APPROXIMATION OF EQUILIBRIUM MEASURES BY DISCRETE MEASURES

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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September, 2012

Basic notions of potential analysis are given. Equilibrium measures can be approximated by discrete measures by means of Fekete points and Leja sequences. We give the sets for which exact locations of Fekete points and Leja sequences are known. An open problem about the location of Fekete points for a Cantor-type set $K(\gamma)$ is presented.

Keywords: potential theory, equilibrium measures, Fekete points, Leja sequences, Cantor-type sets.

ÖZET

DENGE ÖLÇÜLERİNİN AYRIK ÖLÇÜLER YARDIMIYLA YAKLAŞIMI

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Potansiyel analizin temel kavramlar verildi. Denge ölçülerine Fekete noktaları ve Leja dizileri yardımıyla yaklaşabiliriz. Fekete noktaları'nın ve Leja dizileri'nin yerinin tam olarak bilindiği kümeler verildi. Bir Cantor-tipi küme olan $K(\gamma)$'nın Fekete noktaları'nın konumu ile ilgili açık bir problem tanıtıldı.

Anahtar sözcükler: potansiyel analiz, denge ölçüleri, Fekete noktaları, Leja dizileri, Cantor-tipi kümeler .

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Contents

Chapter 1

Introduction

Potential theory originates from the study of gravitation by I. Newton, J. L. Lagrange, A. Legendre and P. S. Laplace in seventeenth and eighteenth centuries. The field of gravitational forces was called 'potential field' by Lagrange. At last this function was called only 'potential' by C. F. Gauss. By using methods of potentials physicists and mathematicians solved problems related to other forces such as electromagnetic and electrostatic forces in the nineteenth century. In order to solve some boundary value problems such as Dirichlet and Neumann problems and the problem of distribution of signed particles with minimal energy, different types of potentials such as single layer potential, double layer potential, Green potential and logarithmic potential are defined. In the twentieth century potential analysis became a branch of mathematics along with the development of theory of harmonic and subharmonic functions.

Fundamental theorem of potential analysis is Frostman's theorem (see 2.2.12) which states that the logarithmic potential with the equilibrium measure for a compact set $K \subset \mathbb{C}$ is constant except on a negligible set. The equilibrium measure μ_K of a compact set $K \subset \mathbb{C}$ is a special measure that minimizes the energy of a system amongst unit Borel measures. The knowledge of μ_K is important since by the knowledge of μ_K we can find the Green function for K. The Green function for K is important since by the knowledge of the Green function Dirichlet problem can be solved. On the other hand, there are a few cases that μ_K has a simple form. For these basic concepts see Chapter 2.

Luckily, it is possible to approximate equilibrium measures by discrete measures. They can be approximated in the weak-star sense which is the natural topology for the space of measures. Many sequences can be used to approximate equilibrium measures. The most important sequences are Leja sequences and sequences generated by n-point Fekete sets. Unfortunately, the exact location of Fekete points is known only for a few cases. They are $\overline{\mathbb{D}(0,1)}$, $[-1,1]$ and $[-1,1]^d$. A Leja sequence is known exactly only in one case, $\overline{\mathbb{D}(0,1)}$. For detailed information see Chapter 3.

For Cantor-type sets $K^{(\alpha)}$ and $K(\gamma)$, it may be easier to find the exact location of Fekete points comparing with simply connected sets. For our attempt to determine the location of Fekete points for these sets see Chapter 4.

Chapter 2

Introduction to Potential Analysis

In this chapter, we will present basic concepts of potential theory which will provide us background information related to Fekete and Leja points.

2.1 Potential and Energy

First, we give a couple of definitions related to logarithmic potential and energy.

Definition 2.1.1. Let (X,T) be a topological space. The *Borel σ-algebra* is defined as the σ-algebra generated by the open sets of X.

Definition 2.1.2. Any measure μ defined on the Borel σ-algebra of X is called a *Borel measure*.

Definition 2.1.3. The support of a positive measure μ denoted by $\mathrm{supp}(\mu)$ consists of all points z such that every open neighborhood of z has positive measure.

It is easy to see that $supp(\mu)$ is a closed set.

Definition 2.1.4. Let $M(K)$ be the collection of all positive unit Borel measures which is supported on the compact set $K \subseteq \mathbb{C}$. Let $\mu \in M(K)$. Then the *logarithmic potential* associated with μ is given by

$$U^\mu(z) = \int \log \frac{1}{|z-t|}\, d\mu(t). \tag{1.1}$$

This integral can take infinite values for $z \in supp(\mu)$.

Definition 2.1.5. Let $u(z) = u(x, y)$ be a real valued function defined in a domain $D \subseteq \mathbb{C}$. Then u is said to be *harmonic* if its second order partial derivatives are continuous and u satisfies the Laplace equation

$$\Delta u(z) = u_{xx}(z) + u_{yy}(z) = 0, \forall z \in D. \tag{1.2}$$

Definition 2.1.6. A function u is said to be *harmonic at a point* z_0 if it is harmonic in some neighborhood centered at z_0.

Theorem 2.1.7. *[1] [Mean value property] If u is harmonic in $|z-a| < r$ and continuous on its closure then we have*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta. \tag{1.3}$$

Theorem 2.1.8. *[1] The potential U^μ is harmonic at each point $z \notin \mathrm{supp}(\mu)$.*

Proof. Fix $t \in \mathrm{supp}(\mu)$ and $z \notin \mathrm{supp}(\mu)$. Then there exists an open ball $B_\delta(z)$ of z such that $t \notin B_\delta(z)$. There exists a branch L of the logarithm in $B_\delta(z)$ by [2]. Both L and $\frac{1}{z-t}$ are analytic on $B_\delta(z)$. Hence we have that $\log \frac{1}{|z-t|}$ is harmonic on $B_\delta(z)$ since it is the real part of $L(\frac{1}{z-t})$. Then we have

$$\Delta U^\mu(z) = \int \Delta \log \frac{1}{|z-t|}\, d\mu(t) = 0 \tag{1.4}$$

since all partial derivatives of $\log \frac{1}{|z-t|}$ are continuous up to degree 2 and we are integrating over a compact set K. So U^μ is harmonic in $\mathbb{C} \setminus \mathrm{supp}(\mu)$. □

Example 2.1.9. Let us show that for each $r > 0$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{|z - re^{i\theta}|} d\theta = \begin{cases} \log \frac{1}{r}, & \text{if } |z| \le r, \\ \log \frac{1}{|z|}, & \text{if } |z| > r. \end{cases} \tag{1.5}$$

If $|z| > r$, then $\log \frac{1}{|z-t|}$ is harmonic with respect to z while $|t| \leq r$. Applying the mean value theorem for harmonic functions we get the result for $|z| > r$.

If $|z| < r$ then we get,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{|z - re^{i\theta}|} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{|ze^{-i\theta} - r|} d\theta \tag{1.6}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{|\bar{z}e^{i\theta} - r|} d\theta = \log \frac{1}{r}. \tag{1.7}$$

For $|z| = r$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{|z - re^{i\theta}|} d\theta = \lim_{\rho \to r^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{|z - \rho e^{i\theta}|} d\theta = \log \frac{1}{r}, \tag{1.8}$$

by the dominated convergence theorem.

This was an example of a logarithmic potential with arc length measure on a circle. Now we give the definition of the (logarithmic) energy.

Definition 2.1.10. Let K be a compact subset of $\mathbb{C}$. Then the *logarithmic energy* $I(\mu)$ for $\mu \in M(K)$ is defined as

$$I(\mu) = \iint \log \frac{1}{|z - t|} \, d\mu(z) \, d\mu(t). \tag{1.9}$$

It is easy to see that $I(\mu) > -\infty$. Since K is a compact set, $\log \frac{1}{|z-t|}$ is bounded below for $z, t \in K$. Moreover, μ is a unit measure. Therefore we have $I(\mu) > -\infty$. As we see in next example, $I(\mu)$ can take infinite value.

Example 2.1.11. Let $K = (a_n)_{n=1}^N$ with $N \leq \infty$ be a compact set in $\mathbb{C}$. Let $\mu \in M(K)$. Then

$$\int \log \frac{1}{|z - t|} \, d\mu(z) = \sum_{n=1}^{N} \log \frac{1}{|a_n - t|} \, \mu(a_n). \tag{1.10}$$

If we put this into the integral we get

$$\int \sum_{n=1}^{N} \log \frac{1}{|a_n - t|} \, \mu(a_n) \, d\mu(t) = \sum_{m=1}^{N} \mu(a_m) \sum_{n=1}^{N} \mu(a_n) \, \log \frac{1}{|a_n - a_m|}. \tag{1.11}$$

In addition to this, μ is a unit measure. Thus, we have $\mu(K) = \mu(\cup_{k=1}^{N} a_k) = \sum_{k=1}^{N} \mu(a_k) = 1$. This means that there exists $k \in \{1, 2, \ldots, N\}$ such that $\mu(a_k) \neq 0$. For $m = k$ and $n = k$ in the sum, $\mu(a_k)^2 \log \frac{1}{|a_n - a_m|} = \infty$. Hence $I(\mu) = \infty$.

As we see in Example 2.1.11, the minimal energy for some compact sets may take infinite values.

2.2 Minimal Energy and Equilibrium Measure

As we see in the previous section, energy of a set K is bounded below just because of compactness of this set. Moreover, it has a common lower bound for any unit Borel measure which implies that $I(\mu)$ has an infimum which is different from $-\infty$, taking over $\mu \in M(K)$.

Definition 2.2.1. Let K be a compact set in $\mathbb{C}$. Then $V_K := \inf \{I(\mu) : \mu \in M(K)\}$ is called *the minimal energy* for K.

Definition 2.2.2. Let μ_n be a sequence of finite positive measures with *supp*$(\mu_n) \subseteq K$ for all n where K is a compact set of $\mathbb{C}$. Then we write $\mu_n \xrightarrow{*} \mu$ if

$$\lim_{n\to\infty} \int f \, d\mu_n = \int f \, d\mu, \quad \forall f \in C(K). \tag{2.1}$$

Theorem 2.2.3. *[3] [Helly's Selection Theorem] If (μ_n) is a sequence on a compact set K with bounded total mass $|\mu_n|(K)$ then we can select a weak star convergent subsequence from this sequence.*

Lemma 2.2.4. *[4] If a sequence $(\mu_n) \subset M(K)$ converges to a measure $\mu \in M(K)$ in weak star sense then $I(\mu) \leq \liminf_n I(\mu_n)$.*

Proof. First, let us define

$$k^\eta(z) = \begin{cases} \log \frac{1}{|z|}, & \text{if } |z| \geq \eta, \\ \log \frac{1}{\eta}, & \text{if } |z| < \eta, \end{cases} \tag{2.2}$$

which is called a truncated kernel.

It has the following properties:

(i) $k^\eta \in C(\mathbb{C})$.

(ii) For $z \in \mathbb{C}$ $k^\eta(z) \leq \log \frac{1}{|z|}$.

(iii) For $z \in \mathbb{C}$, $k^\eta(z) \nearrow \log \frac{1}{|z|}$ as $\eta \searrow 0$.

By (i) we have

$$\lim_n \iint k^\eta(z-t)\, d\mu_n(z)\, d\mu_n(t) = \iint k^\eta(z-t)\, d\mu(z)\, d\mu(t). \tag{2.3}$$

Using (ii) we get

$$\liminf_n \iint \log \frac{1}{|z-t|}\, d\mu_n(z)\, d\mu_n(t) \geq \iint k^\eta(z-t)\, d\mu(z)\, d\mu(t). \tag{2.4}$$

If we let $\eta \searrow 0$ using property (iii) and monotone convergence theorem we reach the inequality

$$I(\mu) \leq \liminf_n I(\mu_n). \tag{2.5}$$

$\square$

Definition 2.2.5. The *logarithmic capacity* of K, denoted by $cap(K)$, is defined as $cap(K) := e^{-V_K}$ where V_K is the minimum energy of the system.

Note that $cap(K) \geq 0$ and it is equal to $0 \iff V_K = +\infty$.

Definition 2.2.6. If $cap(K) = 0$ then K is called a *polar set*.

Theorem 2.2.7. *[3] Suppose $K \subset \mathbb{C}$ is not polar. Then there exists a measure $\mu_K \in M(K)$ such that $I(\mu_K) = V_K$.*

Proof. Let (μ_n) be a sequence of unit Borel measures in K satisfying $\lim_{n\to\infty} I(\mu_n) = V_K$. Then by 2.2.3 there exists a measure μ_K and a subsequence μ_{n_k} of μ_n such that $\mu_{n_k} \xrightarrow{*} \mu_K$. By 2.2.4 $I(\mu_K) \leq \liminf_{n_k} I(\mu_{n_k}) = V_K$. On the other hand by definition $V_K \geq I(\mu_K)$. This implies that $I(\mu_K) = V_K$. In other words, we show that for any compact set there is a measure which minimizes the energy. $\square$

Definition 2.2.8. Let K be a non-polar compact set in $\mathbb{C}$. Then any measure μ_K which satisfies $I(\mu_K) = V_K$ is called an *equilibrium measure* of K.

