A TRI-OBJECTIVE REFORMULATION FOR THE DYNAMIC MEAN-VARIANCE PROBLEM

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

INDUSTRIAL ENGINEERING

By

Muhammed Mustafa Çolak

July 2024

A tri-objective reformulation for the dynamic mean-variance problem

By Muhammed Mustafa Çolak

July 2024

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Çağın Ararat(Advisor)

Firdevs Ulus

Serpil Sayın

Approved for the Graduate School of Engineering and Science:

Orhan Arıkan
Director of the Graduate School

ABSTRACT

A TRI-OBJECTIVE REFORMULATION FOR THE DYNAMIC MEAN-VARIANCE PROBLEM

Muhammed Mustafa Çolak
M.S. in Industrial Engineering
Advisor: Çağın Ararat
July 2024

The classical mean-variance problem aims to find a portfolio that minimizes a linear combination of the expectation and the variance of the terminal wealth. The dynamic version of the problem is known to be time-inconsistent in the classical sense, which makes the scalar dynamic programming approach inapplicable. By decomposing variance into two separate objectives, we introduce a tri-objective formulation in a discrete-time framework that generalizes the scalar problem and can reduce to the original setting. Using a less restrictive concept of time-consistency in a vector-valued sense, we show that the new formulation is time-consistent. Following the literature on set optimization, we develop a set-valued dynamic programming principle with the upper image of the vector-valued problem used as a value function. Finally, we reduce the generalized solutions of the formulation to the classical mean-variance problem using the minimal points of the three-dimensional upper images. We compute portfolios that are optimal for the initial mean-variance problem, and that remain time-consistent with respect to the tri-objective formulation.

Keywords: Mean-variance problem, time-inconsistency, dynamic programming principle, vector optimization, set-valued function.

ÖZET

DİNAMİK BEKLENTİ-DEĞİŞİNTİ PROBLEMİ İÇİN ÜÇ AMAÇLI BİR REFORMÜLASYON

Muhammed Mustafa Çolak
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Çağın Ararat
Temmuz 2024

Klasik beklenti-değişinti problemi, bir yatırımcının yatırım süreci sonundaki varlığının beklentisi ve değişintisinin doğrusal kombinasyonunu enküçükleyen yatırım stratejisini bulmayı hedefler. Bu problemin dinamik versiyonunun zamanda tutarsız olduğu bilinmektedir ve bu durum nedeniyle standart dinamik programlama yöntemlerini uygulamak mümkün değildir. Değişinti fonksiyonunu iki farklı amaç fonksiyonuna ayırarak üç amaç fonksiyonlu bir vektör eniyileme problemi elde ediyoruz. Bu problem, başlangıçtaki problemi genelleştirdiği gibi bu problemin başlangıç zamanındaki halini tekrardan elde etmemizi sağlamaktadır. Zamanda tutarlılığın vektörel tanımını kullanarak bu yeni problemin zamanda tutarlı olduğunu gösteriyoruz. Küme eniyilemesinde bilimsel yazınlarınların ışığında, küme değerli bir dinamik programlama ilkesi geliştirilmiştir ve bu ilke sayesinde yeni problem için uygun çözümler elde edilmiştir. Son olarak, bu vektör eniyileme probleminin genelleştirilmiş çözümlerini kullanarak başlangıçtaki klasik beklenti-değişinti problemini eniyileyen bir yatırım stratejisi bulunmaktadır. Aynı zamanda bu çözümler, vektör eniyleme problemi için zamanda tutarlı olmaktadır.

Anahtar sözcükler: Beklenti-değişinti problemi, zamanda tutarsızlık, dinamik programlama ilkesi, vektör iniyilemesi, küme değerli fonksiyon.

Acknowledgement

My sincerest thanks are for my advisor Professor Çağın Ararat. The three years we spent working together have decided my future, and I am forever thankful for your presence. I left each meeting with gratitude. Your guidance, patience and support have made all of this possible. I will count myself extremely lucky if I run into people half as great as you are.

I would like to thank Professors Firdevs Ulus and Serpil Sayın for sparing their valuable time to review this thesis and providing their helpful feedback.

I would like to acknowledge the financial support provided by TÜBİTAK under project 2210A.

I am very thankful to my friends Deniz Şahin, Deniz Akkaya, Osman, Emre, Göze, Doğuş, İrem, Rana, Ali İlhan, Gülin, Elif Sena, Göksu and Ecenur. I am very lucky to have been accompanied by their presence. They made my last two years bearable. I cannot think of a better friend group, and I will cherish the time we spent forever. Special thanks my roommate Ali Eren. It has been a joy to live with you under the same roof, and I will miss your presence around the house a lot.

I would like to thank Ece for her support, wisdom and affection. You have made me better in every sense I can imagine. Your energy and smile made each day we spent together perfect for me. I feel very lucky and happy to have met you.

Finally, I would like to thank my parents Melahat and Ali, and my sisters Ayşe and Ayça. I learned a lot from you. You always made me feel safe, loved and supported. It is a privilege to have you as a family. It is very comforting to feel that you are always going to be there for me. I love you. This thesis is devoted to you.

Contents

List of Figures

Chapter 1

Introduction

The problem of finding optimal investment strategies in a financial market is a fundamental issue in mathematical finance. The optimality is determined by a utility function, which generally involves the expectation and a measure of risk of the terminal wealth. In 1952, Markowitz introduced the mean-variance portfolio selection problem where a linear combination of the expectation and the variance of terminal wealth in one period is considered [1]. Markowitz discovers efficient mean and variance combinations that are optimal for different risk attitude parameters of the investor. Later, the mean-variance problem was generalized to discrete multi-period and continuous-time cases by Mossin [2], and Merton [3], respectively. In the discrete multi-period setting, the investor is able to change their strategies countably many times as the market changes in each period. In the continuous-time setting, the investor wants to make a time-dependent portfolio selection in a continuous manner. We work on the mean-variance problem in a discrete multi-period setting, where the prices of the assets receive shocks at each $t \in \mathbb{T}$, where $\mathbb{T} := \{0, \dots, T\}$ and $T \in \mathbb{N}$, and a portfolio decision is made right after.

It is well-known in the literature that the multi-period mean-variance problem suffers from time inconsistency, or the lack of dynamic programming principle. This results in cases where an optimal decision at time $t \in \mathbb{T} \setminus \{T\}$ may fail to be viewed as optimal at a later period $s > t$. The time inconsistency is caused by the variance term in the formulation of the problem and the fact that the objective function is a linear combination of mean and variance [4]. We will discuss more in depth why time inconsistency occurs, and how our approach handles it in Section 2.1 and Chapter 3.

There are various approaches in the literature to handle the time inconsistency. The first one is called the precommitment approach, where the investor does not deviate from their initial investment strategy even if it becomes suboptimal at a later time. Some of the studies utilizing precommitment approach in both discrete- and continuous-time can be found in [5], [4], [6], [7], [8].

The second one is the game-theoretic approach, introduced by [9] for general time inconsistent dynamic utility maximization problems, [10] solve the mean-variance problem using an infinite sequential game in continuous-time setting, where the opponents are the investor's future selves. Some examples in literature using game-theoretic approach can be found in [11], [12].

The last one that we will mention is the set optimization approach, where the problem is formulated as a vector optimization problem, and the upper images of the one-step dynamic programming problems are utilized to find an efficient frontier to the mean-variance problem. Although the variance of the terminal wealth cannot directly be implemented to this method due to its quadratic nature, [13] use a different risk measure to satisfy a set-valued Bellman's Principle. Additionally, [14] use an auxiliary formulation to the mean-variance problem using the second moment of the terminal wealth, which is suitable for a set-valued optimization approach, and implement the solutions of that problem to find time consistent mean-variance portfolios. Our approach will use the terms of the mean-variance objective function directly to unveil the dynamic nature, and the time consistent portfolios to the problem.

We consider the following problem where a linear combination of the expectation and the variance of the terminal wealth under a weight vector $\lambda_0 \in \mathbb{R}^2_+$ is minimized as

$$\text{minimize} \quad -\lambda_{0,1}\mathbb{E}(v_T^\pi) + \lambda_{0,2}\text{Var}(v_T^\pi) \quad \text{subject to} \quad \pi \in \Psi_0(v_0)$$

for initial wealth $v_0 \in \mathbb{R}_+$, the set of admissible portfolios $\Psi_0(v_0)$, and terminal wealth $v_T^\pi \in L_T(\mathbb{R}_+)$ under portfolio $\pi \in \Psi_0(v_0)$. This formulation can also be written explicitly as

$$\text{minimize} \quad -\lambda_{0,1}\mathbb{E}(v_T^\pi) + \lambda_{0,2}\left(\mathbb{E}((v_T^\pi)^2) - (\mathbb{E}(v_T^\pi))^2\right) \quad \text{subject to} \quad \pi \in \Psi_0(v_0) \tag{1.0.1}$$

A similar problem with the same weight vector can be formulated using conditional expectation and variance, that is, given $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R})$, it is formulated as

$$\text{minimize} \quad -\lambda_{0,1}\mathbb{E}_t(v_T^\pi) + \lambda_{0,2}\left(\mathbb{E}_t((v_T^\pi)^2) - (\mathbb{E}_t(v_T^\pi))^2\right) \quad \text{subject to} \quad \pi \in \Psi_t(v_t) \tag{1.0.2}$$

where $\Psi_t(v_t)$ is the set of admissible portfolios in time $t \in \mathbb{T}$. However [12] show that a constant weight vector does not reflect the dynamic attitude of an investor throughout the investment horizon. Instead, [14, Theorem 3.3.2] show that there exists a weight process $(\lambda_t)_{t\in\mathbb{T}}$ that makes the family of mean-variance problems time consistent with respect to their definition of time consistency. To that end, similar to the approach in [13], we abandon the scalar problem for a vector optimization problem, where the components of the vector-valued objective function involves each term in the problem (1.0.1) explicitly, that is, we consider the vector-valued function at time $t \in \mathbb{T}$ $\Gamma_t : L_T(\mathbb{R}^3_+) \to L_t(\mathbb{R}^3)$ defined as

$$\Gamma_t(v_T, y_T, z_T) = \begin{pmatrix} -\mathbb{E}_t(v_T) \\ \mathbb{E}_t(y_T) \\ -(\mathbb{E}_t(\sqrt{z_T}))^2 \end{pmatrix}$$

where when taken $t = 0$, $(v_T)^2 = y_T = z_T$, we obtain the objective function in (1.0.1) is vector form as

$$\Gamma_0(v_T, (v_T)^2, (v_T)^2) = \begin{pmatrix} -\mathbb{E}(v_T) \\ \mathbb{E}((v_T)^2) \\ -(\mathbb{E}(v_T))^2 \end{pmatrix}$$

The reason for using this sort of function will be made clear in Chapter 3. We observe that this formulation of the mean-variance problem is time-consistent in a vector-valued sense. Which in return provides a way for us to employ the upper images of the problems in intermediate times, formulated using conditional expectation and variance. In Theorem 3.2.2, we show that our formulation is in fact suitable for a dynamic programming approach, as in [13]. In contrast, we obtain the mean-variance upper images for each time period $t \in \mathbb{T}$ recursively, and utilize them to find efficient portfolios for each period. In [13], they use a dynamic convex risk measure that is less tedious compared to the variance term, and obtain the upper images accordingly. Our formulation is directly related to the mean-variance problem, that is, we do not utilize the mean-second moment auxiliary problem as in [14]. Instead, our approach generalizes the mean-variance problem by considering the three-dimensional vector-valued objective function Γ, which results in upper images that contain the efficient solutions of the mean-variance problem as well. However, this methodology results in nonconvex vector optimization problems that are more difficult to handle computationally. Moreover, the weight process $(\lambda_t)_{t\in\mathbb{T}}$ that makes the conditional version in (1.0.2) time consistent is explicitly given in [14]. However, we fail to provide such an output due to the nonconvex nature of our formulation, and the disadvantages of weighted sum scalarization in nonconvex vector optimization problems. Instead, we directly obtain the efficient portfolios to the problem in (1.0.1).

The vector-valued function Γ_t is closely related to the mean-variance problem as shown in (1). Therefore, we use this connection to obtain a set of portfolios that are optimal for the initial problem. We introduce backward and forward algorithms in Chapter 6 that provide a solution methodology to obtain weakly efficient portfolios.

The organization of the thesis is as follows: In Chapter 2, we introduce the underlying market structure. In Chapter 3, we propose the vector-valued objective function Γ and show how the vector optimization problem satisfies a set-valued Bellman's principle. In Chapter 4, we propose a methodology to compute the upper images of the vector optimization problems. In Chapter 5, we discuss the approximation errors caused by our solution methodology. In Chapter 6, we propose a methodology to find weakly efficient portfolios to the mean-variance problem. In Chapter 7, we apply our methodology to some example markets. Finally in Chapter 8, we compare the result of this thesis to [14] and discuss some future work.

Chapter 2

Preliminaries

On a discrete time horizon $\mathbb{T} := \{0, 1, \ldots, T\}$, $T \in \mathbb{N}$, consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ with $\mathcal{F}_0$ trivial and $\mathcal{F}_T = \mathcal{F}$. Without loss of generality, we assume that any nontrivial event $A \in \mathcal{F}$ has positive probability, that is, $\mathbb{P}(A) > 0$ for all $A \in \mathcal{F}, A \neq \emptyset$. For a subset $D \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, denote the space of all $\mathcal{F}_t$-measurable essentially bounded random vectors taking values in D by $L_t(D) := L_t^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; D)$. For any subset $B \subseteq L_t(\mathbb{R}^d)$, $\operatorname{cl} B, \operatorname{int} B$ denote the closure and the interior of B, respectively.

Let $x, y \in \mathbb{R}^d$. The space $(\mathbb{R}^d, \leq)$ is a partially ordered space where $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in \{1, \ldots, d\}$. In other words, each component of the vector x is less than the corresponding component of the vector y. Similarly, $(\mathbb{R}^d, <)$ is a partially ordered space where $x < y$ if and only if $x_i < y_i$ for all $i \in \{1, \ldots, d\}$ [15]. Let $X, Y \in L_t(\mathbb{R}^d)$. An analogous order for the partially ordered space $(L_t(\mathbb{R}^d), \leq_{L_t(\mathbb{R}^3_+)})$ is given as $X \leq_{L_t(\mathbb{R}^3_+)} Y$ if and only if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$ for all $\omega \in \Omega$, in space $(\mathbb{R}^d, \leq)$. Similarly, $X <_{L_t(\mathbb{R}^3_+)} Y$ if and only if $X(\omega) < Y(\omega)$, in space $(\mathbb{R}^d, <)$. Whenever the choice of ordering cone $L_t(\mathbb{R}^3_+)$ is obvious, we will use $\leq$ and $<$ as these partial orders. Let $A, B \subseteq L_t(\mathbb{R}^d)$. The Minkowski addition is defined as

$$A + B := \{X + Y \mid X \in A, Y \in B\}$$

For $A = \{\xi\}$, $\xi \in L_t(\mathbb{R}^d)$, we will simply write $\xi + B := \{\xi\} + B$. We define the operator $* : L_t(\mathbb{R}^d) \times 2^{L_t(\mathbb{R}^d)}$ as:

$$x * A := \{x \odot y \mid y \in A\}$$

where $\odot$ is the component-wise product. Let $\|\cdot\|_d : \mathbb{R}^d \times \mathbb{R}^d$ be the standard Euclidean norm. Define the Hausdorff distance $H : 2^{\mathbb{R}^d} \times 2^{\mathbb{R}^d}$ as:

$$H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|_d, \sup_{y \in B} \inf_{x \in A} \|x - y\|_d \right\}$$

A market with $d \in \mathbb{N}$ assets is modeled by an d-dimensional adapted process $(S_t)_{t\in\mathbb{T}}$ on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$. We assume that the distribution of the price process is known by the investor. The filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$ is generated by the price process $(S_t)_{t\in\mathbb{T}}$, and has a special structure given by the next assumption.

Assumption 2.0.1. *Let Ω_t denote the set of atoms (nodes) in $\mathcal{F}_t$ for all $t \in \mathbb{T}$. We assume a node-wise market structure where Ω_t satisfies the following:*

- *The elements of Ω_t define a partition for all $t \in \mathbb{T}$.*

- *Let $\omega_t \in \Omega_t$ and $\omega_{t+1} \in \Omega_{t+1}$ for some $t \in \mathbb{T}$. It must be true that either $\omega_{t+1} \subseteq \omega_t$ or $\omega_{t+1} \cap \omega_t = \emptyset$. In other words, either ω_{t+1} is a child node of ω_t or it is impossible to observe ω_t at time t and ω_{t+1} at time $t+1$. We define the successor set of $\omega_t \in \Omega_t$ as*

$$succ(\omega_t) := \{\omega_{t+1} \in \Omega_{t+1} \mid \omega_{t+1} \subseteq \omega_t\}.$$

It holds that $succ(\omega_t) \neq \emptyset$ for all $t \in \{0, \ldots, T-1\}$, $\omega_t \in \Omega_t$, and at time $T \in \mathbb{T}$, $succ(\omega_T) = \emptyset$ for all $\omega_T \in \Omega_T$.

An investor in this market considers the expected return and the variance of her strategies in each period, where a strategy or a portfolio is denoted by $(\pi_t)_{t\in\mathbb{T}\setminus\{0\}}$. In particular, $\pi_{t+1} = (\pi^1_{t+1}, \ldots, \pi^d_{t+1})^\mathsf{T} \in L_t(\mathbb{R}^d)$ denotes how many units of assets $1, \ldots, d$ are held in period $t+1 \in \mathbb{T}$, i.e., the portfolio made at time t, that will be reconsidered at time $t+1$. The investor enters the market with initial wealth $v_0 \in \mathbb{R}_{++}$, and follows a self-financing strategy; i.e., there will be no cash withdrawal or deposit in an intermediate time $t \in \{1, \ldots, T-1\}$. We will give the mathematical description of the feasible set of portfolios below:

Definition 2.0.2. *Under the current market assumptions, the set of feasible portfolios at period $t \in \mathbb{T}$ is defined as:*

$$\begin{aligned}\Psi_t(v_t) \coloneqq \big\{(\pi_s)_{s\geq t+1}\ |v_t = \pi^\mathsf{T}_{t+1}S_t, \pi_{t+1} \in \Phi_t, \pi^\mathsf{T}_s S_s = \pi^\mathsf{T}_{s+1}S_s, v_s \geq 0,\\ \pi_{s+1} \in \Phi_s, \forall s \in \{t+1, \ldots, T-1\}, v_T \geq 0\big\}\end{aligned}$$

where $\Phi_t \subseteq L_t(\mathbb{R}^d)$ is a closed conditionally convex set for all $t \in \mathbb{T}$, including trading restrictions such as no short-selling constraints, and $v_t \in L_t(\mathbb{R}_+)$ denotes the wealth at period $t \in \mathbb{T}$. The wealth at time $s \in \mathbb{T}$ for $s \geq t+1$, starting at time t with wealth v_t is denoted by $v^{t,v_t,\pi}_s \coloneqq v_s \coloneqq \pi^\mathsf{T}_s S_s$, for $\pi \in \Psi_t(v_t)$.

We assume a finite atomic probability space. Therefore, it is important to define the realization of the feasible set under an event.

