is also positively oriented. Using the condition from (3.4) that $\det V \,(\mathbf{v}_1 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{b}) < 0$ is necessary for an intersection with B_1, it follows that (3.11) holds if and only if the first condition in (3.7) holds. And from (3.10), we can provide an explicit expression for t_1:

$$t_1 = \frac{1}{\lambda_1 - \lambda_2} \ln \frac{|(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})|}{|(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b})|}. \tag{3.12}$$

Real and repeated eigenvalues: An intersection time $t_1 > 0$ exists if and only if

$$\mathbf{c}_2^T \mathbf{x}(t_1, \mathbf{b}) = e^{\lambda t_1}[(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b}) + t_1(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_1^T \mathbf{b})] = 0, \quad (3.13)$$

which gives

$$t_1 = -\frac{(\mathbf{c}_2^T \mathbf{v}_1)(\mathbf{w}_1^T \mathbf{b}) + (\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_2^T \mathbf{b})}{(\mathbf{c}_2^T \mathbf{v}_2)(\mathbf{w}_1^T \mathbf{b})} = -\frac{\det V \det S\, \mathbf{c}_2^T \mathbf{b}}{\mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{s}_1} > 0. \quad (3.14)$$

Using the condition from (3.4) that $\det V < 0$ is necessary for an intersection with B_1, it follows that (3.14) holds if and only if the second condition in (3.7) holds. From (3.14),

$$t_1 = \frac{|\det V \det S\, \mathbf{c}_2^T \mathbf{b}|}{|\mathbf{v}_2 \times \mathbf{b} \cdot \mathbf{v}_2 \times \mathbf{s}_1|}. \quad (3.15)$$

Non-real eigenvalues: In this case, $\det V < 0$ is necessary and sufficient for an intersection with B_1, because the trajectories are always foci or centers.

(ii) The proof is analogous to the proof in (i). ■

We will now derive an expression for the value of the trajectory at the border.

Fact 3.1.3. *Let μ denote λ_1 or σ and let $\mathbf{v}$ denote $\mathbf{v}_2$ or $\mathbf{v}_1 + j\mathbf{v}_2$ in the cases of real (distinct or repeated) and non-real eigenvalues, respectively. Then, if (i) and (ii) of Fact 3.1.2 hold, the value of the trajectory at the border can be written as:*

$$\mathbf{x}(t_1, \mathbf{b}) = \mathbf{s}_1 \frac{(\mathbf{v} \times \mathbf{b}) \cdot \mathbf{k}}{(\mathbf{v} \times \mathbf{s}_1) \cdot \mathbf{k}} e^{\mu t_1}, \quad (3.16)$$

$$\mathbf{x}(t_2, \mathbf{b}) = \mathbf{s}_2 \frac{(\mathbf{v} \times \mathbf{b}) \cdot \mathbf{k}}{(\mathbf{v} \times \mathbf{s}_2) \cdot \mathbf{k}} e^{\mu t_2}. \quad (3.17)$$

Proof. Real and distinct eigenvalues: At B_1, from (3.10) we can write,

$$e^{\lambda_2 t_1}(\mathbf{w}_2^T\mathbf{b}) = -\frac{e^{\lambda_1 t_1}(\mathbf{c}_2^T\mathbf{v}_1)(\mathbf{w}_1^T\mathbf{b})}{(\mathbf{c}_2^T\mathbf{v}_2)}.$$

Substituting $e^{\lambda_2 t_1}(\mathbf{w}_2^T\mathbf{b})$ into the first equation in (3.3), we have

$$\mathbf{x}(t_1,\mathbf{b}) = e^{\lambda_1 t_1}\frac{\mathbf{w}_1^T\mathbf{b}}{\mathbf{c}_2^T\mathbf{v}_2}[\mathbf{v}_1(\mathbf{c}_2^T\mathbf{v}_2) - \mathbf{v}_2(\mathbf{c}_2^T\mathbf{v}_1)]$$

By the identity

$$(CV) = \begin{bmatrix} \mathbf{c}_1^T\mathbf{v}_1 & \mathbf{c}_1^T\mathbf{v}_2 \\ \mathbf{c}_2^T\mathbf{v}_1 & \mathbf{c}_2^T\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^T\mathbf{s}_1 & \mathbf{w}_1^T\mathbf{s}_2 \\ \mathbf{w}_2^T\mathbf{s}_1 & \mathbf{w}_2^T\mathbf{s}_2 \end{bmatrix}^{-1} = (WS)^{-1}, \tag{3.18}$$

we can write,

$$\begin{aligned} \mathbf{x}(t_1,\mathbf{b}) &= e^{\lambda_1 t_1}\mathbf{w}_1^T\mathbf{b}(\mathbf{v}_1 + \mathbf{v}_2\frac{\mathbf{w}_2^T\mathbf{s}_1}{\mathbf{w}_1^T\mathbf{s}_1}) \\ &= e^{\lambda_1 t_1}\frac{(\mathbf{v}_2\times\mathbf{b})\cdot\mathbf{k}}{(\mathbf{v}_2\times\mathbf{s}_1)\cdot\mathbf{k}}\,\mathbf{s}_1, \end{aligned} \tag{3.19}$$

which gives (3.16).

Real and repeated eigenvalues: Substituting t_1 obtained form (3.14) into the second equation in (3.3) we have

$$\begin{aligned} \mathbf{x}(t_1,\mathbf{b}) &= e^{\lambda_1 t_1}\frac{\mathbf{w}_1^T\mathbf{b}}{\mathbf{c}_2^T\mathbf{v}_2}[\mathbf{v}_1(\mathbf{c}_2^T\mathbf{v}_2) - \mathbf{v}_2(\mathbf{c}_2^T\mathbf{v}_1)] \\ &= e^{\lambda_1 t_1}\mathbf{w}_1^T\mathbf{b}(\mathbf{v}_1 + \mathbf{v}_2\frac{\mathbf{w}_2^T\mathbf{s}_1}{\mathbf{w}_1^T\mathbf{s}_1}) \\ &= e^{\lambda_1 t_1}\frac{(\mathbf{v}_2\times\mathbf{b})\cdot\mathbf{k}}{(\mathbf{v}_2\times\mathbf{s}_1)\cdot\mathbf{k}}\,\mathbf{s}_1, \end{aligned}$$

which gives (3.16) as in the previous case.

Non-real eigenvalues: At $t = t_1 > 0$, we have

$$\begin{aligned}\mathbf{c}_2^T\mathbf{x}(t,\mathbf{b}) &= e^{\sigma t}\{[\mathbf{w}_1^T\mathbf{b}\cos(\omega t) - \mathbf{w}_2^T\mathbf{b}\sin(\omega t)]\,\mathbf{c}_2^T\mathbf{v}_1 \\ &\qquad + [\mathbf{w}_1^T\mathbf{b}\sin(\omega t) + \mathbf{w}_2^T\mathbf{b}\cos(\omega t)]\,\mathbf{c}_2^T\mathbf{v}_2\} = 0,\end{aligned} \tag{3.20}$$

which gives

$$\begin{aligned} tan(\omega t_1) &= \frac{\mathbf{c}_2^T(\mathbf{v}_1\mathbf{w}_1^T + \mathbf{v}_2\mathbf{w}_2^T)\mathbf{b}}{(\mathbf{c}_2^T\mathbf{v}_1)(\mathbf{w}_2^T\mathbf{b}) - (\mathbf{c}_2^T\mathbf{v}_2)(\mathbf{w}_1^T\mathbf{b})} \\ &= \frac{\mathbf{c}_2^T\mathbf{b}}{(\mathbf{c}_2^T\mathbf{v}_1)(\mathbf{w}_2^T\mathbf{b}) - (\mathbf{c}_2^T\mathbf{v}_2)(\mathbf{w}_1^T\mathbf{b})} \end{aligned} \tag{3.21}$$

and

$$\mathbf{x}(t_1,\mathbf{b}) = \frac{\det V}{\det S}\frac{e^{\sigma t_1}}{\mathbf{c}_2^T\mathbf{v}_2}[\mathbf{w}_1^T\mathbf{b}\cos(\omega t_1) - \mathbf{w}_2^T\mathbf{b}\sin(\omega t_1)]\mathbf{s}_1. \tag{3.22}$$

In obtaining (3.22) from (3.3), we have used the identity

$$\mathbf{v}_1(\mathbf{c}_2^T\mathbf{v}_2) - \mathbf{v}_2(\mathbf{c}_2^T\mathbf{v}_1) = \mathbf{c}_2^T\mathbf{v}_2(\mathbf{v}_1 + \mathbf{v}_2\frac{\mathbf{w}_2^T\mathbf{s}_1}{\mathbf{w}_1^T\mathbf{s}_1}) = \frac{\mathbf{c}_2^T\mathbf{v}_2}{\mathbf{w}_1^T\mathbf{s}_1}\mathbf{s}_1,$$

as in the previous cases.

Noting, with $\Delta := \sqrt{[(\mathbf{w}_1^T\mathbf{b})^2 + (\mathbf{w}_2^T\mathbf{b})^2][(\mathbf{c}_2^T\mathbf{v}_1)^2 + (\mathbf{c}_2^T\mathbf{v}_2)^2]}$, that

$$cos(\omega t_1) = \frac{(\mathbf{c}_2^T\mathbf{v}_1)(\mathbf{w}_2^T\mathbf{b}) - (\mathbf{c}_2^T\mathbf{v}_2)(\mathbf{w}_1^T\mathbf{b})}{\Delta},\ \sin(\omega t_1) = \frac{\mathbf{c}_2^T\mathbf{b}}{\Delta},$$

and substituting in (3.22), we get

$$\begin{aligned}\mathbf{x}(t_1,\mathbf{b}) &= -\frac{\det V}{\det S}e^{\sigma t_1}\sqrt{\frac{(\mathbf{w}_1^T\mathbf{b})^2 + (\mathbf{w}_2^T\mathbf{b})^2}{(\mathbf{c}_2^T\mathbf{v}_1)^2 + (\mathbf{c}_2^T\mathbf{v}_2)^2}}\,\mathbf{s}_1 \\ &= -\frac{\det V}{\det S}\frac{|\det S|}{|\det V|}e^{\sigma t_1}\sqrt{\frac{|\mathbf{v}_2 \times \mathbf{b}|^2 + |\mathbf{v}_1 \times \mathbf{b}|^2}{|\mathbf{s}_1 \times \mathbf{v}_1|^2 + |\mathbf{s}_1 \times \mathbf{v}_2|^2}}\,\mathbf{s}_1,\end{aligned}$$

which can be expressed as in (3.16) since $\det S > 0, -\det V > 0$. ■

The derivation of (3.17) is along the same lines.

3.1.3 Mode Type

Using the conditions (i) and (ii) in Fact 3.1.2, we can define five different types of modes for an LTI system in a cone. The classification of initial conditions made possible by conditions (i) and (ii) of Fact 3.1.2 are given in Figs. (3.2)-(3.4), where basins leading to t_1, t_2, or neither are indicated relative to typical positions of the eigenvectors with respect to the region $\mathbf{S}$.

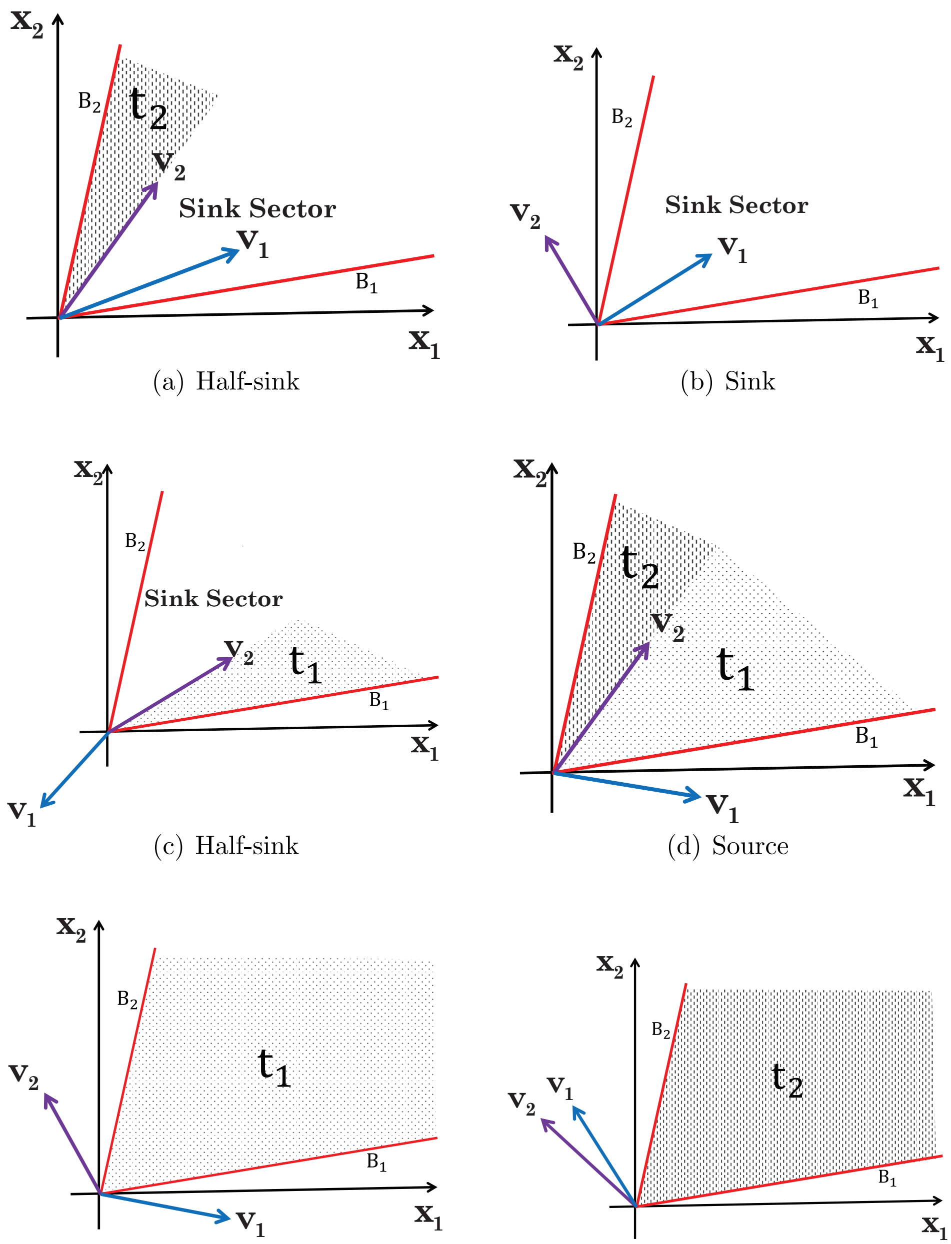
Definition 3.1.1. An eigenvector $\mathbf{v}$ is *interior* to $\mathbf{S}$ if $\mathbf{v}$ or $-\mathbf{v}$ is in $int(\mathbf{S})$; it is *exterior* to $\mathbf{S}$ if neither $\mathbf{v}$ nor $-\mathbf{v}$ is in $\mathbf{S}$. For non-real eigenvalues, the eigenvectors are always exterior. This simply means that,

Fact 3.1.4. *An eigenvector* $\mathbf{v}$ *is interior to* $\mathbf{S}$ *if* $v_p = (\mathbf{s}_1 \times \mathbf{v})(\mathbf{v} \times \mathbf{s}_2) > 0$ *or* $v_n = (\mathbf{s}_1 \times -\mathbf{v})(-\mathbf{v} \times \mathbf{s}_2) > 0$.

Mode type. A mode $(A_i, \mathbf{S}_i)$ is called *transitive* [50] if either the trajectory intersects B_1 (at some finite time) for all $\mathbf{b} \in \mathbf{S}$ or it intersects B_2 for all $\mathbf{b} \in \mathbf{S}$. The mode will be called *negative-transitive* in the former, and *positive-transitive* in the latter case. A mode is a *source* if there exists $\mathbf{n} \in int(\mathbf{S})$ such that for all $\mathbf{b} = \alpha\mathbf{n} + \beta\mathbf{s}_1$ with $\alpha \geq 0, \beta > 0$, the trajectory intersects B_1 and for all $\mathbf{b} = \alpha\mathbf{n} + \beta\mathbf{s}_2$ with $\alpha \geq 0, \beta > 0$, the trajectory intersects B_2.

It is clear from Figs. (3.2)-(3.4) that if a mode is neither transitive nor a source, then it is either a *sink* and all trajectories starting in (or entering into) $\mathbf{S}$ stay in $\mathbf{S}$ or it is a *half-sink*, that is there is a sector of $\mathbf{S}$ that is a sink.

By Fact 3.1.2 (or Figs. (3.2)-(3.4)), it is easy to see that a mode $(A_i, \mathbf{S}_i)$ is transitive if and only if the eigenvector(s) are exterior. It is a source (resp., sink) if and only if there are two eigenvectors such that the one associated with the larger (resp., smaller) eigenvalue is exterior and the other interior. It is a half-sink if and only if the eigenvector(s) are interior.