Notation: $M^*(K)$ denotes the subset of all positive Borel measures with finite energy. $M_1^*(K)$ denotes all positive unit Borel measures with finite energy.

Now we give two lemma, to prove the uniqueness of equilibrium measure.

Lemma 2.2.9. *[1] Let $\mu, \nu \in M^*(K)$. Let $\mu(K) = \nu(K)$. Then $I(\mu - \nu) \geq 0$ and it is equal to zero if and only if $\mu = \nu$.*

Proof. For the proof look at the p.32 of [1]. □

Lemma 2.2.10. *[4] If $\mu, \nu \in M_1^*(K)$. Then*

$$I(\mu - \nu) = 2I(\mu) + 2I(\nu) - 4I\left(\frac{\mu + \nu}{2}\right) \tag{2.6}$$

and

$$I\left(\frac{\mu + \nu}{2}\right) \leq \frac{I(\mu) + I(\nu)}{2}. \tag{2.7}$$

Proof.

$$\begin{aligned} I(\mu - \nu) &= \iint \log \frac{1}{|z-t|} \, [d\mu(z) - d\nu(z)][d\mu(t) - d\nu(t)] && (2.8) \\ &= I(\mu) + I(\nu) - \iint \log \frac{1}{|z-t|} \, d\mu(z) \, d\nu(t) - \iint \log \frac{1}{|z-t|} \, d\nu(z) \, d\mu(t) && (2.9) \end{aligned}$$

and

$$4I\left(\frac{\mu + \nu}{2}\right) = I(\mu) + I(\nu) + \iint \log \frac{1}{|z-t|} \, d\mu(z) \, d\nu(t) + \iint \log \frac{1}{|z-t|} \, d\nu(z) \, d\mu(t). \tag{2.10}$$

From these equalities, we reach the following equality easily:

$$I(\mu - \nu) = 2I(\mu) + 2I(\nu) - 4I\left(\frac{\mu + \nu}{2}\right). \tag{2.11}$$

We can rewrite it as

$$I\left(\frac{\mu + \nu}{2}\right) = \frac{I(\mu) + I(\nu)}{2} - \frac{I(\mu - \nu)}{4}. \tag{2.12}$$

To get the inequality (2.7), we use the first part. Instead of $I\left(\frac{\mu+\nu}{2}\right)$ put $\frac{I(\mu)+I(\nu)}{2} - \frac{I(\mu-\nu)}{4}$. So the inequality is satisfied if and only if $I(\mu - \nu) \geq 0$. Since both μ and ν are unit by 2.2.9, we already get this inequality. □

Theorem 2.2.11. *Equlibrium measure of a compact set is unique.*

Proof. Let μ and ν be equilibrium measures with $\mu \neq \nu$. Then by (2.7), $\frac{\mu+\nu}{2}$ is also an equilibrium measure. Therefore by 2.6 $I(\mu - \nu) = 0$. On the other hand, by 2.2.9, $I(\mu - \nu) = 0 \iff \mu = \nu$ which contradicts our assumption. $\square$

We will denote the equilibrium measure of a compact set $K \subset \mathbb{C}$ by μ_K) after this point.

Theorem 2.2.12. *[5] [Frostman's theorem] Let $K \subset \mathbb{C}$ be a compact set where cap$(K) > 0$. Then we have,*

(a) $U^{\mu_K}(z) \leq V_K$ for all $z \in \mathbb{C}$.

(b) $U^{\mu_K}(z) = V_K$ on K except a set of capacity zero (i.e. quasi-everywhere).

Theorem 2.2.13. *[1] If $\sigma \in M_1^*(K)$ and if $U^\sigma(z)$ coincides with a constant F quasi-everywhere on supp(σ) and it is at least as large as F on K, then $\sigma = \mu_K$.*

Frostman's theorem is also called the fundamental theorem of potential theory due to its importance determining the equilibrium measure. Frostman's theorem and 2.2.13 give us a criterion to find the equilibrium measure in most cases.

Example 2.2.14. [1] Let $K = \overline{D_r(a)}$ while $\overline{D_r(a)}$ is the closed disk centered at a with radius r. Let $d\theta = \frac{ds}{2\pi r}$ where ds is the arc length measure on $\{z : |z-a| = r\}$. Then as we show in 2.1.9,

$$U^\sigma(z) = \begin{cases} \log \frac{1}{r}, & \text{if } |z-a| \leq r, \\ \log \frac{1}{|z-a|}, & \text{if } |z-a| > r. \end{cases} \tag{2.13}$$

Since U^σ is constant on K, by 2.2.13, $d\mu_K = \frac{ds}{2\pi r}$.

Example 2.2.15. Let $K = [-1, 1]$ and let $d\mu = \frac{1}{2}dx$. Then

$$U^\mu(z) = \frac{1}{2} \int_{-1}^{1} \log \frac{1}{|z-t|}\, dt. \tag{2.14}$$

Let $z - t = r$. Hence

$$U^{\mu}(z) = -\frac{1}{2}\int_{z-1}^{z+1} \log|r|dr = 1 - \frac{1}{2}[(1+z)\ln 1 + z + (1-z)\ln 1 - z]. \quad (2.15)$$

Now, we have $U^{\mu}(-1) = U^{\mu}(1) = 1 - \log 2$ and $U^{\mu}(0) = 1$. Let us differentiate $U^{\mu}(x)$ on $[-1, 1]$. $(U^{\mu})'(x) = \ln 1 + x - \ln 1 - x$. The derivative $(U^{\mu})'$ is equal to zero only if $z = 0$ on $[-1, 1]$. Hence (U^{μ}) increases on $[-1, 0]$ and decreases on $[0, 1]$. Thus, it can take same values only for two points. Therefore, by Frostman's theorem μ is not an equilibrium measure.

2.3 The Green Function

Definition 2.3.1. Let $K \subset \mathbb{C}$ and let $f : K \to \overline{\mathbb{R}}$ be an extended real valued function. Then f is said to be lower semi-continuous if for every $x \in K$ and $\alpha \in \mathbb{R}$ with $\alpha < f(x)$ there is a neighborhood O of x such that for every $y \in O \cap K$, we have $f(y) > \alpha$.

Theorem 2.3.2. *[1] The limit function of an increasing sequence of continuous functions is lower semi-continuous.*

Definition 2.3.3. Given a compact set $K \subset \mathbb{C}$ with $cap(K) > 0$, the *Green function of K with pole at infinity* is the function $g_K(z, \infty)$ defined in the unbounded component Ω of $\overline{\mathbb{C}} \setminus K$ with the following properties:

(a) $g_K(z, \infty)$ is harmonic and nonnegative in $\Omega \setminus \{\infty\}$.

(b) $\lim\limits_{|z|\to\infty} (g_K(z, \infty) - \log|z|) = \log \frac{1}{cap(K)}$.

(c) $\lim\limits_{|z|\to z'} g_K(z, \infty) = 0$ with $z \in \Omega$, quasi-everywhere on $z' \in \partial\Omega$.

Lemma 2.3.4. *[1] [Generalized minimum principle] Let $D \subset \overline{\mathbb{C}}$ be a domain and let g be a function such that its first and second partial derivatives are continuous on D and g satisfies $\Delta g \leq 0$ on D. Let g be bounded below and satisfies $\liminf\limits_{z\to z'\, z\in D} g(z) \geq m$ quasi-everywhere $z' \in \partial D$. Then $g(z) > m$, $z \in D$ unless it is constant.*

Proof. For the proof see p. 39 of [1]. □

Lemma 2.3.5. *[1] If u is harmonic and bounded in some punctured disk about z_0 then u can be defined at z_0 so that u is harmonic at z_0.*

Theorem 2.3.6. *The Green function with pole at infinity for a set K with $cap(K) > 0$ exists and is unique.*

Proof. We first show the existence of the Green function. Let $g_K(z, \infty) = V_K - U^{\mu_K}(z)$. Let us show that, this function satisfies (a), (b) and (c) of 2.3.3.

(a) By Frostman's theorem, $U^{\mu_K}(z) \leq V_K$ for all $z \in \mathbb{C}$ which gives nonnegativity. Note that $supp(\mu_K) \subset K$. Hence $U^{\mu_K}(z)$ is harmonic in $\Omega \setminus \{\infty\}$ which gives the harmonicity of $V_K - U^{\mu_K}(z)$.

(b)

$$\lim_{|z|\to\infty} (-U^{\mu_K}(z) - \log|z|) = \lim_{|z|\to\infty} \int [\log|z-t| - \log|z|]\, d\mu(t) \tag{3.1}$$

$$= \lim_{|z|\to\infty} \int \log \frac{|z-t|}{|z|}\, d\mu(t) \tag{3.2}$$

$$\leq \limsup_{t\in K, z\to\infty} \log \frac{|z-t|}{|z|}, \tag{3.3}$$

which is clearly equal to zero since K is a compact set and μ is a unit measure. Therefore, we have $\lim\limits_{|z|\to\infty} (g_K(z, \infty) - \log|z|) = \log \frac{1}{cap(K)}$.

(c) Similarly, by Frostman's theorem, $U^{\mu_K}(z) \leq V_K$ for all $z \in \mathbb{C}$ and we can rewrite $U^{\mu_K}(z)$ as $\lim\limits_{M\to\infty} \int \min\left(M, \log \frac{1}{|z-t|}\right) d\mu(t)$. This implies that U^{μ_K} is limit of a sequence of continuous increasing functions. Therefore it is lower semi-continuous by 2.3.2. Hence we can write

$$U^{\mu_K}(z_0) \leq \liminf_{z\to z_0} U^{\mu_K}(z) \leq \limsup_{z\to z_0} U^{\mu_K}(z) \leq V_K. \tag{3.4}$$

If $U^{\mu_K}(z_0) = V_K$ then $\lim\limits_{z\to z_0} U^{\mu_K}(z) = V_K$ by lower semi-continuity. Since U^{μ_K} is equal to V_K quasi-everywhere on $\partial\Omega$ we have $\lim\limits_{z\to z'} g_K(z, \infty) = 0$ quasi-everywhere for $z' \in \partial\Omega$.

Now we show the uniqueness of this function. Let g' also satisfy a,b and c. Then $g' - g_K(z,\infty)$ is also harmonic on $\Omega \setminus \{\infty\}$. Since g' is nonnegative and $U^{\mu_K}(z)$ is bounded above, $g' - g_K(z,\infty)$ is bounded below on Ω. By our assumptions $g' - g_K(z,\infty)$ vanishes at infinity. Hence by 2.3.5 $g' - g_K(z,\infty)$ is harmonic at infinity. Furthermore $\lim_{z\to z_0}(g' - g_K(z,\infty))(z_0) \equiv 0$ quasi-everywhere on $z_0 \in \partial\Omega$. Then by the generalized minimum principle $g' - g_K(z,\infty) \equiv 0$ which completes the proof. $\square$

Theorem 2.3.7. *[6][Riemann] Let $G \subsetneq \mathbb{C}$ be a simply connected domain and let $w_0 \in G$, $\alpha \leq 2\pi$. Then there is a unique conformal map of D onto G such that $f(0) = w_0$ and $\arg f'(0) = \alpha$.*

A consequence of this theorem which is very useful for us. Let G be a simply connected domain with $\infty \in G$. Then there is unique conformal mapping from G onto the exterior of $\mathbb{D}$ such that

$$\Phi(\varsigma) = b\varsigma + b_0 + \frac{b_1}{\varsigma} + ... \tag{3.5}$$

with $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$.

Theorem 2.3.8. *[3] Let $\Omega = \mathbb{C}\setminus K$ be simply connected with $K \subset \mathbb{C}$ is a compact set. Then $g_K(z,\infty) = \log|\Phi(z)|$, $z \in \Omega$, where $w = \Phi(z)$ is the unique Riemann mapping from Ω to the exterior of the unit disk.*

Proof. As wee see Φ has a Laurent expansion of the form (3.5), with $b > 0$. Now we prove this function satisfies a,b and c of 2.3.3.

(a) Since $\Phi(z)$ maps onto the exterior of the unit disk, $|\Phi(z)| \geq 1$ and $\log|\Phi(z)| \geq 0$ which gives nonnegativity. Since $\Phi(z)$ is holomorphic in $\Omega \setminus \{\infty\}$, $\log\Phi(z)$ is also holomorphic there. This implies that $\log|\Phi(z)|$ which is the real part of $\log\Phi(z)$ is harmonic on $\Omega \setminus \{\infty\}$ which proves the first part.

(b) $\lim_{z\to\infty}(\log|\Phi(z)| - \log|z|) = \lim_{z\to\infty}\left(\log\frac{|z|}{|b|} - \log|z|\right) = \log\frac{1}{b}$. So letting $b = \frac{1}{cap(K)}$ shows that part (b) is also satisfied.

(c) It maps $\partial\Omega$ to unit circle and it is harmonic on Ω which means that it is continuous there. Hence $\lim_{|z|\to z'} |\Phi(z)| = 1$ for quasi-everywhere $z' \in \partial\Omega$ which implies that $\lim_{|z|\to z'} \log|\Phi(z)| = 0$ for quasi-everywhere $z' \in \partial\Omega$.