Definition 2.0.3. *Let $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R}_+)$ and $\omega_t \in \Omega_t$. The feasible set under event ω_t is defined as:*

$$\begin{aligned}\Psi_t(v_t)(\omega_t) \coloneqq \big\{(\pi_s)_{s\geq t+1} \mid v_t(\omega_t) = \pi^\mathsf{T}_{t+1}(\omega_t)S_t(\omega_t), \pi_s(\omega_{s-1})^\mathsf{T}S_s(\omega_s) = \pi_{s+1}(\omega_s)^\mathsf{T}S_s(\omega_s), v_s(\omega_s) \geq 0\\ \pi_{t+1}(\omega_t) \in \Phi_t, \pi_{s+1}(\omega_{s+1}) \in \Phi_s, \forall \omega_s \subseteq \omega_{s-1}, s \in \{t+1, \ldots, T-1\}, v_T(\omega_T) \geq 0\big\}\end{aligned}$$

Remark 2.0.4. The feasible set $\Psi_t(v_t)$ is nonempty for all $v_t \in L_t(\mathbb{R}_+)$, $t \in \mathbb{T}$. Indeed, since the prices of the assets are strictly positive, we can simply invest all the wealth in period $t \in \mathbb{T}$ in the first asset at each period, and make no investments to other assets; i.e., let $\pi^1_{t+1} = \frac{v_t}{S^1_t}$, and $\pi^k_{t+1} = 0$ for all $k \in \{2, \dots, d\}$, and continue to do so for the remaining periods.

Remark 2.0.5. The finiteness of the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ automatically implies $\Psi_t(v_t)(\omega_t) = \Psi_t(v_t(\omega_t)\mathbf{1}_{\omega_t})$.

Lemma 2.0.6. *Let $t \in \mathbb{T} \setminus \{T\}$, $v_t \in L_t(\mathbb{R}_+)$ and $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)$. Then, $(\pi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi^\intercal_{t+1} S_{t+1})$ at all nodes of Ω_t. That is, for $\omega_t \in \Omega_t$, $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)(\omega_t)$ implies $(\pi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi^\intercal_{t+1} S_{t+1})(\omega_{t+1})$ for all $\omega_{t+1} \subseteq \omega_t$. Furthermore, let $(\pi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi^\intercal_{t+1} S_{t+1})$. Then, $(\pi_{t+1}, (\pi_s)_{s\geq t+2}) \in \Psi_t(v_t)$.*

Proof. Let $\omega_t \in \Omega_t$ and $\omega_{t+1} \in \Omega_{t+1}$ with $\omega_{t+1} \subseteq \omega_t$. Consider $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)(\omega_t)$. Then, $(\pi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi^\intercal_{t+1} S_{t+1}(\omega_{t+1})\mathbf{1}_{\omega_{t+1}}) = \Psi_{t+1}(\pi^\intercal_{t+1} S_{t+1})(\omega_{t+1})$ for all $\omega_t \in \Omega_t$ and $\omega_{t+1} \subseteq \omega_t$, by the definition of the atomic feasible set in 2.0.3. Similarly, whenever we have $(\pi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi^\intercal_{t+1} S_{t+1})$, we immediately get $(\pi_{t+1}, (\pi_s)_{s\geq t+2}) \in \Psi_t(v_t)$ by the same definition. □

The wealth at the terminal period $T \in \mathbb{T}$ for a feasible portfolio $\pi \in \Psi_t(v_t)$ is $v_T^{t,v_t,\pi} \in L_T(\mathbb{R}_+)$. Whenever the choices for t and v_t are clear, we will simply denote $v_T^{t,v_t,\pi} := v_T^\pi$.

Finally, for this formulation to be mathematically well-defined, we define the extended square root function as

$$\overline{\sqrt{x}} := \begin{cases} -\infty, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases} := \sqrt{x}$$

and the square function as

$$(x^+)^2 = \begin{cases} 0, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0 \end{cases} := x^2$$

to ensure monotonicity in Chapter 3. Throughout the thesis, we will refer to the extended square root function as the square root function, and the square of the positive part of a real number or a random variable will be denoted simply as the square of that random variable.

2.1 Problem Definition

We consider a multi-period mean variance problem at time $t = 0$ and aim to find optimal trading strategies throughout the investment horizon considering the risk attitude of the investor. The multi-period mean-variance problem with risk attitude $\lambda_0 \in \mathbb{R}^2_{++}$ and initial wealth $v_0 \in \mathbb{R}_{++}$ is given as

$$\text{minimize} \quad -\lambda_{0,1}\mathbb{E}(v_T^\pi) + \lambda_{0,2}\mathrm{Var}(v_T^\pi) \quad \text{subject to} \quad \pi \in \Psi_0(v_0) \qquad (MV_0(v_0, \lambda_0))$$

and the mean-variance problem at an intermediate time $t \in \mathbb{T} \setminus \{T\}$ with wealth $v_t \in L_t(\mathbb{R}_+)$ is given by:

$$\text{minimize} \quad -\lambda_{0,1}\mathbb{E}_t(v_T^\pi) + \lambda_{0,2}\mathrm{Var}_t(v_T^\pi) \quad \text{subject to} \quad \pi \in \Psi_t(v_t) \qquad (MV_t(v_t, \lambda_0))$$

This formulation of the mean-variance problem with constant weight vector is problematic since it is well-known that the problem does not satisfy a dynamic programming principle. Precisely, for the dynamic programming principle to hold, let

$$V_t(v_t) \coloneqq \inf_{\pi \in \Psi_t(v_t)} -\lambda_{0,1}\mathbb{E}_t(v_T^\pi) + \lambda_{0,2}\mathrm{Var}_t(v_T^\pi)$$

be the value function of the problem. The problem is said to satisfy the dynamic programming principle if

$$V_t(v_t) = \inf_{\pi \in \Psi_t(v_t)} -\mathbb{E}_t(V_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1})) + \mathrm{Var}_t(V_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1})).$$

However, we know that

$$\begin{aligned}
&\mathbb{E}_t(-\lambda_{0,1}\mathbb{E}_{t+1}(v_T^\pi) + \lambda_{0,2}\mathrm{Var}_{t+1}(v_T^\pi)) + \mathrm{Var}_t(-\lambda_{0,1}\mathbb{E}_{t+1}(v_T^\pi) + \lambda_{0,2}\mathrm{Var}_{t+1}(v_T^\pi)) \\
&\neq -\lambda_{0,1}\mathbb{E}_{t+1}(v_T^\pi) + \lambda_{0,2}\mathrm{Var}_{t+1}(v_T^\pi)
\end{aligned}$$

in general, due to the variance term. Specifically, the term $(\mathbb{E}_{t+1}(v_T^\pi))^2$ does not satisfy the tower property, which is the property satisfying

$$\mathbb{E}_t(\mathbb{E}_s(X)) = \mathbb{E}_t(X)$$

for $\mathcal{F}$-measurable X, and $t < s$.

One approach to overcome the lack of dynamic programming principle could be the vector-valued formulation of the mean variance problem given as

$$\text{minimize} \quad \begin{pmatrix} -\mathbb{E}_t(v_T^\pi) \\ \mathrm{Var}_t(v_T^\pi) \end{pmatrix} \quad \text{w.r.t.} \quad \leq_{L_t(\mathbb{R}^2_+)} \quad \text{subject to} \quad \pi \in \Psi_t(v_t) \qquad (VMV_t(v_t))$$

However, this formulation is still inadequate. Let $(\pi_s^*)_{s \geq t+2}$ be a desired solution to the problem $(VMV_{t+1}(v_{t+1}))$ (we will provide precise definitions to the desired solutions in a vector-valued setting in Section 2.2). For the dynamic programming principle to hold, this choice should also be desired at time $t \in \mathbb{T}$. With our new vector-valued objective function, we get

$$\begin{pmatrix} -\mathbb{E}_t(\mathbb{E}_{t+1}(v_T^{\pi^*})) \\ \mathrm{Var}_t(\mathrm{Var}_{t+1}(v_T^{\pi^*})) \end{pmatrix} \neq \begin{pmatrix} -\mathbb{E}_t(v_T^{\pi^*}) \\ \mathrm{Var}_t(v_T^{\pi^*}) \end{pmatrix}$$

since $\mathrm{Var}_t(\mathrm{Var}_{t+1}(v_T^{\pi^*})) \neq \mathrm{Var}_t(v_T^{\pi^*})$ in general.

As a solution to lack of 'tower property' for the variance, we will propose using the three-dimensional vector-valued function $\Gamma_t : L_T(\mathbb{R}^3_+) \to L_t(\mathbb{R}^3)$ defined as

$$\Gamma_t(v_T, y_T, z_T) \coloneqq \begin{pmatrix} -\mathbb{E}_t(v_T) \\ \mathbb{E}_t(y_T) \\ -(\mathbb{E}_t(\sqrt{z_T}))^2 \end{pmatrix}. \tag{2.1.1}$$

Here, if one sets $v_T^2 = y_T = z_T$, then we obtain the terms of $MV_t(v_t, \lambda_0)$ in a vector, which can be used to obtain the scalar objective back. The recursiveness, or the 'tower property' of the new vector-valued function Γ_t will be shown in Chapter 3.

Using Γ_t, we will define a new formulation for the mean-variance problem in three-dimensions:

$$\begin{aligned} \text{minimize } & \Gamma_t(v_T, (v_T^\pi)^2, (v_T^\pi)^2) \quad \text{w.r.t.} \quad \leq_{L_t(\mathbb{R}^3_+)} \\ \text{subject to } & v_t = \pi_{t+1}^\mathsf{T} S_t \\ & v_{s+1} = v_s + \pi_{s+1}^\mathsf{T}(S_{s+1} - S_s) \\ & \pi_{s+1} \in \Phi_s \\ & v_s \geq 0 \\ & \forall s \in \{t, \ldots, T-1\} \end{aligned} \tag{$G_t(v_t)$}$$

Remark 2.1.1. The problem $G_0(v_0)$ scalarized with the weight vector $\bar{\lambda} := (\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,2}) \in \mathbb{R}^3$ is equivalent to the problem $MV_0(v_0, \lambda_0)$. Indeed, the feasible sets and the objective functions are the same in both problems.

Since the formulation is now vector-valued, the upper image of the problem carries great importance, especially for the dynamic structure of it. Therefore, we define the upper image here as:

$$\mathcal{P}_t(v_t) := \mathrm{cl}\left(\left\{\Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) \mid \pi \in \Psi_t(v_t)\right\} + L_t(\mathbb{R}^3_+)\right). \tag{2.1.2}$$

2.2 Solution Concepts

The vector-valued structure we will utilize throughout the rest of the thesis requires careful definitions for the solution concepts of the problem. These definitions are borrowed from the literature on vector optimization.

Definition 2.2.1. *[16, Definition 1.41] Let $B \subseteq L_t(\mathbb{R}^m)$. A point $x \in B$ is called a minimal point in B if*

$$(x - L_t(\mathbb{R}^m_+) \setminus \{0\}) \cap B = \emptyset.$$

It is called a weakly minimal point if

$$(x - L_t(\mathbb{R}^m_{++})) \cap B = \emptyset.$$

Definition 2.2.2. *[16, Definition 2.1] We say a feasible portfolio $(\pi_s)_{s \geq t+1} \in \Psi_t(v_t)$ is an efficient portfolio at time $t \in \mathbb{T}$ with wealth $v_t \in L_t(\mathbb{R}_+)$, if there exists no portfolio $(\psi_s)_{s \geq t+1} \in \Psi_t(v_t)$ such that*

$$-\mathbb{E}_t(v_T^\psi) \leq -\mathbb{E}_t(v_T^\pi),\ \mathbb{E}_t((v_T^\psi)^2) \leq \mathbb{E}_t((v_T^\pi)^2),\ -(\mathbb{E}_t(v_T^\psi))^2 \leq -(\mathbb{E}_t(v_T^\pi))^2, \tag{2.2.1}$$

and there exists $\omega_t \in \Omega_t$ where at least one of the inequalities is strict.

A feasible portfolio $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)$ is called a weakly efficient portfolio at time $t \in \mathbb{T}$ with wealth $v_t \in L_t(\mathbb{R}_+)$, if there exists no portfolio $(\psi_s)_{s\geq t+1} \in \Psi_t(v_t)$ with

$$-\mathbb{E}_t(v_T^\psi) < -\mathbb{E}_t(v_T^\pi),\ \mathbb{E}_t((v_T^\psi)^2) < \mathbb{E}_t((v_T^\pi)^2),\ -(\mathbb{E}_t(v_T^\psi))^2 < -(\mathbb{E}_t(v_T^\pi))^2.$$

A feasible portfolio $\pi \in \Psi_t(v_t)$ is called a minimizer of the problem $G_t(v_t)$ if

$$\left(\Gamma_t(V_T^\pi, (V_T^\pi)^2, (V_T^\pi)^2) - L_t(\mathbb{R}^3_+) \setminus \{0\}\right) \cap \left\{\Gamma_t(V_T^\psi, (V_T^\psi)^2, (V_T^\psi)^2) \mid \psi \in \Psi_t(v_t)\right\} = \emptyset$$

and it is called a weak minimizer of the problem $G_t(v_t)$ if

$$\left(\Gamma_t(V_T^\pi, (V_T^\pi)^2, (V_T^\pi)^2) - L_t(\mathbb{R}^3_{++})\right) \cap \left\{\Gamma_t(V_T^\psi, (V_T^\psi)^2, (V_T^\psi)^2) \mid \psi \in \Psi_t(v_t)\right\} = \emptyset.$$

Note that Definition 2.2.2 uses the partial ordering in the space $(L_t(\mathbb{R}^3_+), \leq)$, i.e., a portfolio $\pi \in \Psi_t(v_t)$ is an efficient portfolio at time $t \in \mathbb{T}$ with wealth $v_t \in L_t(\mathbb{R}_+)$, if there exists no portfolio $\psi \in \Psi_t(v_t)$ with

$$\Gamma_t(v_T^\psi, (v_T^\psi)^2, (v_T^\psi)^2) \leq_{L_t(\mathbb{R}^3_+)} \Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) \text{ and } \Gamma(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) \neq \Gamma_t(v_T^\psi, (v_T^\psi)^2, (v_T^\psi)^2).$$

Similarly, a portfolio $\pi \in \Psi_t(v_t)$ is a weakly efficient portfolio at time $t \in \mathbb{T}$ with wealth $v_t \in L_t(\mathbb{R}_+)$ if there exists no portfolio $\psi \in \Psi_t(v_t)$ with

$$\Gamma_t(v_T^\psi, (v_T^\psi)^2, (v_T^\psi)^2) \in \left(\Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) - L_t(\mathbb{R}^3_{++})\right).$$

Remark 2.2.3. Definitions 2.2.1 and 2.2.2 imply that a portfolio $\pi \in \Psi_t(v_t)$ is a (weakly) efficient portfolio at time $t \in \mathbb{T}$ with wealth $v_t \in L_t(\mathbb{R}_+)$ if and only if it is a (weak) minimizer of the problem $G_t(v_t)$. In both cases, its image $\Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$ is a (weakly) minimal element of the set $\{\Gamma_t(v_T^{\psi}, (v_T^{\psi})^2, (v_T^{\psi})^2) \mid \psi \in \Psi_t(v_t)\}$ [13, Lemma 2].

Later, we will also show that any boundary point of the upper image $\mathcal{P}_t(v_t)$ corresponds to some (weakly) efficient portfolio $\pi \in \Psi_t(v_t)$.

Chapter 3

Time-consistency and Set-valued Bellman's Principle

In this chapter, we will show that the problem $G_t(v_t)$ is time-consistent in the vector-valued sense, and there exists a dynamic family of problems that can be solved to obtain the value function of $G_t(v_t)$. Before getting into that, we will explore some of the useful properties of the function Γ_t defined in (2.1.1). For convenience, let us introduce the matrix

$$M := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proposition 3.0.1. *Let $t \in \mathbb{T}$, $v_T, y_T, z_T, \bar{x}_T, \bar{y}_T, \bar{z}_T \in L_T(\mathbb{R}_+)$. The vector-valued function $\Gamma_t : L_T(\mathbb{R}^3_+) \to L_t(\mathbb{R}^3)$ has the following properties:*

1. *Recursiveness:*

$$\Gamma_t(M\Gamma_{t+1}(x_T, y_T, z_T)) = \Gamma_t(x_T, y_T, z_T)$$

2. *Monotonicity: Let* $x_T \geq \bar{x}_T, y_T \leq \bar{y}_T, z_T \geq \bar{z}_T$*. Then,* $\Gamma_t(x_T, y_T, z_T) \leq \Gamma_t(\bar{x}_T, \bar{y}_T, \bar{z}_T)$*.*

3. *Strict monotonicity: Let* $x_T > \bar{x}_T, y_T < \bar{y}_T, z_T > \bar{z}_T$*. Then* $\Gamma_t(x_T, y_T, z_T) < \Gamma_t(\bar{x}_T, \bar{y}_T, \bar{z}_T)$*.*

4. *Continuity:* Γ_t *is continuous.*

Proof. 1. We have

$$\begin{aligned}
\Gamma_t(M\Gamma_{t+1}(x_T, y_T, z_T)) &= \Gamma_t\left(\mathbb{E}_{t+1}(x_T), \mathbb{E}_{t+1}(y_T), (\mathbb{E}_{t+1}(\sqrt{z_T}))^2\right) \\
&= \begin{pmatrix} -\mathbb{E}_t(\mathbb{E}_{t+1}(x_T)) \\ \mathbb{E}_t(\mathbb{E}_{t+1}(y_T)) \\ -(\mathbb{E}_t(\sqrt{(\mathbb{E}_{t+1}(\sqrt{z_T}))^2}))^2 \end{pmatrix} \\
&= \begin{pmatrix} -\mathbb{E}_t(x_T) \\ \mathbb{E}_t(y_T) \\ -(\mathbb{E}_t(\sqrt{z_T}))^2 \end{pmatrix} \\
&= \Gamma_t(x_T, y_T, z_T)
\end{aligned}$$

proving the recursiveness.

2. Immediate by the monotonicity of the expectation and the square function.

3. Similarly, immediate by the strict monotonicity of the expectation and the square function.

4. Continuity is immediate on a finite probability space.

□

3.1 Time-consistency

We first want to show that the family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$ is suitable for an approach involving dynamic programming. For that, we define time consistency in the following way:

Definition 3.1.1. *[13] The family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$ is called time-consistent with respect to (weak) minimizers if for every $t \in \mathbb{T}$ and $v_t \in L_t(\mathbb{R}_+)$, portfolio $(\pi_s)_{s\geq t+1}$ being a (weak) minimizer for problem $G_t(v_t)$ implies the portfolio $(\pi_s)_{s\geq t+2}$ being a (weak) minimizer for problem $(G_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1}))$.*

Remark 3.1.2. Time-consistency with respect to minimizers and weak minimizers do not imply each other in general.

We will now show that the family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$ is time-consistent with respect to both weak minimizers and minimizers. This will imply that a dynamic programming approach involving the vector-valued function Γ_t can be incorporated in order to obtain efficient solutions in a recursive manner.

