DIFFERENTIAL REALIZATIONS OF THE VARIOUS LIE SUPERALGEBRA AND THEIR APPLICATIONS TO THE PHYSICAL PROBLEMS

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ABSTRACT

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Ph. D in Engineering Physics

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Lie algebras have been studied for many years and their roles in solving quantum mechanical problems are well known. In particular, exactly and quasi-exactly solvable quantum mechanical problems are classified according to their Lie algebraic properties. They provide the algebraic basis for a unified language for physics and mathematics which offers many advantages over current techniques. In this thesis, the solutions of the quantum optical problems in the framework of the Lie algebraic technique have been mainly studied and this work consists of three parts.

One of the principle aims of this thesis is the development of a number of new algebraic techniques which serve to broaden the field of applicability of Lie (super) algebra. The spectra of the many quantum optical Hamiltonians have been obtained by introducing a novel similarity transformation technique which transforms the multi-boson systems in the form of the single boson systems. This transformation leads to the determination of the solvability of the associated Hamiltonian.

The other contention of this thesis is that systems of the spin 1/2 particles can be solved by reformulating the Hamiltonians. To support this contention a diagonalization method is developed. In the solution of a general Hamiltonian for spin 1/2 particles, it is argued that the diagonalization method is easily applicable to solve a wide range of quantum mechanical problems and it provides many new insights.

Both methods have been applied to the solution of the quantum optical Hamiltonians as well as the solution of the Dirac equation. In this context second harmonic generation problem, interacting electrons in a quantum dot, Jaynes-Cumings model, quantum dot including spin-orbit coupling and Dirac equation have been solved. It has been shown that these physical systems associated with the algebras of the groups $SU(2)$, $SU(1,1)$, $OSP(2,1)$ and $OSP(2,2)$.

This thesis also includes the solution of Eckart potential in the s-wave Klein-Gordon equation by using super-symmetric quantum mechanical method.

The ultimate goal of the methods developed in this thesis is that they can be extended to solve the other physical Hamiltonians which include multi-boson or multi-fermion-boson systems.

Key words: Lie (super)algebra, (Quasi)exactly solvable potentials, Dirac equation, Klein–Gordon Equation

ÖZET

FARKLI LİE SÜPER CEBİRLERİN DİFFERANSİYEL İFADELERİ VE FİZİKSEL PROBLEMLERDEKİ UYGULAMALARI

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Lie cebirleri yıllardır çalışılmakta ve kuantum mekanik problemlerinin çözümündeki rolleri çok iyi bilinmektedir. Özellikle, tam ve kısmi çözülebilir kuantum mekanik sistemler sahip oldukları Lie cebirsel özelliklerine göre sınıflandırılırlar. Lie cebiri, matematik ve fizikçiler için geçerli tekniklerden daha fazla avantaj sağlayıp ortak bir dil oluşturarak cebirsel bir temel sağlar. Bu tezde, temel olarak Lie cebirsel tekniği çerçevesinde kuantum optik problemlerinin çözümü incelendi ve bu çalışma üç kısımdan oluşur.

Tezin temel amaçlarından bir tanesi Lie (süper) cebirinin uygulanabilirlik alanını genişletmemize olanak sağlayan bir takım yeni cebirsel tekniklerin geliştirilmesidir. Çok sayıda kuantum optiksel Hamiltonyanların spektrumu çoklubozon sistemlerden tek bozonlu sistem formuna dönüştüren yeni bir benzerlik dönüşüm tekniği tanımlanarak elde edildi. Bu dönüşüm ilgili Hamiltonyanın çözülebilirliğinin belirlenmesinde öncelik eder.

Tezin bir diğer iddiasıda spin 1/2 parçacık sistemlerin Hamiltonyanlarının yeniden formülleştirilmesiyle çözülebileceğidir. Bu iddiayı desteklemek için köşegenleştirme metodu geliştirildi. Spin 1/2 parçacık Hamiltonyanlarının çözümünde, köşegenleştirme metodun çok geniş alanda quantum mekanik problemleri çözümünde kolaylıkla uygulanabileceği ve birçok yeni bakış açısı sağladığı tartışıldı.

Her iki metot Dirac denkleminin çözümünün yanında kuantum optiksel Hamiltonyanların çözümünde de uygulandı. Bu bağlamda ikinci harmonik üretimi problemi, kuantum dot'taki elektronların etkileşimi, Jaynes-Cummings modeli, spin-yörünge çiftleşmesini içeren kuantum dot ve Dirac denklemi çözüldü. Bu fiziksel sistemlerin $SU(2)$, $SU(1,1)$, $OSP(2,1)$ ve $OSP(2,2)$ gruplarıyla ilişkili olduğu gösterildi.

Tez ayrıca Ekart potansiyelinin süpersimetrik kuantum mekanik metodu kullanılarak s-dalga Klein-Gordon denklemindeki çözümünüde içermektedir.

Tezde geliştirilen metodların esas amacı çoklu-bozon yada çoklu-fermiyon-bozon sistemleri içeren diğer fizik Hamiltonyanların çözümü için genişletilebilecek olmasıdır.

Anahtar kelimeler: Lie (süper) cebiri, (Kısmi)tam çözülebilen potansiyeller, Dirac denklemi, Klein–Gordon denklemi

To My Wife DİLCAN
and
To My Parents and My Daughter ŞİMAL

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I would like to give my sincere thankfulness to my family for their interest, patient and support through the work presented in this thesis. Especially, I wish to give a very special thank to my wife Dilcan and my daughter Şimal. Without their patients, I doubt that this thesis would ever have been written. For this reason this thesis is dedicated to them.

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LIST OF SYMBOLS

$[.,.]$ Commutator
$\{.,.\}$ Anticommutator
$[.,.]$ Super or $\mathbb{Z}_2$-graded commutator (Lie superbracket)
$\mathbb{R}$ Set of real numbers
$G_{\overline{0}}$ Even part of Lie Superalgebra
$G_{\overline{1}}$ Odd part of Lie Superalgebra
$\overline{0}, \overline{1}$ $\mathbb{Z}_2$ gradation of a superalgebra
a_i Annihilation operator of the field
$a_j^\dagger$ Creation operator of the field
σ_x,σ_y Pauli matrices
$\sigma_{\pm,0}$ Standard Pauli matrices
ϱ Weight function
$|n_1, n_2\rangle$ Eigenstate of the Hamiltonian in the Hilbert space
μ Level separation
κ Coupling constant of the field and atom
λ_1,λ_2 Coupling constant of the field and Kerr medium
m Mass of the electron
m^* Effective mass of the electron
$\hbar$ Planck constant
c Speed of light
p Momentum operator
w_0 Confinement frequency
w Effective frequency
w_c Cyclotron frequency of the electron
ϵ Dielectric constant of the medium
L_{tot} Total angular momentum
λ_R Strength of the spin–orbit coupling
λ_D Dresselhaus parameter
g Total dipole moment
μ Bohr magnetron
$\mathbf{A}$ Vector potential

B	Magnetic field
α,β	The 4×4 Dirac matrices
I	The 2×2 zero and identity matrices
$g_{\mu v}$	The metric tensor
$W(r)$	Superpotentional
$V_{\pm}(r)$	Super partner potential
w_L	Larmor frequency

CHAPTER 1

INTRODUCTION

The Lie (super)algebras have played a crucial role in the study of every branch of physics. Especially, besides improving the understanding of physical structures [1-4], they are associated with the symmetry properties of physical systems. We can see the application of such systems in different fields of physics. Lie algebras are useful tools in these physical systems to find the exact solution.

In physics, many important symmetry properties are described by Lie groups, or more precisely, by their associated Lie algebras. Lie groups of $SU(2)$, $SU(1,1)$, $OSP(2,1)$ and $OSP(2,2)$ are of great interest in many branches of physics. The Lie algebras $su(1,1)$ and $su(2)$ have a great importance in determining spectrum-generating dynamic group for a large number of systems, e.g., the square well, the harmonic oscillator, the Kepler problem, the Coulomb problem, etc.. Bound state problems are solved by using $su(2)$ algebra scattering state problems are solved by $su(1,1)$ algebra. The Lie supergroups $OSP(2,1)$, and $OSP(2,2)$ are the most important symmetry groups candidate for system of two differential equation and $2\times$ 2 matrix Hamiltonians which play an important role in nonlinear quantum optical systems. It is well known that while a Lie algebra can be constructed by using only bosons or fermions, the Lie superalgebra can be constructed by using both bosons and fermions [5].

Every Lie group has an associated Lie algebra. These Lie algebras can be used to write the differential equations of a specific model. To achieve that, it is necessary to realize the required Lie algebra by differential operators specially designed for each model. These algebras are frequently realized by boson operators and a polynomial basis in the creation operators is desired to build their irreducible representations. Under the condition of being only one type of boson [6], only the symmetric and anti-symmetric irreducible representations admit a polynomial basis. Nonetheless, most applications in physics also need the mixed irreducible representations which require more than one type of boson. The use of boson and fermion operators (i.e. harmonic oscillator creation and destruction operators) to provide realizations of various Lie algebras now have quite an extensive history, and these realizations have found application in a number of physical systems [7, 8]. It is not the intention in this introduction to review in any detail this extensive history but just to touch a few portions and provide references which serve a preview view to the wider literature.

Recently, great attention is being paid to examine different quantum optical models [9-13] with Hamiltonians given by nonlinear functions of the bosonic and/or fermionic operators on the condition that they enable to reveal new effects and phenomena. The natural step to relate the quantum optical systems and Lie (super)algebras is to express the generators of the Lie (super)algebra in terms of bosons and/or fermions. The Hamiltonians can be written as combinations of the generators of a relevant symmetry group. Hence, one can be able to compute (a part of) the spectrum by performing a suitable transformation of the generators.

Among the applications of Lie (super)algebra, the most interesting problem in quantum optics is the problem of quantum dots. Recent developments in micro fabrication technology have allowed the creation of quantum dots (QDs) in semiconductor structures [14]. The QDs may be regarded as artificial atoms, with disk-like shapes, in which electrons are confined by an artificial electrostatic potential instead of being attracted to a nucleus. The electron confinement within the QD is described by a Hamiltonian that includes the Coulomb potential, as well as a parabolic potential. Since no general analytical solutions are available, the Hamiltonian has been solved numerically [15] by means of the shifted $N = 1$ expansion [16], the Wentzel-Kramers-Brillouin (WKB) treatment [17, 18] and the non-local potential method [19, 20]. A particular analytical solution of the Hamiltonian has been obtained by Taut [21]. The renormalization perturbation series method has been proposed by Matulis and Peeters to calculate the energy spectrum of the quantum dot [22], and exact solutions for two electrons in a parabolic quantum dot have been obtained by using a power series method [23].

In addition, also the spin–orbit interactions in quantum dots have attracted many authors. Spin–orbit coupling is caused by the effect of electric field due to lack of inversion symmetry. The effects emerging from the presence of this field are called Dresselhaus and Rasha terms. In the literature, most of the studies about the solution of the spin–orbit effects in quantum dots are carried out by means of perturbation theory or numerical methods, because the implementation of the algebraic techniques to solve those problems is not very efficient and most of the other analytical techniques do not yield simple analytical expressions. Analytical solution of the problem has recently been treated by employing various techniques [24-28]. As a related topic the concept of quasi-exactly solvable systems discovered in the 1980s has received much attention in recent years, both from the viewpoint of physical applications and their mathematical beauty. It turns out that in quantum mechanics there exist such systems; a part of their spectrum can be computed using algebraic methods.

Relativistic quantum mechanics is required in order to obtain more accurate results for the particle under the strong potential field. When we consider this condition, a particle in strong potential field should be described with Klein-Gordon equation and Dirac equation [29, 30].

Construction of the Dirac equation including adequate potentials is very important for analyzing relativistic effects on the spectrum of the physical systems. The characteristic of two-dimensional (2D) physical systems can be obtained by using 2D Dirac oscillator [31-33]. Relativistic extension of the various potentials (exactly and quasi-exactly) have also turned out be of importance in description of 2D phenomena. We know that the Dirac equation is an exactly solvable differential equation only for some special potentials. So that, our aim is to extend these potentials of which spectrum's are (quasi-)exactly obtained.

Besides the 2D Dirac equation, the three dimensional Dirac equation for spherically symmetric local interactions has been introduced recently [34, 35]. In this approach, Schrödinger-like equation for the spinor components has been produced and the corresponding radial wave functions for two component spinor have been solved by correspondence with well-known exactly solvable non-relativistic problems. This approach has been facilitated the introduction of relativistic extension of all classes of shape invariant potentials as well as other exactly solvable nonrelativistic problems [36]. The relativistic bound-state spectra and spinor wave functions of a class of shape-invariant potentials [37] have been obtained.

The solutions of the $(2+1)$-dimensional (i.e., 2 dimensions of space and 1 dimension of time) Dirac equation have great importance for physical systems because of the rapid growth in nanofabrication technology that has made possible to confine laterally 2D electron systems. These quantum confined electron systems are referred to as artificial atoms where the potential of the nucleus, in the non relativistic case, is replaced by an effective potential of the form $V = \frac{1}{2}r^2$ which is often used as a realistic approximation [31, 32, 38]. In order to solve the Dirac equation, first of all we transform it into the Schrödinger-like form because of existing a large number of study discussing the 2D, few electron systems, most of which are tackling the problem in the framework of the Schrödinger-like equation. Functional approach which has been applied to solve Schrödinger equation for both of potentials is used to solve the Dirac equation. The transformed differential equation comes in a supersymmetric form (SUSY). By choosing appropriate parameters, we obtain two classes of potentials. These are

(1) Exactly–solvable potentials: Dirac-Oscillator potential, Dirac-Coulomb potential, Dirac-Morse potential, Dirac-Pösch-Teller potential, and Dirac-Eckart-Epstein potential.

(2) Quasi–exactly solvable potentials: Anharmonic oscillator potential, sextic oscillator potential, potentials related with the perturbed Coulomb potential, and deformed Morse oscillator potentials.

These two classes of potentials lead to two systems in quantum mechanics. The first systems for which it is possible to find the number of exact eigenvalues and eigenfunctions in closed manner are called quasi-exactly-solvable (QES) systems [39-44]. In the last decade a lot of effort, has been attracted on the quasi-exactly solvable (QES) equations whose finite number of eigenvector can be obtained algebraically [39, 41, 45, 46, 47]. Lie algebraic methods are useful tools to solve these systems. Dating back over fifteen years there has been a great deal of interest in QES systems. The one dimensional QES systems based on $sl(2, R)$ algebra have been classified by Turbiner [39]. The usual approach to the analysis of the one dimensional QES systems is to express the Hamiltonian of the corresponding physical system as linear and bilinear combination of the generators of the $sl(2, R)$ algebra. The necessary conditions for the normalizabilty of the wavefunction of the QES systems were completely determined in [40]. The QES models, either in the form of differential equations or of the single boson Hamiltonian, have been extended by Dolya and Zaslavskii [6, 48]. The other system is called exactly–solvable systems for which the whole spectrum of potentials can exactly be determined.

The another relativistic equation is Klein-Gordon equation. In recent years, there have been many discussions about Klein-Gordon equation with various type of potentials by using different methods to obtain the spectrum of the system. Some authors have considered the equality of scalar potential and vector potential in solving Klein-Gordon as well as Dirac equations for some potential fields [29, 30, 49]. S-wave bound-state solutions are obtained in [49, 50]. Similarly, the s-wave Klein-Gordon equation with vector potential and scalar Rosen-Morse type potentials [51, 52] have been treated by standard method [53], the same problem with both vector and scalar Hulthen-type potentials have been discussed analytically [54]. Energy spectrum of the s-wave Schrödinger equation with the generalized Hulthen potential has been obtained by using the supersymmetric quantum mechanics (SUSYQM) [55] and supersymmetric WKB approach [56]. We know that, the shape invariance condition in SUSYQM has been shown to have an underlying algebraic structure [57] which leads to the connection of shape invariance to Lie algebra. Thus we can possible to use SUSYQM instead of Lie algebra. In this study, we find the solution of s-wave Klein-Gordon equation including Eckart type potentials using SUSYQM whose operators obey the commutation relations of Lie algebra. The eigenfunctions and eigenvalues are obtained in terms of hypergeometric functions.

In this thesis, especially, we deal with the "Lie (super)algebra" and with the application of it in determining the spectrum of physical systems. For this purpose, first of all we construct the realization of some Lie (super)algebras and apply it to define physical potentials. The structure of the study is organized as follows.

In the second Chapter, we will give the basic definition of Lie (super)algebra in the view of mathematical demonstrations. Angular momentum algebra is given to illustrate the relation of Lie algebra in quantum mechanic.

In Chapter 3, the well known representations of Lie algebras $su(1,1)$, $su(2)$ with bosonic creation and annihilation operators are shown. These are constructed for one–boson and two–boson realizations. Additionally, the construction of boson-fermion representations of $osp(2,1)$, and $osp(2,2)$ superalgebras are presented. Some similarity transformation procedures are given to obtain the conditions of solvability of Hamiltonians. These transformations are necessary for transforming boson and fermion operators from Hilbert space to Bargmann–Fock space.

Chapter 4 deals with the definition of the terms "exactly solvable" and "quasi–exactly solvable". In quantum mechanics, there exist various exactly and quasi–exactly solvable potentials for which it is possible to obtain the full spectrum and to find a finite number of eigenvalues and associated eigenfunctions exactly in closed form. These potentials have important roles in every branch of physics.

The Chapter 5 includes some applications of Lie (super)algebra for some physical potentials. The first application is on the boson–fermion Hamiltonians. In order to solve boson-fermion Hamiltonians, we give a general method to obtain the eigenfunctions and eigenvalues. By this method, two boson Hamiltonian can be reduced to a single variable differential equation in the Bargmann-Fock space with using the solution of number operator or similarity transformations. We are able to determine quasi-exactly solvability of some Hamiltonians that include multi-boson or multi fermion-boson systems. This method provides a unified approach to the various algebraic method. The applicability of the method is demonstrated for some simple physical systems. As a particular case, our model includes the solutions of the MJC Hamiltonian and the Hamiltonian of the systems of photons and bosons expressed in a single mode. These Hamiltonians are not only mathematically interesting but they also have current interest in physics [58].

The another two applications are related with the quantum dots. The first one is the interacting electrons confined in a QD by a parabolic potential in an external magnetic field. We have investigated the quasi-exact solvability of this model. The results obtained here should be useful for checking and assessing numerical and approximate methods.

And the second one is the Hamiltonian of a quantum dot including Rashba coupling. We present a procedure to solve the Schrodinger equation of two interacting electrons in a quantum dot in the presence of an external magnetic field within the context of quasi–exactly solvable spectral problems [42, 43, 60, 59]. We show that the symmetries of the Hamiltonian can be recovered for specific values of the magnetic field, which leads to an exact determination of the eigenvalues and eigenfunctions. We show that the problem possesses a hidden $sl(2,R)$. An important property emerging from quantum dots is the spin–orbit interactions namely Rashba effect and Dresselhaus effect. We briefly review the construction of the Hamiltonian that includes Rashba spin–orbit coupling term. For this case, we try to find a method to solve the problem of Rashba spin–orbit coupling in semiconductor quantum dots, within the context of quasi-exactly solvable spectral problems. We show that the problem possesses a hidden $osp(2,2)$ superalgebra. We constructed a general matrix whose determinant provides exact eigenvalues. Analogous mathematical structures between the Rashba and some of the other spin-boson physical systems are noted.

In Chapter 6, in order to analyze relativistic effects on the spectrum of the physical systems, Dirac equation and Klein-Gordon equation including adequate potentials are constructed and their solutions are obtained. The expression of Dirac oscillator in two dimensions is extended in Dirac equation for constructing the physical potentials. These become exactly and quasi-exactly solvable potentials of non-relativistic quantum mechanics when they are transformed into a Schrödinger-like equation. For the exactly solvable potentials, eigenvalues are calculated and eigenfunctions are given by confluent hypergeometric functions. It is shown that, our formulation also leads to the study of those potentials in the framework of the supersymmetric quantum mechanics. After that, a supersymmetric technique for the bound-state solutions of the s-wave Klein–Gordon equation with equal scalar and vector standard Eckart-type potential [61] is proposed. For obtaining exact solutions, we transform the Klein–Gordon equation in the form of a Schrödinger-like equation, because there are many works to tackle the problem in the framework Schrödinger equations. The eigenvalues and eigenfunctions of the Eckart potential are obtained in terms of the SUSYQM. A possible generalization of our approach is outlined.

A alternative approach to the solutions of the 2×2 matrix Hamiltonians of the physical systems within the context of the $su(2)$ and $su(1,1)$ Lie algebras is considered in Chapter 7. We discuss the bosonization of the physical Hamiltonians whose original forms are given as differential operators. Our technique is relatively simple when compared with those of others and treats those Hamiltonians in a unified framework. With this method, the well known exactly-solvable potentials such as approximated Jaynes-Cummings (JC) Hamiltonian, Dirac oscillator and MJC has been shown that exactly solvable, and quasi-exactly solvable potentials such as JC, and Quantum dot Hamiltonian has been shown that quasi-exactly solvable. The systematic study presented here reproduces a number of earlier results in a natural way as leading to a novel findings.

At the end of this thesis, the conclusion and the further studies are outlined in Chapter 8, and Chapter 9 respectively.

CHAPTER 2

SOME REMARKS ON LIE (SUPER)ALGEBRA

In 1860s, by developing the theory of differential equations, a systematic study of their solutions become possible. In 1870, a Norwegian mathematician Sophus Lie developed a tool to study such algebraic equations which is called group theory. Lie groups which are the groups associated with the study of differential equations are infinite. Sophus Lie introduced the Lie groups in order to study symmetries of differential equations. Lie groups are important in many branches of science such as mathematical analysis, physics and geometry on the account of describing the symmetry structures.

Lie groups are infinite groups which have the extra property that their multiplication law is differentiable. They have manifold properties as well as group properties. Lie groups can be classified regarding their algebraic properties (simple, semisimple, solvable, nilpotent, abelian), their connectedness (connected or simply connected) and their compactness.

Every Lie group can be associated with a Lie algebra. In the view of the mathematics, a Lie algebra is defined as the an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds. Lie algebras were introduced to study the concept of infinitesimal transformations. Lie algebras play a crucial role in the representation theory of Lie groups.

2.1 Lie Algebras

Mathematically, a Lie algebra is an algebra of vector space g over a field $\mathbb{R}$ with a Lie bracket $[X_i, X_j]$ multiplication which must be satisfies the following properties

(1) Antisymmetry:

$$[X_i, X_j] = -[X_j, X_i],$$

for all X_i, $X_j \in g$.

(2) Bilinearity:

$$\begin{aligned} [aX_i + X_j, X_k] &= a[X_i, X_k] + b[X_j, X_k], \\ [X_k, aX_i + X_j] &= a[X_k, X_i] + b[X_k, X_j] \end{aligned}$$

for all $a, b \in \mathbb{R}$ and all X_i, X_j, $X_k \in g$.

(3) The Jacobi identity:

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0$$

We see that the first and second properties together imply $[X_i, X_i] = 0$ for all $X_j \in g$. One sees immediately that the algebra generated by X_i, X_j, X_k with the usual cross product is a Lie algebra. It will be seen that, the Lie bracket will often be the commutator in quantum mechanics.

We basically defined the term Lie algebra. Instead of a detail mathematical expressions, we try to illustrate the connection between Lie algebras and physics with giving well known angular momentum example.

As we know that, in classical mechanics, the angular momentum is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ where $\mathbf{r}$ is the position vector and $\mathbf{p}$ $(md\mathbf{r}/dt)$ is the linear momentum. The angular momentum is used to describe the rotational properties of the system. For example, if the system has rotational symmetry about an axis, the component of angular momentum along that axis is a constant of motion. If the system is invariant under all rotations, the angular momentum $\mathbf{L}$ is conserved.

These results also hold in quantum mechanics except for the definition of the position vector and linear momentum. In cartesian coordinates, $\mathbf{p} = -i\hbar(\partial_x, \partial_y)$ is the linear momentum operator, $\mathbf{r}$ is the position vector operator, and $\mathbf{L}$ is the angular momentum operator. Similarly, if the system possesses full rotational symmetry, the operator $\mathbf{L}$ would commute with the Hamiltonian and would be a constant of motion. Hereafter, we denote the angular momentum operator with $\mathbf{J}$ instead of $\mathbf{L}$ for avoiding the complexity of notation.

In quantum mechanics, the commutator relations between the components of the angular momentum form the Lie algebra for the three-dimensional rotation group, which is homomorphic to special unitary group $SU(2)$ and special orthogonal group $SO(3)$. Consider the generalized angular momentum operators

J_x, J_y, and J_z in the Cartesian coordinate system

$$
\begin{aligned}
J_x &= -i\hbar\left(\widehat{\mathbf{y}}\frac{d}{dz} - \widehat{\mathbf{z}}\frac{d}{dy}\right), \\
J_y &= -i\hbar\left(\widehat{\mathbf{z}}\frac{d}{dx} - \widehat{\mathbf{x}}\frac{d}{dz}\right), \\
J_z &= -i\hbar\left(\widehat{\mathbf{x}}\frac{d}{dy} - \widehat{\mathbf{y}}\frac{d}{dx}\right).
\end{aligned}
$$

These generators satisfy the commutation rules

$$
[J_x, J_y] = iJ_z, \qquad [J_y, J_z] = iJ_x, \qquad [J_z, J_x] = iJ_y, \tag{2.1}
$$

From these three operators, we can form the operator of the square of the modulus of the angular momentum operator J^2,

$$
J^2 = J_x^2 + J_y^2 + J_z^2. \tag{2.2}
$$

This operator commutes with each of the operators J_x, J_y, and J_z

$$
\left[J^2, J_x\right] = 0, \qquad \left[J^2, J_y\right] = 0, \qquad \left[J^2, J_z\right] = 0, \tag{2.3}
$$

Thus, the angular momentum algebra $\mathbf{J}$ is generated by $\{J_x,\ J_y, J_z,\ J^2\}$. $\mathbf{J}$ is an extension in $su(2)$ of the algebra generated by $\{J_x,\ J_y, J_z\}$. It is important to note that, an operator like J^2 which commutes with all the generators of a Lie group is called as a "Casimir operator" for the Lie group. J^2 is also the Casimir operator of $su(2)$ Lie algebra. The Casimir operator J^2 only has meaning for representations, and not as an element of the algebra since products like J_x^2 do not exist in $\mathbf{J}$.

