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On the stability and instability of nonlinear damped wave equations

by

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in the

College of Sciences

Department of Mathematics

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Declaration of Authorship

I, Bilgesu Arif Bilgin, declare that this thesis titled, ‘Blow-up theorems for nonlinear evolutionary equations’ and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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“If you want to find the secrets of the universe, think in terms of energy, frequency and vibration.”

Nikola Tesla

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Abstract

College of Sciences
Department of Mathematics

Ph.D. Thesis

by Bilgesu Arif Bilgin

This thesis is devoted to the investigation of the qualitative behaviours of solutions to the initial-boundary and initial value problems for various nonlinear damped wave equations. Depending on the type of nonlinearities involved these problems exhibit very different dynamics: The ‘unstable’ ones contain trajectories, which develop singularities in finite time, while for the ‘stable’ ones all solutions behave well for all times. In this manuscript we first derive sufficient conditions on the initial functions, which ensure the finite time blow-up of the corresponding solutions to the unstable problems. These results are first derived in an abstract setting, and then applied to equations like the semilinear and quasilinear structurally damped wave equations, damped plate equations, as well as semilinear Boussinesq and Kirchhoff type equations with damping. Also, we show the global nonexistence of solutions to damped quasilinear wave and Boussinesq type equations with a convective term. Later, for the stable structurally damped wave equations with super-critical nonlinear potentials we show the existence of finitely many determining Fourier modes. Finally, we present some simulations carried out for both the stable and the unstable semilinear structurally damped wave equations, and justify our scheme of numerical approximation.

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*I dedicate this thesis to the loving memory of my grandfather,
Hüseyin Uyanık...*

Chapter 1

Introduction

There is no need to emphasize the importance of the role that the partial differential equations (PDE's) play in our quest to understand the workings of the universe that surrounds us. In fact, looking at almost any scientific endeavour men have undertaken, we often see that, one ultimately models the phenomenon investigated via a PDE. Any such model of a dynamical (changing in time) phenomenon will necessarily produce an evolutionary PDE, and an important class of these evolutionary PDE's model '*wave-like*' or oscillatory phenomena. Such phenomena are a common feature of our universe; anything that propagates, electromagnetic, sound or water waves, falls into this class. Also, again a very common thing one observes in these phenomena is the fact that the wave that propagates dims and vanishes in time. For example, if you throw a stone in a still lake, there will be nothing visible left from the waves you have created in sufficient time, or if one hits a string of a guitar, the string will vibrate only for a short time. This effect is referred to as the *damping* of the wave, and is an indication of the fact that the system gradually loses its energy during the evolution, which usually occurs due to some kind of a friction in the system. As an alternative terminology one also says that, the system is *dissipative*, or the system *dissipates energy*.

In this thesis our aim is to investigate the qualitative behaviour of the solutions to the initial value and the initial-boundary value problems for the various equations, which model the aforementioned dissipative wave phenomena. A common feature of all the equations we will consider are that, they are all second order in time, and that the solution is a real valued function of space and time. We analyze these equations with an additional nonlinearity appended to them, which in practice corresponds to an external nonlinear potential being subjected to the system. We consider nonlinearities that are (in some sense) monotone, and with polynomial growth at infinity. It is well-known that, for the class of nonlinearities we are interested in, the solutions can exhibit vastly different behaviours depending on the

sign of the nonlinearity. While in one case all solutions exist globally in time and have nice asymptotic properties such as absorbing balls, global attractors of finite dimension, and finitely many Fourier modes that completely determine the asymptotical behaviour of the dynamics; in the other case there are solutions which escape in finite time from the phase space, in which they are defined. Both aspects of the theory are touched by these notes, and the former class of equations will be referred to as the *stable* equations, while the latter ones will be called *unstable*.

The equations that appear in this thesis are all a relative of one of the following equations: structurally damped wave equations, Klein-Gordon equation, Beam and Plate equations, Boussinesq type equations and Kirchhoff type equations.

Of all the equations that we consider, a special emphasis is given to the structurally damped wave equations. The main reason behind this is that, they illustrate the peculiar effect of the ‘strength’ of the damping, in that a ‘strong’ enough damping can convert the hyperbolic (oscillatory) behaviour of the solutions to a parabolic (purely decaying) one, see Chapter §4 for a discussion. This equation is nowadays most widely used as a model in material sciences to describe the behaviour of the body motions of viscoelastic materials under loading. Such materials display a behaviour that contains the characteristics of both an elastic material, and a viscous one. The strength of the structural damping is related to the deformation memory of the material. The weak damping corresponds to slowly fading memory, while the strong damping models materials with short memory. See [33], and the references therein. Most classically, the weakly damped equation is known to accurately describe the vibrations of a spring with both ends fixed, where the weak damping represents a frictional force proportional to the vertical speed of the spring, acting in the opposite direction to that of the movement of the spring. See §1.1.1 for its derivation. Also, another name for the weakly damped wave equation is the telegraph equation, as it was historically first derived to model the behaviour of the amplitude of the voltage on a telegraph transmission line. The damping of the transmitted wave is due to the energy dissipated in the line and the energy leaked to the surrounding, see §1.1.2. Yet another field, where a very close relative of this equation is used as a model, is the field of population dynamics. There is a discussion of this equation in §1.1.3.

Let us mention a few words on the other equations as well. The Klein-Gordon equation was first proposed by Oskar Klein, and Walter Gordon independently, in 1926 with the aim of describing the relativistic motion of an electron. However, later it turned out that, the motion of the electron was correctly described by the Dirac equation, and that, instead, the Klein-Gordon equation describes the 0-spin particles. Newly discovered Higgs boson is the first known elementary particle of spin 0. The beam and plate equations model the body

motions of elastic beams and plates, respectively. These equations have the same form, the only difference is the dimension of the problem, respectively 1 and 2. There is a wide variety of Boussinesq type equations, and the original equation, which we will consider in these notes, was first proposed by Joseph Boussinesq in his celebrated 1872 paper [5]. It is a model for long waves, and is widely used in simulations of the shallow waters, such as corals and harbours. Finally, the Kirchhoff type equations are non-local equations, first proposed by physicist Gustav Kirchhoff in 1883, see [23]. It is a model, for example for the elastic string, and its derivation differs from that of the classical wave equation of D’Alembert, in that, it takes account the length of the vibrating string during the vibrations as well.

The text is organized as follows. Foremost, the notations are introduced along the text. In the rest of this chapter, first, to provide some motivation and to establish a connection with the physical reality, in section §1.1 we present the derivations of some of the equations that we consider. Then, in §1.2, we briefly present some necessary mathematical tools and results, which will be necessary for the exposition to come. In chapter §2 we investigate the unstable problems posed for semilinear and quasilinear versions of the aforementioned equations. To do this, we first work in an abstract setting in section §2.1, where we derive sufficient conditions for the initial data, that ensure the finite time blow-up of solutions, and also show that, under some restrictions on the nonlinearity all global solutions remain bounded in time. Then, in section §2.2, we apply these abstract results to some concrete problems, which contain the damped wave equations, Klein-Gordon equations, beam and plate equations, and Boussinesq and Kirchhoff type equations. Next, in sections §2.3 and §2.4, we consider the quasilinear strongly damped wave equations and the semilinear Boussinesq type equations with convection, respectively, which do not fit to the abstract setting of §2.1. In chapter §3 we turn our attention to the stable problems, and focus to the structurally damped wave equation with supercritical exponents in the nonlinearity. In section §3.1 we investigate the existence of finitely many determining modes for an initial-boundary value problem for this equation. Finally, in Chapter §4, we present some numerical simulations carried out for the semilinear structurally damped wave equations, and carry out the error analysis to justify the convergence of our numerical scheme.

1.1 Some derivations

In this section we give three different examples, where the structurally damped wave equations serve as a model.

1.1.1 The string in a viscous fluid

Suppose a string of length L and mass M is suspended horizontally in a viscous fluid with a constant tension τ . The viscous fluid is assumed to exert a drag force onto the spring as it moves, which is proportional to the speed of the spring, and is in the opposite direction to the movement. We denote by $u(x, t)$ the upward vertical displacement of the string from the initial rest configuration, where x and t are the space and time variables, respectively. Let us also denote by $T(x, t)$ and $\Theta(x, t)$ the tension and the angle that the spring deviates from the x -axis in the counterclockwise direction, respectively. Note that

$$\tan \Theta(x, t) = \text{the slope of the string at the point } x \text{ and time } t = \frac{\partial u}{\partial x}(x, t). \quad (1.1.1)$$

The first assumption we make is that the amplitude of the oscillations of the string are small compared to the length of the string. This has two consequences; first, we may assume that each part of the string moves vertically, and second, we may ignore the necessary lengthening of the spring during oscillations, which should normally show itself as an increase in the no more constant horizontal tension. Let us now consider an arbitrary small segment of the spring $(x, \Delta x)$. The assumption that each part of the spring moves vertically and Newton's second law imply that, the total upward vertical force acting on this part of the spring is related to the vertical acceleration by the formula

$$F_v = \rho \Delta x \frac{\partial^2 u}{\partial t^2}, \quad (1.1.2)$$

where $\rho = \frac{M}{L}$ is the linear density of the spring. Now, balancing the horizontal components of the forces acting on this piece of string we have

$$T(x + \Delta x, t) \cos \Theta(x + \Delta x, t) = T(x, t) \cos \Theta(x, t) = \tau. \quad (1.1.3)$$

Note that the drag force does not appear here because of the vertical movement assumption. Adding all of the vertical components of the forces we find the total vertical force as

$$F_v = T(x + \Delta x, t) \sin \Theta(x + \Delta x, t) - T(x, t) \sin \Theta(x, t) - b \frac{\partial u}{\partial t}(x, t) - \rho \Delta x g,$$

where g is the gravitational constant, and b is the drag constant. We utilize the horizontal balance relation (1.1.3) in this equality to obtain

$$F_v = \tau (\tan \Theta(x + \Delta x, t) - \tan \Theta(x, t)) - b \frac{\partial u}{\partial t}(x, t) - \rho \Delta x g.$$

After plugging the identities (1.1.1) and (1.1.2) in to this equality and dividing it by $\rho\Delta x$, if we let Δx tend to zero, we end up with

$$\frac{\partial^2 u}{\partial t^2}(x, t) - \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2}(x, t) - \frac{\partial u}{\partial t}(x, t) = -g.$$

This is the so-called weakly damped wave equation. Finally, the fixed endpoints of the spring imply the Dirichlet boundary conditions.

1.1.2 The transmission lines

Cables! That is what we use to transmit energy with our humble technology, and we want to understand the dynamics of the electric potential inside these cables, as it is transmitted. Let us, for this purpose, consider a small piece of this cable. We model this piece as the Figure 1.1 indicates. Here, R and L are the resistance and inductance per unit length of the transmission cable, and we assume that some current is leaked to the ground either through a resistor of conductance $G\Delta x$ or through a capacitor of capacitance $C\Delta x$. Note that the resistance of the leakage resistor is inversely related to the length of the cable, which is natural, as length increases, so does the surface of leakage.