Proposition 3.1.3. *The family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$ is time consistent with respect to weak minimizers.*

Proof. Let $t \in \mathbb{T}\setminus\{T\}$, $v_t \in L_t(\mathbb{R}_+)$ and $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)$ be a weak minimizer for $G_t(v_t)$. Assume $(\pi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1})$ is not a weak minimizer for $G_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1})$. Then, by the definition of a weak minimizer, there exists $(\phi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1})$ such that

$$\Gamma_{t+1}(v_T^{\phi}, (v_T^{\phi})^2, (v_T^{\phi})^2) < \Gamma_{t+1}(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2).$$

Let $\phi_{t+1} \coloneqq \pi_{t+1}$, so that $(\phi_s)_{s\geq t+1} \in \Psi_t(v_t)$ by Lemma 2.0.6. Using the recursiveness and the strict monotonicity of Γ_t, we have

$$\Gamma_t(M\Gamma_{t+1}(v_T^\phi, (v_T^\phi)^2, (v_T^\phi)^2)) < \Gamma_t(M\Gamma_{t+1}(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2))$$
$$\implies \Gamma_t(v_T^\phi, (v_T^\phi)^2, (v_T^\phi)^2) < \Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$$

which contradicts the fact that $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)$ is a weak minimizer. Hence, the family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$ is time consistent with respect to weak minimizers. □

Proposition 3.1.4. *The family of problems $(G_t(v_t))_{t\in\mathbb{T}}$ is time consistent with respect to minimizers.*

Proof. Let $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R}_+)$ and $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)$ be a minimizer for $G_t(v_t)$. Assume that $(\pi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi_{t+1}^\mathsf{T} S_{t+1})$ is not a minimizer for $G_{t+1}(\pi_{t+1}^\mathsf{T} S_{t+1})$. Then, by the definition of a minimizer, there exists a portfolio $(\phi_s)_{s\geq t+2} \in \Psi_{t+1}(\pi_{t+1}^\mathsf{T} S_{t+1})$ such that

$$\Gamma_{t+1}(v_T^\phi, (v_T^\phi)^2, (v_T^\phi)^2) \leq \Gamma_{t+1}(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) \text{ and}$$
$$\Gamma_{t+1}(v_T^\phi, (v_T^\phi)^2, (v_T^\phi)^2) \neq \Gamma_{t+1}(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$$

Let $\phi_{t+1} \coloneqq \pi_{t+1}$. Then, $(\phi_s)_{s\geq t+1} \in \Psi_t(v_t)$ by Lemma 2.0.6. Then, by the recursiveness of the function Γ_t:

$$\Gamma_t(M\Gamma_{t+1}(v_T^\phi, (v_T^\phi)^2, (v_T^\phi)^2)) \leq \Gamma_t(M\Gamma_{t+1}(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2))$$
$$\implies \Gamma_t(v_T^\phi, (v_T^\phi)^2, (v_T^\phi)^2)) \leq \Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$$

and by the strict monotonicity of Γ_t:

$$\begin{aligned}
&\Gamma_{t+1}(v_T^{\phi}, (v_T^{\phi})^2, (v_T^{\phi})^2) \neq \Gamma_{t+1}(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2) \\
\Longrightarrow\ &\Gamma_t(M\Gamma_{t+1}(v_T^{\phi}, (v_T^{\phi})^2, (v_T^{\phi})^2)) \neq \Gamma_t(M\Gamma_{t+1}(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2)) \\
\Longrightarrow\ &\Gamma_t(v_T^{\phi}, (v_T^{\phi})^2, (v_T^{\phi})^2) \neq \Gamma_t(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2)
\end{aligned}$$

which implies that $(\pi_s)_{s\geq t+1} \in \Psi_t(v_t)$ is not efficient, resulting in a contradiction. □

Remark 3.1.5. We have shown that for every efficient portfolio found at time $0 \in \mathbb{T}$, there exist portfolios that stay efficient throughout the investment horizon. This is a very strong result acquired thanks to the strict monotonicity of the objective function. Intuitively, an investor would only be interested in portfolios that are efficient. This property may fail to hold without strict monotonicity, as in [13].

3.2 Set-valued Bellman's Principle

In Section 3.1, the family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$ is shown to be time-consistent with respect to weak minimizers and minimizers. Our next objective is to define an appropriate value function that satisfies a dynamic programming principle in an extended sense. For a generic scalar dynamic programming problem, the value function is obtained by finding the infimum of the objective function at a future time. In the case of a vector-valued objective function, the infimum is not attainable in general [16, Chapter 1]. This requires us to work in a more sophisticated space with a suitable objective function.

Let $\mathbb{F}$ be the set of all closed upper sets defined in [17, Example 2.11]; i.e.

$$\mathbb{F} := \{A \subseteq L_t(\mathbb{R}^3_+) \mid A = \mathrm{cl}(A + L_t(\mathbb{R}^3_+))\}.$$

The partially ordered space $(\mathbb{F}, \supseteq)$ is a complete lattice, i.e., the infimum of a subset $\mathbb{A}$ of $\mathbb{F}$ exists and is given as:

$$\inf \mathbb{A} := \operatorname{cl} \bigcup_{A \in \mathbb{A}} A.$$

Moreover, let $x_T, y_T, z_T \in L_T(\mathbb{R}_+)$ and define the set-valued objective function $\mathcal{S}_t : L_T(\mathbb{R}^3_+) \to \mathbb{F}$ as:

$$\mathcal{S}_t(x_T, y_T, z_T) := \Gamma_t(x_T, y_T, z_T) + L_t(\mathbb{R}^3_+).$$

We will replace the objective function in $G_t(v_t)$ with the set-valued function $\mathcal{S}_t$. The problem with set-valued objective function $\mathcal{S}_t$ is defined as

$$\text{minimize } \mathcal{S}_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) \text{ with respect to } \supseteq \text{ subject to } \pi \in \Psi_t(v_t). \qquad (SOP_t(v_t))$$

We immediately observe that

$$\begin{aligned}
\inf_{\pi \in \Psi_t(v_t)} \mathcal{S}_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) &= \operatorname{cl} \bigcup_{\pi \in \Psi_t(v_t)} \left(\Gamma_t \left(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2 \right) + L_t(\mathbb{R}^3_+) \right) \\
&= \operatorname{cl} \left(\{ \Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) \mid \pi \in \Psi_t(v_t) \} + L_t(\mathbb{R}^3_+) \right) \\
&= \mathcal{P}_t(v_t),
\end{aligned}$$

which is the upper image of problem $G_t(v_t)$ as defined in 2.1.2, and it can be recovered by finding the minimal points of the problem [16, Proposition 4.7]. We also want to use the upper image as a value function to the set optimization problem $SOP_t(v_t)$. In order to show a dynamic programming principle for this value function, we first prove the recursiveness of the set-valued objective function.

Proposition 3.2.1. *Let $t \in \mathbb{T} \setminus \{T\}$ and $x_T, y_T, z_T \in L_T(\mathbb{R}_+)$. The set-valued function $\mathcal{S}_t$*

has the following recursiveness property:

$$\mathcal{S}_t(x_T, y_T, z_T) = \bigcup_{\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)} \mathcal{S}_t(M\xi)$$

.

Proof. To prove the $\subseteq$ part of the claimed equality, let $a \in \mathcal{S}_t(x_T, y_T, z_T)$. Define $\xi := \Gamma_{t+1}(x_T, y_T, z_T)$. Note that $\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)$. Then,

$$\begin{aligned}
\mathcal{S}_t(M\xi) &= \mathcal{S}_t(M\Gamma_{t+1}(x_T, y_T, z_T)) \\
&= \Gamma_t(M\Gamma_{t+1}(x_T, y_T, z_T)) + L_t(\mathbb{R}^3_+) \\
&= \Gamma_t(x_T, y_T, z_T) + L_t(\mathbb{R}^3_+) \\
&= \mathcal{S}_t(x_T, y_T, z_T),
\end{aligned}$$

by the recursiveness of Γ_t. Then, $a \in \mathcal{S}_t(M\xi)$ is automatic. Thus, $a \in \operatorname{cl} \bigcup_{\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)} \mathcal{S}_t(M\xi)$.

For the $\supseteq$ part, since both sets are closed, it is enough to show that $\bigcup_{\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)} \mathcal{S}_t(M\xi) \subseteq \mathcal{S}_t(x_T, y_T, z_T)$. Let $y \in \bigcup_{\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)} \mathcal{S}_t(M\xi)$. Then, there exists $\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)$ such that $y \in \mathcal{S}_t(M\xi)$. Let $\xi = a + b$, where $a = \Gamma_{t+1}(x_T, y_T, z_T)$ and $b \in L_{t+1}(\mathbb{R}^3_+)$. By the monotonicity of Γ_t, we have

$$\Gamma_t(Mx) \geq \Gamma_t(Ma) = \Gamma_t(x_T, y_T, z_T) \in \mathcal{S}_t(x_T, y_T, z_T).$$

Since $\mathcal{S}_t(x_T, y_T, z_T)$ is an upper set, we get $y \in \mathcal{S}_t(x_T, y_T, z_T)$. Moreover, since $\mathcal{S}_t(x_T, y_T, z_T)$

is closed, we get

$$\bigcup_{\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)} \mathcal{S}_t(M\xi) \subseteq \mathcal{S}_t(x_T, y_T, z_T)$$

$$\implies \text{cl} \bigcup_{\xi \in \mathcal{S}_{t+1}(x_T, y_T, z_T)} \mathcal{S}_t(M\xi) \subseteq \mathcal{S}_t(x_T, y_T, z_T),$$

proving the equality. □

We have shown the dynamic structure of $SOP_t(v_t)$, and have found a candidate value function. The next step is to define a one-step optimization problem that will make use of the dynamic structure and satisfy a Bellman's Principle.

Define the following one-step problem:

$$\begin{aligned} \text{min.} \quad & \Gamma_t(M\xi) \quad \text{w.r.t.} \quad \leq_{L_t(\mathbb{R}^3_+)} \\ \text{s.t.} \quad & v_t = \pi_{t+1}^{\mathsf{T}} S_t \\ & \pi_{t+1} \in \Phi_t \\ & \xi \in \mathcal{P}_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1}) \end{aligned} \tag{$\tilde{G}_t(v_t)$}$$

with the feasible set

$$\tilde{\Psi}_t(v_t) := \left\{ (\pi_{t+1}, \xi) \mid v_t = \pi_{t+1}^{\mathsf{T}} S_t, \pi_{t+1} \in \Phi_t, \xi \in \mathcal{P}_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1}) \right\}$$

and the upper image

$$\tilde{\mathcal{P}}_t(v_t) := \text{cl}\left(\left\{ \Gamma_t(M\xi) \mid (\pi_{t+1}, \xi) \in \tilde{\Psi}_t(v_t) \right\} + L_t(\mathbb{R}^3_+) \right).$$

Finally, we will prove that the upper image is in fact suitable for a Bellman's Principle, in other words, the value function at time $t \in \mathbb{T}$ can be obtained using the value function at time $t+1$.

Theorem 3.2.2. *Let $t \in \mathbb{T} \setminus \{T\}$ and $v_t \in L_t(\mathbb{R}_+)$. Then, $\mathcal{P}_t(v_t) = \tilde{\mathcal{P}}_t(v_t)$, i.e.,*

$$\mathcal{P}_t(v_t) = \mathrm{cl}\left(\left\{\Gamma_t(M\xi) \mid v_t = \pi_{t+1}^{\mathsf{T}} S_t, \pi_{t+1} \in \Phi_t, \xi \in \mathcal{P}_{t+1}(\pi_{t+1}^{\mathsf{T}} S_{t+1})\right\} + L_t(\mathbb{R}^3_+)\right)$$

for all $t \in \mathbb{T}$ and $v_t \in L_t(\mathbb{R}_+)$.

Proof. We will first prove $\mathcal{P}_t(v_t) \subseteq \tilde{\mathcal{P}}_t(v_t)$. Let $x = a + b$ with $a \in \{\Gamma_t(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2) \mid \pi \in \Psi_t(v_t)\}$ and $b \in L_t(\mathbb{R}^3_+)$. Also let $\bar{\pi} \in \Psi_t(v_t)$ be such that $a = \Gamma_t(v_T^{\bar{\pi}}, (v_T^{\bar{\pi}})^2, (v_T^{\bar{\pi}})^2)$. Note that $\bar{\pi} = (\bar{\pi}_{t+1}, \bar{\pi}')$ with $\bar{\pi}' := (\bar{\pi}_s)_{s \geq t+2} \in \Psi_{t+1}(v_{t+1})$, and $v_{t+1} := \bar{\pi}_{t+1}^{\mathsf{T}} S_{t+1}$. Define $u := \Gamma_{t+1}(v_T^{\bar{\pi}'}, (v_T^{\bar{\pi}'})^2, (v_T^{\bar{\pi}'})^2)$. We have $(\bar{\pi}_{t+1}, u) \in \tilde{\Psi}_t(v_t)$, since $u \in \mathcal{P}_{t+1}(v_{t+1})$. Then, $\Gamma_t(Mu) \in \tilde{\mathcal{P}}_t(v_t)$ and

$$\Gamma_t(Mu) = \Gamma_t(M\Gamma_{t+1}(v_T^{\bar{\pi}'}, (v_T^{\bar{\pi}'})^2, (v_T^{\bar{\pi}'})^2)) = \Gamma_t(v_T^{\bar{\pi}'}, (v_T^{\bar{\pi}'})^2, (v_T^{\bar{\pi}'})^2) = a,$$

since $v_T^{\bar{\pi}'} = v_T^{\pi}$. Thus, $a \in \tilde{\mathcal{P}}_t(v_t)$. So, $a + b = x \in \tilde{\mathcal{P}}_t(v_t)$.

To prove $\mathcal{P}_t(v_t) \supseteq \tilde{\mathcal{P}}_t(v_t)$, let $y = u + v$ with $u \in \{\Gamma_t(Mx) \mid (\pi_{t+1}, x) \in \tilde{\Psi}_t(v_t)\}$ and $v \in L_t(\mathbb{R}^3_+)$. Let $(\bar{\pi}_{t+1}, \bar{x}) \in \tilde{\Psi}_t(v_t)$ be such that $u = \Gamma_t(M\bar{x})$, $v_{t+1} = \bar{\pi}_{t+1}^{\mathsf{T}} S_{t+1}$ and $\bar{x} \in \mathcal{P}_{t+1}(v_{t+1})$. By the definition of $\mathcal{P}_{t+1}(v_{t+1})$ in 2.1.2, we know that there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\imath\mathbb{N}}$ such that

$$\bar{x}_n := a_n + b_n \in \left\{\Gamma_{t+1}(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2) \mid \pi \in \Psi_{t+1}(v_{t+1})\right\} + L_{t+1}(\mathbb{R}^3_+)$$

with $\lim_{n\to\infty} x_n = \lim_{n\to\infty}(a_n + b_n) = x$, and $a_n \in \{\Gamma_{t+1}(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2) \mid \pi \in \Psi_{t+1}(v_{t+1})\}$ and $b_n \in L_{t+1}(\mathbb{R}^3_+)$ for all $n \in \mathbb{N}$. Finally, let $\pi^n \in \Psi_{t+1}(v_{t+1})$ be such that $a_n =$

$\Gamma_{t+1}(v_T^{\pi^n}, (v_T^{\pi^n})^2, (v_T^{\pi^n})^2)$ and define $\tilde{\pi}^n := (\bar{\pi}_{t+1}, \pi^n) \in \Psi_t(v_t)$ for all $n \in \mathbb{N}$. Thus,

$$\Gamma_t(Ma_n) = \Gamma_t(M\Gamma_{t+1}(v_T^{\pi^n}, (v_T^{\pi^n})^2, (v_T^{\pi^n})^2)) = \Gamma_t(v_T^{\tilde{\pi}^n}, (v_T^{\tilde{\pi}^n})^2, (v_T^{\tilde{\pi}^n})^2) \in \mathcal{P}_t(v_t)$$

for all $n \in \mathbb{N}$. Since Γ_t is monotone, we have $\Gamma_t(Mx_n) \geq \Gamma_t(Ma_n)$ for all $n \in \mathbb{N}$. Using the fact that $\mathcal{P}_t(v_t)$ is an upper set, $\Gamma_t(Mx_n) \in \mathcal{P}_t(v_t)$ for all $n \in \mathbb{N}$. Since Γ_t is continuous, we have that $\lim_{n\to\infty} \Gamma_t(Mx_n) = \Gamma_t(M\bar{x})$ and $\mathcal{P}_t(v_t)$ is closed, we have $\Gamma_t(M\bar{x}) = u \in \mathcal{P}_t(v_t)$. Finally, $u + v \in \mathcal{P}_t(v_t)$, since $\mathcal{P}_t(v_t) = \mathcal{P}_t(v_t) + L_t(\mathbb{R}^3_+)$. □

Theorem 3.2.2 also shows that the upper images of the problems $G_t(v_t)$ and $\tilde{G}_t(v_t)$ are equal.

We have shown that the upper images of the family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$ can be used as a value function for the family of dynamic programming problems $(\tilde{G}_t(\cdot))_{t\in\mathbb{T}}$. The upper images contain the minimal points and associated portfolios that are efficient. Our next step is to compute these upper images.

3.3 Positive Homogeneity of the Upper Images

Using upper images as a value function requires access to all the information related to the set-valued function $\mathcal{P}_t(\cdot)$. Theorem 3.2.2 provides a way to calculate $\mathcal{P}_t(v_t)$, where we require access to uncountably many choices of $\mathcal{F}_{t+1}$-measurable v_{t+1}. From a computational stand point, this is generally impossible. Luckily, the family of upper images $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$ satisfies a scaling property that will provide a method to access any choice of v_{t+1}.

Proposition 3.3.1. *Let $t \in \mathbb{T}$. Let Φ_t be a cone for all $t \in \mathbb{T} \setminus \{T\}$. The set-valued function*

$\mathcal{P}_t(\cdot)$ is positively homogeneous in the following sense:

$$\mathcal{P}_t(v_t) = \begin{pmatrix} v_t \\ v_t^2 \\ v_t^2 \end{pmatrix} * \mathcal{P}_t(\mathbf{1}),$$

for every $v_t \in L_t(\mathbb{R}_{++})$.