Therefore, Eq. (2.1) shows that except for J^2, no two components of $\mathbf{J}$ commute with each other. In any representation, therefore, not more than one component of $\mathbf{J}$ can be diagonalized at a time. By convention, we chose basis states for representation which are simultaneous eingenstates of J^2 and J_z. It is not necessary to obtain the explicit eigenfunctions. The eigenfunctions may be characterized by two indices j and m and an eigenfunction may be denoted in the Dirac's ket notation $|j,m\rangle$. The set of all such eigenfunctions for $j = 0, 1/2, 1, ...$, and $m = -j, -j+1, ..., j-1, j,\ j$ constitutes a complete set in the infinite dimensional Hilbert space of the angular momentum operator. The action of the components of this operator on a basis function is

$$
\begin{aligned}
J_z\,|j,m\rangle &= m\hbar\,|j,m\rangle\,, \qquad (2.4) \\
J^2\,|j,m\rangle &= j(j+1)\hbar^2\,|j,m\rangle\,.
\end{aligned}
$$

From these operations, we can say that J^2 can have eigenvalues $j(j+1)\hbar^2$ with j, and J_z can have eigenvalues $m\hbar$ with m.

We can construct two new operators from Eq. (2.1) $J_{\pm} = J_x \pm iJ_y$ which are Hermitian conjugates of each other have respectively the effect of increasing and decreasing the z component of the angular momentum by unity. The action of these operators on a basis are

$$J_{\pm} |j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle\,, \tag{2.5}$$

The coefficients in the above equations are the elements of the matrices representing $J_{\pm}$ with the basis $|j, m\rangle$. The representation of J^2 and its components (J_x, J_y and J_z or J_+, J_- and J_z) with this basis is called the "standard representation of angular momentum". It is clear that in this representation, only the matrix for J_y is purely imaginary while all the other matrices are real. The matrices for J^2 and J_z are diagonal whereas those for J_x, J_y, J_+ and J_- are block-diagonalized with blocks of dimensions $2j + 1$. The states $|j, m\rangle$ generate the representation of $su(2)$ under generalized rotations in the Hilbert space of the operators J^2 and J_z (J_z notation is also denoted by J_0) with the commutation relations between operator J_+ and J_-

$$[J_0, J_{\pm}] = \pm J_{\pm}, \qquad [J_+, J_-] = 2J_0 \tag{2.6}$$

These operator satisfy the essential properties of Lie algebra.

2.2 Lie Superalgebra

Mathematically, a Lie superalgebra is defined as a generalization of a Lie algebra to include a $\mathbb{Z}_2$-grading. In theoretical physics, Lie superalgebras are important as it is used to describe the mathematics of supersymmetry relating particles of different statistics. In most of these theories, the even elements of the superalgebra correspond to bosons and odd elements to fermions.

A *Lie superalgebra G* over a field $\mathbb{R}$ of characteristic zero is a $\mathbb{Z}_2$-graded Lie algebras or as the physicist call them, $\mathbb{Z}_2$-graded Lie algebras, that is a vector space, direct sum of two vector space $G_{\bar{0}}$ and $G_{\bar{1}}$, in which a product $[,]$ is defined as follows [62]:

(1) $\mathbb{Z}_2$-gradation:

$$[G_i, G_j] \subset G_{i+j}$$

(2) Graded-antisymmetry

$$[X_i, X_j] = -(-1)^{\deg X_i . \deg X_j}[X_j, X_i] = X_iX_j - (-1)^{\deg X_i . \deg X_j} X_jX_i$$

where $\deg X_j$ defines the degree of the vector space. Elements $X \in G_{\bar{0}}$ are called even or degree $\deg X = 0$ while elements $X \in G_{\bar{1}}$ are called odd or degree $\deg X = 1$. $G_{\overline{0}}$ is called the even space and $G_{\overline{1}}$ the odd space. If $\deg X_i . \deg X_j = 0$, the bracket $[,]$ defines the usual commutator, otherwise it is an anticommutator.

(3) Generalized Jacobi identity:

$$(-1)^{\deg X_i . \deg X_k}[X_i, [X_j, X_k]] + (-1)^{\deg X_j . \deg X_i}[X_j, [X_k, X_i]] \\ +(-1)^{\deg X_k . \deg X_j}[X_k, [X_i, X_j]] = 0$$

Notice that $G_{\overline{0}}$ is a Lie algebra-called the even part or bosonic part of G - while $G_{\overline{1}}$-called the odd or fermionic part of G - is not an algebra. Lie superalgebras are a natural generalization of normal Lie algebras to include a $\mathbb{Z}_2$-grading. No explanation needed, the above conditions on the superbracket are exactly those on the normal Lie bracket with modifications made for the grading. The last condition is sometimes called the **super Jacobi identity**.

A superalgebra $G = G_{\overline{0}} \oplus G_{\overline{1}}$ over the field $\mathbb{R}$ is called associative if $(X_1.X_2).X_3 = X_1.(X_2.X_3)$ for all elements $X_1, X_2, X_3 \in G$.A superalgebra G is called commutative if $X_1.X_2 = X_2.X_1$ for two elements $X_1, X_2 \in G$.

If we define the same term from the view of physicist, we can say that Lie superalgebra is the extension of Lie algebra with fermions. $G_{\overline{0}}$ Lie algebra-called the even part or bosonic part and $G_{\overline{1}}$-called the odd or fermionic part of G. Generally, Pauli matrices represents the fermions. For example, if we extent $su(2)$ algebra with Pauli matrices which are fermions, we can construct the orthosymplectic group $osp(2,1)$ Lie superalgebra. These algebra play an important role in the matrix Hamiltonians in quantum mechanic. Therefore, we conclude that a Lie superalgebra is constructed from Lie algebra by extending fermionic operators as Pauli matrices.

CHAPTER 3

CONSTRUCTION OF LIE (SUPER)ALGEBRAS

Lie (super)algebras can be applied in many field such as for obtaining the spectrum of physical systems. The most important is the application in quantum mechanics. The most physical equations such as the Schrödinger, Pauli, Dirac have been solved for a large number of potential by employing variety of approaches. Although the methods usually focus on different aspects of those equations, they are not independent from each other. In fact most of the approaches can be formulated in the framework of Lie (super)algebra. The relation of these methods with certain algebraic technique has been discussed in [63]. In this section we try to formulate Lie (super)algebra in terms of the bosons and fermions. We also introduce some specific realizations of the algebras and discuss the connection between them.

In order to understand realizations, let start from the Schrödinger equation. The Schrödinger equation can be solved by using various methods. One of the well known method is to express Schrödinger equation with bosons. To demonstrate our procedure through this thesis, let us consider the following well known example. We introduce the operators as

$$a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right), \qquad a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \tag{3.1}$$

where these two operators satisfy the following general commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}; \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \tag{3.2}$$

where δ_{ij} is the Kronecker-delta function a_i and $a_j^\dagger$ are the annihilation and creation operators of the field. In the Hilbert space, they act on the state as

$$a_i\,|n_1, n_2, ..., n_i, ...\rangle = \sqrt{n_i}\,|n_0, n_1, n_2, ..., n_i - 1....\rangle \tag{3.3a}$$
$$a_j^\dagger\,|n_1, n_2, ..., n_i, ...\rangle = \sqrt{n_i + 1}\,|n_1, n_2, n_3, ..., n_i + 1, ...\rangle \tag{3.3b}$$

Here, the representations with quantum state can be defined by the two following formulas

$$a\,|0\rangle = 0, \qquad \text{or} \qquad \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)\psi_0 = 0$$

where $|0\rangle$ is the lowest (vacuum) vector. This vector also can be denoted as ψ_0 which equal to $\psi_0 = A\exp(-x^2/2)$ where A is the integral constant. These two operators act on space as

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{3.4}$$

where n takes values $0, 1, 2, \ldots..$ Now, our task is to obtain the corresponding representation which is formed by the linear combinations of vectors. To accomplished that, we must use the relations of these two operators when act on the vacuum state (Eq. (3.4)) as

$$\begin{array}{rcl} a^\dagger\,|0\rangle & = & \sqrt{1}\,|1\rangle \\ a^{\dagger 2}\,|0\rangle & = & \sqrt{1.2}\,|2\rangle \\ \ldots.. & = & \ldots\ldots\ldots \\ a^{\dagger n}\,|0\rangle & = & \sqrt{1.2.3.........n}\,|n\rangle \end{array}$$

Hence, it is easy to obtain the general eigenfunction in Dirac notation from the last equation as

$$|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger n}\,|0\rangle \tag{3.5}$$

If we express the Eq. (3.5) is differential form by substituting the value $a^\dagger$, we can obtain the eigenfunction in the form

$$\psi_n = \frac{1}{\sqrt{n!}}(x - \frac{d}{dx})^n\psi_0 \tag{3.6}$$

where $|n\rangle$ is the eigenstate of any Hamiltonian in Hilbert space.

We must mention here that, the realization in Eq. (3.1) is not unique and a variety of realizations can be obtained as differential operators. This realization can be extended into two dimension. Here, we deal with two different realizations in two dimension. The interesting point here is that, all this realizations can be map into each other with appropriate transformation method. The first realization in the Hilbert space is

$$a_1^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_1} + x_1\right); \quad a_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + x_1\right) \tag{3.7}$$

$$a_2^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_2} + x_2\right); \quad a_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} + x_2\right) \tag{3.8}$$

And, the other form of boson operators can be realized as

$$
\begin{aligned}
a_1^\dagger &= \sqrt{\frac{m^* w}{4\hbar}}(x_1 + i x_2) - \sqrt{\frac{\hbar}{4m^* w}}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \\
a_1 &= \sqrt{\frac{m^* w}{4\hbar}}(x_1 - i x_2) + \sqrt{\frac{\hbar}{4m^* w}}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \\
a_2^\dagger &= \sqrt{\frac{m^* w}{4\hbar}}(x_1 - i x_2) - \sqrt{\frac{\hbar}{4m^* w}}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \\
a_2 &= \sqrt{\frac{m^* w}{4\hbar}}(x_1 + i x_2) + \sqrt{\frac{\hbar}{4m^* w}}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)
\end{aligned}
\tag{3.9}
$$

As we seen that it is possible to express boson operators in different forms. Then let us define momentum operator as $\mathbf{p} \simeq a = \frac{d}{dx}$, and position vector as $\mathbf{x} \simeq a^\dagger = x$ in the Bargmann-Fock space. Then we notice that these operators also obey the commutation relations in Eq. (3.2). After that, we focus on interrelation between these different differential realization forms. Then we realize that if we transform boson operators from Hilbert space to Bargmann space, we can write the eigenstate in the polynomial function form. In this way, the state ψ_0 can be written in the polynomial form.

To accomplish that, there are variety transformations such as similarity transformation, unitary transformation and nonunitary transformation. By developing appropriate transformation procedure, all boson operators can be reduced to operators $\mathbf{p}$ and $\mathbf{x}$. For instance, if we introduce one transformation operator as

$$
\Gamma = \exp\left[\theta\left(a_1^2 + a_1^{\dagger 2} + \lambda(a_2^2 + a_2^{\dagger 2}\right))\right] \tag{3.10}
$$

where θ and λ are constant. If we choose $\lambda = 0$, the operator in Eq. (3.10) becomes

$$
\Gamma = \exp\left[\theta\left(a^2 + a^{\dagger 2}\right)\right]. \tag{3.11}
$$

This operator are the required operator to transform Eq. (3.1). In order to apply similarity transformations we use the definition of the exponential map

$$
\exp[A]\ B\ \exp[-A] = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots.. \tag{3.12}
$$

where A and B are operators. By using this expression, under similarity transformation operators of Eq. (3.1) can be transformed to

$$
\begin{aligned}
\widetilde{a} &= \Gamma a \Gamma^{-1} = a\cos(-2\theta) + a^\dagger \sin(-2\theta) \\
\widetilde{a}^\dagger &= \Gamma a^\dagger \Gamma^{-1} = a\cos(-2\theta) - a^\dagger \sin(-2\theta)
\end{aligned}
\tag{3.13}
$$

If we choose parameter appropriate parameter $\theta = \pi/8$, and substitute into Eq.

(3.13), we get

$$\begin{aligned} \widetilde{a} &= \Gamma a \Gamma^{-1} = \frac{1}{\sqrt{2}}\left(a_1 - a_1^\dagger\right) = \frac{d}{dx} \\ \widetilde{a}^\dagger &= \Gamma a^\dagger \Gamma^{-1} = \frac{1}{\sqrt{2}}\left(a_1 + a_1^\dagger\right) = x \end{aligned} \tag{3.14}$$

In this way, the operator in Hilbert space are transformed to the Bargmann space. During the transformation procedure, the eigenvalue does not change while the eigenfunctions can be related as

$$\psi_0 = \Gamma^{-1}|0\rangle = \exp\left[-\frac{\pi}{8}\left(a^2 + a^{\dagger 2}\right)\right]|0\rangle$$

Therefore, in the new space, the vacuum state is represented by polynomial function.

Similarly, when we choose the parameter $\theta = \pi/8$ and $\lambda = 1$ in Eq. (3.10), then the operators in Eq. (3.7) and Eq. (3.8) can be transformed under the similarity transformation as

$$\begin{aligned} \widetilde{a}_1 &= \Gamma a_1 \Gamma^{-1} = \frac{1}{\sqrt{2}}\left(a_1 - a_1^\dagger\right), \quad \widetilde{a}_1^\dagger = \Gamma a_1^\dagger \Gamma^{-1} = \frac{1}{\sqrt{2}}\left(a_1^\dagger + a_1\right) \\ \widetilde{a}_2 &= \Gamma a_2 \Gamma^{-1} = \frac{1}{\sqrt{2}}\left(a_2 - a_2^\dagger\right), \quad \widetilde{a}_2^\dagger = \Gamma a_2^\dagger \Gamma^{-1} = \frac{1}{\sqrt{2}}\left(a_2^\dagger + a_2\right). \end{aligned} \tag{3.15}$$

The Bargmann-Fock space differential realizations of the operators take the form

$$\widetilde{a}_1 = \frac{\partial}{\partial x_1}; \quad \widetilde{a}_1^\dagger = x_1; \quad \widetilde{a}_2 = \frac{\partial}{\partial x_2}; \quad \widetilde{a}_2^\dagger = x_2 \tag{3.16}$$

In this case, the eigenvalue remains the same while the eigenfunction of the Hamiltonians can be related

$$|n_1, n_2\rangle = \Gamma^{-1} f(x_1, x_2), \tag{3.17}$$

where $|n_1, n_2\rangle$ is the eigenstate of the Hamiltonian in the Hilbert space and $f(x_1, x_2)$ is the its eigenfunction in the Bargmann-Fock space. The function $f(x_1, x_2)$, generally, in the polynomial function of the product of the x_1, and x_2.

Now our task is how to transform the operators of Eq. (3.9) into the Bargman space. First of all let transform operator with Γ. The action of Γ on the operator

$$\begin{aligned} \widetilde{a}_1 &= \Gamma a_1^{-1} \Gamma = a_1 \cos(-2\theta) - a_2 \sin(-2\theta) \\ \widetilde{a}_1^\dagger &= \Gamma a_1^\dagger \Gamma^{-1} = a_1^\dagger \cos(-2\theta) - a_2^\dagger \sin(-2\theta) \\ \widetilde{a}_2 &= \Gamma a_2^{-1} \Gamma = a_2 \cos(-2\theta) + a_1 \sin(-2\theta) \\ \widetilde{a}_1^\dagger &= \Gamma a_2^\dagger \Gamma^{-1} = a_2^\dagger \cos(-2\theta) + a_1^\dagger \sin(-2\theta) \end{aligned} \tag{3.18}$$

The transformation operator of Eq. (3.10) with parameter $\theta = \pi/8$, and $\lambda = -1$ can not transform Eq. (3.9) into the Bargman–Fock space representation. To do this, we must require another similarity transformation operator

$$\Lambda = \exp\left[\phi(a_1^\dagger a_2 + a_2^\dagger a_1)\right]. \tag{3.19}$$

Then, similarity transformation Λ transform the operator as

$$\begin{aligned}
\widetilde{a}_1 &= \Lambda a_1 \Lambda^{-1} = a_1 \cos(\phi) - a_1^\dagger \sin(\phi) \\
\widetilde{a}_1^\dagger &= \Lambda a_1^\dagger \Lambda^{-1} = a_1^\dagger \cos(\phi) + a_1 \sin(\phi) \\
\widetilde{a}_2 &= \Lambda a_2 \Lambda^{-1} = a_2 \cos(\phi) + a_2^\dagger \sin(\phi) \\
\widetilde{a}_2^\dagger &= \Lambda a_2 \Lambda^{-1} = a_2^\dagger \cos(\phi) + a_2 \sin(\phi)
\end{aligned} \tag{3.20}$$

With application of these two operators with parameters $\theta = \pi/8$, $\phi = -\pi/4$ and $\lambda = -1$ and change of variable $y \to iy$ give the following realizations of boson operators of Eq. (3.9)

$$\begin{aligned}
\widetilde{a}_1 &= \Lambda\Gamma a_1 \Gamma^{-1}\Lambda^{-1} = \sqrt{\frac{\hbar}{m^* w}}\frac{\partial}{\partial x_1}, \\
\widetilde{a}_1^\dagger &= \Lambda\Gamma a_1^\dagger \Gamma^{-1}\Lambda^{-1} = \sqrt{\frac{m^* w}{\hbar}} x_1, \\
\widetilde{a}_2 &= \Lambda\Gamma a_2 \Gamma^{-1}\Lambda^{-1} = \sqrt{\frac{\hbar}{m^* w}}\frac{\partial}{\partial x_2}, \\
\widetilde{a}_2^\dagger &= \Lambda\Gamma a_2^\dagger \Gamma^{-1}\Lambda^{-1} = \sqrt{\frac{m^* w}{\hbar}} x_2.
\end{aligned} \tag{3.21}$$

In this case, the eigenfunction of the Hamiltonians can be related

$$|n_1, n_2\rangle = \Gamma^{-1}\Lambda^{-1} f(x_1, x_2).$$

It is seen that writing the differential equation with boson is a useful tool in solving the spectrum of physical systems. For this purpose, in the following section, we will construct Lie algebras with boson operators.

3.1 Bosonic Construction of Lie Algebras

We give some construction of Lie algebras such as $su(2)$ and $su(1,1)$. Bosonic realizations for these Lie algebras with three generators are obtained using bosonic creation and annihilation operators satisfying Heisenberg commutation realizations.

3.1.1 Construction of $su(2)$ Lie Algebra

A convenient way to construct a spectrum-generating algebra for systems with a finite number of bound states is by introducing a set of boson creation and annihilation operators. The number of boson operators needed in this construction, n, is related to the dimension of the space, r, by $n = r + 1$ [7]. Therefore, for one dimensional problems ($r = 1$) we shall introduce two boson operators, a_1 and a_2. These operators obey the usual commutation relations in Eq. (3.2).

The bilinear products $a_1^\dagger a_1$, $a_1^\dagger a_2$, $a_2^\dagger a_1$, and $a_2^\dagger a_2$ generate the group $U(2)$ (the unitary group in two dimensions). It is possible to recast these four generators in a more familiar form by introducing the three parameter group $SU(2)$ (special unitary group in two dimension). The generator of $su(2)$ algebra can be defined as

$$\begin{aligned} J_+ &= a_1^\dagger a_2, \\ J_- &= a_1 a_2^\dagger, \\ J_0 &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \end{aligned} \tag{3.22}$$

which satisfy the commutation rules

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0 \tag{3.23}$$

The representations in Eq. (3.22) are just the Schwinger representations [8] of $su(2)$. The generators of $J_\pm$ can be written in terms of J_x, J_y as $J_\pm = J_x \pm iJ_y$. The fourth generator is the number operator which commutes with the generators of the $su(2)$ algebra given by

$$N = a_1^\dagger a_1 + a_2^\dagger a_2, \tag{3.24}$$

i.e., the total boson number operator. This operator commutes with all the others. The Casimir operator of $su(2)$, C is related to N by

$$C = \frac{1}{4}N\,(N+2)\,; \tag{3.25}$$

therefore if we denote the eigenvalues of the operator C,

$$\langle C \rangle = j(j+1), \tag{3.26}$$

It is obvious that the irreducible representations of $su(2)$ can be characterized by the total boson number

$$N = 2j. \tag{3.27}$$

The application of the realization in Eq. (3.22) on a set of $2j+1$ states lead to the $(2j+1)$-dimensional unitary irreducible representation for each $j = 0, 1/2, 1, ...$

If the basis states $|j, m\rangle$ $(m = j, j-1, ...-j)$, then the action of the operators on the basis states is given in Eqs. (2.4) and (2.5).

Furthermore, the single-mode boson realizations of $su(2)$ algebra can be constructed by defining there operators:

$$\begin{aligned} J'_+ &= \sqrt{2j - N'}a, \\ J'_- &= a^\dagger \sqrt{2j - N'}, \\ J'_0 &= \sqrt{j - N'} \end{aligned}$$

which also satisfy the commutation relations of the $su(2)$ algebra. And the number operator of this realizations can be written as

$$N' = a^\dagger a.$$

3.1.2 Construction of $su(1,1)$ Lie Algebra

One possible way to continue the algebra of $su(2)$ would be by introducing the operators

$$K'_x = iJ_x, \qquad K'_y = iJ_y, \qquad K'_0 = J_0. \tag{3.28}$$

These three operators generate the algebra of $su(1,1)$ [64, 65],

$$\left[K'_+, K'_-\right] = -2K'_0, \qquad \left[K'_0, K'_\pm\right] = \pm 2K'_\pm. \tag{3.29}$$

However, this continuation suffers from the fact that both K'_y and K'_x are not Hermitian and thus cannot correspond to physical observable.

A $su(1,1)$ algebra is more appropriate to the present problem. It can be obtained by using the same set of boson operators as in Eq. (3.1) and Eq.(3.22), and introducing the three operators [66]

$$\begin{aligned} K_+ &= a_1^\dagger a_2^\dagger, \\ K_- &= a_1 a_2, \\ K_0 &= \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \end{aligned} \tag{3.30}$$

The reader can easily check that the K's satisfy $su(1,1)$ commutation relations. Also $K_+^\dagger = K_-$, $K_0^\dagger = K_0$, and thus K_x, K_y, and K_0, are all explicitly Hermitian. To complete the algebra of $su(1,1)$, one needs a fourth operator which is the sum of the boson operators (N), in the present case, the number operator in the Eq. (3.24) takes the form as angular momentum operator;

$$M = a_1^\dagger a_1 - a_2^\dagger a_2, \tag{3.31}$$

i.e., the *difference* of the number operators. The eigenvalues of the operator M characterize the irreducible representations of $su(1,1)$. The Casimir operator of $su(1,1)$, C is related to M by

$$C = \frac{1}{4}\,(1+M)\,(1-M)\,; \tag{3.32}$$

therefore, if we denote the eigenvalues of the operator C,

$$\langle C\rangle = k(1-k), \tag{3.33}$$

we find that

$$M = (1-2k), \tag{3.34}$$

The unitary representations of the non-compact group $SU(1,1)$ are infinite dimensional [67]. The generator K_0 is still compact and has a discrete spectrum in such a representation. However, the generator K_y is non-compact and will have a continuous spectrum [68].

The action of the realization of Eq. (3.30) on the states $|k,n\rangle$ ($n = 0,1,2,......$), leads to infinite dimensional unitary irreducible representation, so-called positive representation $D^+(k)$, which corresponds to any $k = 1/2, 1, 3/2, ...$

$$\begin{aligned}
K_0\,|k,n\rangle &= (k+n)\,|k,n\rangle\,, \\
K_+\,|k,n\rangle &= \sqrt{(2k+n)(n+1)}\,|k,n+1\rangle\,, \\
K_-\,|k,n\rangle &= \sqrt{(2k+n-1)n}\,|k,n-1\rangle\,, \\
C\,|k,n\rangle &= k(1-k)\,|k,n\rangle\,.
\end{aligned}$$

Furthermore, another interesting $su(1,1)$ algebra, generated by three operators

$$\begin{aligned}
T_+ &= \frac{1}{2}(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger), \\
T_- &= \frac{1}{2}(a_1 a_1 + a_2 a_2), \\
T_0 &= \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)
\end{aligned} \tag{3.35}$$

To find the significance of this $su(1,1)$ algebra, we compute its Casimir invariant

$$T^2 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 2a_1^\dagger a_1 a_2^\dagger a_2 - 1) - \frac{1}{4}(a_1^\dagger a_1^\dagger a_2 a_2 + a_2^\dagger a_2^\dagger a_1 a_1) = J_y^2 - \frac{1}{4} \tag{3.36}$$

where $J_y = -i(a_1^\dagger a_2 - a_2^\dagger a_1)$. Thus the Casimir invariant these realization is

$$C = -T^2 - \frac{1}{4} \tag{3.37}$$

on the other hand, the operator T_0 is related to the number operator N , the invariant of $su(2)$, by

$$T_0 = \frac{N+1}{2}. \tag{3.38}$$

Additionally to these different realizations, we can construct a single mode realization. The most popular single-mode bosonic realizations of the $su(1,1)$ can be obtained [6];

$$\begin{aligned} L_- &= \frac{a^2}{2}, \\ L_+ &= \frac{a^{\dagger 2}}{2}, \\ L_0 &= \frac{1}{2}(a^\dagger a + \frac{1}{2}). \end{aligned} \tag{3.39}$$

which satisfy the same commutation relations of $su(1,1)$ lie algebra. In this representations, the Number operator and Casimir operator are

$$\begin{aligned} M' &= a^\dagger a \\ C &= L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+) \equiv -\frac{3}{16}. \end{aligned} \tag{3.40}$$

Here, the Casimir operator is constant. It means that, this representation is infinite-dimensional.

For the single-mode bosonic realization of $su(1,1)$ algebra that we require here, the Bargmann index k ,is equal to either $1/4$ or $3/4$, which splits the Hilbert space of the boson space into two independent subspaces.