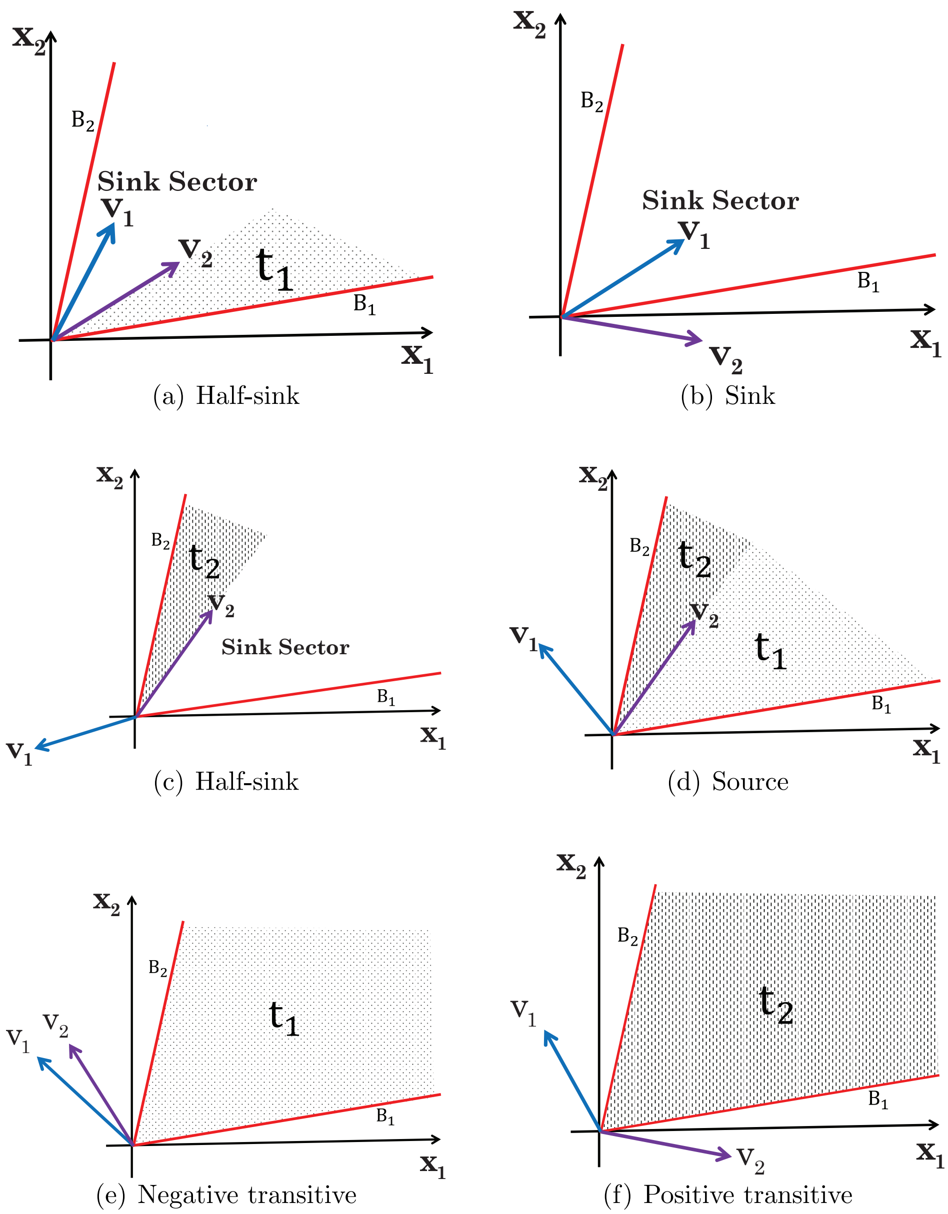


(a) Half-sink (b) Sink

(c) Half-sink (d) Source

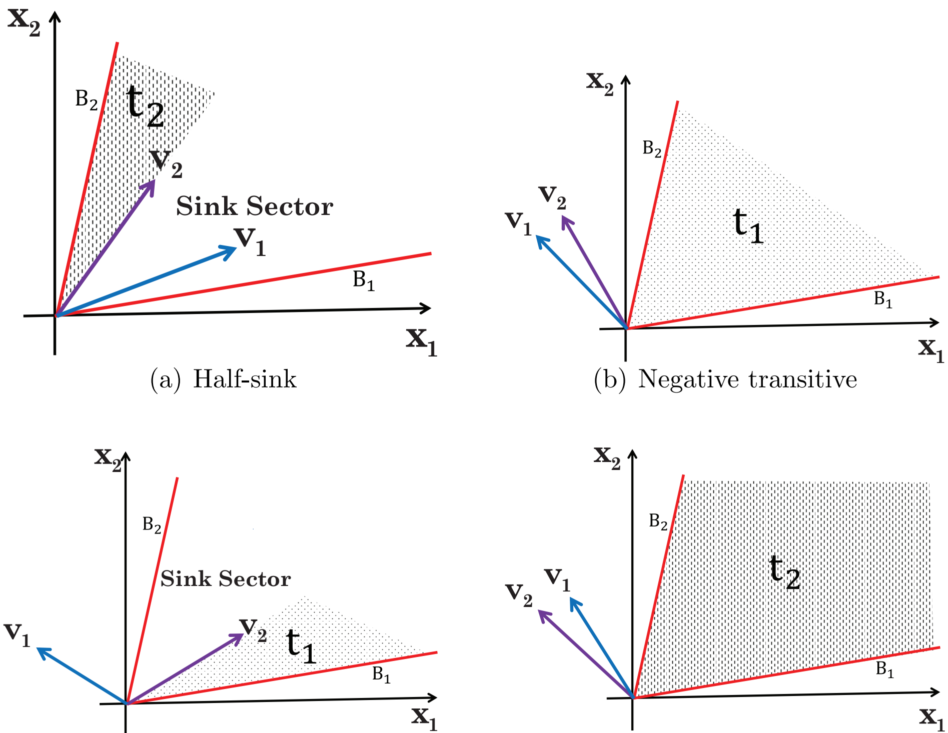
(e) Negative transitive (f) Positive transitive

Figure 3.2: Basins for distinct eigenvalues, $\mathbf{v}_1 \times \mathbf{v}_2$ is positively oriented



(a) Half-sink (b) Sink (c) Half-sink (d) Source (e) Negative transitive (f) Positive transitive

Figure 3.3: Basins for distinct eigenvalues, $\mathbf{v}_1 \times \mathbf{v}_2$ is negatively oriented



(a) Half-sink

(b) Negative transitive

(c) Half-sink

(d) Positive transitive

Figure 3.4: Basins for repeated eigenvalues, $\mathbf{v}_1 \times \mathbf{v}_2$ is positively and negatively oriented

Chapter 4

Well-posedness of Conewise Linear Systems

Well-posedness (existence and uniqueness of solutions) is one of the fundamental issues in the analysis of hybrid systems [38–41]. Unlike linear systems, this issue is nontrivial for hybrid systems. Different necessary and sufficient conditions have been derived for special classes of hybrid systems. The well-posedness of linear complementarity systems has been studied in [30]. In [39], results on well-posedness of switch-driven piecewise-affine systems have been obtained. Imura and Schaft [38] studied the well-posedness of piecewise linear systems in the sense of Carath*é*odory. They derived necessary and sufficient conditions for the well-posedness of piecewise linear systems based on lexicographic inequalities and observability matrices of the subsystems. In [43] and [44], necessary and sufficient conditions for well-posedness of bimodal three-state piecewise linear systems have been derived based on a direct analysis of the vector fields on the switching line .

In this chapter, we will state a new set of necessary and sufficient conditions for the planar conewise linear system to be well-posed in the sense of Carath*é*odory, i.e., there exist a unique solution of the form

$$\mathbf{x}(t, \mathbf{b}) = \mathbf{b} + \int_{t_0}^{t} f(x(\tau))d\tau, \tag{4.1}$$

without any sliding mode, where $f(x(\tau))$ is the discontinuous vector field given by the right hand side of (2.1) and $\mathbf{x}(t_0) = \mathbf{b}$ [38].

Fact 3.1.1 implies a geometric condition for well-posedness of planar conewise linear systems. The condition (ii) below says, in effect, that trajectories of every pair of adjacent modes have the same direction on their common border.

4.1 Well-posedness Condition

Theorem 4.1.1. *The conewise linear system (2.1) is well-posed if and only if for every pair of adjacent modes (i, k) with common border B_{i1}*

i) It holds that

$$\mathbf{v}_{il} \times \mathbf{s}_{i1} \neq 0 \textit{ and } \mathbf{v}_{kl} \times \mathbf{s}_{i1} \neq 0, \textit{ for } l = 1, 2,$$

where $\mathbf{v}_{il}$ and $\mathbf{v}_{kl}$ are real eigenvectors,
and

ii) It holds that

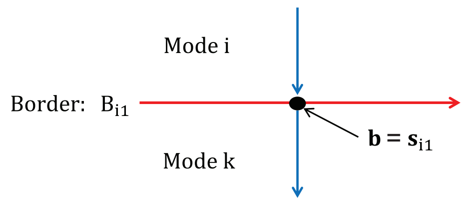
$$\Gamma_i \Gamma_k > 0, \tag{4.2}$$

where

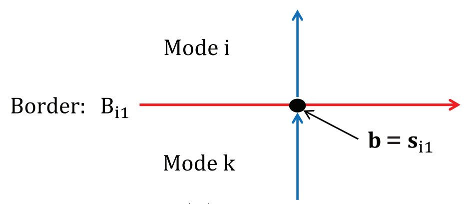
$$\Gamma_i = \begin{cases} \det V_i (\mathbf{v}_{i1} \times \mathbf{s}_{i1} \cdot \mathbf{v}_{i2} \times \mathbf{s}_{i1}) & \textit{if eigenvalues are real and distinct,} \\ \det V_i & \textit{otherwise,} \end{cases}$$

and

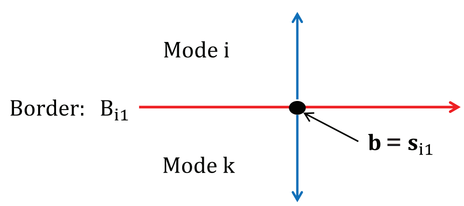
$$\Gamma_k = \begin{cases} \det V_k (\mathbf{v}_{k1} \times \mathbf{s}_{il} \cdot \mathbf{v}_{k2} \times \mathbf{s}_{i1}) & \textit{if eigenvalues are real and distinct,} \\ \det V_k & \textit{otherwise.} \end{cases}$$



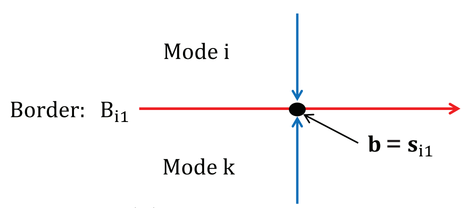
(a) $\Gamma_i \Gamma_k > 0$



(b) $\Gamma_i \Gamma_k > 0$



(c) $\Gamma_i > 0$ and $\Gamma_k < 0$



(d) $\Gamma_i < 0$ and $\Gamma_k > 0$

Figure 4.1: Existence and uniqueness of solution

Proof. From basic theory of differential equations, if $\mathbf{b} \in \mathbf{S}_i(\mathbf{S}_k)$, there exist a solution from the initial condition $\mathbf{b}$ to when the trajectory hits a border. When a trajectory starts from a border, three cases are possible: i) a *unique solution exist*, ii) a *solution does not exist*, iii) *multiple solutions exist*. The first situation occurs if and only if $\Gamma_i \Gamma_k > 0$. However, when $\Gamma_i < 0$ and $\Gamma_k > 0$ any trajectory that starts from the border will stay on the border and in that case there will be no solution in the sense of Carathéodory and what is known as chattering or sliding mode (trajectory stays on the border) occurs. And when $\Gamma_i > 0$ and $\Gamma_k < 0$ there will be two solutions. These three cases are illustrated in Fig. (4.1).

Finally, the first condition in Theorem 4.1.1 ensures that no eigenvector lies on a border. This is necessary because, if an eigenvector lies on a border, any trajectory that hits (or starts on) the border will stay on it (i.e a sliding mode occurs). ■

Remark 4.1.1. The trivial case $A = \lambda I$ is omitted because, in that case the trajectory is given by $\mathbf{x}(t, \mathbf{b}) = e^{\lambda t}\mathbf{b}$. Hence, for i=1,2, whenever $\mathbf{c}_i^T\mathbf{b} = 0$, we have $\mathbf{c}_i^T\mathbf{x}(t, \mathbf{b}) = 0$. This means that any trajectory that starts on the border will remain on the border, even if no eigenvector lies on the border, i.e., a sliding mode exists.

Remark 4.1.2. In splitting a mode of dynamics A into two modes (to satisfy the assumption that all modes are defined on cones), one should take care to choose the common border not to coincide with any eigenvectors of A, since otherwise there will be a sliding mode by Theorem 4.1.1.

The assumption of well-posedness actually puts some serious constraint on the set of systems considered, since any trajectory would have to evolve in one direction only. A thorough study of systems, like [65], in which sliding modes is allowed is thus highly desirable, as pointed out in [38].

A different set of conditions for well-posedness of piecewise linear systems has been independently derived by Imura and Schaft in [38]. The condition is based on the observability matrices of the subsystems. We will show that, their condition is the same as the one in Theorem 4.1.1 (for planar conewise linear systems).

Let ${A_i}^f$ (resp., ${A_k}^f$) and ${A_i}^s$ (resp., ${A_k}^s$) represent the first and second columns of A_i (A_k) respectively.

Theorem 4.1.2. *The conewise linear system (2.1) is well-posed if and only if for every pair of adjacent modes (i, k) with common border B_{i1}*

i) It holds that $({\mathbf{c}_{i2}}^T, A_i)$ and $({\mathbf{c}_{i2}}^T, A_k)$ are observable and *ii) It holds that $T = T_{A_k}{T_{A_i}}^{-1}$ is a lower triangular matrix whose all diagonal elements are positive, where*

$$T_{A_i} := \begin{bmatrix} {\mathbf{c}_{i2}}^T \\ {\mathbf{c}_{i2}}^T A_i \end{bmatrix}, \; T_{A_k} := \begin{bmatrix} {\mathbf{c}_{i2}}^T \\ {\mathbf{c}_{i2}}^T A_k \end{bmatrix}.$$

Proof. T can be computed as

$$T = T_{A_k}{T_{A_i}}^{-1} = \begin{bmatrix} 1 & 0 \\ \hat{\chi} & \dfrac{\chi_k}{\chi_i} \end{bmatrix},$$

where $\chi_i = \det T_{A_i}$, $\chi_k = \det T_{A_k}$ and $\hat{\chi} = \dfrac{({\mathbf{c}_{i2}}^T {A_k}^f)({\mathbf{c}_{i2}}^T {A_i}^s) - ({\mathbf{c}_{i2}}^T {A_k}^s)({\mathbf{c}_{i2}}^T {A_i}^f)}{\chi_i}$. Hence, condition (ii) holds if and only if $\frac{\chi_k}{\chi_i} > 0$.

Let $A_i = V_i \Lambda_i V_i^{-1}$, where Λ_i and V_i are defined as in Chapter 3. Substituting A_i into T_{A_i}, it is easy to see that

$$\chi_i = \det T_{A_i} = \begin{cases} \dfrac{-(\lambda_{i1} - \lambda_{i2})(\det C_i)^2(\mathbf{v}_{i1} \times \mathbf{s}_{i1} \cdot \mathbf{v}_{i2} \times \mathbf{s}_{i1})}{\det V_i}, \\ \dfrac{-(\det C_i)^2(\mathbf{v}_{i1} \times \mathbf{s}_{i1})^2}{\det V_i}, \\ \dfrac{-w_i(\det C_i)^2[(\mathbf{v}_{i1} \times \mathbf{s}_{i1})^2 + (\mathbf{v}_{i2} \times \mathbf{s}_{i1})^2]}{\det V_i}, \end{cases} \tag{4.3}$$

respectively, when the eigenvalues are real and distinct, real and repeated, and non-real.

Analogously,

$$\chi_k = \det T_{A_k} = \begin{cases} \dfrac{-(\lambda_{k1} - \lambda_{k2})(\det C_k)^2(\mathbf{v}_{k1} \times \mathbf{s}_{i1} \cdot \mathbf{v}_{k2} \times \mathbf{s}_{i1})}{\det V_k}, \\ \dfrac{-(\det C_k)^2(\mathbf{v}_{k1} \times \mathbf{s}_{i1})^2}{\det V_k}, \\ \dfrac{-w_k(\det C_k)^2[(\mathbf{v}_{k1} \times \mathbf{s}_{i1})^2 + (\mathbf{v}_{k2} \times \mathbf{s}_{i1})^2]}{\det V_k}, \end{cases} \tag{4.4}$$

respectively, when the eigenvalues are real and distinct, real and repeated, and non-real.

Since, $\lambda_{i1} - \lambda_{i2}$, w_i, $\det C_i$, $(\mathbf{v}_{k1} \times \mathbf{s}_{i1})^2$ and $[(\mathbf{v}_{k1} \times \mathbf{s}_{i1})^2 + (\mathbf{v}_{k2} \times \mathbf{s}_{i1})^2]$ are all greater than zero; $\chi_i > 0$ (resp., $\chi_i < 0$) if and only if $\Gamma_i > 0$ (resp., $\Gamma_i < 0$). Analogously, $\chi_k > 0$ (resp., $\chi_k < 0$) if and only if $\Gamma_k > 0$ (resp., $\Gamma_k < 0$). Therefore, $\frac{\chi_k}{\chi_i} > 0$ holds if and only if $\Gamma_i \Gamma_k > 0$.

The first condition in Theorem 4.1.2 says, in effect, that neither χ_i nor χ_k is equal to zero. Since $\det C_i \neq 0$ (resp., $\det C_k \neq 0$) and $\lambda_{i1} \neq \lambda_{i2}$ (resp., $\lambda_{k1} \neq \lambda_{k2}$); in the real eigenvalue cases, $\chi_i = 0$ (resp., $\chi_k = 0$) if and only if a real eigenvector lies on the border, which is same as the first condition given in Theorem 4.1.1. ■

Theorem 4.1.2 is a generalization of the condition given in [38] for well-posedness of planar bimodal systems.

Remark 4.1.3. It should also be noted that, the observability condition eliminates the trivial case $A = \lambda I$. From the first equation in (4.3) (resp., (4.4)), χ_i (resp., χ_k) is equal to zero if $\lambda_{i1} = \lambda_{i2}$ (resp., $\lambda_{k1} = \lambda_{k2}$).

For a bimodal system (2.2), we have $\mathbf{s}_{21} = -\mathbf{s}_{12}$ and $A_1 = A_2$. Therefore, for well-posedness, it is necessary and sufficient to check the conditions in Theorem 4.1.1 along $\mathbf{s}_{12}$. That is,

Theorem 4.1.3. *A bimodal system (2.2) is well-posed if and only if*

i) It holds that

$$\mathbf{v}_{1l} \times \mathbf{s}_{12} \neq 0 \textit{ and } \mathbf{v}_{2l} \times \mathbf{s}_{12} \neq 0, \textit{ for } l = 1, 2,$$

where $\mathbf{v}_{1l}$ *and* $\mathbf{v}_{2l}$ *are real eigenvectors and*

ii) It holds that

$$\Gamma_1 \Gamma_2 > 0, \tag{4.5}$$

where

$$\Gamma_1 = \begin{cases} \det V_1(\mathbf{v}_{11} \times \mathbf{s}_{12} \cdot \mathbf{v}_{12} \times \mathbf{s}_{12}) & \textit{if eigenvalues of } B_1 \textit{ are real and distinct}, \\ \det V_1 & \textit{otherwise}, \end{cases}$$

$$\Gamma_2 = \begin{cases} \det V_2(\mathbf{v}_{21} \times \mathbf{s}_{12} \cdot \mathbf{v}_{22} \times \mathbf{s}_{12}) & \textit{if eigenvalues of } B_2 \textit{ are real and distinct}, \\ \det V_2 & \textit{otherwise}. \end{cases}$$

Proof. The proof follows the same line as in Theorem 4.1.1. ■

Chapter 5

Stability of Conewise Linear Systems

Two methods are generally used to study the stability of piecewise linear systems (PLS): Lyapunov-based method [45,47–49,53,54,57] and what we will call "Direct Analysis" method [42–44,46,50–52,55,56,58]. While the Lyapunov-based method generally applies to systems with arbitrary order, the method becomes quickly stagnant by the requirement to concoct a Lyapunov function.