Proof. Let $x = a + b$, where $a \in \{\Gamma_t(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2) \mid \pi \in \Psi_t(v_t)\}$ and $b \in L_t(\mathbb{R}_+^3)$. Let $\bar{\pi} \in \Psi_t(v_t)$ be such that $a = \Gamma_t(v_T^{\bar{\pi}}, (v_T^{\bar{\pi}})^2, (v_T^{\bar{\pi}})^2)$. Let $\tilde{\pi} := \frac{\bar{\pi}}{v_t}$. So,

$$\begin{aligned} v_t = \bar{\pi}_{t+1}^{\mathsf{T}} S_t \implies 1 &= (\frac{\bar{\pi}_{t+1}}{v_t})^{\mathsf{T}} S_t = \tilde{\pi}_{t+1}^{\mathsf{T}} S_t \quad \text{and} \\ v_s^{\bar{\pi}} = \pi_s^{\mathsf{T}} S_s \implies \frac{v_s^{\bar{\pi}}}{v_t} &= (\frac{\bar{\pi}_s}{v_t})^{\mathsf{T}} S_s = \tilde{\pi}_s^{\mathsf{T}} S_s \end{aligned}$$

for $s \geq t+1$. Thus, $\tilde{\pi} \in \Psi_t(1)$. Additionally,

$$\begin{aligned} v_T^{\bar{\pi}} = v_t + \sum_{s=t}^{T-1} (v_{s+1}^{\bar{\pi}} - v_s^{\bar{\pi}}) \implies \frac{v_T^{\bar{\pi}}}{v_t} &= 1 + \sum_{s=t}^{T-1} (\frac{v_{s+1}^{\bar{\pi}} - v_s^{\bar{\pi}}}{v_t}) \\ \implies v_T^{\tilde{\pi}} &= 1 + \sum_{s=t}^{T-1} (v_{s+1}^{\tilde{\pi}} - v_s^{\tilde{\pi}}) \end{aligned}$$

where $(v_t^{\tilde{\pi}})_{t \in \mathbb{T}}$ is the wealth process under portfolio $\tilde{\pi} \in \Psi_t(1)$. Thus,

$$a = \begin{pmatrix} -\mathbb{E}_t(v_T^{\bar{\pi}}) \\ \mathbb{E}_t((v_T^{\bar{\pi}})^2) \\ -(\mathbb{E}_t(v_T^{\bar{\pi}}))^2 \end{pmatrix} = \begin{pmatrix} v_t \\ v_t^2 \\ v_t^2 \end{pmatrix} * \begin{pmatrix} -\mathbb{E}_t(v_T^{\tilde{\pi}}) \\ \mathbb{E}_t((v_T^{\tilde{\pi}})^2) \\ -(\mathbb{E}_t(v_T^{\tilde{\pi}}))^2 \end{pmatrix} = \begin{pmatrix} v_t \\ v_t^2 \\ v_t^2 \end{pmatrix} * \Gamma_t(v_T^{\tilde{\pi}}, (v_T^{\tilde{\pi}})^2, (v_T^{\tilde{\pi}})^2)$$

where $\Gamma_t(v_T^{\tilde{\pi}}, (v_T^{\tilde{\pi}})^2, (v_T^{\tilde{\pi}})^2) \in \{\Gamma_t(v_T^{\pi}, (v_T^{\pi})^2, (v_T^{\pi})^2) \mid \pi \in \Psi_t(1)\}$. Finally, since v_t is positive for all $t \in \mathbb{T}$ and the upper images are closed, we have $a + b \in (v_t, v_t^2, v_t^2)^{\mathsf{T}} * \mathcal{P}_t(1)$. Very similarly, it can be shown that for any portfolio $\pi \in \Psi_t(1)$, there exists a feasible portfolio $\bar{\pi} \in \Psi_t(v_t)$

such that $\bar{\pi} = v_t \pi$. Thus, the sets are equal. $\square$

Using this property, we update the problem $\tilde{G}_t(v_t)$ as below:

$$
\begin{aligned}
\text{min.}\ & \Gamma_t(M\xi),\ \text{w.r.t.}\ \ \leq_{L_t(\mathbb{R}^3_+)} \\
\text{s.t.}\ & v_t = \pi_{t+1}^{\mathsf{T}} S_t \\
& v_{t+1} = \pi_{t+1}^{\mathsf{T}} S_{t+1} \\
& \pi_{t+1} \in \Phi_t \\
& \xi \in \begin{pmatrix} v_{t+1} \\ v_{t+1}^2 \\ v_{t+1}^2 \end{pmatrix} * \mathcal{P}_{t+1}(\mathbf{1})
\end{aligned}
\tag{$\bar{G}_t(v_t)$}
$$

with feasible set

$$
\begin{aligned}
\bar{\Psi}_t(v_t) \coloneqq \{(\pi_{t+1}, \xi)\ | & v_t = \pi_{t+1}^{\mathsf{T}} S_t, v_{t+1} = \pi_{t+1}^{\mathsf{T}} S_{t+1}, \pi_{t+1} \in \Phi_t, \\
& \xi \in (v_{t+1}, (v_{t+1})^2, (v_{t+1})^2)^{\mathsf{T}} * \mathcal{P}_{t+1}(\mathbf{1})\}
\end{aligned}
\tag{3.3.1}
$$

Now, it is sufficient to compute the family of sets $(\mathcal{P}_t(\mathbf{1}))_{t\in\mathbb{T}}$, and by component-wise matrix multiplication, we get access to the upper images for any wealth process $(v_t)_{t\in\mathbb{T}}$.[14]

Chapter 4

Solution Methodology

In Chapter 3, we have provided a dynamic characterization for the upper images of family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$. An investor at time $0 \in \mathbb{T}$ with wealth $v_0 \in \mathbb{R}_+$ is naturally interested in the minimal points of the upper image $\mathcal{P}_0(v_0)$. Our goal in this chapter is to provide an algorithm that will compute $\mathcal{P}_0(v_0)$. A methodology to compute $\mathcal{P}_0(v_0)$ for an initial wealth $v_0 \in \mathbb{R}_{++}$ will be provided in Section 4.1. Before that, we will give our algorithm that finds the family of upper images $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$, using the positive homogeneity property in 3.3.1 and the dynamic structure in 3.2.2.

Algorithm 1 Backward recursive computation of the family $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$

Input: A market model satisfying assumptions.
Initialize: $\mathcal{P}_T(\mathbf{1}) \coloneqq -\mathbf{1} + L_T(\mathbb{R}^3_+)$
  for $t = T-1, \dots, 0$ **do**
    Use $\mathcal{P}_{t+1}(\mathbf{1})$ to solve the problem $\bar{G}_t(1)$ to obtain $\mathcal{P}_t(\mathbf{1})$.
  end for
Output: $\mathcal{P}_0(\mathbf{1}), \dots, \mathcal{P}_{T-1}(\mathbf{1})$.

Algorithm 1 aims to compute $\mathcal{P}_t(\cdot)$ for all $t \in \mathbb{T}$ by using the dynamic structure in Theorem 3.2.2. We have already shown in Proposition 3.3.1 that it suffices to find $\mathcal{P}_t(\mathbf{1})$, where we take $v_t = 1$ for all $t \in \mathbb{T}$. Note that the upper image at terminal time $T \in \mathbb{T}$ is taken as

$$\begin{aligned}\mathcal{P}_T(v_T) :=& \Gamma_T(v_T, v_T^2, v_T^2) + L_T(\mathbb{R}^3_+)\\ =& (-v_T, v_T^2, -v_T^2)^\mathsf{T} + L_T(\mathbb{R}^3_+)\end{aligned}$$

as there is no portfolio to construct and v_T in $\mathcal{F}_T-$measurable. After initializing the terminal upper image, we move on to computing $P_{T-1}(\mathbf{1})$ using $\mathcal{P}_T(\mathbf{1})$. The recursion moves down the investment horizon by computing each upper image down to time $0 \in \mathbb{T}$, where we finally obtain $\mathcal{P}_0(\mathbf{1})$. This procedure is going to be the gateway to finding (weakly) efficient trading strategies for an investor with initial wealth $v_0 \in \mathbb{R}_+$, as the upper image at the initial time is found, and can be scaled using Proposition 3.3.1. Before getting into how to find these trading strategies, we will talk about solving for minimal points of the problem $\bar{G}_t(v_t)$ using Pascoletti-Serafini scalarizations. Moreover, since the problem is nonconvex, the upper images are nonconvex as well, which prevents us to use the well-known algorithms in the literature that are suitable for convex vector optimization problems as in [18] and [19].

4.1 Solving the VOP $\bar{G}_t(v_t)$

Algorithm 1 is the recipe to obtain the value functions of the dynamic programming problem $\bar{G}_t(v_t)$. To obtain the upper images at any period $t \in \mathbb{T}$, the nonconvex VOP $\bar{G}_t(v_t)$ should be solved. Since the VOP $\bar{G}_t(v_t)$ is nonconvex, we are going to implement Algorithm 2, which is the nonconvex Benson-type algorithm given in [20]. In order to implement the algorithm, we will make an additional assumption to the problem:

Assumption 4.1.1. *Short-selling is not allowed, i.e.,* $\pi_{t+1} \in L_t(\mathbb{R}^d_+)$ *for all* $t \in \mathbb{T} \setminus \{T\}$.

Remark 4.1.2. Note that the set $L_t(\mathbb{R}^d_+)$ is a closed convex cone, which is sufficient for Proposition 3.3.1 to hold as we take $\Phi_t := L_t(\mathbb{R}^d_+)$.

Lemma 4.1.3. *Assumption 4.1.1 implies that the feasible set $\Psi_t(v_t)$ is compact for any $t \in \mathbb{T}$ and $v_t \in L_t(\mathbb{R}_+)$.*

Proof. The set $\Psi_t(v_t)$ is closed by its defining equalities. It remains to show the boundedness. Let i be one of the assets with price $S_s^i > 0$ at time $s \geq t+1$, and $R_{i,s} := \max\{\frac{S_{s+1}^i(\omega_{s+1})}{S_s^i(\omega_s)} \mid \forall \omega_{s+1} \subseteq \omega_s\}$ be its highest possible return in period $s \in \mathbb{T}$. Since the underlying probability space is finite and the price process is positive, we have $\pi_s \leq R_{i,s}\frac{v_t}{S_s^i}$ for all $s \geq t+1$. Then, there exists an upper bound for the number of assets that can be held in each period. Thus, $\Psi_t(v_t)$ is compact. $\square$

The algorithm in [20] assumes the upper images are closed and the ideal point of the problem $\bar{G}_t(v_t)$ exists for all $t \in \mathbb{T}$. The upper images are defined to be closed in our setting. To define the ideal point of the upper images, let $e_i \in \mathbb{R}^3$ be the vector with all zero entries, other than the entry i, which is 1 for all $i \in \{1, 2, 3\}$. That is, $e_1 = (1, 0, 0)^\mathsf{T}$ and so on. Consider the scalar optimization problem for $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R}_+)$:

$$\text{minimize} \quad \Gamma_t(M\xi)^\mathsf{T} e_i \quad \text{subject to} \quad (\pi_{t+1}, \xi) \in \bar{\Psi}_t(v_t) \tag{$O_t(v_t, i)$}$$

The problem $O_t(v_t, i)$ is a scalar optimization problem, where only one of the components of the objective function vector in problem $\bar{G}_t(v_t)$ is minimized.

Lemma 4.1.4. *Let $v_t \in L_t(\mathbb{R}_+)$. There exists an optimal solution to the problem $O_t(v_t, i)$ for all $i \in \{1, 2, 3\}$.*

Proof. We use the Weierstrass Theorem, however the feasible set $\bar{\Psi}_t(v_t)$ is not compact. Therefore, we add the constraint $\Gamma_t(M\xi) \leq z_t^{ub}$. By the monotonicity of Γ_t, the smaller subset continues to contain an optimal solution for the problem $O_t(v_t, i)$. In addition, the feasible set is now compact. Additionally, the objective function $\Gamma_t(M\xi)$ is continuous for all $i \in \{1, 2, 3\}$. Thus, an optimal solution exists for problem $O_t(v_t, i)$ for all $i \in \{1, 2, 3\}$. $\square$

Let $(\pi^*_{t+1,i}, \xi^*_i)$ be an optimal solution to the problem $O_t(v_t, i)$ for each $i \in \{1, 2, 3\}$. The ideal point of the upper image $\mathcal{P}_t(v_t)$ is defined as:

$$z_t^{ideal} := (\Gamma_t(M\xi_1^*)^\mathsf{T} e_1, \Gamma_t(M\xi_2^*)^\mathsf{T} e_2, \Gamma_t(M\xi_3^*)^\mathsf{T} e_3)^\mathsf{T} \tag{4.1.1}$$

Additionally, define the upper bound point z_t^{ub} as:

$$z_t^{ub} := (z_{t,1}^{ub}, z_{t,2}^{ub}, z_{t,3}^{ub})^\mathsf{T} \tag{4.1.2}$$

where $z_{t,i}^{ub} := \max_{j\in\{1,2,3\}}\{\Gamma_t(M\xi_j^*)^\mathsf{T} e_i\}$. That is, the upper bound point is the maximum of the image of each optimal solution in each component. Finally, we can conclude that the nonconvex Benson-type algorithm in [20] is available for our use. To implement the algorithm, we first scalarize the vector optimization problem $\bar{G}_t(v_t)$ using Pascoletti-Serafini scalarization given in [21]. The scalarization works as follows: Let $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R}_+)$. A starting point $p \in L_t(\mathbb{R}^3)$, usually chosen in the first iteration as the ideal point of the problem $\bar{G}_t(v_t)$, is taken. A vector $d \in L_t(\mathbb{R}^3_{++})$ is chosen as the improvement direction, and the problem minimizes the distance between p, and the upper image of the problem. That is, we find a pair $(\pi_{t+1}, \xi) \in \bar{\Psi}_t(v_t)$ that minimizes the distance $\mu \in L_t(\mathbb{R}_+)$ subject to $\Gamma_t(M\xi) \leq p + \mu d$. Let μ^* be the optimal value. The point $y := p + \mu^* d$ is on the boundary of the upper image $\mathcal{P}_t(v_t)$. By finding boundary points of the upper image, we will find an approximate upper image that will be described in Section 4.1.1.

Let us formulate the Pascoletti-Serafini scalarization of the problem $\bar{G}_t(v_t)$ with $v_t = 1$. Let $t \in \mathbb{T} \setminus \{T\}$, $p \in L_t(\mathbb{R}^3)$ and $d \in L_t(\mathbb{R}^3_{++})$. The scalarized version of the problem $G_t(\mathbf{1})$

is given as:

$$
\begin{aligned}
\text{min.} \quad & \mu \\
\text{s.t.} \quad & 1 = \pi_{t+1}^{\mathsf{T}} S_t \\
& v_{t+1} = \pi_{t+1}^{\mathsf{T}} S_{t+1} \\
& \pi_{t+1} \in L_t(\mathbb{R}^d_+) \\
& \Gamma_t(Mx) \leq p + \mu d \\
& x \in \begin{pmatrix} v_{t+1} \\ v_{t+1}^2 \\ v_{t+1}^2 \end{pmatrix} * \mathcal{P}_{t+1}(\mathbf{1})
\end{aligned}
\tag{$PS_t(p,d)$}
$$

The scalarized problem $PS_t(p,d)$ will be used to operate the nonconvex Benson-type algorithm given in Algorithm 2.

Remark 4.1.5. Although the problem $PS_t(p,d)$ uses the true sets $\mathcal{P}_{t+1}(\mathbf{1})$, for the numerical examples, we use the inner approximations found by Algorithm 2. When solving the scalarizations, we utilize binary variables to represent a polyhedral approximation. Also, the constraint involving the function Γ_t is a nonconvex constraint. Therefore, each scalarization is a nonconvex quadratic mixed integer programming problem, which can be solved to global optimality by the Gurobi solver.

Lemma 4.1.6. *There exists an optimal solution to the problem $PS_t(p,d)$.*

Proof. We again use the Weierstrass Theorem. The feasible set $\bar{\Psi}_t(\mathbf{1})$ is closed by the defining sets and equalities. Moreover, we have shown that π_{t+1} values are bounded in Lemma 4.1.3. To satisfy boundedness, it suffices to argue that x values can be bounded as well. For that, we bound the upper image $\mathcal{P}_{t+1}(\mathbf{1})$ by its upper bound z_{t+1}^{ub} defined in 4.1.2. That is, we add the constraint $x \leq (v_{t+1}, v_{t+1}^2, v_{t+1}^2)^{\mathsf{T}} \odot z_{t+1}^{ub}$. By the monotonicity of Γ_t, we know that the optimal solution is contained in the new feasible set. Moreover, the feasible set is now compact. Since the objective function is continuous, an optimal solution exists to the problem $PS_t(p,d)$. $\square$

4.1.1 The Nonconvex Benson Type Algorithm

We will discuss the nonconvex Benson type Algorithm given in [20]. Before getting into it, the definitions of upper images under certain $\omega_t \in \Omega_t$ should be defined. Computing the family of upper images $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$ is to find it at any $t \in \mathbb{T}$ and any $\omega_t (\in \Omega_t)$. As the values of the assets in the market change, our methodology should be able to provide the investor with proper upper images to make decisions.

Let $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R}_+)$. The upper image $\mathcal{P}_t(v_t)$ under $\omega_t \in \Omega_t$ is defined as:

$$\mathcal{P}_t(v_t)(\omega_t) := \operatorname{cl}\left(\left\{\Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)(\omega_t) \mid \pi \in \Psi_t(v_t)(\omega_t)\right\} + \mathbb{R}^3_+\right)$$

The algorithm uses inner and outer approximations of the upper image to compute 'good enough' solutions. We define them here.

Definition 4.1.7. *Let $t \in \mathbb{T}$, $\omega_t \in \Omega_t$. Let $\mathcal{P}_t(\mathbf{1})(\omega_t)$ be the upper image of the problem $G_t(\mathbf{1})$ for the given ω_t. The finite set $\mathcal{V}_t^{in}(\omega_t)$ is called inner ϵ-solution if*

$$H(\mathcal{P}_t(\mathbf{1})(\omega_t), \mathcal{V}_t^{in}(\omega_t) + \mathbb{R}^3_+) \leq \epsilon \quad \textit{and} \quad \mathcal{V}_t^{in}(\omega_t) \subseteq \mathcal{P}_t(\mathbf{1})(\omega_t),$$

and the finite set $\mathcal{V}_t^{out}(\omega_t)$ is called outer ϵ-solution if

$$H(\mathcal{P}_t(\mathbf{1})(\omega_t), \mathcal{V}_t^{out}(\omega_t) + \mathbb{R}^3_+) \leq \epsilon \quad \textit{and} \quad \mathcal{P}_t(\mathbf{1})(\omega_t) \subseteq \mathcal{V}_t^{out}(\omega_t) + \mathbb{R}^3_+.$$

Let $t \in \mathbb{T}$, $\omega_t \in \Omega_t$, $z_t^{ideal}(\omega_t) \in \mathbb{R}^3$ be defined as in 4.1.1 and $z_t^{ub}(\omega_t) \in \mathbb{R}^3$ be defined as in 4.1.2. A point $v \in \mathcal{L}$ is called a vertex of $\mathcal{L}$ if v cannot be expressed as the convex combination of points taken from $\mathcal{L} \setminus \{v\}$. The set of all vertices of $\mathcal{L}$ is denoted by $\mathrm{vert}(\mathcal{L})$.

For an intermediate time $t \in \{0, \dots, T-1\}$, we will approximate the upper image $\mathcal{P}_t(\mathbf{1})(\omega_t)$. The psuedocode of the nonconvex Benson type algorithm is given in 2.

