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MSc THESIS

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Diffraction of Plane Waves by an Impedance Half-Plane in Cold Plasma

DEPARTMENT OF ELECTRICAL AND
ELECTRONICS ENGINEERING

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ÇUKUROVA ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

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YÜKSEK LİSANS TEZİ

ELEKTRİK-ELEKTRONİK ANA BİLİM DALI

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Bu çalışmada soğuk plazma içindeki bir yarım-düzlemden düzlemsel dalgaların kırınım problemi incelenmiştir. Bir yarım-düzlemden kırınım, kanonik bir problemdir ve çeşitli karmaşık nesnelerden saçılma problemlerinin modellenmesinde kullanılabilmektedir. Bu sınır-değer probleminin Fourier dönüşümü tekniğiyle formüle edilmesiyle matris Wiener-Hopf denklemine ulaşılır. Elde edilen çekirdek matris Daniele-Khrapkov yöntemiyle faktorize edilebilecek formdadır. Dielektrik tensörünün bir elemanı olan ε_2, $\varepsilon_2 \to 0$ yapılırsa matris Wiener-Hopf denklemi daha basit bir forma indirgenir. Bu, frekansın ω_c'ye (cyclotron frekansı) göre çok büyük, ω_p (plazma frekansı) ile ise aynı derecede alınmasıyla veya sadece dc manyetik alan vektörünün sıfır kabul edilmesiyle mümkündür. Alan integrallerinin semer noktası yöntemi ile asimptotik olarak hesaplanması sonucunda yüksek-frekans kırınım katsayıları elde edilmiştir. Burada elde edilen çözümün doğrulanması amacıyla; $\varepsilon_1 = 1$ durumundaki ifadenin, daha önce izotrop ve homojen ortamdaki empedans yarım-düzlem için elde edilen sonuçlarla aynı olduğu gösterilmiştir.

Anahtar Kelimeler : Soğuk Plazma, Empedans Yarım Düzlem, Wiener-Hopf Tekniği, Maliuzhinetz Fonksiyonu

ABSTRACT

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In the present work, the problem of plane wave diffraction by an impedance half-plane in cold plasma is investigated. Diffraction by an impedance half-plane is a canonical configuration and may be used in modelling scattering from a variety of complex objects. The boundary-value problem corresponding to this canonical structure is formulated by Fourier transform technique, and leads to a matrix Wiener-Hopf equation. The resulting matrix is expressed in a form convenient to be factorized by Daniele-Khrapkov method, which yields a formal solution to the problem under consideration. The matrix Wiener-Hopf equation is reduced to a simpler form by assuming $\varepsilon_2 \rightarrow 0$, where ε_2 is an element of the tensor of dielectric permittivity. This occurs; if the operating frequency is assumed to be very large compared to ω_c (the cyclotron frequency), while it is at the same order with ω_p (the plasma frequency), or if the amplitude of the dc magnetic field vector is taken as zero. Asymptotic evaluation of the field integrals yield the high-frequency diffraction coefficients where the field expressions are obtained by the standard saddle-point technique. As a verification of the solution obtained here, it is shown that the expressions for the case $\varepsilon_1 = 1$ is identical to the previously obtained results for an impedance half-plane in an isotropic and homogeneous medium.

Keywords : Cold Plasma, Impedance Half Plane, Wiener-Hopf Technique, Maliuzhinetz Function

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NOTATIONS

D	Diffraction coefficient
E	Electric field intensity
$G(\alpha)$	Kernel Matrix
H_{dc}	DC magnetic field vector
$M_{\pi}(z)$	Maliuzhinetz function
$\Gamma(\alpha)$	Square root function $\sqrt{k^2-\alpha^2}$
u	Scattered field
u^d	Diffracted field
$\hat{\varepsilon}$	Tensor of dielectric permittivity of cold plasma
ω	Operating frequency
ω_c	Cyclotron frequency
ω_p	Plasma frequency

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1. INTRODUCTION

Knowledge of the characteristics of the ionosphere is based largely upon its effect on radio waves, either those that penetrate in the case of signals from rockets and satellites, or those that are reflected from in the case of signals from earth-based transmitters. Experimentally it is found that at night, signals from the earth-based transmitters in the broadcast frequency range are reflected back, but in the daytime the reflected signal is very weak or entirely absent. As the frequency is raised, however, these daytime reflected waves become stronger, and for frequencies between 10 and 30 MHz, they may provide strong signals over distances of several thousand miles. As the frequency is increased still higher, a point is reached where the waves cease to be reflected back, but instead, penetrate to ionosphere to be lost in the outer space. Thus there is a range of frequencies roughly between 3 and 30 MHz where, although the surface wave is greatly attenuated, long-distance transmission between points on the earth's surface may still occur because of reflections from the ionosphere. In general, these "sky-wave" signals are less stable than ground-wave signals, their strength depending upon the frequency, and upon the condition of the ionosphere. Although long-distance ionospheric propagation does not have the stable characteristics of short-distance ground-wave propagation, it does, in general, a predictable, and therefore usable, means of radio communication.

The ionosphere is an upper region of the atmosphere where gas particles are ionized, thereby forming a plasma that contains free electrons and positive ions. The presence of free electrons enables the ionosphere to reflect high-frequency waves and control radio communication around the earth.

The electrons and ions of the ionosphere are produced by shortwave solar radiation and cosmic particles that penetrate the upper atmosphere and ionise molecules and atoms.

It is possible to compute the ion production rate q at each altitude. Since the atmospheric composition varies with altitude, it follows that q varies with altitude. When q is balanced against electron loss processes, such as recombination, there results a steady-state value of electron number density n_e.

Large values of n_e occur from roughly 50 km to several hundred kilometers. This ionospheric range is divided into three principal regions or layers; D, is below about 90 km; E, is between 90 and 150 km; and F, is above 150 km.

These values of n_e vary with the intensity of incident solar radiation and, consequently, vary with time of day and of year. There are also less predictable variations due to solar disturbances. Superimposed on the smooth variations of n_e are small-scale irregularities that are caused by atmospheric waves and plasma waves.

Electromagnetic waves propagate through a plasma in much the same manner as through any dielectric material. To begin an analysis of this wave propagation, let us assume that the plasma is made up of electrons and ions and that the ions are relatively immobile (due to their mass) in comparison with the electrons. Maxwell's equations in time-varying form must be combined with the equation of motion for electrons where the electron motions enter Maxwell's equations in the electron current density and electron charge density. These are given by $\mathbf{J}(\mathbf{r},t) = \mathbf{n}(\mathbf{r},t)e\mathbf{v}(\mathbf{r},t)$ and $\rho(\mathbf{r},t) = \mathbf{n}(\mathbf{r},t)e$ in which $\mathbf{n}(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ are time-varying electron density and electron velocity. If it is assumed that the electron oscillations are small in amplitude and sinusoidal in form, $\mathbf{n}(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ may be written as $\mathbf{n}(\mathbf{r},t) = N + \mathrm{Re}\{\mathbf{n}(\mathbf{r})\exp(-i\omega t)\}$ and $\mathbf{v} = \mathrm{Re}\{\mathbf{v}(\mathbf{r})\exp(-i\omega t)\}$ in which N is the steady (or *ambient*) electron density; the velocity is assumed to have no steady (or *drift*) component. Since the oscillations are small in amplitude, current density expression may be approximated by $\mathbf{J}(\mathbf{r}, t) = Ne\mathrm{Re}\{\mathbf{v}(\mathbf{r})\exp(-i\omega t)\}$ in which the product of two small quantities is assumed to be negligible.

The equation of motion with zero external magnetic field is $e\mathbf{E}(\mathbf{r}) = -i\omega m\mathbf{v}(\mathbf{r})$. By substituting the equation of motion into Maxwell's equations, the following equation can easily be obtained

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\varepsilon_v\left(1 - \frac{Ne^2}{\omega^2 m\varepsilon_v}\right)\mathbf{E}(\mathbf{r}) \qquad (1)$$

which suggests that the plasma can be represented by a dielectric permittivity given by $\varepsilon = \varepsilon_v[1-(\omega_p / \omega)^2]$. Here, ω_p is called the *plasma frequency* which is given by $\omega_p^2 = Ne^2 / m\varepsilon_v$ with ε_v denoting the permittivity where no external magnetic field exists.

The great advantage of the plasma-permittivity concept is that, all the existing solutions for wave propagation in a dielectric can be used in order to study wave propagation in a plasma. Wave propagation in a plasma is similar to wave propagation in an ordinary dielectric, provided that $\omega > \omega_p$. But, if $\omega < \omega_p$, then electromagnetic waves are *attenuated* by a plasma. By analogy with a wave guide, ω_p may be regarded as the *cut-off frequency* of a plasma.

One must appreciate that attenuation in a cut-off plasma as described above is not accompanied by the absorption of power and its subsequent conversion into heat; in this respect the plasma is like a cut-off wave guide. Thus a wave passing from free space into a cut-off plasma is not absorbed or transmitted but rather *totally reflected.* It is this type of reflection which causes radio waves to "bounce" off the ionosphere, thus making long-range radio communication possible.

Of course, some power loss is always present in a plasma due to the fact that the electrons frequently collide with gas molecules, ions, or even other electrons. Such collisions cause some of the power carried by electromagnetic wave to be transformed into heat. At frequencies above the plasma frequency these collisional losses cause a propagating wave to be attenuated, and below the plasma frequency the losses cause total reflection to become partial reflection.

Electrons and ions in the ionosphere are influenced not only by the fields of a passing electromagnetic wave but also by the earth's magnetic field, which causes the charged particles to move in circular or spiral paths. The effects of the earth's magnetic field may be taken into account by a suitable modification in the permittivity, in much the same way as for the case in which no external magnetic field is present. That is, Maxwell's equations may be combined with the equation of motion in such a way as to give a new set of Maxwell's equations with a permittivity involving all physical effects.

The relation between magnetic flux density and magnetic field intensity for the case with earth's magnetic field effect is as follows

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 + \mathrm{Re}\{\mathbf{B}(\mathbf{r})\exp(-i\omega t)\} = \mu_v \mathbf{H}(\mathbf{r},t) \tag{2}$$

and the equation of motion is given as,

$$e\mathbf{E} = -i\omega m(1 + i\frac{v}{\omega})\mathbf{v} + e\mathbf{B}_0 \times \mathbf{v}\,, \tag{3}$$

with v denoting the collision frequency.

By arranging the equations appropriately, the permittivity of the plasma may be obtained in dyadic form as

$$\nabla\times\mathbf{H}(\mathbf{r}) = -i\omega\varepsilon_v \overline{\mathbf{K}}.\mathbf{E}(\mathbf{r}) \tag{4}$$

with

$$\overline{\mathbf{K}} = K'\hat{\mathbf{x}}\hat{\mathbf{x}} + jK''\hat{\mathbf{x}}\hat{\mathbf{y}} - jK''\hat{\mathbf{y}}\hat{\mathbf{x}} + K'\hat{\mathbf{y}}\hat{\mathbf{y}} + K_0\hat{\mathbf{z}}\hat{\mathbf{z}}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ denote the unit vectors in the Cartesian coordinate system.

In matrix form, this equation can be written as

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = j\omega\varepsilon_v \begin{bmatrix} K' & jK'' & 0 \\ -jK'' & K' & 0 \\ 0 & 0 & K_0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{5}$$

with K', K'' and K_0 being certain constants dependent on the constitutive parameters of the medium and the operating frequency.

The displacement density (electric flux density) **D** given by

$$\mathbf{D} = \varepsilon_v \overline{\mathbf{K}} . \mathbf{E} \tag{6}$$

is not parallel to **E**, indicating that the magnetized plasma is an *anisotropic* medium.

The presence of the DC magnetic field which is the earth's magnetic field in the ionosphere causes the plasma to become anisotropic and this is usually called magnetoplasma. The temperature of the electron gas in the ionosphere is finite but the motion of electrons are also affected by the pressure variations of the electron gas. The magnetoionic theory which examines characteristics of anisotropic plasma ignores the effect of the finite temperature and the pressure variations, since they are usually small, and therefore it is equivalent to dealing with the electron gas at low temperature. Thus, the magnetoionic theory may be considered as the theory dealing with a *cold plasma*. The tensor of dielectric permittivity for a cold plasma is defined as (Felsen and Marcuvitz, 1973)

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \tag{7}$$

with

$$\varepsilon_1 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-1} \tag{8}$$

$$\varepsilon_2 = \left(\frac{\omega_p}{\omega}\right)^2 \left[\frac{\omega}{\omega_c} - \frac{\omega_c}{\omega}\right]^{-1} \tag{9}$$

and

$$\varepsilon_3 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \tag{10}$$

Here, ω is the operating frequency and ω_p is the plasma frequency

$$\omega_p = (N_e e^2/m\varepsilon_0)^{1/2} \tag{11}$$

with the N_e being the number of free electrons per unit volume and e and m denoting the charge and mass of an electron, respectively. ω_c is called the *cyclotron frequency*

$$\omega_c = |e|\mu_0 H_{dc}/m \tag{12}$$

which is the frequency of a circular motion of an electron in a plane perpendicular to the DC magnetic field. H_{dc} denotes the magnitude of the DC magnetic field vector, and ε_0 and μ_0 are the permittivity and permeability of vacuum, respectively.

Electromagnetic wave propagation through an ionized gas has received considerable attention for many years. In particular, the reflection of radio waves and the transmission from and through the ionosphere have been studied extensively(Dvorak, Ziolkowski and Dudley, 1997; Tippett and Ziolkowski,1991).

The problem of studying antenna characteristics, wave propagation through the plasma, and the radar cross section are of considerable importance. Also, the antenna and wave propagation characteristics of artificial satellites in the ionosphere are important in communication between the vehicle and the earth station.

In this study, scattering of plane waves by a half-plane in cold plasma is investigated. As known, half-plane is the simplest canonical structure and the solution of such a problem can be used to consider the scattering mechanisms of complex objects.

There are numerous publications about half-plane diffraction problem. First, Sommerfeld (1896) solved the problem of diffraction from the edge of a perfectly conducting half-plane, by expanding the problem to the Riemann surface concerning multivalued functions. Magnus (1941), reduced the half-plane diffraction problem to the solution of a singular integral equation. A short time after Magnus, Schwinger (1944) and Copson (1946) solved this integral equation by Wiener-Hopf technique and obtained the same results with Sommerfeld's solutions.

Clemmov (1951) solved the half-plane problem via Wiener-Hopf technique by reducing the problem to dual integral equations. This method is named as *plane wave spectrum method,* since it is based on the technique of expressing field as a superposition of spectrums of plane waves.

Jones (1952), proposed a simpler method to obtain Wiener-Hopf equation for the solution of half-plane problem directly. The basis of this method is to take the Fourier transform of reduced wave equation and then to apply boundary conditions in complex domain in order to obtain Wiener-Hopf equation. An advantage of this formulation called as *Jones Method* is that it eliminates the derivation of integral equations in real space.

Another problem is the one considered by Senior (1952) for the case of plane wave normally incident on a half-plane, with same impedance on both sides. The extension to oblique incidence case of an *E*-polarized plane wave was made by Senior (1960) and to both *E* and *H* polarizations by Williams (1960). The oblique problem leads to simultaneous Wiener-Hopf equations, which can be solved by elementary transformations of the unknown functions.