$\square$

Example 2.3.9. [1] Let $K = [-1, 1]$. Then $\Psi = \frac{1}{2}\left(w + \frac{1}{w}\right)$ maps the exterior of the unit circle onto $\Omega := \overline{\mathbb{C}} \setminus [-1, 1]$ with $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. And its inverse is given by $w = \Phi(z) = z + \sqrt{z^2 - 1}$. Therefore, by 2.3.8, $g_K(z, \infty) = \log|z + \sqrt{z^2 - 1}|$. Besides, as $|z| \to \infty$, $\sqrt{z^2 - 1}$ will behave as z. Therefore, $\lim_{|z|\to\infty} [\log|z + \sqrt{z^2 - 1}| - \log|z|] = \log 2$. Hence $cap(K) = \frac{1}{2}$.

This example is a good illustration of calculating the Green function and capacity for a continuum K if the map between the exterior of the unit disk and unbounded component of $\overline{\mathbb{C}} \setminus K$ is known or easy to guess.

Our last theorem about the Green function which shows the connection between potential theory and polynomial inequalities is called the Bernstein-Walsh lemma.

Theorem 2.3.10. *If $P_n(z)$ is any polynomial of degree $\leq n$, then*

$$|P_n(z)| \leq \|P_n\|_K e^{ng_K(z,\infty)}\ , z \in \Omega, \tag{3.6}$$

where $\|P_n\|_K := \max_{z\in K} |P_n(z)|$ and Ω is the unbounded component of $\overline{\mathbb{C}} \setminus K$.

2.4 Capacity

In this section, we give some basic results about capacity and calculate capacity of some widely used sets.

Theorem 2.4.1. *[4] Let μ be a Borel measure on $\mathbb{C}$ with compact support with $I(\mu)$ is finite. Then*

$$cap(K) = 0 \implies \mu(K) = 0. \tag{4.1}$$

Theorem 2.4.2. *[4] If $K_1 \subset K_2$ are of positive capacity then $cap(K_1) \leq cap(K_2)$.*

Proof. Let $u(z) = g_{K_1}(z,\infty) - g_{K_2}(z,\infty)$. It is harmonic on $\mathbb{C} \setminus K_2$ clearly. By 2.3.4, in any domain of $\mathbb{C}$ containing ∂K_2, $g_{K_1}(z,\infty) - g_{K_2}(z,\infty)$ is nonnegative since $g_{K_1}(z,\infty) - g_{K_2}(z,\infty)$ is nonnegative on ∂K_2. This implies that $u(z)$ is harmonic and nonnegative on $\mathbb{C} \setminus K_2$. Moreover $\lim\limits_{|z|\to\infty} g_{K_1}(z,\infty) - g_{K_2}(z,\infty)$ is finite. Hence u is harmonic at $\{\infty\}$, by 2.3.5. Applying generalized minimum principle again, we get $u(\infty) \geq 0$. Using the fact that $u(\infty) \geq 0$ and the definition of the Green function, we have $cap(K_1) \leq cap(K_2)$. □

Theorem 2.4.3. *[4] Let K be a compact set and let*

$$p(z) = \sum_{j=0}^{d} a_j z^j, \;\; a_d \neq 0. \tag{4.2}$$

Then

$$cap(p^{-1}(K)) = \left(\frac{cap(K)}{|a_d|}\right)^{\frac{1}{d}} \tag{4.3}$$

where $p^{-1}(K) := \{z \in \mathbb{C} : p(z) \in K\}$.

Proof. Let $K' := p^{-1}(K)$. Let Ω be the exterior domain for K and Ω' be the exterior domain for K'. Then $p(\Omega') = \Omega$ and $p(\partial\Omega') = \partial\Omega$. Let us prove that

$$g_K(p(z),\infty) = d g_{K'}(z,\infty). \tag{4.4}$$

Let $f_1(z) = g_K(p(z),\infty)$ and $f_2(z) = d g_{K'}(z,\infty)$. Then we have $f_1|_{\partial\Omega'} = 0$ quasi-everywhere and f_1 is harmonic on $\Omega \setminus \{\infty\}$. Moreover $\lim\limits_{|z|\to\infty} [f_1(z) - \log|p(z)|]$ is finite. For f_2 similarly we have $f_2|_{\partial\Omega'} = 0$ quasi-everywhere and f_2 is harmonic on $\Omega \setminus \{\infty\}$. In addition, $\lim\limits_{|z|\to\infty} [f_1(z) - \log|p(z)|] = d\,[g_{K'}(z,\infty) - \log|z|] + \log|z|^d - \log|p(z)|$is finite. This implies that $\lim\limits_{|z|\to\infty} (f_1(z) - f_2(z))$ is finite. Therefore, by 2.3.5, $f_1(z) - f_2(z)$ is harmonic at $\{\infty\}$. We have $f_1(z) - f_2(z) = 0$ on $\partial\Omega'$ and harmonicity of $f_1(z) - f_2(z)$ on Ω' including infinity. Therefore, $f_1(z) - f_2(z)$ is also bounded below on Ω'. We have exactly same conditions for $f_2(z) - f_1(z)$. Then by 2.3.4, $f_2(z) - f_1(z) > 0$ and $f_1(z) - f_2(z) > 0$ at infinity unless it is constant. Hence we get $f_1(z) = f_2(z)$. In other words $g_K(p(z),\infty) = g_{K'}(z,\infty)$.

Using this, we get

$$g_K(p(z),\infty) - d\,\log|z| = d\,(g_{K'}(z,\infty) - \log|z|). \tag{4.5}$$

Let us rewrite the right side as

$$(g_K(p(z),\infty) - \log|p(z)|) + \log|p(z)/z^d|. \tag{4.6}$$

Taking the limit of both sides as $|z| \to \infty$ and using the fact that $p(z)$ also goes to ∞, we get

$$\log\frac{1}{cap(K)} + \log|a_d| = d\,\log\frac{1}{cap(K')}, \tag{4.7}$$

which gives the result just by using the basic properties of the logarithm. $\square$

Example 2.4.4. Let $K = [a,b]$ be an interval on real line. At 2.3.9, we prove that $[-1,1]$ has capacity $\frac{1}{2}$. Let $p(z) = \frac{b-a}{2}z + \frac{b+a}{2}$. Then $p^{-1}([a,b]) = [-1,1]$. Using 2.4.3, we get $cap(p^{-1}([a,b])) = cap([-1,1]) = \frac{2cap([a,b])}{b-a}$. Hence $cap([a,b]) = \frac{b-a}{4}$.

Example 2.4.5. Let $K = [-b,-a] \cup [a,b]$. Let $p(z) = z^2$. We clearly have $p^{-1}([a^2,b^2]) = [-b,-a] \cup [a,b]$. Similar to the previous example, by 2.4.3, we have $cap(p^{-1}([a^2,b^2]) = cap([-b,-a] \cup [a,b]) = \sqrt{cap([a^2,b^2])}$. Therefore, $cap([-b,-a] \cup [a,b]) = \sqrt{\frac{b^2-a^2}{4}} = \frac{\sqrt{b^2-a^2}}{2}$.

2.5 Chebyshev Constant

Let $K \subset \mathbb{C}$ be a compact set. Let $M_n(K) := \min\limits_{p\in\mathcal{P}_{n-1}} \|z^n + p(z)\|_K$ where $\mathcal{P}_{n-1}$ denotes the collection of all polynomials of degree less than or equal to $n-1$ and $\|.\|_K$ denotes the sup norm on K. The problem of finding M_n for a given set K is called the minimax problem. Note that this problem is equivalent to finding $\min\limits_{p\in P'_n} \|p(z)\|_K$ where $\mathcal{P}'_n$ denotes the monic polynomials of degree n.

Theorem 2.5.1. *[7] Let K be compact set in the complex plane that contains more than $n+1$ points. Then there exists unique polynomial $p(z)$ such that*

$$M_n = \min_{p\in\mathcal{P}'_n} \|p(z)\|_K \tag{5.1}$$

is satisfied.

Definition 2.5.2. Let K be compact set in the complex plane that contains more than $n+1$ points. Then the polynomial which satisfies

$$M_n = \min_{p \in P'_n} \|p(z)\|_K, \tag{5.2}$$

is called *nth Chebyshev polynomial for* K.

Theorem 2.5.3. *[8] Let $K \in \mathbb{C}$ be a compact set which has infinitely many points. Let $\tau_n = (M_n)^{\frac{1}{n}}$ where M_n is the sup norm of the nth Chebyshev polynomial for K. Then τ_n converges.*

Proof. Since K is compact, there exist disks centered at 0 that contain K. Fix any disk D which satisfies this condition. Let r be the diameter of this disk. Then we have,

$$\tau_n \leq \sqrt[n]{\max_{z \in K}(z - z_0)^n} \leq \max_{z \in K} |z - z_0| \leq r, \tag{5.3}$$

for $z_0 \in K$. This implies that τ_n is bounded above.

Let $\liminf_{n\to\infty}(\tau_n) := \alpha$ and $\limsup_{n\to\infty}(\tau_n) := \beta$. Cleearly, we have $\alpha \leq \beta$. We want to prove the inverse. Fix $\epsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\tau_n < \alpha + \epsilon$. Then for any $k, l \in \mathbb{N}$, we have the inequality

$$|(z - z_0)^l t_n(z)^k| \leq r^l (\alpha + \epsilon)^{nk}, \tag{5.4}$$

on K where $t_n(z)$ is the nth Chebyshev polynomial for K. This implies that $M_{nk+l} \leq r^l(\alpha + \epsilon)^{nk}$. So we have

$$\tau_{nk+l} \leq r^{\frac{l}{nk+l}} (\alpha + \epsilon)^{\frac{nk}{nk+l}}. \tag{5.5}$$

By the definition of lim sup there exists a subsequence of τ_n such that $\lim_{n\to\infty} \tau_{n_\nu} = \beta$. But for any $\nu \in \mathbb{N}$ we have $n_\nu = nk_\nu + l_\nu$ such that $0 < l_\nu \leq n$ where k_ν, l_ν are uniquely determined by ν. Now in the inequality (5.5) put l_ν instead of l and put k_ν instead of k. Therefore we have,

$$\tau_{nk_\nu + l_\nu} \leq r^{\frac{l}{nk_\nu + l_\nu}} (\alpha + \epsilon)^{\frac{nk_\nu}{nk_\nu + l_\nu}}. \tag{5.6}$$

Letting ν go to infinity, we have $\beta \leq \alpha + \epsilon$. Since ϵ is chosen arbitrarily, we have $\beta \leq \alpha$. So we have $\alpha = \beta$. Since τ_n is bounded below by 0 and above by r, this implies that τ_n converges to a real number. $\square$

Definition 2.5.4. Let $K \in \mathbb{C}$ be a compact set which has infinitely many points. Let $t_n(z)$ be nth Chebyshev polynomial for K. Let $\tau_n = (M_n)^{\frac{1}{n}}$ where M_n is the sup norm of t_n on K. Then $\tau := \lim_{n\to\infty}(\tau_n)$ is called the *Chebyshev constant of* K.

Example 2.5.5. [8] Let $K = \overline{\mathbb{D}(0,r)}$ closed disk centered at 0 with radius r. Let $p_n(z) = z^n + a_{n-1}z^{n-1} + \cdots a_0$ be a monic polynomial of degree n. Then for $z = re^{i\theta}$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} |p_n(z)|^2 \, d\theta = r^{2n} + |a_{n-1}r|^{2n-2} + \cdots + |a_0|^2 \geq r^{2n} \tag{5.7}$$

by using $\int_0^{2\pi} e^{ik\theta} \, d\theta = 0$ for $k \in \mathbb{Z}$ and $|p_n(z)|^2 = p_n(z)\overline{p_n(z)}$. This implies that $M_n \geq r^n$. But for the polynomial z^n, we have $\max_{z\in K}(z^n) = r^n$. Therefore $M_n = r^n$. This implies that, $\tau_n = r$, $\forall r$. Hence Chebyshev constant for $K = \overline{\mathbb{D}(0,r)}$ is r.

The last section may give to the reader the impression that the Chebyshev constant and potential theory are quite irrelevant, but as we see in the next chapter, there is a close relationship between the Chebyshev constant and capacity.

Chapter 3

Fekete Points and Leja Sequences

3.1 Transfinite Diameter and Fekete Points

Definition 3.1.1. Let K be a compact set in $\mathbb{C}$. Let $\{z_1, z_2, \ldots, z_n\} \in K$. Then

$$V(z_1, z_2, \ldots, z_n) = \begin{vmatrix} 1 & z_1 & z_1^2 & \ldots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \ldots & z_2^{n-1} \\ 1 & z_3 & z_3^2 & \ldots & z_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \ldots & {z_n}^{n-1} \end{vmatrix} \tag{1.1}$$

is called the *Vandermonde determinant* associated with $\{z_1, z_2, \ldots, z_n\}$.

As we prove in section 3.2.1, $V(z_1, z_2, \ldots, z_n) = \prod_{1 \le i < j \le n} (z_j - z_i)$.