Let $i(x, t)$ and $u(x, t)$ denote the current and the voltage at a point x and time t , respectively. First of all, by Kirchhoff's law we have

$$i(x, t) = i(x + \Delta x, t) + i_G + i_C, \quad (1.1.4)$$

where i_G and i_C are the leakage currents through the resistor and the capacitor, respectively. The voltage drop over the resistor and the inductor is given by the relation

$$u(x, t) - u(x + \Delta x, t) = R\Delta x i(x, t) + L\Delta x i_t(x, t). \quad (1.1.5)$$

Dividing this equation by Δx , and letting Δx go to zero we find

$$u_x(x, t) = Ri(x, t) + Li_t(x, t). \quad (1.1.6)$$

Moreover, we have for the leakage currents

$$i_G = G\Delta x u(x + \Delta x, t),$$

$$i_C = C\Delta x u_t(x + \Delta x, t).$$

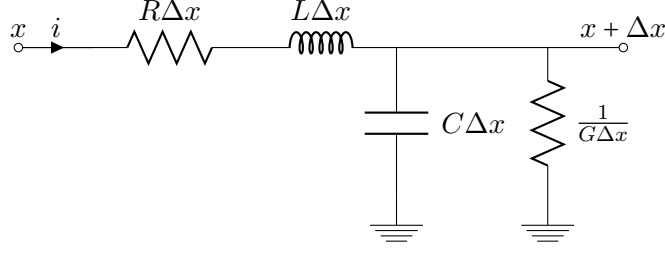


FIGURE 1.1: The circuit model for a small piece of transmission line.

Plugging these identities into the current balance relation (1.1.4) and dividing by Δx we see that

$$\frac{i(x, t) - i(x + \Delta x, t)}{\Delta x} = Gu(x + \Delta x, t) + Cu_t(x + \Delta x, t).$$

Hence, letting Δx tend to zero we arrive at

$$i_x(x, t) = Gu(x, t) + Cu_t(x, t). \quad (1.1.7)$$

Taking time derivative of this equation we get

$$i_{xt}(x, t) = Gu_t(x, t) + Cu_{tt}(x, t). \quad (1.1.8)$$

Finally, differentiating the equation (1.1.6) with respect to x , in view of the equations (1.1.7) and (1.1.8) we conclude that the voltage obeys

$$u_{tt}(x, t) - \frac{1}{LC}u_{xx}(x, t) + \frac{RG}{LC}u(x, t) + \left(\frac{R}{L} + \frac{G}{C}\right)u_t(x, t) = 0.$$

which is again a weakly damped wave equation.

1.1.3 Population dynamics

Let us try to give a simple model of the population dynamics of a habitat composed of herbivorous animals feeding off of the vegetation of the land, as well as carnivorous animals, who hunt the herbivorous animals. For simplicity we assume there is only one species in each group, and they are referred to as the prey and the predator, respectively. We further assume that, the lifespan of the prey species is short, and they can only traverse small distances compared to the size of the habitat. This assures that the prey population of a small region is driven by the local vegetation growth only. To obtain a continuous model we deal with population densities instead of population itself. Let $u(x, t), v(x, t) \geq 0$ denote the population densities of the predators and the preys, respectively. Here, $x \in \Omega$, where $\Omega \subset \mathbb{R}^2$ represents the habitat, and we impose Dirichlet boundary conditions, as the habitat

happens to be a national park surrounded by fences, across which the animals can not pass. We are given the initial densities of the populations, which vanish outside the habitat, and also, we know the amount of the vegetation growth, $h(x, t)$, everywhere in the habitat for times $t \geq 0$. By the locality assumption of the prey species, we may model the variations in its population density by the ordinary differential equation (ODE)

$$v_t(x, t) + \beta v(x, t) + \alpha u(x, t) = h(x, t), \quad (1.1.9)$$

where $\beta > 0$ is a measure of the success of the prey strategy of survival, and $\alpha > 0$ measures the effectiveness of the predator hunting strategy with large α meaning a good one. On the other hand, large β implies that, even in the absence of the predators, in times of scarcity the population plunges rapidly, and in times of abundance the population increases slower and saturates at a lower value compared to lower values of β . In other words, large β indicates a bad strategy of survival for the prey species. Next, the more mobile prey populations dynamics we model by a diffusion mechanism, where the predators are imagined to migrate from regions of low values of the ratio v/u to the regions of higher values:

$$u_t(x, t) - \gamma \Delta u(x, t) = \delta v(x, t). \quad (1.1.10)$$

Here, $\gamma > 0$ increases with the movement speed of the predator species, and $\delta > 0$ is proportional to the number of preys needed to be consumed in order to add a new member to the predator population, or in a similar sense, to the number of preys a predator must catch to have a standardized full meal. Combining the equations (1.1.9) and (1.1.10) we easily arrive at the equation

$$u_{tt}(x, t) - \beta \gamma \Delta u(x, t) + \alpha \delta u(x, t) - \gamma \Delta u_t(x, t) + \beta u_t(x, t) = \delta h(x, t),$$

which is in the linear form of the structurally damped wave equations we consider in §2.2.1.1 and §3. The initial data supplemented to the Dirichlet problem for this equation are

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = \gamma \Delta u_0(x) + \delta v_0(x),$$

where $u_0(x)$ and $v_0(x)$ are the initial population densities of the predator and prey species, respectively, and we have read the second datum from the equation (1.1.10).

1.2 Preliminaries

Here, we collect some well-known facts from various fields of analysis, which will be necessary for our treatment in chapters §2 and §3. These include some facts from the theory of

function spaces, the spectral theory for the Laplacian operator under the Dirichlet boundary conditions, and the definition of the fractional Laplacian operator, as well as a number of inequalities. In addition to these, we also introduce the so-called concavity methods, which is the main tool behind all of the proofs of finite time blow-up theorems of the chapter §2.

1.2.1 Basic function spaces and dual spaces

Function spaces are vector spaces which have functions as their elements. Two most important types of vector spaces are the *Banach spaces*, which simply are complete with respect to their norm, and the *Hilbert spaces*, which are Banach spaces with an additional structure of an inner-product built in. An inner product on a Hilbert space $\mathcal{H}$ is nothing but a bilinear, positive-definite, real or complex valued functional on $\mathcal{H} \times \mathcal{H}$ that is continuous with respect to both of its arguments. These are the most standard objects of the functional analysis, and their properties can be found in any decent book on functional analysis, see [13] or [38] for example.

Given any Banach space $\mathcal{B}$ one can define the *dual space*, denoted by $\mathcal{B}^*$, composed of all continuous linear functionals on $\mathcal{B}$. The dual space $\mathcal{B}^*$ itself is a Banach space with the operator norm

$$\|f\|_{\mathcal{B}^*} := \sup_{\|u\|_{\mathcal{B}}=1} |f(u)|.$$

By the Riesz representation theorem, for each functional f in the dual of a Hilbert space $\mathcal{H}$, one can find a unique element $u_f \in \mathcal{H}$ such that

$$f(v) = (u_f, v)_{\mathcal{H}}, \quad \forall v \in \mathcal{H},$$

and this correspondence defines an isomorphism, a continuous bijection with continuous inverse, of any Hilbert space with its dual, symbolically indicated as $\mathcal{H} \cong \mathcal{H}^*$. This property characterizes the Hilbert spaces, i.e. a Banach space that is not a Hilbert space, can not be isomorphic to its dual. However, many important Banach spaces enjoy the property $\mathcal{B} \cong \mathcal{B}^{**}$. Such spaces are called *reflexive* Banach spaces. The Riesz representation theorem also implies that any separable Hilbert space $\mathcal{H}$ is isomorphic to the Hilbert space of square summable sequences, l^2 . Moreover, given an orthonormal basis $\{e_k\}_{k \geq 1}$ of $\mathcal{H}$, for any $u \in \mathcal{H}$ one has the representation

$$u = \sum_{k \geq 1} (u, e_k)_{\mathcal{H}} e_k,$$

where the equality is understood by means of convergence in $\mathcal{H}$. Finally, from this representation and the orthonormality of the basis immediately follows the celebrated Parseval's

identity

$$\|u\|_{\mathcal{H}}^2 = \sum_{k \geq 1} (u, e_k)_{\mathcal{H}}^2, \quad u \in \mathcal{H}.$$

Throughout our discussion we will always restrict the solutions of the PDE's investigated to belong to some specified function space, and then use the properties of the space in the calculations. The choice of this space, also called the *phase space*, strongly depends on the PDE at hand, and they are always Banach spaces, and frequently Hilbert spaces. Here, we present a list of the function spaces that will appear later. Their definitions require an exposition to the measure theory, which is not undertaken here. For a rigorous exposition on these spaces we refer the reader to the books [1] or [12].

1. L^p Spaces ($1 \leq p \leq +\infty$)

Let $\Omega \subset \mathbb{R}^n$ be a domain and f be a measurable function defined on Ω . If $1 \leq p \leq +\infty$, we say $f \in L^p(\Omega)$ provided that $\|f\|_{L^p(\Omega)}$ is finite, where for $1 \leq p < +\infty$

$$\|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f^p(x)| dx \right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|f\|_{L^\infty(\Omega)} := \text{ess sup}_{x \in \Omega} |f(x)|.$$

All L^p spaces just defined are reflexive Banach spaces with $(L^p(\Omega))^* = L^q(\Omega)$. Evidently, only $L^2(\Omega)$ is a Hilbert space, and it is equipped with the standard inner product

$$(f, g)_{L^2(\Omega)} = \int_{\Omega} f(x)g(x)dx, \quad f, g \in L^2(\Omega).$$

The L^2 -norm and inner product will appear quite often in the text, and for that reason we reserve the notations $\|\cdot\|$ and $(\cdot, \cdot)$ to stand for these quantities, respectively.

2. Sobolev Spaces

In order to define the Sobolev spaces we first need the concept of weak derivatives. Let $\Omega \subset \mathbb{R}^n$ and $C_0^\infty(\Omega)$ denote the set of all infinitely differentiable functions with compact support. Suppose $u \in C^1(\Omega)$. Then for any test function $\phi \in C_0^\infty(\Omega)$ the integration by parts formula gives

$$\int_{\Omega} u \phi_{x_i} dx = - \int_{\Omega} u_{x_i} \phi dx.$$

Here the boundary term vanishes since ϕ has compact support and therefore vanishes near $\partial\Omega$. More generally, for any positive integer k , if $u \in C^k(\Omega)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$

is a multiindex of order $|\alpha| = \alpha_1 + \dots + \alpha_n = k$, then

$$\int_{\Omega} u D^{\alpha} \phi dx = (-1)^{|\alpha|} \int_{\Omega} D^{\alpha} u \phi dx,$$

where

$$D^{\alpha} \phi = \frac{\partial^{|\alpha|} \phi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

This equality follows from the divergence theorem; it is just the application of the first one $|\alpha| = k$ times. This observation provides the motivation for the definition of weak derivatives. If $u, v \in L_{loc}^1(\Omega)$, we say that v is the α^{th} -weak partial derivative of u , written $D^{\alpha} u = v$, provided that

$$\int_{\Omega} u D^{\alpha} \phi dx = (-1)^{|\alpha|} \int_{\Omega} v \phi dx$$

holds for all test functions $\phi \in C_c^{\infty}(\Omega)$. The weak derivative is unique in the sense of almost everywhere equivalence and coincides with the normal derivative for differentiable functions. Now we give the definition of the Sobolev spaces. Let k be a positive integer and $1 \leq p \leq \infty$. Then the Sobolev space, $W^{k,p}(\Omega)$, is the set of all locally summable functions, for which the α^{th} -weak partial derivative exists and belongs to $L^p(\Omega)$ for all multiindices α of order $|\alpha| \leq k$. If $u \in W^{k,p}(\Omega)$, we define its norm as

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^{\alpha} u|^p dx \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \text{ess sup}_{x \in \Omega} |D^{\alpha} u| & \text{if } p = \infty. \end{cases}$$

With respect to this norm all Sobolev spaces are Banach spaces with the dual $(W^{k,p}(\Omega))^* = W^{-k,q}(\Omega)$, and for $p = 2$ they are Hilbert spaces. The space $W^{-k,q}(\Omega)$ is not a function space but a distribution space, that is its elements are not functions but rather distributions, ‘continuous’ linear functionals on the Frechét space $C_0(\Omega)$, or in other words elements of the so-called ‘space of distributions’, $\mathcal{D}(\Omega) := (C_0(\Omega))^*$. We will not give their precise definition here, see the celebrated book by Laurent Schwartz, [41], for a rigorous exposition of the topic. Finally, we denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p}(\Omega)$, which has the same dual space.