Maliuzhinets (1959) studied scalar diffraction by a wedge with different face impedances. The Wiener-Hopf technique was not applicable, but Maliuzhinets obtained a complete solution by means of a function-theoretic technique. When the wedge angle is zero, the solution for a half-plane with different face impedances is obtained. But, if formulated directly as a half-plane problem, two simultaneous Wiener-Hopf equations arise and these cannot be uncoupled in an elementary manner. Hurd (1976) succeeded in solving the Wiener-Hopf problem directly using the Wiener-Hopf-Hilbert technique. For oblique incidence case of the two-impedance half-plane problem, solution is presented by Bucci and Franceschetti (1976) via Maliuzhinetz method and by Lüneburg and Serbest via Wiener-Hopf method.

As known, there still is interest for the half-plane problem with various physical properties of the half-plane and the surrounding medium. But, there was no study about half-plane diffraction in cold-plasma; therefore, it is believed that the results of this study have practical importance due to the effect of the ionosphere in communication between various systems.

2. FORMULATION OF THE PROBLEM

The impedance half-plane is located at y = 0 and $x \in (-\infty, 0)$ as shown in Fig-1. The incident field is assumed to be

$$\mathbf{H}^{\mathbf{i}} = \mathbf{e}_{\mathbf{z}} \exp[-ik(x\cos\phi_0 + y\sin\phi_0)] \tag{13}$$

where the amplitude of the magnetic field is 1 A/m and ϕ_0 denotes the incidence angle with respect to the x-axis. Time dependence is assumed to be *exp(-iωt)* and will be suppressed throughout the analysis.

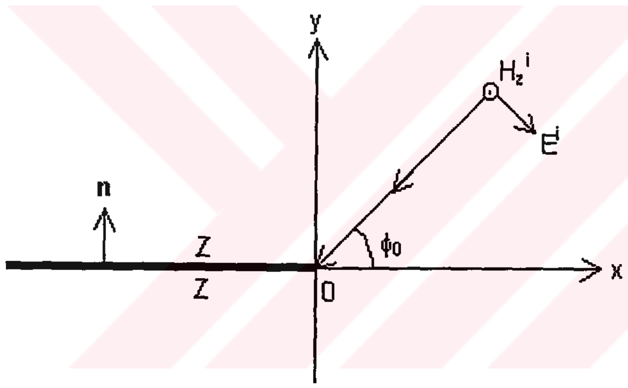

Figure 2.1.The geometry of the canonical structure.

It is well known that Maxwell's equations are valid in plasma with the tensor dielectric permittivity. By using Maxwell's equations along with Eq.(7), the electric field components can be expressed in terms of the magnetic field H_z, as follows :

$$\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}, \tag{14}$$

where $\mathbf{B} = \mu_0\mathbf{H}$, and with $\mathbf{D} = \varepsilon_0\hat{\varepsilon}\mathbf{E}$, then

$$\nabla \times \mathbf{H} = -i\omega\varepsilon_0\hat{\varepsilon}\mathbf{E} \tag{15}$$

where $\mathbf{J} = \sigma\mathbf{E} = 0$ since the conductivity σ of the medium is assumed to be zero.

Now Eqn(15) gives :

$$\mathbf{e}_x \frac{\partial H_z}{\partial y} - \mathbf{e}_y \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0 \tilde{\varepsilon}\mathbf{E} \tag{16}$$

and Eqn(7) is substituted into Eqn(16) :

$$\mathbf{e}_x \frac{\partial H_z}{\partial y} - \mathbf{e}_y \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0 \begin{bmatrix} \varepsilon_1 \mathbf{e}_x \mathbf{e}_x & -i\varepsilon_2 \mathbf{e}_x \mathbf{e}_y & 0 \\ i\varepsilon_2 \mathbf{e}_y \mathbf{e}_x & \varepsilon_1 \mathbf{e}_y \mathbf{e}_y & 0 \\ 0 & 0 & \varepsilon_3 \mathbf{e}_z \mathbf{e}_z \end{bmatrix} \begin{bmatrix} E_x \mathbf{e}_x \\ E_y \mathbf{e}_y \\ E_z \mathbf{e}_z \end{bmatrix}$$

to obtain

$$\mathbf{e}_x \frac{\partial H_z}{\partial y} - \mathbf{e}_y \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0 (\varepsilon_1 E_x \mathbf{e}_x - i\varepsilon_2 E_y \mathbf{e}_x + i\varepsilon_2 E_x \mathbf{e}_y + \varepsilon_1 E_y \mathbf{e}_y).$$

According to the above equation :

$$\frac{\partial H_z}{\partial y} = -i\omega\varepsilon_0\varepsilon_1 E_x - \omega\varepsilon_0\varepsilon_2 E_y \tag{17}$$

and

$$-\frac{\partial H_z}{\partial x} = \omega\varepsilon_0\varepsilon_2 E_x - i\omega\varepsilon_0\varepsilon_1 E_y. \tag{18}$$

is written. If E_x and E_y are derived using Eq(17) and Eq(18),

$$E_x = \frac{i\varepsilon_1}{\omega\varepsilon_0(\varepsilon_1^2 - \varepsilon_2^2)} \frac{\partial H_z}{\partial y} + \frac{\varepsilon_2}{\omega\varepsilon_0(\varepsilon_1^2 - \varepsilon_2^2)} \frac{\partial H_z}{\partial x}, \tag{19}$$

and

$$E_y = \frac{\varepsilon_2}{\omega\varepsilon_0(\varepsilon_1^{\,2}-\varepsilon_2^{\,2})}\frac{\partial H_z}{\partial y} - \frac{i\varepsilon_1}{\omega\varepsilon_0(\varepsilon_1^{\,2}-\varepsilon_2^{\,2})}\frac{\partial H_z}{\partial x}. \tag{20}$$

are obtained.

The following equation which is obtained easily from Maxwell's equations

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_0 H_z \tag{21}$$

is rewritten as, by substituting Eqs.(19-20)

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \omega^2\varepsilon_0\mu_0\left(\frac{\varepsilon_1^{\,2}-\varepsilon_2^{\,2}}{\varepsilon_1}\right)H_z = 0. \tag{22}$$

This equation shows that H_z satisfies Helmholtz's equation where the propagation constant is

$$k^2 = k_0^{\,2}k_p \tag{23}$$

with

$$k_0^{\,2} = \omega^2\varepsilon_0\mu_0 \qquad \text{and} \qquad k_p = \frac{\varepsilon_1^{\,2}-\varepsilon_2^{\,2}}{\varepsilon_1} \tag{24}$$

The total field is taken as the sum of incident and scattered field terms for all y:

$$\mathbf{E}^t(\mathbf{H}^t) = \mathbf{E}^i(\mathbf{H}^i) + \mathbf{E}(\mathbf{H}) \tag{25}$$

The same impedance is imposed on both sides of the half plane and plane surfaces are subject to Leontovich boundary condition, which is given as :

$$\mathbf{n}\times\mathbf{n}\times\mathbf{E}(x,y)=\pm Z_0 Z\mathbf{n}\times\mathbf{H}(x,y),\qquad \text{for } y=\mp 0. \tag{26}$$

Using Leontovich boundary condition, with $\mathbf{n}=\mathbf{y}$

$$E_x{}'(x,y)=\mp Z_0 Z H_z{}'(x,y),\qquad \text{for } y=\mp 0 \tag{27}$$

is obtained for the geometry of the problem. If Eq(19) is substituted into Eq(27),

$$\frac{\partial H_z{}'(x,y)}{\partial y}-i\frac{\varepsilon_2}{\varepsilon_1}\frac{\partial H_z{}'(x,y)}{\partial x}=\pm i\frac{k^2}{\omega\mu_0}Z_0 Z H_z{}'(x,y)\ \text{ for } y=\mp 0, \tag{28}$$

is obtained. To determine $H_z(x,y)$, the solution of the Helmholtz equation for the total field

$$\Delta H'_z(x,y)+k^2 H'_z(x,y)=0,\qquad (x,y)\notin S, \tag{29a}$$

should be obtained such as to satisfy boundary conditions, radiation condition and edge condition with k^2 being defined as in Eq(23). Using Eq(25) and noting that the incident magnetic field H^i_z already satisfies Helmholtz equation, then Eq(29a) becomes

$$\Delta H_z(x,y)+k^2 H_z(x,y)=0,\qquad (x,y)\notin S. \tag{29b}$$

By using the method of separation of variables, the following integral representations can be introduced for the scattered fields :

$$u(x,y)=\begin{cases}\int_L \left[A(\alpha)e^{i\Gamma(\alpha)y}+A_1(\alpha)e^{-i\Gamma(\alpha)y}\right]e^{-i\alpha x}d\alpha, & y>0\\ \int_L \left[B(\alpha)e^{-i\Gamma(\alpha)y}+B_1(\alpha)e^{i\Gamma(\alpha)y}\right]e^{-i\alpha x}d\alpha, & y<0\end{cases} \tag{30}$$

where $u(x,y)$ denotes $H_z(x,y)$.

Here, $\Gamma(\alpha)=\sqrt{k^2-\alpha^2}$ is defined in the appropriately cut complex α-plane with $\Gamma(0)=+k$ and $\Gamma(\alpha)=i|\alpha|$ for $\alpha\to\pm\infty$ (Fig-2.2). The integration line L is any straight line parallel to the real axis in the regularity band, $Im(-k)<Im(\alpha)<Im(k)$. For α approaching infinity in both directions on the integration line L, $\Gamma(\alpha)\to i|\alpha|$ and to satisfy radiation condition for $y>0$ and $y<0$ as $|y|\to\infty$, A_1 and B_1 should be zero. Then Eq(30) is reduced to

$$u=\begin{cases}\int_L A(\alpha)e^{-i\alpha x+i\Gamma(\alpha)y}d\alpha, & y>0\ ,\\ \int_L B(\alpha)e^{-i\alpha x-i\Gamma(\alpha)y}d\alpha, & y<0\ .\end{cases} \tag{31}$$

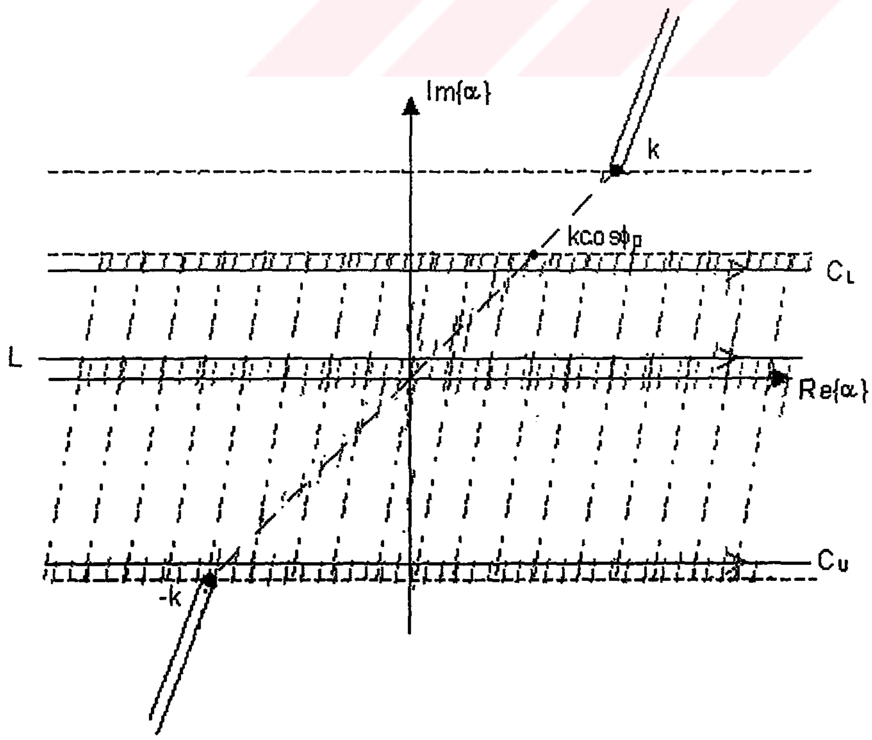

Figure 2.2.The regularity band and the integration lines in the complex α-plane.

The unknown spectral functions $A(\alpha)$ and $B(\alpha)$ must be determined with the aid of the boundary and continuity conditions given for the respective regions of space. In explicit form, the boundary conditions for $x < 0$ are

$$\left\{\frac{\partial u^i}{\partial y}+\frac{\partial u}{\partial y}-i\frac{\varepsilon_2}{\varepsilon_1}\frac{\partial u^i}{\partial x}-i\frac{\varepsilon_2}{\varepsilon_1}\frac{\partial u}{\partial x}+i\frac{k^2}{\omega\mu_0}Z_0Zu^i+i\frac{k^2}{\omega\mu_0}Z_0Zu\right\}\Bigg|_{y=+0}=0 \qquad (32)$$

$$\left\{\frac{\partial u^i}{\partial y}+\frac{\partial u}{\partial y}-i\frac{\varepsilon_2}{\varepsilon_1}\frac{\partial u^i}{\partial x}-i\frac{\varepsilon_2}{\varepsilon_1}\frac{\partial u}{\partial x}-i\frac{k^2}{\omega\mu_0}Z_0Zu^i-i\frac{k^2}{\omega\mu_0}Z_0Zu\right\}\Bigg|_{y=-0}=0 \qquad (33)$$

for y > 0 and y < 0. By substituting the integral representation given by Eq(31) and Eq(13) into Eqs(32-33), the following equations are obtained :

$$\int_L A(\alpha)\left[i\Gamma(\alpha)-\frac{\varepsilon_2}{\varepsilon_1}\alpha+i\frac{k^2}{\omega\mu_0}Z_0Z\right]e^{-i\alpha x}d\alpha=$$

$$\left[ik\sin\phi_0+\frac{\varepsilon_2}{\varepsilon_1}k\cos\phi_0-i\frac{k^2}{\omega\mu_0}Z_0Z\right]e^{-ikx\cos\phi_0}, \qquad \text{for } x<0 \qquad (34)$$

and

$$\int_L B(\alpha)\left[-i\Gamma(\alpha)-\frac{\varepsilon_2}{\varepsilon_1}\alpha-i\frac{k^2}{\omega\mu_0}Z_0Z\right]e^{-i\alpha x}d\alpha=$$

$$\left[ik\sin\phi_0+\frac{\varepsilon_2}{\varepsilon_1}k\cos\phi_0+i\frac{k^2}{\omega\mu_0}Z_0Z\right]e^{-ikx\cos\phi_0}, \qquad \text{for } x<0. \qquad (35)$$

Similarly, the continuity condition of the fields which is given by

$$u^t(x,+0)-u^t(x,-0)=0 \qquad (36)$$

for x > 0, y = 0 yields :

$$\int_L [A(\alpha) - B(\alpha)] e^{-i\alpha x} d\alpha = 0\,, \tag{37}$$

and the continuity condition of the derivative of the fields which is given by

$$\frac{iZ_0}{k}\left[\frac{\partial}{\partial y}u'(x,+0) - \frac{\partial}{\partial y}u'(x,-0)\right] = 0 \tag{38}$$

for $x > 0$, $y = 0$ yields the equation :

$$\int_L i\Gamma(\alpha)[A(\alpha) + B(\alpha)] e^{-i\alpha x} d\alpha = 0\,. \tag{39}$$