The inputs of the algorithm are the ideal and upper bound points z_t^{ideal}, z_t^{ub}; a direction vector for the problem $PS_t(p, d)$ given as $d := \mathbf{1} \in \mathbb{R}^3$ and the user-defined approximation error $\epsilon > 0$. The initial inner and outer approximations will be given as $\mathcal{U}_t^0(\omega_t) := \{z_t^{ub}\} + \mathbb{R}^3_+$ and $\mathcal{L}_t^0(\omega_t) := \{z_t^{ideal}\} + \mathbb{R}^3_+$, respectively. Let $S := \emptyset$. At an arbitrary iteration $k \in \mathbb{N}$, the algorithm will choose a vertex $v \in \mathrm{vert}\mathcal{L}_t^k(\omega_t)$ and check if $v + \mathbf{1}\epsilon \in \mathcal{U}_t^k(\omega_t)$. If not, solve $PS_t(p, d)$ and obtain an optimal value μ^*. The point $y := v + \mu^* d$ will be a boundary point of $\mathcal{P}_t(1)(\omega_t)$. The outer approximation is updated as $\mathcal{L}_t^{k+1}(\omega_t) := \mathcal{L}_t^k(\omega_t) \setminus (\{y\} - \mathbb{R}^3_{++})$ and the inner approximation as $\mathcal{U}_t^{k+1}(\omega_t) := \mathcal{U}_t^k(\omega_t) \cup (\{y\} + \mathbb{R}^3_+)$. The algorithm terminates at a finite iteration $K \in \mathbb{N}$ if $\mathcal{L}_t^K(\omega_t) + \mathbf{1}\epsilon \subseteq \mathcal{U}_t^K(\omega_t)$. The sets $\mathcal{L}_t^K(\omega_t)$ and $\mathcal{U}_t^K(\omega_t)$ are returned as the outer and inner approximations, respectively. By the boundedness of the upper images, [20] guarantees that the algorithm will stop after finitely many iterations. The psuedocode of the algorithm is given as follows:

Algorithm 2 The Nonconvex Benson-Type Algorithm

Input: $t \in \mathbb{T}$, $\omega_t \in \Omega_t$, $z_t^{ideal}(\omega_t)$, $z_t^{ub}(\omega_t)$, the direction vector $d \coloneqq \mathbf{1} \in \mathbb{R}^3$, the approximation error $\epsilon > 0$.

Initialize: $\mathcal{L}_t^0(\omega_t) \coloneqq z_t^{ideal}(\omega_t) + \mathbb{R}^3_+$, $\mathcal{U}_t^0(\omega_t) \coloneqq z_t^{ub}(\omega_t) + \mathbb{R}^3_+$, $S = \emptyset$, $k = 0$.

```
while vert(L_t^k(ω_t)) ⊈ S do
    Choose v ∈ vert(L_t^k(ω_t)) \ S;
    if v + ε1 ∈ U_t^k(ω_t) then
        S ← S ∪ {v};
    else
        Solve problem PS_t(p, d) to get optimal value μ*;
        Set y := v + μ* d;
        L_t^{k+1}(ω_t) := L_t^k(ω_t) \ (y − R^3_{++});
        U_t^{k+1}(ω_t) := U_t^k(ω_t) ∪ (y + R^3_+);
        k ← k + 1;
    end if
end while
```

Output: $K = k, \mathcal{L}_t^K(\omega_t), \mathcal{U}_t^K(\omega_t)$.

The end products of the algorithm are $\mathcal{L}_t^K(\omega_t)$, the outer approximation, and $\mathcal{U}_t^K(\omega_t)$, the inner approximation of the upper image $\mathcal{P}_t(\mathbf{1})$.

Remark 4.1.8. The family of set-valued functions $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$ is related to the market structure and the initial wealth of the investor. Therefore, the same upper images can be used for investors with varying risk attitude parameters.

Chapter 5

Approximation of the Upper Images

In this chapter, we will put an upper bound on the error that will be caused by the approximation technique we will employ. For computational purposes, we will impose no shorting constraints to the problem $\bar{G}_t(v_t)$, that is, add the constraint $\pi_s \in L_s(\mathbb{R}^d_+)$ for all $s \geq t+1$. We will prove some useful properties about the functions and algorithms we use before finding a manageable upper bound for the error.

Lemma 5.0.1. *The maximum value of wealth $v_{t+1} \in L_{t+1}(\mathbb{R}_+)$ for problem $\bar{D}_t(1)$ at node ω_t can be achieved by investing to the stock with highest possible rate of return; i.e.,*

$$M(\omega_{t+1}) := \|v_{t+1}(\omega_{t+1})\|_\infty = \|\frac{S^k_{t+1}(\omega_{t+1})}{S^k_t(\omega_t)}\|_\infty.$$

Similarly, the minimum value is found by

$$m(\omega_{t+1}) := \|v_{t+1}\|_\infty = \min_{\substack{\omega_{t+1}\in\Omega_{t+1} \\ k\in\{1\ldots,d\}}} \frac{S^k_{t+1}(\omega_{t+1})}{S^k_t}.$$

Proof. With the addition of no shorting constraints, this is trivial. □

Note that Lemma 5.0.1 implies $v_t \in L_t^\infty(\mathbb{R}_+)$ for all $t \in \mathbb{T}$.

Lemma 5.0.2. *Let $f : L^\infty(\mathbb{R}_+^d) \to \mathbb{R}_+$ be defined as*

$$f(\xi) \coloneqq \|\xi\|_{\frac{1}{2}} = \left(\sum_{\omega\in\Omega} p_\omega \sqrt{\xi(\omega)}\right)^2$$

where $p_\omega \coloneqq \mathbb{P}\{\omega\}$ for $\omega \in \Omega$. Let $\xi \in L_t^\infty(\mathbb{R}_+)$ be wealth as in Lemma 5.0.1. Then, $f(T\xi + \mathbf{1}T\epsilon) - f(T\xi) \leq \|T\|_\infty \sqrt{\frac{M}{m}}\epsilon$, for $\epsilon > 0$, $T \in L^\infty(\mathbb{R}_+)$.

Proof. Since f is a concave, using its supergradient we get

$$\begin{aligned}
f(T\xi + \mathbf{1}T\epsilon) - f(T\xi) &\leq \nabla f(T\xi)^\mathsf{T} \mathbf{1}T\epsilon \\
&= \left(2\left(\sum_{\omega\in\Omega} p_\omega \sqrt{T(\omega)\xi(\omega)}\right)\frac{p}{2\sqrt{T\xi}}\right)^\mathsf{T} \mathbf{1}T\epsilon \\
&= \left(\mathbb{E}(\sqrt{T\xi})\frac{p}{\sqrt{T\xi}}\right)^\mathsf{T} \mathbf{1}T\epsilon \\
&= \mathbb{E}(\sqrt{T\xi})\left(\frac{p}{\sqrt{T\xi}}\right)^\mathsf{T} \mathbf{1}T\epsilon \\
&= \mathbb{E}(\sqrt{T\xi})\mathbb{E}\left(\frac{T}{\sqrt{T\xi}}\right)\epsilon \\
&= \mathbb{E}(\sqrt{T\xi})\mathbb{E}\left(\frac{\sqrt{T}}{\sqrt{\xi}}\right)\epsilon \\
&\leq \|T\|_\infty \sqrt{\frac{M}{m}}\epsilon.
\end{aligned}$$

□

Lemma 5.0.2 serves as a bound to the third component of the vector-valued objective function Γ_t.

Lemma 5.0.3. *Let $\omega_t \in \Omega_t$, $H : 2^{\mathbb{R}_+^3} \times 2^{\mathbb{R}_+^3} \to \mathbb{R}_+$ be the Hausdorff distance, $\mathcal{V}_{t+1}^{in}, \mathcal{V}_{t+1}^{out}$ be random sets denoting ϵ-inner and outer approximations of the upper image of the problem*

$D_{t+1}(1)$ and $H(\mathcal{V}^{out}_{t+1}(\omega_{t+1}), \mathcal{V}^{in}_{t+1}(\omega_{t+1})) \leq \delta$ for all $\omega_{t+1} \subseteq \omega_t$. Let $r : L^{\infty}_{t+1}(\mathbb{R}_+) \to L^{\infty}_{t+1}(\mathbb{R}^3_+)$ be defined as $r(X) = (X, X^2, X^2)^{\mathsf{T}}$. Define

$$\mathcal{P}^{in}_t(\omega_t, 1) \coloneqq \left\{\Gamma_t(Mx) \mid 1 = \pi^{\mathsf{T}}_{t+1} S_t(\omega_t), V_{t+1} = \pi^{\mathsf{T}}_{t+1} S_{t+1}, x \in r(V_{t+1})(\mathcal{V}^{in}_{t+1} + L_{t+1}(\mathbb{R}^3_+))\right\}$$
$$\mathcal{P}^{out}_t(\omega_t, 1) \coloneqq \left\{\Gamma_t(Mx) \mid 1 = \pi^{\mathsf{T}}_{t+1} S_t(\omega_t), V_{t+1} = \pi^{\mathsf{T}}_{t+1} S_{t+1}, x \in r(V_{t+1})(\mathcal{V}^{out}_{t+1} + L_{t+1}(\mathbb{R}^3_+))\right\}$$

where multiplication by $r(V_{t+1})$ is component-wise. The following hold:

1. *Consider the problem $\bar{D}_t(1)$. We have*

   $$\|\Gamma_t(Mr(X)x) - \Gamma_t(Mr(X)(x + \mathbf{1}\delta))\|_2 \leq \delta \sqrt{\|X^2\|_\infty + \|X^4\|_\infty + \left(\|X^2\|_\infty \sqrt{\frac{\|x_3\|_\infty}{m}}\right)^2}$$

   for $\delta > 0$ and for any $x \in \mathcal{P}_{t+1}(\omega_t, 1)$ and $X \in L^{\infty}_{t+1}(\mathbb{R}_+)$.

2. *Let $x \in \mathcal{V}^{out}_{t+1}(\omega_{t+1}) + \mathbb{R}^3_+$. Then, $x + \mathbf{1}\delta \in \mathcal{V}^{in}_{t+1}(\omega_{t+1}) + \mathbb{R}^3_+$. Moreover, for $X \in L^{\infty}_{t+1}(\mathbb{R}_+)$, $r(X)x \in r(X)(\mathcal{V}^{out}_{t+1}(\omega_{t+1}) + \mathbb{R}^3_+)$ implies $r(X)(x + \mathbf{1}\delta) \in r(X)(\mathcal{V}^{in}_{t+1}(\omega_{t+1}) + \mathbb{R}^3_+)$ for all $\omega_{t+1} \subseteq \omega_t$.*

3. $H(\mathcal{P}^{out}_t(\omega_t, 1), \mathcal{P}^{in}_t(\omega_t, 1)) \leq \sup\limits_{\substack{x \in r(V_{t+1})\mathcal{V}^{out}_{t+1} \\ \pi_{t+1} \in \tilde{\Psi}_t(1)}} \|\Gamma_t(Mx) - \Gamma_t(M(x + \mathbf{1}\delta))\|_2.$

4. *Let $\mathcal{V}^{out}_t(\omega_t)$ be the ϵ-outer approximation of $\mathcal{P}^{out}_t(\omega_t, 1)$ and $\mathcal{V}^{in}_t(\omega_t)$ be the ϵ-inner approximation of $\mathcal{P}^{in}_t(\omega_t, 1)$ at time $t \in \mathbb{T}$. Then, $H(\mathcal{V}^{out}_t(\omega_t), \mathcal{V}^{in}_t(\omega_t)) \leq H(\mathcal{V}^{out}_t(\omega_t) + \mathbb{R}^3_+, \mathcal{P}^{out}_t(\omega_t, 1)) + H(\mathcal{V}^{in}_t(\omega_t) + \mathbb{R}^3_+, \mathcal{P}^{in}_t(\omega_t, 1)) + H(\mathcal{P}^{in}_t(\omega_t, 1), \mathcal{P}^{out}_t(\omega_t, 1))$.*

Proof. 1. We have

$$\begin{aligned}
\|\Gamma_t(Mr(X)x) - \Gamma_t(Mr(X)(x+\mathbf{1}\delta))\|_2 &= \| \begin{pmatrix} -\mathbb{E}_t(-Xx_1) + \mathbb{E}_t(-X(x_1+\delta)) \\ \mathbb{E}_t(X^2x_2) - \mathbb{E}_t(X^2(x_2+\delta)) \\ -(\mathbb{E}_t(\sqrt{-X^2x_3}))^2 + \mathbb{E}_t(\sqrt{-X^2(x_3+\delta)})^2 \end{pmatrix} \|_2 \\
&\leq \| \begin{pmatrix} \delta\|X\|_\infty \\ \delta\|X^2\|_\infty \\ \delta\sqrt{\frac{M}{m}}\|X^2\|_\infty \end{pmatrix} \|_2 \\
&= \delta\sqrt{\|X\|_\infty^2 + \|X\|_\infty^4 + \left(\|X^2\|_\infty\sqrt{\frac{M}{m}}\right)^2} \\
&= \delta\sqrt{\|X\|_\infty^2 + \|X\|_\infty^4 + \left(\|X^2\|_\infty\sqrt{\frac{M}{m}}\right)^2}.
\end{aligned}$$

2. Let $x \in \mathcal{V}_{t+1}^{out}(\omega_{t+1}) + \mathbb{R}_+^3$. Then, $x + \mathbf{1}\delta \in \mathcal{V}_{t+1}^{in}(\omega_{t+1}) + \mathbb{R}_+^3$ holds since $H(\mathcal{V}_{t+1}^{out}(\omega_{t+1}), \mathcal{V}_{t+1}^{in}(\omega_{t+1})) \leq \delta$. Now let $X \in L_{t+1}^\infty(\mathbb{R}_+)$. Then, $r(X)x \in r(X)(\mathcal{V}_{t+1}^{out}(\omega_{t+1}) + \mathbb{R}_+^3)$ and

$$r(X)(x+\mathbf{1}\delta) \in r(X)(\mathcal{V}_{t+1}^{out}(\omega_{t+1}) + \mathbb{R}_+^3 + \mathbf{1}\delta) \subseteq r(X)(\mathcal{V}_{t+1}^{in}(\omega_{t+1}) + \mathbb{R}_+^3)$$

since $r(X) \geq 0$ almost surely, showing that $r(X)(x+\mathbf{1}\delta) \in r(X)(\mathcal{V}_{t+1}^{in}(\omega_{t+1}) + \mathbb{R}_+^3)$ holds for any $\omega_{t+1} \subseteq \omega_t$.

3.

$$
\begin{aligned}
H(\mathcal{P}_t^{out}(\omega_t,1),\mathcal{P}_t^{in}(\omega_t,1)) &= \sup_{\substack{x\in r(v_{t+1})(\mathcal{V}_{t+1}^{out}(\omega_{t+1})+\mathbb{R}_+^3)\\ \pi_{t+1}\in\tilde{\Psi}_t(1)}} \inf_{\substack{y\in r(v_{t+1})(\mathcal{V}_{t+1}^{in}(\omega_{t+1})+\mathbb{R}_+^3)\\ \pi_{t+1}\in\tilde{\Psi}_t(1)}} \|\Gamma_t(Mx)-\Gamma_t(My)\|_2 \\
&\leq \sup_{\substack{x\in r(v_{t+1})\mathcal{V}_{t+1}^{out}(\omega_{t+1})\\ \pi_{t+1}\in\tilde{\Psi}_t(1)}} \inf_{\substack{y\in r(v_{t+1})(\mathcal{V}_{t+1}^{in}(\omega_{t+1})+\mathbb{R}_+^3)\\ \pi_{t+1}\in\tilde{\Psi}_t(1)}} \|\Gamma_t(Mx)-\Gamma_t(My)\|_2 \\
&\leq \sup_{\substack{x\in r(v_{t+1})\mathcal{V}_{t+1}^{out}(\omega_{t+1})\\ \pi_{t+1}\in\tilde{\Psi}_t(1)}} \|\Gamma_t(Mx)-\Gamma_t(M(x+\mathbf{1}\delta))\|_2
\end{aligned}
$$

for all $\omega_{t+1}\subseteq\omega_t$, using item 1 of this lemma and the Hausdorff distance given for the approximations.

4. This follows directly from the fact that

$$
\mathcal{V}_t^{in}(\omega_t)+\mathbb{R}_+^3 \subseteq \mathcal{P}_t^{in}(\omega_t,1) \subseteq \mathcal{P}_t^{out}(\omega_t,1) \subseteq \mathcal{V}_t^{out}(\omega_t)+\mathbb{R}_+^3
$$

□

Proposition 5.0.4. *Let $H(\mathcal{V}_{t+1}^{out}(\omega_{t+1}),\mathcal{V}_{t+1}^{in})\leq\delta$ for all $\omega_{t+1}\subseteq\omega_t$. Then,*

$$
H(\mathcal{V}_t^{out}(\omega_t),\mathcal{V}_t^{in}(\omega_t)) \leq 2\epsilon+\delta\sqrt{M^2+M^4+\frac{M^5}{m}}.
$$

Proof. Let $\bar{x}\in\mathcal{P}_t^{in}(\omega_t,1)$. Then, there exists $z\in r(v_{t+1})(\mathcal{V}_{t+1}^{out}+\mathbb{R}_+^3)$ and π_{t+1} such that $\bar{x}=\Gamma_t(Mz)$ and $v_{t+1}=\pi_{t+1}^{\mathsf{T}}S_{t+1}$. Since $H(\mathcal{V}_{t+1}^{out}(\omega_{t+1}),\mathcal{V}_{t+1}^{in}(\omega_{t+1})+\mathbb{R}_+^3)\leq\delta$, we have

$z + r(v_{t+1})\delta \in r(v_{t+1})(\mathcal{V}_t^{in}(\omega_t) + \mathbb{R}^3_+)$ using Lemma 5.0.3 item 2. Then,

$$\begin{aligned} H(\mathcal{P}_t^{out}(\omega_t, 1), \mathcal{P}_t^{in}(\omega_t, 1)) &\leq \sup_{\substack{z \in r(v_{t+1})\mathcal{V}_{t+1}^{out}(\omega_{t+1}) \\ \pi_{t+1} \in \tilde{\Psi}_t(1)}} \|\Gamma_t(Mz) - \Gamma_t(M(z + \mathbf{1}\delta))\|_2 \\ &= \sup_{\substack{x \in \mathcal{V}_{t+1}^{out}(\omega_{t+1}) \\ \pi_{t+1} \in \tilde{\Psi}_t(1)}} \|\Gamma_t(Mr(v_{t+1})x) - \Gamma_t(Mr(v_{t+1})(x + \mathbf{1}\delta))\|_2 \\ &\leq \delta \sqrt{\|v_{t+1}\|_\infty^2 + \|v_{t+1}\|_\infty^4 + \left(\|v_{t+1}^2\|_\infty \sqrt{\frac{M}{m}}\right)^2} \end{aligned}$$

Finally,

$$H(\mathcal{V}_t^{out}(\omega_t), \mathcal{V}_t^{in}(\omega_t)) \leq 2\epsilon + \delta\sqrt{M^2 + M^4 + \frac{M^5}{m}}$$

using Lemma 5.0.3. □

Proposition 5.0.4 provides a bound for the approximation error that can be made during the computation of the upper images for each $t \in \mathbb{T}$. Corollary 5.0.5 below provides the bound on the total approximation error that can be made during calculations.

Corollary 5.0.5. *Let* $c := \sqrt{M^2 + M^4 + \frac{M^5}{m}}$. *The error in approximation at each period is bounded by a constant in 5.0.4. For a* T *period problem, Let* $H(\mathcal{V}_{T-1}^{out}(\omega_{T-1}), \mathcal{V}_{T-1}^{in}(\omega_{T-1})) \leq 2\epsilon$. *Then, the total error in approximation will be given by* $2\epsilon\frac{c^T - 1}{c-1}$.