3.2 Boson-Fermion Realizations of the Lie Superalgebra

It is well known that the boson and fermion operators are tightly associated to the field modes and interaction between the levels in the multilevel physical systems. In usual, bosons are realized as differential operators and fermions are realized with matrices. For the two level systems the fermions can be realized by Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.41}$$

and they obey the anti commutation relation;

$$\{\sigma_i, \sigma_j\} = \delta_{ij} \tag{3.42}$$

In general, the Hamiltonian of a physical system can be expressed as combinations of the bosons and fermions. The solvability of the Hamiltonian depends on the form of the combinations of the bosons and the fermions.

Lie superalgebras have become increasingly important in nuclear physics, supernification, and in supergranity. Any Lie superalgebra is only an isomorphism of a quotient subalgebra of the universal enveloping algebra of the Heisenberg Weyl superalgebra, we can regard a representation of this quotient subalgebra as its representation, which is called boson-fermion realizations of representations of the Lie superalgebra. It is well known that, Lie superalgebra is the extension of Lie algebras with fermions which are realized by Pauli matrices. Due to realized fermions by matrices, Lie superalgebra become efficient method to describe matrix Hamiltonians. In the next section, we construct some boson-fermion realizations of the Lie superalgebras.

3.2.1 Construction of $osp(2,1)$ Superalgebra

A convenient way to construct a spectrum generating superalgebra for systems with a finite number of bound states is by introducing a set of boson and fermion operators. The superalgebra $osp(2,1)$ might be constructed by extending $su(2)$ algebra with the fermionic generators

$$V_+ = \sigma_+ a_2, V_- = -\sigma_+ a_1, W_+ = \sigma_- a_1^\dagger, W_- = \sigma_- a_2^\dagger \tag{3.43}$$

where σ_+ and σ_-

$$\sigma_+ = \frac{1}{2}\left(\sigma_1 + i\sigma_2\right), \qquad \sigma_- = \frac{1}{2}\left(\sigma_1 - i\sigma_2\right) \tag{3.44}$$

which lead to the matrices

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{3.45}$$

and obey the usual commutation relations

$$[\sigma_0, \sigma\pm] = \pm 2\sigma_0, \qquad [\sigma_+, \sigma_-] = \sigma_0 \tag{3.46}$$

The superalgebra $osp(2,1)$ can be constructed with the generators of Eq. (3.22) and Eq. (3.43). It is easily seen that the generators given in Eq. (3.43) can be mapped on to each others by a change of the fermionic creation and annihilation operators. The number operator of this superalgebra is the same with that of $su(2)$ algebra. As discussed in [70], the generators of the $osp(2,1)$ algebra are written as follows:

$$\{J_\pm, J_0, J \in osp(2,1)_{\bar{0}} \mid V_\pm, W_\pm \in osp(2,1)_{\bar{1}}\}, \tag{3.47}$$

where J is the total number operator of the system and is given by

$$J = \frac{1}{2}N + \sigma_+\sigma_- \tag{3.48}$$

for the generators of Eq. (3.22) and Eq. (3.43). The generators of the $osp(2,1)$ superalgebra satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
[J_+, J_-] &= 2J_0, \qquad [J_0, J_\pm] = \pm J_\pm, \qquad [J, J_{\pm,0}] = 0, \\
[J_0, V_\pm] &= \pm\frac{1}{2}V_\pm, \qquad [J_0, W_\pm] = \pm\frac{1}{2}W_\pm, \qquad [J, V_\pm] = \frac{1}{2}V_\pm \\
[J_\pm, V_\mp] &= V_\pm, \qquad [J_\pm, W_\mp] = W_\pm, \qquad [J, W_\pm] = -\frac{1}{2}W_\pm, \\
[J_\pm, V_\pm] &= 0, \qquad [J_\pm, W_\pm] = 0 \\
\{V_\pm, W_\pm\} &= \pm J, \qquad \{V_\pm, W_\mp\} = -J_0 \pm J \\
\{V_\pm, V_\pm\} &= \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0.
\end{aligned} \tag{3.49}$$

Another realization form of the $osp(2,1)$ algebra can be obtained from some basic matrix operations. Let introduce a matrix

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.50}$$

Then multiplying these matrix with both sides of each generators in Eq. (3.43), we have a new realization of $osp(2,1)$ algebra

$$\begin{aligned}
V_+ &= Q\sigma_+ a_2 Q = \sigma_- a_2, \qquad V_- = Q(-\sigma_+ a_1)Q = -\sigma_- a_1 \\
W_+ &= Q\sigma_- a_1^\dagger Q = \sigma_+ a_1^\dagger, \qquad W_- = Q\sigma_- a_2^\dagger Q = \sigma_+ a_2^\dagger
\end{aligned}$$

which obey the commutation rules in Eq. (3.49). And the total number operator of the new system is given by

$$J = \frac{1}{2}N + \sigma_-\sigma_+ \tag{3.51}$$

3.2.2 Construction of $osp(2,2)$ Superalgebra

One of the major symmetry group candidates for spin one-half particles is the supergroup $osp(2,2)$ which has four even and four odd generators. Its even generators can be represented by bosons while odd generators are represented by combinations of the fermions and bosons [70]. The superalgebra $osp(2,2)$ might be constructed by extending $su(1,1)$ algebra with the fermionic generators. It is possible to express a set of fermionic generators to extend the $su(1,1)$ algebra to the $osp(2,2)$ algebra. It can be given as

$$V_+ = \sigma_- a_2^\dagger, V_- = \sigma_- a_1, W_+ = \sigma_+ a_1^\dagger, W_- = \sigma_+ a_2. \tag{3.52}$$

The total number operator of the system is given by

$$J = \frac{1}{2}\left(N - \sigma_0\right). \tag{3.53}$$

The conservative quantity of a physical system possesses $osp(2,2)$ superalgebra can be written as

$$K = N - \frac{1}{2}\sigma_0 \tag{3.54}$$

which commutes with the operators of the $osp(2,2)$ superalgebra and is the number operator of $su(1,1)$ algebra. The generators of the $osp(2,2)$ superalgebra satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
[K_+, K_-] &= -2K_0, \qquad [K_0, K_\pm] = \pm K_\pm, \\
[J, K_\pm] &= 0, \qquad [J, K_0] = 0 \\
[K_0, V_\pm] &= \pm\frac{1}{2}V_\pm, \quad [K_0, W_\pm] = \pm\frac{1}{2}W_\pm, \\
[K_\pm, V_\mp] &= V_\pm, \qquad [K_\pm, W_\mp] = W_\pm, \\
[J, W_\pm] &= -\frac{1}{2}W_\pm, \qquad [J, V_\pm] = \frac{1}{2}V_\pm \\
[K_\pm, V_\pm] &= 0, \qquad [J_\pm, W_\pm] = 0 \\
\{V_\pm, W_\pm\} &= J_\pm, \qquad \{V_\pm, W_\mp\} = \pm K_0 - J \\
\{V_\pm, V_\pm\} &= \{V_\pm, V_\mp\} = 0 \\
\{W_\pm, W_\pm\} &= \{W_\pm, W_\mp\} = 0.
\end{aligned} \tag{3.55}$$

In order to obtain the second realization of this algebra, we can apply the same matrix operation procedure with $osp(2,1)$ algebra. Therefore the new realization of $osp(2,2)$ algebra can be expressed as

$$\begin{aligned}
V_+ &= Q\sigma_+ a_2^\dagger Q = \sigma_- a_2^\dagger, \qquad V_- = Q\sigma_+ a_1 Q = \sigma_- a_1 \\
W_+ &= Q\sigma_- a_1^\dagger Q = \sigma_+ a_1^\dagger, \qquad W_- = Q\sigma_- a_2 Q = \sigma_+ a_2
\end{aligned}$$

where the new expressed operators obey the commutation rules in Eq. (3.55).

3.3 Transformation of the Bosons and Fermions

Before going further, we introduce the following another transformation procedure which plays a key role in the solution of the boson-fermion Hamiltonians by introducing two operators S and T operator. S operator can be defined as

$$S = (a_2^\dagger)^{c a_1^\dagger a_1 + d\sigma_+\sigma_-} \tag{3.56}$$

where c and d are constants. This operator can be written as

$$S = (a_2^+)^{c a_1^\dagger a_1 + d\sigma_+\sigma_-} \equiv d(\sigma_-\sigma_+ + \sigma_+\sigma_- a_2^\dagger)(a_2^\dagger)^{c a_1^\dagger a_1}$$

The action of operator S on two component state

$$S\left(|n_1,n_2\rangle|\uparrow\rangle+|n_1,n_2\rangle|\downarrow\rangle\right)=\sqrt{\frac{(n_2+cn_1)!}{n_2!}}$$
$$\times\left(\sqrt{(n_2+cn_1+1)}\,|n_1,n_2+cn_1+1\rangle|\uparrow\rangle+|n_1,n_2+cn_1\rangle|\downarrow\rangle\right) \tag{3.57}$$

where $|\uparrow\rangle$ stands for up state and $|\downarrow\rangle$ stands for down state. Since a_1, and a_2 commute and $\sigma_{\pm,0}$ also commute with the bosonic operators, the transformation a_1, and $a_1^\dagger$ under S can be obtained by writing $a_2^\dagger=e^b$, with $[a_1,b]=[a_1^\dagger,b]=0$. Then all boson operators can be transformed under S operator

$$\begin{aligned}
S^{-1}a_1S &= a_1(a_2^\dagger)^c\\
S^{-1}a_1^\dagger S &= a_1^\dagger(a_2^\dagger)^{-c}\\
S^{-1}a_2S &= a_2+(ca_1^\dagger a_1+d\sigma_+\sigma_-)(a_2^\dagger)^{-1}\\
S^{-1}a_2^\dagger S &= a_2^\dagger\\
S^{-1}\sigma_\pm S &= \sigma_\pm(a_2^\dagger)^{\pm d}.
\end{aligned} \tag{3.58}$$

In a similar manner and for the same purpose, let us consider the other transformation operator T

$$T=(a_2)^{\varepsilon a_1^\dagger a_1+\eta\sigma_+\sigma_-} \tag{3.59}$$

where ε and η are constants. By using the similar arguments given in the previous transformation operations one can easily obtain the following transformations:

$$\begin{aligned}
Ta_1^\dagger T^{-1} &= a_1^\dagger(a_2)^{+\varepsilon}\\
Ta_1T^{-1} &= a_1(a_2^\dagger)^{-\varepsilon}\\
Ta_2T^{-1} &= a_2\\
Ta_2^\dagger T^{-1} &= a_2^\dagger+(\varepsilon a_1^\dagger a_1+\eta\sigma_+\sigma_-)(a_2^\dagger)^{+\varepsilon}\\
T\sigma_\pm T^{-1} &= \sigma_\pm(a_2^\dagger)^{\pm\eta}
\end{aligned} \tag{3.60}$$

3.4 Transformation of Lie (Super)algebra under S and T Operators

At the beginning of this section, firstly we will look at the transformation of Lie algebra $su(1,1)$ and $su(2)$ under T and S operators. When we take the parameter $\varepsilon=1$ and $\eta=0$, the operator becomes

$$T=(a_2)^{a_1^\dagger a_1}$$

Then T transformation operator on the $su(1,1)$ generators of Eq. (3.30) gives

$$\begin{aligned} K''_+ &= TK_+T^{-1} = a_1^\dagger a_2^\dagger a_2 + a_1^\dagger a_1^\dagger a_1 \\ K''_- &= TK_-T^{-1} = a_1, \\ K''_0 &= TK_0T^{-1} = \frac{1}{2}(2a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \\ M'' &= TMT^{-1} = -a_2^\dagger a_2, \end{aligned} \tag{3.61}$$

In the final form of transformation, the number operator M is transformed to the one type of boson -$a_2^\dagger a_2$ which characterizes the system. So that, we can characterize the algebra by a fixed number $a_2^\dagger a_2 = -j$ in the primed representation. The difference between the representations of $su(1,1)$ given in Eq. Eq. (3.30) and primed representations in Eq. (3.61) is that, while in the first representations two bosons characterize the system, in second it is only the number of a_2 bosons that characterize the system. By this way, the generators can be expressed in terms of one-boson operator a_1 and yield the following realization:

$$\begin{aligned} K''_+ &= TK_+T^{-1} = -ja_1^\dagger + a_1^\dagger a_1^\dagger a_1 \\ K''_- &= TK_-T^{-1} = a_1, \\ K''_0 &= TK_0T^{-1} = \frac{1}{2}(2a_1^\dagger a_1 - j + 1) \end{aligned} \tag{3.62}$$

Now, we try to investigate the action of S operator on $su(2)$ algebra operators in Eq. (3.22). In this operation, we set the parameters in Eq. (3.56) $c = 1$, and $d = 0$, then S operator becomes

$$S = (a_2^\dagger)^{a_1^\dagger a_1}.$$

When we act on the related algebra generator, we get

$$\begin{aligned} J'_+ &= SJ_+S^{-1} = -a_1^\dagger a_2^\dagger a_2 + a_1^\dagger a_1^\dagger a_1, \\ J'_- &= SJ_-S^{-1} = a_1, \\ J'_0 &= SJ_0S^{-1} = \frac{1}{2}(2a_1^\dagger a_1 - a_2^\dagger a_2) \\ N' &= SNS^{-1} = a_2^\dagger a_2 \end{aligned} \tag{3.63}$$

Similarly, the number operator N is transformed to the one type of boson $a_2^\dagger a_2$ which characterizes the system. If we fixed to number $a_2^\dagger a_2 = j$, the generators can be expressed with one type of boson as

$$\begin{aligned} J'_+ &= -ja_1^\dagger + a_1^{\dagger 2} a_1, \\ J'_- &= a_1, \\ J'_0 &= \frac{1}{2}(2a_1^\dagger a_1 - j) \\ N' &= j \end{aligned} \tag{3.64}$$

Here, the interesting point is that the transformed generators J'_+, J'_-, and J'_0 is became the generators of $sl(2,R)$ algebra which is the linearization of the $su(2)$ algebra and satisfy the commutation relations

$$\left[J'_+, J'_-\right] = -2J_0, \qquad \left[J'_0, J'_\pm\right] = \pm J'_\pm \tag{3.65}$$

Up to now, we completed the transformation of Lie algebra under two transformation operator. These two operators not only transform the boson operators, an also transform the Lie algebra into the another algebra such as $sl(2,R)$ algebra. In the following, we will do the same procedure for the transformation of the Lie superalgebra $osp(2,1)$.

Let us consider the generators of $osp(2,1)$ algebra under the S transformation when the parameter $c = d = 1$. After the transformation we get;

$$\begin{aligned}
J'_+ &= SJ_+S^{-1} = -a_1^\dagger a_1^\dagger a_1 + a_1^\dagger(a_2^\dagger a_2 - \sigma_+\sigma_-) \\
J'_- &= SJ_-S^{-1} = a_1 \\
J'_0 &= SJ_0S^{-1} = \frac{1}{2}(2a_1^\dagger a_1 - a_2^\dagger a_2 + \sigma_+\sigma_-) \\
J' &= SJS^{-1} = \frac{1}{2}(a_2^\dagger a_2 + \sigma_+\sigma_-) \\
V'_+ &= SV_+S^{-1} = \sigma_+(a_2^\dagger a_2 - a_1^\dagger a_1 - \sigma_+\sigma_-) \\
V'_- &= SV_-S^{-1} = -\sigma_+ a_1 \\
W'_+ &= SW_+S^{-1} = \sigma_- a_1^\dagger \\
W'_- &= SW_-S^{-1} = \sigma_- \\
N' &= SNS^{-1} = a_2^\dagger a_2
\end{aligned} \tag{3.66}$$

As it is shown in above equation, the number operator N becomes to the one type of boson $a_2^\dagger a_2$. Therefore, we can characterize the algebra by a fixed number $a_2^\dagger a_2 = j$ in the primed representation. The representations of $osp(2,1)$ given in Eq. (3.43) and transformed representations in Eq. (3.66) are different in the characterization form. That is, in the first representations two bosons and fermions characterize the system, in second it is only the number of a_2 bosons that characterize the system. By this way, the generators can be expressed in terms of one-boson operator a_1 and yield the following realization:

$$
\begin{aligned}
J'_+ &= -a_1^\dagger a_1^\dagger a_1 + a_1^\dagger(j - \sigma_+\sigma_-) \\
J'_- &= a_1 \\
J'_0 &= \frac{1}{2}(2a_1^\dagger a_1 - j + \sigma_+\sigma_-) \\
J' &= \frac{1}{2}(j + \sigma_+\sigma_-) \\
V'_+ &= -\sigma_+(a_1^\dagger a_1 - j) \\
V'_- &= -\sigma_+ a_1 \\
W'_+ &= \sigma_- a_1^\dagger \\
W'_- &= \sigma_- \\
N' &= j
\end{aligned}
\tag{3.67}
$$

These generators play an important role in the quasi-exact solution of the matrix Schrödinger equation.

Consequently, after the transformation of Lie (super)algebra, the Hamiltonian of a physical systems which $osp(2,1)$ and $osp(2,2)$ symmetry can be characterized only with one type of boson.

CHAPTER 4

EXACT AND QUASI–EXACT SOLVABILITY

Exact–solvability and quasi–exact solvability terms have a special place in physics. The physical potentials can be classified into the some classes. The most interesting class potentials are exactly–solvable potentials and quasi–exactly solvable potentials. For these two class potentials, scientists have introduce some models which are called exactly and quasi–exactly models.

The exactly solvable model is a model which describes the entire spectrum whereas the quasi–exactly solvable models describes only a finite part of spectrum. In other words, exactly solvable model provides to obtain for all its energy levels and corresponding wave functions in explicit form. Nevertheless, only a few potentials in physics can be described with the exactly–solvable model.

In order to define the quasi–exactly solvable potentials, scientists have tried to introduce new approach which is based on the some constructive expansion of the concept of exact solvability. This approach was proposed several years ago. It was based on the finding a new models for which the spectral problem could be solved exactly only for certain limited parts of the spectrum but not for the whole spectrum. This idea has gradually crystallized in the last decade. The critical impetus has given in the works of authors [71, 72, 59], and especially Turbiner [39] and Ushveridze [73]. Turbiner and Ushveridze have introduced these models "quasi–exactly solvable" referring to their "order" as the number of states for which exact results can be obtained.

4.1 Exact Solvability

Any physical potentials in Schrödinger equation is exactly solvable potential if the whole spectrum can be obtained algebraically that generating the sets of eigenvalues and eigenfunctions, E_n. Exactly solvable potentials in nonrelativistic quantum mechanics can be classified into two classes as type I and type II potentials. For the type I, the Schrödinger equation can be reduced to a hypergeometric equation by using some transformation methods. For these type potentials, the corresponding eigenfunctions can be written as Laguerre polynomials. These type potentials are Generalized Pösch–Teller, Scarf (trigonometric and hyperbolic), Eckart and Rosen–Morse I potentials. However, for the type II potentials, the Schrödinger equation can be reduced to a confluent hypergeometric equation. For these type II potentials, the corresponding eigenfunctions can be written as associated Legendre functions. Type II potentials include Harmonic oscillator, Morse and Coulomb potentials.

These two type potentials can be mapped into each other by some transformation procedures. However, here we will not deal with this procedure, we only deal with the basic definition of exactly solvable potentials.

4.2 Quasi–exact Solvability

Now, lets take our attention to the quasi–exactly solvable problems. Recently, the quasi–exactly solvable problems of quantum mechanics are shown to be related to the existence of the finite-dimensional representations of the algebra $sl(2,R)$ and $osp(2,1)$ algebra. The operators of these algebras have a finite–dimensional invariant subspace with an explicit basis in functions. If this is so, the operators of algebras can be presented in the explicit diagonal form which is called quasi–exactly solvable.

Let us introduce the realization of the algebra $sl(2,R)$ in the Eq. (3.64) in differential equation form

$$\begin{aligned} J_+ &= z^2\frac{d}{dz} - jz, \\ J_- &= \frac{d}{dz}, \\ J_0 &= z\frac{d}{dz} - \frac{j}{2}. \end{aligned} \tag{4.1}$$

where there generators satisfy the commutation relations of the $sl(2,R)$-algebra for any value of the parameter j. If n is a positive integer, the algebra in Eq. (4.1) possess a $(n+1)-$ dimensional irreducible representation:

$$P_{n+1} = \left\langle 1, z, z^2, ..., z^{n+1} \right\rangle . \tag{4.2}$$

Linear and bilinear combinations of the operators given in Eq. (4.1) are quasi-exactly solvable when the space is defined as in Eq. (4.2). These three generator act on the space as

$$\begin{aligned} J_+ P_{n+1} &= (n-j) P_{n+2}, \\ J_- P_{n+1} &= n P_n, \\ J_0 P_{n+1} &= (n - \frac{j}{2}) P_{n+1}. \end{aligned} \tag{4.3}$$

We seek to find the quasi–exactly solvable operator in terms of $sl(2,R)$-algebra generators. To find the relation between quasi–exact solvability and Lie algebra, let takes the eigenvalue problem for the Schrödinger differential operators of in one variable

$$H\varphi(x) = E\varphi(x), \qquad H = -\frac{d^2}{dx^2} + V(x) \tag{4.4}$$

These eigenvalue problem has $(n+1)$ linearly independent eigenfunctions in the form of P_{n+1} if the H operator is quasi–exactly solvable, and has an infinite sequences of polynomial eigenfunctions if the H operator is exactly solvable. Here, the bilinear form of H operator can be derived from [39]

$$H = \sum_{j,i=+,-,0} d_{ij} J_i J_j + \sum_{i=+,-,0} d_i J_i \tag{4.5}$$

where d_{ij} and $d_i \in \mathbb{R}$ and J's are the generators of the $SL(2,R)$ group. If we write this operator in two dimension as

$$\begin{aligned} H &= d_{++} J_+ J_+ + d_{+0} J_+ J_0 + d_{+-} J_+ J_- + d_{0-} J_0 J_- + d_{--} J_- J_- \\ &\quad + d_+ J_+ + d_0 J_0 + + d_- J_- + d \end{aligned} \tag{4.6}$$

If H possess a $(n+1)-$ dimensional irreducible representation P_{n+1} base function, it is called quasi–exactly solvable operator, and if H possess possess a $n-$ dimensional irreducible representation P_n for all n values under the conditions $d_{++} = d_{+0} = d_+ = 0$, it is called exactly solvable operator.

Now, if we substitute the differential form of z dependent $sl(2,R)$ generators of Eq. (4.1) into Eq. (4.6) and then into Eq. (4.4), we can obtain

$$\left(-F_4(z) \frac{d^2}{dz^2} + F_3(z) \frac{d}{dz} + (F_2(z) - E) \right) \varphi(z) = 0 \tag{4.7}$$

where the $F_i(z)$ are polynomials of ith order

$$\begin{aligned} F_4 &= d_{++} z^4 + d_{+0} z^3 + (d_{+-} + d_{00}) z^2 + d_{0-} z + d_{--}, \\ F_3 &= 2(2j-1) d_{++} z^3 + \left[(3j-1) d_{+0} + d_+ \right] z^2 \\ &\quad \left[2j(d_{+-} + d_{00}) + d_{00} + d_0 \right] z + j d_{0-} + d_-, \\ -F_2 &= 2j(2j-1) d_{++} z^2 + 2jz(j d_{+0} + d_+) + d_{00} j^2 + d_0 j. \end{aligned} \tag{4.8}$$

Now let us introduce a new variable $x = x(z)$ and a new function $\psi = \varphi \exp(-g)$. Taking

$$g = \frac{1}{2}\left(\frac{F_3}{F_4} - \frac{x''}{x'}\right) dz \tag{4.9}$$

and then dividing the both sides of the Eq. (4.7) by $F_4.(x')^2$, we can arrive at the Schrödinger form equation

$$-\frac{d^2\psi}{dx^2} + \left[g'^2 - g'' + \frac{F_2 - E}{F_4.(x')^2}\right]\psi(x) = 0 \tag{4.10}$$

where $1/F_4.(x')^2$ is called weight function ϱ. For example if the weight function is equal to 1, then

$$x = \pm\frac{dz}{\sqrt{F_4}} \tag{4.11}$$

Therefore after these algebraic calculation, we get the standard Schrödinger equation in which we know $N = 2j + 1$ solutions in the form of polynomials of the jth power in z multiplied by $\exp(-\varphi)$.

In order to understand explicitly, we give a quasi–exactly solvable potential which is associated to the harmonic oscillator. If we take the weight function $\varrho = 1$ and changing the variable as $z = x^2$, we obtain from Eq. (4.9)

$$g = a\frac{x^4}{4} + b\frac{x^2}{2} \tag{4.12}$$

If we substitute Eq. (4.12) into Eq. (4.10), we obtain the potential in the form of perturbed harmonic oscillator as

$$V = a^2x^6 + 2abx^4 + (b^2 - (2k + 3)a)x^2 \tag{4.13}$$

which can be represented in the form of algebraic operator as

$$H = -4J_0J_- + 4aJ_+ + 4bJ_0 - 2(2j + 1)J_- \tag{4.14}$$

Similarly, by choosing appropriate parameters, it can be possible to find the other quasi–exactly solvable potentials. Here, we will give some of them. If we choose $\varrho = x$, $z = x$ with $g = ax + bx^2/2 + (c - l)\ln(r)$, we will reach to quasi–exactly solvable Coulomb potential

$$V = b^2x^2 + 2abx - [a(2l + d - 1 - 2c) + \lambda]x^{-1} + c(c - 2l - d + 2)x^{-2}. \tag{4.15}$$

With parameters of the sets $z = \exp(-\alpha x)$, $\varrho = 1/\alpha$, $g = a\exp(-\alpha x)/\alpha + bx + \exp(\alpha x)/\alpha$ and $z = \exp(-\alpha x)$, $\varrho = \exp(-\alpha x)/\alpha$, $g = d\exp(-2\alpha x)/2\alpha + a\exp(\alpha x)/\alpha - bx$, it can be obtained the quasi–exactly solvable Morse oscillator forms. And, the quasi–exactly solvable Pösch-Teller potentials can be constructed with two set of parameters $z = \cosh(\alpha x)^{-2}$, $\varrho = 1/\alpha$, $g = -c\cosh(2\alpha x)/4 - a/\alpha\ln(\cosh(\alpha x)$ and $z = \cosh(\alpha x)^{-2}$, $\varrho = \cosh(\alpha x)^2/\alpha$, $g = b\tanh(\alpha x)^2/2\alpha - (a + b)\ln(\cosh(\alpha x/2)$. These examples can be extended by choosing appropriate parameters in order to obtain other type of quasi–exactly solvable potentials.