Now, application of the inverse Fourier transform to Eqs(34-35) and Eqs(37-39) gives :

$$A(\alpha)\left[i\Gamma(\alpha) - \frac{\varepsilon_2}{\varepsilon_1}\alpha + i\frac{k^2}{\omega\mu_0}Z_0 Z\right] = U_1(\alpha) + F_1(\alpha)\,, \tag{40}$$

$$B(\alpha)\left[-i\Gamma(\alpha) - \frac{\varepsilon_2}{\varepsilon_1}\alpha - i\frac{k^2}{\omega\mu_0}Z_0 Z\right] = U_2(\alpha) + F_2(\alpha)\,, \tag{41}$$

$$A(\alpha) - B(\alpha) = L_1(\alpha)\,, \tag{42}$$

$$i\Gamma(\alpha)[A(\alpha) + B(\alpha)] = L_2(\alpha)\,. \tag{43}$$

Here, $F_{1,2}(\alpha)$ denote the contributions of the source functions, defined as :

$$F_1(\alpha)=\frac{1}{2\pi}\left[ik\sin\phi_0+\frac{\varepsilon_2}{\varepsilon_1}k\cos\phi_0-i\frac{k^2}{\omega\mu_0}Z_0Z\right]\int_{-\infty}^{0}e^{i(\alpha-k\cos\phi_0)x}dx\,, \tag{44}$$

resulting with

$$F_1(\alpha)=\frac{1}{2\pi i(\alpha-k\cos\phi_0)}\left[ik\sin\phi_0+\frac{\varepsilon_2}{\varepsilon_1}k\cos\phi_0-i\frac{k^2}{\omega\mu_0}Z_0Z\right], \tag{45}$$

and

$$F_2(\alpha)=\frac{1}{2\pi i}\left[ik\sin\phi_0+\frac{\varepsilon_2}{\varepsilon_1}k\cos\phi_0+i\frac{k^2}{\omega\mu_0}Z_0Z\right]\int_{-\infty}^{0}e^{i(\alpha-k\cos\phi_0)x}dx\,, \tag{46}$$

resulting with

$$F_2(\alpha)=\frac{1}{2\pi i(\alpha-k\cos\phi_0)}\left[ik\sin\phi_0+\frac{\varepsilon_2}{\varepsilon_1}k\cos\phi_0+i\frac{k^2}{\omega\mu_0}Z_0Z\right]. \tag{47}$$

While accomplishing the integrations here, it was required to deform the regularity band into $Im(-k) < Im(\alpha) < Im(k\cos\phi_0)$ as shown in Fig-2.2.

The unknown spectral functions $L_{1,2}$ and $U_{1,2}$ are regular in the lower half of the complex α-plane ($Im\alpha < Imk\cos\phi_0$) and in the upper half of the complex α-plane ($Im\alpha > Im(-k)$), respectively, whose integral expressions are :

$$U_1(\alpha)=\frac{1}{2\pi}\int_0^{\infty}\left(\frac{\partial}{\partial y}-i\frac{\varepsilon_2}{\varepsilon_1}\frac{\partial}{\partial x}+i\frac{k^2}{\omega\mu_0}Z_0Z\right)u'(x,y)\big|_{y=+0}\,e^{i\alpha x}dx\,, \tag{48}$$

$$U_2(\alpha)=\frac{1}{2\pi}\int_0^{\infty}\left(\frac{\partial}{\partial y}-i\frac{\varepsilon_2}{\varepsilon_1}\frac{\partial}{\partial x}-i\frac{k^2}{\omega\mu_0}Z_0Z\right)u'(x,y)\big|_{y=-0}\,e^{i\alpha x}dx\,, \tag{49}$$

$$L_1(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{0}\left(u'(x,+0)-u'(x,-0)\right)e^{i\alpha x}dx\,, \tag{50}$$

$$L_2(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{0}\left(\frac{\partial}{\partial y}u'(x,y)\,|_{y=+0}-\frac{\partial}{\partial y}u'(x,y)\,|_{y=-0}\right)e^{i\alpha x}dx\,. \tag{51}$$

$A(\alpha)$ and $B(\alpha)$ are eliminated from Eqs.(42-43), then $A(\alpha)$ and $B(\alpha)$ are expressed as

$$A(\alpha)=\frac{1}{2}\left[L_1(\alpha)+\frac{1}{i\Gamma(\alpha)}L_2(\alpha)\right], \tag{52}$$

$$B(\alpha)=\frac{1}{2}\left[-L_1(\alpha)+\frac{1}{i\Gamma(\alpha)}L_2(\alpha)\right], \tag{53}$$

and Eqs.(52-53) are substituted into Eqs.(40-41). Then the following system of functional equations is obtained :

$$\frac{1}{2}\left[i\Gamma(\alpha)-\frac{\varepsilon_2}{\varepsilon_1}\alpha+i\frac{k^2}{\omega\mu_0}Z_0Z\right]L_1(\alpha)$$

$$+\frac{1}{2i\Gamma(\alpha)}\left[i\Gamma(\alpha)-\frac{\varepsilon_2}{\varepsilon_1}\alpha+i\frac{k^2}{\omega\mu_0}Z_0Z\right]L_2(\alpha)=U_1(\alpha)+F_1(\alpha)\,, \tag{54}$$

$$-\frac{1}{2}\left[-i\Gamma(\alpha)-\frac{\varepsilon_2}{\varepsilon_1}\alpha-i\frac{k^2}{\omega\mu_0}Z_0Z\right]L_1(\alpha)$$

$$+\frac{1}{2i\Gamma(\alpha)}\left[-i\Gamma(\alpha)-\frac{\varepsilon_2}{\varepsilon_1}\alpha-i\frac{k^2}{\omega\mu_0}Z_0Z\right]L_2(\alpha)=U_2(\alpha)+F_2(\alpha)\,, \tag{55}$$

The addition and subtraction of Eqs(54-55) yield,

$$\left[i\Gamma(\alpha)+i\frac{k^2}{\omega\mu_0}Z_0 Z\right]L_1(\alpha)+\left[-\frac{1}{i\Gamma(\alpha)}\frac{\varepsilon_2}{\varepsilon_1}\alpha\right]L_2(\alpha)=\hat{U}_1(\alpha)+\hat{F}_1(\alpha) \tag{56}$$

$$-\left[\frac{\varepsilon_2}{\varepsilon_1}\alpha\right]L_1(\alpha)+\left[1+\frac{k^2}{\omega\mu_0\Gamma(\alpha)}Z_0 Z\right]L_2(\alpha)=\hat{U}_2(\alpha)+\hat{F}_2(\alpha), \tag{57}$$

where

$$\hat{U}_1(\alpha)=U_1(\alpha)+U_2(\alpha), \tag{58}$$

$$\hat{U}_2(\alpha)=U_1(\alpha)-U_2(\alpha), \tag{59}$$

and

$$\hat{F}_1(\alpha)=F_1(\alpha)+F_2(\alpha)=\frac{ik\sin\phi_0+(\frac{\varepsilon_2}{\varepsilon_1})k\cos\phi_0}{i\pi(\alpha-k\cos\phi_0)}, \tag{60}$$

$$\hat{F}_2(\alpha)=F_1(\alpha)-F_2(\alpha)=\frac{-\frac{k^2}{\omega\mu_0}Z_0 Z}{\pi(\alpha-k\cos\phi_0)}. \tag{61}$$

The system of equations given by Eqs(56-57) is known as a *matrix Wiener-Hopf system* and can be expressed in matrix form as :

$$G(\alpha)L(\alpha)=\hat{U}(\alpha)+\hat{F}(\alpha), \tag{62}$$

with

$$G(\alpha)=\begin{bmatrix} i\Gamma(\alpha)+i\dfrac{k^2}{\omega\mu_0}Z_0Z & i\dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{\alpha}{\Gamma(\alpha)} \\ -\dfrac{\varepsilon_2}{\varepsilon_1}\alpha & 1+\dfrac{1}{\Gamma(\alpha)}\dfrac{k^2}{\omega\mu_0}Z_0Z \end{bmatrix}. \tag{63}$$

Here, $L(\alpha), \hat{U}(\alpha)$ and $\hat{F}(\alpha)$ denote the following column vectors :

$$L(\alpha)=\begin{bmatrix} L_1(\alpha) \\ L_2(\alpha) \end{bmatrix}, \qquad \hat{U}(\alpha)=\begin{bmatrix} \hat{U}_1(\alpha) \\ \hat{U}_2(\alpha) \end{bmatrix} \quad \text{and} \quad \hat{F}(\alpha)=\begin{bmatrix} \hat{F}_1(\alpha) \\ \hat{F}_2(\alpha) \end{bmatrix}. \tag{64}$$

3. SOLUTION OF THE MATRIX WIENER-HOPF EQUATION

As known, the solution of a matrix Wiener-Hopf equation is formally identical to the solution procedure of a scalar Wiener-Hopf equation. The classical Wiener-Hopf procedure in solving Eq(63) requires the factorization of the kernel matrix $G(\alpha)$ in the following form :

$$G(\alpha) = G_U(\alpha)G_L(\alpha), \tag{65}$$

where $G_U(\alpha)$ and $G_L(\alpha)$ and their inverses are regular functions of α with algebraic behaviour for $\alpha \to \infty$ in an upper and lower half of the complex α-plane, respectively.

Now, assuming that the kernel matrix $G(\alpha)$ is factorized,

$$G_U(\alpha)G_L(\alpha)L(\alpha) = \hat{U}(\alpha) + \hat{F}(\alpha), \tag{66}$$

is obtained, and simple matrix algebra yields :

$$G_L(\alpha)L(\alpha) = G_U^{-1}(\alpha)\hat{U}(\alpha) + G_U^{-1}(\alpha)\hat{F}(\alpha). \tag{67}$$

Next step is the additive decomposition of the second term on the right-hand side of Eq(67) as :

$$G_U^{-1}(\alpha)\hat{F}(\alpha) = H(\alpha) = H_U(\alpha) + H_L(\alpha) \tag{68}$$

with

$$H_{U,L}(\alpha) = \pm\frac{1}{2\pi i}\int_{C_{U,L}} \left[G_U^{-1}(\tau)\hat{F}(\tau)\right]\frac{d\tau}{(\tau-\alpha)}, \tag{69}$$

where the integration lines C_U and C_L are to be taken as parallel to the real axis (see Fig.-2.2). Then, by substituting (68) into Eq(67)

$$G_L(\alpha)L(\alpha) - H_L(\alpha) = G_U^{-1}(\alpha)\hat{U}(\alpha) + H_U(\alpha), \tag{70}$$

and by principle of analytic continuation, an entire matrix $Q(\alpha)$ can be defined as

$$Q(\alpha) = \begin{cases} G_U^{-1}(\alpha)\hat{U}(\alpha) + H_U(\alpha), & \alpha \in R_U \\ \mathrm{G}_L(\alpha)L(\alpha) - H_L(\alpha), & \alpha \in R_L \end{cases} \tag{71}$$

where R_U denotes the upper half of the complex α-plane defined by $Im\alpha > Im(-k)$ and R_L denotes the lower half of the complex α-plane with $Im\alpha < Im(k\cos\phi_0)$.

By applying the extension of Liouville's theorem together with the asymptotic behaviour of the elements for $\alpha \to \infty$, to each element of the matrices on the right-hand side of Eq(71), it will be seen that the entire matrix $Q(\alpha)$ must be a polynomial matrix. After determining $Q(\alpha)$ uniquely, and explicitly, the solution can be written as follows :

$$\hat{U}(\alpha) = G_U(\alpha)[Q(\alpha) - H_U(\alpha)], \qquad \alpha \in R_U, \tag{72}$$

and

$$L(\alpha) = G_L^{-1}(\alpha)[Q(\alpha) + H_L(\alpha)], \qquad \alpha \in R_L. \tag{73}$$

This completes the formal solution of the matrix Wiener-Hopf problem, and the field expressions can easily be obtained by using these solutions.

3.1. Factorization of the Kernel Matrix

The factorization of the kernel matrix in Eq(63) will be accomplished by *Daniele-Khrapkov method*, which relies on the logarithmic approach being suggested by analogy with the factorization of a scalar function. This approach, in general, derives the multiplicative factors of a scalar function from the additive splitting of the logarithm. This concept is applied to the class of matrices that can be expressed in the following form :

$$G(\alpha) = a_1(\alpha)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_2(\alpha)\begin{bmatrix} l_1(\alpha) & m(\alpha) \\ n(\alpha) & l_2(\alpha) \end{bmatrix}, \tag{74}$$

where $a_{1,2}(\alpha)$ are scalar functions and $l_{1,2}(\alpha)$, $m(\alpha)$ and $n(\alpha)$ are polynomials.

So the first step is to express the kernel matrix in Eq(63) in the form shown by Eq(74). For this purpose, the kernel matrix is written as :

$$G(\alpha) = \begin{bmatrix} i\Gamma(\alpha) & -\dfrac{1}{i\Gamma(\alpha)}\dfrac{\varepsilon_2}{\varepsilon_1}\alpha \\ 0 & \dfrac{1}{\Gamma(\alpha)}\dfrac{k^2}{\omega\mu_0}Z_0 Z \end{bmatrix} + \begin{bmatrix} i\dfrac{k^2}{\omega\mu_0}Z_0 Z & 0 \\ -\dfrac{\varepsilon_2}{\varepsilon_1}\alpha & 1 \end{bmatrix}. \tag{75}$$

and by simple manipulations

$$G(\alpha) = \begin{bmatrix} i\dfrac{k^2}{\omega\mu_0}Z_0 Z & 0 \\ -\dfrac{\varepsilon_2}{\varepsilon_1}\alpha & 1 \end{bmatrix}$$

$$\times\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\dfrac{k^2}{\omega\mu_0}Z_0 Z\Gamma(\alpha)}\begin{bmatrix} \Gamma^2(\alpha) & \dfrac{\varepsilon_2}{\varepsilon_1}\alpha \\ \dfrac{\varepsilon_2}{\varepsilon_1}\alpha\Gamma^2 & \left(\dfrac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2 + \left(\dfrac{k^2}{\omega\mu_0}Z_0 Z\right)^2 \end{bmatrix}\right\} \tag{76}$$

is obtained. So,

$$G(\alpha)=G_1(\alpha)G_2(\alpha), \tag{77}$$

is written, with

$$G_1(\alpha)=\begin{bmatrix} i\dfrac{k^2}{\omega\mu_0}Z_0Z & 0 \\ -\dfrac{\varepsilon_2}{\varepsilon_1}\alpha & 1 \end{bmatrix}, \tag{78}$$

and

$$G_2(\alpha)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\frac{1}{\dfrac{k^2}{\omega\mu_0}Z_0Z\Gamma(\alpha)}\begin{bmatrix} \Gamma^2(\alpha) & \dfrac{\varepsilon_2}{\varepsilon_1}\alpha \\ \dfrac{\varepsilon_2}{\varepsilon_1}\alpha\Gamma^2(\alpha) & \left(\dfrac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2+\left(\dfrac{k^2}{\omega\mu_0}Z_0Z\right)^2 \end{bmatrix}. \tag{79}$$

$G_1(\alpha)$ is a simple polynomial matrix with constant determinant and $G_2(\alpha)$ is in the form of the matrix given in Eq(74) where

$$a_1(\alpha)=1, \qquad a_2(\alpha)=\frac{1}{\dfrac{k^2}{\omega\mu_0}Z_0Z\Gamma(\alpha)},$$

$$l_1(\alpha)=k^2-\alpha^2, \qquad l_2(\alpha)=\left(\frac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2+\left(\frac{k^2}{\omega\mu_0}Z_0Z\right)^2,$$

$$m(\alpha)=\frac{\varepsilon_2}{\varepsilon_1}\alpha^2, \qquad n(\alpha)=\frac{\varepsilon_2}{\varepsilon_1}\alpha(k^2-\alpha^2) \tag{80}$$

Now, for the subsequent factorization process, it is better to express the matrix $G_2(\alpha)$ in a more convenient form, as :

$$G_2(\alpha)=s(\alpha)I+a_2(\alpha)J(\alpha) \tag{81}$$

Here, $s(\alpha)$ denote the half of the trace of $G_2(\alpha)$ and is obtained as

$$s(\alpha)=a_1(\alpha)+\frac{1}{2}a_2(\alpha)[l_1(\alpha)+l_2(\alpha)], \tag{82}$$

which, for the present case yields :

$$s(\alpha)=1+\frac{1}{2}\frac{\omega\mu_0}{k^2Z_0Z\Gamma(\alpha)}\left\{k^2\left[1+\left(\frac{k}{\omega\mu_0}Z_0Z\right)^2\right]-\alpha^2\left[1-\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]\right\}. \tag{83}$$

$J(\alpha)$ is a polynomial matrix, named as the commutant of the matrix $G_2(\alpha)$,

$$J(\alpha)=\begin{bmatrix} l(\alpha) & m(\alpha) \\ n(\alpha) & -l(\alpha) \end{bmatrix}, \tag{84}$$

with

$$l(\alpha)=\frac{1}{2}[l_1(\alpha)-l_2(\alpha)]. \tag{85}$$

For the present case, it gives

$$l(\alpha)=\frac{1}{2}\left\{k^2\left[1-\left(\frac{k}{\omega\mu_0}Z_0Z\right)^2\right]-\alpha^2\left[1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]\right\}. \tag{86}$$

and $m(\alpha)$, $n(\alpha)$ are given in Eq(80).