Proof. At time $T - 1 \in \mathbb{T}$, we have an error of 2ϵ. Then, at time $T - 2 \in \mathbb{T}$, the total error is

$$2\epsilon + 2\epsilon c = 2\epsilon\frac{c^2 - 1}{c - 1}$$

Let $t \in \mathbb{T}$ and the error at this time be $2\epsilon\frac{c^{T-t}}{c-1}$. Then, the total error at time $t - 1$ will be

$$2\epsilon + 2\epsilon\frac{c^{T-t} - 1}{c - 1}c = 2\epsilon\left(1 + \frac{c^{T-t} - 1}{c - 1}c\right) = 2\epsilon\frac{c^{T-t+1} - 1}{c - 1}$$

Thus, the total error at initial time is $2\epsilon\frac{c^T-1}{c-1}$. $\square$

The importance of Corollary 5.0.5 is that the total approximation error is manageable for a market with less volatility and long investment horizon, or a volatile market with short investment horizon. The vector-valued function Γ_t is the main reason for the given total approximation error. Although the function is linear in the first two components, its third component is difficult to bound.

Chapter 6

Finding (Weakly) Efficient Portfolios

After computing the upper images for each period $t \in \mathbb{T}$, our next objective is to find the corresponding optimal portfolios for an investor with a given risk attitude. Until now, we worked on the family of problems $(\bar{G}_t(\cdot))_{t\in\mathbb{T}}$, and the upper images of these problems contain minimal points that are not necessarily mean-variance solutions. Therefore, we will only work with a subset of the family of upper images $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$. In Section 6.1.1, we will propose a methodology to find mean-variance efficient portfolios by using the upper images in a forward manner. That is, starting from time $0 \in \mathbb{T}$, we will find portfolios that are optimal for problem $MV_0(v_0, \lambda_0)$, and remain weakly efficient for the intermediate time vector optimization problems $VMV_t(\cdot)$ for all $t \in \mathbb{T}$. First, we will show that a minimizer of the problem $\bar{G}_t(v_t)$, which is a boundary point of the corresponding upper image can be found using an efficient portfolio. For that, we will state a very useful theorem called vectorial Weierstrass theorem:

Theorem 6.0.1. *[16, Theorem 2.40] Let $n, k \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}^k$ be continuous and $S \subseteq \mathbb{R}^n$ be compact. Let $C \subseteq \mathbb{R}^k$ be closed, convex pointed cone with $\emptyset \neq C \neq \mathbb{R}^k$. Consider the* *problem*

$$\text{minimize} \quad f(x) \quad \text{with respect to} \quad \leq_C \quad x \in S \tag{V}$$

Then, there exists a solution to the problem (V), *where a solution is a nonempty set* $X \subseteq S$ *such that*

$$\{f(x) \mid x \in X\} = Min\{f(x) \mid x \in S\}$$

and $Min\{f(x) \mid x \in S\}$ *is the set of minimal points of problem* (V).

Remark 6.0.2. By the characterization in Remark 2.2.3, a solution $X \subseteq S$ is comprised of efficient solutions.

Lemma 6.0.3. *Let* $t \in \mathbb{T}$ *and* $v_t \in L_t(\mathbb{R}_+)$. *If* x^* *is a minimal point of* $\mathcal{P}_t(v_t)$, *then there exists* $\pi \in \Psi_t(v_t)$ *such that* $\Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2) = x^*$, *that is,* π *is an efficient solution.*

Proof. The compactness of the feasible set $\Psi_t(v_t)$ is stated in Remark 4.1.3, and the vector-valued objective function Γ_t is continuous by Proposition 3.0.1. Thus, the vectorial Weierstrass theorem applies. $\square$

Lemma 6.0.3 shows that a minimal point of the upper image $\mathcal{P}_t(\cdot)$ corresponds to a solution $\pi \in \Psi_t(\cdot)$. The next step is to connect the family of upper images $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$ with the mean-variance problem.

6.1 Relation Between the Mean-Variance Upper Image and $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$

Define

$$\mathcal{I}_t(v_t) \coloneqq \{(x_1, x_2 + x_3) \mid x \in \mathcal{P}_t(v_t),\ x_1^2 = -x_3\},$$

and consider the upper set below:

$$TD_t(v_t) \coloneqq \left\{ \begin{pmatrix} x_1 \\ x_2 + x_3 \end{pmatrix} \mid x \in \mathcal{P}_t(v_t),\ x_1^2 = -x_3 \right\} + L_t(\mathbb{R}^2_+), \tag{$TD_t(v_t)$}$$

where the mean-variance objectives are constructed using $\mathcal{P}_t(v_t)$. Also define

$$\tilde{\mathcal{I}}_t(v_t) \coloneqq \{(-\mathbb{E}_t(v_T^\pi), \mathrm{Var}_t(v_T^\pi))^\mathsf{T} \mid \pi \in \Psi_t(v_t)\},$$

and consider the upper image of the vector-valued mean-variance problem given in $VMV_t(v_t)$:

$$PMV_t(v_t) \coloneqq \left\{ \begin{pmatrix} -\mathbb{E}_t(v_T^\pi) \\ \mathrm{Var}_t(v_T^\pi) \end{pmatrix} \mid \pi \in \Psi_t(v_t) \right\} + L_t(\mathbb{R}^2_+).$$

Remark 6.1.1. The closure in the definition of an upper image can be dropped when the feasible set is compact and the vector-valued function Γ_t is continuous.

Proposition 6.1.2. *Let $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R}_+)$. Using $\mathcal{P}_t(v_t)$, the mean-variance upper image can be recovered using the nonlinear transformation in $(TD_t(v_t))$:*

$$PMV_t(v_t) = TD_t(v_t)$$

Proof. For the $(\subseteq)$ part, let $x = a + b$, $a = (-\mathbb{E}_t(v_T^\pi), \mathrm{Var}_t(v_T^\pi))$, $b \in L_t(\mathbb{R}^2_+)$ for some

$\pi \in \Psi_t(v_t)$. Then, we can write

$$a = \begin{pmatrix} -\mathbb{E}_t(v_T^\pi) \\ \mathrm{Var}_t(v_T^\pi) \end{pmatrix} = \begin{pmatrix} -\mathbb{E}_t(v_T^\pi) \\ \mathbb{E}_t((v_T^\pi)^2) - (\mathbb{E}_t(v_T^\pi))^2 \end{pmatrix}$$

Then,

$$(a_1)^2 = (-\mathbb{E}_t(v_T^\pi))^2 = (\mathbb{E}_t(v_T^\pi))^2 = -a_3.$$

Moreover, define $\tilde{a} := (-\mathbb{E}_t(v_T^\pi), \mathbb{E}_t((v_T^\pi)^2), -(\mathbb{E}_t(v_T^\pi))^2)^\mathsf{T}$. Then, $\tilde{a} \in \mathcal{P}_t(v_t)$ and $a = (\tilde{a}_1, \tilde{a}_2 + \tilde{a}_3)^\mathsf{T}$. Thus, $a \in \mathcal{I}_t(v_t)$. Since $TD_t(v_t)$ is an upper set, $a + b \in TD_t(v_t)$.

Conversely, for the $(\supseteq)$ part, let $z = c + d$, $c = (x_1, x_2 + x_3)$ for some $x \in \mathcal{P}_t(v_t)$ satisfying $x_1^2 = -x_3$, and $d \in L_t(\mathbb{R}^2_+)$. Since $x \in \mathcal{P}_t(v_t)$ is from a closure of a set, take a sequence $x^n := a^n + b^n$, $a^n = \Gamma_t(v_T^{\pi^n}, (v_T^{\pi^n})^2, (v_T^{\pi^n})^2)$, for some $\pi^n \in \Psi_t(v_t)$, $b^n \in L_t(\mathbb{R}^3_+)$ for all $n \in \mathbb{N}$, where $x^n \to x$ as $n \to \infty$. Define $\bar{a}^n := (a_1^n, a_2^n + a_3^n)$ and $\bar{b}^n := (b_1^n, b_2^n + b_3^n)$. Similar to the first part, we see that

$$(a_1^n)^2 = \left(-\mathbb{E}_t(v_T^{\pi^n})\right)^2 = \left(\mathbb{E}_t(v_T^{\pi^n})\right)^2 = -a_3^n$$

for all $n \in \mathbb{N}$. Thus, $\bar{a}^n \in \tilde{\mathcal{I}}_t(v_t)$, implying that $\bar{a}^n + \bar{b}^n \in MV_t(v_t)$ for all $n \in \mathbb{N}$, since $MV_t(v_t)$

is an upper set. Furthermore, we have

$$
\begin{aligned}
\lim_{n\to\infty}\left(\bar{a}^n+\bar{b}^n\right) &= \lim_{n\to\infty}\left(\begin{pmatrix} a_1^n \\ a_2^n+a_3^n \end{pmatrix}+\begin{pmatrix} b_1^n \\ b_2^n+b_3^n \end{pmatrix}\right)\\
&= \lim_{n\to\infty}\begin{pmatrix} x_1^n \\ x_2^n+x_3^n \end{pmatrix}\\
&= \begin{pmatrix} x_1 \\ x_2+x_3 \end{pmatrix}\\
&= c\in PMV_t(v_t),
\end{aligned}
$$

since $c_1^2=-c_3$ by its definition, and $PMV_t(v_t)$ is closed. Finally, by the fact that $PMV_t(v_t)$ is an upper set, $z\in MV_t(v_t)$ as well. □

We have shown that the mean-variance upper images can be obtained using $(\mathcal{P}_t(\cdot))_{t\in\mathbb{T}}$ and a linear transformation in the objective function, and a nonlinear transformation in the feasible set.

6.1.1 Finding Mean-Variance Efficient Solutions

We will use Proposition 6.1.2 to find mean-variance efficient solutions. First, define the three-dimensional mean-variance correspondence set

$$\overline{TD}_t(v_t) \coloneqq \left\{x\in\mathcal{P}_t(v_t)\mid x_1^2=-x_3\right\}. \tag{6.1.1}$$

We will prove a one-sided recursion related to $\overline{TD}_t(v_t)$ that will be useful later.

Lemma 6.1.3. $\overline{TD}_t(v_t)\supseteq\left\{\Gamma_t(Mx)\mid x\in\overline{TD}_{t+1}(v_{t+1})\right\}$.

Proof. Let $x \in \overline{TD}_{t+1}(v_{t+1})$. Then, $x \in \mathcal{P}_{t+1}(v_{t+1})$ and $x_1^2 = -x_3$ by definition. Then, $\Gamma_t(Mx) \in \mathcal{P}_t(v_t)$ by the Bellman's Principle and

$$\Gamma_t(Mx) = \begin{pmatrix} -\mathbb{E}_t(-x_1) \\ \mathbb{E}_t(x_2) \\ -(\mathbb{E}_t(x_1))^2 \end{pmatrix} = \begin{pmatrix} \mathbb{E}_t(x_1) \\ \mathbb{E}_t(x_2) \\ -(\mathbb{E}_t(x_1))^2 \end{pmatrix}.$$

Thus, $\Gamma_t(M_x) \in \overline{TD}_t(v_t)$. □

Let $t \in \mathbb{T}$, $v_t \in L_t(\mathbb{R}_+)$ and $x^* \in \mathcal{P}_t(v_t)$. Define the feasibility problem $F_t(x_t^*, v_t)$ as

$$\begin{aligned} \text{minimize } & 0 \\ \text{subject to } & v_t = \pi_{t+1}^{\mathsf{T}} S_t \\ & v_{t+1} = \pi_{t+1}^{\mathsf{T}} S_{t+1} \\ & \pi_{t+1} \in \Phi_t \\ & x_{t+1} \in \overline{TD}_{t+1}(v_{t+1}) \\ & \Gamma_t(Mx_{t+1}) \leq x_t^*. \end{aligned} \tag{$F_t(x_t^*, v_t)$}$$

We will use the problem $F_t(x_t^*, v_t)$ starting at time $0 \in \mathbb{T}$ to find a point from the set $\overline{TD}_t(v_t)$. Let $\lambda_0 \in \mathbb{R}^2_{++}$ be the risk attitude and $v_0 > 0$ be the initial wealth of the investor at time $0 \in \mathbb{T}$. Define the weighted-sum scalarization using $\overline{TD}_0(v_0)$ with weight $\bar{\lambda}_0 := [\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,2}]^{\mathsf{T}}$

$$\inf_{x \in \overline{TD}_0(v_0)} \bar{\lambda}_0^{\mathsf{T}} x. \tag{6.1.2}$$

In a market satisfying Assumptions 4.1.1, let the family of upper images $(\mathcal{P}_t(\cdot))_{t \in \mathbb{T}}$ be available. The sets $(\overline{TD}_t)_{t \in \mathbb{T}}$ can be obtained via a nonlinear transformation using the upper images. Let $x_0^* \in \arg\min\{\bar{\lambda}_0^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$, and (π_1^*, x_1^*) be a feasible solution to the problem $F_0(x_0^*, v_0)$. Then, π_1^* is the portfolio of the investor at time 0. Using x_1^*, we may find the remaining portfolios and corresponding x_t^* values for all $t \in \mathbb{T}$. The set of portfolios $(\pi_t^*)_{t\in\mathbb{T}}$ will be efficient for the given investor. The psuedocode of the forward processes is below:

Algorithm 3 Forward Processes

Input: $\lambda_0 \in \mathbb{R}^2_{++}, v_0 \in \mathbb{R}_{++}$, a market satisfying the Assumption 4.1.1
Initialize: $x_0^* \in \arg\min\{\bar{\lambda}_0^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$.

```
1: for t = 0, ..., T − 1 do
2:     Solve the problem F_t(x_t^*, v_t);
3:     Obtain (π_{t+1}^*, x_t^*).
4: end for
```

Output: The set of portfolios $(\pi_{t+1}^*)_{t\in\mathbb{T}}$.

Lemma 6.1.4. *Let $v_0 \in \mathbb{R}_{++}$. $x_0^* \in \arg\min\{\bar{\lambda}_0^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$. Then, $y^* := (x_{0,1}^*, x_{0,2}^* + x_{0,3}^*)^\mathsf{T}$ is a minimal point of the mean variance upper image $PMV_0(v_0)$.*

Proof. Since the weight λ_0 is strictly positive, the point x_0^* is minimal in $\overline{TD}_0(v_0)$ [15, Theorem 11.17]. Assume there exists $y \in PMV_0(v_0)$ such that $y \leq y^*$ and $y \neq y^*$. Then, since $\lambda_0 \in \mathbb{R}^2_{++}$, we have $\lambda_0^\mathsf{T} y < \lambda_0^\mathsf{T} y^*$. Furthermore, since $y \in MV_0(v_0)$, there exist $a \in \tilde{\mathcal{I}}_0(v_0)$ and $b \in L_0(\mathbb{R}^2_+)$ such that $y = a + b$. Define $\bar{a} := (a_1, a_2 + a_1^2, -a_1^2)^\mathsf{T}$, $\bar{b} := (b_1, \frac{b_2}{2}, \frac{b_2}{2}) \in L_0(\mathbb{R}^3_+)$ and $\bar{\lambda}_0 := (\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,2})$. We have $\bar{a}_1^2 = a_1^2 = -(-a_1^2) = -\bar{a}_3$. Additionally, $a = (-\mathbb{E}(v_T^\pi), \mathrm{Var}(v_T^\pi))$ for some $\pi \in \Psi_0(v_0)$ since $a \in \tilde{\mathcal{I}}_0(v_0)$. Thus, $\bar{a} = (-\mathbb{E}(v_T^\pi), \mathbb{E}((v_T^\pi)^2), -(\mathbb{E}(v_T^\pi))^2) \in \mathcal{P}_0(v_0)$. Then, $\bar{a} \in \overline{TD}_0(v_0)$. Since $\overline{TD}_0(v_0)$ is an upper set, $\bar{a} + \bar{b} \in \overline{TD}_0(v_0)$. Moreover,

$$\begin{aligned}
\bar{\lambda}_0^\mathsf{T}\left(\bar{a} + \bar{b}\right) &= \bar{\lambda}_{0,1}\left(\bar{a}_1 + \bar{b}_1\right) + \bar{\lambda}_{0,2}\left(\bar{a}_2 + \bar{b}_2\right) + \bar{\lambda}_{0,3}\left(\bar{a}_3 + \bar{b}_3\right) \\
&= \lambda_{0,1}(a_1 + b_1) + \lambda_{0,2}(a_2 + a_1^2 + \frac{b_2}{2} - a_1^2 + \frac{b_2}{2}) \\
&= \lambda_{0,1}(a_1 + b_1) + \lambda_{0,2}(a_2 + b_2) \\
&= \lambda_0^\mathsf{T} y
\end{aligned}$$

and $\lambda_0^\mathsf{T} y < \lambda_0^\mathsf{T} y^* = \bar{\lambda}_0^\mathsf{T} x^*$. Which is a contradiction to the fact that x^* is a minimal point. Thus, there exists no such y and $y^* \in MV_0(v_0)$ is a minimal point. $\square$

Lemma 6.1.5. *Let $v_0 \in \mathbb{R}_{++}$. $x_0^* \in \arg\min\{\bar{\lambda}_0^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$, and $y_0^* := [x_{0,1}^* \; x_{0,2}^* + x_{0,3}^*] \in PMV_0(v_0)$ be the corresponding mean-variance minimal point from Lemma 6.1.4. Then, there exists $(\pi_s^*)_{s\geq 1} \in \Psi_0(v_0)$ such that $y_0^* = (-\mathbb{E}(v_T^{\pi^*}), \mathrm{Var}(v_T^{\pi^*}))^\mathsf{T}$. In other words, 6.1.2 generates a minimal point of the mean-variance problem and there exists an optimal solution that corresponds to that minimal point.*

Proof. The existence of an optimal solution to a minimal point is immediate by the compactness of the feasible set shown in Lemma 4.1.3, continuity of Γ_t shown in 3.0.1 and Theorem 6.0.1. $\square$

Lemma 6.1.6. *let $v_0 \in \mathbb{R}_{++}$. The problem $F_0(x_0^*, v_0)$ is feasible for $x_0^* \in \arg\min\{\bar{\lambda}_0^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$. Furthermore, for any $t \in \mathbb{T} \setminus \{T\}$, let $v_t \in L_t(\mathbb{R})$, $x_t^* \in \overline{TD}_t(v_t)$, and (π_{t+1}, x_{t+1}^*) be feasible for $F_t(x_t^*, v_t)$. Then, $F_{t+1}(x_{t+1}^*, \pi_{t+1}^\mathsf{T} S_{t+1})$ is feasible.*

Proof. We will use the fact that if portfolio $(\pi_t^*)_{t\in\mathbb{T}}$ is optimal at time $0 \in \mathbb{T}$ with weight vector $\lambda_0 \in \mathbb{R}^2_{++}$, then there exists a weight process $(\lambda_t)_{t\in\mathbb{T}}$ such that $(\pi_s)_{s\geq t+1}$ remains optimal for all $t \in \mathbb{T}$ [14, Theorem 3.14]. Let $x_0^* \in \arg\min\{\bar{\lambda}_0^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$. Then, $y := [x_{0,1}^* \; x_{0,2}^* + x_{0,3}^*]^\mathsf{T}$ is minimal for $MV_0(v_0)$ by Lemma 6.1.5. Since y is minimal, by Theorem 6.0.1, there exists an optimal solution, say $(\pi_t)_{t\in\mathbb{T}}$ with $y = (-\mathbb{E}(v_T^{\pi^*}), \mathrm{Var}(v_T^{\pi^*}))^\mathsf{T}$. Consider problem $F_0(x_0^*, v_0)$. We know that there exists a weight $\lambda_1 \in L_1(\mathbb{R}^2)$ such that $(\pi_s^*)_{s\geq 2}$ is optimal for the problem $PMV_1(\pi_1^\mathsf{T} S_1, \lambda_1)$, with objective function vector $\Gamma_1(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2)$. Then,

$$\Gamma_0\left(M\Gamma_1(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2)\right) = \Gamma_0(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2) = x_0^* \tag{6.1.3}$$

by the recursiveness of Γ_t. Furthermore, $(\pi_1^*, \Gamma_1(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2))$ is feasible for $F_0(x_0^*, v_0)$, by (6.1.3) and $\Gamma_1(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2) \in \overline{TD}_1((\pi_1^*)^\mathsf{T} S_1)$.