Definition 3.1.2. Let K be a compact set in $\mathbb{C}$. Let $V_n(K) := \max_{z_1, z_2, \ldots, z_n \in K} \prod_{1 \le i < j \le n} (|z_j - z_i|)$ and $d_n(K) := \max_{z_1, z_2, \ldots, z_n \in K} \prod_{1 \le i < j \le n} (|z_j - z_i|)^{\frac{2}{n(n-1)}}$. By compactness argument, there are points $z'_1, z'_2, \ldots, z'_n \in K$ such that $|V(z'_1, z'_2, \ldots, z'_n)^{\frac{2}{n(n-1)}}| = d_n(K)$. Then $F_n := \{z'_1, z'_2, \ldots, z'_n\}$ is called an *n-point Fekete set of* K.

Theorem 3.1.3. *[8] Let $K \in \mathbb{C}$ be a compact set. Let $d_n(K)$ be defined as in the definition above. Then $(d_n(K))$ converges to a real number $d(K)$.*

Proof. Let $z_1, z_2, \dots z_{n+1} \in K$ such that $\{z_1, z_2, \dots z_{n+1}\}$ be an $(n+1)$-point Fekete set for K. Then

$$V_{n+1}(K) = |(z_1 - z_2)(z_1 - z_3) \dots (z_1 - z_{n+1}) V(z_2, z_3, \dots, z_{n+1})| \tag{1.2}$$

$$\leq |(z_1 - z_2)(z_1 - z_3) \dots (z_1 - z_{n+1})| V_n(K). \tag{1.3}$$

Similarly we have

$$V_{n+1}(K) = |(z_2 - z_1)(z_2 - z_3) \dots (z_2 - z_{n+1}) V(z_1, z_3, \dots, z_{n+1})| \tag{1.4}$$

$$\leq |(z_2 - z_1)(z_2 - z_3) \dots (z_2 - z_{n+1})| V_n(K). \tag{1.5}$$

And for $n+1$ we have

$$V_{n+1}(K) = |(z_{n+1} - z_1)(z_{n+1} - z_2) \dots (z_{n+1} - z_n) V(z_1, z_2, \dots, z_n)| \tag{1.6}$$

$$\leq |(z_{n+1} - z_1)(z_{n+1} - z_2) \dots (z_{n+1} - z_n)| V_n(K). \tag{1.7}$$

Hence multiplying these $n+1$ inequalities we get $(V_{n+1}(K))^{n+1} \leq (V_n(K))^2 (V_n)^{n+1}$. Therefore, we have $(V_{n+1}(K))^{n-1} \leq (V_n(K))^{n+1}$. Taking $\frac{2}{(n-1)n(n+1)}$ th power of both sides, we get $(V_{n+1}(K))^{\frac{2}{n(n+1)}} \leq (V_n(K))^{\frac{2}{n(n-1)}}$. This is equivalent to saying that $d_{n+1}(K) \leq d_n(K)$. Clearly $(d_n(K))_{n=1}^{\infty}$ is a decreasing sequence bounded above and below by 0. Therefore $(d_n(K))$ is convergent. □

Definition 3.1.4. Let $K \in \mathbb{C}$ be a compact set. Let $d(K) := \lim_{n \to \infty} d_n(K)$. Then $d(K)$ is called the *transfinite diameter* of K.

The next result is one of the most important results in potential theory.

Theorem 3.1.5. *[3] Let $K \in \mathbb{C}$ be a compact set. Then the capacity, the Chebyshev constant and the transfinite diameter of K coincides i.e.*

$$\tau(K) = cap(K) = d(K). \tag{1.8}$$

Proposition 3.1.6. *Let K be a compact set in $\mathbb{C}$. Let $F_n(K) = \{z_1, z_2, \dots, z_n\}$ be an n-point Fekete set of K. Then $\{z_1, z_2, \dots, z_n\} \in \partial K$.*

Proof. Suppose $\exists k$, $1 \leq k \leq n$, such that $z_k \notin K$. Let $p_k(z) := (z - z_1)(z - z_2) \dots (z - z_{k-1})(z - z_{k+1} \dots (z - z_n)$ with $z \in \mathbb{C}$. Clearly, $p_k(z)$ is analytic on $\mathbb{C}$. Then by maximum modulus principle $|p_k(z)|$ takes its maximum value for K on ∂K. Let $w \in K$ be such that $|p_k(w)| = \max_{z\in K}(|p_k(z)|)$. Then $|p_k(w)\, V(z_1, z_2, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)| > |p_k(z_k)\, V(z_1, z_2, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)|$. But, this implies that

$$|V(z_1, z_2, \ldots, z_{k-1}, z_w, z_{k+1}, \ldots, z_n| > |V(z_1, z_2, \ldots, z_{k-1}, z_k, z_{k+1}, \ldots, z_n)|, \quad (1.9)$$

which contradicts the fact that $\{z_1, z_2, \ldots, z_n\}$ is an n-point Fekete set. □

Corollary 3.1.7. *Let K be a compact set in $\mathbb{C}$. Then $cap(K) = cap(\partial K)$ and $\mu_K = \mu_{\partial K}$.*

Proof. Since Fekete points lie on the boundary of K we get $d(K) = d(\partial K)$. From 3.1.5, we get $\tau(K) = cap(K)$ and $\tau(\partial K) = cap(\partial K)$. Hence, we have $cap(K) = cap(\partial K)$. By 2.2.12, $U^{\mu_K}(z) = V_K$ quasi-everywhere on K. Then $U^{\mu_K}(z) = V_K$ on ∂K quasi-everywhere. Then by 2.2.13, we get $\mu_K = \mu_{\partial K}$. □

Proposition 3.1.8. *[6]Let K be a compact set in $\mathbb{C}$. Then*

(i) If $z^* = az + b$ maps K to K^*, then $cap(K^*) = |a|cap(K)$.

(ii) If $|\Phi(z) - \Phi(z')| \leq |z - z'|\ \forall z, z' \in K$, then $cap(\Phi(K)) \leq cap(K)$.

Proof. (i) If $\{z_1, z_2, \ldots, z_n\}$ is an n-point Fekete set of K then $\{az_1 + b, az_2 + b, \ldots, az_n + b\}$ is an n-point Fekete set for K^*. Hence $V_n(K) = |V(z_1, z_2, \ldots, z_n)|$ and $V_n(K^*) = |V(az_1+b, az_2+b, \ldots, az_n+b)|$. Therefore, $V_n(K^*) = |a|^{\frac{n(n-1)}{2}} V_n(K)\ \forall n \in \mathbb{N}$. This implies that $d_n(K^*) = |a|d_n(K)$. Letting $n \to \infty$ we have $d(K^*) = |a|d(K)$. By 3.1.5, we reach the result.

(ii) Let $\{\Phi(z_1), \Phi(z_2), \ldots, \Phi(z_n)\}$ be an n-point Fekete set for $\Phi(K)$. Then $V_n(\Phi(K)) = \prod_{1\leq i<j\leq n} (|\Phi(z_j) - \Phi(z_i)|) \leq \prod_{1\leq i<j\leq n} (z_j - z_i) = |V(z_1, z_2, \ldots, z_n)| \leq V_n(K)$. Thus, $V_n(\Phi(K) \leq V_n(K)$.

□

3.2 Applications of Fekete Points to Approximation Theory

Before giving information about the applications of Fekete points to approximation theory, we give a couple of basic theorems which make the connection between Fekete points and approximation theory.

Theorem 3.2.1. *[7] Given $n+1$ distinct points of complex plane $z_0, z_1, \ldots, z_n$ and complex values $w_0, w_1, \ldots, w_n$, there exists a unique complex valued polynomial $p_n(z) \in \mathcal{P}_n$ such that*

$$p_n(z_i) = w_i, \ \forall i \in \{0, 1, \ldots, n\} \tag{2.1}$$

where $\mathcal{P}_n$ is the set of all polyonimals which has degree less than or equal to n defined on the complex plane.

Proof. Let $p_n(z) = a_n z^n + z_{n-1} z^{n-1} + \ldots + a_0$. Then $p_n(z)$ satisfies (2.1) if and only if

$$a_0 + a_1 z_i + \ldots + a_n {z_i}^n = w_i \ \ , \forall i \in \{0, 1, \ldots, n\}. \tag{2.2}$$

This system of $n+1$ linear equations has unique solution if and only if

$$V(z_0, z_1, z_2, \ldots, z_n) = \begin{vmatrix} 1 & z_0 & z_0^2 & \ldots & z_0^n \\ 1 & z_1 & z_1^2 & \ldots & z_1^n \\ 1 & z_2 & z_2^2 & \ldots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \ldots & {z_n}^n \end{vmatrix} \neq 0. \tag{2.3}$$

Letting $z_n = z$ we get,

$$V(z_0, z_1, z_2, \ldots, z) = \begin{vmatrix} 1 & z_0 & z_0^2 & \ldots & z_0^n \\ 1 & z_1 & z_1^2 & \ldots & z_1^n \\ 1 & z_2 & z_2^2 & \ldots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z & z^2 & \ldots & z^n \end{vmatrix}. \tag{2.4}$$

Vandermonde determinant $V(z_0, z_1, z_2, \ldots, z)$ is clearly in $\mathcal{P}_n$ and it vanishes at the points $z_0, z_1, z_2, \ldots z_{n-1}$. Hence we have,

$$V(z_0, z_1, z_2, \ldots, z) = A(z - z_0)(z - z_1) \ldots (z - z_{n-1}) \ , A \in \mathbb{C}. \tag{2.5}$$

By expanding the determinant, we easily see that the coefficient of z_n is $V(z_0, z_1, \ldots, z_{n-1})$. Therefore,

$$V(z_0, z_1, z_2, \ldots, z) = V(z_0, z_1, \ldots, z_{n-1})(z - z_0)(z - z_1) \ldots (z - z_{n-1}). \tag{2.6}$$

Letting $z = z_n$ we get the following recursion relation:

$$V(z_0, z_1, z_2, \ldots, z_n) = V(z_0, z_1, \ldots, z_{n-1})(z_n - z_0)(z_n - z_1) \ldots (z_n - z_{n-1}). \tag{2.7}$$

It is easy to see that $V(z_0, z_1) = z_1 - z_0$ and $V(z_0, z_1, z_2) = (z_1 - z_0)(z_2 - z_0)(z_2 - z_1)$. Using the recursion formula above and induction we get

$$V(z_0, z_1, z_2, \ldots, z_n) = \prod_{0 \le i < j \le n} (z_j - z_i). \tag{2.8}$$

Since $z_i \neq z_j$, $\forall i \neq j$, then $V(z_0, z_1, z_2, \ldots, z_n) = \prod_{0 \le i < j \le n} (z_j - z_i) \neq 0$, which prove the uniqueness of the polynomial $p_n(z)$. $\square$

Definition 3.2.2. Let $l_k(z) := \frac{(z-z_0)(z-z_1)\ldots(z-z_{k-1})(z-z_{k+1})(z-z_{k+2})\ldots(z-z_n)}{(z_k-z_0)(z_k-z_1)\ldots(z_k-z_{k-1})(z_k-z_{k+1})(z_k-z_{k+2})\ldots(z_k-z_n)}$, $k = 0, 1, \ldots, n$, be polynomials of degree n defined on the complex plane where $z_0, z_1, \ldots, z_n \in \mathbb{C}$ which satisfy

$$l_k(z_j) = \delta_{kj} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{2.9}$$

Then $l_k(z)$, $k = 0, 1, \ldots, n$, are called *fundamental Lagrange polynomials* for pointwise interpolation.

Theorem 3.2.3. *Let $z_0, z_1, \ldots, z_n, w_0, w_1, \ldots, w_n \in \mathbb{C}$. Then the unique polynomial which has degree at most n defined on the complex plane satisfying $p_n(z_k) = w_k$, for $k = 0, 1, 2, \ldots, n$, is given by $p_n(z) = \sum_{k=0}^{n} w_k l_k(z)$.*

Proof. It is easy to see that $\sum_{k=0}^{n} w_k l_k(z)$ is a polynomial have degree at most n satisfies $\sum_{k=0}^{n} w_k l_k(z) = w_k$. It is unique by 3.2.1. $\square$

Definition 3.2.4. Let $z_0, z_1, \ldots, z_n, w_0, w_1, \ldots, w_n \in \mathbb{C}$. Then the polynomial satisfies $p_n(z) = \sum_{k=0}^{n} w_k l_k(z)$ is called the *Lagrange interpolating polynomial* associated with $z_0, z_1, \ldots, z_n, w_0, w_1, \ldots, w_n \in \mathbb{C}$.

Theorem 3.2.5. *Let $F_{n+1} = \{z_0, z_1, \ldots, z_n\}$ be an $(n+1)$-point Fekete set of a compact set $K \subset \mathbb{C}$. Let $l_k(z)$, $k = 0, 1, \ldots, n$, be fundamental Lagrange polynomials. Then $|l_k(z)| \leq 1$ for $k = 0, 1, \ldots, n$ and $z \in K$. Moreover, if $p_n(z) \in \mathcal{P}_n$ where $\mathcal{P}_n$ is the set of all polynomials at most degree n, then*

$$\|p_n\|_K \leq (n+1)\|p_n\|_{F_n} \tag{2.10}$$

where $\|p_n\|_K = \max_{z \in K} p_n(z)$.