Three main approaches have been developed to study the stability of PLS using the Lyapunov-based techniques: Common Quadratic (CQ) Lyapunov approach [54], Piecewise Quadratic (PWQ) Lyapunov approach [48], [53], [57] and Surface Lyapunov (SuL) approach [45]. The weakest among the three is the CQ Lyapunov approach, because to apply this method it is necessary to require that all the modes are asymptotically stable. A less conservative approach is the Piecewise Quadratic Lyapunov approach, where the piecewise linear property of the system is exploited in searching for a Lyapunov function. In this method, one searches for a Lyapunov function in a particular region of the state space, and whenever a mode is active, the energy is required to decrease along the trajectory and it does not have to decrease when the mode is inactive. However, a major drawback in applying this method is that, a repartitioning of the state space may be required to find a PWQ Lyapunov function. In the Surface Lyapunov approach, a Lyapunov function that decreases along the impact maps is sought. However this method becomes unattractive when the number of switching surfaces increases.

5.1 Conservatism of Lyapunov-based Techniques

A major drawback in applying Lyapunov-based techniques is that, they only provide sufficient conditions for the stability of the system. The following example shows the need for better approaches for the stability analysis of piecewise linear systems. Let us first recall the following definition [66].

Definition 5.1.1. The equilibrium point $\mathbf{x}_{eq} = 0$ of the nonlinear system defined by (2.1) is globally asymptotically stable (GAS) if $\forall \mathbf{x}(t_0) \in \mathbf{R}^n$, $||\mathbf{x}(t)|| \to \mathbf{0}$ as $t \to \infty$.

Example 5.1.1. [48] Consider the Lure's system:

$$\dot{\mathbf{x}} = B_1\mathbf{x} + Bu; u = -\phi(y); y = C_p x,$$

$$\phi(y) = \begin{cases} 0 & if\, y \geq 0, \\ y & if\, y \leq 0. \end{cases}$$

$$G(s) = C_p(sI - B_1)^{-1}B, B_1 = \begin{bmatrix} -3 & 7 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -2 \\ 7 \end{bmatrix} \text{ and } C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The system can be represented as a bimodal piecewise linear system, where

$$\dot{\mathbf{x}} = \begin{cases} B_1 x & if\, \mathbf{x}_1 \geq 0, \\ B_2 x & if\, \mathbf{x}_1 \leq 0, \end{cases}$$

with $B_2 = B_1 - BC_p$.

Theorem 5.1.1. [54] *A necessary and sufficient condition for the systems B_1 and B_2 to have a common quadratic Lyapunov function is that the pencils $A_c = \alpha\ B_1 + (1-\alpha)\ B_2$ and $A_{c_1} = \alpha\ B_1 + (1-\alpha)\ {B_2}^{-1}$ are both Hurwitz,* $\forall\alpha \in [0\ 1]$.

As shown in Fig. (5.1), the system in Example 5.1.1 is globally asymptotically stable. However, by applying Theorem 5.1.1, it is easy to see that, there exist no common quadratic Lyapunov function for the system (Fig. (5.2) plots the eigenvalue-locus of the pencils A_c and A_{c_1} as a function of α). So the CQ Lyapunov approach has failed in this case. The Nyquist plot in Fig. (5.3) of $G(s)$ also shows that circle criterion [66] cannot be applied since $Re[G(jw)] > -1$.

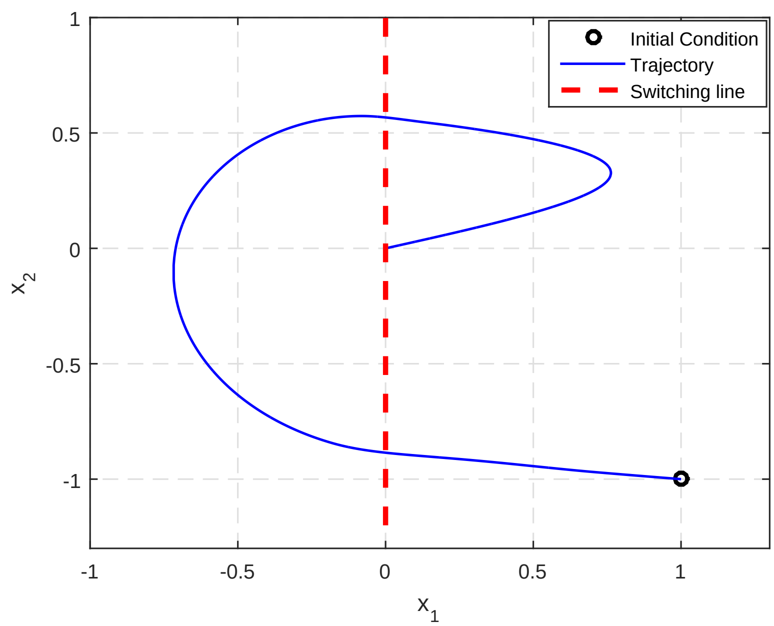


Figure 5.1: Phase portrait of Example 5.1.1 with $\mathbf{b}^T = [\,1\ \ -1\,]$

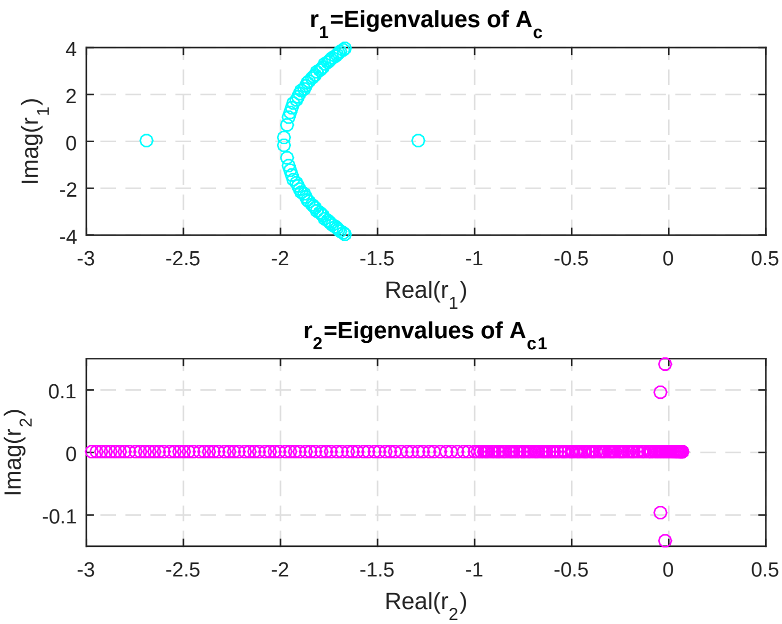


Figure 5.2: Eigenvalue-locus of the pencil as a function of α

5.2 Stability Analysis of Conewise Linear Systems

The above example has shown that, even for planar piecewise linear systems, stability analysis may not be possible with Lyapunov based techniques. Motivated by this, we will use the results obtained in previous chapters to derive a new set of necessary and sufficient conditions to check the stability of a planar piecewise linear system. Any planar piecewise linear system is shown to be globally asymptotically stable just in case each linear mode satisfies certain conditions that only depend on how its eigenvectors stand relative to the cone on which it is defined. The condition is in terms of the eigenvalues, eigenvectors, and the cones. In addition, the known condition for bimodal piecewise linear systems is obtained as an easy corollary of our main result.

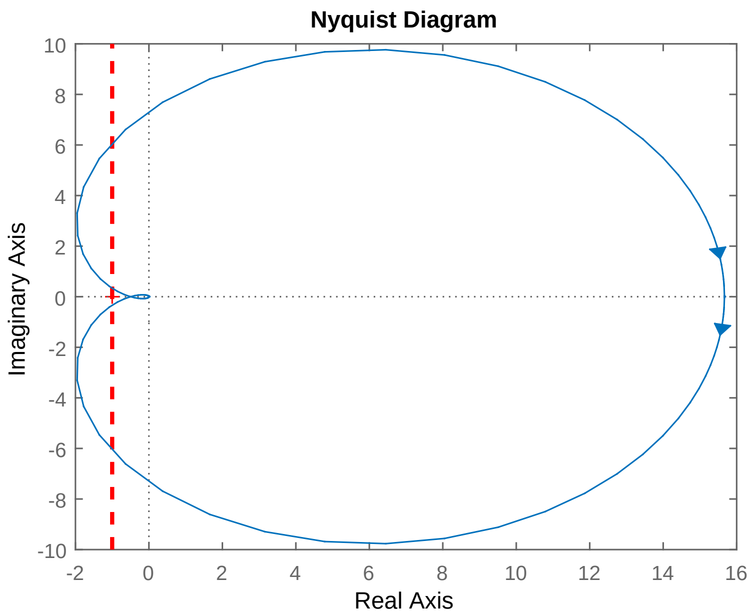


Figure 5.3: Circle criterion: Nyquist plot of G(s)

5.2.1 Factor of Expansion (FEX) and Regime Time

We now present a measure of closeness to the origin $\mathbf{x}_{eq} = 0$ of a trajectory when it hits one border B_{i1} or B_{i2} of the cone.

Definition 5.2.1. If a mode i is transitive, then its *factor of expansion (FEX)* is

$$F_i := \begin{cases} \ln \frac{|\mathbf{v}^i \times \mathbf{s}_{i1}|}{|\mathbf{v}^i \times \mathbf{s}_{i2}|} + \mu_i t_{i2} & \textit{if it is positive-transitive,} \\ \ln \frac{|\mathbf{v}^i \times \mathbf{s}_{i2}|}{|\mathbf{v}^i \times \mathbf{s}_{i1}|} + \mu_i t_{i1} & \textit{if it is negative-transitive,} \end{cases}$$

where μ_i denote λ_{i1} or σ_i, $\mathbf{v}^i$ is $\mathbf{v}_{i2}$ or $\mathbf{v}_{i1} + j\mathbf{v}_{i2}$ in case of real or non-real eigenvalues, respectively and; $t_{i1}(t_{i2})$ is called *the regime time*. That is, the time it takes the trajectory to move from $\mathbf{s}_{i2}$ to $\mathbf{s}_{i1}$ ($\mathbf{s}_{i1}$ to $\mathbf{s}_{i2}$).

In view of Fact 3.1.3, the factor of expansion is the natural logarithm of the gain $\frac{|\mathbf{x}|}{|\mathbf{s}_k|}$ a trajectory goes through when it starts at a border B_l and traverses the whole sector hitting the other border B_k.

Let log denote the complex principal logarithm, where $\log z = \ln |z| + j\angle z$ with

$-\pi < \angle z \leq \pi$.

Fact 5.2.1. *Let, for $i = 1, ..., m$,*

$$E_i := \frac{\lambda_{i1}}{\lambda_{i1} - \lambda_{i2}} \log \frac{(\hat{\mathbf{v}}^i \times \mathbf{s}_{i1}) \cdot \mathbf{k}}{(\hat{\mathbf{v}}^i \times \mathbf{s}_{i2}) \cdot \mathbf{k}} - \frac{\lambda_{i2}}{\lambda_{i1} - \lambda_{i2}} \log \frac{(\mathbf{v}^{\mathbf{i}} \times \mathbf{s}_{i1}) \cdot \mathbf{k}}{(\mathbf{v}^{\mathbf{i}} \times \mathbf{s}_{i2}) \cdot \mathbf{k}},$$

where $\mathbf{v}^i$ is $\mathbf{v}_{i2}$ or $\mathbf{v}_{i1} + j\mathbf{v}_{i2}$ and $\hat{\mathbf{v}}^i$ is $\mathbf{v}_{i1}$ or $\mathbf{v}_{i1} - j\mathbf{v}_{i2}$ in case of real or non-real eigenvalues, respectively, and the right hand side is computed as $\lim(\lambda_{i1} - \lambda_{i2}) \to 0$ in case of repeated eigenvalues. Then, $F_i = E_i$ when mode i is positive-transitive and $F_i = -E_i$, when negative-transitive.

Proof. We omit 'i' whenever it is clear from the context. Let us consider the negative-transitive case.

Real and distinct eigenvalues:

$$\begin{aligned} F_i &= \frac{\lambda_2}{\lambda_1 - \lambda_2} \log \frac{(\mathbf{v}_2 \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v}_2 \times \mathbf{s}_2) \cdot \mathbf{k}} - \frac{\lambda_1}{\lambda_1 - \lambda_2} \log \frac{(\mathbf{v}_1 \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v}_1 \times \mathbf{s}_2) \cdot \mathbf{k}} \\ &= \frac{\lambda_2}{\lambda_1 - \lambda_2} \ln \frac{|\mathbf{v}_2 \times \mathbf{s}_1|}{|\mathbf{v}_2 \times \mathbf{s}_2|} - \frac{\lambda_1}{\lambda_1 - \lambda_2} \ln \frac{|\mathbf{v}_1 \times \mathbf{s}_1|}{|\mathbf{v}_1 \times \mathbf{s}_2|} \\ &= \ln \frac{|\mathbf{v}_2 \times \mathbf{s}_2|}{|\mathbf{v}_2 \times \mathbf{s}_1|} + \frac{\lambda_1}{\lambda_1 - \lambda_2} \ln \frac{|\mathbf{v}_1 \times \mathbf{s}_2||\mathbf{v}_2 \times \mathbf{s}_1|}{|\mathbf{v}_1 \times \mathbf{s}_1||\mathbf{v}_2 \times \mathbf{s}_2|} \\ &= \ln \frac{|\mathbf{v}_2 \times \mathbf{s}_2|}{|\mathbf{v}_2 \times \mathbf{s}_1|} + \lambda_1 t_1, \end{aligned} \tag{5.1}$$

where the first equality is by $\mathbf{v} = \mathbf{v}_2, \hat{\mathbf{v}} = \mathbf{v}_1$. The second equation follows by noting, for any transitive mode, that

$$z_l := \frac{(\mathbf{v}_l \times \mathbf{s}_1) \cdot \mathbf{k}}{(\mathbf{v}_l \times \mathbf{s}_2) \cdot \mathbf{k}} > 0 \; for \; l = 1, 2,$$

since eigenvectors are exterior to $\mathbf{S}$ and that $\log z_i = \ln z_i$ for any real, positive z_i. The last equality follows by the expression for t_1 in (3.12) since $\mathbf{b} = \mathbf{s}_2$ and $(\mathbf{c}_2^T \mathbf{v}_2)\mathbf{k} = -\mathbf{v}_2 \times \mathbf{s}_1$, $\det V \, (\mathbf{w}_1^T \mathbf{s}_2)\mathbf{k} = -\mathbf{v}_2 \times \mathbf{s}_2$, $\det V \, (\mathbf{w}_2^T \mathbf{s}_2)\mathbf{k} = \mathbf{v}_1 \times \mathbf{s}_2$.

Real and repeated eigenvalues: To derive (3.16) as the limit of $F_i = -E_i$

in the case of repeated eigenvalues, consider the third expression in (5.1) with $\delta := \lambda_1 - \lambda_2$. By (3.12),

$$\lim_{\delta\to 0}\frac{\lambda_1}{\delta}\ln\frac{|\mathbf{v}_1\times\mathbf{s}_2||\mathbf{v}_2\times\mathbf{s}_1|}{|\mathbf{v}_1\times\mathbf{s}_1||\mathbf{v}_2\times\mathbf{s}_2|} = \lim_{\delta\to 0}\frac{\lambda_1}{\delta}\ln e^{\delta t_1} = \lambda_1 t_1.$$

Thus,

$$\lim_{\delta\to 0} F_i = \ln\frac{|\mathbf{v}_2\times\mathbf{s}_2|}{|\mathbf{v}_2\times\mathbf{s}_1|} + \lambda_1 t_1,$$

where t_1, *we interpret*, is given by (3.15) with $\mathbf{b} = \mathbf{s}_2$.

Non-real eigenvalues: Finally, suppose the eigenvalues are non-real with $\lambda_1 = \sigma + j\omega = \bar{\lambda_2}$. With $\mathbf{v} = \mathbf{v}_1 + j\mathbf{v}_2$ and $\hat{\mathbf{v}} = \mathbf{v}_1 - j\mathbf{v}_2$, let us first note that

$$\log\frac{(\mathbf{v}\times\mathbf{s}_1)\cdot\mathbf{k}}{(\mathbf{v}\times\mathbf{s}_2)\cdot\mathbf{k}} = \ln\frac{|\mathbf{v}\times\mathbf{s}_1|}{|\mathbf{v}\times\mathbf{s}_2|} + j\theta, \tag{5.2}$$

$$\log\frac{(\hat{\mathbf{v}}\times\mathbf{s}_1)\cdot\mathbf{k}}{(\hat{\mathbf{v}}\times\mathbf{s}_2)\cdot\mathbf{k}} = \ln\frac{|\mathbf{v}\times\mathbf{s}_1|}{|\mathbf{v}\times\mathbf{s}_2|} - j\theta, \tag{5.3}$$

where

$$\begin{aligned}
\theta \;&:=\; \angle\frac{(\mathbf{v}\times\mathbf{s}_1)\cdot\mathbf{k}}{(\mathbf{v}\times\mathbf{s}_2)\cdot\mathbf{k}} = \angle\frac{(\mathbf{v_1}\times\mathbf{s_1}+\mathbf{jv_2}\times\mathbf{s}_1)\cdot\mathbf{k}}{(\mathbf{v_1}\times\mathbf{s_2}+\mathbf{jv_2}\times\mathbf{s}_2)\cdot\mathbf{k}} \\
&=\; \angle\frac{-\mathbf{c}_2^T\mathbf{v}_1 - j\mathbf{c}_2^T\mathbf{v}_2}{\mathbf{c}_1^T\mathbf{v}_1 + j\mathbf{c}_1^T\mathbf{v}_2} \\
&=\; \arctan\{\frac{(\mathbf{c}_2^T\mathbf{v}_2)(\mathbf{c}_1^T\mathbf{v}_1) - (\mathbf{c}_2^T\mathbf{v}_1)(\mathbf{c}_1^T\mathbf{v}_2)}{(\mathbf{c}_2^T\mathbf{v}_1)(\mathbf{c}_1^T\mathbf{v}_1) + (\mathbf{c}_2^T\mathbf{v}_2)(\mathbf{c}_1^T\mathbf{v}_2)}\} = \omega t_1,
\end{aligned}\tag{5.4}$$

where the last equality follows upon setting $\mathbf{b} = \mathbf{s}_2$ in (3.21) and noting that

$\mathbf{c}_1^T\mathbf{v}_2 = -\frac{\det V}{\det S}\mathbf{w}_1^T\mathbf{s}_2$, $\mathbf{c}_1^T\mathbf{v}_1 = \frac{\det V}{\det S}\mathbf{w}_2^T\mathbf{s}_2$. Therefore,

$$\begin{aligned}
F_i &= \frac{\lambda_2}{\lambda_1-\lambda_2}\log\frac{(\mathbf{v}\times\mathbf{s}_1)\cdot\mathbf{k}}{(\mathbf{v}\times\mathbf{s}_2)\cdot\mathbf{k}} - \frac{\lambda_1}{\lambda_1-\lambda_2}\log\frac{(\hat{\mathbf{v}}\times\mathbf{s}_1)\cdot\mathbf{k}}{(\hat{\mathbf{v}}\times\mathbf{s}_2)\cdot\mathbf{k}} \\
&= \frac{\sigma-j\omega}{2j\omega}(\ln\frac{|\mathbf{v}\times\mathbf{s}_1|}{|\mathbf{v}\times\mathbf{s}_2|}+j\theta) - \frac{\sigma+j\omega}{2j\omega}(\ln\frac{|\mathbf{v}\times\mathbf{s}_1|}{|\mathbf{v}\times\mathbf{s}_2|}-j\theta) \\
&= -\ln\frac{|\mathbf{v}\times\mathbf{s}_1|}{|\mathbf{v}\times\mathbf{s}_2|}+\frac{\sigma}{\omega}\theta = \ln\frac{|\mathbf{v}\times\mathbf{s}_2|}{|\mathbf{v}\times\mathbf{s}_1|}+\sigma t_1.
\end{aligned}$$

It follows that F_i is as in Definition 5.2.1 for the case of non-real eigenvalues as well. The expression for positive-transitive case is similarly derived. ∎

Admittedly, the limit argument in case of repeated eigenvalues is heuristic and needs to be done more rigorously. Nevertheless, the mere fact that the factor of expansion in all three cases can be expressed by a single formula is very appealing.