Now consider $F_t(x_t^*, v_t)$ for an arbitrary $t \in \mathbb{T}$, where x_t^* is found by the same procedure as in the first part. That is, using a feasible solution (π_1^*, x_1^*) of $F_0(x_0^*, v_0)$ to find a feasible solution (π_2^*, x_2^*) to $F_1(x_1^*, (\pi_1^*)^\mathsf{T} S_1)$. Then, using (π_2^*, x_2^*) to find a feasible solution (π_3^*, x_3^*) to $F_2(x_2^*, (\pi_2^*)^\mathsf{T} S_2)$, and so on, up to time $t \in \mathbb{T}$. Then, we have $x_t^* = \Gamma_t(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2)$. There exists $\lambda_{t+1} \in L_{t+1}(\mathbb{R}^2)$ such that $(\pi_s^*)_{s \geq t+2}$ is an optimal solution to the problem $MV_{t+1}((\pi_{t+1}^*)^\mathsf{T} S_{t+1}, \lambda_{t+1})$. Then, similarly

$$\Gamma_t(M\Gamma_{t+1}(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2)) = \Gamma_t(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2) = x_t^* \tag{6.1.4}$$

by the recursiveness of Γ_t. Furthermore, $(\pi_{t+1}^*, \Gamma_{t+1}(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2))$ is feasible for $F_t(x_t^*, v_t)$, by (6.1.4) and $\Gamma_{t+1}(v_T^{\pi^*}, (v_T^{\pi^*})^2, (v_T^{\pi^*})^2) \in \overline{TD}_{t+1}((\pi_{t+1}^*)^\mathsf{T} S_{t+1})$. Hence, $F_t(x_t^*, v_t)$ is feasible for all $t \in \mathbb{T}$. □

Theorem 6.1.7. *Let $\lambda_0 \in \mathbb{R}^2_{++}$, $v_0 \in \mathbb{R}_{++}$. Algorithm 3 finds an optimal portfolio for the problem $MV_0(v_0, \lambda_0)$.*

Proof. We have shown in Lemma 6.1.4 that the problem in 6.1.2 can be solved to obtain a minimal point $y_0 \coloneqq (x_{0,1}, x_{0,2} + x_{0,3})$ of the mean-variance upper image $PMV_0(v_0)$ where $x_0 \in \arg\min\{\bar{\lambda}_0^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$. Our aim is to find a portfolio $\pi \in \Psi_0(v_0)$ such that $x_0 = \Gamma_0(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$, which we know to exists by Lemma 6.1.5. Additionally, we know that $F_t(x_t^*, v_t)$ is feasible for all $t \in \mathbb{T}$ by Lemma 6.1.6. Let $(\pi_{t+1}^*, x_{t+1})_{t \in \mathbb{T}\setminus\{T\}}$ be feasible for the family of problems $(F_t(x_t, (\pi_t)^\mathsf{T} S_t))_{t \in \mathbb{T}\setminus\{0\}}$. Then, we have

$$\Gamma_t(Mx_{t+1}) \leq x_t$$

for all $t \in \mathbb{T} \setminus \{0, T\}$ by feasibility. Furthermore, by the monotonicity and recursiveness of Γ_t, we get

$$\Gamma_0(M\Gamma_1(\ldots(M\Gamma_t(Mx_{t+1})))) = \Gamma_0(Mx_{t+1}) \leq x_0 = \Gamma_0(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$$

for all $t \in \mathbb{T} \setminus \{T\}$. Since $x_0 \in \overline{TD}_0(v_0)$ is a minimal point, we get $\Gamma_0(Mx_{t+1}) = \Gamma_0(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$ for all $t \in \mathbb{T} \setminus \{T\}$. Then, π can be chosen as the associated portfolio $(\pi_{t+1}^*)_{t\in\mathbb{T}}$ from the feasible pairs $(\pi_{t+1}^*, x_{t+1})_{t\in\mathbb{T}\setminus\{T\}}$. Thus, Algorithm 3 finds an optimal portfolio. □

Theorem 6.1.8. *Let $x_0^* \in \arg\min\{\bar{\lambda}^\mathsf{T} x \mid x \in \overline{TD}_0(v_0)\}$, $(\pi_1^*, x_1^*), \ldots, (\pi_T^*, x_T^*)$ be feasible for the problems $F_0(x_0^*, v_0), \ldots, F(x_{T-1}^*, (\pi_{T-1}^*)^\mathsf{T} S_{T-1})$, respectively. Then, $x_t^* \in \overline{TD}_t((\pi_t^*)^\mathsf{T} S_t)$ is a minimal point for all $t \in \mathbb{T}$. Furthermore, x_t^* is minimal for $\mathcal{P}_t((\pi_t^*)^\mathsf{T} S_t)$ as well.*

Proof. The minimality of $x_0^* \in \overline{TD}_0(v_0)$ is guaranteed by the strictly positive weight $\bar{\lambda} \in \mathbb{R}^3_{++}$, by [15, Theorem 11.17]. Assume $x_1^* \in \overline{TD}_1((\pi_1^*)^\mathsf{T} S_1)$ is not minimal. Then, there exists $x \in \overline{TD}_1((\pi_1^*)^\mathsf{T} S_1)$ such that $x \leq x_1^*$ and $x \neq x_1^*$. By feasibility of (π_1^*, x_1^*), we have

$$\Gamma_0(Mx_1^*) \leq x_0^*.$$

Additionally, by Lemma 6.1.3, we know that $\Gamma_0(Mx) \in \overline{TD}_0(v_0)$. Then, the hypothesis and the strict monotonicity of Γ_0 implies

$$\Gamma_0(Mx) \leq \Gamma_0(Mx_1^*) \leq x_0^* \quad \text{and} \quad \Gamma_0(Mx) \neq \Gamma_0(Mx_1^*),$$

contradicting the fact that x_0^* is minimal. Thus, $x_1^* \in \overline{TD}_1((\pi_1^*)^\mathsf{T} S_1)$ is a minimal point.

Let $x_t^* \in \overline{TD}_t((\pi_t^*)^\mathsf{T} S_t)$ be a minimal point. Assume $x_{t+1}^* \in \overline{TD}_{t+1}((\pi_{t+1}^*)^\mathsf{T} S_{t+1})$ is not minimal. Then, there exists $x \in \overline{TD}_{t+1}((\pi_{t+1}^*)^\mathsf{T} S_{t+1})$ such that $x \leq x_{t+1}^*$ and $x \neq x_{t+1}^*$. Similarly to the first part, $\Gamma_t(Mx) \in \overline{TD}_t((\pi_t^*)^\mathsf{T} S_t)$ by Lemma 6.1.3. Then, by the strict monotonicity and the hypothesis, we get

$$\Gamma_t(Mx) \leq \Gamma_t(Mx_{t+1}^*) \leq x_t^* \quad \text{and} \quad \Gamma_t(Mx) \neq \Gamma_t(Mx_{t+1}^*),$$

contradicting the fact that $x_t^* \in \overline{TD}_t((\pi_t^*)^\mathsf{T} S_t)$ is minimal. Thus, $x_{t+1}^* \in \overline{TD}_{t+1}((\pi_{t+1}^*)^\mathsf{T} S_{t+1})$ is a minimal point.

For the last part, assume x_t^* is not minimal for $\mathcal{P}_t((\pi_t^*)^\mathsf{T} S_t)$. Then, there exists $x \in \mathcal{P}_t((\pi_t^*)^\mathsf{T} S_t)$ such that $x \leq x_t^*$ and $x \neq x_t^*$. Let $x = a + b$, $a = \Gamma_t(v_T^\pi, (v_T^\pi)^2, (v_T^\pi)^2)$ for some $\pi \in \Psi_t((\pi_t^*)^\mathsf{T} S_t)$, and $b \in L_t(\mathbb{R}_+^3)$. Then, $a \in \overline{TD}_t((\pi_t^*)^\mathsf{T} S_t)$, as it is a mean-variance image. By the monotonicity of Γ_t, we have

$$\Gamma_{t-1}(Ma) \leq \Gamma_{t-1}(Mx) \leq \Gamma_{t-1}(Mx_t^*).$$

However, since $x_t^* \in \overline{TD}_t((\pi_t^*)^\mathsf{T} S_t)$ is a minimal point, this cannot hold as $a \in \overline{TD}_t((\pi_t^*)^\mathsf{T} S_t)$. Thus, x_t^* is minimal for $\mathcal{P}_t((\pi_t^*)^\mathsf{T} S_t)$ for all $t \in \mathbb{T}$. $\square$

Theorem 6.1.7 shows that a portfolio found by using Algorithm 3 finds an optimal portfolio for the mean-variance problem at time 0, that is, a portfolio that fits the precommitment approach. Furthermore, Theorem 6.1.8 shows that this portfolio is efficient for the family of problems $(G_t(\cdot))_{t\in\mathbb{T}}$.

Chapter 7

Examples

In this Chapter, the Algorithms 1 and 3 are implemented to two sample markets.

Example 7.0.1. In the first example, a binomial model is considered. Over $T = 100$ periods, the market is made up of one risky asset with up factor 1.0025, down factor 0.999 and one bond with 0.5% return in each period. We take initial wealth $v_0 = 10$.

Example 7.0.2. In the second example, consider a five-dimensional Black-Scholes model, where the initial stock price is taken as 1 for each asset, that is, $S_0 = \mathbf{1} \in \mathbb{R}^5_+$, and W^i_t is a Wiener process for each $i \in \{1, \ldots, 5\}$. The continuous-time price dynamics are given by

$$dS^i_t = S^i_t(rdt + \sigma_i dW^i_t)$$

for each $i \in \{1, \ldots, 5\}$. Moreover, we let the Brownian motions be correlated with a constant correlation coefficient $\rho_{ij} \neq 0$ for every $i, j \in \{1, \ldots, 5\}$ where $i \neq j$. Take $r = 0.1$ as the drift rate, and the rest of the parameters are

$$\sigma = (0.2, 0.5, 0.15, 0.3, 0.25)^{\mathsf{T}}, \rho = \begin{pmatrix} 1 & 0.5 & -0.2 & -0.1 & -0.3 \\ 0.5 & 1 & -0.4 & 0.4 & -0.1 \\ -0.2 & -0.4 & 1 & 0.25 & -0.3 \\ -0.1 & 0.4 & 0.25 & 1 & -0.1 \\ -0.3 & -0.1 & -0.3 & -0.1 & 1 \end{pmatrix}$$

where σ is the volatility rate of the stocks. The choice of the parameters are taken from [14], and the method to transition the price process into a discrete-time process is given in [22]. We solve for $T = 10$, with initial wealth $v_0 = 1$.

In the examples, we take $\lambda_0 = (1 - \lambda_{0,2}, \lambda_{0,2}, \lambda_{0,2})$. In Figures 7.1 and 7.4, the stock price process $(S_t)_{t\in\mathbb{T}}$, the portfolio process $(\pi^\star_{t+1})_{t\in\mathbb{T}\setminus\{0\}}$, the wealth process $(v_t)_{t\in\mathbb{T}}$ and the conditional expectation process $(\mathbb{E}_t(v_T^{\pi^\star}))$ are given for Examples 7.0.1 and 7.0.2, respectively. We observe for Example 7.0.1 that as the risk attitude of the investor decreases, that is, $\lambda_{0,2}$ parameter increases, the investor tends to buy less of the risky asset. This results in a more volatile wealth process for a risk-seeking investor. For Example 7.0.2, the least risky assets are the first and the third assets. Therefore, a risk-averse investor leans towards those assets more compared to a risk-seeking investor. We again see that the wealth process for a risk-seeking investor is more volatile.

In Figures 7.2 and 7.5, the process $(x_t^\star)_{t\in\mathbb{T}}$, where $x_t^\star \in VMV_t(v_t)$, and the intermediate time mean-variance curves are given for Examples 7.0.1 and 7.0.2, respectively. We observe that $x_0^\star \in VMV_0(v_0)$ is a minimal point for both examples, but the subsequent points fail to remain minimal due to the time-inconsistency of the mean-variance problem. Finally, in Figures 7.3 and 7.6, the three-dimensional upper images are given for each example in different times. One may observe the nonconvex nature of these upper sets as the problem $G_t(v_t)$ is nonconvex for all $t \in \mathbb{T} \setminus \{0\}$.

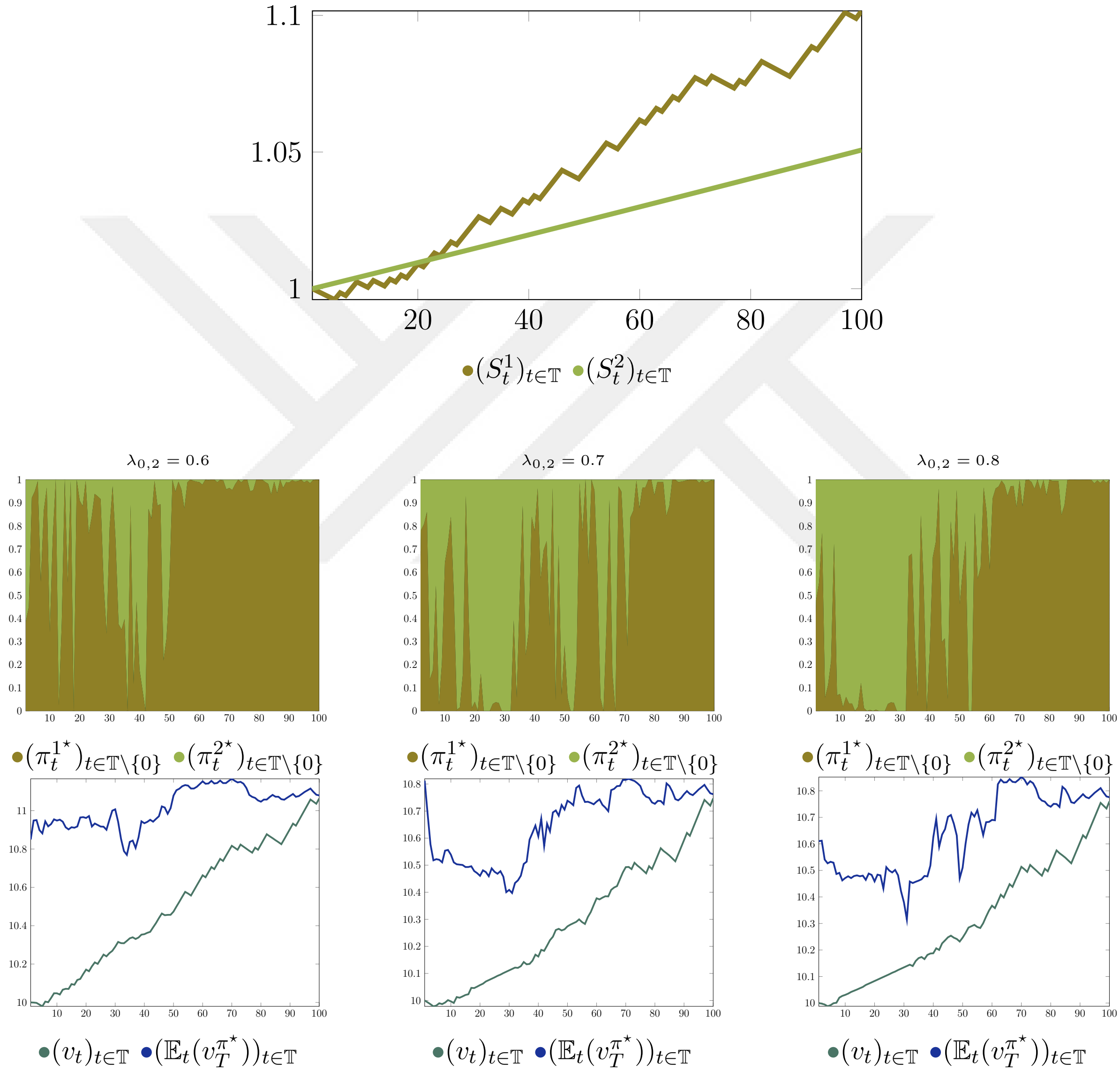
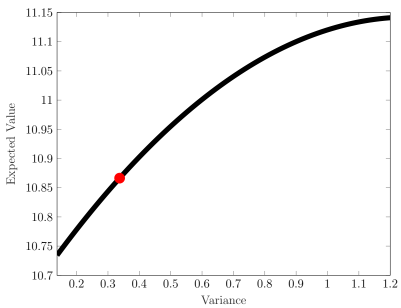
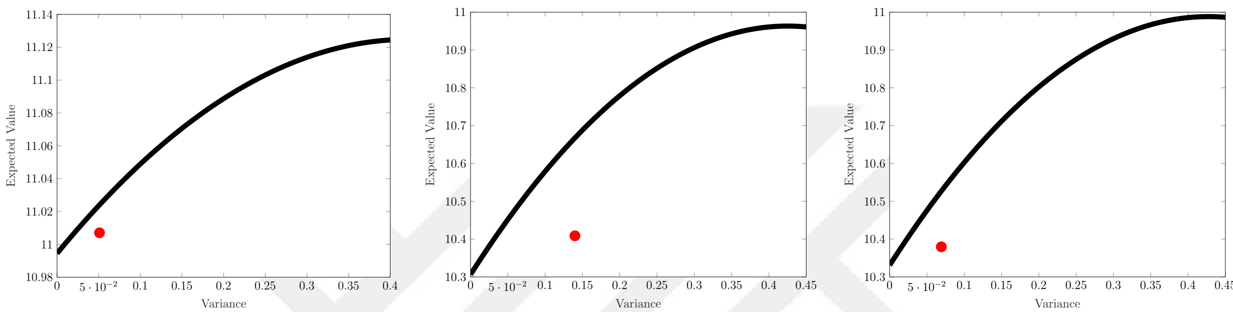


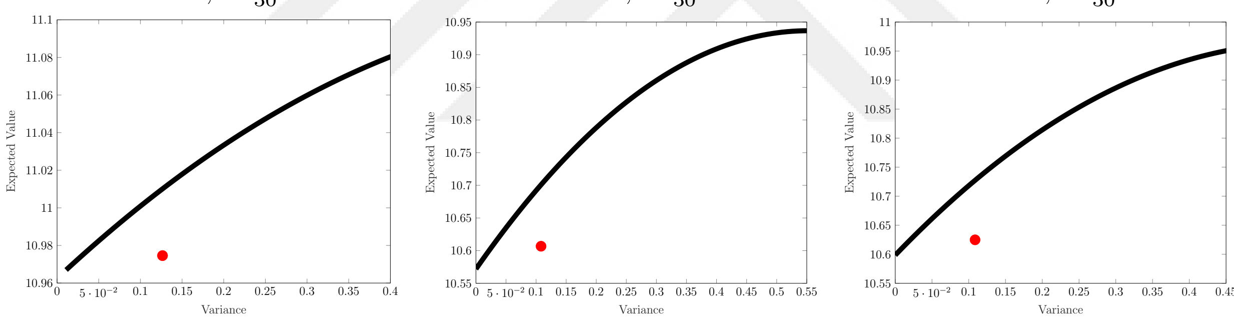
Figure 7.1: The processes $(S_t)_{t\in\mathbb{T}}$, $(\pi^\star_t)_{t\in\mathbb{T}\setminus\{0\}}$, $(v_t)_{t\in\mathbb{T}}$ and $(\mathbb{E}_t(v_T^{\pi^\star}))_{t\in\mathbb{T}}$ for $\lambda_0 \in \{0.6, 0.7, 0.8\}$ along one path for Example 7.0.2.