In order to express the split matrices explicitly, it is convenient to express the matrix $G_2(\alpha)$ in the following form :

$$G_2(\alpha) = \sqrt{g_2(\alpha)}\left[I\cosh(\in) + \frac{1}{\sqrt{f}}J(\alpha)\sinh(\in)\right]. \tag{87}$$

Here, $g_2(\alpha)$ denotes the determinant of the matrix $G_2(\alpha)$ which can be expressed in terms of the eigenvalues $\lambda_{1,2}(\alpha)$ as follows :

$$g_2(\alpha) = \lambda_1(\alpha)\lambda_2(\alpha), \tag{88}$$

with

$$\lambda_{1,2}(\alpha) = a_1(\alpha) \pm a_2(\alpha)\sqrt{f}, \tag{89}$$

$$f(\alpha) = l^2(\alpha) + m(\alpha)n(\alpha), \tag{90}$$

and

$$\in(\alpha) = \frac{1}{2}\ln\left[\frac{\lambda_1(\alpha)}{\lambda_2(\alpha)}\right]. \tag{91}$$

$F(\alpha)$ is defined as

$$F(\alpha) = \frac{\in(\alpha)}{\sqrt{f(\alpha)}} = \frac{1}{2\sqrt{f(\alpha)}}\ln\{\lambda_1(\alpha)/\lambda_2(\alpha)\}. \tag{92}$$

Now, above definitions yield the following for the present case :

$$\lambda_{1,2}(\alpha) = 1 \pm \frac{\omega\mu_0}{2k^2 Z_0 Z\sqrt{k^2-\alpha^2}}$$

$$\times\sqrt{\left\{k^2\left[1-\left(\frac{k}{\omega\mu_0}Z_0 Z\right)^2\right]-\alpha^2\left[1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]\right\}^2+4\left(\frac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2(k^2-\alpha^2)} \quad (93)$$

$$f(\alpha) = \frac{1}{4}\left\{k^2\left[1-\left(\frac{k}{\omega\mu_0}Z_0 Z\right)^2\right]-\alpha^2\left[1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]\right\}^2+\left(\frac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2(k^2-\alpha^2) \quad (94)$$

and

$$F(\alpha) = \frac{1}{\sqrt{\left\{k^2\left[1-\left(\frac{k}{\omega\mu_0}Z_0 Z\right)^2\right]-\alpha^2\left[1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]\right\}^2+4\left(\frac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2(k^2-\alpha^2)}}$$

$$\times\ln\frac{1+\dfrac{\omega\mu_0\sqrt{\left\{k^2\left[1-\left(\frac{kZ_0 Z}{\omega\mu_0}\right)^2\right]-\alpha^2\left[1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]\right\}^2+4\left(\frac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2(k^2-\alpha^2)}}{2k^2 Z_0 Z\sqrt{k^2-\alpha^2}}}{1-\dfrac{\omega\mu_0\sqrt{\left\{k^2\left[1-\left(\frac{kZ_0 Z}{\omega\mu_0}\right)^2\right]-\alpha^2\left[1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]\right\}^2+4\left(\frac{\varepsilon_2}{\varepsilon_1}\alpha\right)^2(k^2-\alpha^2)}}{2k^2 Z_0 Z\sqrt{k^2-\alpha^2}}}. \quad (95)$$

Now, the explicit expressions of the split matrices can easily be written as :

$$G_{2U,L}(\alpha) = \sqrt{g_{2U,L}(\alpha)}\left\{I\cosh\left[\sqrt{f}F_{U,L}(\alpha)\right]+\frac{1}{\sqrt{f}}J(\alpha)\sinh\left[\sqrt{f}\,F_{U,L}(\alpha)\right]\right\} \quad (96)$$

3.2. Modification of the Theory for the Exponential Behaviour at Infinity

A difficulty which sometimes arises in the application of Daniele's or Khrapkov's method is that the split matrices may be exponentially large in their respective regions of regularity. This prohibits the use of Liouville's theorem at a later stage of the Wiener-Hopf procedure. Khrapkov has pointed out the possibility of such a difficulty and also has given a condition to determine whether the split matrices have an exponential behaviour or not. This condition, named as regularity condition, is given by Khrapkov as follows :

$$\int_{-\infty}^{\infty} t^{\varsigma-1} \in (t) \frac{dt}{\sqrt{f(t)}}, \qquad (\varsigma = 1, 2, \ldots) \tag{97}$$

where ς is the greatest integer in $[(q-1)/2]$ and q denotes the order of the polynomial $f(\alpha)$.

It is supposed that

$$f(\alpha) = O(\alpha^q) \text{ as } |\alpha| \to \infty. \tag{98}$$

Then the following asymptotic expansions are valid

$$\sqrt{f} = \alpha^{q/2}\left[c_0 + \frac{c_1}{\alpha} + \frac{c_2}{\alpha^2} + \cdots\right] \text{ as } |\alpha| \to \infty \tag{99}$$

and

$$F_U(\alpha) = \frac{1}{2\pi i}\int_{L_U} \frac{F(\tau)}{\tau - \alpha} d\tau = -\frac{1}{2\pi i \alpha}\int_{L_U} \frac{F(\tau)}{1 - \tau/\alpha} d\tau$$

$$= -\frac{1}{2\pi i \alpha}\sum_{j=0}^{\infty} \alpha^{-j} I_j \text{ as } |\alpha| \to \infty \tag{100}$$

where

$$I_j = \int_{L_U} \tau^j F(\tau) d\tau . \tag{101}$$

The expansion (100) holds as long as the integrals in (101) converge. Combining (99) and (100)

$$\sqrt{f} F_U(\alpha) = -\frac{1}{2\pi i} \alpha^{q/2-1} \left[c_0 I_0 + (c_1 I_0 + c_0 I_1)\alpha^{-1} + \cdots \right] \tag{102}$$

is written. The factor matrices in (96) behave algebraically for $q \leq 2$, which means that they are of the order $O(\alpha^q)$ as $\alpha \to \infty$ for any q real. However, for $q > 2$, the factor matrices display an exponential behaviour. So, when $2 < q \leq 4$, it requires the coefficient I_0 to vanish, but not the higher terms. Similarly, $(c_1 I_0 + c_0 I_1)$ is required to vanish in addition to the above condition when $4 < q \leq 6$, and so on. In general, they do not vanish, and the theory should be modified for $q > 2$ to omit exponentially growing factor matrices.

Since $f(\alpha)$ is a fourth order polynomial, the factor matrices will have an exponential behaviour. The next step is to introduce a polynomial matrix of the form to cancel this exponential growth.

Daniele showed that the exponentially growing terms appearing in the split matrices can be cancelled by finding a polynomial matrix of the form

$$\overline{P}(\alpha) = p_1(\alpha) I + p_2(\alpha) J(\alpha) \tag{103}$$

where p_1, p_2 are polynomials in α with undetermined coefficients and J is proportional to the commutant of the $G(\alpha)$ matrix. To cancel the exponentially growing terms, $G(\alpha)$ will be multiplied by the polynomial matrix and its inverse in the following form

$$G(\alpha) = G(\alpha)\overline{P}^{-1}(\alpha)\overline{P}(\alpha) = G_P(\alpha)\overline{P}(\alpha)\,. \tag{104}$$

It is also assumed that the multiplicative factors of $\overline{P}(\alpha)$ have the same exponential behaviour as the factors of $G(\alpha)$.

For the polynomial matrix $\overline{P}(\alpha)$ given by (103), the Khrapkov factors can be found as

$$\overline{P}_U(\alpha) = \frac{\sqrt{(\det \overline{P})_u}}{f_U g_U^{\ 2}}\left[\cosh(\sqrt{f}F_{PU})I + g\sqrt{f}\sinh(\sqrt{f}F_{PU})J(\alpha)\right] \tag{105}$$

$$\overline{P}_L(\alpha) = \frac{\sqrt{(\det \overline{P})_L}}{f_L g_L^{\ 2}}\left[\cosh(\sqrt{f}F_{PL})I + g\sqrt{f}\sinh(\sqrt{f}F_{PL})J(\alpha)\right] \tag{106}$$

with

$$\epsilon_P = (1/2)\ln[\lambda_{P1}/\lambda_{P2}] \tag{107}$$

$$\lambda_{P1}/\lambda_{P2} = p_1 \pm p_2 g\sqrt{f} \tag{108}$$

and

$$F_P = \epsilon_P/\sqrt{f}\,. \tag{109}$$

The polynomials $p_1(\alpha)$ and $p_2(\alpha)$ have undetermined constants which should be determined in such a way that the regularity conditions will be satisfied. Keeping this in mind, (104) can be written as

$$G(\alpha) = G_{PL}(\alpha)\,\overline{P}(\alpha)\,G_{PU}(\alpha) \tag{110}$$

since $G(\alpha)$, $P(\alpha)$ and their inverses and the multiplicative factors have the same commutant.

Thus, the factors are as follows :

$$G_{PU}(\alpha) = \overline{P}_U^{\,-1}(\alpha) G_U(\alpha) = \sqrt{\frac{(\det G)_U}{(\det \overline{P})_U}} \times$$

$$\left(\cosh[\sqrt{f}(F_U - F_{PU})]I + g\sqrt{f}\sinh[\sqrt{f}(F_U - F_{PU})]J(\alpha)\right) \tag{111}$$

$$G_{PL}(\alpha) = \overline{P}_L^{\,-1}(\alpha) G_L(\alpha) = \sqrt{\frac{(\det G)_L}{(\det \overline{P})_L}} \times$$

$$\left(\cosh[\sqrt{f}(F_L - F_{PL})]I + g\sqrt{f}\sinh[\sqrt{f}(F_L - F_{PL})]J(\alpha)\right) \tag{112}$$

where

$$F_{PU}(\alpha) = (-1/2\pi i\alpha)\sum_{j=0}^{\infty} \alpha^{-j} I_{P_j} \tag{113}$$

with

$$I_{P_j} = \int_{L_U} \tau^j F_P(\tau) d\tau \tag{114}$$

The gist of this approach is that the multiplicative factors of the matrix $G(\alpha)$ satisfies the regularity conditions by proper choice of the constants in the polynomials p_1 and p_2. So, the integrals I_{P_j} must be chosen to satisfy the conditions that will be obtained from $\sqrt{f}[F_U(\alpha) - F_{PU}(\alpha)]$. For example, when $2 < q \le 4, j = 0$ gives

$$I_0 - I_{P0} = 0$$

and when $4 < q \leq 6$, in addition to the above condition $j = 0$ gives

$$c_1(I_0 - I_{P0}) + c_0(I_1 - I_{P1}) = 0.$$

These conditions on I_{Pj} lead, in turn, to conditions on the matrix $\overline{P}(\alpha)$, more precisely on the coefficients $p_1(\alpha)$ and $p_2(\alpha)$ in (103). Daniele showed that a solution can always be found, but this is usually far away from finding an explicit solution. Because, in general, the integrals I_j, I_{P_j} can not be expressed in terms of elementary or even known transcendental functions, and I_{P_j} involve the undetermined coefficients of $p_{1,2}(\alpha)$ in a highly non-linear manner.

Now, if the above mentioned integrals are calculated and the unknown constants are determined, it remains only to factor $\overline{P}(\alpha)$ in an elementary way, say

$$\overline{P}(\alpha) = \overline{P}_L{}'(\alpha)\overline{P}_U{}'(\alpha). \tag{115}$$

this gives the final factorization of $G(\alpha)$ as

$$G(\alpha) = [G_{PL}(\alpha)\overline{P}_L{}'(\alpha)][\overline{P}_U{}'(\alpha)G_{PU}(\alpha)]. \tag{116}$$

The factors $\overline{P}_L{}'(\alpha)$ and $\overline{P}_U{}'(\alpha)$ will in general not commute, so that the new factors of $G(\alpha)$ will not commute.

4. SOLUTION FOR THE HIGH FREQUENCY CASE

If the operating frequency is assumed to be very large compared to the cyclotron frequency, while it is at the same order with the plasma frequency, then ω is taken to be very large than ω_c. In the limit case Eqs.(8) and (9) yield

$$\varepsilon_1 = 1-(\omega_p/\omega)^2 \text{ and } \quad \varepsilon_2 \to 0 . \tag{117}$$

The same result can also be obtained if the amplitude of the DC magnetic field vector is taken as zero.