It should be noted that, (3.12), (3.15) and (5.4) provide an explicit expression for the regime time

$$t_{i1} = \begin{cases} \dfrac{1}{\lambda_{i1}-\lambda_{i2}}\ln\dfrac{|(\mathbf{c}_{i2}^T\mathbf{v}_{i2})(\mathbf{w}_{i2}^T\mathbf{s}_{i2})|}{|(\mathbf{c}_{i2}^T\mathbf{v}_{i1})(\mathbf{w}_{i1}^T\mathbf{s}_{i2})|} & \textit{if eigenvalues are real and distinct,} \\ \dfrac{|\det V_i \det S_i\, \mathbf{c}_{i2}^T\mathbf{s}_{i2}|}{|\mathbf{v}_{i2}\times\mathbf{s}_{i2}\cdot\mathbf{v}_{i2}\times\mathbf{s}_{i1}|} & \textit{if eigenvalues are real and repeated,} \\ \dfrac{\theta_i}{w_i} & \textit{if eigenvalues are non-real} \end{cases} \tag{5.5}$$

where,

$$\theta_i := \angle\frac{(\mathbf{v}^i\times\mathbf{s}_{i1})\cdot\mathbf{k}}{(\mathbf{v}^i\times\mathbf{s}_{i2})\cdot\mathbf{k}}. \tag{5.6}$$

Analogously,

$$t_{i2} = \begin{cases} \dfrac{1}{\lambda_{i1}-\lambda_{i2}}\ln\dfrac{|(\mathbf{c}_{i1}^T\mathbf{v}_{i2})(\mathbf{w}_{i2}^T\mathbf{s}_{i1})|}{|(\mathbf{c}_{i1}^T\mathbf{v}_{i1})(\mathbf{w}_{i1}^T\mathbf{s}_{i1})|} & \textit{if eigenvalues are real and distinct,} \\ \dfrac{|\det V_i \det S_i\, \mathbf{c}_{i1}^T\mathbf{s}_{i1}|}{|\mathbf{v}_{i2}\times\mathbf{s}_{i1}\cdot\mathbf{v}_{i2}\times\mathbf{s}_{i2}|} & \textit{if eigenvalues are real and repeated,} \\ \dfrac{\theta_i}{w_i} & \textit{if eigenvalues are non-real} \end{cases} \tag{5.7}$$

where,

$$\theta_i := \angle \frac{(\mathbf{v}^i \times \mathbf{s}_{i2}) \cdot \mathbf{k}}{(\mathbf{v}^i \times \mathbf{s}_{i1}) \cdot \mathbf{k}}. \tag{5.8}$$

5.2.2 Stability Criteria

We can now state and prove a geometric condition for stability of a conewise linear system (2.1). The condition is "geometric" since a mode being transitive, source, or (half-)sink is characterized solely in terms of the eigenvectors associated with the mode and how they stand in the phase-plane relative to the sector on which the mode is defined.

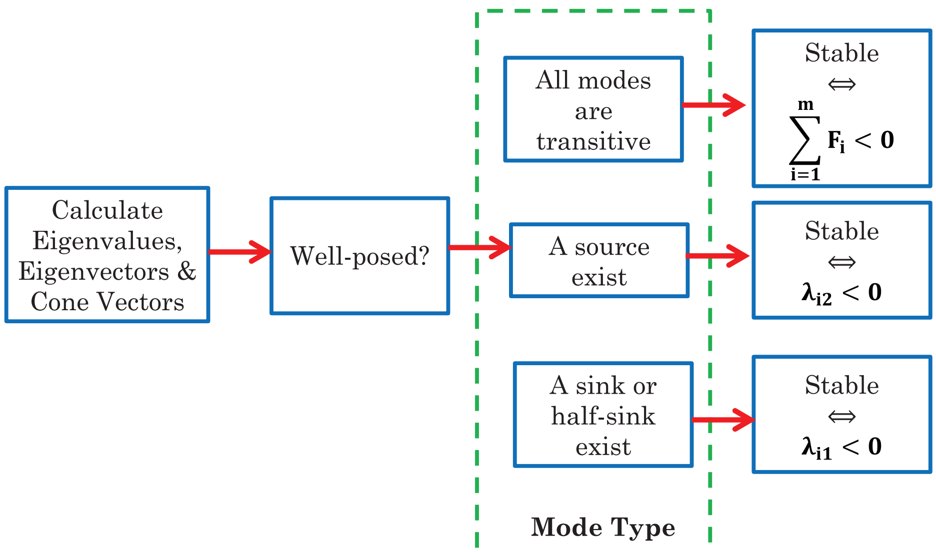


Figure 5.4: Condition for stability and the proposed method for stability check

Theorem 5.2.1. *A well-posed conewise linear system (2.1) is globally asymptotically stable if and only if*

$$\begin{aligned}
&F = \sum_{i=1}^{m} F_i < 0, \ \textit{when all modes } i = 1, ..., m \textit{ are transitive},\\
&\lambda_{i2} < 0, \ \textit{when a source } i \textit{ exists},\\
&\lambda_{i1} < 0, \ \textit{when a sink or half-sink } i \textit{ exists}.
\end{aligned}$$

Proof. Suppose all modes are transitive. Then, by well-posedness, all are positive or all are negative transitive, since otherwise a solution will not exist (or the solution will not be unqiue). In either case, for any $\mathbf{b} \in \mathbf{R}^2$, we must have $\mathbf{x}(t(\mathbf{b}), \mathbf{b}) = \gamma(\mathbf{b})\mathbf{b}$ for some $t(\mathbf{b}) > 0$ and $\gamma(\mathbf{b}) > 0$, i.e., the trajectory comes back to the ray passing through $\mathbf{b}$ after going through an expansion or contraction of size $\gamma(\mathbf{b})$ (see Fig. (5.5)). We might as well consider the case $\mathbf{b} = \mathbf{s}_{12}$, which is taking the initial state to be on the left-hand border of the first mode, without loss of generality. If the modes are negative transitive, then by (3.16) of Fact 3.1.3,

$$\gamma(\mathbf{s}_{12}) = \prod_{i=1}^{m} \frac{|\mathbf{v}^i \times \mathbf{s}_{i2}|}{|\mathbf{v}^i \times \mathbf{s}_{i1}|} e^{\mu_i t_{i1}},$$

where μ_i denotes λ_{i1} or σ_i and $\mathbf{v}^i$ denotes $\mathbf{v}_{i2}$ or $\mathbf{v}_{i1} + j\mathbf{v}_{i2}$ in the cases of mode i having real or non-real eigenvalues. The system is globally asymptotically stable if and only if $\gamma(\mathbf{s}_{12}) < 1$, which is equivalent to $F < 0$ in view of $\ln[\gamma(\mathbf{s}_{12})] < 0$ and the expressions for t_{i1}. Note that if all modes are positive transitive, then starting the trajectory at $\mathbf{b} = \mathbf{s}_{11}$, we have $\gamma(\mathbf{s}_{11}) = 1/\gamma(\mathbf{s}_{12})$, and the same condition is again obtained. Note that F can be thought of as the "Eigenvalue of the conewise linear system" and, the trajectory converges to zero faster as F gets smaller.

Suppose now that not all modes are transitive and a mode i with $\mathbf{n} = \mathbf{v}_{i2}$ is a source. In this case there must be a mode k that is a sink or half-sink since otherwise a sliding mode or chattering would exist. If the system is stable, then $\lambda_{i2} < 0$ must clearly hold for trajectories starting along that eigenvector to converge. Conversely, $\lambda_{i2} < 0$ implies that trajectories starting in mode i converge to the origin if they start along $\mathbf{v}_{i2}$ or they go outside $\mathbf{S}_i$ and enter mode k. For such trajectories to converge to the origin, it is necessary that $\lambda_{k1}, \lambda_{k2} < 0$. The necessity of $\lambda_{k1}, \lambda_{k2} < 0$ for any sink or half-sink mode k is also clear by considering trajectories that start inside the sink-sector of k. Conversely, the sufficiency of the condition follows by the fact that any trajectory starting in a transitive mode or a non-sink sector of a half-sink must end up in either a sink or in the sink sector of a half-sink. ■

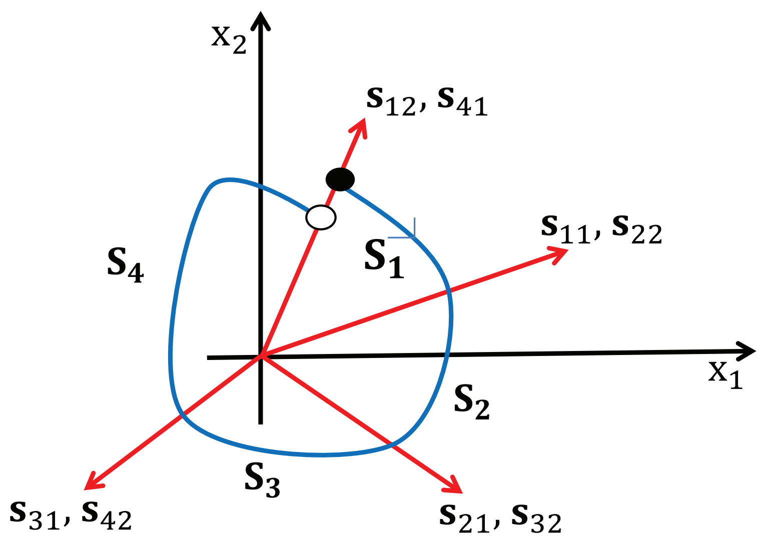


Figure 5.5: Contraction (Expansion) size; $\mathbf{b} : \bullet$, $\mathbf{x}(t(\mathbf{b}), \mathbf{b}) = \gamma(\mathbf{b})\mathbf{b} : \circ$

In summary, stability of a planar conewise linear system can be checked by the steps shown in Fig. (5.4). Firstly, the eigenvalues, eigenvectors and cone vectors are calculated. Then, well-posedness can be checked using Theorem 4.1.1. Finally, the stability criteria in Theorem 5.2.1 can be applied to check for stability.

The following result has been first obtained by [56] and then extended to (2.2) by [50].

Corollary 5.2.1. *A well-posed bimodal system (2.2) is globally asymptotically stable if and only if*

$$\begin{array}{l} \frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} < 0, \ \textit{when both modes have non-real eigenvalues,} \\ \lambda_{i1} < 0, \ \textit{when a mode, say } i\textit{, has real eigenvalues } \lambda_{i1} \geq \lambda_{i2}. \end{array}$$

Proof. In order to be able to apply Theorem 5.2.1, we let $\mathbf{c}_0$ be any vector that is not perpendicular to any of the real eigenvectors of B_1 and B_2, if any, and such that $d := \det[\mathbf{c}\ \mathbf{c}_0]$ is positive. Let $C_1^T := [\mathbf{c}\ \mathbf{c}_0]$. Note that $S_1 := C_1^{-1}$ satisfies $\det S_1 > 0$ and neither columns are in the direction of the real eigenvectors of B_1 or B_2, if any. The four modal system $A_1 = A_2 = B_1$, $A_3 = A_4 = B_2$, $C_2^T := [-\mathbf{c}_0 \quad \mathbf{c}]$, $C_3 = -C_1$, $C_4 = -C_2$, is in the framework of (2.1), and is equivalent to (2.2). Since by assumption, the bimodal system (2.2) is well-posed, the above choice of $\mathbf{c}_0$ ensures that the four-modal system is also well-posed.

Suppose, first that, say, B_1 has real eigenvalues so that modes 1 and 2 both have those eigenvalues with the same corresponding eigenvector(s). If, say, mode 1 is a source, then eigenvalues must be distinct and $\mathbf{v}_1$ of mode 1 must be exterior. This implies, since mode 2 complements 1 in a half plane, that $\mathbf{v}_1$ is interior to mode 2 and $\mathbf{v}_2$ is exterior, that is mode 2 is a sink. By Theorem 1, the system is stable if and only if both eigenvalues of B_1 are negative. If mode 1 is transitive, then both eigenvectors are exterior to mode 1. Thus, the eigenvector(s) are interior to mode 2 so that mode 2 is a half-sink, which again implies that eigenvalues being negative is necessary and sufficient for stability. The other possibilities for mode 1 clearly give the same result.

Suppose, second, that both B_1 and B_2 have non-real eigenvalues. It must be that all four modes are transitive in the same direction, say negative, by $A_1 = A_2 = B_1$, $A_3 = A_4 = B_2$ and by well-posedness of (2.2). It is easy to compute, by Fact 5.2.1 and by the expression in (5.5), that

$$F_1 + F_2 = \frac{\sigma_1}{\omega_1}\pi, F_3 + F_4 = \frac{\sigma_2}{\omega_2}\pi,$$

which implies by Theorem 5.2.1 that the four-modal system, and therefore (2.2), is stable if and only if $\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} < 0$. ■

5.3 Examples

In this section, we will apply the proposed method to the stability analysis of a three-modal system and a selector control system.

5.3.1 Three-modal System

Example 5.3.1. Consider the three-modal system with free parameters γ_1, γ_2 $\in \mathbf{R}$ and with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -80 & 2\gamma_1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -15 & 4 \end{bmatrix},\ A_3 = \begin{bmatrix} \frac{\gamma_2}{2} - 7.5 & \frac{-7.5}{\gamma_2} - 0.5 \\ -\frac{\gamma_2{}^2}{2} - 7.5\gamma_2 & \frac{\gamma_2}{2} - 7.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},\ C_3 = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}.$$

For each mode (A_i, C_i), the eigenvalues, eigenvectors and cone vectors can be calculated as follows.

Mode 1: $\lambda_{11} = \gamma_1 + j\sqrt{\gamma_1{}^2 - 20}$, $\lambda_{12} = \gamma_1 - j\sqrt{\gamma_1{}^2 - 20}$, $\mathbf{v}_{11} = \begin{bmatrix} \frac{\gamma_1}{20} \\ 1 \end{bmatrix}$, $\mathbf{v}_{12} = \begin{bmatrix} \frac{\sqrt{\gamma_1{}^2-20}}{20} \\ 0 \end{bmatrix}$, $\mathbf{s}_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{s}_{12} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Mode 2: $\lambda_{21} = 2 + j3.3166$, $\lambda_{22} = 2 - j3.3166$, $\mathbf{v}_{21} = \begin{bmatrix} 0.1291 \\ 0.9682 \end{bmatrix}$, $\mathbf{v}_{22} = \begin{bmatrix} 0.2141 \\ 0 \end{bmatrix}$, $\mathbf{s}_{21} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\mathbf{s}_{22} = \mathbf{s}_{11}$.

Mode 3: $\lambda_{31} = \gamma_2$, $\lambda_{32} = -15$, $\mathbf{v}_{31} = \begin{bmatrix} -1 \\ \gamma_2 \end{bmatrix}$, $\mathbf{v}_{32} = \begin{bmatrix} -1 \\ -\gamma_2 \end{bmatrix}$, $\mathbf{s}_{31} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{s}_{32} = \mathbf{s}_{21}$.

We consider four alternative values for γ_1 and γ_2, where $\gamma_1 = Re[\lambda_{11}]$ and

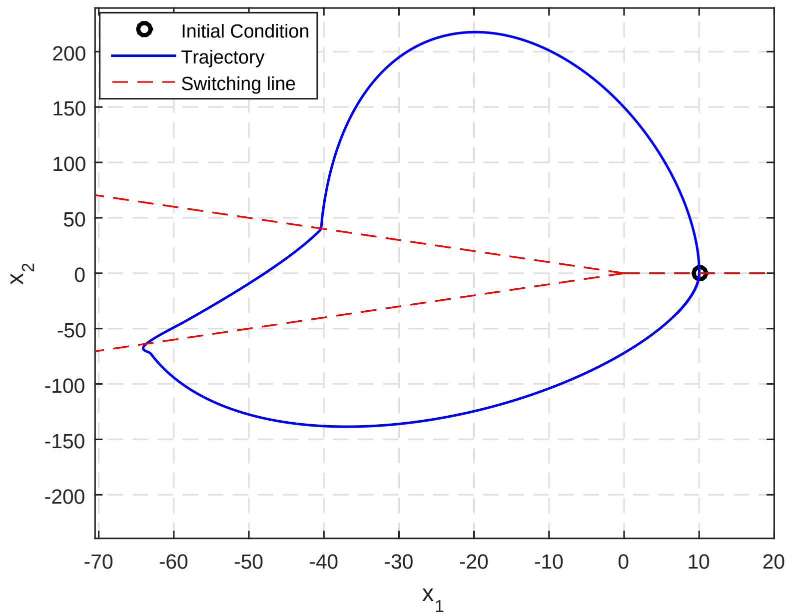


Figure 5.6: Phase portrait of the three-modal system for $\gamma_1 = -3.641$, $\gamma_2 = 3$, $\mathbf{b}^T = [\ 10\ 0\ ]$

$\gamma_2 = \lambda_{31}$.