1.2.2 The fractional Laplacian and the fractional Sobolev spaces

It is well-known that the Laplacian operator

$$\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

is a continuous operator from $H^2(\Omega)$ onto $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ we take to be a bounded domain. However, it is clearly not well-defined, as it sends all constants functions to the zero function. Imposing the Dirichlet boundary conditions to the functions in the domain, the new operator, which we now by Δ_0 , is now well-defined, and also injective, from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$. The injectiveness and surjectiveness of Δ_0 is equivalent to the statement that, for each $f \in L^2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\Delta u = f.$$

This equation is called the Poisson's equation, and the above mentioned result is a classical result of elliptic PDE theory, see [28] for instance. It is also well-known that considered as an operator in $L^2(\Omega)$, $-\Delta_0$ is an unbounded positive self-adjoint operator with compact inverse, and as a consequence by the celebrated Hilbert-Schmidt theorem, see [37], it has a discrete positive spectrum that increases to infinity. In other words, the eigenvalues $\{\lambda_j\}_{j \geq 1}$ of $-\Delta$ satisfy $0 < \lambda_1 \leq \lambda_j$ for all $j \geq 1$, and $\lambda_j \rightarrow +\infty$. The corresponding eigenfunctions, that are orthonormal in $L^2(\Omega)$, will be denoted by $\{e_j\}_{j \geq 1}$, and they are smooth provided that the boundary of the domain, $\partial\Omega$ is smooth. The span of these eigenfunctions is dense in all of the Sobolev spaces $W_0^{k,p}(\Omega)$, and they form a basis for $H_0^k(\Omega)$ for each $k \geq 0$. Moreover, by the Parseval's identity and the Poincaré inequality, see §1.2.4, the norm defined by

$$\|u\|_k^2 := \sum_{j \geq 1} \lambda_j^k (u, e_j)^2$$

is equivalent to the standard norm in $H_0^k(\Omega)$ as defined in §1.2.1. Moreover, the above statement can be extended to hold for all $k \in \mathbb{Z}$ by duality, that is the same definition defines a norm on $H^{-k}(\Omega)$, which is equivalent to the standard norm, that is the operator norm derived from the standard norm of $H^k(\Omega)$.

As can be done for any positive operator, the fractional powers of the Laplacian operator can be defined in the following way. For any $s \in \mathbb{R}$ let $(-\Delta)^s$ denote the unique linear operator, that has the same eigenfunctions as the Laplacian, but has the eigenvalues $\{\lambda_j^s\}_{j \geq 1}$. It is easy to see that, this definition is consistent with the standard definition for integer values of s . As a consequence we define the fractional Sobolev spaces, $H_0^s(\Omega)$ when $s \geq \frac{1}{2}$, and $H^s(\Omega)$ when $s < \frac{1}{2}$, as those functions or distributions who have the norm

$$\|u\|_s^2 := \sum_{j \geq 1} \lambda_j^s (u, e_j)^2$$

finite. The change in notation at the treshold value is due to the trace theorem, and is of small significance for us. In fact, to avoid cumbersome notation each time, we will knowingly make the mistake of talking about the space $H_0^s(\Omega)$ for $s < \frac{1}{2}$. It is also convenient to note

that, all the operators $(-\Delta)^s$, $s \in \mathbb{R}$ are self-adjoint, and we have the generalized integration by parts formula

$$((-\Delta)^s u, v) = (u, (-\Delta)^s v)$$

for all u, v in the appropriate spaces.

A final remark is that, all of the previous discussion has a similar version for the case of the Neumann boundary conditions, that is $\nabla u \cdot \vec{n}|_{\partial\Omega} = 0$, where $\vec{n}$ is the outward normal unit vector, as well.

1.2.3 Some useful inequalities

Inequalities are the fundamental tools used in the analysis. Here, we list the inequalities, which we will frequently use in the main discussion without any reference. The proofs of all these inequalities are very basic, and are omitted.

1. *Young- ε inequality*

Let $1 < p, q < \infty$ be given numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $a, b \geq 0$ and any $\varepsilon > 0$ we have the inequality

$$ab \leq \varepsilon \frac{a^p}{p} + \frac{b^q}{q\varepsilon^{\frac{q}{p}}}.$$

2. *Hölder's inequality*

Let $\Omega \subset \mathbb{R}^n$, and suppose that $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, where $1 \leq p, q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, $fg \in L^1(\Omega)$, and we have the inequality

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

3. *Cauchy-Schwarz inequality*

Let $\mathcal{H}$ be a Hilbert space. Then, for any $x, y \in \mathcal{H}$ we have

$$|(x, y)_{\mathcal{H}}| \leq \|x\|_{\mathcal{H}} \|y\|_{\mathcal{H}}.$$

4. *L^p -embedding theorem*

Suppose that Ω is a bounded domain in $\mathbb{R}^n$, and $1 \leq p \leq q \leq \infty$ are given numbers. Then, $L^q(\Omega) \subset L^p(\Omega)$, and we have the embedding inequality

$$\|u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q(\Omega)}, \quad \forall u \in L^q(\Omega).$$

5. *L^p -interpolation inequality*

Let $\Omega \subset \mathbb{R}^n$ be any domain, and $1 \leq p \leq q \leq r \leq \infty$ be given numbers. Then, $L^p(\Omega) \cap L^r(\Omega) \subset L^q(\Omega)$, and the inequality

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^p(\Omega)}^\theta \|u\|_{L^r(\Omega)}^{1-\theta},$$

holds for all $u \in L^q(\Omega)$, where $\theta = \frac{r-q}{r-p} \in [0, 1]$.

6. *Poincaré inequality (version 1)*

Let Ω be a bounded domain in $\mathbb{R}^n$, and $r \leq s \in \mathbb{R}$ be a given numbers. Then, $H_0^s(\Omega)$ is continuously embedded into $H_0^2(\Omega)$, and we have the embedding inequality

$$\|u\|_r \leq \lambda_1^{r-s} \|u\|_s, \quad \text{for all } u \in \mathcal{H}_0^s(\Omega),$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian operator Δ_0 . Moreover, if $r < s$, then this embedding is compact.

7. *Poincaré inequality (version 2)*

Let Ω be a bounded domain in $\mathbb{R}^n$, and $p \geq 1$ be a given number. Assume that $u \in W_0^{1,p}$. Then, there exists C depending only on Ω and p such that

$$\int_{\Omega} |u|^p dx \leq C \int_{\Omega} |\nabla u|^p dx.$$

8. *Sobolev embedding theorem*

Suppose that $1 \leq p < n$, $p^* = \frac{np}{n-p}$, and $u \in W^{1,p}(\mathbb{R}^n)$. Then, $u \in L^{p^*}(\mathbb{R}^n)$, and there exists $C \geq 0$ such that

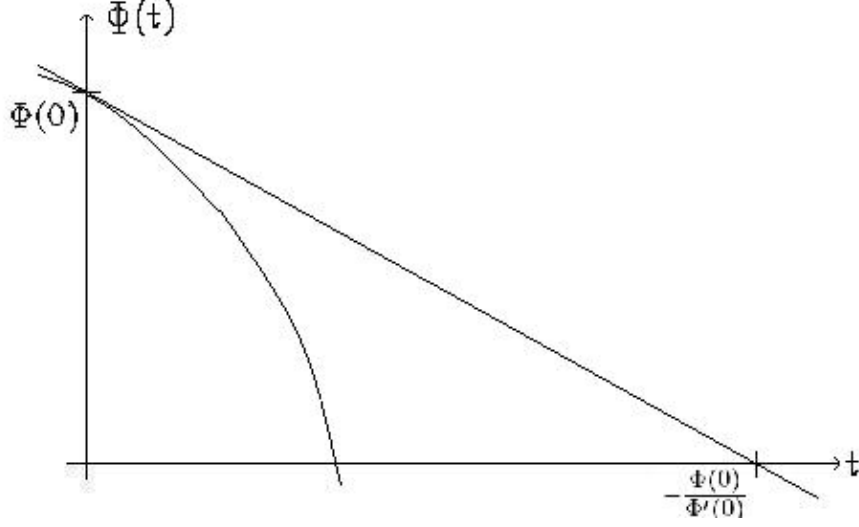
$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}.$$

On the other hand, suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain. If $1 \leq p < n$, then for all $1 \leq q \leq p^*$; or if $p = n$, then for all $1 \leq q < \infty$; or if $p > n$, then for all $1 \leq q \leq \infty$ we have $W^{k,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$, and we have the embedding inequality

$$\|u\|_{L^q} \leq C \|\nabla u\|_{L^p}.$$

1.2.4 **Concavity methods**

Suppose $\Psi(t)$ is a twice differentiable, positive function of time and we want to show that it blows up in finite time. If we define a new function $\Phi(t) := \Psi^{-\alpha}(t)$ with $\alpha > 0$, then it suffices to show that $\Phi(t)$ becomes zero in finite time. This is done by showing that

FIGURE 1.2: $\Phi(t)$ crosses zero.

$\Phi(t)$ satisfies a certain differential inequality with certain initial conditions. In both of the concavity methods described below this is the main idea.