Thus, according to this assumption Eq.(63) is reduced to

$$G(\alpha)=\begin{bmatrix} i\Gamma(\alpha)+i\dfrac{k^2}{\omega\mu_0}\mathrm{Z}_0\mathrm{Z} & 0 \\ 0 & 1+\dfrac{k^2}{\omega\mu_0}\mathrm{Z}_0\mathrm{Z} \end{bmatrix} \tag{118}$$

which yields

$$\begin{bmatrix} i\Gamma(\alpha)+i\dfrac{k^2}{\omega\mu_0}\mathrm{Z}_0\mathrm{Z} & 0 \\ 0 & 1+\dfrac{k^2}{\omega\mu_0}\mathrm{Z}_0\mathrm{Z} \end{bmatrix}\begin{bmatrix} L_1(\alpha) \\ L_2(\alpha) \end{bmatrix}=\begin{bmatrix} \hat{U}_1(\alpha) \\ \hat{U}_2(\alpha) \end{bmatrix}+\begin{bmatrix} \hat{F}_1(\alpha) \\ \hat{F}_2(\alpha) \end{bmatrix} \tag{119}$$

and this is equal to the following uncoupled equations :

$$\left[i\Gamma(\alpha)+i\frac{k^2}{\omega\mu_0}Z_0Z\right]L_1(\alpha)=\hat{U}_1(\alpha)+\hat{F}_1(\alpha), \tag{120}$$

$$\left[1+\frac{1}{\Gamma(\alpha)}\frac{k^2}{\omega\mu_0}Z_0Z\right]L_2(\alpha)=\hat{U}_2(\alpha)+\hat{F}_2(\alpha). \tag{121}$$

The expression in parenthesis on the left hand side of Eq(121) can be rewritten in the form

$$1+\frac{1}{\Gamma(\alpha)}\frac{k^2}{\omega\mu_0}Z_0Z=\frac{\Gamma(\alpha)+(\frac{k^2}{\omega\mu_0})Z_0Z}{\Gamma(\alpha)}=\frac{k}{\omega\mu_0}Z_0Z\left[\frac{\frac{\omega\mu_0}{k}\frac{1}{Z_0Z}\Gamma(\alpha)+k}{\Gamma(\alpha)}\right]$$

to obtain

$$=\sqrt{\varepsilon_1}Z\left[\frac{k+\frac{1}{\sqrt{\varepsilon_1}Z}\Gamma(\alpha)}{\Gamma(\alpha)}\right] \tag{122}$$

in which

$$\sqrt{\varepsilon_1}=\frac{k}{\omega\mu_0}Z_0\ , \tag{123}$$

where $k=\omega\sqrt{\varepsilon_0\mu_0\varepsilon_1}$ with $\varepsilon_2\to 0$.

The following variable changes are made

$$\eta=\frac{1}{\sqrt{\varepsilon_1}Z}, \tag{124}$$

and

$$h(\alpha)=\frac{\Gamma(\alpha)}{k+\eta\Gamma(\alpha)}. \tag{125}$$

Then Eqs(120-121) take the form

$$i\frac{1}{\eta}\Gamma(\alpha)h^{-1}(\alpha)L_1(\alpha)=\hat{U}_1(\alpha)+\frac{k\sin\phi_0}{\pi(\alpha-\cos\phi_0)}, \tag{126}$$

$$\frac{1}{\eta}h^{-1}(\alpha)L_2(\alpha)=\hat{U}_2(\alpha)-\frac{k\sqrt{\varepsilon_1}Z}{\pi(\alpha-k\cos\phi_0)} . \tag{127}$$

The factorization of $h(\alpha)$ is assumed to be as follows

$$h(\alpha)=h_U(\alpha)h_L(\alpha), \tag{128}$$

where $h_U(\alpha)$ and $h_L(\alpha)$ are regular functions in R_U and R_L , respectively. This gives

$$i\frac{1}{\eta}\frac{\Gamma_U(\alpha)\Gamma_L(\alpha)}{h_U(\alpha)h_L(\alpha)}L_1(\alpha)=\hat{U}_1(\alpha)+\hat{F}_1(\alpha), \tag{129}$$

and

$$\frac{1}{\eta}\frac{1}{h_U(\alpha)h_L(\alpha)}L_2(\alpha)=\hat{U}_2(\alpha)+\hat{F}_2(\alpha), \tag{130}$$

or

$$\frac{i}{\eta}\frac{\Gamma_L(\alpha)}{h_L(\alpha)}L_1(\alpha)=\frac{h_U(\alpha)}{\Gamma_U(\alpha)}\hat{U}_1(\alpha)+\frac{h_U(\alpha)}{\Gamma_U(\alpha)}\hat{F}_1(\alpha), \tag{131}$$

and

$$\frac{1}{\eta}\frac{1}{h_L(\alpha)}L_2(\alpha)=h_U(\alpha)\hat{U}_2(\alpha)+h_U(\alpha)\hat{F}_2(\alpha)\,. \tag{132}$$

The multiplicative factors of $h(\alpha)$ can be obtained in terms of the Maliuzhinetz function. The following variable changes

$$\alpha=\text{-}k\cos\phi \text{ and } \eta=\sec\lambda \tag{133}$$

with

$$\arcsin(\alpha/k)=\phi-\pi/2 \text{ and } \sqrt{1-\eta^2}=-i\tan\lambda\,, \tag{134}$$

give the factors as

$$h_U(-k\cos\phi)=\sqrt{\frac{2}{\eta}}\sin(\phi/2)f(\pi-\phi+\lambda)f(\pi-\phi-\lambda)\,. \tag{135}$$

The function $f(z)$ is defined as

$$f(z)=\frac{2^{3/4}}{1+\sqrt{2}\cos(z/2)}\left[\frac{M_\pi(z)}{M_\pi(\pi/2)}\right]^2 \tag{136}$$

with $M_\pi(z)$ being the Maliuzhinetz function.

Now, the next step is to write the additive split of the terms

$$f_1(\alpha)=\frac{h_U(\alpha)}{\Gamma_U(\alpha)}\hat{F}_1(\alpha)\,, \tag{137}$$

$$f_2(\alpha)=h_U(\alpha)\hat{F}_2(\alpha)\,, \tag{138}$$

which appear on the right-hand sides of Eq(131) and Eq(132). According to the standard procedure

$$f_{1,2U,L}(\alpha) = \pm\frac{1}{2\pi i}\int_{C_{U,L}} f_{1,2}(t)\frac{dt}{(t-\alpha)} \tag{139}$$

is written where $C_{U,L}$ are the lines shown in Fig-2.2. For the present case $\hat{F}_{1,2}(\alpha)$ are as follows :

$$\hat{F}_1(\alpha) = \frac{k\sin\phi_0}{\pi(\alpha - k\cos\phi_0)}, \tag{140}$$

$$\hat{F}_2(\alpha) = -\frac{(\frac{k^2}{\omega\mu_0})Z_0 Z}{\pi(\alpha - k\cos\phi_0)}, \tag{141}$$

and the integration lines for the upper factors are shown in Fig-4.1.

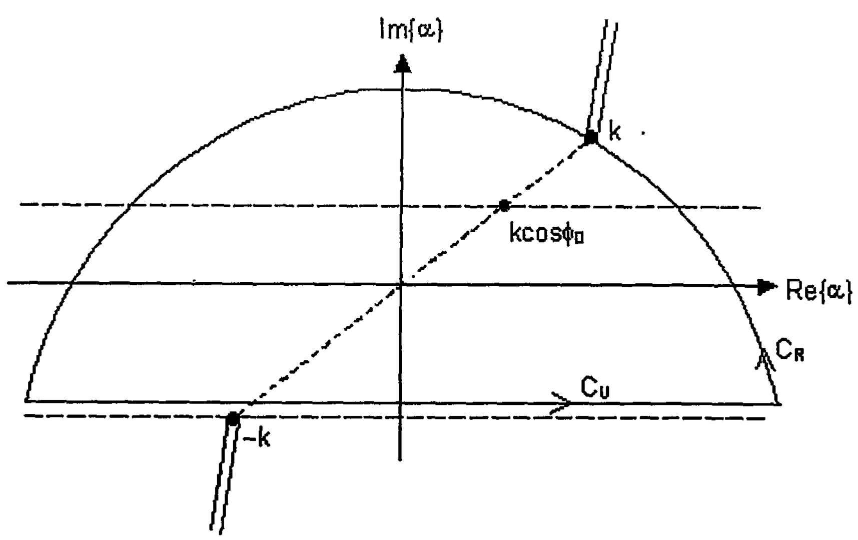

Figure 4.1. Integration lines for the upper factors

This gives

$$f_{1U}(\alpha)=\frac{1}{2\pi i}\int_{C_U}\frac{k\sin\phi_0}{\pi(t-k\cos\phi_0)}\frac{h_U(t)}{\Gamma_U(t)}\frac{dt}{(t-\alpha)} \quad , \tag{142}$$

$$f_{2U}(\alpha)=\frac{1}{2\pi i}\int_{C_U}\left(\frac{-k\sqrt{\varepsilon_1}Z}{\pi(t-k\cos\phi_0)}\right)\frac{h_U(t)}{(t-\alpha)}dt\,, \tag{143}$$

which requires C_R to be located in the upper half of the complex α-plane. The application of the residue theorem to (142) and (143) results :

$$f_{1U}(\alpha)=2\pi i\sum \mathrm{Rez} f_{1U}(t)$$

$$=2\pi i\frac{1}{2\pi i}\frac{1}{\pi}k\sin\phi_0\left[\left.\frac{h_U(t)}{(t-k\cos\phi_0)\Gamma_U(t)}\right|_{t=\alpha}+\left.\frac{h_U(t)}{(t-\alpha)\Gamma_U(t)}\right|_{t=k\cos\phi_0}\right]$$

or

$$f_{1U}=\frac{k\sin\phi_0}{\pi(\alpha-k\cos\phi_0)}\left[\frac{h_U(\alpha)}{\Gamma_U(\alpha)}-\frac{h_U(k\cos\phi_0)}{\Gamma_U(k\cos\phi_0)}\right], \tag{144}$$

and similarly for $f_2(\alpha)$ it results :

$$f_{2U}=2\pi i\sum \mathrm{Re\,z} f_{2U}(t)$$

$$=2\pi i\frac{1}{2\pi i}\frac{-k\sqrt{\varepsilon_1}Z}{\pi}\left[\left.\frac{h_U(t)}{(t-k\cos\phi_0))}\right|_{t=\alpha}+\left.\frac{h_U(t)}{(t-\alpha)}\right|_{t=k\cos\phi_0}\right]$$

or

$$f_{2U} = -\frac{k\sqrt{\varepsilon_1}Z}{\pi(\alpha - k\cos\phi_0)}\left[h_U(\alpha) - h_U(k\cos\phi_0)\right]. \tag{145}$$

The integration line for the lower factors is shown in Fig-4.2.

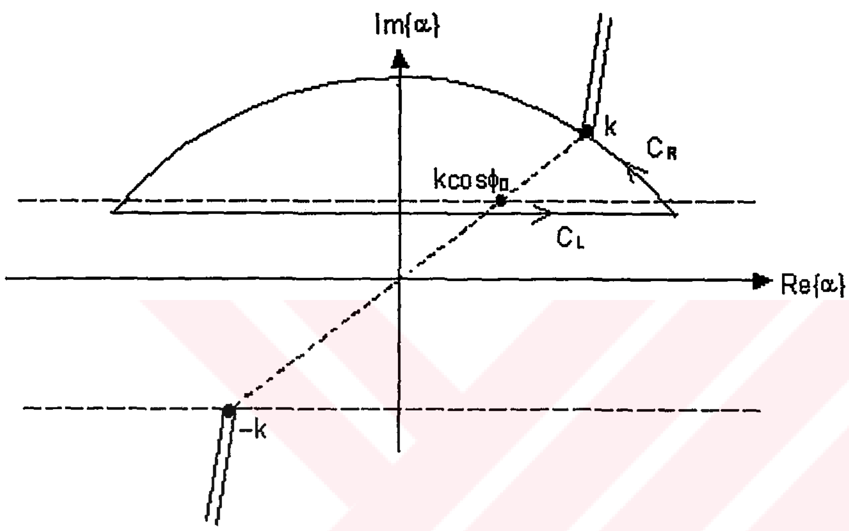

Figure 4.2. Integration lines for the lower factors

The integral expression is

$$f_{1L} = -\frac{1}{2\pi i}\int_{C_L}\frac{k\sin\phi_0}{\pi(t - k\cos\phi_0)}\frac{h_U(t)}{\Gamma_U(\alpha)}\frac{dt}{(t-\alpha)} \quad , \tag{146}$$

$$f_{2L} = -\frac{1}{2\pi i}\int_{C_L}\frac{(-k\sqrt{\varepsilon_1}Z)}{\pi(t - k\cos\phi_0)}\frac{h_U(t)}{(t-\alpha)} \tag{147}$$

By residue theorem, it gives

$$f_{1L} = 2\pi i\left(-\frac{1}{2\pi i}\right)\frac{k\sin\phi_0}{\pi}\left[\left.\frac{h_U(t)}{\Gamma_U(\alpha)(t-\alpha)}\right|_{t=k\cos\phi_0}\right]$$

or

$$f_{1L} = \frac{k\sin\phi_0}{\pi(\alpha - k\cos\phi_0)} \cdot \frac{h_U(k\cos\phi_0)}{\Gamma_U(k\cos\phi_0)}, \tag{148}$$

and for $f_2(\alpha)$ it gives

$$f_{2L} = 2\pi i(-\frac{1}{2\pi i})\frac{-k\sqrt{\varepsilon_1}Z}{\pi}\left[\left.\frac{h_U(t)}{(t-\alpha)}\right|_{t=k\cos\phi_0}\right]$$

or

$$f_{2L} = -k\sqrt{\varepsilon_1}Z\frac{h_U(k\cos\phi_0)}{\pi(\alpha - k\cos\phi_0)}. \tag{149}$$

By combining Eqs.(131-132) and Eqs.(137-138)

$$\frac{i}{\eta}\frac{\Gamma_L(\alpha)}{h_L(\alpha)}L_1(\alpha) = \frac{h_U(\alpha)}{\Gamma_U(\alpha)}\hat{U}_1(\alpha) + f_{1U}(\alpha) + f_{1L}(\alpha) \tag{150}$$

and

$$\frac{1}{\eta}\frac{1}{h_L(\alpha)}L_2(\alpha) = h_U(\alpha)\hat{U}_2(\alpha) + f_{2U}(\alpha) + f_{2L}(\alpha) \tag{151}$$

is written and it gives

$$q_1(\alpha) = \begin{cases} \dfrac{i}{\eta}\dfrac{\Gamma_L(\alpha)}{h_L(\alpha)} L_1(\alpha) - f_{1L}(\alpha); & \mathrm{Im}\alpha < \mathrm{Im}(\mathrm{k}\cos\phi_0) \\ \\ \dfrac{h_U(\alpha)}{\Gamma_U(\alpha)} \hat{U}_1(\alpha) + f_{1U}(\alpha); & \mathrm{Im}\alpha > \mathrm{Im}(-k) \end{cases} \tag{152}$$

and

$$q_2(\alpha) = \begin{cases} \dfrac{1}{\eta}\dfrac{1}{h_L(\alpha)} L_2(\alpha) - f_{2L}(\alpha); & \mathrm{Im}\alpha < \mathrm{Im}(k\cos\phi_0) \\ \\ h_U(\alpha)\hat{U}_2(\alpha) + f_{2U}(\alpha); & \mathrm{Im}\alpha > \mathrm{Im}(-k) \end{cases} . \tag{153}$$

Here, $q_1(\alpha)$ and $q_2(\alpha)$ are entire functions which will be determined by using the asymptotic behaviour of the functions for $\alpha \to \infty$ that appear in their expressions which correspond to the edge conditions of the field components.