CHAPTER 5

APPLICATIONS IN PHYSICS

In this Chapter we present the applications of theoretical models given in the previous chapter on various physical Hamiltonians in the framework of Lie (super)algebras.

It is shown that the solution of the two boson Hamiltonians can be transformed into the one dimensional QES differential equation by applying suitable transformations given in Chapter 2. This method is useful to study nonlinear quantum–optical Hamiltonians.

5.1 Solution of the Hamiltonian of the Second Harmonic Generation

In order to solve the solution of the second harmonic generation Hamiltonian, we introduce a general two-mode Hamiltonian form which plays an important role in nonlinear quantum optical systems.

Recently, there has been a great deal of interest in quantum optical systems which reveal new physical phenomena described by the Hamiltonians expressed as nonlinear functions of Lie algebra generators or boson operators [74, 75, 10, 76, 77]. Such systems have often been analyzed by using numerical methods, because the implementation of the Lie algebraic techniques in order to solve those problems is not very efficient and most of the other analytical techniques do not yield simple analytical expressions. They require tedious calculations [9, 78, 79].

However, a new algebraic approach essentially improving both analytical and numerical solution of the problems, has been suggested and developed for some nonlinear quantum optical systems [80, 81, 82, 83]. Most of such developments are mainly based on linear Lie algebras, but it is evident that there is no physical reason for symmetries to be only linear. Nonlinear Lie algebra techniques and their relations to the nonlinear quantum optical systems have been discussed [84, 85, 86, 87]. In both cases finite part of spectrum of the corresponding Hamiltonian can be exactly obtained in closed forms. Recently it has been proven that the single boson Hamiltonians also lead to a QES under some certain constraints [48, 6].

In this section we try to determine quasi-exact-solvability of the two-boson spectrum generating Hamiltonians. To accomplished that, first of all we write the two-mode Hamiltonian in general form, and try to solve in algebraic form. After that, by choosing appropriate parameters, our introduced Hamiltonian can be reduced to well known physical systems such as second harmonic generation, modified Jayness-Cumming Hamiltonian, the Hamiltonian of the systems of photons and bosons expressed in a single mode form and so on. These Hamiltonians are not only mathematically interesting but they have also potential interest in physics [58].

Let start from writing the general form of the two-mode Hamiltonian. It can be generalized as follows:

$$H = \sum_{m_i} \alpha_{m_1,m_2,m_3,m_4} (a_1^\dagger)^{m_1} (a_1)^{m_2} (a_2^\dagger)^{m_3} (a_2)^{m_4} \tag{5.1}$$

where α_i is a constant and m_i determines order of the interaction. We assume that the conserved quantity of the physical system described by the Hamiltonian H, correspond to the operator:

$$K = s a_1^\dagger a_1 + p a_2^\dagger a_2. \tag{5.2}$$

Clearly the Hamiltonian conserves the number of particles in the system, when H commutes with the K. Meanwhile, we have mention here, in general, conserved quantity of Eq. (5.1) type appears more general contexts than the number operator. Indeed a convenient quantum mechanical treatment of the m^{th} order interaction of the two-boson model exploits the $su(2)$ algebra structure of the problem, by expressing the Hamiltonian as a polynomial of the angular momentum operators whose degree depends on the degree of interaction. In this case the conserved quantity is the total angular momentum, $J = j(j+1)$, which can be related to the number operator $j = N/2$. Therefore the conserved quantity J crucially depends on the conservation of the number of particles. The classical motion of the particle takes place in the space of angular momentum on the sphere of radius $j(j+1)$. To have a physical insight in to K, from other perspective, when the motion of the particle takes the place on an ellipsoid the conserved quantity K takes the form Eq. (5.1). The operator K and bosonic operators satisfy the commutation relations

$$[K, a_1^\dagger] = s a_1^\dagger, \quad [K, a_1] = -s a_1, \quad [K, a_2^\dagger] = p a_2^\dagger, \quad [K, a_2] = -p a_2. \tag{5.3}$$

If K and H commute, then they have the same eigenfunction. Therefore it is worth to seek the conditions for the commutation of H and K. This can easily be done by using the commutation relations and we obtain the following relation

$$[K,H]=\sum_{m_i}[(s(m_1-m_2)+p(m_3-m_4)]\alpha_{m_1m_2m_3m_4}(a_1^\dagger)^{m_1}(a_1)^{m_2}(a_2^\dagger)^{m_3}(a_2)^{m_4}. \tag{5.4}$$

It is obvious that the constant of motion K and H commute when the following set of equation is satisfied

$$s(m_1-m_2)+p(m_3-m_4)=0. \tag{5.5}$$

The action of the operator K on the state $|n_1,n_2\rangle$ is given by

$$K\,|n_1,n_2\rangle=(sn_1+pn_2)\,|n_1,n_2\rangle\,. \tag{5.6}$$

In this section we seek the solution of the eigenvalue equation of Eq. (5.6) in the Bargmann-Fock space. The bosonic operators in Hilbert space is transformed to the Bargmann-Fock space by the operator Γ in (3.10) with parameters $\theta=\pi/8$ and $\lambda=1$. Thus, in the Bargmann-Fock space the eigenvalue problem in Eq. (5.6) leads to the following solution:

$$\psi(x_1,x_2)=x_1^k\phi\left(x_2x_1^{-\frac{s}{p}}\right). \tag{5.7}$$

where k is given by

$$k=n_1+\frac{p}{s}n_2. \tag{5.8}$$

The eigenfunction of the Hamiltonian can be obtained from the relation

$$|n_1,n_2\rangle=\Gamma^{-1}\psi(x_1,x_2). \tag{5.9}$$

We know that, the general two-mode Hamiltonian of Eq. (5.1) includes various physical Hamiltonians. The Hamiltonian of second harmonic generation

$$H=\omega_1a_1^\dagger a_1+\omega_2a_2^\dagger a_2+\kappa(a_1^\dagger)^2a_2+\overline{\kappa}a_2^\dagger(a_1)^2 \tag{5.10}$$

which can be obtained from the Hamiltonian of Eq. (5.1), when the parameters

$$\alpha_{m_1m_2,m_3,m_4}=0 \tag{5.11}$$

except that

$$\alpha_{1,1,0,0}=\omega_1,\alpha_{0,0,1,1}=\omega_2,\alpha_{2,0,0,1}=\kappa,\alpha_{0,2,1,0}=\overline{\kappa} \tag{5.12}$$

The commutation condition for K and H in Eq. (5.5) is satisfied when $p=2s$. The constants ω_1 and ω_2 are the angular frequencies of two independent harmonic oscillators characterized by annihilation and creation operators, κ and

$\overline{\kappa}$ are the coupling coefficients that determine the strength of the interaction of the oscillators. In the following we use the Bargmann-Fock realization, where creation and annihilation operators $a_i^\dagger$ and a_i are replaced by $\widetilde{a}_i^\dagger$ and $\widetilde{a}_i$ in Eq. (3.16), respectively. The eigenfunction is of the form

$$\psi(x_1, x_2) = (x_1)^k \phi(z) \tag{5.13}$$

where $z = x_2 x_1^{-1/2}$. The solution of this system describes a quantum mechanical state of H provided that $\phi(z)$ belongs to the Bargmann-Fock space. The eigenvalue equation of the second harmonic generation Hamiltonian can be written as

$$H\psi(x_1, x_2) = E\psi(x_1, x_2) \tag{5.14}$$

Insertion of Eq. (5.13) into Eq. (5.14) yields the following differential equation

$$\begin{array}{c} [4\overline{\kappa}z^3 \frac{d^2}{dz^2} + (\kappa + z(\omega_2 - 2\omega_1 + 2\overline{\kappa}z(3 - 2k))\frac{d}{dz} \\ +k\omega_1 + \overline{\kappa}zk(k-1)]\phi(z) = E\phi(z) \end{array} \tag{5.15}$$

According to Turbiner [59], the Hamiltonian in Eq. (5.15) can be written in terms of $sl(2, R)$ algebra.

$$H = (4\overline{\kappa}J_+J_0 + (\omega_2 - 2\omega_1)J_0 + \kappa J_- + \overline{\kappa}J_+)$$

Therefore it can be say that, this Hamiltonian is quasi-exactly solvable. The eigenvalue equation of Eq. (5.15) can be obtained as follows. Let us assume that the function $\phi(z)$ is a polynomial in z with the coefficients being functions of energy:

$$\phi(z) = \sum_{m=0}^{\infty} P_m(E) z^m. \tag{5.16}$$

When inserted $\phi(z)$ in Eq. (5.15) then we obtain the following three-term recurrence relation:

$$\begin{aligned} \overline{\kappa}(k-2m)(k-2m-1)P_{m+1}(E)+ \\ (\omega_2 + \kappa\omega_1 + m(\omega_2 - 2\omega_1) - E)P_m(E) + \kappa m P_{m-1}(E) &= 0. \end{aligned} \tag{5.17}$$

The function $P_m(E)$ terminates when $m > k/2$, and therefore the roots of the $P_m(E)$ belong to the spectrum of the Hamiltonian. For example, for $k = m = 1$, the polynomial has an eigenvalues

$$E = \frac{1}{2}\left((-2 + \kappa\omega_1) + 2\omega_2 \pm \sqrt{4\kappa + (-2 + \kappa\omega_1 + 2\omega_2)^2}\right)$$

For the corresponding problem, Beckers, Brihaye and Debergh [84] have obtained three different Hamiltonians in the framework of nonlinear algebra. Two of the Hamiltonians are PT- symmetric and they related to the sextic harmonic oscillator. The third Hamiltonian is exactly the same one in Eq. (5.15) which was obtained in a different method. The Hamiltonian of Eq. (5.15) can also be transformed in the form of the Schrödinger equation by changing variable $z = -1/\overline{\kappa}y^2$ and defining the wave function

$$\phi(z) = e^{-\int W(y)dy}\psi(y) \tag{5.18}$$

where

$$W(y) = \frac{k}{y} + \frac{(\omega_2 - 2\omega_1)y}{4} - \frac{\kappa\overline{\kappa}y^3}{4} \tag{5.19}$$

Substituting Eq. (5.18) in Eq. (5.15) we obtain the following equation:

$$\begin{aligned} H &= -\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{4}\left((2k+5)\omega_2 - 2\omega_1)\right) + \frac{1}{16}\Big((\omega_2 - 2\omega_1)^2 - 4\kappa\overline{\kappa}(2k+3)\Big)\, y^2 \\ &\quad -\frac{1}{8}\kappa\overline{\kappa}(\omega_2 - 2\omega_1)y^4 + \frac{1}{16}\kappa^2\overline{\kappa}^2 y^6. \end{aligned} \tag{5.20}$$

The last equation is the Hamiltonian of the sextic Harmonic oscillator. Consequently the procedure given here can be applied to obtain eigenfunction and eigenvalues of the various physical Hamiltonians. The validity of the procedure depends on the choice of the $\alpha_{m_1 m_2, m_3, m_4}$. One can easily be obtain various physical Hamiltonians by appropriate choice of $\alpha_{m_1 m_2, m_3, m_4}$ and by considering the conditions given in Eq. (5.5). For instance one can easily obtain n^{th} harmonic generation Hamiltonian by setting:

$$\alpha_{1,1,0,0} = \omega_1, \alpha_{0,0,1,1} = \omega_2, \alpha_{n,0,0,1} = \kappa, \alpha_{0,n,1,0} = \overline{\kappa} \tag{5.21}$$

and the remaining values of the $\alpha_{m_1 m_2, m_3, m_4} = 0$. Then the Hamiltonian of Eq. (5.1) takes the form

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \kappa(a_1^\dagger)^n a_2 + \overline{\kappa} a_2^\dagger (a_1)^n. \tag{5.22}$$

which is quasi-exactly solvable Hamiltonians.

5.2 Solution of the Modified Jaynes-Cummings Hamiltonian

JC model is the one of the major model describing atom–field interactions in quantum optics. This model was introduced as a highly simplified model to explain especially the remarkable features of the interaction of matter with a quantized radiation in a cavity. Despite the fact that it considers matter in the highly simplified form of a single-mode field, it demonstrates quite a few uniquely quantum mechanical features. Because of this fact that the JC models have been generalized to the case of multi-level atoms interacting with one or more discrete field modes, or to the case involving a multiphoton transition through the intermediary of virtual levels. One of this model is called the modified Jaynes-Cummings model (MJCM).

The MJCM can be used to describe two-level atom placed in the common domain of two cavities, where there exists a "free–exchange" of excitations between two modes. It is an example of two boson and one fermion hamiltonian. The Hamiltonian for such a physical systems given by,

$$H = w(a_1^\dagger a_1 + a_2^\dagger a_2) + \frac{1}{2}w_0\sigma_0 + \lambda_1(a_1^\dagger\sigma_- + a_1\sigma_+) + \lambda_2(a_2^\dagger\sigma_- + a_2\sigma_+), \quad (5.23)$$

where λ_1 and λ_2 the coupling constants of the field and atom. The eigenvalue equation can be written as

$$H\psi = E\psi \quad (5.24)$$

Our task is now to express the Hamiltonian of Eq. (5.23) in terms of the generators of the $osp(2,1)$ algebra. In terms of the $osp(2,1)$ generators and number operator N, the Hamiltonian can be written as

$$H = wN + \frac{1}{2}w_0(J - 1 - \frac{N}{2}) + \lambda_1(W_+ - V_-) + \lambda_2(W_- - V_+) \quad (5.25)$$

By considering the transformation procedures S for $osp(2,1)$ algebra given in Chapter 3, one can obtain Eq. (5.25) in the single variable differential equation form which in the Bargmann-Fock space reads:

$$H' = w(j - 1 - \sigma_0) + \frac{w_0}{2}\sigma_0 + \lambda_1(x\sigma_- + \frac{d}{dx}\sigma_+) + \lambda_2(\sigma_- + (x\frac{d}{dx} - j)\sigma_+) \quad (5.26)$$

The action of the Hamiltonian on the two component spinor

$$P_{n,m}(x) = \begin{pmatrix} \phi_n(x) \\ \phi_m(x) \end{pmatrix} \quad (5.27)$$

gives us the following spinor wave functions for the Hamiltonian in Eq. (5.26):

$$\phi_1(x) = C_1(\lambda_2 + \lambda_1 x)^n(\lambda_1 - \lambda_2 x)^{j-n} \quad (5.28)$$

$$\phi_2(x) = C_2(\lambda_2 + \lambda_1 x)^{n-1}(\lambda_1 - \lambda_2 x)^{j-n} \quad (5.29)$$

where C_1 and C_2 are constants and $n = 0, 1, ..., j$; and eigenvalues of the Hamiltonian are given by

$$E = w(2j - \frac{1}{2}) \pm \frac{1}{2}\sqrt{(w_0 - 2w)^2 + 4n(\lambda_1^2 + \lambda_2^2)} \quad (5.30)$$

Consequently we have obtained the exact result for the eigenvalues of the modified Jaynes-Cummings Hamiltonian. The procedure given here can be applied to obtain eigenfunction and eigenvalues of various physical Hamiltonians.

5.3 Solution of Hamiltonian of Quantum Dots with Rashba Spin-orbit Coupling

The optical and electrical properties of confined electrons in semiconductor quantum wells, quantum dots and quantum wires depend on Rashba spin–orbit coupling [91, 92, 93]. The analysis of the behavior of spins in semiconductors leads to the construction of new spintronic devices [94, 90, 95, 96]. It is practical to contemplate semiconductor devices based on electron spins, because spin is not coupled to electromagnetic noise and hence should have much longer coherence times than charge [97]. The spin–orbit interaction in a confined geometry has also interesting consequences for the electron spectrum [24].

Spin–orbit interactions can arise in quantum dots by various mechanisms related to the electron confinement and symmetry breaking. In semiconductors, spin–orbit coupling from the relativistic effect is caused by the electric field due to the lack of the inversion symmetry of the certain alloys. Depending on the particular origin of the electric field, the spin–orbit interaction presents two distinct contributions: these are Dresselhaus and Rashba terms. Dresselhaus term is obtained from the electric field produced by the bulk inversion asymmetry of the material and Rashba term is generated due to the structural asymmetry of the heterostructure. Rashba splitting has been observed in many experiments and it constitutes the basis of the proposed electronic nanostructures.

Our formulation of the Hamiltonian associated with the quantum dots leads to an interesting consequence: Hamiltonians of the $E \otimes \varepsilon$ Jahn–Teller [98, 99, 100, 101, 102], two-mode bosonic Jaynes–Cummings [103, 105, 106] and quantum dot Hamiltonians with spin–orbit interaction are constructed for similar spin-boson models, though not identical, and can be solved with the same mathematical techniques. The connection between the Hamiltonians plays crucial roles to the solution and analysis of the Rashba Hamiltonian, because the solutions of the Jahn–Teller and the Jaynes–Cummings problems have been a focus of interest, both from the mathematical and physical point of view, over the years and their exact solution has been treated by various authors.

The origin of Rashba spin–orbit coupling in quantum dots is due to the lack of inversion symmetry which causes a local electric field perpendicular to the plane of heterostructure. The Hamiltonian representing Rashba spin–orbit coupling for an electron in a quantum dot can be expressed as

$$H_R = \frac{\lambda_R}{\hbar}(p_{x_2}\sigma_1 - p_{x_1}\sigma_2) \tag{5.31}$$

where λ_R represents the strength of the spin–orbit coupling, and it can be adjusted by changing the asymmetry of the quantum well via external electric field.

The Hamiltonian dominated by the bulk inversion symmetry term is known as Dresselhaus spin–orbit coupling and it is given by

$$H_D = \frac{\lambda_D}{\hbar}(p_{x_2}\sigma_2 - p_{x_1}\sigma_1) \tag{5.32}$$

where λ_D is the Dresselhaus parameter that depends on the material properties, device design and external electric field. In a centrosymmetric crystal-like materials it becomes zero, because of the non-existence of bulk inversion asymmetry. In the case of both Rashba and Dresselhaus, interactions exist in the system, the Hamiltonian cannot exactly be solved. The typical ratio of the coupling strengths is $\lambda_R/\lambda_D = 6$ [26]. Because of this condition, in our treatment we have included the Rashba term rather than the Dresselhaus term, since the Rashba term may be dominant. We mention here that the results will only need a trivial modification, when a solo Dresselhaus term is present, because Rashba and Dresselhaus terms transform into each other under spin rotation. To this end, we assume that the electron is confined in a parabolic potential

$$V = \frac{1}{2}m^* w_0^2(x_1^2 + x_2^2). \tag{5.33}$$

The Hamiltonian describing an electron in a two-dimensional quantum dot takes the form

$$H = \frac{1}{2m^*}(P_{x_1}^2 + P_{x_2}^2) + \frac{1}{2}g\mu B\sigma_0 + V + H_R \tag{5.34}$$

The term $\frac{1}{2}g\mu B\sigma_0$ introduces the Zeeman splitting between the $(+)x$-polarized spin up and $(-)x$-polarized spin down. The factors g is the gyromagnetic ratio and μ is the Bohr magnetron. The kinetic momentum $P = p + eA$ is expressed with canonical momentum $\mathbf{p} = -i\hbar(\partial x_1, \partial x_2, 0)$ and the vector potential $\mathbf{A}$ can be related with the magnetic field $\mathbf{B} = \nabla \times A$. The choice of symmetric gauge vector potential $\mathbf{A} = B/2(-x_2, x_1, 0)$ leads to the following Hamiltonian:

$$\begin{aligned} H &= \frac{\hbar^2}{2m^*}(P_{x_1}^2 + P_{x_2}^2) + \frac{1}{2}m^* w^2(x_1^2 + x_2^2) \\ &\quad + \frac{1}{2}i\hbar w_c\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right) + \frac{1}{2}g\mu B\sigma_0 + H_R \end{aligned} \tag{5.35}$$

From now on we restrict ourselves to the solution of Eq. (5.35). It will be shown that the Hamiltonian in Eq. (5.35) without the Dresselhaus term is one of the recently discovered quasi-exactly solvable operators [59, 42]. It is well known that the underlying idea behind the quasi-exact solvability is the existence of a hidden algebraic structure. In the following section, we obtain an algebraic expression for the Hamiltonian H and discuss its quasi-exact solvability.

Algebraic form of Rashba Hamiltonian in terms of boson operators in Eq. (3.9)

$$\begin{aligned} H &= \hbar w(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) + \frac{\hbar w_c}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) \\ &\quad -\sqrt{\frac{m^* w}{4\hbar}}\lambda_R\left[(a_2^\dagger - a_1)\sigma_+ + (a_2 - a_1^\dagger)\sigma_-\right] + \frac{1}{2}g\mu B\sigma_0 \end{aligned} \qquad (5.36)$$

Before we turn to exploring the solvability of the Hamiltonian H, we seek the relation of this Hamiltonian with Lie superalgebra.

The natural step to relate a Hamiltonian with a Lie (super)algebra is to express the Hamiltonian with the generators of the relevant symmetry group. One of the major symmetry group candidates for spin one-half particles is the supergroup $OSP(2,2)$ which has four even and four odd generators. Its even generators can be represented by bosons while odd generators are represented by combinations of the fermions and bosons [70].

The Hamiltonian in Eq. (5.36) has been expressed in terms of the operators of $osp(2,2)$ algebra (generalized in Chapter 3)

$$H = 2\hbar w J_0 + \frac{\hbar w_c}{2}N - \kappa[V_+ - V_- + W_- + W_+] + g\mu B(N-K) \qquad (5.37)$$

where $\kappa = \sqrt{\frac{m^* w}{\hbar}}\lambda_R$. Consequently, we have shown that the Rashba Hamiltonian possesses the $osp(2,2)$ symmetry. The Hamiltonian of Eq. (5.37) has $(2N+1)$ linearly independent eigenfunctions, and it is quasi-exactly solvable.

Now we our aim is to obtain the solution of the Hamiltonian of Eq. (5.36) in the framework of the quasi-exactly solvable problem. A simple connection between the Hilbert space and the Bargmann–Fock space can be obtained by transforming the differential realizations of the boson operators in the Eq. (3.9). By applying the similarity transformation operators in Eq. (3.56) with parameters $c = 1$, and $d = 1$, the Hamiltonian takes the form

$$\begin{aligned} \widetilde{H} &= S^{-1}HS = \hbar w(2a_1^\dagger a_1 + a_2^\dagger a_2 + \sigma_+\sigma_- + 1) - \frac{\hbar w_c}{2}\left(a_2^\dagger a_2 + \sigma_+\sigma_-\right) \\ &\quad -\kappa\left[(1-a_1)\sigma_+ + (a_2 a_2^\dagger + a_1^\dagger a_1 + \sigma_+\sigma_- - a_1^\dagger)\sigma_-\right] + \frac{1}{2}g\mu B\sigma_0 \end{aligned} \qquad (5.38)$$

It is obvious that $\widetilde{H}$ can be characterized by a fixed number $a_2^\dagger a_2 = j-1$. Here, j takes integer values. The transformed Hamiltonian $\widetilde{H}$ includes a one-boson operator and possessing infinitely many finite-dimensional invariant subspaces with a basis function

$$\phi_{n,n+1}(z) = \begin{pmatrix} p_n(z) \\ q_{n+1}(z) \end{pmatrix},$$

where $\phi(z)$ is two-component spinor and $p_n(z)$ and $q_{n+1}(z)$ are polynomials of degree n and $n+1$, respectively. Substituting Bargmann–Fock space realizations of the bosonic operators of Eq. (3.21) by changing variable $z = \sqrt{\frac{m^*w}{4\hbar}}x$, we obtain the following single-variable differential equation

$$\begin{aligned}
\widetilde{H} &= \hbar w(2z\frac{d}{dz} - j + \sigma_+\sigma_-) + \frac{\hbar w_c}{2}(j + 1 - \sigma_+\sigma_-) \\
&\quad -\kappa\left[\left(1 - \frac{d}{dz}\right)\sigma_+ + \sigma_-\left(z\frac{d}{dz} - j - z\right)\right] + \frac{1}{2}g\mu B\sigma_0. \qquad (5.39)
\end{aligned}$$

The eigenvalue problem can be expressed as

$$\widetilde{H}\phi(z) = E\phi(z).$$

The action of the $\widetilde{H}$ on the basis function $\phi(z)$ gives the following recurrence relation

$$\begin{aligned}
[2n\hbar w - E + \epsilon_j + \epsilon_b]\, p_n(E) - \kappa[q_{n+1}(E) - (n+1)q_n(E)] &= 0 \\
[2n\hbar w - E + \epsilon_j - \epsilon_b]\, q_{n+1}(E) + \kappa[p_{n+1}(E) + (j-n)p_n(E)] &= 0 \qquad (5.40)
\end{aligned}$$

where $p_n(E)$ and $q_n(E)$ are the coefficients of the polynomials $p_n(z)$ and $q_n(z)$, respectively. The justified parameters are given by

$$\begin{aligned}
\epsilon_j &= \hbar w\left(\frac{1}{2} - j + \frac{(3+2j)w_c}{4w}\right) \\
\epsilon_b &= \left(\frac{\hbar w_c}{4} - \frac{1}{2}g\mu B + \frac{\hbar w}{2}\right). \qquad (5.41)
\end{aligned}$$

It is necessary that the determinant of these sets be equal to zero giving the compatibility conditions that establish the locations of the exact eigenvalues on the energy baseline. The recurrence relation implies that the wave function is itself the generating function of the energy polynomials. If E_k is the roots of the Eq. (5.40), the eigenfunction truncated at $n = j$ and it is the exact eigenvalues of the Hamiltonian $\widetilde{H}$. These recurrence relations can be written in the matrix form:

$$\begin{bmatrix}
E_+ & -\kappa & 0 & 0 & 0 & . \\
j\kappa & E_- & \kappa & 0 & 0 & . \\
0 & 2\kappa & E_+ + 2\hbar w & -\kappa & 0 & . \\
0 & 0 & (j-1)\kappa & E_- + 2\hbar w & \kappa & . \\
0 & 0 & 0 & 3\kappa & E_+ + 3\hbar w & . \\
. & . & . & . & . & \ddots
\end{bmatrix}
\begin{bmatrix} p_0 \\ q_1 \\ p_1 \\ q_2 \\ p_2 \\ \vdots \end{bmatrix} = 0$$

where $E_\pm = \epsilon_j \pm \epsilon_b - E$. It is obvious that the determinant of the matrix forms polynomials in E and the first few of them are given by

$$\begin{aligned}
D_0(E) &= E_+E_- \\
D_1(E) &= (E_- + 2\hbar w)[E_+E_-(E_+ + 2\hbar w) - \kappa^2(E_+ - 2\hbar w)] \qquad (5.42) \\
D_2(E) &= (E_- + 4\hbar w)[E_+E_-(E_+ + 2\hbar w)(E_- + 2\hbar w)(E_+ + 4\hbar w) + HO
\end{aligned}$$

for the values $j = 0, 1$ and 2, respectively with $HO = +\kappa^2(2\hbar w E_+(E_- + 4\hbar w) + 162\hbar^2 w^2(E_- + 2\hbar w) - 2E_+^2 E_-) + 2\kappa^4(E_+ - 2\hbar w)]$. The eigenvalues E of the Hamiltonian of Eq. (5.36) can be determined by finding the roots of the polynomials in the Eq. (5.42).