Concavity method:

The first idea is due to H. A. Levine [29], [30] and it is illustrated in Figure 1.2. What we require is that $\Phi(0) > 0$, $\Phi'(0) < 0$ and $\Phi''(t) \leq 0$ when $t > 0$. If $\Phi(t)$ satisfies these conditions, then its curve will lie under the line with slope equal to $\Phi'(0)$ and hence $\Phi(t)$ will be zero at a time $t \leq -\frac{\Phi(0)}{\Phi'(0)}$. Now, since $\Phi(t) := \Psi^{-\alpha}(t)$, we have

$$\Phi'(t) = -\alpha \Psi^{-(1+\alpha)}(t) \Psi'(t),$$

and

$$\begin{aligned} \Phi''(t) &= \alpha(1+\alpha) \Psi^{-(2+\alpha)}(t) (\Psi'(t))^2 - \alpha \Psi^{-(1+\alpha)}(t) \Psi''(t) \\ &= \alpha \Psi^{-(2+\alpha)}(t) \left[(1+\alpha) (\Psi'(t))^2 - \Psi(t) \Psi''(t) \right]. \end{aligned}$$

Therefore, the conditions stated above in terms of $\Phi(t)$ can be restated in terms of $\Psi(t)$ as $\Psi(0) > 0$, $\Psi'(0) > 0$, and $\Psi(t) \Psi''(t) - (1+\alpha) (\Psi'(t))^2 \geq 0$. Also, the blow-up occurs in a time $t \leq \frac{\Psi(0)}{\alpha \Psi'(0)}$. Thus, we have the following lemma.

Lemma 1.1. *Let $\Psi(t)$ be a twice differentiable, positive function, which satisfies, for $t > 0$, the inequality*

$$\Psi(t) \Psi''(t) - (1+\alpha) (\Psi'(t))^2 \geq 0$$

with some $\alpha > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then there exists a time $t_1 \leq \frac{\Psi(0)}{\alpha \Psi'(0)}$ such that $\Psi(t) \rightarrow +\infty$ as $t \rightarrow t_1$.

Generalized concavity method:

The second concavity method is a generalization of the first one. This is achieved by generalizing the differential inequality that $\Phi(t)$ satisfies, that is one works with a differential inequality of the form

$$\Phi''(t) + C_1\Phi'(t) + C_2\Phi(t) \leq 0,$$

and investigates under which initial conditions this inequality ensures that $\Phi(t)$ becomes zero in finite time. Then, by a similar passage, as done above, one finds the equivalent differential inequality and initial conditions on $\Psi(t)$. The resulting conclusion was first discovered by V. K. Kalantarov and O. A. Ladyzhenskaya [20] and we present it with the following lemma.

Lemma 1.2. *Assume that a twice differentiable, positive function $\Psi(t)$ satisfies for all $t \geq 0$ the inequality*

$$\Psi(t)\Psi''(t) - (1 + \alpha)(\Psi')^2(t) \geq -2C_1\Psi(t)\Psi'(t) - C_2\Psi^2(t) \quad (1.2.1)$$

where $\alpha > 0$ and $C_1, C_2 \geq 0$. Then,

1. if $\Psi(0) > 0$, $\Psi'(0) + \alpha_2\alpha^{-1}\Psi(0) > 0$, and $C_1 + C_2 > 0$, we have $\Psi(t) \rightarrow +\infty$ as

$$t \rightarrow t_1 \geq t_2 = \frac{1}{2\sqrt{C_1^2 + \alpha C_2}} \ln \left(\frac{\alpha_1\Psi(0) + \alpha\Psi'(0)}{\alpha_2\Psi(0) + \alpha\Psi'(0)} \right),$$

$$\text{where } \alpha_1 = -C_1 + \sqrt{C_1^2 + \alpha C_2}, \alpha_2 = -C_1 - \sqrt{C_1^2 + \alpha C_2},$$

2. if $\Psi(0) > 0$, $\Psi'(0) > 0$, and $C_1 = C_2 = 0$, then $\Psi(t) \rightarrow +\infty$ as

$$t \rightarrow t_1 \geq t_2 = \frac{\Psi(0)}{\alpha\Psi'(0)}.$$

Proof. Set $\Phi(t) := \Psi^{-\alpha}(t)$. Then we have

$$\Phi'(t) = -\alpha \frac{\Psi(t)}{\Psi'(t)}$$

and

$$\Phi''(t) = -\frac{\alpha}{\Psi^{2+\alpha}(t)} \left[\Psi(t)\Psi''(t) - (1 + \alpha)(\Psi')^2(t) \right].$$

Using the inequality (2.3) we obtain the differential inequality

$$\begin{aligned}\Phi''(t) &\geq -\frac{\alpha}{\Psi^{2+\alpha}(t)} [-2C_1\Psi(t)\Psi'(t) - C_2\Psi^2(t)] \\ &= -2C_1\Phi'(t) + \alpha C_2\Phi(t).\end{aligned}$$

Hence we have for all $t \geq 0$

$$\Phi''(t) + 2C_1\Phi'(t) - \alpha C_2\Phi(t) = f(t) \geq 0. \quad (1.2.2)$$

For the first case, where $C_1 + C_2 > 0$, the solution of (1.2.2) is

$$\begin{aligned}\Phi(t) &= \beta_1 e^{\alpha_1 t} + \beta_2 e^{\alpha_2 t} + (\alpha_1 + \alpha_2)^{-1} \int_0^t f(\tau) [e^{\alpha_1(t-\tau)} - e^{\alpha_2(t-\tau)}] d\tau \\ &\geq \beta_1 e^{\alpha_1 t} + \beta_2 e^{\alpha_2 t} \quad \forall t \geq 0,\end{aligned} \quad (1.2.3)$$

where β_1 and β_2 are defined by

$$\begin{cases} \beta_1 + \beta_2 = \Phi(0) \\ \beta_1 \alpha_1 + \beta_2 \alpha_2 = \Phi'(0). \end{cases}$$

From this we obtain

$$\begin{aligned}\beta_1 &= -(\alpha_1 - \alpha_2)^{-1} [\alpha_2 \Psi(0) + \alpha \Psi'(0)] \Psi^{-1-\alpha} \\ \beta_2 &= (\alpha_1 - \alpha_2)^{-1} [\alpha_1 \Psi(0) + \alpha \Psi'(0)] \Psi^{-1-\alpha}\end{aligned}$$

Note that $\alpha_1 > 0$ and $\alpha_2 < 0$, in particular $\alpha_1 > \alpha_2$. In view of this the assumption, $\Psi'(0) + \alpha_2 \alpha^{-1} \Psi(0) > 0$, implies that $\beta_1 < 0$ and $\beta_2 > 0$. Combining these with (1.2.3) we conclude that $\Phi(t)$ vanishes at the time

$$t_2 = \frac{1}{2\sqrt{C_1^2 + \alpha C_2}} \ln \left(\frac{\alpha_1 \Psi(0) + \alpha \Psi'(0)}{\alpha_2 \Psi(0) + \alpha \Psi'(0)} \right)$$

and becomes negative afterwards. This implies that $\Phi(t) \rightarrow +\infty$ as $t \rightarrow t_1 \leq t_2$.

For the second case, $C_1 = C_2 = 0$ and equation (1.2.2) imply

$$\Phi''(t) \leq 0.$$

Integrating both sides over $(0, t)$ twice we have

$$\Phi(t) \leq \Phi(0) + \Phi'(0)t$$

and by using the definition of $\Phi(t)$ we obtain

$$\Psi^{-\alpha}(t) \leq \Psi^{-\alpha}(0) - \alpha \frac{\Psi'(0)}{\Psi^{1+\alpha}(0)}$$

and since $\Psi(t) > 0$ for all $t \geq 0$ we have

$$\Psi^\alpha(t) \geq \Psi^\alpha(0) [1 - \alpha t \Psi^{-1}(0) \Psi'(0)]^{-1}.$$

Since $\Psi(0) > 0$ and $\Psi'(0) > 0$ we deduce

$$\Psi(t) \rightarrow +\infty \text{ as } t \rightarrow t_1 \leq t_2 = \frac{\Psi(0)}{\alpha \Psi'(0)}.$$

□

Chapter 2

Unstable Equations

The aim of this chapter is to investigate into the behaviour of solutions to damped hyperbolic PDE's, which exhibit the phenomenon of finite time blow-up. The literature is rich with such results, and here we give a chronological review of the progress. The first results were, of course, for the undamped wave equation, and the most important of these include the papers [29], [30] due to H. A. Levine, in which he works in an abstract setting, and first time introduces the celebrated concavity method to give the characterization that all solutions with negative initial energy blow up in finite time. A little later, in [42], the author slightly improves these results and for the first time considers the damped wave equations. In the same year, in [35], the authors introduce the concept of the potential well, and obtain a characterization for the initial data with initial energy less than or equal to the depth of the potential well. Then, the authors of [20] present a generalization of the concavity method, which they apply to obtain improved sufficient conditions on the initial data with blow-up. After about a decade later, in 1985, T. Cazenave published [7] on the solutions to the initial value problem for the Klein-Gordon equation, where on top of the global nonexistence results he proves that all global solutions must remain uniformly (in time) bounded in the energy phase space. The interest in the damped hyperbolic equations rises yet another decade later. Some of the works in that direction include the papers [17], [34], [36], [14], [24], [25], [26], [46], [9] for the case of linear damping, and the papers [15], [18], [31], [45], [43], [32], [44], when the damping is nonlinear. Here, [31], [36], [45] are presented in an abstract setting, and contain many equations, while [24], [26] deal with the Klein-Gordon equation and a system of wave equations, respectively, [34], [25] deal with Kirchhoff type equations, and all of the rest investigate the wave equation. It is worth noting that, all the papers mentioned so far deal with semilinear equations, except [31] and [45], in which the authors work in an abstract setting, that contains quasilinear and fully nonlinear equations as well. Out of these, the only results that can handle initial data

with arbitrarily positive initial energies are [32],[44], [14], [24], [25], [26]. Furthermore, [14] also establishes the uniform boundedness of global solutions for the linearly damped wave equations.

Another important aspect of the problem we haven't mentioned is the nature of the domain. Most of the results mentioned above study the Dirichlet initial-boundary value problem in a bounded domain, some for the Cauchy initial value problem in the whole space, and those studying the abstract problem have applications to both problems, depending on the problem. The Cauchy problem is a much harder one, as one no more has neither the L^p -embedding inequality, nor the Poincaré inequality. A curious fact is that, although in the case of nonlinear damping the papers [43], [32] and [44] manage to handle the Cauchy problem, none of the aforementioned results, except those that are formulated in an abstract setting, can do, or does so.

So far, in all of the problems considered in the articles mentioned above, both the nonlinear external potential, which is the cause of finite time blow-up, and the damping were applied on the whole of the interior domain. However, one can also consider problems, where the damping and/or the potential are applied through the boundary, or even through a part of the boundary and/or the domain, the so-called 'degenerate damping' and/or 'degenerate potential', see [6], [3] and the references therein.

In this dissertation we will always take the damping to be linear. Our method of proof is based on the concavity and the generalized concavity methods, which do not apply to the case of nonlinear damping. Also, although the methods of our proofs have straightforward adaptations to the problems with boundary damping and potential, we will always work with internal damping and potentials. In the first section we work with an abstract equation. First, in §2.1.1 we derive conditions on the initial data that ensure finite time blow-up of corresponding solutions, and later in §2.1.2 we prove the uniform boundedness of global solutions. In section §2.2 we apply the abstract results to concrete problems, and deduce corresponding theorems for the solutions of equations like the semilinear and quasilinear structurally damped wave equations, damped semilinear Klein-Gordon equations, semilinear beam and plate equations, semilinear damped Boussinesq type equations, and the Kirchhoff type equations. All of our blow-up results for the Dirichlet initial-boundary value problems (as well as Neumann or mixed boundary value problems, see Remark 2.15) and the Cauchy initial value problems for these equations allow the initial energy to be arbitrarily positive. We give the construction of initial data with energy as large as one desires in the examples of §2.2.1.1 and §2.3. Moreover, except for the Boussinesq type equations, we prove the uniform boundedness of global solutions for the solutions of all problems considered. It is important to note that, this result we can not extend to the Cauchy problem

in general. Next, in the following two sections we establish global nonexistence of solutions to the quasilinear damped wave equations and strongly damped Boussinesq type equations with convection. The content of §2.3 is already published in [2], see also [11], and the theorem of §2.4 has a very similar idea of proof, and is also new. Finally, for the sake of completeness, in §2.5 we present the potential well theory, which was first developed in [35], in the special case of the initial-boundary value problem for the strongly damped wave equation. This theory gives a characterization on the behaviour of the solutions for small energies, and provides examples of nontrivial initial data, which yield to global solutions.