It gives $q_1(\alpha) = q_2(\alpha) = 0$ and from Eqs.(152-153) :

$$L_1(\alpha) = -i\eta \frac{h_L(\alpha)}{\Gamma_L(\alpha)} f_{1L}(\alpha)$$

$$\hat{U}_1(\alpha) = -\frac{\Gamma_U(\alpha)}{h_U(\alpha)} f_{1U}(\alpha)$$

$$L_2(\alpha) = \eta h_L(\alpha) f_{2L}(\alpha)$$

$$\hat{U}_2(\alpha) = -\frac{1}{h_U(\alpha)} f_{2U}(\alpha)$$

are written. $A(\alpha)$ and $B(\alpha)$ are derived from Eqs.(42-43) as

$$A(\alpha) - B(\alpha) = L_1(\alpha)$$

$$A(\alpha) + B(\alpha) = -i\frac{1}{\Gamma(\alpha)}L_2(\alpha)$$

which gives

$$A(\alpha) = \frac{1}{2}[L_1(\alpha) - i\frac{1}{\Gamma(\alpha)}L_2(\alpha)] \tag{154}$$

and

$$B(\alpha) = \frac{1}{2}[-L_1(\alpha) - i\frac{1}{\Gamma(\alpha)}L_2(\alpha)]. \tag{155}$$

5. ANALYSIS OF THE FIELD

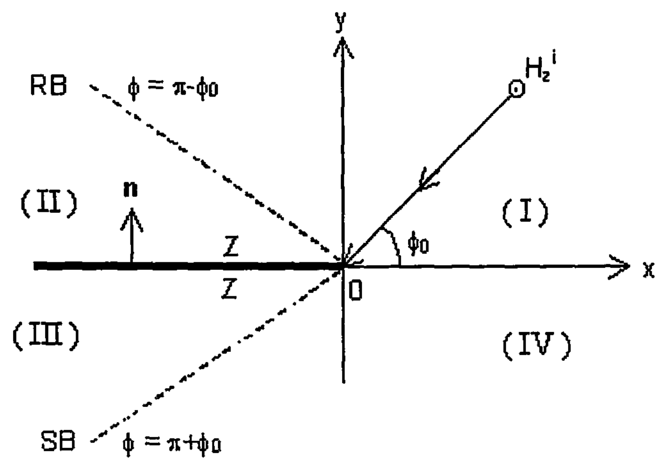

Figure 5.1. Reflection and shadow boundaries

The scattered field was given by Eq.(31)

$$u(x,y) = \begin{cases} \int_L A(\alpha) e^{-i\alpha x + i\Gamma(\alpha) y} d\alpha, & y > 0 \\ \int_L B(\alpha) e^{-i\alpha x - i\Gamma(\alpha) y} d\alpha, & y < 0. \end{cases}$$

and when the expressions of $A(\alpha)$ and $B(\alpha)$ in Eqs(154-155) are substituted into above equation, an integral expression of the scattered field is obtained. The evaluation of these integrals will yield the analytical expressions for the scattered field components. Geometrical optics gives the information about the optical boundaries and the components that should be expected in these regions (Fig-5.1). For $k \to \infty$, asymptotic expression of the integral in Eq(31) can be obtained by using *the steepest descent path method* (see A.1).

Let the following variable change made,

$$\alpha = \text{-}k\cos t, \quad d\alpha = k\sin t dt, \quad x = r\cos\phi, \quad y = r\sin\phi .$$

This change constitutes a mapping of the cut complex α-plane into the t plane of $0 < \mathrm{Re}t < \pi$ band. According to this mapping, integration line L in the α-plane is replaced with curve T in the band $0 < \mathrm{Re}t < \pi$ as shown in Fig-5.2.

With the new variables

$$-i\alpha x \pm i\Gamma(\alpha)y = ikr\cos(t \mp \phi),$$

is written, then the field expression becomes

$$u = \begin{cases} \int_C A(-k\cos t)e^{ikr\cos(t-\phi)}k\sin t\,dt, & y > 0, \\ \int_C B(-k\cos t)e^{ikr\cos(t+\phi)}k\sin t\,dt, & y < 0. \end{cases} \tag{156}$$

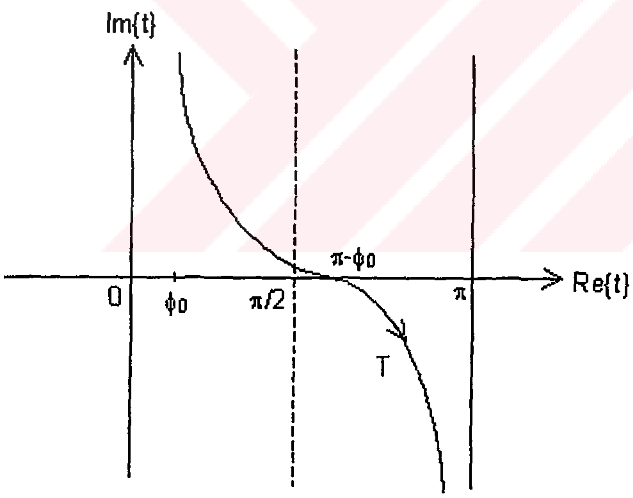

Figure 5.2. Complex t-plane obtained with α = -kcosφ

As previously mentioned, by using the saddle-point technique, an asymptotic expression will be obtained for the integral given in Eq.(156) for $k \to \infty$.

Let the exponential function be denoted as

$q(t) = \cos(t \pm \phi)$

and

$$q'(t) = -\sin(t \pm \phi) = 0 \quad t_s - \phi = n\pi, \quad n=0,1,\ldots$$

will be calculated. This gives for $y > 0$, or $0 < \phi < \pi$,

$t_s - \phi = n\pi$, $n = 0$ and yields $t_s = \phi$

and for $y < 0$ or $\pi < \phi < 2\pi$,

$t_s = n\pi - \phi$, $n = 2$ and yields $t_s = 2\pi - \phi$.

So, the second order derivative is

$$q''(t_s) = -\cos(t_s - \phi) = -\cos(\phi - \phi) = -1 < 0, \tag{157}$$

for $y > 0$, and

$$q''(t_s) = -\cos(t_s + \phi) = -\cos(2\pi - \phi + \phi) = -1 < 0, \tag{158}$$

for $y < 0$. This gives

$$u^d(r,\phi) \approx \begin{cases} \sqrt{\dfrac{2\pi}{kr|q''(t_s)|}} A(-k\cos t_s) k \sin t_s e^{ikr\cos(t_s-\phi)-i\pi/4}, & y > 0 \\ \sqrt{\dfrac{2\pi}{kr|q''(t_s)|}} B(-k\cos t_s) k \sin t_s e^{ikr\cos(t_s+\phi)-i\pi/4}, & y < 0. \end{cases} \tag{159}$$

In Region-I, defined with $0 < \phi < \pi - \phi_0$, the saddle point of the integrand for $y > 0$ is at $t = \phi$, and the expression of the scattered field is given as

$$u(r,\phi) = u^d(r, \phi)$$

where $u^d(r, \phi)$ is the diffracted field and there is no pole contribution. The diffracted field is determined by using the saddle point technique :

$$u^d(r,\phi) \sim u^i(0) D(\phi_0,\phi) \frac{e^{ikr}}{\sqrt{kr}} \tag{160}$$

in which

$$D(\phi_0,\phi) = \frac{e^{i\pi/4}}{\sqrt{2\pi}} \eta \frac{h_L(-k\cos\phi) h_U(k\cos\phi_0)}{(\cos\phi + \cos\phi_0)} \cdot \left[\sqrt{1-\cos\phi}\sqrt{1-\cos\phi_0} - \frac{1}{\eta} \right]. \tag{161}$$

In Region-II, defined with $\pi\text{-}\phi_0 < \phi < \pi$, saddle point of the integrand for $y > 0$ is at $t = \phi$. The scattered field is given as

$$u(r, \phi) = u^r(r, \phi) + u^d(r, \phi)$$

where $u^r(r, \phi)$ is the contribution of the pole $t = \pi - \phi_0$ corresponding to the reflected field.

In Region-III, where $\pi < \phi < \pi + \phi_0$, saddle point is given as $t = 2\pi\text{-}\phi$. The scattered field is as follows:

$$u(r,\phi) = -u^i(r, \phi) + u^d(r, \phi)$$

and the negative of the incident field is obtained as the residue contribution of the pole at $(\pi\text{-}\phi_0)$. $D(\phi_0,\phi)$ is given as

$$D(\phi_0,\phi) = \frac{e^{i\pi/4}}{\sqrt{2\pi}} \eta \frac{h_L(-k\cos\phi) h_U(k\cos\phi_0)}{(\cos\phi + \cos\phi_0)} \cdot \left[\sqrt{1-\cos\phi}\sqrt{1-\cos\phi_0} + \frac{1}{\eta} \right]. \tag{162}$$

In Region-IV, where $\pi+\phi_0 < \phi < 2\pi$, saddle point is $t = 2\pi-\phi$. In this case, scattered field is given as

$$u(r,\phi) = u^d(r,\phi)$$

without any pole contribution where the expression of $D(\phi_0,\phi)$ is given as in Eq(162).

Now, to write the diffraction coefficient in a form convenient for numerical analysis $h_{U,L}(\alpha)$ are written explicitly.

$h_L(-k\cos\phi)$ can be derived using the relations $h(\alpha) = h_U(\alpha)h_L(\alpha)$ and $h_U(\alpha) = h_L(-\alpha)$ yielding

$$h(-k\cos\phi) = h_U(-k\cos\phi)h_L(-k\cos\phi)$$

and

$$h_L(-k\cos\phi) = h(-k\cos\phi).h_U^{-1}(-k\cos\phi). \tag{163}$$

On the other hand, $h_U(-k\cos\phi)$ is written in terms of Maliuzhinetz function $M_\pi(z)$ as follows

$$h_U(-k\cos\phi) = \frac{\frac{4}{\sqrt{\eta}}\sin(\phi/2)}{\left[1+\sqrt{2}\cos\left(\frac{\pi-\phi+\arccos(1/\eta)}{2}\right)\right]^2\left[1+\sqrt{2}\cos\left(\frac{\pi-\phi-\arccos(1/\eta)}{2}\right)\right]^2}$$

$$\times\frac{[M_\pi(\pi-\phi+\arccos(1/\eta))M_\pi(\pi-\phi-\arccos(1/\eta))]^2}{\left[M_\pi(\frac{\pi}{2})\right]^4} \tag{164}$$

Then, using (163), $h_L(-k\cos\phi)$ can be written as

$$h_L(-k\cos\phi) = \sqrt{\eta}\,\frac{\cos\frac{\phi}{2}[M\pi(\pi/2)]^4}{2(1+\eta\sin\phi)}$$

$$\times\frac{\left[1+\sqrt{2}\cos(\frac{\pi-\phi-\arccos(1/\eta)}{2})\right]\left[1+\sqrt{2}\cos(\frac{\pi-\phi+\arccos(1/\eta)}{2})\right]}{[M\pi(\pi-\phi-\arccos(1/\eta))]^2[M\pi(\pi-\phi+\arccos(1/\eta))]^2} \quad (165)$$

and in a similar way, $h_u(k\cos\phi_0)$ can be written as

$$h_U(k\cos\phi_0) = \eta\,\frac{\cos\frac{\phi_0}{2}[M\pi(\pi/2)]^4}{2(1+\eta\sin\phi_0)}$$

$$\times\frac{\left[1+\sqrt{2}\cos(\frac{\pi-\phi_0-\arccos)(1/\eta)}{2})\right]\left[1+\sqrt{2}\cos(\frac{\pi-\phi_0+\arccos(1/\eta)}{2})\right]}{[M\pi(\pi-\phi_0-\arccos(1/\eta))]^2[M\pi(\pi-\phi_0+\arccos(1/\eta)]^2} \quad (166)$$

in which $h_L(-k\cos\phi_0) = h_u(k\cos\phi_0)$.

Now, putting (165) and (166) into (161) and (162), the expression of diffracted field u^d is obtained in the upper and lower media, respectively.

$$u^d(r,\phi) = u^i(0)D(\phi_0,\phi)\,\frac{e^{ikr}}{\sqrt{kr}}$$

$$D(\phi_0,\phi) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}}\eta^2\,\frac{\cos(\phi/2)\cos(\phi_0/2)[M_\pi(\pi/2)]^8}{4(1+\eta\sin\phi)(1+\eta\sin\phi_0)}\,\frac{\left[\sqrt{1-\cos\phi}\sqrt{1-\cos\phi_0}-(1/\eta)\right]}{\cos\phi+\cos\phi_0}$$

$$\times\frac{\left[1+\sqrt{2}\cos(\frac{\pi-\phi+\arccos(1/\eta)}{2})\right]\left[1+\sqrt{2}\cos(\frac{\pi-\phi-\arccos(1/\eta)}{2})\right]}{[M_\pi(\pi-\phi+\arccos(1/\eta))]^2[M_\pi(\pi-\phi-\arccos(1/\eta)]^2}$$

$$\times \frac{\left[1+\sqrt{2}\cos(\frac{\pi-\phi_0+\arccos(1/\eta)}{2})\right]\left[1+\sqrt{2}\cos(\frac{\pi-\phi_0-\arccos(1/\eta)}{2})\right]}{\left[M_\pi(\pi-\phi_0+\arccos(1/\eta))\right]^2\left[M_\pi(\pi-\phi_0-\arccos(1/\eta)\right]^2} \quad (167)$$

for $y > 0$, and

$$D(\phi_0,\phi)=\frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}}\eta^2\frac{\cos(\phi/2)\cos(\phi_0/2)\left[M_\pi(\pi/2)\right]^8}{4(1+\eta\sin\phi)(1+\eta\sin\phi_0)}\frac{\left[\sqrt{1-\cos\phi}\sqrt{1-\cos\phi_0}+(1/\eta)\right]}{\cos\phi+\cos\phi_0}$$

$$\times \frac{\left[1+\sqrt{2}\cos(\frac{\pi-\phi+\arccos(1/\eta)}{2})\right]\left[1+\sqrt{2}\cos(\frac{\pi-\phi-\arccos(1/\eta)}{2})\right]}{\left[M_\pi(\pi-\phi+\arccos(1/\eta))\right]^2\left[M_\pi(\pi-\phi-\arccos(1/\eta)\right]^2}$$

$$\times \frac{\left[1+\sqrt{2}\cos(\frac{\pi-\phi_0+\arccos(1/\eta)}{2})\right]\left[1+\sqrt{2}\cos(\frac{\pi-\phi_0-\arccos(1/\eta)}{2})\right]}{\left[M_\pi(\pi-\phi_0+\arccos(1/\eta))\right]^2\left[M_\pi(\pi-\phi_0-\arccos(1/\eta)\right]^2} \quad (168)$$

for $y < 0$. Then the total field $u^t = u + u^i$ is written as

$$u^t = \begin{cases} u^i + u^d, & \text{Region I} \\ u^i + u^r + u^d, & \text{Region II} \\ u^d, & \text{Region III} \\ u^i + u^d, & \text{Region IV.} \end{cases} \quad (169)$$