Proof. First, let us prove that $|l_k(z)| \leq 1$ on K. Let $k \in \{0, 1, \ldots, n\}$ and $z \in K$ be chosen arbitrarily. Then

$$\frac{|V(z_0, z_1, \ldots, z_{k-1}, z, z_{k+1}, z_{k+2}, \ldots, z_n)|}{|V(z_0, z_1, \ldots, z_{k-1}, z_k, z_{k+1}, z_{k+2}, \ldots, z_n)|} = \tag{2.11}$$

$$\frac{|(z - z_0)(z - z_1) \ldots (z - z_{k-1})(z - z_{k+1})(z - z_{k+2}) \ldots (z - z_n)|}{|(z_k - z_0)(z_k - z_1) \ldots (z_k - z_{k-1})(z_k - z_{k+1})(z_k - z_{k+2}) \ldots (z_k - z_n)|} = |l_k(z)| \leq 1 \tag{2.12}$$

since $\{z_0, z_1, \ldots, z_n\}$ is an $(n+1)$-point Fekete set for K.

Let $p_n(z) \in \mathcal{P}_n$. Then by the uniqueness of the Lagrange interpolating polynomial, we have $p_n(z) := \sum_{k=0}^{n} p_n(z_k) l_k(z)$. Then for any $z \in K$, we have

$$|p_n(z)| \leq \sum_{k=0}^{n} |p_n(z_k)||l_k(z)| \leq \sum_{k=0}^{n} |p_n(z_k)| \leq (n+1) \max_{z \in F_{n+1}} (p_n(z)) \tag{2.13}$$

since $|l_k(z)| \leq 1$, by the first part. Therefore, we get

$$\|p_n\|_K \leq (n+1)\|p_n\|_{F_n}. \tag{2.14}$$

$\square$

Theorem 3.2.6. *[7] Let K be a compact set in $\mathbb{C}$ and f be a continuous function on K. Let $\mathcal{P}_n$ be the set of all holomorphic polynomials defined on the complex plane which have degree at most n. Then the problem of finding $\min\limits_{p\in\mathcal{P}_n}\max\limits_{z\in K}|f(z)-p(z)|$ has a unique solution.*

Definition 3.2.7. Let f be a continuous function on a compact set $K \in \mathbb{C}$. Then the polynomial solving the problem $\min\limits_{p\in P_n}\max\limits_{z\in K}|f(z)-p(z)|$, is called the *best uniform approximation* to f on K out of $\mathcal{P}_n$.

Definition 3.2.8. Let K be a compact set in $\mathbb{C}$ and let $l_k(z)$, $k = 0, 1, \ldots, n$, be fundamental Lagrange polynomials associated with $z_0, z_1, \ldots, z_n \in K$. Then $\lambda_n(z) := \sum_{k=0}^{n}|l_k(z)|$ is called the *Lebesgue function* and $\Lambda_n = \max_{z\in K}\lambda_n(z)$ is called *Lebesgue constant* associated with $\{z_0, z_1, \ldots, z_n\}$.

The Lebesgue constant is of extreme importance since it links the best uniform approximation to the Lagrange interpolating polynomial.

Theorem 3.2.9. *[9] Let $p_n^* \in \mathcal{P}_n$ be the best uniform appriximation out of $\mathcal{P}_n$ to a function f which is continuous on a compact set $K \subset \mathbb{C}$. Let $\{z_o, z_1, \ldots, z_n\}$ be in K and let $p_n(f; z)$ be Lagrange interpolating polynomial such that $p_n(f; z) := \sum_{k=0}^{n} f(z_k)l_k(z)$. Then we have*

$$\|f(z) - p_n(f; z)\|_K \leq (1 + \Lambda_n)\|f(z) - p_n^*\|_K. \tag{2.15}$$

Corollary 3.2.10. *Let f be a continuous function on a compact set $K \subset \mathbb{C}$ and $p_n(f; z)$ be a polynomial at most degree n that interpolates f on an $(n + 1)$-point Fekete set $F_{n+1} = \{z_0, z_1, \ldots, z_n\} \in K$, i.e. $p_n(f; z) := \sum_{k=0}^{n} f(z_k)l_k(z)$. Then*

$$\|f(z) - p_n(f; z)\|_K \leq (n + 2)\|f(z) - p_n^*\|_K \tag{2.16}$$

where p_n^* is the best uniform approximation out of $\mathcal{P}_n$ to f.

Proof. Since $\{z_0, z_1, \ldots, z_n\}$ is an $(n+1)$-point Fekete set for K by 3.2.5, $l_k(z) \leq 1$ on K for every $k \in \{0, 1, \ldots, n\}$. Thus by the definition of Lebesgue function, we have $\lambda_n(z) := \sum_{k=0}^{n}|l_k(z)| \leq n + 1$. Therefore, by 3.2.9 we get

$$\|f(z) - p_n(f; z)\|_K \leq (n + 2)\|f(z) - p_n^*\|_K. \tag{2.17}$$

$\square$

This corollary implies that choosing Fekete points as interpolation nodes is suitable since it does not grow fast (which is almost optimal in general). It has polynomial growth in this case. For example choosing equally spaced nodes may lead to terrible Lebesgue constants which makes interpolation less sensitive.

There are some results for Lebesgue constants related to Fekete points on some special sets. We mention only one. For example, as you may see on [10], on $[-1, 1]$, the Lebesgue constant grows $O(\log(n))$ which is best possible for $[-1, 1]$.

3.3 Examples Of Known Locations of Fekete Points

In the previous section, we see that Fekete points can be used for interpolation and approximation. In spite of the fact that they are excellent nodes for interpolation there are only a few cases(only three) that we know the location of Fekete points analytically. These are the unit disk in $\mathbb{C}$, $[-1, 1]$, and $[-1, 1]^d$ which is the d-dimensional unit cube.

Lemma 3.3.1. *[7] Let $D = [a_{ij}]$,$i, j = 1, 2, \ldots, n$, be an $n \times n$ matrix with complex entries. Then we have*

$$|\det(D)|^2 \leq \prod_{k=1}^{n}(|a_{k1}|^2 + |a_{k2}|^2 + \ldots + |a_{kn}|^2). \tag{3.1}$$

Theorem 3.3.2. *Let $\overline{\mathbb{D}(0,1)}$ be the unit closed disk in $\mathbb{C}$. Then the solution of $z^n = 1$ constitutes an n-point Fekete set for $\overline{\mathbb{D}(0,1)}$.*

Proof. Let $a_1, a_2, \ldots, a_n \in K$. Since $|a_i| \leq 1$ for all $i \in \{1, 2, \ldots, n\}$ by 3.3.1, we get

$$|V(a_0, a_1, \ldots, a_n)| \leq (\prod_{k=1}^{n}(n))^{\frac{1}{2}} \leq n^{\frac{n}{2}}. \tag{3.2}$$

Now, we show that the set of n-th root of unity will maximize Vandermonde determinant. Let $z_j := e^{\frac{i2\pi j}{n}}$, $\forall j \in 1, 2, \ldots, n$. Hence $\overline{z_j} := e^{\frac{-i2\pi j}{n}}$, $\forall j \in 1, 2, \ldots, n$.

Thus,

$$|V(z_1, z_2, \ldots, z_n)|^2 = |\det{(z_j^{k-1})_{j,k=1}^n}||\det{(\overline{z_j^{k-1}})_{j,k=1}^n}| \tag{3.3}$$

$$= |\det{((z_j^{k-1})_{j,k=1}^n)^T} \det{(\overline{z_j^{k-1}})_{j,k=1}^n}| \tag{3.4}$$

$$= \det{(\sum_{l=1}^{n} z_j^{l-1} z_k^{1-l})_{j,k=1}^n} \tag{3.5}$$

where T is the transpose of the matrix. But powers of unity are orthogonal in the following sense: $\sum_{l=0}^{n-1} z_j^l z_k^{-l} = n\delta_{jk}$. Therefore, we have a diagonal $n \times n$ matrix which takes the value n along the diagonal. Therefore,

$$|V(z_1, z_2, \ldots, z_n)|^2 = n^n. \tag{3.6}$$

Thus, $|V(z_1, z_2, \ldots, z_n)| = n^{\frac{n}{2}}$. But as we see before, it is the upper bound for $|V|$. Therefore, the solution of $z^n = 1$ constitutes an n-point Fekete set for $\overline{\mathbb{D}(0,1)}$. □

Note that n-point Fekete set for $\overline{\mathbb{D}(0,1)}$ is not unique. If each root of unity of some degreen n is rotated with the same angle θ then pairwise distances of each root stay unchanged. Then these rotated points also constitute an n-point Fekete set for $\overline{\mathbb{D}(0,1)}$.

Our second example for the exact locations of Fekete points is $[-1, 1]$.

Definition 3.3.3. *Jacobi polynomials* are orthogonal polynomials with respect to the weight $(1-x)^{\alpha}(1-x)^{\beta}$ for $[-1, 1]$ where $\alpha, \beta > -1$.

There are a variety of ways of characterizing Jacobi polynomials. We give the characterization with Rodriguez formula: Given $\alpha, \beta > -1$, the n-th degree Jacobi polynomial is given by

$$(1-x)^{\alpha}(1-x)^{\beta} p_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n [(1-x)^{1+\alpha}(1+x)^{n+b}]. \tag{3.7}$$

There is an electrostatic problem which is stated as follows: Let $p, q > 0$. If n unit masses, $n \geq 2$, are located at $x_1, x_2, \ldots, x_n$ in $[-1, 1]$ and fixed masses p, q

at -1 and 1 then how can we maximize the following expression?

$$T(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} (1 - x_k)^p (1 + x_k)^q \prod_{1 \leq k < j \leq n} |x_j - x_k|. \qquad (3.8)$$

The answer is stated as a theorem in Szegö's book.

Theorem 3.3.4. *[11] Let $p, q > 0$ and let $\{x_k\}_{k=1}^{n}$ where $-1 \leq x_k \leq 1$, $1 \leq k \leq n$, be a system of values which maximizes $T(x_1, x_2, \ldots, x_n)$ which is defined as above. Then $\{x_k\}_{k=1}^{n}$ are the zeros of Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$ where $\alpha = 2p - 1$ and $\beta = 2q - 1$.*

Proof. For a maximum, we have $\frac{\partial T}{\partial x_\nu} = 0$ or in other saying we get the following equality:

$$\frac{1}{x_\nu - x_1} + \ldots + \frac{1}{x_\nu - x_{\nu-1}} + \frac{1}{x_\nu - x_{\nu+1}} \ldots + \frac{1}{x_\nu - x_n} + \frac{p}{x_\nu + 1} + \frac{q}{x_\nu + 1} = 0. \quad (3.9)$$

Let $f(x)$ be defined as $(x - x_1)(x - x_2) \ldots (x - x_n)$. Thus, we get

$$\frac{1}{2} \frac{f''(x_\nu)}{f'(x_\nu)} + \frac{p}{x_\nu + 1} + \frac{q}{x_\nu + 1} = 0. \qquad (3.10)$$

Equivalently, we have

$$(1 - {x_\nu}^2) f''(x_\nu) + \{2q - 2p - (2q + 2p)x_\nu\} + f'(x_\nu) = 0. \qquad (3.11)$$

The last equation means that $(1 - {x_\nu}^2) f''(x) + \{\beta - \alpha - (\alpha + \beta + 2)x\} f'(x)$ is a polynomial of degree n which vanishes for all the zeros of $f(x)$. Hence $f(x)$ is a constant times Jacobi polynomial $p_n^{(\alpha,\beta)}(x)$ by the differential equation it satisfies. $\square$

Theorem 3.3.5. *The $(n + 2)$-point Fekete set of $[-1, 1]$ are the zeros of $p_n^{(1,1)}$ together with $\{-1, 1\}$ where p_n is the Jacobi polynomial of degree n with $\alpha, \beta = 1$. Moreover, this set is unique.*

Proof. Let $p = 1$ and $q = 1$. Then $T(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n}(1 - x_k)(1 + x_k)\prod_{1\leq k<j\leq n}|x_j - x_k|$ is maximized by the zeros of $p_n^{(1,1)}$ by 3.3.4. Let $n \geq 2$. Then it is easy to see that n-point Fekete set of $[-1, 1]$ must contain the points -1 and 1. Note that, $\prod_{k=1}^{n}(1-x_k)(1+x_k)\prod_{1\leq k<j\leq n}|x_j - x_k| := V(x_1, x_2, \ldots, x_n, -1, 1)$. But, this is uniquely maximized by the roots of $p_n^{(1,1)}$. Therefore the zeros of $p_n^{(1,1)}$ together with $\{-1, 1\}$ uniquely gives $(n + 2)$- point Fekete set of $[-1, 1]$. □

For the set $[-1, 1]^d$, $(n + 1)^d$-point Fekete sets are also known. To prove this result, generalized Vandermonde determinant and multivariate polynomials are used to calculate the Vandermonde determinant. For each n, the Vandermonde determinant is the determinant of an $n^d \times n^d$ matrix. The n^d-Fekete set of $[-1, 1]^d$ is given by the tensor product of n-point Fekete set (See [12]).