2.1 Abstract problem

We study the Cauchy problem for a second order differential-operator equation of the form

$$\begin{cases} Pu_{tt} + Qu_t + A(u) = F(u), \\ u(0) = u_0, u_t(0) = u_1, \end{cases} \quad (2.1.1)$$

in a separable Hilbert space $\mathcal{H}$ with the inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$. Suppose that P, Q are self-adjoint, positive definite linear operators defined in $\mathcal{H}$, and with domains $D(P), D(Q)$, respectively, both dense in $\mathcal{H}$. By the Riesz representation theorem we identify $\mathcal{H}$ with its dual, and all the duals hereafter will be taken with respect to $\mathcal{H}$. Consequently, for $X = P, Q$, the operator $X^{\frac{1}{2}} : D(X^{\frac{1}{2}}) \rightarrow \mathcal{H}$ is well-defined, $D(X) \subset D(X^{\frac{1}{2}}) \subset \mathcal{H}$, and $D(X), D(X^{\frac{1}{2}})$ are themselves Hilbert spaces with the inner products

$$(u, v)_{D(X)} = (Xu, Xv), \quad \text{and} \quad (\hat{u}, \hat{v})_{D(X^{\frac{1}{2}})} = (X^{\frac{1}{2}}\hat{u}, X^{\frac{1}{2}}\hat{v}),$$

for any $u, v \in D(X)$ and $\hat{u}, \hat{v} \in D(X^{\frac{1}{2}})$, respectively. Moreover, one has

$$\langle Xu, v \rangle_{D(X^{\frac{1}{2}})^*, D(X^{\frac{1}{2}})} = (X^{\frac{1}{2}}u, X^{\frac{1}{2}}v), \quad \forall u, v \in D(X^{\frac{1}{2}}), \quad (2.1.2)$$

where $\langle \cdot, \cdot \rangle_{D(X^{\frac{1}{2}})^*, D(X^{\frac{1}{2}})}$ stands for the dual pairing between $D(X^{\frac{1}{2}})$ and its dual $D(X^{\frac{1}{2}})^*$.

We suppose that $A : D(A) \rightarrow D(A)^*$ is a continuous (possibly) nonlinear operator, which possibly depends on time, and with $D(A) \subset \mathcal{H}$ dense. We assume that there exists a potential $\mathcal{A} \in C^1(D(A) \rightarrow \mathbb{R}^+)$ such that

$$\frac{d}{dt}\mathcal{A}(u(t)) = \langle A(u(t)), u_t(t) \rangle_{D(A)^*, D(A)}, \quad (2.1.3)$$

for almost every $t \geq 0$, and $u(t) \in C(0, T; D(A)) \cap C^1(0, T; D(A)^*)$. Moreover, we impose the following structural condition on A : There exists numbers $\beta, \geq 0$ and $L_1, R_1 \in \mathbb{R}$ such

that

$$(A(v), v) \leq 2(1 + 2\beta)\mathcal{A}(v) + R_1, \quad \forall v \in D(A), \quad (2.1.4)$$

and

$$-L_1 \leq \mathcal{A}(v), \quad \forall v \in D(A). \quad (2.1.5)$$

We take $F : D(F) \rightarrow D(F)^*$ to be a continuous nonlinear gradient operator with the potential functional $\mathcal{F} \in C^1(D(F); \mathbb{R})$, where $D(F)$ is a Banach space with $D(A) \subset D(F) \subset \mathcal{H}$, that is to say we have

$$\frac{d}{dt}\mathcal{F}(u(t)) = \langle F(u(t)), u_t(t) \rangle_{D(F)^*, D(F)}, \quad (2.1.6)$$

for almost every $t \geq 0$, and $u(t) \in C(0, T; D(F)) \cap C^1(0, T; D(F)^*)$. Let us here note that the assumption $D(A) \subset D(F)$ is natural, and is in fact necessary even to show the local existence of solutions to problems that we consider in this section. The finite time blow-up phenomenon will be totally due to the nonlinearity F , and it is the following structural growth assumption that will provide this property to F . Namely, we will suppose that there exists some $\alpha > \beta \geq 0$ and $R_0 \in \mathbb{R}$ such that F verifies

$$(F(v), v) \geq 2(1 + 2\alpha)\mathcal{F}(v) - R_0, \quad \forall v \in D(F). \quad (2.1.7)$$

Now, let us make clear what we mean by a solution to the abstract equation (2.1.1). Given $(u_0, u_1) \in D(A) \times D(P^{\frac{1}{2}})$ and $T > 0$, we define a local solution to (2.1.1) to be a function

$$u \in \mathcal{E}_T := C(0, T; D(A)) \cap C^1(0, T; D(P^{\frac{1}{2}})) \cap H^1(0, T; D(Q^{\frac{1}{2}})),$$

that verifies the initial data, and satisfies, firstly, the distribution relation

$$\begin{aligned} \langle Pu_t(\tau), \phi(\tau) \rangle \Big|_{\tau=0}^{\tau=t} = \\ \int_0^t \{ \langle Pu_t(\tau), \phi_t(\tau) \rangle - \langle A(u(\tau)), \phi(\tau) \rangle - \langle Qu_t(\tau), \phi(\tau) \rangle + \langle F(u(\tau)), \phi(\tau) \rangle \} d\tau, \end{aligned} \quad (2.1.8)$$

for all $\phi \in \mathcal{E}_T$, and all $t \in [0, T]$. The other important relation one expects u to fulfill for it to be a local solution is the so-called energy identity

$$E(t) = E(0) - \int_0^t \|Q^{\frac{1}{2}}u_t(\tau)\|^2 d\tau, \quad \forall t \in [0, T],$$

where the energy is defined as

$$E(t) = E(u(t), u_t(t)) := \frac{1}{2} \|P^{\frac{1}{2}} u_t(t)\|^2 + \mathcal{A}(u(t)) - \mathcal{F}(u(t)). \quad (2.1.9)$$

Utilizing the conditions (2.1.3) and (2.1.3) this identity is formally derived by multiplying the equation (2.1.1) by u_t and integrating. However, for our arguments we will only need the weaker energy inequality

$$E(t) \leq E(0) - \int_0^t \|Q^{\frac{1}{2}} u_t(\tau)\|^2 d\tau, \quad \forall t \in [0, T], \quad (2.1.10)$$

and thus we take the fulfillment of this inequality as the final ingredient in the definition of a local solution to (2.1.1). Of course, to be able to proceed further we suppose that such a local solution exists for each initial data.

It is easy to see that, for $u \in \mathcal{E}_T$, all the terms in (2.1.6), (2.1.8), (2.1.9) and (2.1.10) make sense with the assumptions given above on the domains of the aforementioned operators. Hence, for sake of easier writing now on we use $(\cdot, \cdot)$ for pairings between dual spaces as well.

At last, we may start with our treatment of the problem (2.1.1). To this end let us define for $t \geq 0$ the following nonnegative functional

$$\Psi(t) = \|P^{\frac{1}{2}} u(t)\|^2 + \int_0^t \|Q^{\frac{1}{2}} u(s)\|^2 ds + C_0 \quad (2.1.11)$$

where $C_0 \geq 0$ is to be chosen (many times) later in the sequel. Consequently, we have

$$\begin{aligned} \Psi'(t) &= 2(Pu_t(t), u(t)) + \|Q^{\frac{1}{2}} u(t)\|^2 \\ &= 2(Pu_t(t), u(t)) + 2 \int_0^t (Qu_s(s), u(s)) ds + \|Q^{\frac{1}{2}} u_0\|^2 \end{aligned} \quad (2.1.12)$$

and

$$\Psi''(t) = 2(Pu_{tt}(t) + Qu_t(t), u(t)) + 2\|P^{\frac{1}{2}} u_t(t)\|^2. \quad (2.1.13)$$

Observe that the distribution identity (2.1.8) is absolutely continuous with respect to t . Hence, we may differentiate it, and in doing so in view of (2.1.13) we obtain

$$\Psi''(t) = -2(A(u(t)), u(t)) + 2(F(u(t)), u(t)) + 2\|P^{\frac{1}{2}} u_t(t)\|^2. \quad (2.1.14)$$

Utilizing the structural assumptions (2.1.4) and (2.1.7) in this equality we get

$$\begin{aligned}\Psi''(t) &\geq -4(1+2\beta)\mathcal{A}(u(t)) + 2\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(1+2\alpha)\mathcal{F}(u(t)) - 2(R_0 + R_1) \\ &= 4(1+2\alpha)\left[-\mathcal{A}(u(t)) - \frac{1}{2}\|P^{\frac{1}{2}}u_t(t)\|^2 + \mathcal{F}(u(t))\right] \\ &\quad + 4(1+\alpha)\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha-\beta)\mathcal{A}(u(t)) - 2(R_0 + R_1),\end{aligned}$$

which, in view of the definition of the energy (2.1.9), we may rewrite as

$$\Psi''(t) \geq -4(1+2\alpha)E(t) + 4(1+\alpha)\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha-\beta)\mathcal{A}(u(t)) - 2(R_0 + R_1).$$

Finally, thanks to the energy inequality (2.1.10) we deduce from this inequality the following one:

$$\Psi''(t) \geq 4(1+2\alpha) \int_0^t \|Q^{\frac{1}{2}}u_s(s)\|^2 ds + 4(1+\alpha)\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha-\beta)\mathcal{A}(u(t)) - B_0, \quad (2.1.15)$$

where

$$B_0 := 4(1+2\alpha)E(0) + 2(R_0 + R_1). \quad (2.1.16)$$

The inequality (2.1.15) is the main inequality from which we will derive most of our results.

2.1.1 Blowing up solutions

In this subsection we derive sufficient conditions on the initial functions that ensure the finite time blow-up of the corresponding solutions to the abstract equation (2.1.1). Although all theorems presented here eventually make use of the concavity method, the proofs contain different approaches, and all results have their own virtue. In the next section we compare these results on the concrete example of a weakly damped wave equation in 1-dimension. The first approach is a direct one, and is encapsulated in the following theorem. Although the conditions on the initial data obtained in this theorem will be improved upon by the other approaches, it is still meaningful to present the theorem, as it completely characterizes the case when the initial data satisfy the condition $B_0 + 4(\alpha - \beta)L_1 < 0$, and requires no hierarchy between the operators P , A and Q .