6. NUMERICAL RESULTS

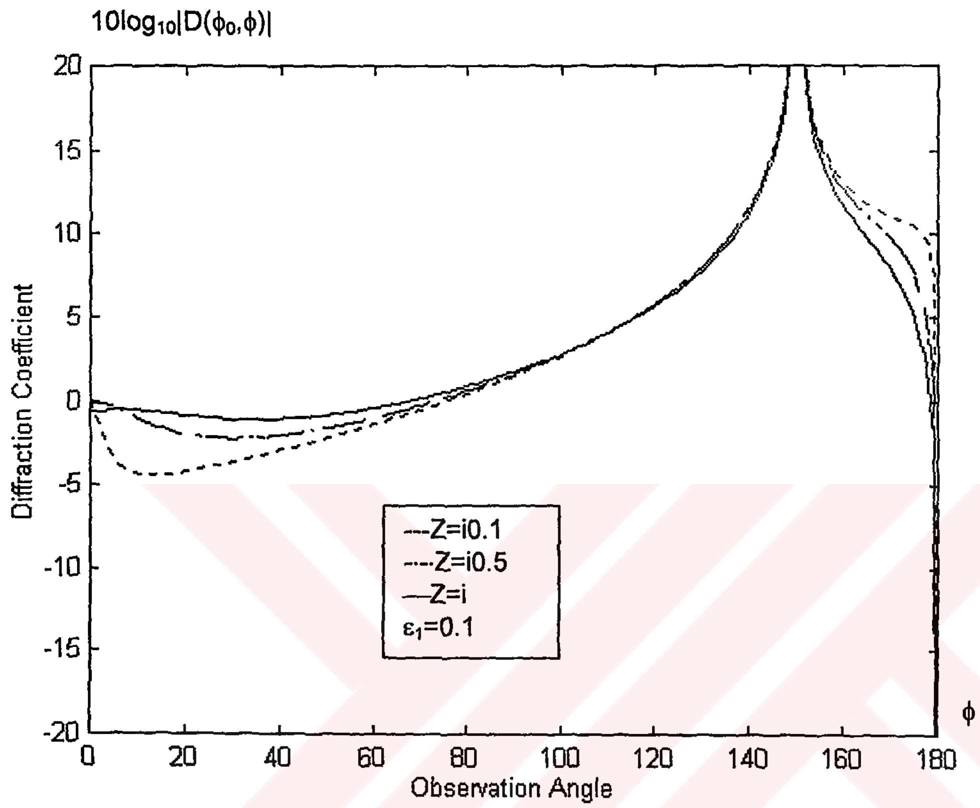

Figure 6.1. $10\log_{10}|D(\phi_0,\phi)|$ with $\phi_0=30°,\varepsilon_1=0.1$

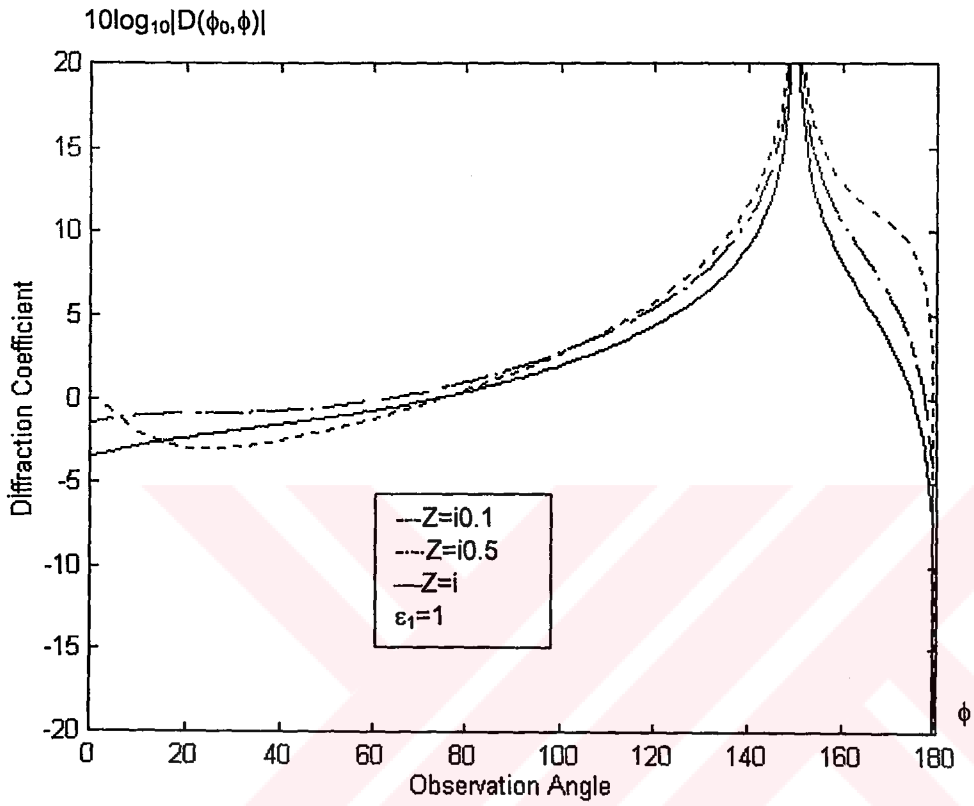

Figure 6.2. $10\log_{10}|D(\phi_0,\phi)|$ with $\phi_0=30°,\varepsilon_1=1$

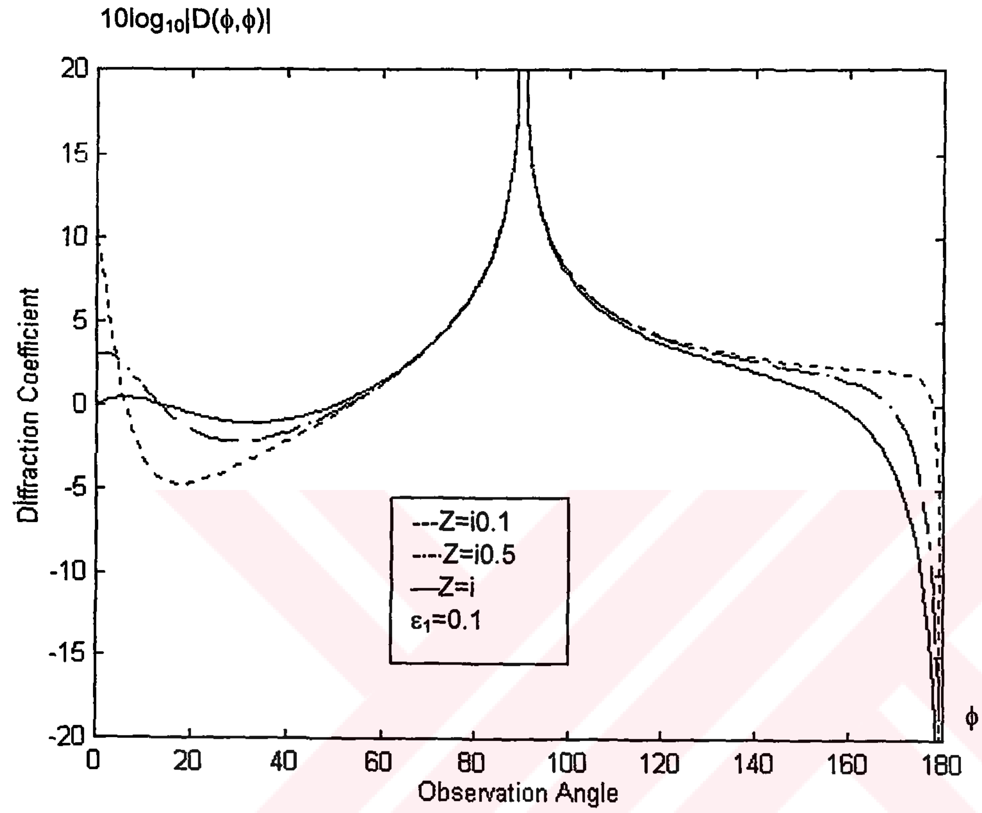

Figure 6.3. $10\log_{10}|D(\phi,\phi)|$ in monostatic case with $\varepsilon_1=0.1$

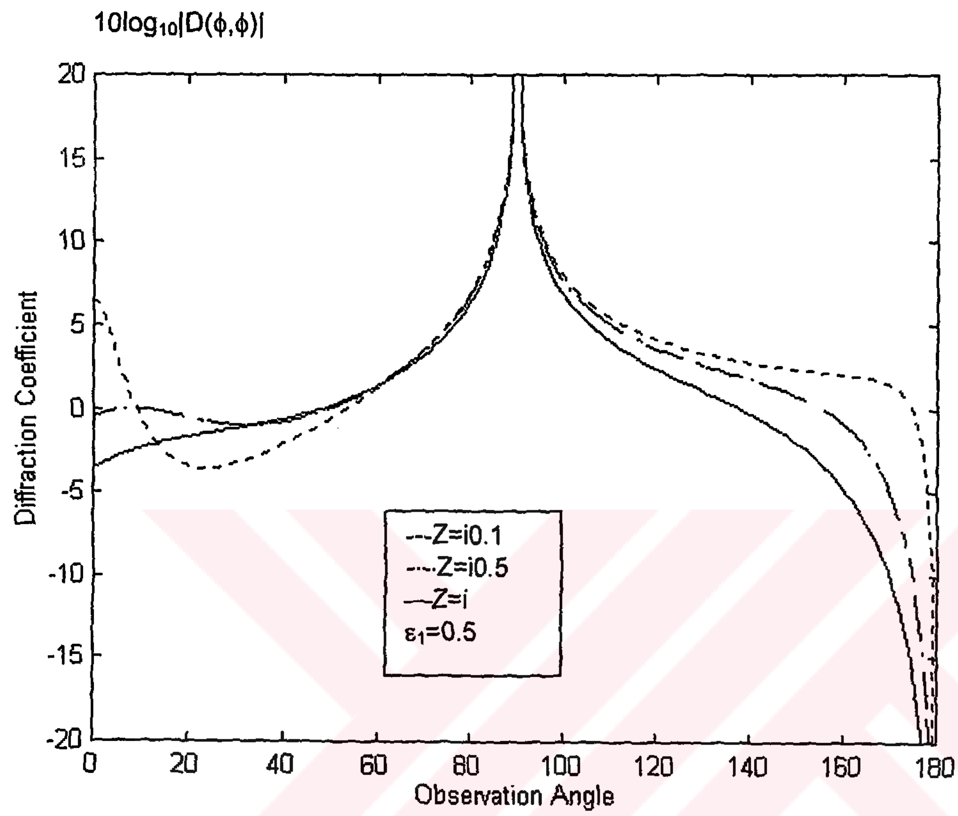

Figure 6.4. $10\log_{10}|D(\phi,\phi)|$ in monostatic case with $\varepsilon_1=0.5$

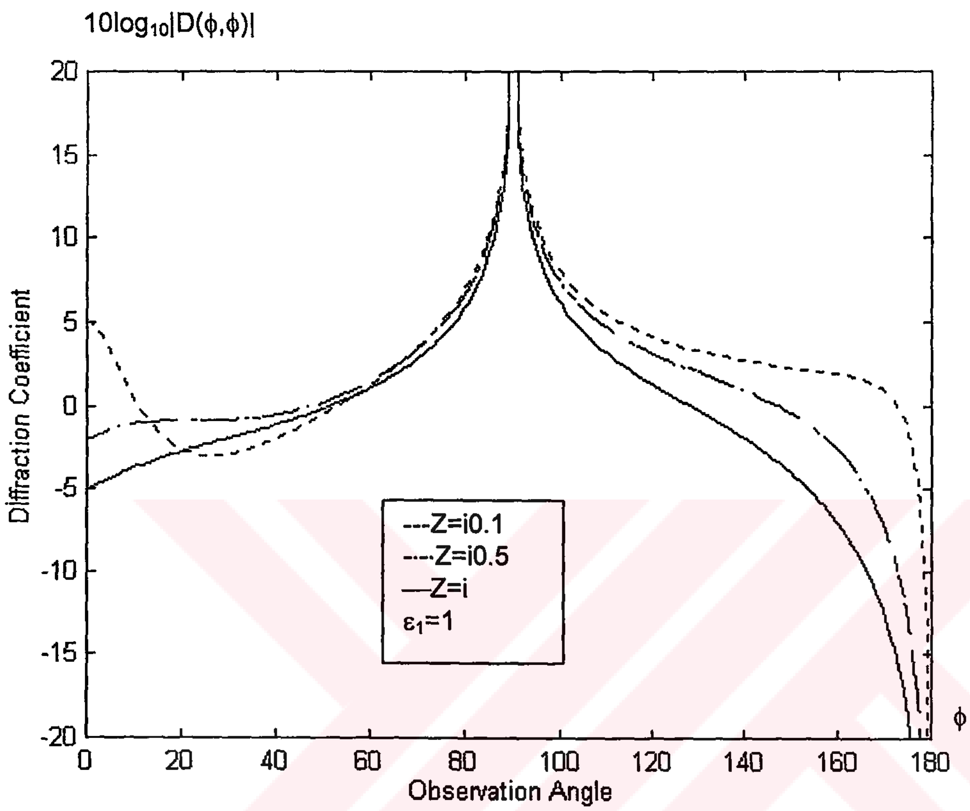

Figure 6.5. $10\log_{10}|D(\phi,\phi)|$ in monostatic case with $\varepsilon_1=1$