3.4 Counting Measure Associated with Fekete Points and Fekete Potential

Definition 3.4.1. Let $A_n = \{z_1, z_2, \ldots, z_n\}$ be an n-point set in $\mathbb{C}$. Then the *normalized counting measure* associated with $\{z_1, z_2, \ldots, z_n\}$ is given by

$$\nu(A_n) := \frac{1}{n}\sum_{k=1}^{n}\delta_{z_k} \tag{4.1}$$

where δ_z is the unit mass at point z.

As we stated before, an n-point Fekete set is the solution of problem of maximizing $\prod\limits_{1\leq i<j\leq n}|z_i - z_j|$ where all points are chosen from a compact set $K \subset \mathbb{C}$. Note that, this problem is equivalent to minimizing $\prod\limits_{1\leq i<j\leq n}\dfrac{1}{|z_i - z_j|}$. Since log is an increasing function on $\mathbb{R}$, minimizing the following expressions also give an n-point Fekete set for K:

$$\log\prod_{1\leq i<j\leq n}\frac{1}{|z_i - z_j|} = \sum_{1\leq i<j\leq n}\log\frac{1}{|z_i - z_j|}. \tag{4.2}$$

Theorem 3.4.2. *[3] Let $F_n := \{z_1, z_2, \dots, z_n\}$ be an n-point Fekete set of a compact set K of $\mathbb{C}$ such that cap$(K) > 0$. Then $\nu(F_n) \xrightarrow{*} \mu_K$ where $\nu(F_n)$ is the normalized counting measure on Fekete points and μ_K is the equilibrium measure of K.*

Proof. Let $\hat{\mu}$ be a weak star limit of $\nu(F_n)$. Let $\log_M x := \min(M, \log x)$. Then by monotone convergence theorem, we have

$$I(\hat{\mu}) = \iint \log \frac{1}{|z-t|}\, d\hat{\mu}(z)\, \hat{\mu}(t) = \lim_{M\to\infty} \iint \log_M \frac{1}{|z-t|}\, d\hat{\mu}(z)\, \hat{\mu}(t). \tag{4.3}$$

By using the fact that $\nu(F_n) \xrightarrow{*} \hat{\mu}$ implies $\nu(F_n) \times \nu(F_n) \xrightarrow{*} \hat{\mu} \times \hat{\mu}$, we get

$$\begin{aligned}
I(\hat{\mu}) &= \lim_{M\to\infty} \lim_{n\to\infty} \iint \log_M \frac{1}{|z-t|}\, d\nu(z)\, d\nu(t) && (4.4)\\
&= \lim_{M\to\infty} \lim_{n\to\infty} \frac{1}{n^2} \sum_{l=1}^{n} \sum_{k=1}^{n} \log_M \frac{1}{|z_k - z_l|} && (4.5)\\
&= \lim_{M\to\infty} \lim_{n\to\infty} \sum_{1\le l<k\le n} \frac{2}{n^2} \log_M \frac{1}{|z_k - z_k|} + \sum_{l=k=1}^{n} \frac{2}{n^2} \log_M \frac{1}{|z_k - z_l|} && (4.6)\\
&\le \lim_{M\to\infty} \lim_{n\to\infty} \left(\frac{2}{n^2} \log \frac{1}{V_n(K)} + \frac{nM}{n^2} \right) && (4.7)\\
&= \lim_{M\to\infty} \log \frac{1}{d} = V_k, && (4.8)
\end{aligned}$$

where $V_n(K) = V(z_1, z_2, \dots, z_n)$, d is the transfinite diameter and V_K is the minimal energy. Thus, we reach that $V_K \le I(\hat{\mu}) \le V_K$. Hence, we have $I(\hat{\mu}) = V_K$. By the uniqueness of the equilibrium measure, we get $\hat{\mu} = \mu_K$ which concludes the proof. □

Definition 3.4.3. Let $F_n := \{z_1, z_2, \dots, z_n\}$ be an n-point Fekete set of a compact set $K \subset \mathbb{C}$. Then $U^{\nu(F_n)}(z) := \frac{1}{n}\sum_{k=1}^{n} \log \frac{1}{|z-z_k|}$ is called the *Fekete potential* corresponding to F_n.

There are a couple of results on how fast $U^{\nu(F_n)}(z)$ converges to $U^{\mu_K}(z)$. We give Korevaar's result.

Theorem 3.4.4. *[13] Let K be a continuum in $\mathbb{C}$. Then there are constants c_1, c_2 such that then for every $n \geq 2$ we have*

$$\log \min \{n^{-3}, d(z, \partial K)\} - c_1 \leq n(|U^{\mu_K}(z) - U^{\nu(F_n)}(z)|) \tag{4.9}$$

$$\leq \log n + \log \log n + c_2, \ \forall z \in \mathbb{C} \tag{4.10}$$

where $d(z, \partial K)$ is the smallest distance from z to the outer boundary of K.

3.5 Spacing of Fekete Points

This is very related to our research topic. There are some results for the distribution of n-point Fekete sets for continuum sets in $\mathbb{C}$. The asymptotic distribution changes with smoothness of the outer boundary of the continuum.

Theorem 3.5.1. *[13] Let K be a continuum in $\mathbb{C}$. Then $g_K(z, \infty)$ and $U^{\mu_K}(z)$ are of class $Lip\frac{1}{2}$ on a neighborhood of the outer boundary Γ of K.*

Recall that, f is of class $Lip\alpha$ around Γ if for every $z \in \Gamma$, there exists a neighborhood $B_\epsilon(z)$ of z such that $|f(z_n) - f(z)| < |z_n - z|^\alpha$ with $z_n \in B_\epsilon(z)$.

Theorem 3.5.2. *[13] Let $K \in \mathbb{C}$ be a continuum set. Let $g_K(z, \infty)$ is in $Lip\alpha$ around the outer boundary Γ of K with $\frac{1}{2} \leq \alpha \leq 1$. Then there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and n-point Fekete set $\{z_1, z_2, \dots, z_n\}$ we have*

$$\min_{j \neq k} |z_j - z_k| \geq \frac{\delta}{n^{\frac{1}{\alpha}}}. \tag{5.1}$$

For an arbitrary continua, the Green function is of $Lip\frac{1}{2}$. Hence we can always get n^2 in the denominator, for the theorem above.

Definition 3.5.3. Suppose $K \in \mathbb{C}$ be a compact set. Then $dist_D(a, b) := \sup_{\|p\|_K, \deg p \geq 1} \left(\frac{1}{\deg p} |\arccos(p(b)) - \arccos(p(a))| \right)$ is called the *Dubiner distance* on K.

Theorem 3.5.4. *[10] Suppose that $K \subset \mathbb{C}$ be a compact set. Let $F_n = \{z_1, z_2, \ldots, z_n\}$ be an n-point Fekete set for K. Then for all $z_k \in F_n$, we have*

$$\frac{\pi}{2n} \leq \min_{z_j \in F_n} dist_D(z_j, z_k), \tag{5.2}$$

with $j \neq k$.

Proof. Let $l_k(z) = \frac{(z-z_1)(z-z_2)\ldots(z-z_{k-1})(z-z_{k+1})(z-z_{k+2})\ldots(z-z_n)}{(z_k-z_1)(z_k-z_2)\ldots(z_k-z_{k-1})(z_k-z_{k+1})(z_k-z_{k+2})\ldots(z_k-z_n)}$ be a fundamental Lagrange polynomial. Putting $p(z) = l_k(z)$ in the definition of the Dubiner distance, we have

$$\begin{aligned} dist_D(z_j, z_k) &\geq \frac{1}{n}\left(\frac{1}{\deg p}|\arccos(l_k(z_k)) - \arccos(l_k(z_j))|\right) && (5.3)\\ &= \frac{1}{n}|\arccos(1) - \arccos(0)| && (5.4)\\ &= \frac{\pi}{2n}. && (5.5) \end{aligned}$$

□

3.6 Leja Sequences

In the previous section, by using Fekete points, we approximate the equilibrium measure. In this section, we present Leja sequences which are similar to Fekete points and do the same job, but locally.

Definition 3.6.1. Let $K \subset \mathbb{C}$ be a compact set. Let $z_0 \in K$ chosen arbitrarily. We say $E_n := (z_0, z_1, \ldots, z_{n-1})$ is an *n-Leja section* for K if for all $j = 1, 2, \ldots, n-1$, we have

$$\prod_{m=0}^{j-1} |z_{n-1} - z_m| = \max_{z \in K} \prod_{m=0}^{j-1} |z - z_m|, \tag{6.1}$$

The sequence $E := (z_n : n \in \mathbb{N})$ such that $E_n = (z_0, z_1, \ldots, z_{n-1})$ is an n-Leja section for every $n \in \mathbb{N}$, is called a *Leja sequence* for K.

Leja sequences are similar to Fekete points because they locally maximize Vandermonde determinant. If we are given an n-Leja section $(z_0, z_1, \ldots, z_{n-1})$, then $V(z_0, z_1, \ldots, z_{n-1}, z_n) = \max_{z \in K} V(z_0, z_1, \ldots, z_{n-1}, z)$ gives us an $(n+1)$-Leja section where $z_n \in K$.

Theorem 3.6.2. *[1] Let $(z_0, z_1, \ldots, z_n)$ is an $(n+1)$-Leja section for a compact set K in $\mathbb{C}$. Then*

$$\sigma_n := \frac{1}{n+1} \sum_{i=0}^{n} \delta_{z_i} \xrightarrow{*} \mu_K. \tag{6.2}$$

Proof. Let $s_n := \sum_{0 \leq j < k \leq n} \log \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \sum_{j=0}^{k-1} \log \frac{1}{z_k - z_j}$. By the definition of $(n+1)$-Leja section, we get

$$\prod_{m=0}^{j-1} \frac{1}{z_n - z_m} = \min_{z \in K} \prod_{m=0}^{j-1} \frac{1}{z_- z_m}, \ \forall j = 1, 2, \ldots, n. \tag{6.3}$$

Hence $\sum_{0 \leq j < k \leq n} \log \frac{1}{z - z_j}$ is minimized by z_k for all $k = 1, 2, \ldots, n$. Thus,

$$s_n \leq \sum_{k=1}^{n} \sum_{j=0}^{k-1} \log \frac{1}{z - z_j}, \ z \in K. \tag{6.4}$$

Integrating the right side with respect to μ_K we get

$$s_n \leq \sum_{k=1}^{n} \sum_{j=0}^{k-1} U^{\mu_K}(z_j) \tag{6.5}$$

But by Frostman's theorem, $U^{\mu_K} \leq V_K$ on K. Hence,

$$s_n \leq \frac{n(n-1)}{2} V_K. \tag{6.6}$$

Let $\mathcal{N}$ be a subsequence of $\mathbb{N}$. Then $(\sigma_n)_{n \in \mathcal{N}}$ has a subsequence $(\sigma_n)_{n \in \mathcal{N}_1}$ such that σ_n converges to a unit measure σ of compact support in the weak star sense.

Now, let $\log_M(z) = \min\{\log |z|, M\}$ and let $n \in \mathcal{N}_1$. Then

$$s_n = \frac{1}{2} \sum_{j \neq k} \log \frac{1}{z_j - z_k} \tag{6.7}$$

$$\geq \frac{(n+1)^2}{2} \iint \log_M \frac{1}{|z-t|} \, d\sigma_n(z) \, d\sigma_n(t) - M \frac{n+1}{2}. \tag{6.8}$$

Then by (6.6) and (6.8),

$$I(\mu_K) = V_K \geq \lim_{n\to\infty} \inf_{n\in\mathcal{N}_1} \frac{2}{(n+1)^2} s_n \tag{6.9}$$

$$\geq \liminf_{n\to\infty} \left(\iint \log_M \frac{1}{|z-t|}\ d\sigma_n(z)\ d\sigma_n(t) - M\frac{n+1}{(n+1)^2} \right) \tag{6.10}$$

$$\geq \lim_{m\to\infty} \iint \log_M \frac{1}{|z-t|}\ d\sigma_n(z)\ d\sigma_n(t) \tag{6.11}$$

$$= \iint \log \frac{1}{|z-t|}\ d\sigma_n(z)\ d\sigma_n(t) = I(\sigma). \tag{6.12}$$

Thus, we have $I(\mu_K) \leq I(\sigma) \leq I(\mu_K)$, which implies that $\sigma = \mu_K$, by the uniqueness of the equilibrium measure. Since $\mathcal{N}$ and $\mathcal{N}_1$ are chosen arbitrarily $(\sigma_n)_{n=1}^{\infty}$ converges to μ_K in the weak star sense. □

Notation: The k-th section of a Leja sequence (a_n) is denoted by $a^k = (a_0, a_1, \ldots, a_n)$. If the sequences (a_n) and (b_n) are given, then (a^s, b^q) is equal to $(a_0, a_1, \ldots, a_{s-1}, b_0, b_1, \ldots, b_{q-1})$.