Theorem 2.1. *Suppose that the operators P, Q, A and F are as described above and let $u(t)$ be a local solution to the problem (2.1.1) in the sense above. In addition assume F satisfies the growth condition (2.1.7). If the initial elements satisfy:*

$$B_0 + 4(\alpha - \beta)L_1 < 0,$$

or

$$B_0 + 4(\alpha - \beta)L_1 \geq 0, \quad \text{and}$$

$$2(P^{\frac{1}{2}}u_0, P^{\frac{1}{2}}u_1) + \|Q^{\frac{1}{2}}u_0\|^2 > \sqrt{\frac{1}{\alpha}} \left(\|P^{\frac{1}{2}}u_0\|^2 + B_0 + 4(\alpha - \beta)L_1 + \sqrt{1 + \alpha}\|Q^{\frac{1}{2}}u_0\|^2 \right), \quad (2.1.17)$$

where B_0 is given by (2.4.31), and L_1 is as in the assumption (2.1.5). Then,

$$\lim_{t \rightarrow t_1^-} \left(\|P^{\frac{1}{2}}u(t)\|^2 + \int_0^t \|Q^{\frac{1}{2}}u(s)\|^2 ds \right) = \infty,$$

for some $t_1 \geq 0$. In the latter case we have the bound

$$t_1 \leq \frac{1}{2\sqrt{\alpha}} \ln \frac{2(P^{\frac{1}{2}}u_0, P^{\frac{1}{2}}u_1) + \|Q^{\frac{1}{2}}u_0\|^2 + \sqrt{\frac{1}{\alpha}} \left(\|P^{\frac{1}{2}}u_0\|^2 + B_0 + 4(\alpha - \beta)L_1 + \sqrt{1 + \alpha}\|Q^{\frac{1}{2}}u_0\|^2 \right)}{2(P^{\frac{1}{2}}u_0, P^{\frac{1}{2}}u_1) + \|Q^{\frac{1}{2}}u_0\|^2 - \sqrt{\frac{1}{\alpha}} \left(\|P^{\frac{1}{2}}u_0\|^2 + B_0 + 4(\alpha - \beta)L_1 + \sqrt{1 + \alpha}\|Q^{\frac{1}{2}}u_0\|^2 \right)}.$$

Proof. First, suppose that $B_0 + 4(\alpha - \beta)L_1 \geq 0$. Thanks to the assumption (2.1.5) on $\mathcal{A}$, from the inequality (2.1.15) we easily deduce the following inequality

$$\Psi''(t) \geq 4(1 + \alpha) \left[\int_0^t \|Q^{\frac{1}{2}}u_s(s)\|^2 ds + \|P^{\frac{1}{2}}u_t(t)\|^2 \right] - B_0 - 4(\alpha - \beta)L_1.$$

Multiplying this inequality by $\Psi(t)$, due to the identity (2.1.12) satisfied by $\Psi'(t)$, and the fact that $\mathcal{A} \geq 0$, we obtain

$$\begin{aligned} & \Psi''(t)\Psi(t) - (1 + \alpha) [\Psi'(t)]^2 \geq \\ & 4(1 + \alpha) \left[\|P^{\frac{1}{2}}u_t(t)\|^2 + \int_0^t \|Q^{\frac{1}{2}}u_s(s)\|^2 ds + \frac{\|Q^{\frac{1}{2}}u_0\|^4}{4C_0} \right] \left[(Pu, u) + \int_0^t \|Q^{\frac{1}{2}}u(s)\|^2 ds + C_0 \right] - \\ & 4(1 + \alpha) \left[(Pu_t, u) + \int_0^t (Qu_s, u) ds + \frac{1}{2}\|Q^{\frac{1}{2}}u_0\|^2 \right]^2 - \left[\frac{1 + \alpha}{C_0} \|Q^{\frac{1}{2}}u_0\|^4 + B_0 + 4(\alpha - \beta)L_1 \right] \Psi(t) \end{aligned}$$

Successive applications of the Cauchy-Schwarz inequality show that the difference composed of the first two terms on the right hand side of the inequality above is nonnegative. Consequently, the following inequality is valid for all $t \geq 0$:

$$\Psi''(t)\Psi(t) - (1 + \alpha) [\Psi'(t)]^2 \geq - \left[\frac{1 + \alpha}{C_0} \|Q^{\frac{1}{2}}u_0\|^4 + B_0 + 4(\alpha - \beta)L_1 \right] \Psi(t). \quad (2.1.18)$$

Note that

$$\frac{1 + \alpha}{C_0} \|Q^{\frac{1}{2}}u_0\|^4 + B_0 + 4(\alpha - \beta)L_1 \leq C_0 \leq \Psi(t),$$

when

$$C_0 \geq \frac{1}{2} \left((B_0 + 4(\alpha - \beta)L_1) + \sqrt{(B_0 + 4(\alpha - \beta)L_1)^2 + 4(1 + \alpha)\|Q^{\frac{1}{2}}u_0\|^4} \right) \geq 0.$$

Let us choose

$$C_0 = B_0 + 4(\alpha - \beta)L_1 + \sqrt{1 + \alpha}\|Q^{\frac{1}{2}}u_0\|^2.$$

It is easy to see by the triangular inequality in $\mathbb{R}^2$ that C_0 verifies the inequality above. Hence, we see that the inequality (2.1.18) yields

$$\Psi''(t)\Psi(t) - (1 + \alpha)[\Psi'(t)]^2 \geq -\Psi^2(t).$$

So, we have proved that for all $t \geq 0$ the function $\Psi(t)$ satisfies the main inequality (1.2.1) of Lemma 1.2 with numbers $C_1 = 0$ and $C_2 = 1$. The assumption (2.1.17) of the theorem on the initial data translates to $\Psi'(0) > \frac{1}{\sqrt{\alpha}}\Psi(0)$, which is one of the necessary conditions of the lemma with the aforementioned numbers. The other condition, namely $\Psi(0) > 0$, also follows from the same assumption, as this assumption forces that $u_0 \neq 0$. Hence, all the conditions of the Lemma 1.2 are fulfilled by $\Psi(t)$, and the second part of the conclusion of the theorem follows from the statement of this lemma.

Next, suppose $B_0 + 4(\alpha - \beta)L_1 < 0$. Note that the inequality (2.1.18) is valid also in this case. Consequently, if we choose $C_0 = -\frac{(1+\alpha)\|Q^{\frac{1}{2}}u_0\|^4}{B_0+4(\alpha-\beta)L_1} \geq 0$, we may easily deduce from this inequality that

$$\Psi''(t)\Psi(t) - (1 + \alpha)[\Psi'(t)]^2 \geq 0$$

for all $t \geq 0$. Moreover, in view of the assumption (2.1.5) on $\mathcal{A}$, it is easy to see from (2.1.15) that, in this case $\Psi(t)$ is uniformly convex for $t \geq 0$. This implies that $\Psi(t) \rightarrow \infty$ and $\Psi'(t) \rightarrow \infty$ as $t \rightarrow \infty$. Thus, we conclude that after some time $t_0 \geq 0$ all assumptions of Lemma 1.1 will be satisfied, and the first part of the conclusion of the theorem follows. □

Remark 2.2. Because of the first result of Theorem 2.1, throughout the rest of this section we will only consider initial data that obey $B_0 + 4(\alpha - \beta)L_1 \geq 0$.

In the proof of Theorem 2.1, in the case when $B \geq 0$, the finite time blow-up result is directly derived from the inequality (2.1.15) via the use of Lemma 1.2. Thanks to this, from the statement of the lemma we could immediately write an upper bound for the blow-up time of the solutions to the problem (2.1.1), given that the initial data satisfy the condition (2.1.17). It is worth noting that we could find upper bounds for the blow-up time in the case $B < 0$ as well, and this is also true for all blow-up theorems of this subsection, however

such undertakings are not included here, partly because the resulting expressions are too long.

Now, we focus on indirect methods for deriving sufficient conditions on initial data with corresponding solution blowing up. A common feature of these methods is, that we first derive sufficient conditions under which an appropriate quantity depending on the solution becomes unbounded as time increases. Once we achieve this, we then show that these solutions necessarily blow up in finite time. However, such results can only be proven provided that we impose some hierarchy between the operators P , A and Q . Hence, throughout the rest of this section we will interchangeably assume that the following embedding inequalities hold:

$$\|P^{\frac{1}{2}}v\|^2 \leq d_1\mathcal{A}(v) + D_1 \quad \forall v \in D(A), \quad (2.1.19)$$

$$\|Q^{\frac{1}{2}}v\|^2 \leq d_2\mathcal{A}(v) + D_2 \quad \forall v \in D(A), \quad (2.1.20)$$

$$\|P^{\frac{1}{2}}v\|^2 \leq d_3\|Q^{\frac{1}{2}}v\|^2 \quad \forall v \in D(Q^{\frac{1}{2}}), \quad (2.1.21)$$

for some numbers $d_i, D_j \geq 0$. Further assuming (2.1.19), (2.1.20) we may derive the following sufficient conditions on the initial data that ensure finite time blow-up of solutions to (2.1.1).

Theorem 2.3. *Assume that the operators $P, Q, A, F(\cdot)$ satisfy the conditions of Theorem 2.1 together with (2.1.19) and (2.1.20). Let $u(t)$ be a local solution to the problem (2.1.1). Suppose that the initial functions verify*

$$2(P^{\frac{1}{2}}u_0, P^{\frac{1}{2}}u_1) + \|Q^{\frac{1}{2}}u_0\|^2 > \kappa B_0 + \frac{D_1}{4\kappa(1+\alpha)} + D_2, \quad (2.1.22)$$

where

$$\kappa = \frac{8(\alpha - \beta)}{d_2} \left(1 + \sqrt{1 + \frac{4(\alpha - \beta)d_1}{(1 + \alpha)d_2^2}} \right)^{-1}. \quad (2.1.23)$$

Then, there exists $t_1 < \infty$ such that

$$\|P^{\frac{1}{2}}u(t)\|^2 + \int_0^t \|Q^{\frac{1}{2}}u(s)\|^2 ds \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

Proof. First see that, utilizing the Cauchy-Schwarz and Young- ε inequalities in the definition of Ψ' , (2.1.12), and employing the embedding inequalities (2.1.19), (2.1.20) for any $\varepsilon > 0$ we have

$$\begin{aligned} |\Psi'(t)| &\leq \varepsilon \|P^{\frac{1}{2}}u_t(t)\|^2 + \frac{1}{\varepsilon} \|P^{\frac{1}{2}}u(t)\|^2 + \|Q^{\frac{1}{2}}u(t)\|^2 \\ &\leq \varepsilon \|P^{\frac{1}{2}}u_t(t)\|^2 + \left(\frac{d_1}{\varepsilon} + d_2\right)\mathcal{A}(u(t)) + \frac{D_1}{\varepsilon} + D_2. \end{aligned} \quad (2.1.24)$$

In view of the inequality (2.1.15), since $\mathcal{A}$ is not necessarily positive, we would like to choose $\varepsilon > 0$ such that we have

$$\kappa_\varepsilon^{-1} := \frac{4(1+\alpha)}{\varepsilon} = \frac{4(\alpha-\beta)}{\frac{d_1}{\varepsilon} + d_2}. \quad (2.1.25)$$