Consequently, we have obtained the quasi-exact solution for the eigenvalues of the problem of an electron in a quantum dot in the presence of both the magnetic field and spin–orbit coupling. Our formulation gives a unified treatment of the solution of the spin-boson physical systems. It has been shown that the mathematical structures of the quantum dot Hamiltonian including Rashba spin–orbit interaction and some of the spin-boson physical systems are identical. We have also shown that the Hamiltonian possesses $osp(2, 2)$ hidden symmetries. The suggested approach can be modified to solve the quantum dot Hamiltonian including Dresselhaus interaction.

It is obvious that the algebraic techniques have been used in a variety of problems to compute their spectra. We have presented the steps towards an extension of the algebraic formulation of the quantum dot Hamiltonians. This method can be extended in several ways. The Hamiltonian of a quantum dot including position-dependent effective mass may be formulated and solved within the procedure given here. We hope that our method leads to interesting results on the spin–orbit effects in quantum dots in future research.

5.4 Spectra of Interacting Electrons in a Quantum Dot

Quantum dots, also called a semiconductor nanocrystal are fabricated, nanoscale structures. Typical dimensions for quantum dots range from nanometers to a few micrometers, and the number of electrons can be tuned from zero to up to thousands. Due to the quantum dot having discrete energy levels, much like an atom, they are sometimes called "artificial atoms". The energy levels can be controlled by changing the size and shape of the quantum dot, and the depth of the potential. They are able to capture and localize charge carriers, such as electrons , holes, or electron-hole pairs or so-called excitons to zero dimensions to a region on the order of the electrons' Compton wavelength.

The electron confinement within the QD is described by a Hamiltonian that includes the Coulomb potential, as well as a parabolic potential. Consider a QD containing two electrons in the presence of a magnetic field B perpendicular to the dot and a lateral potential $m^* w_0^2 r^2/2$ that confines the electrons. The

Hamiltonian of the present system is

$$H = \sum_{i=1}^{2} \left(\frac{1}{2m_i^*} \left(P_i + eA(r_i)\right)^2 + \frac{1}{2} m_i^* w_0^2 r_i^2 \right) + \frac{e^2}{\epsilon \left| r_2 - r_1 \right|} \tag{5.43}$$

where m^*, w_0, and ϵ are the effective mass of the electrons, the confinement frequency and the dielectric constant of the medium, respectively. $A(r_i)$ denotes the vector potential at the i^{th} particle's location. For the perpendicular magnetic field ($\overrightarrow{B} \| z$) we choose the gauge symmetric gauge vector potential described by the vector $A(r_i) = \left[B \times r_i\right]/2 = B/2(-x_{2i}, x_{1i}, 0)$.When we substitute these parameters and arrange the Hamiltonian in Eq. (5.43), we obtain

$$H = \sum_{i=1}^{2} \left(\frac{P_i^2}{2m_i^*} + \frac{1}{2} m_i^* w_0^2 r_i^2 \right) + \frac{e^2}{\epsilon \left| r_2 - r_1 \right|} + \frac{1}{2} w_c L_{tot}, \tag{5.44}$$

where $w_c = eB/m^*$ stands for the cyclotron frequency of the electron, $w = \sqrt{w_0^2 + \left(\frac{w_c}{2}\right)^2}$ is the effective frequency and $L_{tot} = i\hbar \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right)$ is the total angular momentum along z- direction with Planck's constant $\hbar$. The Hamiltonian in Eq. (5.44) seems to be separable equation. In order to show that, we just need a straightforward analysis.

Now we introduce relative and center-of-mass coordinates:

$$r = r_2 - r_1, \qquad R = \frac{1}{2}(r_1 + r_2) \tag{5.45}$$

In the new coordinates, the Hamiltonian in Eq. (5.44) can be separated into the center-of-mass (CM) and relative-motion (RM) terms as

$$H = 2H_r + \frac{1}{2} H_R, \tag{5.46}$$

where

$$H_r = \frac{p^2}{2m^*} + \frac{1}{2} m^* w^2 r^2 + \frac{e^2}{2\epsilon r} + \frac{1}{2} w_c L_r \tag{5.47a}$$

$$H_R = \frac{P^2}{2m^*} + \frac{1}{2} m^* w^2 R^2 + \frac{1}{2} w_c L_R \tag{5.47b}$$

It is obvious that $L_{tot} = L_r + L_R$. Eq. (5.47b) is the Hamiltonian of the harmonic oscillator, and its eigenvalues are given by

$$E_R = (2N + |M| + 1)\hbar w + M \frac{\hbar w_c}{2},$$

where $N = 0, 1, 2, \cdots$ denotes the radial quantum number and $M = 0, \pm 1, \pm 2, \cdots$ is the azimuthal quantum number. Let us turn our attention to the solution of the Hamiltonian H_r. In the polar coordinate $r = (r, \alpha)$, if the eigenfunction

$$\phi = r^{-\frac{1}{2}} e^{im\alpha} u(r), \tag{5.48}$$

is introduced, the Schrödinger equation $H_r\phi = E_r\phi$, can be expressed as

$$\left(-\frac{\hbar^2}{2m^*}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m^*}(m^2+\frac{1}{4})\frac{1}{r^2} + \frac{1}{2}m^*w^2r^2 + \frac{e^2}{2\epsilon r} + \frac{1}{2}w_cL_r\right)u(r) = E_ru(r). \tag{5.49}$$

Now our aim is to solve above Hamiltonian in the framework of Lie algebra.

5.4.1 Method of Solution

In this section we show that the Schrödinger Equation given in Eq. (5.49) is one of the recently discovered quasi-exactly solvable operators [59, 42, 88].

It is well known that the underlying idea behind the quasi-exact solvability is the existence of a hidden algebraic structure. In order to show that the Schrödinger equation has a $sl(2,R)$ symmetry, let us consider the following combinations of the operators in Eq. (4.1):

$$T = -J_-J_0 - \frac{1}{2}(4m+j)J_- + \hbar wJ_+. \tag{5.50}$$

Then, the eigenvalue problem becomes

$$TP_k(r) = \lambda P_k(r) \tag{5.51}$$

where $P_k(r)$ is the k^{th} degree polynomial in r. After changing the variable $r \to \frac{\hbar}{\sqrt{2m^*}}r$ and substituting

$$u(r) = r^{m+\frac{1}{2}}e^{-\frac{\hbar w}{4}r^2}P_k(r), \tag{5.52}$$

we can show that the eigenvalue equations, Eqs. (5.49) and (5.51), are identical when the following holds:

$$\begin{aligned} E_r &= (j+|m|+1)\hbar w + \frac{1}{2}m\hbar w_c, \\ \lambda &= -\frac{e^2\sqrt{m^*}}{\sqrt{2}\epsilon\hbar}. \end{aligned} \tag{5.53}$$

When the generators act on the polynomial in Eq. (4.2), we can obtain the following recurrence relation:

$$\hbar w(k-j)P_{k+1}(\lambda) - \lambda P_k(\lambda) - k(k+2m)P_{k-1}(\lambda) = 0 \tag{5.54}$$

with the initial condition $P_0(1) = 1$. If λ_i is a root of the polynomial $P_{k+1}(\lambda)$, the wave function is truncated at $k = j$ and belongs to the spectrum of the Hamiltonian T. This property implies that the wave function is itself the generating function of the energy polynomials.

The roots of the recurrence relation in Eq. (5.54) can be computed, and the first few of them are given by

$$
\begin{aligned}
P_1(\lambda) &= \lambda \\
P_2(\lambda) &= \lambda^2 - j(1+2m)\hbar w \\
P_3(\lambda) &= \lambda[\lambda^2 - (5j+6jm-4m-4)\hbar w] \\
P_4(\lambda) &= \lambda^4 - 2\lambda^2[j(6m+7)-8m-11]\hbar w + 3j(j-2)(1+2m)(3+2m)(\hbar w)^2
\end{aligned}
\tag{5.55}
$$

The function $P_j(r)$ forms a basis for $sl(2,R)$ algebra, and it can be written in the form

$$
P_j(r) = \sum_{k=0}^{j} \frac{j!(2m)!P_k(\lambda)}{(j-k)!k!(2m+k)!}(\hbar w r)^k. \tag{5.56}
$$

Therefore, we can obtain the eigenfunction of Eq. (5.43) in a closed form, at least, the first few values of j.

5.4.2 Results

The polynomial $P_k(\lambda)$ vanishes for $k = j+1$, and the roots of the polynomials are given by

$$
\begin{aligned}
\lambda &= 0 \\
\lambda &= \pm\sqrt{(1+2m)\hbar w} \\
\lambda &= 0, \pm\sqrt{(6+8m)\hbar w} \\
\lambda &= \pm\sqrt{(10+10m \pm \sqrt{73+128m+64m^2})\hbar w}
\end{aligned}
\tag{5.57}
$$

for $j = 0, 1, 2$ and 3, respectively. It is obvious that the effective frequency w can be expressed as

$$
w = \frac{\lambda^2}{\hbar\eta(j,m)}, \tag{5.58}
$$

where $\eta(j,m)$ is the coefficients of $\hbar w$ in Eq. (5.57). The relation in Eq. (5.58) leads to the following conclusion: The Coulomb interaction destroys the general symmetry of the QD, but the magnetic field can restore the symmetries which are common for a harmonic oscillator and Coulomb system. This feature implies that for some specific values of the magnetic field B, the Hamiltonian in Eq. (5.43) can be solved exactly. Since the effective frequency w is a function of B, the relation in Eq. (5.58) can be written as

$$
w_c \equiv \frac{eB}{m^*} = \sqrt{\left(\frac{e^4 m^*}{\eta\epsilon^2\hbar^3}\right)^2 - 4w_0^2}. \tag{5.59}
$$

In order to find the exact eigenvalues and eigenfunctions of Eq. (5.43), it is enough to perform the calculation in Eq. (5.59). The accuracy of the approximate and the numerical results can be checked by means of the exact solution given in the previous section. Let us consider the $GaAs$ heterostructure. The material parameters of this structure are chosen to be $m^* = 0.067m_0$ and $\epsilon = 12.4$. The magnetic field is calculated and tabulated in Table 5.1. The confinement energy $\hbar w_0$ is given as $4meV$.

	$\hbar w_c$ (in eV)			E_r (in eV)			
j	$m=0$	$m=1$	$m=2$	$m=0$	$m=1$	$m=2$	N_r
1	1.870092	0.623318	0.373936	1.870109	1.246710	1.121980	0
2	0.311582	0.133339	0.084627	0.467527	0.333828	0.297140	0
3	1.100529	0.050923	0.033572	0.201694	0.154332	0.137109	0
3	1.284394	0.502495	0.321600	2.568837	1.507640	1.286700	1
4	0.043551	0.024225	0.015929	0.110700	0.088650	0.078318	0
4	0.240664	0.111710	0.074112	0.601993	0.391842	0.335014	1
5	0.021771	0.012144	0.006844	0.069584	0.056970	0.048959	0
5	0.082677	0.043767	0.029690	0.249192	0.177607	0.152686	1
5	1.005801	0.427792	0.284926	3.0017498	1.711430	1.425080	2

Table 5.1: The value of $\hbar w_c$ and the corresponding eigenvalues in eV

We have shown that the destroyed symmetry of the electrons in a quantum dot can be recovered by adjusting the magnetic field and that there is an infinite set of exact analytical solutions of the Schrödinger equation for two electrons in an external homogeneous magnetic field. It is easy to see that the confinement region of the electrons in the dot can be expressed as

$$l = \left(\frac{\hbar}{m^* w} \right)^{1/2} . \tag{5.60}$$

This may be interpreted as follows: The size of the QD depends on the quantum number of the angular momentum and the external magnetic field. When the magnetic field increases, the dot size will decrease, and the electrons have higher angular momentum.

Finally, we have mention that the method presented for quantum dot can be extended in various directions. The algebra can be modified to investigate the hidden symmetries of the interacting electrons in quantum dots [89]. As it is known that spin-related phenomena are the key ingredient in the emerging field of spintronics [90], the Hamiltonian representing the Rashba and Dresselhaus spin-orbit coupling can be treated by using the $osp(2,1)$ or the $osp(2,2)$ algebra within the framework of quasi–exact solvability.

CHAPTER 6

SOLVABLE POTENTIALS IN DIRAC AND KLEIN-GORDON EQUATIONS

Relativistic equations are very important for obtaining the effects of particles with velocity close to velocity of light. Two of them for spin 1/2 and spin 0 cases are Dirac equation and Klein–Gordon equation, respectively. They are used to determine the spectrum of these particles under some physical potentials. This chapter includes the applications of Dirac equation and Klein-Gordon equation in physical systems in the context of solvability. Some physical potentials in these equations are solved for obtaining the eigenfunctions and eigenvalues.

6.1 Dirac Equation

The relativistic quantum mechanical wave equation invented by Paul Dirac in 1928 in order to describe elementary spin-1/2 particles is called Dirac equation. These elementary spin-1/2 particles are fully consistent with the principles of quantum mechanics and largely consistent with the theory of special relativity, such as electrons, positrons, neutrinos etc. Additionally, this equation is required for the nature of particle spin and the existence of antiparticles.

It can be also used to describe the probability amplitudes for a single electron. This single-particle theory gives a fairly good prediction of the spin and magnetic moment of the electron and explains much of the fine structure observed in atomic spectral lines. Additionally, it makes the peculiar prediction that there exists an infinite set of quantum states in which the electron possesses negative energy.

The Dirac equation can be applied to other types of elementary spin-1/2 particles, such as neutrinos. A modified Dirac equation can be used to approximately describe protons and neutrons, which are made of smaller particles called quarks and are therefore not elementary particles.

The Dirac equation is for free electron

$$E\psi(x,t) = \left(\sum_{j=1}^{3}\alpha_j \mathbf{p}_j c + \beta mc^2\right)\psi(x,t) \tag{6.1}$$

where c is the speed of light, x and t are the space and time coordinates respectively, and $\psi(x,t)$ is a four-component wave function. (The wave function has to be formulated as a four-component spinor, rather than a simple scalar, due to the demands of special relativity.) The α and β are linear operators that act on the wave function, written as a column matrix, as 4×4 matrices known as Dirac matrices. There is more than one way to choose a set of Dirac matrices, a convenient representation can be more compactly written as

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad \beta = \alpha_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{6.2}$$

where 0 and I are the 2×2 zero and identity matrices, respectively, and the σ_j's $(j = 1,2,3)$ are the Pauli matrices (Eq. (3.41)) .

The Hamiltonian in the Eq. (6.1),

$$H = \sum_{j=1}^{3}\alpha_j \mathbf{p}_j c + \beta mc^2 \tag{6.3}$$

is called the Dirac Hamiltonian.

And relativistically covariant form of the Dirac Equation, in which the derivatives with time and space are treated on the same footing

$$\left(i\hbar c\sum_{j=1}^{3}\gamma^\mu \partial_\mu - \gamma^0 mc^2\right)\psi = 0 \tag{6.4}$$

where γ-matrices are anti-Hermitian, while γ_0 is Hermitian

$$\gamma^0 = \beta \tag{6.5a}$$
$$\gamma^\mu = \beta\alpha_i, \qquad \mu = 1,2,3. \tag{6.5b}$$

The convenient covariant behavior of γ-matrices comes from anti-commutator property

$$\{\gamma^\mu, \gamma^v\} = 2g_{\mu v} \tag{6.6}$$

where $g_{\mu v}$ is the metric tensor

$$g_{\mu v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.7}$$

These relations define a Clifford algebra called the Dirac algebra. This property means that γ-matrices transform like four vectors. Since the Dirac equation in Eq. (6.4) contains the gamma-matrices, it is obvious that the solution of this equation, the state vectors ψ must reflect this feature and should be represented by a column vector of dimension 4.

The wave function in Eq. (6.4) ψ is an 4 component vector wave function solution,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{6.8}$$

where the wave function ψ_+ is called particle spin wave function and ψ_- is the anti-particle spin wave function. They are given by

$$\psi_+ = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \qquad \text{and} \qquad \psi_- = \begin{pmatrix} g_+ \\ g_- \end{pmatrix} \tag{6.9}$$

where the subscripts $(+,-)$ in the f and g functions show the the spin up and spin down wave functions respectively. This kind of a four component object is called bispinor or Dirac spinor.

6.2 Construction of (Quasi)Exactly Solvable Potentials

In this section, the method which we use in construction of quantum exactly-solvable and quasi-exactly solvable potentials for relativistic particles are explained in detail. Some examples of potentials are given.

6.2.1 Method

The $(2+1)$ dimensional Dirac equation for free particle of mass m in terms of two-component spinors ψ, can be written as

$$E\psi = \left[\sum_{i=1}^{2} c\alpha_i p_i + \beta mc^2\right]\psi \tag{6.10}$$

Since we are using only two component spinors, the matrices β and α_i are conveniently defined in terms of the Pauli spin matrices.

In $(2+1)$ dimensions, in the presence of the magnetic field it is replaced by momentum operator $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$, where $\mathbf{A}$ is the vector potential, and the 2D Dirac oscillator can be constructed by changing the momentum $\mathbf{p} \to \mathbf{p} - imw\sigma_3 r\widehat{\mathbf{r}}$. We are now seeking for a certain form of the momentum operator that can be interpreted as exactly solvable Schrödinger equation in the non-relativistic limit. For this purpose we introduce the following momentum operator

$$\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A} + i\sigma_3 v(r)\widehat{\mathbf{r}} \tag{6.11}$$

Then Dirac equation in Eq. (6.10) with the momentum operator of Eq. (6.11) takes the form

$$\begin{aligned}
[E - \sigma_0 mc^2]\psi &= c\sigma_+ \left[p_x - ip_y - e(A_x - iA_y) - (v_x - iv_y)\right]\psi + \\
&\quad c\sigma_- \left[p_x + ip_y - e(A_x + iA_y) + (v_x - iv_y)\right]\psi
\end{aligned} \tag{6.12}$$

where A_i and v_i are the i^{th} components of the momentum, vector potential and the potential in Cartesian coordinate system, respectively. In polar coordinate, $x = r\cos\phi$, $y = r\sin\phi$, the 2D Dirac equation in Eq. (6.12) can be written as

$$E_-\psi_- = ce^{-i\phi}[-i\hbar\frac{\partial}{\partial r} - \frac{\hbar}{r}\frac{\partial}{\partial\phi} + ieA(r) + iv(r)]\psi_+ \tag{6.13a}$$

$$E_+\psi_+ = ce^{i\phi}[-i\hbar\frac{\partial}{\partial r} + \frac{\hbar}{r}\frac{\partial}{\partial\phi} - ieA(r) - iv(r)]\psi_- \tag{6.13b}$$

where $E_\pm = E \pm mc^2$ and $\psi_\pm = \psi_+(r, \phi)$ are the upper and lower components of the spinor ψ. The substitution of the wave functions

$$\psi_\pm = \frac{e^{-i(l\pm\frac{1}{2})\phi}}{\sqrt{r}} f_\pm(r) \tag{6.14}$$

leads to the following set of coupled differential equations

$$E_-f_-(r) = ic(-\hbar\frac{\partial}{\partial r} + \frac{\hbar l}{r} + eA(r) + v(r))f_+(r) \tag{6.15a}$$

$$E_+f_+(r) = -ic(\hbar\frac{\partial}{\partial r} + \frac{\hbar l}{r} + eA(r) + v(r))f_-(r) \tag{6.15b}$$

Our task is now to transform Eq. (6.15a) and Eq. (6.15b) in the form of the Schrödinger-like equation. For this purpose we substitute

$$W(r) = \frac{\hbar l}{r} + \frac{eA(r) + v(r)}{\hbar}; \qquad E_\pm = ic\hbar\varepsilon_\pm; \qquad \varepsilon^2 = \varepsilon_+\varepsilon_- \tag{6.16}$$

then we obtain

$$\varepsilon_-f_-(r) = (-\frac{\partial}{\partial r} + W(r))f_+(r) \tag{6.17a}$$

$$\varepsilon_+f_+(r) = (\frac{\partial}{\partial r} + W(r))f_-(r) \tag{6.17b}$$

Notice that the results of Eq. (6.17a) and Eq. (6.17b) seem to hint at a supersymmetric treatment of the Dirac equation, because the supersymmetric operators can be expressed as $A^\pm = (\mp\frac{\partial}{\partial r} + W(r))$. It is obvious that the functional form of the superpotential $W(r)$ and $v(r)$ are the same except that the radial function $\frac{\hbar l}{r}$. Thus the superpotential of the non-relativistic quantum mechanics can be recognized as the potential of the relativistic quantum mechanics. It is obvious that the expressions of Eq. (6.17a) and Eq. (6.17b) can be written in the form of the Schrödinger-like equation, by eliminating $f_+(r)$ and/or $f_-(r)$ between Eq. (6.17a) and Eq. (6.17b) provides the following expressions

$$(-\frac{\partial^2}{\partial r^2} + W^2(r) - W'(r) + \varepsilon^2)f_-(r) = 0 \qquad (6.18a)$$

$$(-\frac{\partial^2}{\partial r^2} + W^2(r) + W'(r) + \varepsilon^2)f_+(r) = 0 \qquad (6.18b)$$

This is indeed a nice result considering the potentials $V_\pm = W^2(r) \pm W'(r)$ yield a common spectrum, thus forming an isospectral system.

In the following section, we investigate the construction of the exactly solvable potentials.

6.2.2 Exactly Solvable Potentials

Appropriate choices of the superpotential $W(r)$ permit the construction of the exactly solvable potentials. In this section we illustrate that for a large class of potentials in the form of the Schrödinger type equations that provide Dirac equation are exactly solvable. These potentials are shown in Table 6.1 which shows the appropriate parameters of exactly solvable potentials with partner potentials and eigenvalues.

6.2.3 Solution of the Exactly Solvable Hamiltonians

6.2.3.1 Dirac-Oscillator

Let us start with the well known problem, namely Dirac oscillator. From the appropriate parameters as in Table 6.1, we can construct the Dirac oscillator. The Schrödinger type equations take the form:

$$(-\frac{\partial^2}{\partial r^2} + \frac{(l+1)(l+2)}{r^2} + V_+)f_+(r) = 0 \qquad (6.19a)$$

$$(-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} + V_-)f_-(r) = 0 \qquad (6.19b)$$

where the frequency w_T can be expressed in terms of the Larmor frequency as follows

$$w_T = w + w_L = w + \frac{eB}{2m}$$

and

$$V_\pm = (\frac{m}{\hbar}w_T)^2 r^2 \mp \frac{m}{\hbar}w_T(2l + 2 \mp 1) - \varepsilon^2$$

Potentials	$W(r)$	V_-	ε^2
Dirac-Oscillator	$-\frac{l+1}{r} + \frac{m}{\hbar} w_T r$	$\frac{l(l+1)}{r^2} + (\frac{m}{\hbar} w_T)^2 r^2 + \frac{m}{\hbar} w_T (2l+3)$	$4n \frac{m}{\hbar} w_T$
Dirac-Coulomb	$\frac{me^2}{4\pi\varepsilon_0 \hbar^2 (l+1)} - \frac{\hbar(2l+1)}{r}$	$\frac{l(l+1)}{r^2} - \frac{me^2}{2\pi\varepsilon_0 \hbar^2 (l+1)} + \left(\frac{me^2}{4\pi\varepsilon_0 \hbar^2 (l+1)}\right)^2$	$\left(\frac{me^2}{4\pi\varepsilon_0 \hbar^2 (l+1)}\right)^2 - \left(\frac{me^2}{4\pi\varepsilon_0 \hbar^2 (n+l+1)}\right)^2$
Dirac-Morse	$b - a\exp(-\alpha r)$	$a^2 \exp(-2\alpha r) - a(\alpha + 2b)\exp(-\alpha r) + b^2$	$\alpha n(\alpha n - 2b)$
Dirac-Pösch-Teller	$a\tan(r\alpha) - b\cot(r\alpha)$	$-(a+b)^2 + a(a-\alpha)\sec(r\alpha)^2 + b(b-\alpha)\csc(r\alpha)^2$	$n(n+a+b)$
Dirac-Eckart-Epstein	$\frac{c}{1+\exp(-ar)} + \frac{b\exp(ar)}{1+\exp(ar)}$	$\frac{(b+c)\exp(ar)(-a+(b+c)\exp(ar))}{(1+\exp(ar))^2}$	$\varepsilon_0 - \frac{V_0}{2} - [\delta' - (n+\frac{1}{2})]^2 D - \left[\frac{1}{4[\delta' - (n+\frac{1}{2})]}\right]^2 \frac{V_0^2}{D}$

Table 6.1: Appropriate parameters and eigenvalues of exactly-solvable potentials

In order to solve Eq. (6.19b), we change the variable $z = \frac{m}{\hbar} w_T r^2$, and introduce the wave function

$$f_-(z) = A z^{\frac{l+1}{2}} \exp(-\frac{z}{2}) g_-(z),$$

where A is the normalization constant, then Eq. (6.19b) takes the form

$$\left[z \frac{\partial^2}{\partial z^2} + \left(l + \frac{3}{2} - z \right) + n \right] g_-(z) = 0. \tag{6.20}$$

The natural number n and energy satisfy the relation

$$\varepsilon^2 = \frac{E^2 - m^2 c^4}{\hbar^2 c^2} = 4n \frac{m}{\hbar} w_T. \tag{6.21}$$

By solving Eq. (6.21), we find that

$$E = \pm \sqrt{m^2 c^4 + 4nm\hbar w_T c^2} \tag{6.22}$$

Note that the nonrelativistic limit of the energy is obtained by setting $E = E_{nr} + mc^2$ and considering $E_{nr} << mc^2$, we obtain the nonrelativistic energy

$$E_{nr} = 4n\hbar w_T.$$

We now investigate the dependence of the energy on the spin. In order to analyze spin effects, it is worth to obtain nonrelativistic form of the Eq. (6.19b):

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{l(l+1)\hbar^2}{2mr^2} + \frac{1}{2} m {w_T}^2 r^2 + \hbar w_T \left(l + \frac{3}{2} \right) - E_{nr} \right) f_-(r) = 0.$$

The corresponding Hamiltonian is the Hamiltonian of a harmonic oscillator with an additional spin-dependent term, $\hbar w_T \left(l + \frac{3}{2} \right)$.