Theorem 3.6.3. *[14] Let $a_0 = 1$ where (a_n) is a Leja sequence of $\overline{\mathbb{D}(0,1)} \subset \mathbb{C}$. Then 2^n-Leja section for unit disk consists of 2^n-th roots of unity and if 2^n-Leja section is known then the 2^{n+1}-Leja section is given by*

$$(a^{2^n}, \rho b^{2^n}) \tag{6.13}$$

where ρ is any solution of $z^{2^n} = -1$ and b^{2^n} is the Leja section for $\overline{\mathbb{D}(0,1)}$ with $b_0 = 1$.

Proof. We use double induction to prove the theorem. If $n = 0$, then $(-1, 1)$ is a 2-Leja section for $\overline{\mathbb{D}(0,1)}$, clearly. We assume that $\{a_k : k < 2^n\}$ is the set of 2^n-th roots of unity. Then by induction on n, we show that 2^{n+1}-Leja section is given by (6.13). Let $m = 2^n$. Then we have to show that

$$a_{m+k} = \rho b_k,\ \ k = 0, 1, \ldots, 2^{n-1}. \tag{6.14}$$

We prove this by induction on k. Let $k = 0$. According to our induction hypothesis on n, 2^n-Leja section is given by 2^n-th roots of 1. Hence, a_m should maximize

$|z^{2^n}-1|$. But, to maximize this expression on the unit disk, z^{2^n} must be equal to -1. Thus, a_m is given by $\rho = \rho b_0$, which is one of the 2^n-th roots of -1.

Let $0 \leq k < 2^n - 1$. Now, assume that $a_{m+j} = \rho b_j$ for $j = 0, 1, \ldots, k$ where $(b_0, b_1, \ldots, b_k)$ is $(k+1)$- Leja section for $\overline{\mathbb{D}(0,1)}$ with $b_0 = 1$. Our aim is to prove that

$$\prod_{j=0}^{k} |b_{k+1} - b_j| = \max_{|z|=1} \prod_{j=0}^{k} |z - b_j| \tag{6.15}$$

with $a_{m+k+1} = \rho b_{k+1}$. Now, define $W_k(z) = (z - b_0) \ldots (z - b_k)$. Hence, we are looking for a point that satisfies

$$|W_k(b_{k+1}| = \max_{|z|=1} |W_k(z)|. \tag{6.16}$$

Let us define $w_{m+k}(z)$ as $(z-a_0)\ldots(z-a_{2^n-1})(z-a_m)(z-z_{m+1})\ldots(z-a_{m+k})$. By our induction hypothesis on n, $a_0, a_1, \ldots, a_{2^n-1}$ forms a complete set of 2^n-th roots of unity. Thus,

$$w_{m+k}(z) = (z^{2^n} - 1)(z - a_m)(z - a_{m+1}) \ldots (z - a_{m+k}). \tag{6.17}$$

By induction hypothesis on k, $a_{m+j} = \rho b_j$ for $j = 0, 1, \ldots, k$. Therefore,

$$|w_{m+k}(z)| = |z^{2^n} - 1||z - \rho b_0| \ldots |z - \rho b_k|. \tag{6.18}$$

Observing that the rotation $z \to \rho z$ leaves the unit disk invariant, we get

$$\max_{|z|=1} |w_{n+k}(z)| = \max_{|z|=1} |w_{n+k}(\rho z)| \tag{6.19}$$

$$= \max_{|z|=1} |(\rho z)^{2^n-1}||\rho z - \rho b_0| \ldots |\rho z - \rho b_k| \tag{6.20}$$

$$= \max_{|z|=1} \{|W_k(z)||z^{2^n} + 1|\}. \tag{6.21}$$

Clearly, $|z^{2^n} + 1|$ is maximized on $z^{2^n} = 1$. By our induction hypothesis on n, 2^n-Leja section consists of the roots of unity. Since $k < 2^n - 1$, $|W_k(z)|$ is maximized on $z^{2^n} = 1$. Hence,

$$\max_{|z|=1} |w_{m+k}(z)| = 2 \max_{|z|=1} |W_k(z)|. \tag{6.22}$$

Clearly, if $a_{m+k+1} = \rho b_{k+1}$ maximizes $|W_{m+k}(z)|$, then $\frac{a_{m+k+1}}{\rho} = b_{k+1}$ maximizes $|W_k(z)|$. Therefore, we have

$$\prod_{j=0}^{k} |b_{k+1} - b_j| = \max_{|z|=1} \prod_{j=0}^{k} |z - b_j| \tag{6.23}$$

which shows that our assumption on k is true. This also implies the assumption on n is also true which concludes the proof. $\square$

The next theorem is an example of Leja sequence for $\overline{\mathbb{D}(0,1)}$. There are no sets other than $\overline{\mathbb{D}(0,1)}$ for which a Leja sequence is known analytically.

Theorem 3.6.4. *[14] Let* $c_n = \exp\left(i\pi \sum_{k=0}^{s} j_k 2^{-k}\right)$ *for* $n = \sum_{k=0}^{s} j_k 2^k$, $j_k \in \{0,1\}$ *with* $c_0 = 1$. *Then* (c_n) *is a Leja sequence for* $\overline{\mathbb{D}(0,1)}$.

Proof. First, observe that $c_1 = e^{-i\pi} = -1$ and $\{c_0, c_1\}$ is a 2-Leja section for $\overline{\mathbb{D}(0,1)}$. By 3.6.3 we can say that if (a^{2^n}) is a 2^n-Leja section, then $\left(a^{2^n}, e^{\frac{i\pi}{2^n}} a^{2^n}\right)$ is a 2^{n+1}-Leja section for $\overline{\mathbb{D}(0,1)}$. Now, let us show that $c_{2^n+t} = e^{\frac{i\pi}{2^n}} c_t$ for $1 \leq t < 2^n$:

$$e^{\frac{i\pi}{2^n}} c_t = e^{\frac{i\pi}{2^n}} \exp\left(i\pi \sum_{k=0}^{s} j_k 2^{-k}\right) = \exp\left(i\pi \left(\sum_{k=0}^{s} j_k 2^{-k} + 2^{-n}\right)\right) = c_t \tag{6.24}$$

where $t = \sum_{k=0}^{n-1} j_k 2^k$.

Assume that c_{2^n} is a 2^n-Leja sequence. Then

$$c_{2^{n+1}} = (c_0, c^1, \ldots, c_{2^{n+1}-1}) = (c_0, c_1, \ldots, c_{2^n-1}, e^{\frac{i\pi}{2^n}} c_0, \ldots, e^{\frac{i\pi}{2^n}} c_{2^n-1}) \tag{6.25}$$

is a 2^{n+1}-Leja section.

Therefore, by induction, for any n, (c_0, c_1, c_{2^n}) is a 2^n-Leja section. Letting $n \to \infty$, (e_n) is a Leja sequence for $\overline{\mathbb{D}(0,1)}$. $\square$

3.7 Spacing and Distribution of Leja Sequences

In the previous section, we show that by using Leja sequences with discrete measures we can get the equilibrium measure of a set. If we get the equilibrium measure, then by definition, we get the logarithmic potential and then the Green function associated with this set. This shows that the distribution of a Leja sequence and the Green function of a set is somehow related to each other. Leja showed that a Leja sequence for a compact set $K \subset \mathbb{C}$ is uniformly distributed with respect to $\frac{1}{2\pi}\frac{\partial}{\partial n}g_K(z,\infty)$ on the outer boundary of K where n is used for normal derivative.

Example 3.7.1. [15] Let $K = \overline{\mathbb{D}(0,r)}$. Then $g_K(z,\infty) = g_K(x,y,\infty) = \log\frac{|z|}{r}$ with $z = x+iy$. As we show before, K has capacity r. Then $\frac{\partial}{\partial n}g_K(x,y,\infty) = \frac{1}{r}$ on the boundary. Let(a_n) be a Leja sequence for K. Then as $n \to \infty$, on each piece γ_1, γ_2 with the same length on the boundary, there will be the same number of a_n since normal derivative is constant on the boundary.

The next lemma is called Markov's inequality.

Lemma 3.7.2. *Let p_n be a holomorphic polynomial of degree n. Then*

$$|p_n'(t)| \le n^2\|p_n\|_{[-1,1]}. \tag{7.1}$$

Theorem 3.7.3. *[16] Let (a_n) be a Leja sequence on $[-1,1]$. Let $n > j$. Then*

$$|a_n - a_j| \ge \frac{1}{n^2}. \tag{7.2}$$

Proof. Let $p_{n-1}(x) = \prod_{i=1}^{n-1}(x - x_i)$. Then by Markov's inequality,

$$|p_{n-1}'(t)| \le n^2\|p_{n-1}\|_{[-1,1]}. \tag{7.3}$$

Hence by the mean value theorem, we get

$$\frac{|p_{n-1}(a_j) - p_{n-1}(a_n)|}{a_j - a_n} \le n^2\|p_{n-1}\|_{[-1,1]}. \tag{7.4}$$

Clearly $p_{n-1}(a_j) = 0$ and by the definition of n-Leja section, $p_{n-1}(a_n) = \|p_{n-1}\|_{[-1,1]}$. Therefore, by (7.4), we have $|a_j - a_n| \ge \frac{1}{n^2}$. □

Definition 3.7.4. A set Γ in $\mathbb{C}$ is called C^2-arc if there exists a twice continuously differentiable function $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma'(t) \neq 0$ on $[0,1]$ and $\Gamma = \{\gamma(t) : t \in [0,1]\}$.

The next result is quite important since it gives a characterization for spacing of Leja points for some general class of sets.

Theorem 3.7.5. *[16] Let $K \subset \mathbb{C}$ be a compact set whose boundary is the union of finitely many C^2-arcs. Let (a_j) be a Leja sequence on K. Then there is a constant c_K depending only on K such that for $i, j \leq n$, with $i \neq j$,*

$$|a_i - z_j| \geq c_K \frac{1}{n^2}. \tag{7.5}$$

Chapter 4

Spacing of Fekete Points for Some Special Compact Sets

4.1 The Cantor-Type Set $K^{(\alpha)}$

First, let us define $K^{(\alpha)}$. Let $\alpha > 1$, $0 < l_1 < \frac{1}{2}$ and $2l_1^{\alpha-1} < 1$. Then $K^{(\alpha)} = \cap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0, 1]$, E_s is a union of 2^s closed basic intervals $I_{j,s}$ of length $l_s = l_{s-1}^{\alpha}$ and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$ with $j = 1, 2, 3, \ldots, 2^s$. The Cantor-type set $K^{(\alpha)}$ is non-polar if and only if $\alpha < 2$ by [17].

Definition 4.1.1. Let $\alpha > 1$. Then 2^n-point Fekete sets of $K^{(\alpha)}$ are *uniformly distributed* if there exists a 2^n-point Fekete set such that on each subinterval of $\cap_{s=0}^{n} E_s$, there is exactly one point from this Fekete set for all $n \in \mathbb{N}$.

The next result belongs A. Goncharov.

Theorem 4.1.2. *For $a > 2$, 2^n-point Fekete sets of $K^{(\alpha)}$ are uniformly distributed. For $\alpha = 2$, 2^n-point Fekete sets of $K^{(\alpha)}$ are uniformly distributed if $l_1 \leq \frac{1}{4}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. Let $N = 2^n$. Now, let $Y = (y_j)_{j=1}^{N}$ be an N-point

Fekete set for $\cap_{s=0}^n E_s$ and $N_{j,s}$ denote the number of points from Y on $I_{j,s}$. Thus, $\sum_{j=1}^{2^s} N_{j,s} = 2^n$ for all s.

We fix two sequences of nested basic intervals. Let $j_1 \in \{1, 2\}$ be such that $N_{j_1,1} = \min_{1\leq j\leq 2} N_{j,1}$. If $N_{1,1} = N_{1,2} = 2^{n-1}$, then let $j_1 = 1$. Next, we fix j_2 with $2j_1 - 1 \leq j_2 \leq 2j_1$ such that $N_{j_2,2} = \min N_{j_2}$. Thus, $N_{j_{k+1},k+1} = \min N_{j,k+1}$ where minimum is taken for $2j_k - 1 \leq j \leq 2j_k$.

If $(y_j)_{j=1}^N$ are uniformly distributed, then $N_{j,s} = N_{i,s}$ for all $i, j \leq 2^s$ and $s \leq n$. Suppose, by contradiction there exists s_0 and $i, j \leq 2^{s_0}$ with $N_{i,s_0} \neq N_{j,s_0}$. Without loss of generality, let $s_0 = 1$ and so $N_{1,1} \neq N_{2,1}$. Let $N_{1,1} < N_{2,1}$. Then $j_1 = 1$ and $N_{j_1,1} < 2^{n-1} < N_{2,1}$ with $N_{j_2,2} < 2^{n-2}$ and in general $N_{j_k,k} < 2^{n-k}$ for $1 \leq k \leq n$. In this way we get

$$I_{j_n,n} \subset I_{j_{n-1},n-1} \subset \ldots I_{j_2,2} \subset I_{j_1,1}. \tag{1.1}$$

In particular, $I_{j_n,n}$ does not contain points from Y.