Mode 3: Negative Transitive

Mode 3 is negative transitive for all $\gamma_2 > 1$ (i.e., $\mathbf{v}_{31}$ and $\mathbf{v}_{32}$ will be exterior for all $\gamma_2 > 1$).

i) With $\gamma_1 = -3.641$ and $\gamma_2 = 3$; $\det V_1 = -0.1009$ and $\det V_2 = -0.2073$, which implies that mode 1 and 2 are both negative transitive and, the system eigenvalue $F = \sum_{i=1}^{3} F_i \simeq 0$. By Theorem 5.2.1, the system is unstable and the trajectory forms a closed orbit as shown in Fig. (5.6).

ii) Suppose, next, that $\gamma_1 = -4$ and $\gamma_2 = 3$ so that $\det V_1 = -0.0988$, $\det V_2 = -0.2073$ and mode 1 and 2 are both negative transitive with $F = \sum_{i=1}^{3} F_i = -0.1708$. By Theorem 5.2.1, the system is GAS as Fig. (5.7) illustrates. It should be noted that, both mode 2 and 3 are unstable, however the conewise linear system is GAS. This is a well-known property of systems with

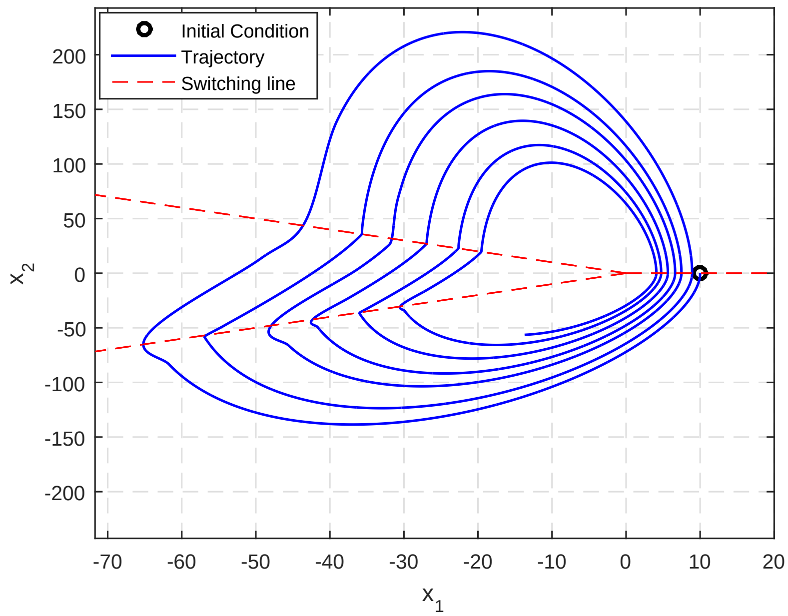


Figure 5.7: Phase portrait of the three-modal system for $\gamma_1 = -4$, $\gamma_2 = 3$, $\mathbf{b}^T = [\ 10\ 0\ ]$

switched dynamics [9], where switching between unstable modes results in a stable trajectory.

iii) Let $\gamma_1 = -3.5$ and $\gamma_2 = 3$. With $\det V_1 = -0.1016$ and $\det V_2 = -0.2073$, mode 1 and 2 are both negative transitive and $F = \sum_{i=1}^{3} F_i = 0.064$. By Theorem 5.2.1, the system is unstable as shown in Fig. (5.8).

We have seen that in all the three cases, by simply changing the parameter γ_1 of mode 1, the stability of the conewise linear system is affected. In all the three cases, mode 1 is stable, but depending on the choice of γ_1, stability or instability can be achieved.

Mode 3: Half-sink

Mode 3 is a half-sink for all $\gamma_2 < 1$ (i.e., $\mathbf{v}_{31}$ and $\mathbf{v}_{32}$ will be interior for all $\gamma_2 < 1$). However, we assume that, $\gamma_2 < 0$ to ensure well-posedness.

iv) If we choose $\gamma_1 = -4$ and $\gamma_2 = -0.5$ then $\det V_1 = -0.0988$ and $\det V_2 =$

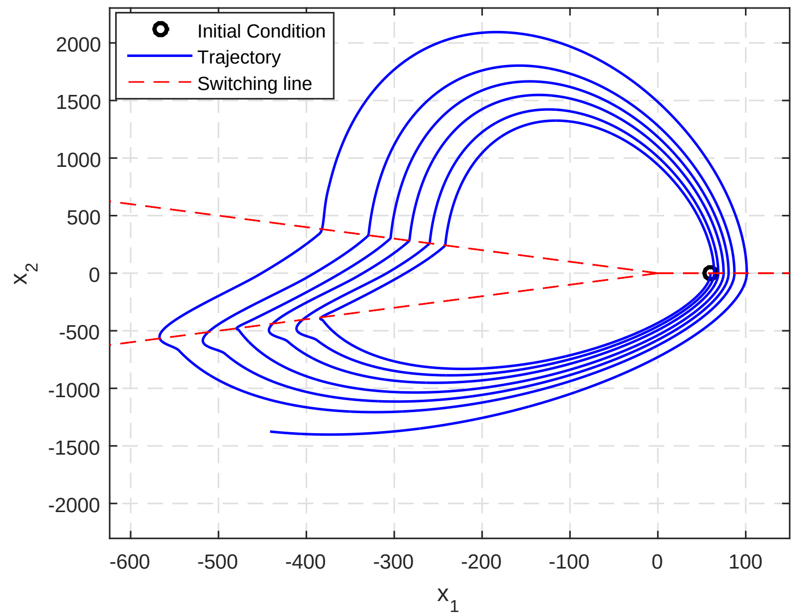


Figure 5.8: Phase portrait of the three-modal system for $\gamma_1 = -3.5$, $\gamma_2 = 3$, $\mathbf{b}^T = [\ 50\ 0\ ]$

-0.2073, and it follows that mode 1 and 2 are both negative transitive. Therefore, by Theorem 5.2.1, the system is GAS $\forall\ \gamma_2 < 0$ as shown in Fig. (5.9).

5.3.2 Selector Control System

Consider the system in Fig. (5.10) whose stability analysis was done in [12] using Lyapunov approach:

$$G\ :=\ \dot{\mathbf{x}} = A\mathbf{x} + Bu,\ u = \max\,(\mathbf{k}_1{}^T\mathbf{x}, \mathbf{k}_2{}^T\mathbf{x}), \tag{5.9}$$

where $x \in \mathbf{R}^2$, $A \in \mathbf{R}^{2\times 2}$, $B \in \mathbf{R}^2$ $\mathbf{k}_1 \in \mathbf{R}^2$, $\mathbf{k}_2 \in \mathbf{R}^2$.

The system in (5.9) is called a selector control system in [12]. Such systems appear in the control of longitudinal dynamics of an aircraft, where the control law is designed to track a command signal without exceeding a certain limit [49].

As simple as (5.9) may appear, stability analysis of the system is not trivial. For instance, how can we select $\mathbf{k}_1$ and $\mathbf{k}_2$ to achieve stability?

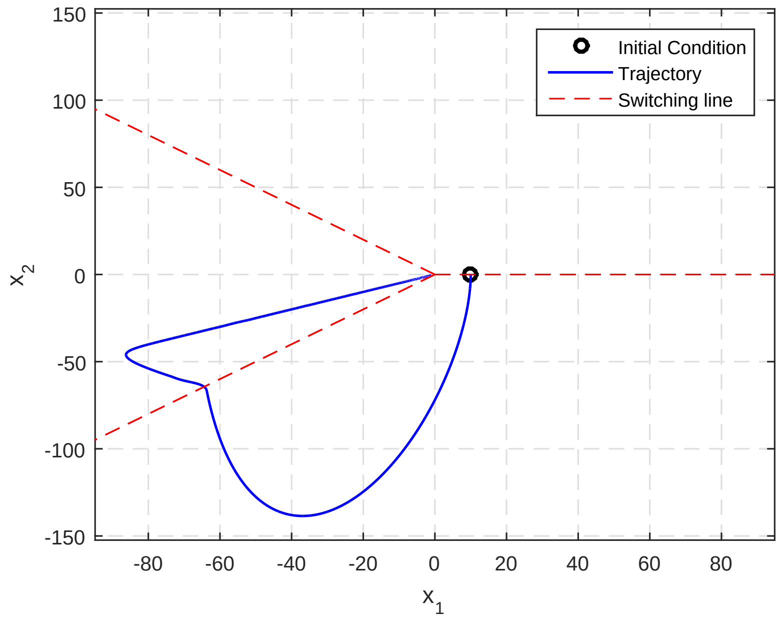


Figure 5.9: Phase portrait of the three-modal system for $\gamma_1 = -4$, $\gamma_2 = -0.5$, $\mathbf{b}^T = [\,10\ 0\,]$

Note that, (5.9) can be represented as a bimodal system

$$\dot{\mathbf{x}} = \begin{cases} B_1\mathbf{x} & if \quad \mathbf{k}^T\mathbf{x} \geq 0, \\ B_2\mathbf{x} & if \quad \mathbf{k}^T\mathbf{x} \leq 0, \end{cases} \tag{5.10}$$

with $\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2$, $B_1 = B_2 + B\mathbf{k}^T$ and $B_2 = A + B\mathbf{k}_2^T$.

As (5.10) has the same form as (2.2) with $\mathbf{c}^T = \mathbf{k}^T$, Corollary 5.2.1 can be used for the stability analysis of (5.9) as the following example shows.

Example 5.3.2. Consider (5.9), where $\mathbf{k}_1$ and $\mathbf{k}_2$ are chosen such that B_1 and B_2 have non-real eigenvalues:

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{k}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{k}_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

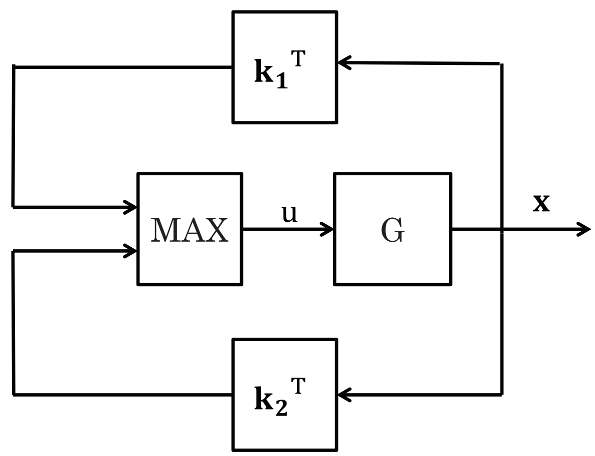


Figure 5.10: Selector control system

which results in (5.10) with

$$B_1 = \begin{bmatrix} 0.5 & 1 \\ -1 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} -0.5 & 1 \\ -1 & 0 \end{bmatrix}, \; \mathbf{k} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalues of B_1 and B_2 can be calculated as: $\lambda_{11} = -0.25 + j0.968$, $\lambda_{12} = -0.25 - j0.968$; $\lambda_{21} = 0.25 + j0.968$, $\lambda_{22} = 0.25 - j0.968$. Since $\det V_1 = \det V_2 = -0.4841$, both B_1 and B_2 are negative transitive and $\frac{\sigma_1}{w_1} + \frac{\sigma_2}{w_2} = \frac{0.25}{0.968} - \frac{0.25}{0.968} = 0$. By Corollary 5.2.1, the system is unstable and the trajectory forms a closed orbit as shown in Fig. (5.11). As seen in Fig. (5.12), this results in a sinusoidal-like motion of the states.

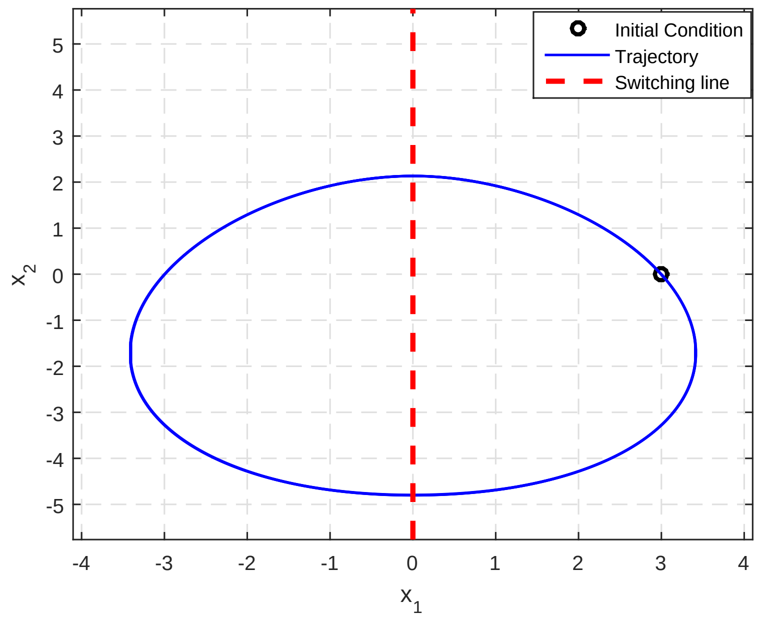


Figure 5.11: Phase portrait of selector control system with $\mathbf{b}^T = [\ 3\ 0\ ]$

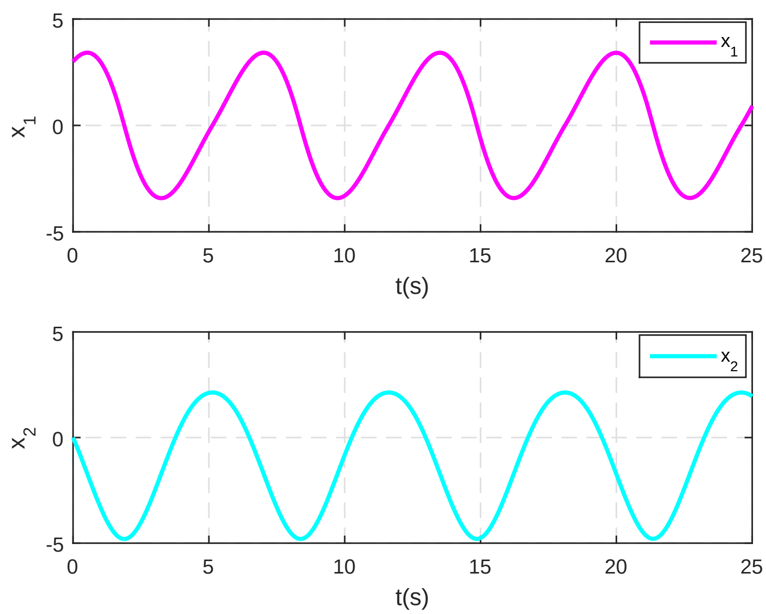


Figure 5.12: States of selector control system with $\mathbf{b}^T = [\ 3\ 0\ ]$

Chapter 6

Hybrid Output Feedback Controller Design for Planar Linear Systems

When all the states of a system are not available for measurement and when dynamic controllers are costly, then a static output feedback controller is desirable [67], [68]. On the other hand, not all LTI systems can be stabilized by a static output feedback controller. With recent developments in switched systems literature, design of hybrid output feedback controllers for LTI systems has attracted the attention of researchers. This problem (design of hybrid output feedback controllers for LTI systems) was raised in [69] and [70]. And [71] established the link between this problem and the problem of stabilization of switched linear systems. Under assumptions of controllability and observability, [72] proposed a discrete automaton for the stabilization of linear systems. A periodic hybrid output feedback controller based on averaging theory was proposed in [73], for stabilizing controllable and observable LTI systems. In [74], a multirate sampling scheme was proposed for the design of hybrid output feedback controllers for LTI systems with single output. The stability criterion in [26] was deployed in [75] to design a hybrid output feedback controller for stabilizing harmonic oscillating systems.

State-dependent switching rules have also been proposed to perform the same task. In [76], the conic switching rule proposed in [27] for stabilization of second order switched systems was applied to stabilize the linear time-invariant system below. A sliding mode control scheme was used in [77] for asymptotic stabilization of second order LTI systems.

In this chapter, the results obtained in previous chapters will be applied to design a hybrid output feedback controller for asymptotic stabilization of planar LTI systems. The effectiveness of the proposed controller will be illustrated on a magnetic levitation system.

6.1 A Hybrid Output Feedback Controller for a Second Order Linear System

Consider the second order unstable plant ($p_1 \geq 0$) with transfer function as follows:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{n_o}{s^2 - p_1 s - p_o}. \tag{6.1}$$

The system can be represented in the following form where, position is taken as the output

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ p_o & p_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ n_o \end{bmatrix} u, \; y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \tag{6.2}$$

Let us define the control law as

$$u = \begin{cases} \kappa_1 y & if \quad y \geq 0, \\ \kappa_2 y & if \quad y \leq 0. \end{cases} \tag{6.3}$$

Fact 6.1.1. *The system (6.1) cannot be stabilized by (6.3).*

Proof. The control law (6.3) transforms the system (6.2) to

$$\dot{\mathbf{x}} = \begin{cases} B_1\mathbf{x} & if \quad y \geq 0, \\ B_2\mathbf{x} & if \quad y \leq 0, \end{cases} \tag{6.4}$$

with $B_1 = A + B\kappa_1 C_p = \begin{bmatrix} 0 & 1 \\ q_1 & p_1 \end{bmatrix}, B_2 = A + B\kappa_2 C_p = \begin{bmatrix} 0 & 1 \\ q_2 & p_1 \end{bmatrix}, q_1 = p_o + \kappa_1 n_o$ and $q_2 = p_o + \kappa_2 n_o$.

The eigenvalues of B_1 and B_2 can be calculated as

$$B_1 : \lambda_{11} = \frac{p_1 + \sqrt{p_1^2 + 4q_1}}{2}, \lambda_{12} = \frac{p_1 - \sqrt{p_1^2 + 4q_1}}{2}.$$

$$B_2 : \lambda_{21} = \frac{p_1 + \sqrt{p_1^2 + 4q_2}}{2}, \lambda_{22} = \frac{p_1 - \sqrt{p_1^2 + 4q_2}}{2}.$$

It is easy to see that, (6.4) represents a bimodal piecewise linear system with $\mathbf{c}^T = C_p$. Suppose, first that, say, q_1 (resp., q_2) is such that B_1 (resp., B_2) has real eigenvalues. This implies that $\lambda_{11} > 0$ (resp., $\lambda_{21} > 0$), then by Corollary 5.2.1, stability cannot be achieved.

Suppose, second, that q_1 and q_2 are such that B_1 and B_2 have non-real eigenvalues. This will imply that, $\sigma_1 = \sigma_2 = \frac{p_1}{2}$. So $\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} < 0$ cannot be satisfied (Corollary 5.2.1) where $\omega_1 = \frac{\sqrt{|p_1^2+4q_1|}}{2}, \omega_2 = \frac{\sqrt{|p_1^2+4q_2|}}{2}$ ■

Remark 6.1.1. It should be noted that, since $p_1 \geq 0$, the system cannot be stabilized by a static output feedback controller (i.e., $\kappa_1 = \kappa_2$).

We can thus ask, the question of whether it can be stabilized by a feedback control law of the form

$$u = \begin{cases} \kappa_1 y & if \quad \mathbf{x}_1\mathbf{x}_2 \geq 0, \\ \kappa_2 y & if \quad \mathbf{x}_1\mathbf{x}_2 \leq 0. \end{cases} \tag{6.5}$$

This question is now shown to have an affirmative answer. Using the control law in (6.5), the system will be transformed into a conewise linear system. In Section 6.1.1, we will characterize the modes of the conewise linear system under different possible choices of κ_1 and κ_2. Our main result will be stated in Section 6.1.2.

The feedback law (6.5) transforms (6.2) into a bimodal quadratic piecewise linear system (2.4)

$$\dot{\mathbf{x}} = \begin{cases} B_1\mathbf{x} & if \quad \mathbf{x}_1\mathbf{x}_2 \geq 0, \\ B_2\mathbf{x} & if \quad \mathbf{x}_1\mathbf{x}_2 \leq 0, \end{cases} \tag{6.6}$$

with $B_1 = A + B\kappa_1 C_p = \begin{bmatrix} 0 & 1 \\ q_1 & p_1 \end{bmatrix}, B_2 = A + B\kappa_2 C_p = \begin{bmatrix} 0 & 1 \\ q_2 & p_1 \end{bmatrix}, q_1 = p_o + \kappa_1 n_o$ and $q_2 = p_o + \kappa_2 n_o$.