To complete our analysis, we turn our attention to the normalization of the wave function. It is easy to see that the solution of Eq. (6.20) is the associated Laguerre polynomials, $L_n^{l+\frac{1}{2}}(z)$, then $f_-(z)$ can be written as

$$f_-(z) = A z^{\frac{l+1}{2}} \exp(-\frac{z}{2}) L_n^{l+\frac{1}{2}}(z)$$

The upper component, $f_+(z)$, of spinor $\psi(r)$, can be obtained from the relation of Eq. (6.17a) and it is given by

$$f_+(z) = -\frac{2A\sqrt{w_T}}{\varepsilon_-} z^{\frac{l+2}{2}} \exp(-\frac{z}{2}) L_{n-1}^{l+\frac{3}{2}}(z).$$

Normalization condition in polar coordinate is given by

$$\langle \psi \mid \psi \rangle = \int_0^\infty (|f_+(r)|^2 + |f_-(r)|^2) dr = 1. \tag{6.23}$$

Associated Laguerre polynomials satisfy the orthogonality condition:

$$\int_0^\infty z^l \exp(-z) L_n^l(z) L_k^l(z) dr = \frac{\Gamma(n+l+1)}{n!}\delta_{nk}.$$

Thus, we finally obtain an expression for the normalization constant A:

$$A = \left[\frac{\sqrt{w_T}\varepsilon_-^2\Gamma(n+1)}{(nw_T + \varepsilon_-^2\Gamma(n+1))\Gamma(n+l+\frac{3}{2})}\right]$$

We have solved Dirac oscillator in two dimensional space permits a series of interesting results. The Dirac oscillator has various physical applications particularly in semiconductor physics [107].

6.2.3.2 Dirac-Coulomb Potential

Another well known examples of the exactly solvable Dirac equation is that relativistic Hydrogen atom. The problem can be solved exactly when the magnetic field is zero and in the presence of the Coulomb interaction [108, 109]. By substituting the parameters from the Table 6.1 and then the Schrödinger-like Eqs. (6.18a) and (6.18b) take the form

$$(-\frac{\partial^2}{\partial r^2} + \frac{(l+1)(l+2)}{r^2} - \frac{e^2}{r} + \left(\frac{me^2}{4\pi\varepsilon_0\hbar^2(l+1)}\right)^2 - \varepsilon^2)f_+(r) = 0 \tag{6.24a}$$

$$(-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} - \frac{me^2}{2\pi\varepsilon_0\hbar^2(l+1)} + \left(\frac{me^2}{4\pi\varepsilon_0\hbar^2(l+1)}\right)^2 - \varepsilon^2)f_-(r) = 0 \tag{6.24b}$$

Following the similar developments used in the construction of the harmonic oscillator problem, Eq. (6.24b) can be transformed in the form

$$\left[z\frac{\partial^2}{\partial z^2} + (2l+2-z) + n\right] g_-(z) = 0. \tag{6.25}$$

by changing the variable $r = 2\pi\varepsilon_0\hbar^2(n+l+1)z/me^2$ and the wave function

$$f_-(z) = Az^{l+1}\exp(-\frac{z}{2})g_-(z),$$

The natural number n and energy of the Hamiltonian satisfy the relation

$$\varepsilon^2 = \frac{E^2 - m^2c^4}{\hbar^2c^2} = \left(\frac{me^2}{4\pi\varepsilon_0\hbar^2(l+1)}\right)^2 - \left(\frac{me^2}{4\pi\varepsilon_0\hbar^2(n+l+1)}\right)^2. \tag{6.26}$$

Solution of the Eq. (6.25) leads to the following expressions for the wave function

$$f_-(z) = Az^{l+1}\exp(-\frac{z}{2})L_n^{2l+1}(z) \tag{6.27}$$

The upper component, $f_+(z)$ can be obtained from the relation of Eq. (6.17a) and it is given by

$$f_+(z) = -\frac{A}{\varepsilon_-}z^{l+1}\exp(-\frac{z}{2})\left[2(l+1)L_{n-1}^{2l+2}(z) + \left(l+1-\frac{me^2}{2\pi\varepsilon_0\hbar^2}(2l+1)\right)L_n^{2l+1}(z)\right].$$

Using the identity

$$\int_0^\infty z^{l+1}\exp(-z)L_n^l(z)L_n^l(z)dr = (2n+l+1)^{l+2}\frac{\Gamma(n+l+1)}{n!}, \tag{6.28}$$

after some straightforward calculation we obtain the normalization constant

$$A = \left[\frac{2\pi\varepsilon_0\hbar^2}{me^2}\left(\frac{4(l+1)^2}{\varepsilon_-^2\Gamma(n)} + (2n+2l+2)^{2l+3}B\right)\Gamma(n+2l+2)\right] \tag{6.29}$$

where K is given by

$$B = \left(1+\frac{1}{\varepsilon_-^2 n!}\left(l+1-\frac{me^2}{2\pi\varepsilon_0\hbar^2}(2l+1)\right)\right)^2. \tag{6.30}$$

Due to the recent interest in the 2D field theory and condensed matter physics, the 2D Coulomb potential is physically relevant and the results obtained in 2D exhibit some new features [110, 111].

6.2.3.3 Dirac-Morse Potential

The Morse oscillator is exactly solvable quantum mechanical problem and it is used to model the interaction of the atoms in the diatomic molecules. In order to obtain its relativistic form we apply the standard procedure. Then the equations take the form

$$(-\frac{\partial^2}{\partial r^2} + V_+ - \varepsilon^2)f_+(r) = 0 \tag{6.31a}$$

$$(-\frac{\partial^2}{\partial r^2} + V_- - \varepsilon^2)f_-(r) = 0 \tag{6.31b}$$

where

$$V_\pm = a^2\exp(-2\alpha r) \pm a(\alpha - 2b)\exp(-\alpha r) + b^2$$

In the order to solve Eq. (6.31b) we change the variable $\exp(-\alpha r) = \frac{\alpha}{2a}z$ and then introduce the wave function

$$f_-(z) = z^{\frac{b}{\alpha}-n}\exp(-\frac{z}{2})g_-(z),$$

then we obtain

$$\left[z\frac{\partial^2}{\partial z^2} + \left(\frac{2b}{\alpha} - 2n + 1 - z\right) + n\right]g_-(z) = 0. \tag{6.32}$$

Energy expression for the Morse oscillator is given by

$$\varepsilon^2 = \alpha n(\alpha n - 2b). \tag{6.33}$$

The wave functions can be obtained from Eqs. (6.32) and (6.17a) as

$$\begin{aligned} f_-(z) &= Az^{\frac{b}{\alpha}-n}\exp(-\frac{z}{2})L_n^{\frac{2b}{\alpha}-2n}(z), \\ f_+(z) &= \frac{A\alpha}{\varepsilon_-}z^{\frac{b}{\alpha}-n}\exp(-\frac{z}{2})\left[nL_n^{\frac{2b}{\alpha}-2n}(z)+zL_{n-1}^{\frac{2b}{\alpha}-2n+1}(z)\right]. \end{aligned}$$

Normalization condition yields the following expression for the normalization constant

$$A=\left[\frac{\varepsilon_-^2\Gamma(n)\Gamma(n-1)}{\Gamma(\frac{2b}{\alpha}+1-n)(\varepsilon_-^2\Gamma(n)+2\alpha^2\Gamma(n+1))}\right]^{-1/2}.$$

6.2.3.4 Dirac-Pöschl-Teller Potential

Pöschl-Teller potentials were originally introduced in a molecular physics context. The Pöschl-Teller potentials are analytical exactly solvable problems in nonrelativistic quantum mechanics for which all the energy values and eigenfunctions are explicitly known. Here we construct the relativistic form of Pöschl-Teller problems by choosing the parameter as in Table 6.1 provides the following potential

$$(-\frac{\partial^2}{\partial r^2}+V_++\varepsilon^2)f_+(r) = 0 \tag{6.34a}$$

$$(-\frac{\partial^2}{\partial r^2}+V_-+\varepsilon^2)f_-(r) = 0 \tag{6.34b}$$

where

$$V_\pm=-(a+b)^2+a(a\pm\alpha)\sec(r\alpha)^2+b(b\pm\alpha)\csc(r\alpha)^2 \tag{6.35}$$

The energy eigenvalues and corresponding eigenstates are solutions to the Schrödinger equation. After straightforward calculation we obtain an expression for the eigenfunction of the Pösch-Teller potentials. The wave function for lower spinor can be given by confluent hypergeometric function [112]:

$$f_-(r)=(c_n(b,a))^{-1/2}(\cos(r\alpha))^a(\sin(r\alpha))^b{}_2F_1(-n,n+a+b;b+\frac{1}{2};\sin(r\alpha)^2). \tag{6.36}$$

where where $c_n(b,a)$ is a normalization factor that can be given analytically when a and b are positive integers and ${}_2F_1(a,b,c,y)$ is a confluent hypergeometric functions which can be written as

$${}_2F_1(a,b,c,y)=1+\frac{ab}{1\times c}y+1\frac{a(a+1)b(b+1)}{1\times 2\times c(c+1)}y^2+........ \tag{6.37}$$

From the relation of confluent hypergeometric function with Jacobi polynomials, the Eq. (6.36) can be written as

$$f_-(r)=(c_n(b,a))^{-1/2}(\cos(r\alpha))^a(\sin(r\alpha))^b\frac{n!}{(b+1/2)_n}P_n^{(b-1/2,a+1/2)}(1-2\sin(r\alpha)^2). \tag{6.38}$$

Then, the energy can be found as

$$\varepsilon^2 = n(n+a+b) \tag{6.39}$$

in $a, b > 1$.

6.2.3.5 Dirac-Eckart-Epstein Potential

Recently, the mathematical discussions have all been based either on a potential function which has discontinuous derivatives, or else on approximate treatments involving asymptotic (i.e., divergent) series. It is therefore of some interest to note that there is an analytic function which has been discussed and whose associated Shrödinger is soluble. This function or potential is called Eckart-Epstein potential. An Eckart-Epstein potential is used for which the Schrödinger equation has exact analytic solutions.

The Eckart-Epstein potential [61, 119, 120] with parameters V_0 and V_1 is given by

$$V_{ground}(r) = V(\xi(r)) = \frac{V_0\xi}{1+\xi} + \frac{V_1\xi}{(1+\xi)^2}, \qquad \xi = \exp(2\pi r/l) \tag{6.40}$$

for the ground state, in which r is the polar coordinate, and V_0, V_1, and l are constants. And an inverted Eckart potential,

$$V_{excited}(r) = \varepsilon_0 - V(\xi(r)) \tag{6.41}$$

And the potential in Eq. (6.40) describes an asymmetric barrier with conditions

$$V(-\infty) = 0, \tag{6.42a}$$

$$V(\infty) = V_0. \tag{6.42b}$$

It is seen to approach zero for large negative values of r and a constant value V_0 for large positive values. The width of the transition region is, practically speaking, $2l$. When $|V_1|$ is greater than $|V_0|$ it possesses an extremum at

$$r_m = \frac{1}{2\pi}\log[\frac{V_1+V_0}{V_1-V_0}], \tag{6.43}$$

whose height is

$$V(r_m) = V_m = \frac{(V_1+V_0)^2}{4V_1}. \tag{6.44}$$

In the following, it will be assumed that $V_1 \geq 0$ so that the extremum is a maximum, when it exists at all.

Now, our aim is to find the general solution of Hamiltonian including Eckart-Epstein potential in Eq. (6.40) with obtaining $W(r)$ function. In order to obtain the relativistic form of this potential using the parameters in Table

6.1. Consequently the Schrödinger type equations takes the form

$$\left(-\frac{\partial^2}{\partial r^2} + V_- + \varepsilon^2\right) f_-(r) \;=\; 0 \tag{6.45a}$$

$$\left(-\frac{\partial^2}{\partial r^2} + V_+ + \varepsilon^2\right) f_+(r) \;=\; 0 \tag{6.45b}$$

where $a = \frac{2\pi}{l}$, b, and c are constants and

$$V_- = \frac{(b+c)\exp(ar)(-a+(b+c)\exp(ar))}{(1+\exp(ar))^2}$$

$$V_+ = \frac{(b+c)\exp(ar)(a+(b+c)\exp(ar))}{(1+\exp(ar))^2}$$

Now, lets try to find the solution of the Hamiltonian in Eq. (6.45b). This equation under the conditions

$$(b+c)^2 = ab = V_1$$

$$ca = V_0$$

returns to Schrödinger equation including Eckart-Epstein potential

$$\frac{d^2 f_+(r)}{dr^2} - \frac{8\pi^2 m}{h^2}\left\{\frac{V_0\exp(ar)}{1+\exp(ar)} + \frac{V_1\exp(ar)}{(1+\exp(ar))^2} - E_{new}\right\} f_+(r) = 0, \tag{6.46}$$

where $E_{new} = \varepsilon^2$. Now, with making substitution $y^{-1} = 1+\exp(ar)$, the equation is reduced to

$$\left(-a^2\left[y(y-1)\frac{d}{dy}\right]\left[y(y-1)\frac{d}{dy}\right] - (V_0(1-y) + V_1(1-y)y - E_{new})\right) f_+(x) = 0 \tag{6.47}$$

Then putting

$$f_+(x) = y^{\beta}(1-y)^{\alpha} f(y) \tag{6.48}$$

leads to the hypergeometric equations

$$y(y-1)f''(y) + (-1-2\beta+2y(1+\beta+\alpha))\, f'(y) + (1+\beta+\alpha-\delta)(\beta+\alpha+\delta) f(y) = 0 \tag{6.49}$$

where

$$\alpha = \sqrt{\frac{E_{new}}{4C}} = \frac{1}{2\pi}k_1, \qquad \beta = \sqrt{\frac{E_{new}-V_0}{4C}} = \frac{1}{2\pi}k_1, \tag{6.50}$$

$$\delta = \sqrt{\frac{B-C}{4C}}, \qquad C = \frac{\hbar^2}{2m}.$$

This type hypergeometric equation has an eigenfunctions in terms of hypergeometric functions [61, 121] as

$$f_+(y) = A(1-\xi)^{i\beta}\left(\frac{\xi}{\xi-1}\right)^{i\alpha} {}_2F_1[\frac{1}{2}+i(\alpha-\beta+\delta), \frac{1}{2}+i(\alpha-\beta-\delta), 1-2i\beta, \frac{1}{1-\xi}] \tag{6.51}$$

The function $f_+(y)$ obviously approaches $(-\xi)^{i\beta}$ for large values of ξ, because of the properties of ${}_2F_1(a,b,c,0) = 1$. It is evident that the asymptotic condition for large positive y (or large ξ) Eq. (6.51) is fulfilled,

$$\lim_{x\to\infty}(f_+(\xi)) = A(1-\xi)^{i\beta} = A\exp(ik_2x). \tag{6.52}$$

But, its value for very small values of ξ cannot be determined at once, however, since the series ${}_2F_1(a,b,c,1)$ diverges. It is necessary to have rewritten the analytic extension of the hypergeometric functions. To determine the asymtotic behavior for large negative x (or small values of $|\xi|$), one can use the property

$$\begin{aligned}{}_2F_1(a,b,c,y) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a,b,a+b-c+1,1-y) \\ &+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)G(b)}(1-y)^{c-a-b}\ {}_2F_1(c-a,c-b,c-a-b+1,1-y).\end{aligned} \tag{6.53}$$

By using this property, the functions in Eq. (6.51) results in an expression in terms of two hypergeometric series that converge for small ξ values,

$$\begin{aligned}f_+(\xi) &= C_1(1-\xi)^{i\beta}\,(z)^{i\alpha}\ {}_2F_1[\frac{1}{2}+i(\alpha-\beta+\delta),\frac{1}{2}+i(\alpha-\beta-\delta),1+2i\beta,z] \\ &+C_2(1-\xi)^{i\beta}\,(z)^{-i\alpha}\ {}_2F_1[\frac{1}{2}+i(-\alpha-\beta-\delta),\frac{1}{2}+i(-\alpha-\beta+\delta),1-2i\beta,z]\end{aligned} \tag{6.54}$$

where $z = \frac{\xi}{\xi-1}$, and

$$C_1 = \frac{\Gamma(1-2i\beta)\Gamma(-2i\alpha)}{\Gamma(\frac{1}{2}+i(-\alpha-\beta-\delta))\Gamma(\frac{1}{2}+i(-\alpha-\beta+\delta))} \tag{6.55}$$

and

$$C_2 = \frac{\Gamma(1-2i\beta)\Gamma(+2i\alpha)}{\Gamma(\frac{1}{2}+i(\alpha-\beta-\delta))\Gamma(\frac{1}{2}+i(\alpha-\beta+\delta))} \tag{6.56}$$

For large negative values x (or small values of $|\xi|$), we have

$$\lim_{r\to-\infty}(f_+(\xi)) = C_1(-\xi)^{i\alpha} + C_2(-\xi)^{-i\alpha} = C_1\exp(ik_1r) + C_2\exp(-ik_1r).$$

The two series on the right side of the Eq. (6.54) converge when $\left|\frac{\xi}{\xi-1}\right| \leq 1$ hence certainly when $|\xi| < \frac{1}{2}$. For very small values of ξ the value of $f_+(\xi)$ may be computed from this equation. And also Eqs. (6.51) and (6.54) thus define the function $f_+(\xi)$ for all real values of r; it may readily be shown that it satisfies Eq. (6.47), and this function is finite, continuous, and possesses continuous derivatives throughout this range. Therefore, it is the wave-function we are trying to obtain.

The intermediate state is taken to be one of the vibrational states of an inverse Eckart given by Eq. (6.41). Bound state energies for the time-independent Schrödinger equation with this potential are given as

$$E_{new,n} = \varepsilon_0 - \frac{V_0}{2} - [\delta' - (n + \frac{1}{2})]^2 D - \left[\frac{1}{4[\delta' - (n + \frac{1}{2})]}\right]^2 \frac{V_0^2}{D}, \tag{6.57}$$

where $\delta' = \sqrt{(V_1 + D)/4D}$, $D = h^2/8ml^2$, and

$$n = 0, 1, 2......, < \delta' - \frac{1}{2} - \sqrt{V_0/4D}. \tag{6.58}$$

Corresponding eigenfunctions are given as

$$\psi_n(r) = N(1-\xi)^{-\beta_n} \left(\frac{\xi}{\xi - 1}\right)^{-\alpha_n} F[-n, 2\delta' - n, 1 + 2\beta_n, \frac{1}{1-\xi}], \tag{6.59}$$

where

$$\alpha_n = \left(\frac{\varepsilon_0 - E_{new,n}}{4D}\right)^{1/2}; \qquad \beta_n = \left(\frac{\varepsilon_0 - E_{new,n} - A}{4D}\right)^{1/2}. \tag{6.60}$$

We have constructed relativistic version of the Eckart-Epstein potentials whose eigenfunctions are associated to the hypergeometric functions.

The potentials we have derived here is significant from the both physical and mathematical points of view. For instance the relativistic quark model requires the solution of the Dirac equation containing single quark potential. The potential behaving like r at a large distances and $1/r$ at a short distances has been recently treated by Muci [113]. We mention that Morse oscillator potential can also be used as a large distance potential in some various physical systems. Consequently we have constructed relativistic version of the three well known potentials whose eigenfunctions are associated to the confluent hypergeometric functions. In the following section we construct QES potentials.

6.2.4 Solution of the Quasi-Exactly Solvable Potentials

The formulation given in section 6.2 is also useful to construct the relativistic version of the QES potentials. In this section we construct three of them. At this stage we mention that the underlying idea behind the quasi-exact solvability is the existence of a hidden algebraic structure. Our task is now to demonstrate by appropriate choices of relativistic potential we can construct QES Dirac equations. Our examples includes anharmonic oscillator potential, radial sextic oscillator potential and perturbed Coulomb potential.

6.2.4.1 Harmonic Oscillator Type Potentials

These quasi-exactly solvable potentials are associated to the Harmonic oscillator potentials.

(1) First examples is the anharmonic oscillator potential. The anharmonic oscillator potential have been widely used in many physical and chemical applications. In order to construct anharmonic oscillator potential we introduce

$$\begin{aligned} v(r) &= -\frac{\hbar l}{r} + \hbar w_T r + \hbar b r^2 + \hbar a, \\ A(r) &= \hbar B r, \\ W(r) &= a + w_T r + c r^2 \end{aligned} \tag{6.61}$$

With these choices we obtain the following Schrödinger-like equations

$$(-\frac{\partial^2}{\partial r^2} + V_+ + w_T + a^2 + \varepsilon^2) f_+(r) = 0 \tag{6.62a}$$

$$(-\frac{\partial^2}{\partial r^2} + V_- - w_T + a^2 + \varepsilon^2) f_-(r) = 0 \tag{6.62b}$$

where $V_\pm$ are given by

$$V_\pm = 2(a w_T \pm b) r + (2ab + w_T^2) r^2 + 2 b w_T r^3 + b^2 r^4 \tag{6.63}$$

and its the ground state wavefunction is given by

$$f_-^{(0)}(r) = \exp(-\frac{1}{3} b r^3 - \frac{1}{2} w_T r^2 - a r). \tag{6.64}$$

A part of the spectrum of the anharmonic oscillator potential can be obtained in the framework of the QES problem. For the wavefunction in Eq. (6.64), the eigenvalue has been obtained as [114]

$$\varepsilon^2 = -(\gamma + 5) w_T \tag{6.65}$$

where γ is constant which can be determined from analytic method.

(2) Another well known QES potential is the sextic oscillator potential with a centrifugal barrier. Its relativistic form can be obtained by the choices of the

$$\begin{aligned} v(r) &= -\frac{2\hbar l}{r} + \hbar w r + \hbar b r^3, \\ A(r) &= \hbar B r, \\ W(r) &= -\frac{l}{r} + w_T r + b r^3. \end{aligned} \tag{6.66}$$

In this case we obtain the following equations;

$$\begin{aligned}(-\frac{\partial^2}{\partial r^2}+\frac{l(l+1)}{r^2}+V_{+}+\varepsilon^2)f_{+}(r) &= 0 &\quad (6.67a)\\ (-\frac{\partial^2}{\partial r^2}+\frac{l(l-1)}{r^2}+V_{-}+\varepsilon^2)f_{-}(r) &= 0 &\quad (6.67b)\end{aligned}$$

where

$$V_{\pm}=(w_T^2\pm b(2l\mp 3))r^2+2bw_Tr^4+b^2r^6-w_T(2l\mp 1) \qquad (6.68)$$

Its ground state wave function are given by

$$f_{-}^{(0)}(r)=r^{4l}\exp(-\frac{1}{2}w_Tr^2-\frac{1}{4}br^4). \qquad (6.69)$$

with an energy

$$\varepsilon^2=w_T(2l+1) \qquad (6.70)$$

(3) Another potential associated with harmonic oscillator is the sextic oscillator potential having the parameters,

$$\begin{aligned}v(r) &= -\frac{\hbar l}{r}+\hbar w_Tr+\hbar br^3-\hbar Br,\\ A(r) &= \hbar Br, \qquad (6.71)\\ W(r) &= w_Tr+br^3.\end{aligned}$$

and yields the following differential equations;

$$\begin{aligned}(-\frac{\partial^2}{\partial r^2}+V_{+}+\varepsilon^2)f_{+}(r) &= 0 &\quad (6.72a)\\ (-\frac{\partial^2}{\partial r^2}+V_{-}+\varepsilon^2)f_{-}(r) &= 0 &\quad (6.72b)\end{aligned}$$

where

$$V_{\pm}=(w_T^2\pm 3b)r^2+2bw_Tr^4+b^2r^6\pm w_T \qquad (6.73)$$

Its ground state wave function are given by

$$f_{-}^{(0)}(r)=\exp(-\frac{1}{2}w_Tr^2-\frac{1}{4}br^4). \qquad (6.74)$$

with an energy

$$\varepsilon^2=w_T \qquad (6.75)$$

6.2.4.2 Coulomb Type Potentials

In this type, we deal with perturbed Coulomb potential which has an additional terms to exactly-solvable Coulomb potentials. choosing appropriate parameters as

$$\begin{aligned}v(r) &= \frac{e^2\hbar}{2(l+1)}-\frac{\hbar(2l+1)}{r},\\ A(r) &= \hbar Br, \qquad (6.76)\\ W(r) &= \frac{e^2}{2(l+1)}-\frac{l+1}{r}.\end{aligned}$$

In this case, the potentials takes the form

$$(-\frac{\partial^2}{\partial r^2}+\frac{(l+1)(l+2)}{r^2}+V_+-w_T(2l+1)+\varepsilon^2)f_+(r) = 0 \quad (6.77a)$$

$$(-\frac{\partial^2}{\partial r^2}-\frac{l(l+1)}{r^2}+V_--w_T(2l+3)+\varepsilon^2)f_-(r) = 0 \quad (6.77b)$$

where

$$V_\pm=-\frac{e^2}{r}+w_T^2r^2+\frac{e^2w_T}{l+1}r+\frac{e^4}{4(l+1)^2} \quad (6.78)$$

Its ground state wave function are given by

$$f_-^{(0)}(r)=r^{l+1}\exp(-\frac{w_T}{2}r^2-\frac{e^2r}{2(l+1)}). \quad (6.79)$$

with an eigenvalue

$$\varepsilon^2=-\frac{e^4}{4(l+1)^2}+2w_T(l+2j+\frac{3}{2}). \quad (6.80)$$

In this type of potentials, it is possible to obtain another forms of perturbed potential by changing the parameters. The non-relativistic two dimensional Hamiltonian described Coulomb interaction between charged particles, i. e. the interaction between conduction electron and donor impurity center when a constant magnetic field is applied perpendicular to the plane of motion has been discussed in the literature [108, 115, 116, 117]. We have obtained relativistic interaction and it can be solve quasi-exactly. Recently its spectrum and wave function has been computed numerically [38].