In other words, we want to choose ε as the positive root of the following second order polynomial

$$p(r) := r^2 - \frac{(1+\alpha)d_2}{\alpha-\beta}r - \frac{(1+\alpha)d_1}{\alpha-\beta},$$

which is possible, and the desired value is

$$\varepsilon_0 = \frac{(1+\alpha)d_2}{2(\alpha-\beta)} \left(1 + \sqrt{1 + \frac{4(\alpha-\beta)d_1}{(1+\alpha)d_2^2}} \right).$$

Note that, by the definition of κ_ε , (2.1.25), our choice (2.1.23) of κ in the statement of the theorem is exactly such that $\kappa = \kappa_{\varepsilon_0}$. Consequently, multiplying the inequality (2.1.24) by κ^{-1} , and also using (2.1.25) to eliminate dependence on ε , we deduce that, the inequality

$$\kappa^{-1}|\Psi'(t)| \leq 4(1+\alpha)\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha-\beta)\mathcal{A}(u(t)) + \kappa^{-1}\left(\frac{D_1}{4\kappa(1+\alpha)} + D_2\right)$$

holds for all $t \geq 0$. Thus, we may deduce from the inequality (2.1.15) that

$$\Psi''(t) \geq \kappa^{-1}|\Psi'(t)| - B_0 - \kappa^{-1}\left(\frac{D_1}{4\kappa(1+\alpha)} + D_2\right), \quad t \geq 0. \quad (2.1.26)$$

Integrating this inequality we easily deduce

$$\Psi'(t) - \kappa B_0 - \frac{D_1}{4\kappa(1+\alpha)} - D_2 \geq e^{\kappa^{-1}t} \left(\Psi'(0) - \kappa B_0 - \frac{D_1}{4\kappa(1+\alpha)} - D_2 \right)$$

holds for all $t \geq 0$. Hence, the assumption (2.1.22) implies that $\Psi'(t) \rightarrow \infty$ as $t \rightarrow \infty$. Moreover, similar to the proof of Theorem 2.1 we may deduce from (2.1.15) that

$$\begin{aligned} \Psi''(t)\Psi(t) - \left(1 + \frac{\alpha}{2}\right)[\Psi'(t)]^2 \geq \\ \left[2\alpha\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha-\beta)\mathcal{A}(u(t)) - \frac{2+\alpha}{2C_0}\|Q^{\frac{1}{2}}u_0\|^4 - B_0 \right] \Psi(t). \end{aligned}$$

Pick $C_0 = 1$. In view of (2.1.24) and the fact that $\Psi'(t) \rightarrow \infty$ as $t \rightarrow \infty$ we see that there exists $t_* \geq 0$ such that the right hand side of this inequality is nonnegative for all $t \geq t_*$. We conclude from this inequality that there exists $t_* \geq 0$ such that $\Psi(t)$ satisfies all assumptions of Lemma 1.1 for $t \geq t_*$, and the conclusion of the theorem immediately follows from the statement of this lemma.

□

The next theorem employs the same idea as the previous theorem, that is one first finds sufficient conditions that ensure $\Psi'(t)$ increases to infinity as time increases. To do that we will use an idea first employed by H. A. Levine in [29], where it was done for the abstract undamped equations. Again we will need the embedding inequalities (2.1.19) and (2.1.20) to hold, however, in contrast to the previous theorem, the conditions obtained by the method of this theorem do not depend on the embedding constants that appear in these inequalities.

Theorem 2.4. *Assume that the operators $P, Q, A, F(\cdot)$ satisfy the conditions of Theorem 2.1 together with the embedding inequalities (2.1.19) and (2.1.20). Let $u(t)$ be a local solution to the problem (2.1.1). If the initial data satisfy*

$$2(P^{\frac{1}{2}}u_0, P^{\frac{1}{2}}u_1) + \|Q^{\frac{1}{2}}u_0\|^2 > \sqrt{\frac{2}{1+2\alpha}} \left(\|P^{\frac{1}{2}}u_0\| (B_0 + 4(\alpha - \beta)L_1)^{\frac{1}{2}} + \sqrt{1+\alpha} \|Q^{\frac{1}{2}}u_0\|^2 \right), \quad (2.1.27)$$

then

$$\lim_{t \rightarrow t_1^-} \left(\|P^{\frac{1}{2}}u(t)\|^2 + \int_0^t \|Q^{\frac{1}{2}}u(s)\|^2 ds \right) = \infty,$$

for some $t_1 \geq 0$.

Proof. We argue as in the proof of Theorem 2.1 to obtain the inequality (2.1.18), with $C_0 \geq 0$ to be determined, which we rewrite for convenience as:

$$\Psi''(t)\Psi(t) - (1+\alpha)[\Psi'(t)]^2 \geq -D_0\Psi(t), \quad \forall t \geq 0,$$

where

$$D_0 = \frac{1+\alpha}{C_0} \|Q^{\frac{1}{2}}u_0\|^4 + B_0 + 4(\alpha - \beta)L_1. \quad (2.1.28)$$

Let us choose

$$C_0 = \frac{\sqrt{1+\alpha} \|Q^{\frac{1}{2}}u_0\|^2 \|P^{\frac{1}{2}}u_0\|}{(B_0 + 4(\alpha - \beta)L_1)^{\frac{1}{2}}}. \quad (2.1.29)$$

Though, as will be apparent, the proof works for any value of $C_0 > 0$, this particular choice gives the best assumptions on the initial data, see the remark after the proof. Set $H(t) := \Psi^{-\alpha}(t)$. Upon multiplying (2.1.18) by $-\alpha\Psi^{-2-\alpha}(t) < 0$, we see that $H(t)$ satisfies

$$H''(t) \leq \alpha D_0 H^{\frac{1+\alpha}{\alpha}}(t), \quad \forall t \geq 0. \quad (2.1.30)$$

It is easy to see after a bit of algebra that, by the choice (2.1.29) of C_0 , and in view of (2.1.28), the assumption (2.1.27) on the initial data exactly reads

$$\Psi'(0) > \left(\frac{2D_0}{1+2\alpha} \right)^{\frac{1}{2}} \Psi^{\frac{1}{2}}(0) \quad (2.1.31)$$

which translates in terms of H as

$$H'(0) < -\alpha \left(\frac{2D_0}{1+2\alpha} \right)^{\frac{1}{2}} H^{\frac{1+2\alpha}{2\alpha}}(0). \quad (2.1.32)$$

Moreover, the assumption (2.1.27) also forces that $u_0 \neq 0$, which ensures that $\Psi(0) > 0$. Consequently, $H(0) > 0$, and due to the assumption (2.1.32) we have $H'(0) < 0$. Hence, by the assumed regularity of the solutions, there exists a maximal time $\delta \in (0, T_{max}]$, such that $H'(t) < 0$ for all $t \in [0, \delta)$, and $H'(\delta) = 0$ if $\delta \in (0, T_{max})$. Here, T_{max} denotes the lifespan of the local solution at hand. Suppose for contradiction that $\delta \in (0, T_{max})$. Multiplying the inequality (2.1.30) by $H'(t)$ we obtain for $t \in [0, \delta]$

$$\frac{1}{2} \frac{d}{dt} \{ [H'(t)]^2 \} \geq \frac{\alpha^2 D_0}{1+2\alpha} \frac{d}{dt} \left\{ H^{\frac{1+2\alpha}{\alpha}}(t) \right\}.$$

Integrating this inequality and rearranging we get for any $t \in [0, \delta]$

$$\begin{aligned} & \left(H'(t) - \tilde{D}_0 H^{\frac{1+2\alpha}{2\alpha}}(t) \right) \left(H'(t) + \tilde{D}_0 H^{\frac{1+2\alpha}{2\alpha}}(t) \right) \\ & \geq \left(H'(0) - \tilde{D}_0 H^{\frac{1+2\alpha}{2\alpha}}(0) \right) \left(H'(0) + \tilde{D}_0 H^{\frac{1+2\alpha}{2\alpha}}(0) \right), \end{aligned} \quad (2.1.33)$$

where $\tilde{D}_0 = \alpha \left(\frac{2D_0}{1+2\alpha} \right)^{\frac{1}{2}}$. Observe that the assumption (2.1.32), together with $H(0) > 0$ and $H'(0) < 0$, implies that the right hand side of (4.3.6) is positive, which implies that the left hand side is strictly positive for each $t \in [0, \delta]$, and in particular when $t = \delta$. But, $H'(\delta) = 0$ implies that the left hand side of (4.3.6) is nonpositive at $t = \delta$, that is we have reached a contradiction, and we deduce that $\delta = T_{max}$. Hence, $H'(t) < 0$ for all $t \in [0, T_{max})$, and it follows that the inequality (4.3.6) is valid for all $t \in [0, T_{max})$. Now, the same argument shows that the left hand side of (4.3.6) is positive for all $t \in [0, T_{max})$, and thanks to the continuity assumptions we deduce that

$$H'(t) \leq -\tilde{D}_0 H^{\frac{1+2\alpha}{2\alpha}}(t), \quad \forall t \in [0, T_{max}). \quad (2.1.34)$$

It is easy to see upon multiplying this inequality by $-\frac{1}{\alpha} H^{-1-\frac{1}{\alpha}}(t)$, that we may write it back in terms of Ψ as

$$\Psi'(t) \geq \left(\frac{2D_0}{1+2\alpha} \right)^{\frac{1}{2}} \Psi^{\frac{1}{2}}(t), \quad t \in [0, T_{max}).$$

Integrating this differential inequality, under the condition (2.1.27) on the initial data, which is equivalent to (2.1.31), we see that $\Psi(t) \rightarrow +\infty$ and $\Psi'(t) \rightarrow +\infty$ as $t \rightarrow T_{max}$. After this point the rest of the proof follows the exact same lines as that of Theorem 2.3, and this finishes the proof. $\square$

Remark 2.5. As noted in the proof of the Theorem 2.4, the proof can be carried out for any choice of $C_0 > 0$. Correspondingly, according to the formula (2.1.31) with D_0 as in (2.1.28),

there would be a change in the assumptions on the initial data. Observe that the left hand side of (2.1.31) does not depend on the value of C_0 . The particular choice (2.1.29) for the value of C_0 is the one that minimizes the right hand side of (2.1.31).

2.1.2 Uniform boundedness of global solutions

The techniques of the previous subsection, devised for deriving sufficient conditions on the initial functions which ensure the global nonexistence of corresponding solutions to the abstract problem (2.1.1), included first showing that the initial data at hand gives rise to a presumably global solution that becomes (in some sense) unbounded as time increases. This brings to the mind the following question: If the initial data are such that the corresponding solution exists globally in time, is then this solution necessarily ‘bounded’ uniformly in time? Here, by ‘bounded’ we mean that the norm in an appropriate Hilbert space is bounded. As its name suggests this subsection is devoted to this question, and we answer it in the affirmative direction. The property that all global solutions to an unstable problem remain bounded in time was first shown by T. Cazenave in [7] to hold true for the solutions to the Cauchy initial value problem for the Klein-Gordon equation. The treatment that we present here for the abstract problem (2.1.1) to a great extent imitates the methods used in that paper.