As we have shown in Section 2.3, the system in (6.6) can in turn be represented as a conewise linear system

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad C_1\mathbf{x} \geq 0, \\ A_2\mathbf{x} & if \quad C_2\mathbf{x} \geq 0, \\ A_3\mathbf{x} & if \quad C_3\mathbf{x} \geq 0, \\ A_4\mathbf{x} & if \quad C_4\mathbf{x} \geq 0, \end{cases} \tag{6.7}$$

with $A_1 = A_3 = B_1,\ A_2 = A_4 = B_2$ and $C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ C_3 = -C_1,\ C_4 = -C_2$.

6.1.1 Mode Type Detection

In what follows, we will characterize the modes for (6.7) under different possible choices of q_1 and q_2 (that is, κ_1 and κ_2). Recall that, $A_3 = A_1$, $A_4 = A_2$, $S_3 = -S_1$ and $S_4 = -S_2$. Therefore, the properties of mode 1 (resp., mode 2) in (6.7) is same with the one of mode 3 (resp., mode 4).

Mode 1

The eigenvalues of B_1 can be computed as $\lambda_{11} = \frac{p_1 + \sqrt{p_1^2 + 4q_1}}{2}$ and $\lambda_{12} = \frac{p_1 - \sqrt{p_1^2 + 4q_1}}{2}$. Clearly, q_1 can be selected such that the eigenvalues are real and distinct, real and repeated, or non-real.

Case 1a (Real and Distinct Eigenvalues): $q_1 > -\frac{p_1^2}{4}$

$q_1 > -\frac{p_1^2}{4}$ implies that B_1 has real and distinct eigenvalues ($\lambda_{11} > \lambda_{12}$) and two real eigenvectors:

$$\lambda_{11} = \frac{p_1 + \sqrt{|p_1^2 + 4q_1|}}{2}, \ \mathbf{v}_{11} = \begin{bmatrix} \frac{-p_1 + \sqrt{|p_1^2 + 4q_1|}}{2q_1} \\ 1 \end{bmatrix}.$$

$$\lambda_{12} = \frac{p_1 - \sqrt{|p_1^2 + 4q_1|}}{2}, \ \mathbf{v}_{12} = \begin{bmatrix} \frac{-p_1 - \sqrt{|p_1^2 + 4q_1|}}{2q_1} \\ 1 \end{bmatrix}.$$

From Fig. (6.1), for all $q_1 > 0$, $\mathbf{v}_{11}$ will be interior of mode 1 and $\mathbf{v}_{12}$ will be exterior. Also, as seen in Fig. (6.2), for all $-\frac{p_1^2}{4} < q_1 < 0$, $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$ will be interior. Therefore, $\forall q_1 > 0$ ($\forall -\frac{p_1^2}{4} < q_1 < 0$), mode 1 is a sink (half-sink).

Case 1b (Real and Repeated Eigenvalues): $q_1 = -\frac{p_1^2}{4}$ $(p_1 \neq 0)$

When $q_1 = -\frac{p_1^2}{4}$, the eigenvalues of B_1 will be real and repeated:

$$\lambda_{11} = \lambda_{12} = \frac{p_1}{2}, \ \mathbf{v}_{12} = \begin{bmatrix} 1 \\ \lambda_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{p_1}{2} \end{bmatrix}, \ \mathbf{v}_{11} = \begin{bmatrix} 1 \\ 1 + \lambda_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 + \frac{p_1}{2} \end{bmatrix}$$

where $\mathbf{v}_{12}$ and $\mathbf{v}_{11}$ denote the real and generalized eigenvector respectively. Since $p_1 > 0$, $\mathbf{v}_{12}$ is interior to mode 1 (Fig. 6.3). Therefore, mode 1 is a half-sink. Note that, when $p_1 = 0$, $\mathbf{v}_{12}$ lies on the border $\mathbf{s}_{22}$.

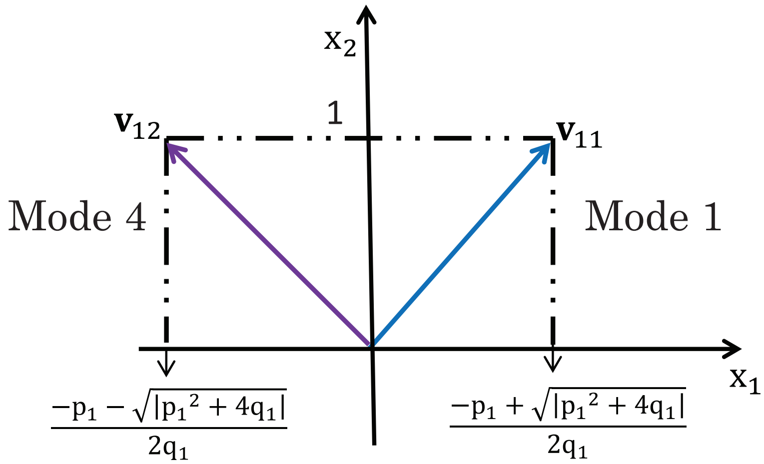


Figure 6.1: Eigenvectors of B_1 with $q_1 > 0$

Case 1c (Non-real Eigenvalues): $q_1 < -\frac{p_1^2}{4}$

Clearly, when $q_1 < -\frac{p_1^2}{4}$ the eigenvalues will be non-real. Therefore,

$$\lambda_{11} = \sigma_1 + jw_1,\ \lambda_{12} = \sigma_1 - jw_1,\ \mathbf{v}_{11} = \begin{bmatrix} -\frac{\sigma_1}{q_1} \\ 1 \end{bmatrix},\ \mathbf{v}_{12} = \begin{bmatrix} -\frac{w_1}{q_1} \\ 0 \end{bmatrix}$$

where $\sigma_1 = \frac{p_1}{2}$, $w_1 = \frac{\sqrt{|p_1^2+4q_1|}}{2}$ and $\mathbf{v}_{11} + j\mathbf{v}_{12}$ is the eigenvector associated with λ_{12}. This implies that, $V_1 = \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} \end{bmatrix}$ and $\det V_1 = \frac{w_1}{q_1}$. Since $q_1 < 0$, by (i) in Fact 3.1.2, mode 1 is negative transitive. By Fact 5.2.1 and by the expression in (5.5), the FEX of mode 1 can be calculated as

$$F_1 = \ln|\frac{\lambda_{11}}{q_1}| + \sigma_1 t_{11}, \tag{6.8}$$

where

$$t_{11} = \frac{\arctan[\frac{w_1}{-\sigma_1}]}{w_1}. \tag{6.9}$$

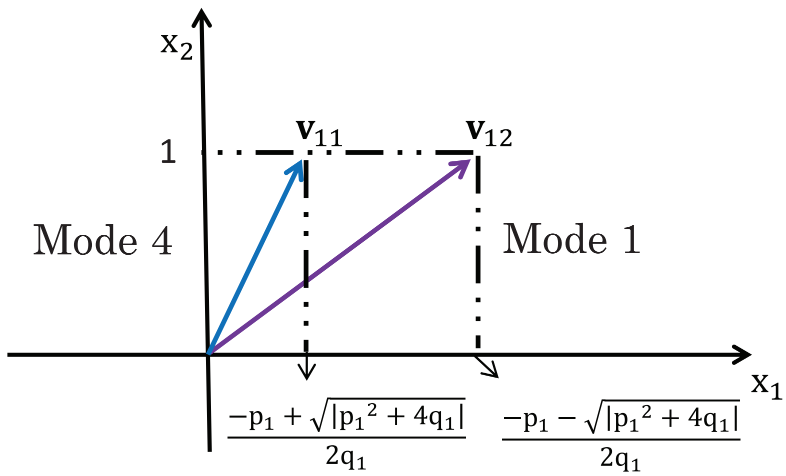


Figure 6.2: Eigenvectors of B_1 with $-\frac{p_1^2}{4} < q_1 < 0$

Mode 2

The eigenvalues of B_2 can be computed as $\lambda_{21} = \frac{p_1+\sqrt{p_1^2+4q_2}}{2}$ and $\lambda_{22} = \frac{p_1-\sqrt{p_1^2+4q_2}}{2}$. As in mode 1, q_2 can be selected such that the eigenvalues are real and distinct, real and repeated, or non-real.

Case 2a (Real and Distinct Eigenvalues): $q_2 > -\frac{p_1^2}{4}$

$q_2 > -\frac{p_1^2}{4}$ implies that B_2 has real and distinct eigenvalues ($\lambda_{21} > \lambda_{22}$) and two real eigenvectors:

$$\lambda_{21} = \frac{p_1 + \sqrt{|p_1^2 + 4q_2|}}{2}, \ \mathbf{v}_{21} = \begin{bmatrix} \frac{-p_1+\sqrt{|p_1^2+4q_2|}}{2q_2} \\ 1 \end{bmatrix}.$$

$$\lambda_{22} = \frac{p_1 - \sqrt{|p_1^2 + 4q_2|}}{2}, \ \mathbf{v}_{22} = \begin{bmatrix} \frac{-p_1-\sqrt{|p_1^2+4q_2|}}{2q_2} \\ 1 \end{bmatrix}.$$

From Fig. (6.4), for all $q_2 > 0$, $\mathbf{v}_{21}$ will be exterior of mode 2 and $\mathbf{v}_{22}$ will

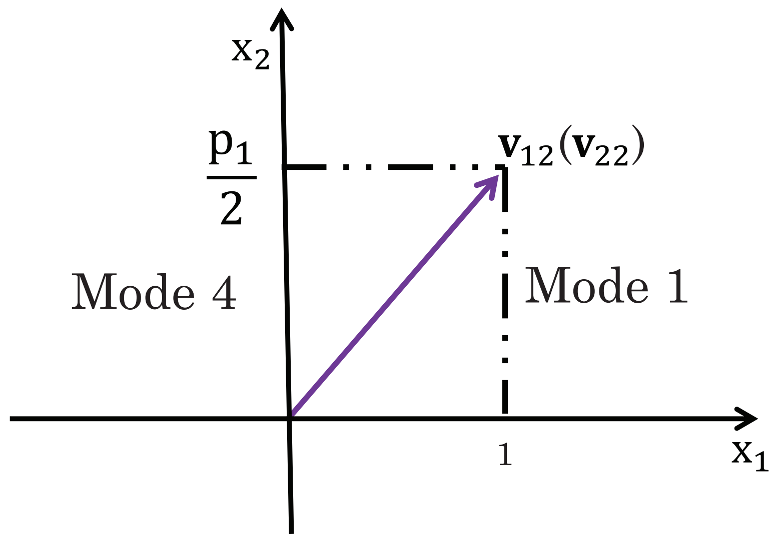


Figure 6.3: Eigenvector of B_1 (B_2) with $q_1 = -\frac{p_1^2}{4}$ ($q_2 = -\frac{p_1^2}{4}$)

be interior. Therefore, for all $q_2 > 0$, mode 2 is source. Also, as seen in Fig. (6.5), for all $-\frac{p_1^2}{4} < q_2 < 0$, both $\mathbf{v}_{21}$ and $\mathbf{v}_{22}$ will be exterior. Therefore, for all $-\frac{p_1^2}{4} < q_2 < 0$, mode 2 is transitive. This implies that, $V_2 = \begin{bmatrix} \mathbf{v}_{21} & \mathbf{v}_{22} \end{bmatrix}$ and $\det V_2 = \frac{\sqrt{|p_1^2+4q_2|}}{q_2}$. Since, $-\frac{p_1^2}{4} < q_2 < 0$, $\mathbf{v}_{21} \times \mathbf{v}_{22}$ is negatively oriented. By (i) in Fact 3.1.2, mode 2 is negative transitive for all $-\frac{p_1^2}{4} < q_2 < 0$. By Fact 5.2.1 and by the expression in (5.5), the FEX of mode 2 can be calculated as

$$F_2 = \ln|\frac{q_2}{\lambda_{21}}| + \lambda_{21}t_{21}, \tag{6.10}$$

where

$$t_{21} = \frac{1}{\lambda_{21} - \lambda_{22}} \ln|\frac{\lambda_{21}}{\lambda_{22}}|. \tag{6.11}$$

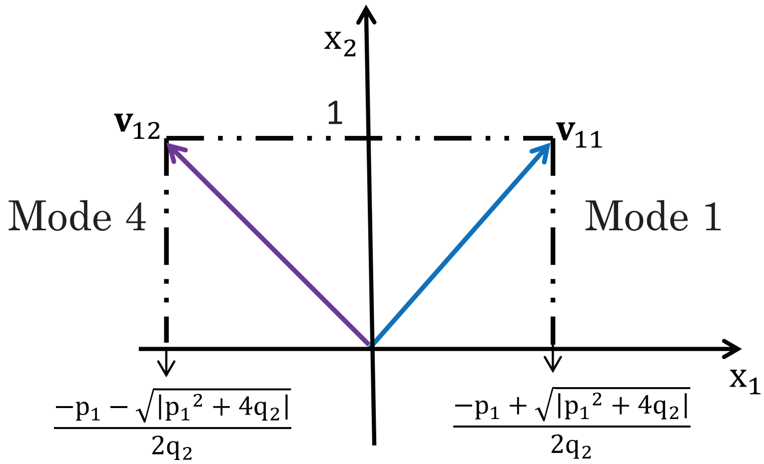


Figure 6.4: Eigenvectors of B_2 with $q_2 > 0$

Case 2b (Real and Repeated Eigenvalues): $q_2 = -\frac{p_1^2}{4}$ $(p_1 \neq 0)$

When $q_2 = -\frac{p_1^2}{4}$, the eigenvalues of B_2 will be real and repeated:

$$\lambda_{21} = \lambda_{22} = \frac{p_1}{2},\ \mathbf{v}_{22} = \begin{bmatrix} 1 \\ \lambda_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{p_1}{2} \end{bmatrix},\ \mathbf{v}_{21} = \begin{bmatrix} 1 \\ 1 + \lambda_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 + \frac{p_1}{2} \end{bmatrix},$$

where $\mathbf{v}_{22}$ and $\mathbf{v}_{21}$ denote the real and generalized eigenvector respectively. Since $p_1 > 0$, $\mathbf{v}_{22}$ is exterior to mode 2 (Fig. 6.3). Therefore, mode 2 is transitive where $V_2 = \begin{bmatrix} \mathbf{v}_{21} & \mathbf{v}_{22} \end{bmatrix}$, $\det V_2 = -1$ and $\mathbf{v}_{21} \times \mathbf{v}_{22}$ is negatively oriented because, $\det V_2 < 0$. By (i) in Fact 3.1.2, mode 2 is negative transitive and by Fact 5.2.1 and by the expression in (5.5), the FEX of mode 2 can be calculated as

$$F_2 = \ln|\frac{q_2}{\lambda_{21}}| + \lambda_{21} t_{21}, \tag{6.12}$$

where

$$t_{21} = |\frac{1}{\lambda_{21}}|. \tag{6.13}$$

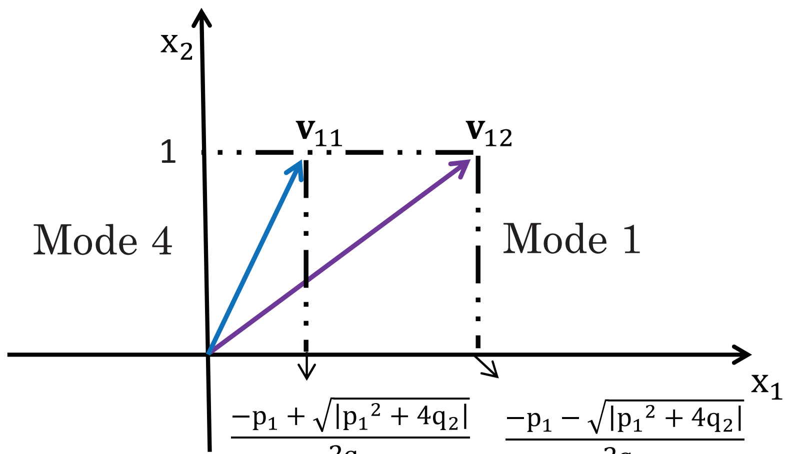


Figure 6.5: Eigenvectors of B_2 with $-\frac{p_1^2}{4} < q_2 < 0$

Note. When $p_1 = 0$, $\mathbf{v}_{22}$ lies on the border $\mathbf{s}_{22}$.

Case 2c (Non-real Eigenvalues): $q_2 < -\frac{p_1^2}{4}$

Clearly, when $q_2 < -\frac{p_1^2}{4}$ the eigenvalues will be non-real. Therefore,

$$\lambda_{21} = \sigma_2 + jw_2,\ \lambda_{22} = \sigma_2 - jw_2,\ \mathbf{v}_{21} = \begin{bmatrix} -\frac{\sigma_2}{q_2} \\ 1 \end{bmatrix},\ \mathbf{v}_{22} = \begin{bmatrix} -\frac{w_2}{q_2} \\ 0 \end{bmatrix}.$$

where $\sigma_2 = \frac{p_1}{2}$, $w_2 = \frac{\sqrt{|p_1^2+4q_2|}}{2}$ and $\mathbf{v}_{21} + j\mathbf{v}_{22}$ is the eigenvector associated with λ_{22}. Therefore, $V_2 = \begin{bmatrix} \mathbf{v}_{21} & \mathbf{v}_{22} \end{bmatrix}$ and $\det V_2 = \frac{w_2}{q_2}$. Since $q_2 < 0$, by (i) in Fact 3.1.2, mode 2 is negative transitive. By Fact 5.2.1 and by the expression in (5.5), the FEX of mode 2 can be calculated as

$$F_2 = \ln|\frac{q_2}{\lambda_{21}}| + \sigma_2 t_{21}, \tag{6.14}$$

where

$$t_{21} = \frac{\arctan\left[\frac{w_2}{\sigma_2}\right]}{w_2}. \tag{6.15}$$

6.1.2 Hybrid Output Feedback Stabilization

We will now state and prove the main result of this chapter

Theorem 6.1.1. *i) The set of* (κ_1, κ_2) *that stabilize (6.1) are such that all the modes in (6.7) are transitive. ii) The system (6.1) can always be stabilized by the feedback law given in (6.5).*

Proof. i) From Section 6.1.1, mode 1 (mode 3) is either a sink, half-sink or negative transitive mode and mode 2 (mode 4) is either a source or negative transitive mode. Recall from Theorem 5.2.1 that, when a half-sink or sink mode exist λ_{i1} must be less than zero to ensure global asymptotic stability. Mode 1 (Mode 3) cannot be selected to be a sink or half-sink mode because, λ_{11} is always greater than zero. Mode 2 (Mode 4) cannot be selected to be a source because, a combination of a transitive mode and a source mode will lead to ill-posedness. Therefore, all modes must be transitive.

ii) From our analysis in Section 6.1.1, mode 1 (mode 3) is negative transitive when $q_1 < -\frac{p_1^2}{4}$ (i.e., when the eigenvalues are non-real). Mode 2 (Mode 4) is negative transitive when one of the following holds: a) $q_2 < -\frac{p_1^2}{4}$ (i.e., when the eigenvalues are non-real) b) $q_2 = -\frac{p_1^2}{4}$ $(p_1 \neq 0)$ (i.e., when the eigenvalues are real and repeated) c) $-\frac{p_1^2}{4} < q_2 < 0$.