6.2.4.3 Morse Type Potentials

When the Morse oscillator potentials become deformed or has an additional terms which are called perturbation terms, the Schrödinger equation describing this additional terms becomes quasi-exactly solvable problems. We can give three types of potentials which is called deformed Morse oscillator problems.

(1) Here, we select the proper choices parameters for applying the standard method. When we choose

$$\begin{aligned} v(r) &= \hbar(b-a\exp(-2r\alpha)+c\exp(-r\alpha)-\frac{l}{r}-Ber), \\ A(r) &= \hbar Br, \\ W(r) &= b-a\exp(-2r\alpha)+c\exp(-r\alpha) \end{aligned} \quad (6.81)$$

the Schrödinger equation becomes,

$$(-\frac{\partial^2}{\partial r^2}+V_++\varepsilon^2)f_+(r) = 0 \quad (6.82a)$$

$$(-\frac{\partial^2}{\partial r^2}+V_-+\varepsilon^2)f_-(r) = 0 \quad (6.82b)$$

where

$$V_{\pm} = b^2 - 2ac\exp(-3r\alpha) + a^2\exp(-4r\alpha) \\ +\exp(-2r\alpha)(-2ab + c^2 \pm 2a\alpha) + \exp(-r\alpha)(2bc \mp c\alpha) \tag{6.83}$$

Its ground state wave function are given by

$$f_-^{(0)}(r) = \exp(-\frac{a}{2\alpha}\exp(-2r\alpha) - br + \frac{c}{\alpha}\exp(-r\alpha)). \tag{6.84}$$

at $\alpha > 0$, $a \geq 0$, $b > 0$ and for all c with energy

$$\varepsilon^2 = -b^2 \tag{6.85}$$

(2) Similarly, by choosing

$$\begin{aligned} v(r) &= -\hbar(b + a\exp(2r\alpha) - c\exp(r\alpha) + \frac{l}{r} - Ber), \\ A(r) &= \hbar Br, \\ W(r) &= b - a\exp(2r\alpha) + c\exp(r\alpha) \end{aligned} \tag{6.86}$$

the Schrödinger equation becomes,

$$(-\frac{\partial^2}{\partial r^2} + V_+ + \varepsilon^2)f_+(r) = 0 \tag{6.87a}$$

$$(-\frac{\partial^2}{\partial r^2} + V_- + \varepsilon^2)f_-(r) = 0 \tag{6.87b}$$

where

$$V_{\pm} = b^2 - 2ac\exp(3r\alpha) + a^2\exp(4r\alpha) \\ +\exp(2r\alpha)(-2ab + c^2 \mp 2a\alpha) + \exp(r\alpha)(2bc \pm c\alpha) \tag{6.88}$$

Its ground state wave function is given by

$$f_-^{(0)}(r) = \exp(-\frac{c}{2\alpha}\exp(2r\alpha) + br - \frac{a}{\alpha}\exp(r\alpha)). \tag{6.89}$$

at $\alpha > 0$, $c \geq 0$, $b > 0$ and for all a with energy eigenvalues

$$\varepsilon^2 = -b^2 - \alpha k(\alpha + 2b) \tag{6.90}$$

We have obtained analytical solutions of the (2 + 1)- dimensional Dirac equation for a set of potentials in two dimensions with the hope that they could be useful in the low dimensional field theory and condensed matter physics. The potentials for Dirac equation have been obtained by extending the notion of the Dirac oscillator. In a similar manner one can construct the exactly and QES Dirac equation including hyperbolic and trigonometric potentials. It has been shown that the (2 + 1)-dimensional Dirac equation can be transformed in the form of a Schrödinger-like equation and for all exactly and QES Schrödinger equations one can find potentials of the Dirac equation which are also exactly solvable or QES.

Therefore, besides its importance as a new treatment of the construction of the various potentials for Dirac equation, the potentials obtained here might be relevant to model some physical problems. Finally, it is expected that the models presented here to construct Dirac equation including various potentials may provide a good starting point for the study of more realistic models for the low dimensional of field theory and condensed matter physics.

6.3 Klein-Gordon Equation

The Klein-Gordon equation (Klein-Fock-Gordon equation or sometimes Klein-Gordon-Fock equation) is the relativistic version of the Schrödinger equation. The Schrödinger equation suffers from not being relativistically covariant, meaning it does not take into account Einstein's special theory of relativity.

The special-relativistic expression for the kinetic energy may be used to form the classical free particle Hamiltonian

$$E = \sqrt{m^2c^4 + p^2c^2} \tag{6.91}$$

The treatment of the Klein–Gordon equation by the analogous quantum mechanical expression may be constructed by replacing the classical momentum, $\mathbf{p}$, with its quantum mechanical operator, $\widehat{\mathbf{p}}$ which leads to the free particle wave equation

$$\left(i\hbar\frac{d}{dt}\right)\psi = \sqrt{m^2c^4 + \widehat{\mathbf{p}}^2c^2}\psi \tag{6.92}$$

When we look at this equation, we see that it does not satisfy some of the conditions required by special relativity. And the wave equation is not invariant to a Lorentz transform, and the square root term introduces ambiguity. The Klein-Gordon equation rectifies both of these problems simply by taking the square of the original energy expression and extending the result to a quantum mechanical wave equation:

$$E^2\psi = \left(i\hbar\frac{d}{dt}\right)^2\Psi = (m^2c^4 + \widehat{\mathbf{p}}^2c^2)\psi \tag{6.93}$$

The resulting wave equation is Lorentz covariant and well defined, but suffers from other problems. Negative energy solutions to this equation are possible, which do not have a readily obvious explanation, and the probability density, $\psi^*\psi$, fluctuates with time as does its integral over all space.

When the Eq. (6.93) is rearranged in the presence of vector and scalar potentials the $(1+1)$-dimensional time-independent Klein–Gordon equation for a spinless particle of rest mass m reads [124]

$$\left(-\hbar^2 c^2 \frac{d^2}{dx^2} + (mc^2 + V(x))^2\right)\psi = (E - S(x))^2\psi \tag{6.94}$$

where $V(x)$ and $S(x)$ are the vector and scalar potentials, respectively. In the non-relativistic approximation (potential energies smaller compared to mc^2 and $E \simeq mc^2$), the Eq. (6.94) reduces to

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + (S(x) + V(x))\right)\psi = (E - mc^2)\psi \tag{6.95}$$

When we look at this equation, we can say that ψ obeys the Schrödinger equation with binding energy $E - mc^2$.

In this equation, the case $V(x) = -S(x)$ presents bounded solutions in the relativistic approach, although it reduces to the free-particle problem in the nonrelativistic limit. The attractive vector potential for a particle is, of course, repulsive for its corresponding antiparticle, and vice versa. However, the attractive (repulsive) scalar potential for particles is also attractive (repulsive) for antiparticles. For $V(x) = S(x)$ and an attractive vector potential for particles, the scalar potential is counterbalanced by the vector potential for antiparticles as long as the scalar potential is attractive and the vector potential is repulsive. As a consequence there is no bounded solution for antiparticles. For $V(x) = 0$ and a pure scalar attractive potential, one finds energy levels for particles and antiparticles arranged symmetrically about $E = 0$. For $V(x) = -S(x)$ and a repulsive vector potential for particles, the scalar and the vector potentials are attractive for antiparticles but their effects are counterbalanced for particles. Thus, there is no bound state solution for particles in this last case of mixing.

In the following section, we seek the solution of Eckart potential in the Klein–Gordon equation with SUSYQM approach to the bound state solution.

6.3.1 Standard Eckart Potential in the Klein-Gordon Equation

In this part, we try to find the exact solution of the s-wave Klein-Gordon equation under the influence of Eckart potential [125]. For obtaining the spectrum of this potential, we transform the Klein-Gordon equation into well known form of Schrödinger equation, because there exist large body of papers tackle the problem in the framework of Schrödinger equation. The full spectrum of Eckart potential are obtained with method of SUSYQM.

6.3.1.1 SUSYQM Formulation

Supersymmetric quantum mechanics starts with the factorization of one-dimensional Hamiltonians

$$H_\pm = A^\mp A^\pm = -\frac{1}{2}\frac{d^2}{dr^2} + V_\pm(r) \tag{6.96}$$

where $V_\pm(r)$ are the supersymmetric partner potentials. They are related to the superpotential $W(r)$ with the expression

$$V_\pm(r) = W^2(r) \pm W'(r) \tag{6.97}$$

The ladder operators in Eq. (6.96) are

$$A^\mp = \frac{1}{\sqrt{2}}\left(\pm\frac{d}{dr} + W(r)\right) \tag{6.98}$$

The ground state wave function of H_- in the SUSY is

$$f_0(r) = D\exp\left[\int W(r)dr\right] \tag{6.99}$$

where D is a normalization constant.

If the potentials are shape invariant, that is

$$V_+(r, a_0) = V_-(r, a_1) + R(a_1) \tag{6.100}$$

where a_0 and a_1 stand for the potential parameters in the supersymmetric partner potentials that specify space independent properties of the potentials (such as strength, range, and diffuseness), a_1 is a function of a_0, and $R(a_1)$ is a constant.

The eigenvalues and eigenfunctions of the Hamiltonians H_+ and H_- are related by

$$E_0^- = 0, \quad E_1^- = R(a_0), \quad E_{n+1}^- = E_n^+, \quad E_n^- = \sum_{k=0}^{n-1} R(a_k) \tag{6.101}$$
$$f_n^-(r, a_0) = A^+(r, a_0) f_{n-1}^-(r, a_1)$$

If the operators can be introduced as [57],

$$\begin{aligned} J_\pm &= e^{\pm i\phi}\left(\pm\frac{d}{dx} \pm W(x, -i\partial_\phi \pm 1/2)\right) \\ J_0 &= -i\frac{d}{d\phi} \end{aligned}$$

The factors $e^{\pm i\phi}$ in $J_\pm$ ensure that they indeed behave as ladder operators for the quantum number m. Operators $J_\pm$ have the same form as the $A^\mp$ operators of SUSYQM, except that the parameter m of the superpotential W is replaced by operators $(J_0 + 1/2)$ with the basis function

$$f_{jm}(x) = e^{im\phi} f_{jm}(x)$$

The shape invariance enable us to write the commutation relations of these operator as

$$\begin{aligned} [J_+, J_-] &= -R(J_0 + 1/2) \qquad (6.102) \\ [J_0, J_\pm] &= \pm J_\pm \end{aligned}$$

Now, if the function $R(J_0)$ is linear in J_0, the algebra of Eq. (6.102) reduces to that of $so(3)$ or $so(1,1)$ algebra.

6.3.1.2 SUSYQM Approach to Bound State Solution

Generally, the s-wave Klein-Gordon equation with scalar potential $S(r)$ and vector potential $V(r)$ can be written as [123, 125] $(\hbar = 1, c = 1)$

$$\left\{ \frac{d^2}{dr^2} + [E - V(r)]^2 - [m + S(r)]^2 \right\} f(r) = 0 \qquad (6.103)$$

where E is the energy. Indeed, the original wavefunction is expressed as $\psi(r) = f(r)/r$. We consider the standard Eckart potential as in the form

$$V(r) = V_1 \mathrm{sech}^2(\alpha r) - V_2 \tanh(\alpha r) \qquad (6.104)$$

When we consider the vector potential and scalar potential are equals $V(r) = S(r)$, Eq.(6.103) becomes a well-known form of Schrödinger equation

$$\left\{ \frac{d^2}{dr^2} + (E^2 - m^2) - 2(E + m)[V_1 \mathrm{sech}^2(\alpha r) - V_2 \tanh(\alpha r)] \right\} f(r) = 0, \qquad (6.105)$$

with the effective potential

$$V_{\text{eff}}(r) = 2(E + m)[V_1 \mathrm{sech}^2(\alpha r) - V_2 \tanh(\alpha r)] \qquad (6.106)$$

Then Eq. (6.105) takes the form

$$\left\{ -\frac{d^2}{dr^2} + V_{\text{eff}}(r) \right\} f(r) = \lambda f(r) \qquad (6.107)$$

where $\lambda = E^2 - m^2$ is the redefined energy parameter. In order to solve Eq. (6.107), in the framework of SUSYQM, substituting Eq. (6.99) into Eq. (6.107), we obtain

$$W^2(r) - W'(r) = V_{\text{eff}}(r) - \lambda_0 \qquad (6.108)$$

where λ_0 is the ground-state energy, and Eq. (6.108) is a nonlinear Riccati equation which gives the wave function of the system.

Our task is now, to obtain the super potential $W(r)$, which helps to express the super partner potentials $V_+(r)$ and $V_-(r)$. After some straightforward calculation, we obtain the super potential $W(r)$ which can be written as

$$W(r) = A - B\tanh(\alpha r) \tag{6.109}$$

where A and B are constant coefficients. Notice that the result Eq. (6.109) shows that the problem can be treated in the framework of SUSYQM. The supersymmetric partner potentials are obtained by substituting Eq. (6.109) into Eq. (6.100), we obtain the following partner potentials

$$\begin{aligned} V_+(r) &= A^2 + B^2 - B(B+\alpha)\mathrm{sech}^2(\alpha r) - 2AB\tanh(\alpha r) && (6.110a) \\ V_-(r) &= A^2 + B^2 - B(B-\alpha)\mathrm{sech}^2(\alpha r) - 2AB\tanh(\alpha r). && (6.110b) \end{aligned}$$

In order to obtain λ_0 and the relations between A, B and V_1 and V_2, we compare Eqs. (6.105,6.108,6.110a,6.110b). As a consequence, one can easily obtain the following relations between parameters

$$\begin{aligned} \lambda_0 &= A^2 + B^2 && (6.111a) \\ B &= \frac{1}{2}\left[\alpha \pm \sqrt{8(E+m)V_1 + \alpha^2}\right] && (6.111b) \\ A &= \frac{2(E+m)V_2}{\alpha \pm \sqrt{8(E+m)V_1 + \alpha^2}} && (6.111c) \end{aligned}$$

It is well known that the potentials are shape invariant., that is $V_+(r)$ has the same functional form with $V_-(r)$ but different parameters except for an additive constant. The shape invariant condition permits an immediate analytical determination of eigenvalues and eigenfunctions. It is obvious that the potentials are invariant when the following conditions hold

$$\begin{aligned} a_0 &= -\frac{AB}{\alpha - B} \\ a_1 &= -\alpha + B \\ R(a_1) &= a_0^2 + a_1^2 - (A^2 + B^2) \end{aligned}$$

Thus, the energy eigenvalues of Hamiltonian which includes $V_-(r)$ partner potential are given by Eq. (6.101)

$$\begin{aligned} \lambda_0^{(-)} &= 0 && (6.112) \\ \lambda_n^{(-)} &= \sum R(a_k) = \left(-\frac{AB}{\alpha n - B}\right)^2 + (-\alpha n + B)^2 - (A^2 + B^2) && (6.113) \end{aligned}$$

Therefore, the complete energy spectrum are obtained as

$$\begin{aligned}\lambda_n &= \lambda_n^{(-)} + \lambda_0 = \left(-\frac{AB}{\alpha n - B}\right)^2 + (-\alpha n + B)^2 \\ n &= 1, 2, 3, ...\end{aligned} \tag{6.114}$$

Substituting the values of coefficients A, B, and λ_0 into Eq. (6.114), we obtained the required relativistic bound-state energy spectrum

$$m^2 - E_n^2 = \frac{(E_n + m)^2 V_2^2}{\alpha^2} \frac{1}{(n+\delta)^2} + \alpha^2 (n+\delta)^2 \tag{6.115}$$

where the parameters δ is defined as $\delta = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8(E_n+m)V_1}{\alpha^2}}$.

The corresponding unnormalized ground-state wave function is determined from Eq. (6.99)

$$f_0(r) = D \exp(-Ar)(\cosh(\alpha r))^{B/\alpha} \tag{6.116}$$

By inserting the parameters of A, B, and $f_0(r)$, we obtain the wave function as in the form

$$\psi_0(r) = \frac{1}{r}[\cosh(\alpha r)]^{(p+w)} \exp(\alpha[w - p]r) \tag{6.117}$$

where

$$\begin{aligned}p &= \frac{1}{2}\left[n + \delta + \frac{2(E+m)V_2}{\alpha^2}\frac{1}{n+\delta}\right] \\ w &= \frac{1}{2}\left[n + \delta - \frac{2(E+m)V_2}{\alpha^2}\frac{1}{n+\delta}\right]\end{aligned}$$

The unnormalized wave function of related Hamiltonian can be obtained by a similar mathematical procedure presented in [56]. By using this method, the wavefunction can be expressed in terms of the Jacobi polynomial as

$$\psi(r) = \frac{1}{r}[\cosh(\alpha r)]^{(p+w)} \exp[\alpha(w-p)r] \times P_n^{-2p,-2w}[-\coth(\alpha r)] \tag{6.118}$$

This equation gives the required wavefunction of standard Eckart potential with Klein-Gordon equation.

In summary, we have discussed the exact solution of the Klein–Gordon equation including the equal scalar and vector Eckart potentials by using the SUSYQM. We have shown that both the eigenvalues and eigenfunctions of the Klein–Gordon equation can be obtained in the closed form for the Eckart potential. Finally, we emphasize that the method discussed here can be generated for other potentials.

CHAPTER 7

AN ALTERNATIVE METHOD TO SOLVE PHYSICAL HAMILTONIANS: DIAGONALIZATION OF HAMILTONIAN

Up to now, we have prepared a general method to obtain the solution of two boson Hamiltonian. By using either solution of number operator or similarity transformation, we have been able to provide a QES of the various physical Hamiltonians. Furthermore, it has been given that two boson Hamiltonian can be reduced to single variable differential equation in the Bargmann-Fock space. It is also important to mention here that the methods given here can be used to solve higher order differential equations. Additionally, this method can easily be extended to solve the Hamiltonians that include multi-boson or fermion-boson systems. This extension leads to the solution of the various physical problems, such as Pauli equation, Rabi Hamiltonian, Jaynes-Cummings Hamiltonian.

Now, we consider solutions of the 2×2 matrix Hamiltonians of the physical systems within the context of $su(2)$ and $su(1,1)$ Lie algebras. The systematic study presented here reproduces a number of earlier results in a natural way as well as leads to a novel finding.

During the last decade, a great deal of attention has been paid to examine different quantum optical models. Recently, some algebraic techniques which improve both analytical and numerical solutions of the problems have been suggested and developed for some quantum optical systems [80, 81, 10, 87, 126, 127, 128, 70, 129]. In general, the study of two-level systems, in a one- and two-dimensional geometry, coupled to bosonic modes has been the subject of intense attention because of its extensive applicability in the various fields of physics [98, 99, 132, 133, 130, 105, 103, 91].

The most general form of the Hamiltonian of a two-level system in two-dimensional geometry can be expressed as

$$\begin{aligned} H &= H_0 + \beta\sigma_0 + (\kappa_1 a_1 + \kappa_2 a_1^\dagger + \kappa_3 a_2 + \kappa_4 a_2^\dagger)\sigma_+ \\ &\quad +(\gamma_1 a_1 + \gamma_2 a_1^\dagger + \gamma_3 a_2 + \gamma_4 a_2^\dagger)\sigma_- \end{aligned} \tag{7.1}$$

where $H_0 = \hbar w_1 a_1^\dagger a_1 + \hbar w_2 a_2^\dagger a_2$ and w_i, β, κ_i, and γ_i are physical constants. The Hamiltonian in Eq. (7.1) includes various physical Hamiltonians depending on the choice of the parameters. For instance, when $w_1 = w_2 = w$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$, and $\kappa_1 = -\kappa_4 = \gamma_2 = -\gamma_3 = \kappa$, the Hamiltonian, H, reduces to $E \times \varepsilon$ Jahn–Teller (JT) Hamiltonian [99, 102], when $w_1 = w_2 = w$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$, and $\kappa_1 = \kappa_4 = \gamma_2 = \gamma_3 = \kappa$, the Hamiltonian, H reduces to the Hamiltonians of quantum dots including spin-orbit coupling [91, 4]. One can also obtain JC Hamiltonian [103], MJC Hamiltonian [104], as well as many other interesting physical Hamiltonians by an appropriate choice of the parameters w_i, κ_i, and γ_i in Eq. (7.1) There exist a relatively large number of different approaches for the solutions of the eigenvalue problems in the literature; however, we present here a systematic and a unified treatment for the determination of the eigenvalues and eigenfunctions of Eq. (7.1), in the context of the $su(2)$ and $su(1,1)$ Lie algebras. Furthermore, we develop an algorithm and routines to implement an exact solution scheme for determining eigenvalues and eigenfunctions of the Hamiltonian H.

It is well known that the Lie algebraic techniques are very powerful in describing many physical problems while improving both analytical and numerical solutions as well as understanding the nature of physical structures. In this paper, we concentrate our attention on the solution of the Hamiltonian, H, by constructing the proper realization of the algebras $su(2)$ and $su(1,1)$. In a straightforward way, furthermore, we shall see that, the Hamiltonian in Eq. (7.1). automatically leads to $su(2)$ or $su(1,1)$ algebras depending on the choice of the parameters κ_i, and γ_i.

7.1 Method

In our method, the bosonized Hamiltonians are connected with $su(1,1)$ and $su(2)$ Lie algebras. This connection opens the way to an algebraic treatment of a large class of physical Hamiltonians of practical interest. In this section, we present a general procedure to solve the Hamiltonian in Eq. (7.1). Consider the eigenvalue equation

$$H\psi = E\psi \tag{7.2}$$

where ψ is a two component wavefunction and E is the eigenvalue of the Hamiltonian H. Consequently, the Hamiltonian can be written as:

$$\begin{aligned}(H_0 - E - \beta)\psi_1 + (\kappa_1 a_1 + \kappa_2 a_1^\dagger + \kappa_3 a_2 + \kappa_4 a_2^\dagger)\psi_2 &= 0 \qquad (7.3a)\\ (H_0 - E + \beta)\psi_2 + (\gamma_1 a_1 + \gamma_2 a_1^\dagger + \gamma_3 a_2 + \gamma_4 a_2^\dagger)\psi_1 &= 0 \qquad (7.3b)\end{aligned}$$

These coupled equations may be solved by using various techniques. Herein we follow a new strategy. In the first step, we eliminate ψ_2 (or ψ_1) between the above equation

$$\psi_2 = -(H_0 - E + \beta)^{-1}(\gamma_1 a_1 + \gamma_2 a_1^\dagger + \gamma_3 a_2 + \gamma_4 a_2^\dagger)\psi_1 \tag{7.4}$$

Substituting Eq. (7.4) into Eq. (7.3a) we obtain the following equation:

$$\begin{aligned}&(H_0 - E - \beta)\psi_1 - (\kappa_1 a_1 + \kappa_2 a_1^\dagger + \kappa_3 a_2 + \kappa_4 a_2^\dagger)\\ &(H_0 - E + \beta)^{-1}(\gamma_1 a_1 + \gamma_2 a_1^\dagger + \gamma_3 a_2 + \gamma_4 a_2^\dagger)\psi_1 = 0,\end{aligned} \tag{7.5}$$

The last equation can be solved by performing a suitable realization of the $su(1,1)$ and $su(2)$ algebras. In our formalism, in order to obtain an adequate form of Eq. (7.5), in the next step, we use the relations:

$$\begin{aligned}a_1(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta - w_1)^{-1}a_1\\ a_1^\dagger(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta + w_1)^{-1}a_1^\dagger\\ a_2(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta - w_2)^{-1}a_2\\ a_2^\dagger(H_0 - E + \beta)^{-1} &= (H_0 - E + \beta + w_2)^{-1}a_2^\dagger\end{aligned} \tag{7.6}$$

and by setting $w_1 = w_2 = w$, we obtain the following general expression:

$$\begin{aligned}&(H_0 - E + \beta + \hbar w)(H_0 - E + \beta - \hbar w)(H_0 - E - \beta)\psi_1\\ =\ &\kappa_1(H_0 - E + \beta + \hbar w)(\gamma_1 a_1^2 + \gamma_2 a_1 a_1^\dagger + \gamma_3 a_1 a_2 + \gamma_4 a_1 a_2^\dagger)\psi_1\\ &+\kappa_2(H_0 - E + \beta - \hbar w)(\gamma_1 a_1^\dagger a_1 + \gamma_2 a_1^{\dagger 2} + \gamma_3 a_1^\dagger a_2 + \gamma_4 a_1^\dagger a_2^\dagger)\psi_1\\ &+\kappa_3(H_0 - E + \beta + \hbar w)(\gamma_1 a_2 a_1 + \gamma_2 a_2 a_1^\dagger + \gamma_3 a_2^2 + \gamma_4 a_2 a_2^\dagger)\psi_1\\ &+\kappa_4(H_0 - E + \beta - \hbar w)(\gamma_1 a_2^\dagger a_1 + \gamma_2 a_1^\dagger a_2^\dagger + \gamma_3 a_2^\dagger a_2 + \gamma_4 a_2^{\dagger 2})\psi_1.\end{aligned} \tag{7.7}$$

It can be easily checked that for certain values of κ_i, γ_i, and β, the equation can be solved in the framework of $su(1,1)$ or $su(2)$ Lie algebra techniques and the Hamiltonian can be related to the various physical Hamiltonians of interest. The resulting Hamiltonians are summarized in Table 7.1. The corresponding Hamiltonians are illustrated in the first column, and their algebras and appropriate realizations are illustrated in the last column.