Similarly, we find

$$I_{i_n,n} \subset I_{i_{n-1},n-1} \subset \ldots I_{i_2,2} \subset I_{i_1,1} \tag{1.2}$$

with $N_{i_{k+1},k+1} = \max N_{i,k}$, for $2i_k - 1 \leq i \leq 2i_k$. Since $N_{2,1} > 2^{n-1}$, we get $\nu_k := N_{i_k,k} > 2^{n-k}$ for $1 \leq k \leq n$. In particular, $I_{i_n,n}$ contains at least 2 points from Y. Take one of them for some m. Let it be y_m. Let us compare $|V(y_1, y_2, \ldots, y_N)|$ and $|V(y_1, y_2, \ldots, y_{m-1}, z, y_{m+1}, \ldots, y_N)|$. We will show that the second value is larger. This will mean that Y is not a 2^n-point Fekete set which leads us to a contradiction.

Let $w(x) := \prod_{j=1, j\neq m}^{N} (x - y_j)$. Our aim is t show that $\frac{|w(y)|}{|w(z)|} < 1$. There are $N_{i_n,n} - 1$ points from $Y \setminus \{y_m\}$ on $I_{i_n,n}$. So, distance from them to y_m is not larger than l_n. Also there are $N_{i_{n-1},n-1} - N_{i_n,n}$ points from Y on the interval of length l_n adjacent to $I_{i_n,n}$. Continuing in this way, we get

$$|w(y_m)| < {l_n}^{N_{i_n}-1} {l_{n-1}}^{N_{i_{n-1},n-1} - N_{i_n,n-1}} \ldots {l_1}^{N_{i_1,1} - N_{i_2,2}} {l_0}^{N - N_{i_1,1}}. \tag{1.3}$$

Let us show that

$$l_n{}^{\nu_n-1}l_{n-1}{}^{\nu_{n-1}-\nu_n}\dots l_1{}^{\nu_1-\nu_2}l_0{}^{N-\nu_1} \leq l_n l_{n-1} l_{n-2}{}^2 \dots l_1{}^{2^{n-2}}. \tag{1.4}$$

Clearly, $\nu_n \geq 2$, so we have $l_n{}^{\nu_n-1} \leq l_n$. Also, $\nu_{n-1} \geq 3$ implies

$$l_n{}^{\nu_n-2} l_n{}^{\nu_{n-1}-\nu_n-1} \leq l_{n-1}{}^{\nu_{n-1}-3} \leq 1 \tag{1.5}$$

Applying the same technique for $\nu_{n-2} \geq 5$ we get $l_{n-2}^{\nu_{n-2}-5} \leq 1$. Doing the same calculations for other ν_k we get (1.4). Thus, we have

$$|w(y_m)| < l_n l_{n-1} l_{n-2}{}^2 \dots l_1{}^{2^{n-2}}. \tag{1.6}$$

On the other hand, let $\mu_k := N_{j_k,k}$. We have $\mu_n = 0$, $\mu_{n-1} \leq 1$, …, $\mu_k \leq 2^{n-k}-1$. Distance from z to the nearest point from Y is not smaller than $l_{n-1}-l_n$. Distance from z to the second nearest distance from Y is bigger than $l_{n-2}-2l_{n-1}$. Thus, we have

$$|w(z)||z-y_m| > (l_{n-1}-l_n)^{\mu_n-1}(l_{n-2}-2l_{n-1})^{\mu_{n-2}-\mu_{n-1}}\dots(l_1-2l_2)^{\mu_1-\mu_2}(1-2l_1)^{N-\mu_1} \tag{1.7}$$

Arguing as above, we see that the right hand side of (1.7) is larger than

$$(l_{n-1}-l_n)(l_{n-2}-2l_{n-1})^2\dots(l_1-2l_2)^{2^{n-2}}(1-2l_1)^{2^{n-1}+1}. \tag{1.8}$$

Thus we get

$$\frac{|w(y_m)|}{|w(z)|} < \frac{l_n l_{n-1} l_{n-2}{}^2 \dots l_1{}^{2^{n-2}} l_0{}^{2^{n-1}-1}|z-y_m|}{(l_{n-1}-l_n)(l_{n-2}-2l_{n-1})^2\dots(l_1-2l_2)^{2^{n-2}}(l_0-2l_1)^{2^{n-1}+1}} \tag{1.9}$$

$$= \frac{l_n|z-y_m|}{(1-2l_1)^2}\Pi \tag{1.10}$$

where $\Pi = \frac{l_{n-1}}{l_{n-1}-l_n}\left(\frac{l_{n-2}}{l_{n-2}-2l_{n-1}}\right)^2\dots\left(\frac{1}{1-2l_1}\right)^{2^{n-1}}$.

Now, we show the following inequalities are valid:

$$\frac{l_{n-1}}{l_{n-1}-2l_n} < \frac{l_{n-2}}{l_{n-2}-2l_{n-1}} < \dots \frac{l_1}{l_1-2l_2} \leq \frac{1}{1-2l_1}. \tag{1.11}$$

Let us show that $\frac{l_{s+1}}{l_{s+1}-2l_{s+2}} < \frac{l_s}{l_s-2l_{s+1}}$ for $s > 0$. To have this inequality, the inequalities $l_s l_{s+1} - 2l_{s+1}^2 < l_s l_{s+1} - 2l_s l_{s+2}$ and $l_s l_{s+2} < l_{s+1}{}^2$ must be valid. We have $l_{s+2} = l_{s+1}^{\alpha}$ and $l_{s+1} = {l_s}^{\alpha}$. If $l_s^{\alpha^2+1} < {l_s}^{2\alpha}$ is true, then all inequalities hold. It is clearly true since for $\alpha \geq 2$, $\alpha^2 + 1 - 2\alpha = (\alpha - 1)^2 > 0$.

For $s = 0$, we check the inequality $\frac{l_1}{l_1 - 2l_2} \leq \frac{1}{1-2l_1}$. It is valid if $l_2 \leq {l_1}^2$. But since $l_2 = {l_1}^{\alpha}$ with $\alpha \geq 2$, we have this inequality also. Therefore, we get

$$\Pi < \left(\frac{l_1}{(l_1 - 2l_2)(1 - 2l_1)} \right)^{2^{n-1}-1}. \tag{1.12}$$

For $\alpha > 2$ we have

$$\frac{|w(y_m)|}{|w(z)|} < {l_1}^{\alpha^{n-1}} \left(\frac{l_1}{(l_1 - 2l_2)(1 - 2l_1)} \right)^{2^{n-1}-1} \frac{1}{(1 - 2l_1)^2} \tag{1.13}$$

$$\leq {l_1}^{\alpha^{n-1}} \left(\frac{l_1}{1 - 2l_1} \right)^{2^n - 4} < 1. \tag{1.14}$$

The last line is true since $l_1 < 1 - 2l_1$ for $l_1 < 1/2$ which comes from the definition of K^{α}.

For $\alpha = 2$ and $l_1 \leq \frac{1}{4}$, we have

$$\frac{|w(y_m)|}{|w(z)|} < \left(\frac{{l_1}^2}{(l_1 - 2l_2)(1 - 2l_1)} \right)^{2^{n-1}} \tag{1.15}$$

$$\leq \left(\frac{l_1}{(1 - 2l_1)^2} \right)^{2^{n-1}} \tag{1.16}$$

since $l_2 = {l_1}^{\alpha} = {l_1}^2$, in this case. Since for $l_1 \leq \frac{1}{4}$, $\left(\frac{l_1}{(1-2l_1)^2} \right)$ is less than or equal to 1 we get $\frac{|w(y_m)|}{|w(z)|} < 1$. Therefore, we get the contradiction for $a \geq 2$ which completes the proof. $\square$

4.2 The Cantor-Type Set $K(\gamma)$

A. Goncharov, in his unpublished article "Equilibrium Cantor-Type Sets", defines a new class of Cantor-type sets depending on $\gamma = (\gamma_s)_{s=1}^{\infty}$ which possess fabulous properties in terms of logarithmic potential theory. Before mentioning these properties let us show how they are defined. Let $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < \frac{1}{4}$

for all s. Letting $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for all $s \in \mathbb{N}$, define a sequence of real polynomials inductively: Let $P_2(x) := x(x-1)$ and $P_{2^{s+1}} := P_{2^s}(P_{2^s} + r_s)$ for $s \in \mathbb{N}$. It is easy to see that P_{2^s} has 2^{s-1} points of minimum with equal values $P_{2^s} = \frac{-r_{s-1}^2}{4}$. The polynomial $P_{2^{s+1}}$ is zero at the roots of P_{2^s} by definition. Moreover, $P_{2^{s+1}}$ have 2^s more zeros exactly at the intersection of $y = -r_s$ and $y = P_{2^s}$. Now, define $E_s := \{x \in \mathbb{R} : P_{2^{s+1}}(x) \leq 0\}$. Since $r_s < \frac{r_{s-1}^2}{4}$, the set E_s consists of 2^s disjoint intervals $I_{j,s}$. The length of intervals of the same level are generally different. On the other hand, by construction, $\max_{1 \leq j \leq 2^s} l_{j,s} \to 0$ as $s \to \infty$. Clearly, we have $E_{s+1} \subset E_s$. Set $K(\gamma) = \cap_{s=0}^{\infty} E_s$.

In potential theory, as we see in previous chapters, a set should be non-polar (has positive capacity) in order to work on it. This is because of the fact that important theorems of logarithmic potential theory are valid quasi-everywhere and most of the time a definition is meaningful only for non-polar sets. Thus, in spite of the fact that a nice technique is used in 4.1.2, the result of this theorem is valid only for polar cases. This bitter fact makes this theorem uninteresting in terms of potential theory.

Now, we give some remarkable properties of $K(\gamma)$ without proof.

Theorem 4.2.1. *[18] The set $K(\gamma)$ is polar if and only if $\lim_{s \to \infty} 2^{-s} \log \dfrac{2}{r_s} = \infty$. If this limit is finite and $z \notin K(\gamma)$, then we have*

$$g_{K(\gamma)}(z, \infty) = \lim_{s \to \infty} 2^{-s} \log \frac{|P_{2^s}(z)|}{|r_s|}. \tag{2.1}$$

This theorem shows that $K(\gamma)$ is interesting enough since it gives a precise criterion for polarity. By just defining $(\gamma_s)_{s=1}^{\infty}$, we can define infinitely many non-polar sets.

The next theorem is important for our research topic since it provides a benchmark to compare distances of the intervals of the same level.

Theorem 4.2.2. *[18] Let $\gamma_s \leq \frac{1}{32}$ for $s \in \mathbb{N}$. Then*

$$\gamma_1 \gamma_2 \ldots \gamma_s < l_{i,s} < \gamma_1 \gamma_2 \ldots \gamma_s \exp\,(16 \sum_{k=1}^{\infty} \gamma_k). \tag{2.2}$$

Let $s \in \mathbb{N}$ be given. Let us distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \leq j \leq 2^s$. Let λ_s be the normalized Lebesgue measure on the set E_s i.e. $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$. The next theorem actually inspires us to study the location of Fekete points on $K(\gamma)$.

Theorem 4.2.3. *[18] Let $\gamma_s \leq \frac{1}{32}$ for $s \in \mathbb{N}$ and cap$(K(\gamma)) > 0$. Then $\lambda_s \xrightarrow{*} \mu_{K(\gamma)}$.*

Our hypothesis is that 2^n-point Fekete sets of $K(\gamma)$ is uniformly distributed by the theorem above. We can make this hypothesis because of the approximation property of sequences generated by Fekete points to equilibrium measure. Actually, our main hypothesis is that roots of P_{2^n} form a 2^n-point Fekete set for $K(\gamma)$. The next theorem implies that P_{2^n} possesses extremal property for $K(\gamma)$ which is an underpinning argument for our main hypothesis.

Theorem 4.2.4. *[18] The polynomial $P_{2^s} + \frac{r_s}{2}$ is the Chebyshev polynomial for the set $K(\gamma)$.*

Our hypothesis, if we could prove, would be interesting enough for the people who are working potential or approximation theory since, as we mention in the previous chapter, there are only three cases for which we know the exact location of Fekete points. The last example is shown only twelve years ago. Thus, if we show location of Fekete points only for one $\gamma = (\gamma_s)_{s=1}^{\infty}$ which makes $K(\gamma)$ non-polar, it would be enough. Unfortunately, with the technique used in 4.1.2, we get equidistribution of 2^n-Fekete points of $K(\gamma)$ only for some polar sets which are trivial sets for potential theory. Actually, there are three possibilities. First, our hypothesis is wrong for all γ which make $K(\gamma)$ non-polar. The second possibility is that our hypothesis is valid only for some γ which makes $K(\gamma)$ non-polar. The last possibility is that the roots of P_{2^n} are 2^n-point Fekete sets for all n, for all non-polar $K(\gamma)$. This problem is still open and we strogly believe (since we could not prove it yet) that the second or third possibility is true by the arguments above.

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