$t = 0$, $\bullet x_0^\star$ $\quad$ $t = 0$, $\bullet x_0^\star$ $\quad$ $t = 0$, $\bullet x_0^\star$

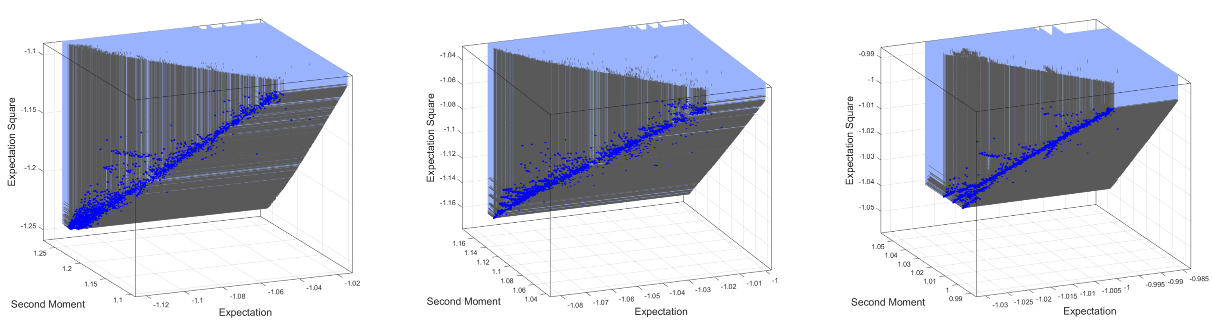


$t = 30$, $\bullet x_{30}^\star$ $\quad$ $t = 30$, $\bullet x_{30}^\star$ $\quad$ $t = 30$, $\bullet x_{30}^\star$



$t = 40$, $\bullet x_{40}^\star$ $\quad$ $t = 40$, $\bullet x_{40}^\star$ $\quad$ $t = 40$, $\bullet x_{40}^\star$

Figure 7.2: The points $x_t^\star \in VMV_t(v_t)$ for $t \in \{0, 30, 40\}$ for Example 7.0.1.



$t = 0$ $\quad$ $t = 40$ $\quad$ $t = 80$

Figure 7.3: $\mathcal{P}_t(v_0)$ graphs for the binomial market model for $t \in \{0, 40, 80\}$ for Example 7.0.1.

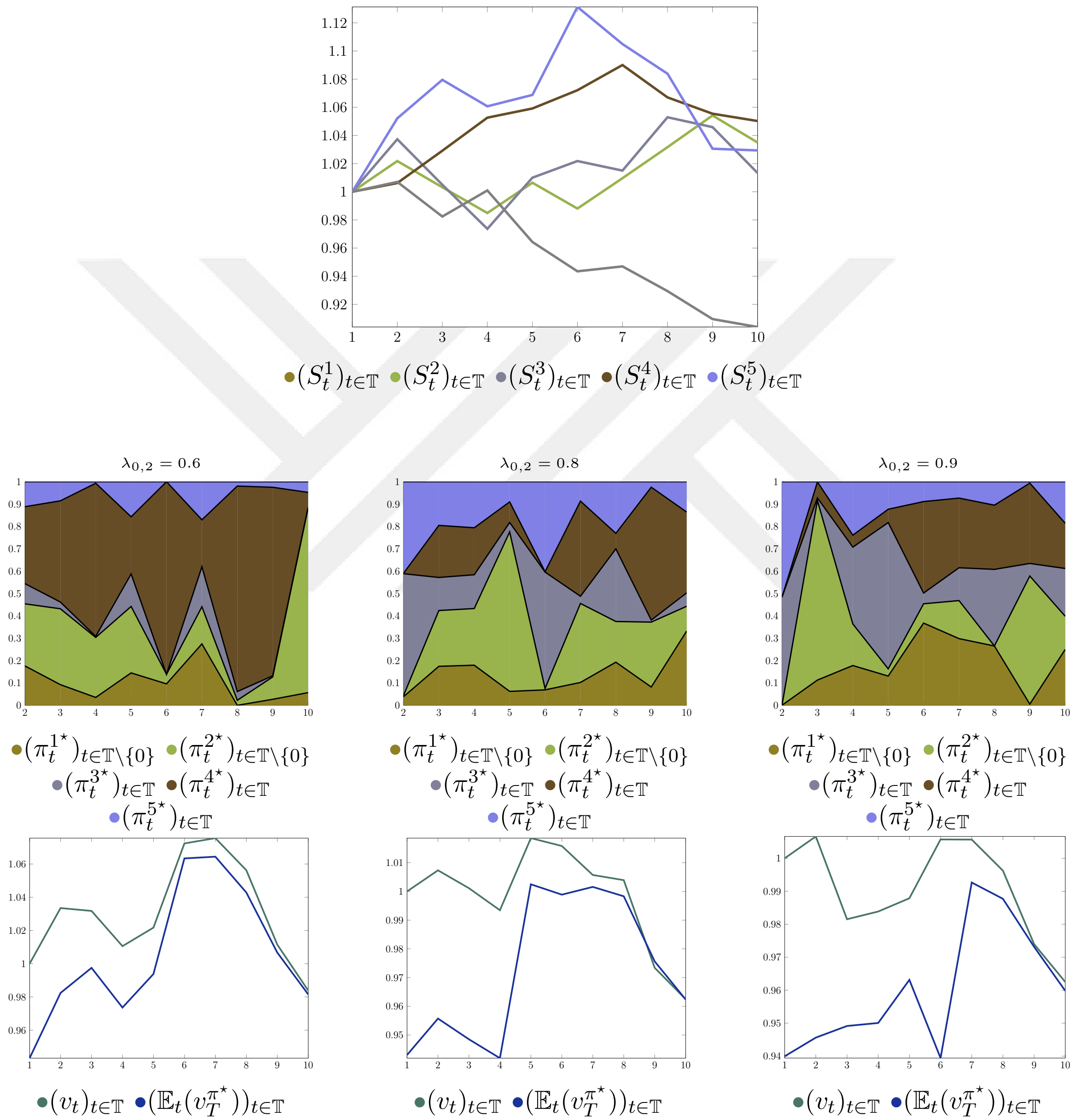


Figure 7.4: The processes $(S_t)_{t\in\mathbb{T}}$, $(\pi_t^\star)_{t\in\mathbb{T}\setminus\{0\}}$, $(v_t)_{t\in\mathbb{T}}$ and $(\mathbb{E}_t(v_T^{\pi^\star}))_{t\in\mathbb{T}}$ for $\lambda_0 \in \{0.6, 0.8, 0.9\}$ along one path for Example 7.0.2.

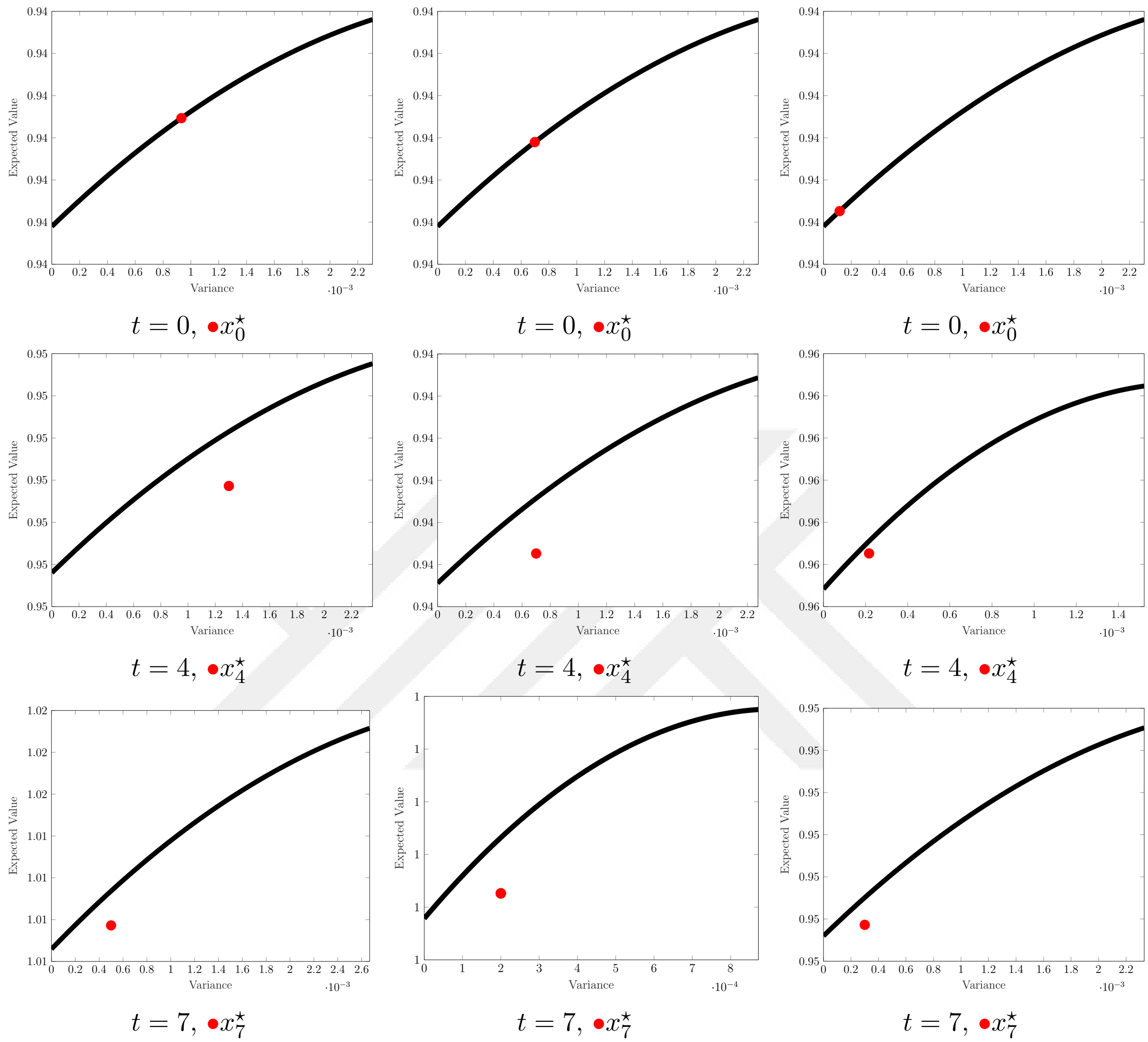


Figure 7.5: The points $x_t^\star \in VMV_t(v_t)$ for $t \in \{0, 4, 7\}$ for Example 7.0.2.

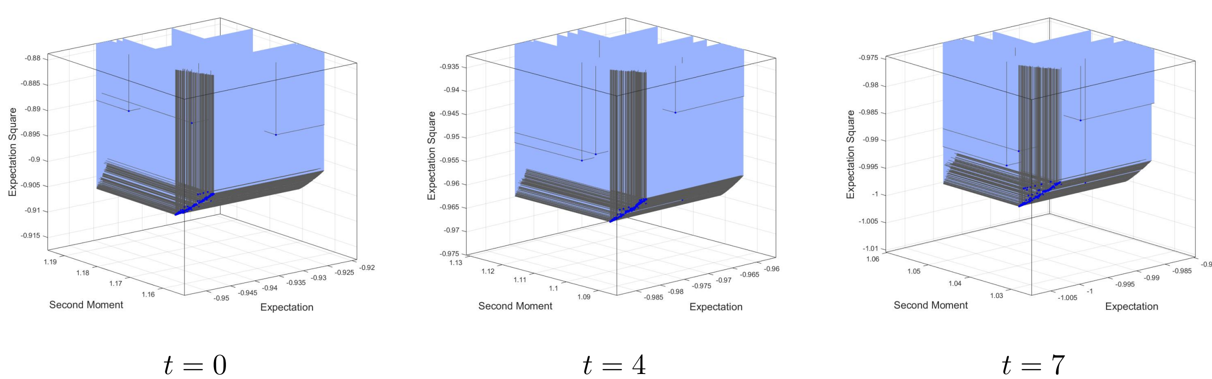


Figure 7.6: $\mathcal{P}_t(v_t)$ graphs for the 5 asset market model for $t \in \{0, 4, 7\}$ for Example 7.0.2.

Chapter 8

Conclusion

In this thesis, we reformulate the mean-variance problem as a tri-dimensional multi objective problem, where the scalarization of the reformulated problem is equivalent to the mean-variance problem under the given risk attitude of the investor. Next, we show that the reformulated problem is time-consistent with respect to Definition 3.1.1. Using a set-valued optimization problem on the space of closed upper sets, we show that the upper images of the reformulated problem can be used as a value function to solve the problem dynamically. We unveil the dynamic structure in Theorem 3.2.2.

We propose a methodology to compute the upper images of the reformulated problem, where we use the upper images iteratively to compute the next upper image in a backward manner. We use the nonconvex Benson-type Algorithm in [20]. Alternatively, one may also use the solution methodologies in [23] and [24]. After computing the upper images throughout the investment horizon, we are able to use them to find investment strategies for the investor entering the market with a certain risk attitude. We show that the investment strategies we find are optimal for the initial mean-variance problem in Theorem 6.1.7. Additionally, we show that the strategies are time-consistent with respect to minimizers for the reformulated problem in Theorem 6.1.8. However, we fail to obtain time-consistency for the mean-variance problem.

This thesis is comparable to [14]. There, they are able to find time-consistent investment strategies for the mean-variance problem using the mean-second moment auxiliary problem. Additionally, they provide a method to obtain a weight process that makes the family of mean-variance problem time-consistent. However, the time-consistent weight process they find may take negative values, which intuitively implies that it is reasonable to consider investment strategies that minimize the expected wealth, making the problem lose its purpose. In our methodology, although we are able to find time-consistent strategies for the reformulated problem, we fail to provide a such weight process as the problem is nonconvex.

One important distinction to note compared to [14] is that their problem is convex in nature, and the intermediate time problems they solve use a convex polyhedral approximation, which turns into a linear programming problem. In our methodology, we work with nonconvex quadratic mixed integer programming problems, making it quite difficult to solve the problem for longer periods of time, or in a market with many assets.

In the future, this methodology can be altered in a way that it provides time-consistent investment strategies for the mean-variance problem as well. The computation of the upper images in the backwards algorithm includes the images of the mean-variance time-consistent strategies. However, the forward processes should be revised to unveil them. Additionally, the way the mean-variance problem is originally formulated using risk attitude parameters in $MV_0(v_0, \lambda_0)$ may be counter intuitive, as an investor may not be able to provide such parameters. It may be more realistic for the investor to provide a minimum expected return or a maximum variance at the end of investment horizon, as it is more likely that they already have a certain expectation from their investments. Therefore, this methodology can be extended to a slightly different formulation of the mean-variance problem as well.

Bibliography

[1] H. Markowitz, "Portfolio selection," *The Journal of Finance*, vol. 7, no. 1, pp. 77–91, 1952.

[2] J. Mossin, "Optimal multiperiod portfolio policies," *The Journal of Business*, vol. 41, no. 2, pp. 215–229, 1968.

[3] R. C. Merton, "Lifetime portfolio selection under uncertainty: The continuous-time case," *The Review of Economics and Statistics*, vol. 51, no. 3, pp. 247–257, 1969.

[4] D. Li and W.-L. Ng, "Optimal dynamic portfolio selection: Multiperiod mean-variance formulation," *Mathematical Finance*, vol. 10, no. 3, pp. 387–406, 2000.

[5] I. Bajeux-Besnainou and R. Portrait, "Dynamic asset allocation in a mean-variance framework," *Management Science*, vol. 44, pp. 79–95, 1998.

[6] X. Zhou and D. Li, "Continuous-time mean-variance portfolio selection: A stochastic lq framework," *Appl. Math. Optim.*, vol. 42, pp. 19–33, 12 2000.

[7] X. Li, X. Zhou, and A. Lim, "Dynamic mean-variance portfolio selection with no-shorting constraints," *SIAM J. Control and Optimization*, vol. 40, pp. 1540–1555, 01 2002.

[8] A. E. B. Lim and X. Y. Zhou, "Mean-variance portfolio selection with random parameters in a complete market," *Mathematics of Operations Research*, vol. 27, no. 1, pp. 101–120, 2002.

[9] R. H. Strotz, "Myopia and inconsistency in dynamic utility maximization," *The Review of Economic Studies*, vol. 23, no. 3, pp. 165–180, 1955.

[10] S. Basak and G. Chabakauri, "Dynamic Mean-Variance Asset Allocation," *The Review of Financial Studies*, vol. 23, pp. 2970–3016, 05 2010.

[11] C. Czichowsky, "Time-consistent mean-variance portfolio selection in discrete and continuous time," *Finance and Stochastics*, vol. 17, pp. 227 – 271, 2012.

[12] T. Björk, A. Murgoci, and X. Y. Zhou, "Mean–variance portfolio optimization with state-dependent risk aversion," *Mathematical Finance*, vol. 24, no. 1, pp. 1–24, 2014.

[13] G. Kovacova and B. Rudloff, "Time consistency of the mean risk problem," *Operations Research*, vol. 69, 03 2021.

[14] C. Ararat and E. Duzoylum, "Dynamic mean variance problem recovering time inconsistency," 2021.

[15] J. Jahn, *Vector Optimization*. Springer, 2011.

[16] A. Löhne, *Vector Optimization with Infimum and Supremum*. Springer, 2011.

[17] A. Hamel, F. Heyde, A. Lohne, B. Rudloff, and C. Schrage, *Set Optimization and Applications - The State of the Art: From Set Relations to Set-Valued Risk Measures*, vol. 151. 01 2015.

[18] H. Benson, "An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem," *Journal of Global Optimization*, vol. 13, pp. 1–24, 1998.

[19] A. Lohne, B. Rudloff, and F. Ulus, "Primal and dual approximation algorithms for convex vector optimization problems," *Journal of Global Optimization*, vol. 60, pp. 713–736, 2014.

[20] S. Nobakhtian and N. Shafiei, "A benson type algorithm for nonconvex multiobjective programming problems," *TOP*, vol. 25, pp. 271–287, 2017.

[21] A. Pascoletti and P. Serafini, "Scalarizing vector optimization problems," *Journal of Global Optimization*, vol. 42, pp. 499–524, 1984.

[22] R. Korn and S. Müller, "The decoupling approach to binomial pricing of multi-asset options," *The Journal of Computational Finance*, vol. 3, 03 2009.

[23] G. Eichfelder, O. Stein, and L. Warnow, "A solver for multiobjective mixed-integer convex and nonconvex optimization," *Journal of Optimization Theory and Applications*, pp. 1–31, 09 2023.

[24] Q. Zhu, L. P. Tang, and X. M. Yang, "A piecewise convexification method for non-convex multi-objective optimization programs with box constraints," 2022.