Related Hamiltonian	Parameters							Symmetry Group
	κ_1	κ_4	γ_2	γ_3	w_1	w_2	β	
H_{dot}	λ'	$-\lambda'$	λ'	$-\lambda'$	w	w	$\frac{\mu}{2}gB$	$su(1,1)$
H_{JC}	κ	0	κ	0	w	0	$\frac{\hbar w_0}{2}$	$su(1,1)$
H_{JC}^{RWA}	κ	0	κ	0	w	0	$\frac{\hbar w_0}{2}$	$su(1,1)$
H_{MJC}	λ_1	0	λ_1	0	w	w	$\hbar w_0$	$su(2)$
H_{Dirac}	κ''	0	κ''	0	0	0	mc^2	$su(1,1)$

Table 7.1: List of the Hamiltonians depending on the choices of the parameters of Hamiltonian (7.1). The parameters $\kappa_4 = \gamma_2 = 0$ except for H_{JC}, $\kappa_4 = \gamma_2 = \kappa$, $\kappa_3 = \gamma_4 = 0$ except for H_{MJC}, $\kappa_3 = \gamma_4 = \lambda_2$, $\kappa' = \sqrt{\frac{mw}{4\hbar}}\kappa$, $\kappa'' = 2ic\sqrt{mw\hbar}$, and $\lambda' = \sqrt{\frac{mw}{4\hbar}}\lambda_R$.

7.2 Applications of the Method

The Hamiltonians whose list is given in Table 7.1, are demonstrated in two categories: exactly [136, 134] and quasi-exactly solvable Hamiltonians [59, 42, 135, 43]. It will be shown that, the rotating wave approximated Jayness–Cumming (JC) Hamiltonian, Dirac oscillator, and modified JC Hamiltonian, as they expected, exactly solvable, while JC, and quantum dot Hamiltonians are quasi-exactly solvable.

7.2.1 Exactly Solvable Hamiltonians

7.2.1.1 Dirac Oscillator

The (2+1)- dimensional Dirac equation for free particle of mass m in terms of two component spinors ψ can be written as [132, 133]

$$E\psi = \left[\sum_{i=1}^{2} c\sigma_i p_i + mc^2\sigma_0\right]\psi \tag{7.8}$$

In $(2+1)$ dimensions, the momentum operator p_i for free particle and 2D Dirac oscillator can be constructed by changing the momentum $\mathbf{p} \to \mathbf{p} - imw\sigma_0\mathbf{r}$. Then Dirac equation in Eq. (6.10) takes the form

$$\begin{aligned}[E - \sigma_0 mc^2]\psi &= c\sigma_+\left[p_{x_1} - ip_{x_2} - imw(x_1 - ix_2)\right]\psi + \\ &\quad c\sigma_-\left[p_{x_1} + ip_{x_2} - imw(x_1 + ix_2)\right]\psi \end{aligned} \tag{7.9}$$

After some straightforward treatment, we obtain the bosonic form of the Dirac oscillator:

$$[E - \sigma_0 mc^2]\psi = 2ic\sqrt{mw\hbar}[a_1\sigma_+ + a_1^\dagger\sigma_-] \tag{7.10}$$

An immediate practical consequence of these results is that the Lie algebraic structure of the Hamiltonians can be easily determined.

The exactly solvable problem, Dirac oscillator can be obtained by setting the parameters $w_1 = w_2 = \kappa_2 = \kappa_3 = \kappa_2 = \gamma_1 = \gamma_3 = \gamma_4 = 0$, and $\kappa_1 = \gamma_2 = 2ic\sqrt{mw\hbar}$, and $\beta = mc^2$ in Eq. (7.7), yielding the following expression:

$$(-E+mc^2)(-E+mc^2)(-E-mc^2)\psi_1 = -4\hbar wmc^2(-E+mc^2)a_1a_1^\dagger\psi_1 \quad (7.11)$$

The algebraic form of the equation, with the realization of Eq. (3.40), is given by

$$(-E+mc^2)(-E-mc^2)\psi_1 = -4\hbar wmc^2(M'+1)\psi_1 \quad (7.12)$$

Then we obtain the following energy values for the Dirac oscillator:

$$E = \pm\sqrt{m^2c^4 + 4\hbar wmc^2(n+1)} \quad (7.13)$$

which is the exact eigenvalue of the Dirac oscillator. When we compare the energies in Eq. (7.13) with the energies found by the method developed in Chapter 6 in Eq. (6.22), we can say that these two energies are identical when the relationship between quantum numbers $n \to n+1$. This results is confirmation of our these two methods.

7.2.1.2 Jaynes–Cumming Hamiltonian with RWA

One type of Hamiltonians given in terms of bosons and fermions are JC Hamiltonians. One of them is known as JC Hamiltonian with rotating wave approximation (RWA).

The RWA method is widely used in quantum optics. This approximation can be applied when the frequency associated with the free evolution of the system is essentially bigger than transition frequencies. Transition frequencies are induced by the interaction between subsystems and/or some external source. The RWA basically consists in neglecting rapidly oscillating (counter rotating) terms in such a way that in the rotating frame the system Hamiltonian becomes time independent or it depends slowly on time. In semiclassical models, when a quantum oscillator and/or two- (or multilevel) level atoms are excited by an external force, RWA is commonly used .

JC Hamiltonian with RWA can be expressed as

$$H_{JC}^{RWA} = \hbar wa_1^\dagger a_1 + \frac{\hbar}{2}w_0\sigma_0 + \kappa(a_1^\dagger\sigma_- + a_1\sigma_+), \quad (7.14)$$

which can be exactly solved. When single two-level atom is placed in the common domain of two cavities interacting with two quantized modes, the Hamiltonian of such a system can be obtained from the modification of the JC Hamiltonian Eq. (5.23).

The JC Hamiltonian with the rotating wave approximation can be obtained by setting $\kappa_2 = \kappa_3 = \kappa_4 = \gamma_1 = \gamma_3 = \gamma_4 = 0$, $\kappa_1 = \gamma_2 = \kappa$, and $\beta = \frac{\hbar w_0}{2}$. In this case, the formula in Eq. (7.7) takes the form:

$$\begin{aligned}&(H_0 - E + \frac{\hbar w_0}{2} + \hbar w)(H_0 - E + \frac{\hbar w_0}{2} - \hbar w)(H_0 - E + \frac{\hbar w_0}{2})\psi_1 \\ &= \kappa^2 (H_0 - E + \beta + \hbar w) a_1 a_1^\dagger \psi_1 \end{aligned} \tag{7.15}$$

The algebraic form of the Hamiltonian can be obtained by combining Eq. (7.15) and the $su(1,1)$ realization, given in Eq. (3.40), yields

$$(\hbar w M' - E + \frac{\hbar w_0}{2} - \hbar w)(\hbar w M' - E + \frac{\hbar w_0}{2})\psi_1 = \kappa^2 (M' + 1)\psi_1. \tag{7.16}$$

If the wave function $\psi_1 = \left|\frac{1}{4}, n\right\rangle$, then the action of Eq. (7.16) on the state leads to the following expression for the eigenvalues of the JC Hamiltonian:

$$E = \left(n - \frac{1}{2}\right)\hbar w + \frac{\hbar w_0}{2} \pm \frac{1}{2}\sqrt{\hbar^2 w^2 + 4\kappa^2 (n+1)}\} \tag{7.17}$$

It is obvious that when the coupling constant κ is zero, then the result is the eigenvalues of the simple harmonic oscillator.

7.2.1.3 Modified Jaynes–Cumming Hamiltonian

The another exactly solvable problem we consider here is the MJC model. The Hamiltonian can be obtained by choosing the parameters: $\kappa_2 = \kappa_4 = \gamma_1 = \gamma_3 = 0$, $\kappa_1 = \gamma_2 = \lambda_1$, and $\kappa_3 = \gamma_4 = \lambda_2$, and $\beta = \hbar w_0$. Then Eq. (7.7) takes the form:

$$\begin{aligned}&(H_0 - E + \hbar w_0 + \hbar w)(H_0 - E + \hbar w_0 - \hbar w)(H_0 - E + \hbar w_0)\psi_1 \\ &= \lambda_1 (H_0 - E + \hbar w_0 + \hbar w)(\lambda_1 a_1 a_1^\dagger + \lambda_2 a_1 a_2^\dagger)\psi_1 + \\ &+ \lambda_2 (H_0 - E + \hbar w_0 + \hbar w)(\lambda_1 a_2 a_1^\dagger + \lambda_2 a_2 a_2^\dagger)\psi_1 \end{aligned} \tag{7.18}$$

We insert Eq. (3.22) and Eq. (3.24), the realization of $su(2)$, into the Eq. (7.18) and we obtain the following expressions:

$$\begin{aligned}&(\hbar w N - E + \hbar w_0 - \hbar w)(\hbar w N - E - \hbar w_0)\psi_1 \\ &= \left((\lambda_1^2 + \lambda_2^2)\left(1 + \frac{N}{2}\right) + (\lambda_1^2 - \lambda_2^2) J_0 + \lambda_1 \lambda_2 (J_+ + J_-)\right)\psi_1. \end{aligned} \tag{7.19}$$

The above equation is not diagonal, but it can be diagonalized by similarity transformation, by the operator

$$O = e^{\frac{\alpha}{2}(J_+ + J_-)}. \tag{7.20}$$

The action of the operator on the generators of $su(2)$ is given by:

$$\begin{aligned} O(J_+ + J_-)O^\dagger &= (J_+ + J_-)\cos(\alpha) + 2J_0\sin(\alpha), \\ OJ_0O^\dagger &= J_0\cos(\alpha) - \frac{J_+ + J_-}{2}\sin(\alpha), \\ ONO^\dagger &= N \end{aligned} \tag{7.21}$$

The transformations of the generators of $su(2)$ by the operator O are given by

$$\begin{aligned} &(\hbar wN - E + \hbar w_0 - \hbar w)(\hbar wN - E - \hbar w_0)\psi_1 \\ &= (\lambda_1^2 + \lambda_2^2)\left(1 + \frac{N}{2}\right) + \left((\lambda_1^2 - \lambda_2^2)\cos(\alpha) + 2\lambda_1\lambda_2\sin(\alpha)\right) J_0\psi_1 \\ &+ \left(\lambda_1\lambda_2\cos(\alpha) - \frac{(\lambda_1^2 - \lambda_2^2)}{(\lambda_1^2 + \lambda_2^2)}\sin(\alpha)\right)(J_+ + J_-)\psi_1. \end{aligned} \tag{7.22}$$

The Hamiltonian takes the diagonal form when the following condition holds:

$$\alpha = \cos^{-1}\left(\frac{(\lambda_1^2 - \lambda_2^2)}{(\lambda_1^2 + \lambda_2^2)}\right). \tag{7.23}$$

The final form of the diagonal Hamiltonian under the condition of Eq. (7.23) is given by

$$\begin{aligned} &(\hbar wN - E + \hbar w_0 - \hbar w)(\hbar wN - E - \hbar w_0)\psi_1 \\ &= (\lambda_1^2 + \lambda_2^2)\left(1 + \frac{N}{2}\right) + (\lambda_1^2 - \lambda_2^2)J_0\psi_1 \end{aligned} \tag{7.24}$$

It is obvious that when $\psi_1 = |j, m\rangle$, which is the eigenstate of the operators J_0, and N, we obtain the following expression for the eigenvalues of the MJC Hamiltonian:

$$E = \left(2j - \frac{1}{2}\right)\hbar w \pm \frac{1}{2}\sqrt{\hbar^2(w - 2w_0)^2 + 4\lambda_1^2\rho_+ + \lambda_2^2\rho_-} \tag{7.25}$$

where

$$\rho_\pm = (j \pm m + 1)$$

The eigenvalue of the MJC Hamiltonian in Eq. (7.25) can be reduced to the same form of eigenvalues of MJC Hamiltonian in Eq. (5.30) when $\lambda_1 = \lambda_2 = 0$. The difference between these two eigenvalues come from quantum numbers.

7.2.2 Quasi-Exactly-Solvable Hamiltonians

7.2.2.1 Hamiltonians of Quantum Dots Including Spin–orbit Coupling

The Hamiltonian H_{dot} describing a two-level fermionic subsystem coupled to two boson modes can be expressed as:

$$\begin{aligned} H_{dot} &= \hbar w(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) + \frac{\hbar w_c}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \\ &-\sqrt{\frac{mw}{4\hbar}}\lambda_R[(a_2^\dagger - a_1)\sigma_+ + (a_2 - a_1^\dagger)\sigma_-] + \frac{1}{2}g\mu B\sigma_0 \end{aligned}$$

QES Hamiltonian of the quantum dot including spin–orbit coupling can be obtained from the formula of Eq. (7.7) by setting the parameters: $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$, $\kappa_4 = -\kappa_1 = -\gamma_2 = \gamma_3 = -\sqrt{\frac{mw}{4\hbar}}\lambda_R$, and $\beta = \frac{1}{2}g\mu B$, then we obtain

$$\begin{aligned}
&(H_0 - E + \frac{1}{2}g\mu B + \hbar w)(H_0 - E + \frac{1}{2}g\mu B - \hbar w)(H_0 - E - \frac{1}{2}g\mu B)\psi_1 \\
&= \frac{mw}{4\hbar}\lambda_R^2(H_0 - E + \frac{1}{2}g\mu B + \hbar w)(a_1 a_1^\dagger + a_1 a_2)\psi_1 + \\
&+ \frac{mw}{4\hbar}\lambda_R^2(H_0 - E + \frac{1}{2}g\mu B - \hbar w)(a_2^\dagger a_1^\dagger + a_2^\dagger a_2)\psi_1
\end{aligned} \tag{7.26}$$

The appropriate Lie algebra of this system is $su(1,1)$ and it can be written in terms of the generators of Eq. (3.30):

$$\begin{aligned}
&(2\hbar w K_0 - E + \frac{1}{2}g\mu B)(2\hbar w K_0 - E + \frac{1}{2}g\mu B - 2\hbar w)(2\hbar w K_0 - E - \frac{1}{2}g\mu B - \hbar w)\psi_1 \\
&= \frac{mw}{4\hbar}\lambda_R^2(2\hbar w K_0 - E + \frac{1}{2}g\mu B)(K_0 + K_- + \frac{M}{2})\psi_1 + \\
&+ \frac{mw}{4\hbar}\lambda_R^2(2\hbar w K_0 - E + \frac{1}{2}g\mu B - 2\hbar w)(K_0 + K_+ - \frac{M+1}{2})\psi_1
\end{aligned} \tag{7.27}$$

The basis function of Eq. (7.27) is $\psi_1 = |k,n\rangle$, and by using the action of the generators on the state, we obtain the three term recurrence relation

$$\begin{aligned}
&(2\hbar w(k+n) - E_+)(2\hbar w(k+n) - E_+ - 2\hbar w)(2\hbar w(k+n) - E_- - \hbar w)\,|k,n\rangle \\
&= \frac{mw}{4\hbar}\lambda_R^2(2\hbar w(k+n) - E_+)\left(\left(n + \frac{1}{2}\right)|k,n\rangle + \sqrt{(2k+n-1)n}\,|k,n-1\rangle\right) + \\
&+ \frac{mw}{4\hbar}\lambda_R^2(2\hbar w(k+n) - E_+ - 2\hbar w)\left((2k+n-1)\,|k,n\rangle + \sqrt{(2k+n)(n+1)}\,|k,n+1\rangle\right)
\end{aligned} \tag{7.28}$$

where $E_\pm = E \mp \frac{1}{2}g\mu B$. This recurrence relation can be written as on polynomial space as

$$\begin{aligned}
&(2\hbar w(k+n) - E_+)(2\hbar w(k+n) - E_+ - 2\hbar w)(2\hbar w(k+n) - E_- - \hbar w)P[k,n] \\
&= \frac{mw}{4\hbar}\lambda_R^2(2\hbar w(k+n) - E_+)\left(\left(n + \frac{1}{2}\right)P[k,n] + \sqrt{(2k+n-1)n}P[k,n-1]\right) + \\
&+ \frac{mw}{4\hbar}\lambda_R^2(2\hbar w(k+n) - E_+ - 2\hbar w)\left((2k+n-1)\,P[k,n] + \sqrt{(2k+n)(n+1)}P[k,n]\right)
\end{aligned} \tag{7.29}$$

The solution of the recurrence relation for each value of $k = 0,1,2,...$ and $n = 0,1,2,...,k$ gives an expression for the eigenvalues of the Hamiltonian H_{dot}. The eigenvalues can be obtained by solve the recurrence relation in Eq. (7.29) for some values of k and r. They are shown in Table 7.2 with appropriate parameters [26].

k	B	n	$\kappa = 0$	$\kappa = 0.25$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 1$
	0	1	2.0	1.99237	1.97141	1.94177	1.90853
1	3/2	1	2.5	2.50388	2.51527	2.53340	2.55719
		1	2.0	1.99237	1.97141	1.94177	1.90853
2	0	2	3.0	2.88655	2.74951	2.57526	2.29013
	3/2	1	2.5	2.50388	2.51527	2.53340	2.55719
		2	4.5	4.50872	4.53364	4.57170	4.61921
		1	3.0	3.01545	3.05983	3.12838	3.21566
	0	2	4.0	3.98895	3.96023	3.29593	3.4065
3		3	5.0	4.84907	4.65625	4.35509	4.15525
		1	4.5	4.50391	4.51562	4.53515	4.56247
	3/2	2	6.5	6.50782	6.53136	6.57083	6.62648
		3	8.5	8.51174	8.5472	8.60698	8.69173

Table 7.2: The eigenvalues of Hamiltonian of quantum dots including spin–orbit coupling with parameters $\hbar = 1; m = 1; w = 1; \mu = 1; g = 2$.

7.2.2.2 Jaynes–Cumming Hamiltonian without RWA

Another type of JC Hamiltonian is the JC Hamiltonian without rotating wave approximation (RWA) and can be expressed as

$$H_{JC} = \hbar w a_1^\dagger a_1 + \frac{\hbar}{2} w_0 \sigma_0 + \kappa(\sigma_- + \sigma_+)(a_1^\dagger + a_1), \tag{7.30}$$

The JC Hamiltonian without rotating wave approximation is one of such QES problems that can be associated to the Hamiltonian of Eq. (7.1) with the choices of the parameters: $w_2 = \kappa_3 = \kappa_4 = \gamma_3 = \gamma_4 = 0$, $\kappa_1 = k_2 = \gamma_1 = \gamma_2 = \kappa$, $\beta = \frac{\hbar w_0}{2}$, and $w_1 = w$. Thus, the general expression of Eq. (7.7) takes the form:

$$\begin{aligned} &(H_0 + E_+)(H_0 + E_-)(H_0 + E_+ - \hbar w)\psi_1 \\ &= \kappa^2 (H_0 + E_+)(a_1^2 + a_1 a_1^\dagger)\psi_1 + \kappa^2 (H_0 + E_-)(a_1^\dagger a_1 + a_1^{\dagger 2})\psi_1 \end{aligned} \tag{7.31}$$

where $E_\pm = \left(-E + \frac{\hbar w_0}{2} \pm \hbar w\right)$. The JC Hamiltonian can be expressed in terms of the $su(1,1)$ generators given in Eqs. (3.39,3.40):

$$\begin{aligned} &(\hbar w M' + E_+)(\hbar w M' + E_-)(\hbar w M' + E_+ - \hbar w)\psi_1 \\ &= \kappa^2 (\hbar w M' + E_+)(2L_0 + M' + 1)\psi_1 + \kappa^2 (\hbar w M' + E_-)(2L_+ + M')\psi_1. \end{aligned} \tag{7.32}$$

Here the Bargmann index k is $\frac{1}{4}$ for the even state $\psi_1 = \left|\frac{1}{4}, 2n\right\rangle$ and $k = \frac{1}{4}$ for odd state $\psi_1 = \left|\frac{3}{4}, 2n+1\right\rangle$. Under the action of the operators on the even state $\left|\frac{1}{4}, 2n\right\rangle \equiv |2n\rangle$ we obtain the following recurrence relation:

$$\begin{aligned} &(2\hbar w n + E_+)(2\hbar w n + E_-)(2\hbar w n + E_+ - \hbar w)|2n\rangle \\ = \;& \kappa^2 (\hbar w M' + E_+)\left(\sqrt{2n(2n-1)}\,|2n-2\rangle + (2n+1)|2n\rangle\right) \\ &+\kappa^2 (\hbar w M' + E_-)\left(\sqrt{(2n+1)(2n+2)}\,|2n+2\rangle + 2n\,|2n\rangle\right) \end{aligned} \tag{7.33}$$

The eigenvalues can be obtained from the recurrence relation for each value of $n = 0, 1, 2, ..., k$ gives an expression for the eigenvalues of the Hamiltonian H_{JC}. From the recurrence relations in Eq. (7.33) eigenvalues are obtained for some value of n Table 7.3.

n	$\kappa = 0$	$\kappa = 0.25$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 1.0$
0	0.5	0.559017	0.707107	0.901388	1.11803
1	1.5	1.62943	1.67962	1.71663	1.76318
2	2.5	2.70686	2.61815	2.32082	2.4826
3	3.5	3.76281	4.04238	4.45241	4.71838
4	4.5	4.77238	4.47296	4.45971	4.29421
5	5.5	5.87409	6.22353	6.59523	6.54727

Table 7.3: The eigenvalues of Jaynes–Cumming Hamiltonian without RWA with corresponding parameters $\hbar = 1; m = 1; w = 1; w_0 = 1$

Here we have systematically discussed exact and QES of the Hamiltonian of Eq. (7.1), within the framework of $su(2)$ and $su(1, 1)$ Lie algebras. The technique given here can be used in determining the spectrum of the variety of physical systems. We have shown that our formulation leads to the exact or quasi-exact solution of the problems of the various physical systems.

CHAPTER 8

CONCLUSION

In this thesis, we have given the connection of the Lie algebraic techniques to the definition spectra of physical systems. For this goal, the well known boson realizations of $su(2)$, and $su(1,1)$ Lie algebras in the literature have been used and boson-fermion realizations of $osp(2,1)$, and $osp(2,2)$ Lie superalgebra have been constructed. These algebra types consist of very large physical potential Hamiltonians. The realizations of this algebra are firstly constructed with creation and annihilation operators in Hilbert space. Lie algebras have been constructed with bosons whereas the Lie superalgebra have been constructed with fermions (Pauli matrices). In other words, the Pauli matrices represent $\mathbb{Z}_2$-grading.

After realizations of Lie (super)algebra, the similarity transformations of boson and fermion operator have been constructed to reduce these Hamiltonian systems, which include multi boson operators or multi boson-fermion operators, to single-boson Hamiltonian. These transformations are useful methods to find the full spectrum of exactly solvable and quasi–exactly solvable potentials.

In the applications of the Lie (super)algebra, interacting electrons in a QD have a special place due to the application of it in nanotechnology. A different treatment of interacting electrons in a QD in the presence of an external magnetic field has been done. For this problem, an algebraic framework is presented to show that the corresponding Hamiltonian possesses a hidden $sl(2,R)$ algebraic structure and we have obtained somewhat simpler recurrence relations which are leading to exact eigenvalues of the Hamiltonian. Additionally, we have obtained the spectrum of an electron in quantum dot in the presence of two phenomena, magnetic field and spin–orbit coupling, in quasi–exact frame. It is shown that under these condition these Hamiltonian possesses $osp(2,2)$ hidden symmetries.

Also, we have constructed a set of potentials by defining the expression of Dirac oscillator in two dimensions. These potentials are transformed into Shrödinger-like differential equations to find the eigenfunctions and eigenvalues.

And surprisingly, the transformed equation looks like similar to the equation of supersymmetric quantum mechanics systems. After this transformation, the function $W(r)$ has been obtained in the form of Schrödinger equation form. This function leads to interesting results. We have multiple choices of vector potential $A(r)$ and scalar potential $V(r)$ to generate the Shrödinger equation including some potentials. These calculations have been done in two categories as exactly solvable potentials and quasi–exactly solvable potentials. The solution and the energy spectrum of the (2+1)-dimensional Dirac equation including these potentials have been analyzed. From the SUSYQM view, these method can be extended to find the solutions of the other physical potentials to find the whole spectrum.

We have also discussed the exact solution of Klein-Gordon equation including equal scalar and vector Eckart potential by using SUSYQM. We have shown that both eigenvalues and eigenfunctions of the related Klein-Gordon equation can be obtained in the closed form for the Eckart potential.

Finally, an alternative method has been generated to obtain the solutions of some physical Hamiltonians for both relativistic and nonrelativistic particles moving in a potential.

CHAPTER 9

FURTHER STUDIES

By this thesis, we have completed the realization of $su(2)$, $su(1,1)$, $osp(2,1)$, $osp(2,2)$ with some applications in physics. The introduced work can be extended to involve other quasi–exactly and exactly solvable potentials leading to their spectrum. By using the generators of $su(2)$ and $su(1,1)$ one can have transitions either from one bound state to another, or from one scattering state to another. Nevertheless, none of these generators connect a bound state to a scattering states. To overcome this problem we need a larger group that includes both the bound and the scattering group as a subgroup which are $SU(2)$ and $SU(1,1)$ respectively. This group is called $SP(4,R)$ group. The group $Sp(4,R)$ thus provides a unified framework within which both bound and scattering states can be discussed. We plan to construct this interesting group that contains both generators of $su(2)$ and $su(1,1)$ algebra with bosons.

Another outlook is to consider the boson-fermion realizations of other Lie superalgebra and especially exceptional lie superalgebras (G_2, G_3, F_4, $D(2,1;\alpha)$, ...) with application in physics.

Moreover, we also intend to investigate application of Lie (super)algebra to relativistic equations involving the Dirac equation, and the Klein–Gordon equation since in literature, there are no more studies on the relation of these equation within Lie superalgebra. In this study, we have discussed the exact solution of Klein-Gordon equation including equal scalar and vector Eckart potential by using SUSYQM. The present technique may be used in treating some interesting problems with position-dependent mass moving in relativistic frame. Finally, we would like to mention that the method discussed here can be generated for other potentials with the context of the quasi–exact solvability.

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