Having selected (q_1, q_2) such that all modes are transitive (negative), by Theorem 5.2.1, we must also ensure that the selected (q_1, q_2) make $F < 0$ (to achieve global asymptotic stablity). Recall that, $A_3 = A_1$, $A_4 = A_2$, $S_3 = -S_1$ and $S_4 = -S_2$. Therefore, the properties of mode 3 (resp., mode 4) in (6.7) is same with the one of mode 1 (resp. mode 2). Hence, $F_3 = F_1$, $F_4 = F_2$ and $F = 2(F_1 + F_2)$. By Theorem 5.2.1, $F_1 + F_2 < 0$ must be satisfied for stability.

Case A: $q_1 < -\frac{p_1^2}{4}$ and $q_2 < -\frac{p_1^2}{4}$

From (6.8) and (6.14), the factor of expansion can be written as

$$F = F_1 + F_2 = \ln|\frac{q_2\lambda_{11}}{q_1\lambda_{21}}| + \sigma_1(t_{11} + t_{21}).$$

Therefore, $F < 0$ if and only if

$$\ln|\frac{q_2\lambda_{11}}{q_1\lambda_{21}}| < -\sigma_1(t_{11} + t_{21}),$$

which is the same as

$$\frac{q_2}{q_1} < |\frac{\lambda_{21}}{\lambda_{11}}| \, \exp\left(-\sigma_1 t_{11} - \sigma_1 t_{21}\right). \tag{6.16}$$

Recall from (6.9) and (6.15) that, $t_{11} = \frac{\arctan\left[\frac{w_1}{-\sigma_1}\right]}{w_1}$ and $t_{21} = \frac{\arctan\left[\frac{w_2}{\sigma_2}\right]}{w_2}$. Since, $\omega_1 = \frac{\sqrt{|p_1^2+4q_1|}}{2} > 0$, $\omega_2 = \frac{\sqrt{|p_1^2+4q_2|}}{2} > 0$ and $\sigma_1 = \sigma_2 = \frac{p_1}{2} > 0$. This implies that $t_{11} \leq \frac{\pi}{w_1}$ and $t_{21} \leq \frac{\pi}{2w_2}$. Substituting $\lambda_{11} = \sigma_1 + jw_1$ and $\lambda_{21} = \sigma_2 + jw_2$ to (6.16)

$$\frac{q_2}{q_1} < \sqrt{\frac{{\sigma_2}^2 + {w_2}^2}{{\sigma_1}^2 + {w_1}^2}} \, \exp\left(-\frac{\sigma_1\pi}{w_1} - \frac{\sigma_1\pi}{2w_2}\right). \tag{6.17}$$

Substituting $\sigma_1 = \sigma_2 = \frac{p_1}{2}$, $\omega_1 = \frac{\sqrt{|p_1^2+4q_1|}}{2}$, and $\omega_2 = \frac{\sqrt{|p_1^2+4q_2|}}{2}$ into (6.17), it is easy to see that (6.17) can be written as

$$\frac{q_2}{q_1} < \sqrt{\frac{q_2}{q_1}} \, \exp\left(-\frac{p_1\pi}{\sqrt{|p_1^2+4q_1|}} - \frac{p_1\pi}{2\sqrt{|p_1^2+4q_2|}}\right)$$

which implies that,

$$q_2 > q_1 \exp\left(-\frac{2p_1\pi}{\sqrt{|p_1^2+4q_1|}} - \frac{p_1\pi}{\sqrt{|p_1^2+4q_2|}}\right). \tag{6.18}$$

It is clear that (q_1, q_2) that satisfies (6.18) will always exist. Since

$$\begin{aligned} \kappa_1 &= \frac{q_1 - p_o}{n_o}, \\ \kappa_2 &= \frac{q_2 - p_o}{n_o}, \end{aligned} \tag{6.19}$$

existence of (q_1, q_2) implies that (κ_1, κ_2) exist.

Case B: $q_1 < -\frac{p_1^2}{4}$ and $q_2 = -\frac{p_1^2}{4}(p_1 \neq 0)$

From (6.8) and (6.12), the factor of expansion can be written as

$$F = F_1 + F_2 = \ln|\frac{q_2\lambda_{11}}{q_1\lambda_{21}}| + \sigma_1 t_{11} + 1.$$

Therefore, $F < 0$ if and only if

$$\ln|\frac{q_2\lambda_{11}}{q_1\lambda_{21}}| < -\sigma_1 t_{11} - 1,$$

which is the same as

$$\frac{q_2}{q_1} < |\frac{\lambda_{21}}{\lambda_{11}}| \exp\left(-\sigma_1 t_{11} - 1\right). \tag{6.20}$$

Recall from (6.9) that, $t_{11} = \frac{\arctan\left[\frac{w_1}{-\sigma_1}\right]}{w_1}$. Since, $\omega_1 = \frac{\sqrt{|p_1^2+4q_1|}}{2} > 0$ and $\sigma_1 = \frac{p_1}{2} > 0$. This implies that $t_{11} \leq \frac{\pi}{w_1}$. Substituting $\lambda_{11} = \sigma_1 + jw_1$ and $\lambda_{21} = \frac{p_1}{2}$ to (6.20)

$$\frac{q_2}{q_1} < \frac{p_1}{2}\sqrt{\frac{1}{{\sigma_1}^2 + {w_1}^2}} \exp\left(-1 - \frac{\sigma_1\pi}{w_1}\right). \tag{6.21}$$

Substituting $\sigma_1 = \frac{p_1}{2}$ and $\omega_1 = \frac{\sqrt{|p_1^2+4q_1|}}{2}$ into (6.21), it is easy to see that (6.21) can be written as

$$\frac{q_2}{q_1} < \frac{p_1}{2}\sqrt{\frac{1}{q_1}}\ \exp\left(-1 - \frac{p_1\pi}{\sqrt{|p_1^2 + 4q_1|}}\right),$$

with $q_2 = -\frac{p_1^2}{4}$ we can write,

$$q_1 > \frac{p_1^2}{4}\exp\left(2 + \frac{2p_1\pi}{\sqrt{|p_1^2 + 4q_1|}}\right). \tag{6.22}$$

It is clear that q_1 that satisfies (6.22) will always exist. Since

$$\begin{aligned} \kappa_1 &= \frac{q_1 - p_o}{n_o}, \\ \kappa_2 &= \frac{q_2 - p_o}{n_o}, \end{aligned} \tag{6.23}$$

existence of (q_1, q_2) implies that (κ_1, κ_2) exist.

Case C: $q_1 < -\frac{p_1^2}{4}$ and $-\frac{p_1^2}{4} < q_2 < 0$

From (6.8) and (6.10), the factor of expansion can be written as

$$F = F_1 + F_2 = \ln|\frac{q_2\lambda_{11}}{q_1\lambda_{21}}| + \sigma_1 t_{11} + \lambda_{21}t_{21}.$$

Therefore, $F < 0$ if and only if

$$\ln|\frac{q_2\lambda_{11}}{q_1\lambda_{21}}| < -\sigma_1 t_{11} - \lambda_{21}t_{21},$$

which implies

$$\frac{q_2}{q_1} < |\frac{\lambda_{21}}{\lambda_{11}}|\exp\left(-\sigma_1 t_{11} - \lambda_{21}t_{21}\right) \tag{6.24}$$

where $\sigma_1 = \frac{p_1}{2}$, $\omega_1 = \frac{\sqrt{|p_1^2+4q_1|}}{2}$, $\lambda_{21} = \frac{p_1+\sqrt{|p_1^2+4q_2|}}{2}$, $\lambda_{22} = \frac{p_1-\sqrt{|p_1^2+4q_2|}}{2}$ and; $t_{11} =$

$\frac{\arctan[\frac{w_1}{-\sigma_1}]}{w_1}$, $t_{21} = \frac{1}{\lambda_{21}-\lambda_{22}} \ln |\frac{\lambda_{21}}{\lambda_{22}}|$. ■

The proof above is constructive and shows that the controller gains κ_1 and κ_2 that stabilize (6.1) can be calculated by one of the following methods

Method 1: Select $q_1 < -\frac{p_1^2}{4}$ and $q_2 < -\frac{p_1^2}{4}$ (which ensures that all modes in (6.7) are negative transitive) such that (6.18) is satisfied (which ensures that $F < 0$). Then, compute (κ_1, κ_2) from (6.23).

Method 2: Select $q_1 < -\frac{p_1^2}{4}$ and $q_2 = -\frac{p_1^2}{4}(p_1 \neq 0)$ (which ensures that all modes in (6.7) are negative transitive) such that (6.22) is satisfied (which ensures that $F < 0$). Then, compute (κ_1, κ_2) from (6.23).

Method 3: Select $q_1 < -\frac{p_1^2}{4}$ and $-\frac{p_1^2}{4} < q_2 < 0$ (which ensures that all modes in (6.7) are negative transitive) such that (6.24) is satisfied (which ensures that $F < 0$). Then, compute (κ_1, κ_2) from (6.23).

6.2 Stabilization of a Magnetic Levitation System

In this section, we will apply the proposed method to design a controller that stabilizes a magnetic levitation system.

The magnetic levitation system considered consist of a light source-photodetector pair (sensor), ferromagnetic ball, electromagnetic coil and a controller (Fig. 6.6). The sensor produces a feedback signal (V_s) as the position of the ball varies. V_s is then fed to a controller, which regulates the system. This way, the levitated ferromagnetic ball is maintained at y_o. A detailed description of the system can be found in [1, 78, 79]. The transfer function of the magnetic

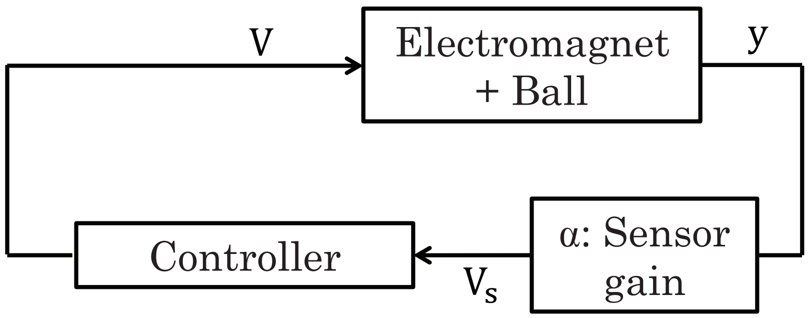


Figure 6.6: Magnetic levitation system

levitation system linearized at an equilibrium point, (y_o, i_o) is given by [1]

$$G(s) = \alpha \frac{Y(s)}{V(s)} = \frac{V_s(s)}{V(s)} = \frac{n_o}{s^2 - p_o}, \tag{6.25}$$

where $n_o = -\dfrac{2\alpha k i_o}{R_e m {y_o}^2}$ and $p_o = \dfrac{2k{i_o}^2}{m{y_o}^3}$.

Parameters	**Value**
Equilibrium distance, y_o	0.016m
Equilibrium current, i_o	0.7A
Mass of ball, m	0.02kg
Sensor gain, α	359.375V/m
Force constant, k	$1.025 \times 10^{-4} \frac{Nm^2}{A^2}$
Voltage-current gain of electromagnetic coil, R_e	6.5V/A

Table 6.1: Parameters of the magnetic levitation system of [1]

Consider the magnetic levitation system with parameters given in Table 6.1

$$G(s) = \frac{V_s(s)}{V(s)} = \frac{-1549.7}{s^2 - 1226.3}, \tag{6.26}$$

where

$$\begin{bmatrix} \dot{V}_s \\ \ddot{V}_s \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1226.3 & 0 \end{bmatrix} \begin{bmatrix} V_s \\ \dot{V}_s \end{bmatrix} + \begin{bmatrix} 0 \\ -1549.7 \end{bmatrix} V, \; V_s = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} V_s \\ \dot{V}_s \end{bmatrix}. \tag{6.27}$$

From (6.26), the system is unstable and stability cannot be achieved by a static output feedback controller ($p_1 = 0$). Therefore, we will design a hybrid output feedback controller to achieve stability. Using method 1, select (q_1, q_2)=(-1500, -1000), this satisfies (6.18) and the controller gains can be computed from (6.23), (κ_1, κ_2)=(1.759, 1.437). Fig. (6.7) illustrates a typical phase portrait for $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ which shows that (6.1) is stable. Time evolution of the states are also plotted in Fig. (6.8). Moreover, the smaller the value of F, the faster the system converges to the equilibrium point. As illustrated in Figs. (6.9)-(6.10) the system converges faster to the equilibrium point with (q_1, q_2)=(-3000, -1000) and (κ_1, κ_2)=(2.727, 1.437).

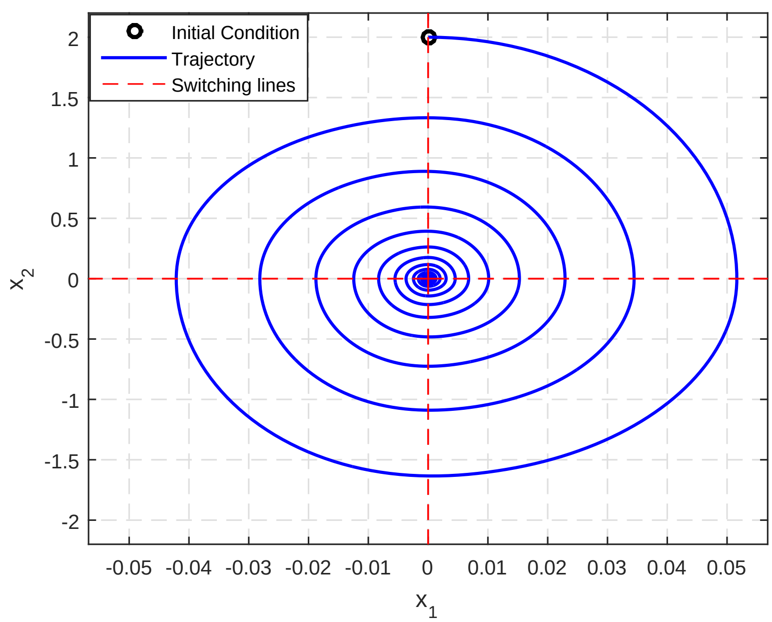


Figure 6.7: Hybrid output feedback stabilization of magnetic levitation system with (κ_1, κ_2)=(1.759, 1.437), $\mathbf{b}^T = [\,0\ 2\,]$

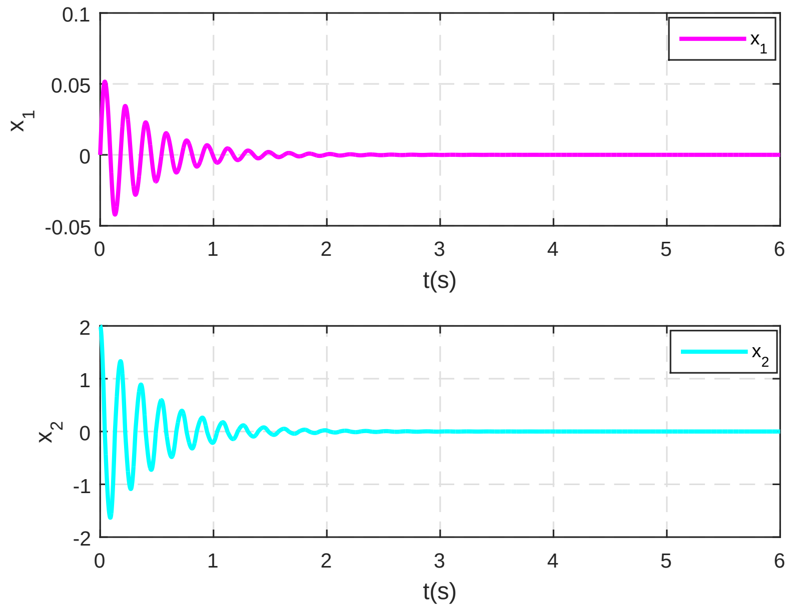


Figure 6.8: States of magnetic levitation system with (κ_1, κ_2)=(1.759, 1.437), $\mathbf{b}^T = [\ 0\ 2\ ]$

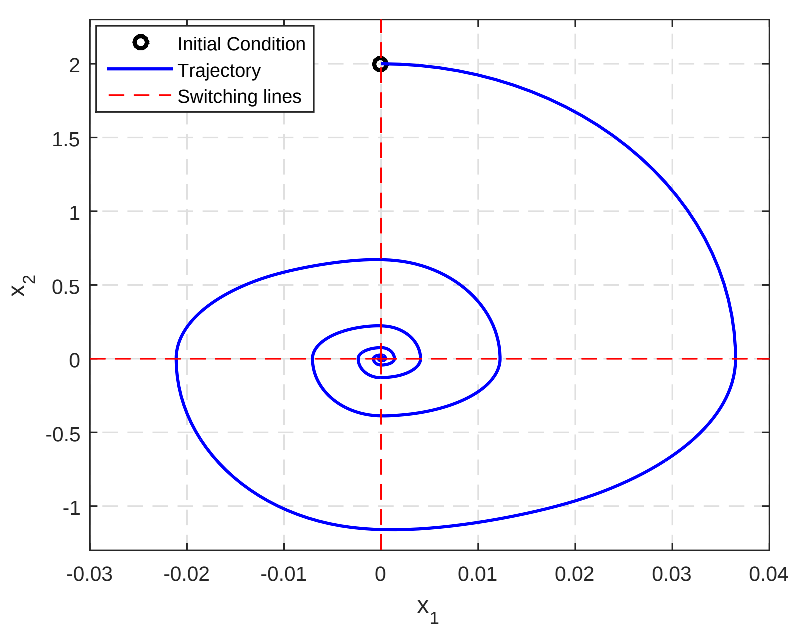


Figure 6.9: Hybrid output feedback stabilization of magnetic levitation system with (κ_1, κ_2)=(2.727, 1.437), $\mathbf{b}^T = [\ 0\ 2\ ]$

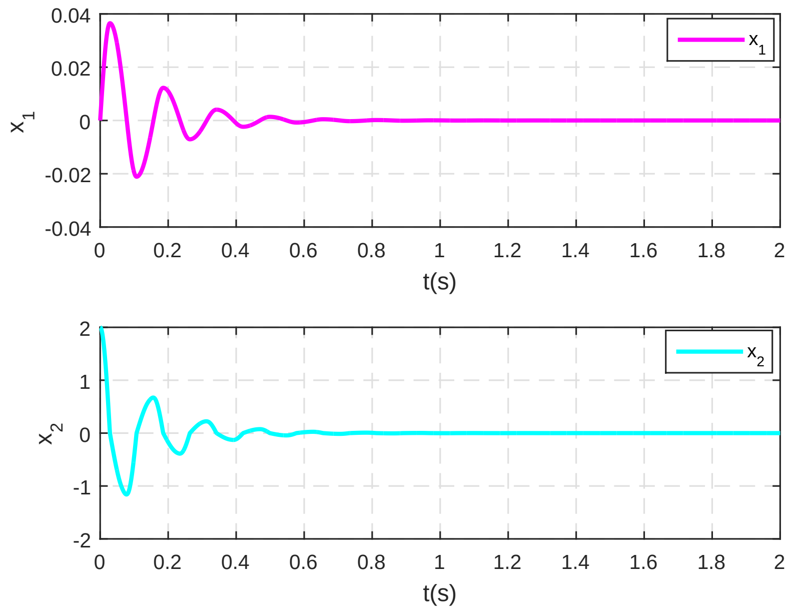


Figure 6.10: States of magnetic levitation system with (κ_1, κ_2)=(2.727, 1.437), $\mathbf{b}^T = [\ 0\ 2\ ]$

Chapter 7

Conclusion

In this thesis, stability of planar piecewise linear systems and stabilization of a class of planar LTI systems have been investigated. Using a geometric approach, we have derived a new set of necessary and sufficient conditions for the well-posedness and stability of planar conewise linear systems, which are directly in terms of the "givens" of the problem. A hybrid output feedback controller has also been designed for global asymptotic stabilization of a class of planar LTI systems.