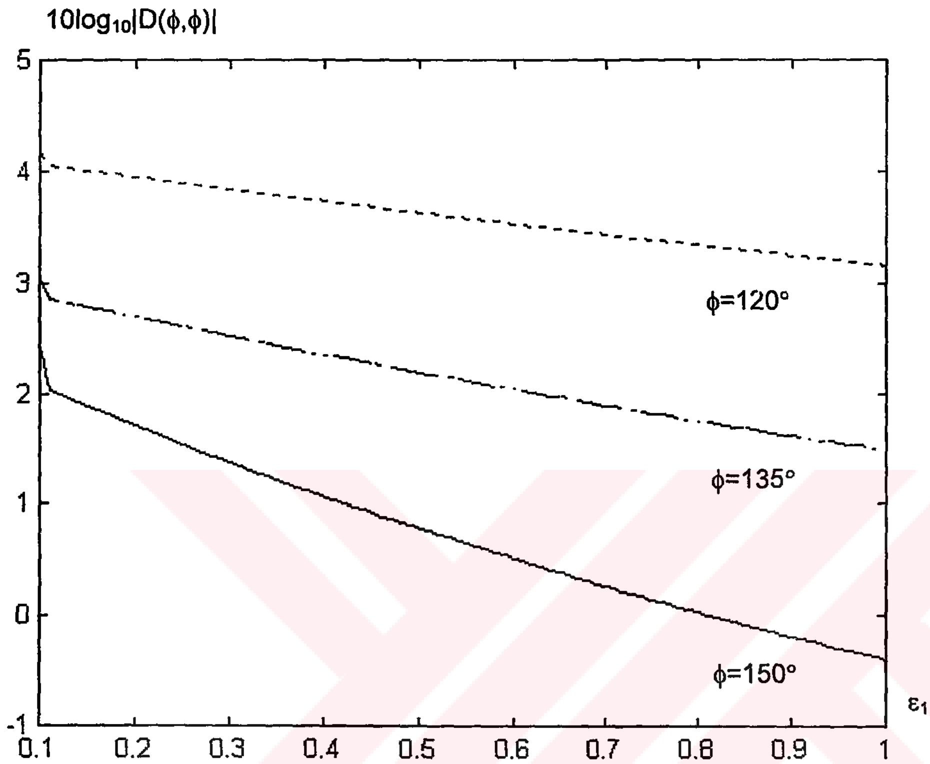

Figure 6.6. $10\log_{10}|D(\phi,\phi)|$ versus ε_1 in monostatic case

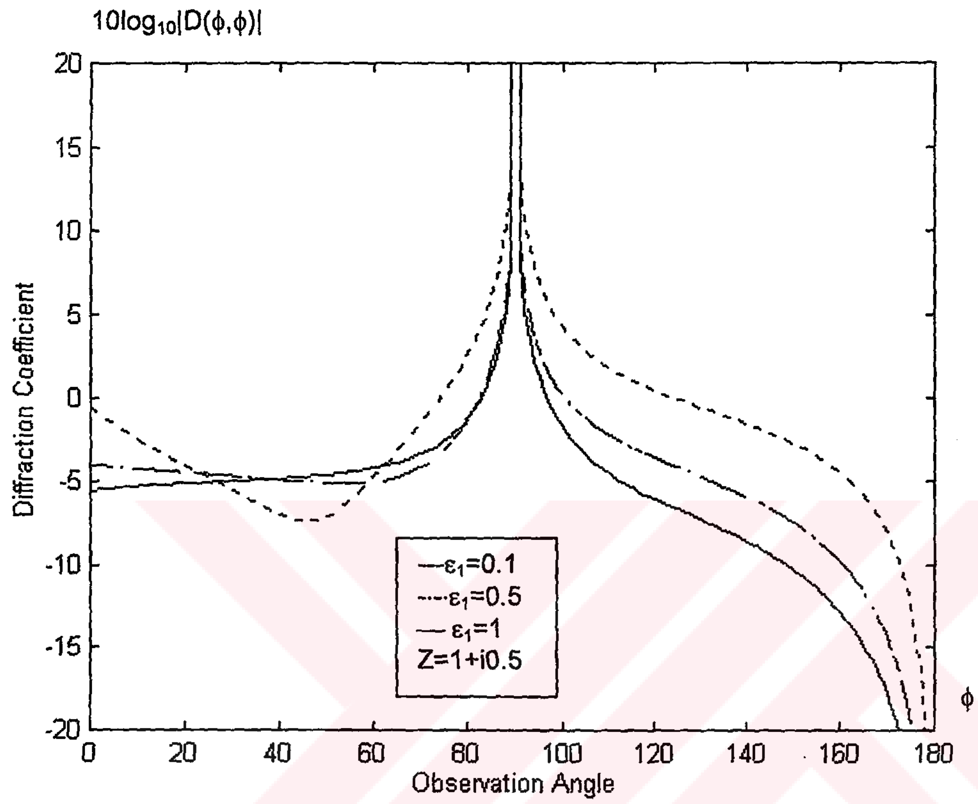

Figure 6.7. $10\log_{10}|(D(\phi,\phi)|$ in monostatic case for different values of ε_1.

7. CONCLUSION

The scattered field expressions obtained for the problem of diffraction by an impedance half plane in cold plasma are relatively simple. For numerical computation, only difficulty will arise from the presence of the multiplicative factor of the kernel function $h(\alpha)$. As given before, the factors $h_U(\alpha)$ and $h_L(\alpha)$ are expressed in terms of Maliuzhinetz function. The numerical calculation of the Maliuzhinetz function, which is a meromorphic transcendental function, was first accomplished by Bucci (1975). In the present study, approximate expressions introduced by Senior and Volakis (1985) are used.

The figures from (6.1) to (6.5) show the variation of the diffraction coefficient with respect to the observation angle. The impedance values are taken to be pure imaginary as (i0.1), (i0.5) and (i1.0). Also the monostatic diffraction coefficient is calculated with respect to ε_1 for the range $0 < \varepsilon_1 < 1$ (Figure6.6).

For $Z = 1+i0.5$ the variation of the monostatic diffraction coefficient with respect to the observation angle is shown in Fig(6.7) for various values of ε_1. The effect of the impedance and ε_1 is easily seen from the figures such that the diffraction coefficient is weakened as the value of impedance or ε_1 is increased. As is obvious, the case $\varepsilon_1 = 1$ corresponds to the case where the surrounding medium is isotropic and homogeneous.

The diffraction of plane waves by an impedance half-plane problem in cold plasma is being considered for the first time in this study. Therefore, it is not possible to verify the results obtained here with some previously derived solutions. But, as a verification of the present investigation the diffraction coefficient can be reduced to the impedance half-plane problem in simple medium. The reduction can be achieved by simple writing $\varepsilon_1 = 1$ and it gives $\eta = \frac{1}{\sqrt{\varepsilon_1} Z} = \frac{1}{Z}$, resulting with

$$D(\phi_0,\phi)=\frac{e^{i\pi/4}}{Z\sqrt{2\pi}}\frac{\pm Z+\sqrt{1-\cos\phi}\sqrt{1-\cos\phi_0}}{\cos\phi+\cos\phi_0}h_U(k\cos\phi_0)h_U(k\cos\phi),\ y \lessgtr 0 \tag{170}$$

This expression is identical to the solution given by Büyükaksoy and Uzgören. Because of the difference in the geometry of the problem, it is necessary to make the following substitutions : $\phi \rightarrow \pi+\phi$ and $\phi_0 \rightarrow \pi+\phi_0$ in the expressions obtained here. Then Eq(170) becomes

$$D(\phi_0,\phi)=\frac{e^{i\pi/4}}{Z\sqrt{2\pi}}\frac{\mp Z-\sqrt{1+\cos\phi}\sqrt{1+\cos\phi_0}}{\cos\phi+\cos\phi_0}h_U(-k\cos\phi_0)h_U(-k\cos\phi),\ y \lessgtr 0 \tag{171}$$

The solution of the present problem is given in GTD form, yielding the diffraction coefficient corresponding an impedance edge in cold plasma. Since cold plasma is considered as a medium modelling ionosphere, it is possible to investigate the diffraction mechanism of the complex scatterers in the ionosphere such as airplanes or satellites.

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BIOGRAPHY

I was born in Adana, August 12, 1975. I graduated from high school in 1993 and entered to Electrical and Electronics Engineering Department of Çukurova University at the same year. I received my B. S. Degree in 1997 and joined to the same department as an MS student. I have been working there as a research assistant since then.

APPENDIX

A.1. Asymptotic Evaluation of Integrals

Consider the integral representation of the form

$$I(k) = \int_C f(z)e^{kq(z)}dz \tag{1}$$

where k is very large parameter. $f(z)$ and $q(z)$ are analytic functions of the complex variable z along a path of integration C with endpoints at infinity, and where the large parameter k is assumed to be positive.

Suppose that at the point z_s on C, Re$q(z)$ has a maximum value so that Re$q(z)$ < Re$q(z_s)$ on the remainder of the path. Since k is very large, it follows that A = $|\exp[kq(z)]|$ likewise has a maximum at z_s and decreases very rapidly away from z_s.

The point(s) z_s, where there occur maximum contributions to the integral, are *saddle* or *stationary points* of the function $q(z)$, defined by the vanishing of one or more of the derivatives of $q(z)$. The desired path C_z is one on which *Im*$q(z)$ remains constant. A finite number of terms in the power series expansion of $q(z)$ about z_s characterizes the asymptotic evaluation for large k. This feature may be exploited in a rigorous manner by transforming the given integral into a *canonical* form, wherein the function $q(z)$ is replaced by another function, a polynomial that describes in the simplest fashion the relevant saddle-point arrangement as z_s. The transformation will be phrased as :

$$\tau(s) = q(z) \tag{2}$$

the point z_s in the complex z-plane being chosen to correspond to $s = 0$ in the complex s-plane. Thus, Eq(1) leads to

$$I(k) = \int_{C'} G(s) e^{k\tau(s)} ds \tag{3}$$

where

$$G(s) = f(z)\frac{dz}{ds} \text{ and } \frac{dz}{ds} = \frac{\tau'(s)}{q'(s)} \tag{3a}$$

C' represents the mapping onto the s plane of the path C in the z plane.

$I(k)$ in the s-plane is approximated most simply on a contour $\overline{C}$, which passes in the vicinity of the origin and along which the magnitude of $\exp[k\tau(s)]$ decreases rapidly away from $s=0$. The desired contour $\overline{C}$ in the s-plane, along which $\operatorname{Im}\tau(s) =$ constant and $\operatorname{Re}\tau(s) < \operatorname{Re}\tau(0)$ away from a region about $s = 0$, constitutes a mapping into the s-plane of the path C_z in the z-plane.Generally C' will not be identical with $\overline{C}$, and in any ensuing deformation of C' into $\overline{C}$, the presence of singularities of the integrand in Eq(3) may have to be taken into account.

Assume that the integral in Eq(3) is to be evaluated over the desired contour $\overline{C}$ and that $G(s)$ is regular and slowly varying in the vicinity of $s = 0$. For large values of k the dominant contribution to the integral in Eq(3) arises from the vicinity of the origin since the exponential term decreases rapidly away from the region $s \approx 0$. Then

$$I(k) = \int_{\overline{C}} G(s) e^{k\tau(s)} ds \sim G(0) \int_{\overline{C}} e^{k\tau(s)} ds\,,\ k \to \infty \tag{5}$$

where the symbol ~ means *asymptotically represented by* and $G(0)$ is the value of the regular function $G(0)$ at $s = 0$. The last integral in Eq(5) represents a *canonical* integral that provides a first-order approximation to the unknown integral $I(k)$ of Eq(1).

A.1.1. Saddle Points

Consider the analytic function $q(z)$

$$q(z) = u(x, y) + iv(x, y), \qquad z = x + iy \tag{6}$$

where u, v, x and y are real. A stationary point z_s of $q(z)$ is determined by

$$\frac{dq}{dz} = 0 \quad \text{at } z = z_s, \tag{7}$$

in view of the analytic behaviour of q about z_s, it is thereby implied that the functions u and v are also stationary at (x_s, y_s),

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \text{ at } x = x_s,\ y = y_s. \tag{8}$$

Although Eqs(8) are satisfied, the surfaces $u(x, y)$ = constant and $v(x, y)$ = constant do not have absolute maximum or minimum values at (x_s, y_s). This is seen readily from the Cauchy-Riemann equations, satisfied by u and v,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{9}$$

from which it follows that for a first-order saddle-point ($d^2q/dz_s^2 \neq 0$),

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2} \tag{10}$$

Thus, if at (x_s, y_s) the curvature of the surface $u(x, y)$ = constant, or $v(x, y)$ = constant, is positive along the x direction, it is negative along the (perpendicular) y direction.

The stationary points z_s are therefore *saddle points*. Depending on the choice of path through z_s, the magnitude of $u(x,y)$ and therefore of $\exp|kq(z)|$ may increase, decrease, or remain constant along the path.

A.1.2. Paths of Constant Level and Constant Phase

The *steepest paths* through a saddle point will be the paths along which $u(x,y)$ changes most rapidly. In view of Eq(6), these will be the paths along which $u(x, y)$ changes most rapidly. v = costant is along the the path on which u changes most rapidly, so the steepest path is a constant-phase path. Similarly, the magnitude of the exponential is constant along the paths of most rapid phase variation.

A.1.3. First Order Approximation

If in the integrand of

$$I(k) = \int_{SDP} f(z) e^{kq(z)} dz$$

the function $f(z)$ has no singularities near an isolated first-order saddle point z_s of $q(z)$, where $q'(z_s) = 0,\ \ q''(z_s) \neq 0,$ the asymptotic approximation of $I(k)$ is given by

$$I(k) \sim \sqrt{\frac{-2\pi}{kq''(z_s)}} f(z_s) e^{kq(z_s)},\ \ k \to \infty \tag{11}$$

When $q(z) = i\hat{q}(z)$, with $\hat{q}(z)$ denoting a real function of z, and z_s is real, Eq(11) may be written as

$$I(k)=\int_{SDP} f(z)e^{ik\hat{q}(z)}dz \sim \begin{cases} \sqrt{\dfrac{2\pi}{k\,|\,\hat{q}''(z_s)\,|}}\,f(z_s)e^{ik\hat{q}(z_s)+i\pi/4}, & \hat{q}''(z_s)>0 \\ \sqrt{\dfrac{2\pi}{k\,|\,\hat{q}''(z_s)\,|}}\,f(z_s)e^{ik\hat{q}(z_s)-i\pi/4}, & \hat{q}''(z_s)<0 \end{cases} \qquad (12)$$

A.2. Maliuzhinets Function

The functions of the form

$$\chi(\alpha)=\frac{\sqrt{k^2-\alpha^2}}{k+\beta\sqrt{k^2-\alpha^2}} \tag{13}$$

can be expressed in terms of Maliuzhinetz functions.

The following variable changes

$$\alpha=-k\cos\phi \quad \text{and} \quad \beta=\sec\lambda \tag{14}$$

with

$$\arcsin(\alpha/k)=\phi-\pi/2 \quad \text{and} \quad \sqrt{1-\beta^2}=-i\tan\lambda \tag{15}$$

gives the factors as

$$\chi_U(-k\cos\phi)=\sqrt{2/\beta}\sin(\phi/2)f(\pi-\phi+\lambda)f(\pi-\phi-\lambda) \tag{16}$$

The function $f(z)$ is defined as

$$f(z)=\frac{2^{3/4}}{1+\sqrt{2}\cos(z/2)}\left[\frac{M_\pi(z)}{M_\pi(\pi/2)}\right]^2 \tag{17}$$

with $M_\pi(z)$ being the Maliuzhinetz function.

Maliuzhinetz function $M_\pi(z)$ is an even meromorphic function of z and some of its properties are given by Bowmann (1967).

An alternative description of Maliuzhinetz function with $z=x+jy$ is given as

$$M_\pi(z) = \exp\left\{-\frac{1}{8\pi}\int_0^z \frac{\pi \sin u - 2\sqrt{2}\pi \sin(u/2) + 2u}{\cos u} du\right\} \tag{18}$$

and a recurrence relation satisfied by $M_\pi(z)$ is written as

$$M_\pi(z) = \left[M_\pi(\frac{\pi}{2})\right]^2 \frac{\cos(\frac{z}{4} - \frac{\pi}{8})}{M_\pi(z-\pi)} \tag{19}$$

in which $M_\pi(\frac{\pi}{2})$ is given by

$$M_\pi(\frac{\pi}{2}) = 0.96562. \tag{20}$$

In addition,

$$M_\pi(-z) = M_\pi(z) \tag{21}$$

$$M_\pi(-z^*) = M_\pi^*(z) \tag{22}$$

are also satisfied by Maliuzhinetz function.

For $x = 0$ or $y = 0$, Maliuzhinetz function is real-valued. And its approximate expressions in the corresponding strip was given by Senior and Volakis as below. Taylor series expansion of the function in the neighbourhood of $z = 0$ is given as

$$M_\pi(z) = 1 - az^2 + o(z^4) \tag{23}$$

where

$$a = \frac{1}{16}\left(1 - \sqrt{2} + \frac{2}{\pi}\right) = 0.01390 . \tag{24}$$

Then, for small values of argument,

$$M_{\pi}(z) = 1\text{-}1.01390z^{2}, \ |z| \to 0 \tag{25}$$

can be written.

On the other hand, for $y >> 0$,

$$b = \sqrt{\frac{2}{\sqrt{2}+1}}\exp\left(\frac{K}{2\pi}\right), \tag{26}$$

where

$$K = 0.91596\ldots \qquad (\textit{Catalan constant}) \tag{27}$$

then,

$$M_{\pi}(z) = b\sqrt{\cos\left(\frac{z - i\log z}{4}\right)}\exp\left\{\frac{iz}{2\pi}e^{iz}\right\} \tag{28}$$

can be written.