In Chapter 3, we studied the structural properties of an LTI system in a cone. Five different mode types were identified. The mode type is solely characterized by how the eigenvectors stand relative to the cone on which the subsystem is defined. In Chapter 4, we derived a new set of necessary and sufficient conditions for well-posedness (in the sense of Carathéodory) of planar piecewise linear systems. The conditions are in terms of the eigenvectors and the cone vectors.

Using the results obtained in Chapter 3 and 4, a simple stability criteria for planar piecewise linear systems was provided in Chapter 5. The conditions are computationally tractable. Only the eigenvalues, eigenvectors and cone vectors are required for the stability test. In addition, the known condition for bimodal piecewise linear systems was obtained as an easy corollary of our main result. The results were validated on a three-modal system and a selector control system.

Finally, taking advantage of the stability criteria derived in Chapter 5, in Chapter 6 we designed a hybrid output feedback controller for the stabilization of a class of planar LTI systems. The controller was shown to stabilize the system, just in case, the controller gains satisfy certain algebraic condition. The effectiveness of the proposed controller was illustrated on a magnetic levitation system.

The new approach proposed provides more insight into the stability analysis of piecewise linear systems and hybrid output feedback controller synthesis for planar LTI systems. An obvious next step is to investigate the stability of 3D or higher dimensional conewise linear systems by extending the techniques developed here. It is expected that the interaction among the cone vectors and eigenvectors of the modes should also play a crucial role in the characterization of stability of the higher dimensional cases.

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Appendix A

Matlab Codes

```
%EXAMPLE 5.1.1
%This code plots the root-locus of the matrix pencils
%given in Example 5.1.1,
%it also plots the Nyquist diagram of G(s).
clc
clear all
figure

B1=[-3  7;0  -1];
B=[-2;7];
Cp=[1  0];
D=0;
B2=B1-(B*Cp);
%Eigenvalue locus
alpha=0;
for  i=1:100
    Ac=(alpha*(B1))+((1-alpha)*(B2));
    eigenvalues(:,i)=eig(Ac);
    B2s=inv(B2);
    Ac1=(alpha*(B1))+((1-alpha)*(B2s));
```

```
    eigenvaluesb(:,i)=eig(Ac1);
    if alpha==1
    break
else
    alpha = alpha + 0.01;
end
end
subplot(2,1,1);
plot(real(eigenvalues(:,:)),imag(eigenvalues(:,:)),'oc')
axis([-3 0.5 -4 4])
xlabel('Real(r_1)')
ylabel('Imag(r_1)')
title('r_1=Eigenvalues of A_c')
grid
set(gca,'GridLineStyle','--')
set(gcf,'color','white');
subplot(2,1,2);
plot(real(eigenvaluesb(:,:)),imag(eigenvaluesb(:,:)),'om')
axis([-3 0.5 -0.15 0.15])
xlabel('Real(r_2)')
ylabel('Imag(r_2)')
title('r_2=Eigenvalues of A_c_1')
grid
set(gca,'GridLineStyle','--')
set(gcf,'color','white');
%Nyquist plot
figure
G=ss(B1,B,Cp,D);
Co=ctrb(B1,B);
Ob = obsv(B1,Cp);
rCo=rank(Co)
rOb=rank(Ob)
G=tf(G);%G(s)
```

```
nyquist(G)
hold on
plot([-1 -1],[-10 10],'--r','LineWidth',2)
set(gcf,'color','white');

%WELL-POSEDNESS OF CONEWISE LINEAR SYSTEMS
%This code implements Theorem 4.1.1 to check for
%well-posedness of a planar conewise linear system.
clc
clear all

%Define system and cones
m=3; %Number of modes
a=0.05;
A{1}=[6 3;-3 6];
A{2}=[-1 1;-1 0];
A{3}=[-2 1/a;a -2];
C{1}=[1 0.5/a;0 0.5/a]; S{1}=inv(C{1});
C{2}=[0 -0.5/a;1 -0.5/a] ;S{2}=inv(C{2});
C{3}=[-0.5 0.25/a;-0.5 -0.25/a] ; S{3}=inv(C{3});
for i=1:m
[E{i} D{i}]=eig(A{i});
detS(i)=det(S{i});
s1{i}=[S{i}(1);S{i}(2)]; s2{i}=[S{i}(3);S{i}(4)];
c1t{i}=[C{i}(1) C{i}(3)];
c2t{i}=[C{i}(2) C{i}(4)];
if (imag(D{i}(1))==0) && (real(D{i}(1))~=real(D{i}(4)))
[E{i} D{i}]=eig(A{i});
if (max(D{i}(1),D{i}(4)))==D{i}(1)
lamda1(i)=D{i}(1);lamda2(i)=D{i}(4);
v1{i}=[E{i}(1);E{i}(2)];v2{i}=[E{i}(3);E{i}(4)];
else
lamda1(i)=D{i}(4);
lamda2(i)=D{i}(1);
```

```
v1{i}=[E{i}(3);E{i}(4)];v2{i}=[E{i}(1);E{i}(2)];
end
V{i}=[v1{i} v2{i}];
p(i)=(det(V{i})*det([v1{i} s1{i}])*det([v2{i} s1{i}]));
else if (imag(D{i}(1))==0) && (real(D{i}(1))==real(D{i}(4)))
[E{i} D{i}]=jordan(A{i});
lamda(i)=D{i}(1);
v1{i}=[E{i}(3);E{i}(4)];
v2{i}=[E{i}(1);E{i}(2)];
V{i}=[v1{i} v2{i}];
p(i)=det(V{i});
else
[E{i} D{i}]=eig(A{i});
sigma(i)=real(D{i}(1));
wimag(i)=imag(D{i}(1));
w(i)=abs(wimag(i));
vn{i}=[E{i}(3);E{i}(4)];
v1{i}=real(vn{i});v2{i}=imag(vn{i});
V{i}=[v1{i} v2{i}];
p(i)=det(V{i});
end
end
end
for i=1:m
if i~=m
k=i+1;
else
k=1;
end
if imag(D{k}(1))==0 && real(D{k}(1))~=real(D{k}(4))
pp(k)=(det(V{k})*det([v1{k} s1{i}])*det([v2{k} s1{i}]));
else
pp(k)=det(V{k});
```

```
end
end
detS
for i=1:m
if i~=m
k=i+1;
else
k=1;
end
if ((imag(D{i}(1))==0) && (real(D{i}(1))~=real(D{i}(4))) ...
&& ((det([v1{i} s1{i}])*det([v2{i} s1{i}]))==0)) || ...
((imag(D{k}(1))==0) && (real(D{k}(1))~=real(D{k}(4))) ...
&& ((det([v1{k} s1{i}])*det([v2{k} s1{i}]))==0)) || ...
((imag(D{i}(1))==0) && (real(D{i}(1))==real(D{i}(4))) ...
&& (det([v2{i} s1{i}])==0)) ||((imag(D{k}(1))==0) ...
&& (real(D{k}(1))==real(D{k}(4))) && (det([v2{k} s1{i}])==0))
swpc(i)=0;
disp('Eigenvector lies on the border ')
break
else
wpc(i)=p(i)*pp(k);
swpc(i)=sign(wpc(i));
end
end
grtzero = sum((swpc ==1))
if grtzero==m
disp('CLS is well-posed')
else if swpc(i)~=0
disp('CLS is ill-posed')
 end
end

%STABILITY OF CONEWISE LINEAR SYSTEMS
%This code implements Theorem 5.2.1 to check the stability
```

```
% of a planar conewise linear system.
clc
clear all
figure

%Define system and cones
m=3; %Number of modes
a=0.05;
A{1}=[6 3;-3 6];
A{2}=[-1 1;-1 0];
A{3}=[-2 1/a;a -2];
C{1}=[1 0.5/a;0 0.5/a]; S{1}=inv(C{1});
C{2}=[0 -0.5/a;1 -0.5/a] ;S{2}=inv(C{2});
C{3}=[-0.5 0.25/a;-0.5 -0.25/a] ; S{3}=inv(C{3});
for i=1:m
[E{i} D{i}]=eig(A{i});
detS(i)=det(S{i});
s1{i}=[S{i}(1);S{i}(2)]; s2{i}=[S{i}(3);S{i}(4)];
b=s1{i}+s2{i};
c1t{i}=[C{i}(1) C{i}(3)];
c2t{i}=[C{i}(2) C{i}(4)];
if (imag(D{i}(1))==0) && (real(D{i}(1))~=real(D{i}(4)))
disp('REAL_AND_DISTINCT_EIGENVALUES')
[E{i} D{i}]=eig(A{i});
if (max(D{i}(1),D{i}(4)))==D{i}(1)
lamda1(i)=D{i}(1);lamda2(i)=D{i}(4);
v1{i}=[E{i}(1);E{i}(2)];v2{i}=[E{i}(3);E{i}(4)];
else
lamda1(i)=D{i}(4);
lamda2(i)=D{i}(1);
v1{i}=[E{i}(3);E{i}(4)];v2{i}=[E{i}(1);E{i}(2)];
end
V{i}=[v1{i} v2{i}];
```

```
W{i}=inv(V{i});
w1t{i}=[W{i}(1) W{i}(3)];
w2t{i}=[W{i}(2) W{i}(4)];
mv1(i)=det([s1{i} v1{i}])*det([v1{i} s2{i}]);
mv1n(i)=det([s1{i} -1*v1{i}])*det([-1*v1{i} s2{i}]);
mv2(i)=det([s1{i} v2{i}])*det([v2{i} s2{i}]);
mv2n(i)=det([s1{i} -1*v2{i}])*det([-1*v2{i} s2{i}]);
% Plot: Basins
subplot(m,1,i)
plotv([v1{i}],'b'); hold on ; plotv([-v1{i}],'g');
hold on;plotv([v2{i}],'c');hold on;plotv([-v2{i}],'m');
hold on;plotv([s1{i}],'r');hold on;plotv([s2{i}],'r')
grid
set(gca,'GridLineStyle','--')
set(gcf,'color','white');
%Mode type
if ((mv2(i)>0)||(mv2n(i)>0))&&((mv1(i)<0)&&(mv1n(i)<0))
disp('Source: check eigenvalue for stability')
lamda2=lamda2(i)
tr(i)=-1;
else if ((mv1(i)>0)||(mv1n(i)>0))&&((mv2(i)>0)||(mv2n(i)>0))
disp('Half-Sink: check eigenvalue for stability')
lamda1=lamda1(i)
tr(i)=-1;
else if ((mv1(i)>0)||(mv1n(i)>0))&&...
((mv2(i)<0)&&(mv2n(i)<0))
disp('Sink: check eigenvalue for stability')
lamda1=lamda1(i)
tr(i)=-1;
else if ((det(V{i})*det([v1{i} b])*det([v2{i} b]))<0) ...
&& ((det([v1{i} b])*det([v1{i} s1{i}]))>0)
disp('Negative Transitive')
t1x(i)=1/(lamda1(i)-lamda2(i));
```

```
t1y(i)=-1*((((c2t{i})*(v2{i}))*((w2t{i})*(s2{i})))...
/(((c2t{i})*(v1{i}))*((w1t{i})*(s2{i}))));
t1(i)=t1x(i)*log(abs(t1y(i)));
F(i)=log((abs(det([v2{i} s2{i}])))/(abs(det([v2{i} s1{i}]))))+...
(lamda1(i)*t1(i));
tr(i)=1;
else
disp('Positive Transitive')
t2x(i)=1/(lamda1(i)-lamda2(i));
t2y(i)=-1*((((c1t{i})*(v2{i}))*...
((w2t{i})*(s1{i})))/(((c1t{i})*(v1{i}))*((w1t{i})*(s1{i}))));
t2(i)=t2x(i)*log(abs(t2y(i)));
F(i)=log((abs(det([v2{i} s1{i}])))/(abs(det([v2{i} s2{i}]))))+...
(lamda1(i)*t2(i)) ;
tr(i)=1;
end
end
end
end
else if (imag(D{i}(1))==0) && (real(D{i}(1))==real(D{i}(4)))
disp('REAL[REPEATED] EIGENVALUES')
[E{i} D{i}]=jordan(A{i});
lamda1(i)=D{i}(1);
v1{i}=[E{i}(3);E{i}(4)];
v2{i}=[E{i}(1);E{i}(2)];
V{i}=[v1{i} v2{i}];
W{i}=inv(V{i});
w1t{i}=[W{i}(1) W{i}(3)];
w2t{i}=[W{i}(2) W{i}(4)];
mv2(i)=det([s1{i} v2{i}])*det([v2{i} s2{i}]);
mv2n(i)=det([s1{i} -1*v2{i}])*det([-1*v2{i} s2{i}]);
%Plot: Basins
subplot(m,1,i)
```

```
plotv([v1{i}],'b'); hold on ; plotv([-v1{i}],'g');
hold on;plotv([v2{i}],'c');hold on;plotv([-v2{i}],'m');
hold on;plotv([s1{i}],'r');hold on;plotv([s2{i}],'r')
grid
set(gca,'GridLineStyle','--')
set(gcf,'color','white');
%Mode type
if ((mv2(i)>0)||(mv2n(i)>0))
disp('Half-Sink: check eigenvalue for stability')
lamda1=lamda1(i)
tr(i)=-1;
else if (det(V{i})<0)&&((det([v2{i} b])*det([v2{i} s1{i}]))>0)
disp('Negative Transitive')
t1n(i)=-1*(det(V{i}))*(det(S{i}))*((c2t{i})*(s2{i}));
t1d(i)=(det([v2{i} s2{i}]))*(det([v2{i} s1{i}]));
t1(i)=abs(t1n(i)/t1d(i));
F(i)=log((abs(det([v2{i} s2{i}])))/(abs(det([v2{i} s1{i}]))))+...
(lamda1(i)*t1(i));
tr(i)=1;
else
disp('Positive Transitive')
t2n(i)=-1*(det(V{i}))*(det(S{i}))*((c1t{i})*(s1{i}));
t2d(i)=(det([v2{i} s1{i}]))*(det([v2{i} s2{i}]));
t2(i)=abs(t2n(i)/t2d(i));
F(i)=log((abs(det([v2{i} s1{i}])))/(abs(det([v2{i} s2{i}]))))...
+(lamda(i)*t2(i));
tr(i)=1;
end
end
else
disp('NONREAL EIGENVALUES')
[E{i} D{i}]=eig(A{i});
sigma(i)=real(D{i}(1));
```

```
wimag(i)=imag(D{i}(1));
w(i)=abs(wimag(i));
vn{i}=[E{i}(3);E{i}(4)];
v1{i}=real(vn{i});v2{i}=imag(vn{i});
V{i}=[v1{i} v2{i}];
W{i}=inv(V{i});
w1t{i}=[W{i}(1) W{i}(3)];
w2t{i}=[W{i}(2) W{i}(4)];
%Mode type
if det(V{i})<0
disp('Negative Transitive')
theta(i)=phase((det([vn{i} s1{i}]))/(det([vn{i} s2{i}])));
t1(i)=abs(theta(i)/w(i));
F(i)=log((abs(det([vn{i} s2{i}])))/(abs(det([vn{i} s1{i}]))))+...
(sigma(i)*t1(i));
tr(i)=1;
else
disp('Positive Transitive')
theta(i)=phase((det([vn{i} s2{i}]))/(det([vn{i} s1{i}])));
t2(i)=abs(theta(i)/w(i));
F(i)=log((abs(det([vn{i} s1{i}])))/(abs(det([vn{i} s2{i}]))))+...
(sigma(i)*t2(i));
tr(i)=1;
end
end
end
end
legend('v_1','-v_1','v_2','-v_2','s_1','s_2')
detS
FEX=0;
  for i=1:m
if tr(i)<0
disp('Not all modes are transitive')
```

```
break
else
FEX=FEX+F(i);
end
end
if (FEX <0) && (tr(i)>0)
FEX=FEX
disp('CLS is stable')
else if (FEX >=0) && (tr(i)>0)
FEX=FEX
disp('CLS is unstable')
end
end

%MULTIMODAL TRAJECTORY
%This code plots the trajectory of a planar conewise linear system.
clc
clear all
figure

global A
global C
tf=20;
m=3; %Number of modes
a=0.05;
A{1}=[6 3;-3 6];
A{2}=[-1 1;-1 0];
A{3}=[-2 1/a;a -2];
C{1}=[1 0.5/a;0 0.5/a]; S{1}=inv(C{1});
C{2}=[0 -0.5/a;1 -0.5/a] ;S{2}=inv(C{2});
C{3}=[-0.5 0.25/a;-0.5 -0.25/a] ; S{3}=inv(C{3});
x1b=3;
x2b=0;
x0=[x1b;x2b];
```

```
plot(x1b,x2b,'ok','LineWidth',2);
hold on
[t x]=ode45('slidd2t',[0:0.001:tf],x0);
plot(x(:,1),x(:,2),'b','LineWidth',1.5);

% Switching Lines
ax1=1.1*max(abs(x(:,1)));
ax2=1.1*max(abs(x(:,2)));
for i=1:m
    S{i}=inv(C{i});
    s1{i}= S{i}(:,1); s2{i}= S{i}(:,2);
    hold on
    plotv([10^70*s1{i}],'--r')
    hold on
    plotv([10^70*s2{i}],'--r')
end
xlabel('x_1');
ylabel('x_2');
axis([-ax1 ax1 -ax2 ax2])
grid
set(gca,'GridLineStyle','--')
set(gcf,'color','white');
legend('Initial Condition','Trajectory','Switching lines')

function xdot=slidd2t(t,x)
global A
global C
if ((C{1})*[x(1) x(2)]')>=0
xdot=A{1}*[x(1) x(2)]';
else if ((C{2})*[x(1) x(2)]')>=0
xdot=A{2}*[x(1) x(2)]';
else if ((C{3})*[x(1) x(2)]')>=0
xdot=A{3}*[x(1) x(2)]';
end
```

end
end
end