Let us start with our exposition. We first start by proving some auxillary propositions. Although the next proposition is valid for any local solution, it is a necessary step to study the boundedness property of global solutions. It shows that, for any solution to (2.1.1), $\Psi'(t)$ is bounded from below by a constant.

Proposition 2.6. *Suppose that the operators $P, Q, A, F(\cdot)$ satisfy the conditions of Theorem 2.1 together with the embedding inequalities (2.1.19) and (2.1.20). Then, for any local solution to the problem (2.1.1) we have*

$$\Psi'(t) \geq \min \{ \Psi'(0), -\kappa B_0 - D_1 - D_2 \} \quad \forall t \geq 0. \quad (2.1.35)$$

Proof. Observe that, since all assumptions of Theorem 2.3 are fulfilled the inequality (2.1.26) is satisfied by the solution with κ as in (2.1.23). From that inequality we deduce

$$\Psi''(t) \geq -\kappa^{-1} \Psi'(t) - B_0 - \kappa^{-1}(D_1 + D_2), \quad \forall t \geq 0.$$

Integrating this inequality we see that

$$\Psi'(t) + \kappa B_0 + D_1 + D_2 \geq e^{-\kappa^{-1}t} (\Psi'(0) + \kappa B_0 + D_1 + D_2)$$

holds for all $t \geq 0$. This inequality immediately implies (2.1.35). □

In the next proposition we show that, if we a priori assume that the solution is global, then the average time integral of the total energy of the solution remains bounded in time. Here, by the total energy we mean the sum of the kinetic energy ($\frac{1}{2}\|P^{\frac{1}{2}}u_t(t)\|^2$), the internal potential energy ($\mathcal{A}(u)$) and the external nonlinear potential energy ($\mathcal{F}(u(t))$).

Proposition 2.7. *Suppose that the operators $P, Q, A, F(\cdot)$ satisfy the conditions of Theorem 2.1 together with the embedding inequalities (2.1.19) and (2.1.20). Assume that $u(t)$ is a global solution to the problem (2.1.1). Then, for any $t, s \geq 0$ we have*

$$\int_t^{t+s} H(\tau) d\tau \leq (4(1 + \alpha + \beta)E(0) + 2(R_0 + R_1))s + 2(\kappa B_0 + D_1 + D_2) + |\Psi'(0)|, \quad (2.1.36)$$

where

$$H(t) = 4(1 + \frac{\alpha + \beta}{2})\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha - \beta)\mathcal{A}(u(t)) + 4(\alpha - \beta)\mathcal{F}(u(t)).$$

Proof. First note that, since $u(t)$ is a global solution to (2.1.1), it immediately follows from Theorem 2.3 that

$$\Psi'(t) \leq \kappa B_0 + D_1 + D_2, \quad \forall t \geq 0. \quad (2.1.37)$$

Next observe that, instead of estimating $\Psi''(t)$ as was done when obtaining the inequality (2.1.15) from the identity (2.1.14), we may proceed from that identity in the following similar way:

$$\begin{aligned} \Psi''(t) &\geq -4(1 + 2\beta)\mathcal{A}(u(t)) + 2\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(1 + 2\alpha)\mathcal{F}(u(t)) - 2(R_0 + R_1) \\ &= 4(1 + \frac{\alpha + \beta}{2})\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha - \beta)\mathcal{A}(u(t)) + 4(\alpha - \beta)\mathcal{F}(u(t)) \\ &\quad - 4(1 + \alpha + \beta)E(t) - 2(R_0 + R_1) \\ &\geq 4(1 + \frac{\alpha + \beta}{2})\|P^{\frac{1}{2}}u_t(t)\|^2 + 4(\alpha - \beta)\mathcal{A}(u(t)) + 4(\alpha - \beta)\mathcal{F}(u(t)) \\ &\quad - 4(1 + \alpha + \beta)E(0) - 2(R_0 + R_1) \\ &= H(t) - 4(1 + \alpha + \beta)E(0) - 2(R_0 + R_1). \end{aligned}$$

Integrating this inequality over $(t, t + s)$, $t, s \geq 0$ we have

$$\int_t^{t+s} H(\tau) d\tau \leq (4(1 + \alpha + \beta)E(0) + 2(R_0 + R_1))s + \Psi'(t + s) - \Psi'(t).$$

By Propostion 2.6 we have

$$-\Psi'(t) \leq \kappa B_0 + D_1 + D_2 + |\Psi'(0)|, \quad \forall t \geq 0.$$

Utilizing this and the estimate (2.1.37) in the previous inequality we deduce (2.1.36).

□

Remark 2.8. It is easy to see that, in the proof above, instead of reestimating $\Psi''(t)$, if we continue from the inequality (2.1.15), the same arguments yield

$$\int_t^{t+s} \tilde{H}(\tau) d\tau \leq B_0 s + 2(\kappa B_0 + D_1 + D_2) + |\Psi'(0)| \quad \forall t, s \geq 0,$$

where

$$\tilde{H}(t) = 4(1 + \alpha) \|P^{\frac{1}{2}} u_t(t)\|^2 + 4(\alpha - \beta) \mathcal{A}(u(t)).$$

Remark 2.9. It is easy to see from the proofs of Theorem 2.1, Theorem 2.3, Theorem 2.4, Proposition 2.6 and Proposition 2.7 (including Remark 2.8) that, after obvious modifications on the assumptions and the conclusions (i.e. erase all terms with Q , and set $d_2 = 0$), they hold also for the solutions to the undamped problem

$$\begin{cases} Pu_{tt} + Au = F(u), \\ u(0) = u_0, u_t(0) = u_1. \end{cases} \quad (2.1.38)$$

Next result is an immediate consequence of Theorem 2.3 and the embedding inequality (2.1.21), which we will assume for the first time. It states that the weak norm ($\|P^{\frac{1}{2}} u(t)\|$) of any global solution $u(t)$ to (2.1.1) remains bounded in time.

Proposition 2.10. *Suppose that the operators $P, Q, A, F(\cdot)$ satisfy the conditions of Theorem 2.1 together with the embedding inequalities (2.1.19), (2.1.20) and (2.1.21). Assume that $u(t)$ is a global solution to the problem (2.1.1). Then, the estimate*

$$\|P^{\frac{1}{2}} u(t)\|^2 \leq \max \left\{ \|P^{\frac{1}{2}} u_0\|^2, d_3(\kappa B_0 + D_1 + D_2) \right\} \quad (2.1.39)$$

holds for all $t \geq 0$.

Proof. Since $u(t)$ is a global solution to (2.1.1), we infer from Theorem 2.3 that

$$\Psi'(t) = \frac{d}{dt} \|P^{\frac{1}{2}} u(t)\|^2 + \|Q^{\frac{1}{2}} u(t)\|^2 \leq \kappa B_0 + D_1 + D_2, \quad \forall t \geq 0.$$

Thanks to the embedding inequality (2.1.21) we deduce from this inequality

$$\frac{d}{dt} \|P^{\frac{1}{2}} u(t)\|^2 + \frac{1}{d_3} \|P^{\frac{1}{2}} u(t)\|^2 \leq \kappa B_0, \quad \forall t \geq 0.$$

It follows that

$$\left(\|P^{\frac{1}{2}} u(t)\|^2 - d_3(\kappa B_0 + D_1 + D_2) \right) \leq e^{-d_3^{-1}t} \left(\|P^{\frac{1}{2}} u_0\|^2 - d_3(\kappa B_0 + D_1 + D_2) \right), \quad \forall t \geq 0,$$

which implies the estimate (2.1.39).

□

Note that this method of proof can not be extended to contain the global solutions of the undamped problem (2.1.38), because it assumes (2.1.21). But, a similar result still holds, and for the sake of completeness we prove it in the following proposition.

Proposition 2.11. *Suppose that the operators $P, A, F(\cdot)$ satisfy the conditions of Theorem 2.1 together with the embedding inequality (2.1.19). Let $u(t)$ be a global strong solution to the undamped problem (2.1.38). Then,*

$$\|P^{\frac{1}{2}} u(t)\|^2 \leq \max \left\{ \|P^{\frac{1}{2}} u_0\|^2, 1 + \frac{d_1}{4(\alpha - \beta)D_1} B_0 \right\} \quad (2.1.40)$$

for all $t \geq 0$.

Proof. Let $u(t)$ be a global solution to (2.1.38). As noted in Remark 2.9 the calculations of Theorem 2.1 and Theorem 2.3 hold for $u(t)$ after apparent modifications. After these modifications we have

$$\Psi(t) = \|P^{\frac{1}{2}} u(t)\|^2,$$

and instead of (2.1.15) there holds for all $t \geq 0$

$$\Psi''(t) \geq 4(1 + \alpha) \|P^{\frac{1}{2}} u_t(t)\|^2 + 4(\alpha - \beta) \mathcal{A}(u(t)) - B_0. \quad (2.1.41)$$

Thanks to the embedding inequality (2.1.19) we have

$$\mathcal{A}(u(t)) \geq \frac{1}{d_1} \|P^{\frac{1}{2}} u(t)\|^2 - \frac{D_1}{d_1} = \frac{1}{d_1} \Psi(t) - \frac{D_1}{d_1}.$$

Thus, we infer from the inequality (2.1.41) that

$$\Psi''(t) \geq \frac{4(\alpha - \beta)D_1}{d_1} \Psi(t) - \frac{4(\alpha - \beta)D_1}{d_1} - B_0. \quad (2.1.42)$$

To prove the estimate (2.1.40) we will show that $\Psi(t)$ is nonincreasing when it is greater than

$$l_0 := 1 + \frac{d_1}{4(\alpha - \beta)D_1} B_0.$$

Suppose for contradiction that this not the case, i.e. there is $t_0 \geq 0$ such that $\Psi(t_0) > l_0$ and $\Psi'(t_0) > 0$. Set

$$\Phi(t) := \Psi(t) - l_0.$$

Then, in view of (2.1.42) we have

$$\Phi''(t) \geq \frac{4(\alpha - \beta)D_1}{d_1} \Phi(t) \quad \forall t \geq 0,$$

with $\Phi(t_0) > 0$ and $\Phi'(t_0) > 0$. It follows that $\Phi(t)$ is uniformly convex for $t \geq t_0$, and $\Phi(t) \rightarrow \infty$, $\Phi'(t) \rightarrow \infty$ as $t \rightarrow \infty$. Thus, due to the embedding inequality (2.1.19) we may further deduce from (2.1.41) that there is $t_* \geq 0$ such that $\Psi(t_*) > 0$, $\Psi'(t_*) > 0$, and

$$\Psi''(t) \geq 4(1 + \alpha) \|P^{\frac{1}{2}} u_t(t)\|^2 \quad \forall t \geq t_*,$$

which implies thanks to the Cauchy-Schwarz inequality that $\Psi(t)$ satisfies

$$\Psi''(t)\Psi(t) - (1 + \alpha)[\Psi'(t)]^2 \geq 0 \quad \forall t \geq t_*.$$

Therefore, by Lemma 1.1 we arrive at a contradiction to the assumption that $u(t)$ is global. This finishes the proof. □

At last we have come to the point, where we can prove the uniform boundedness of the total energy of any global solution to (2.1.1), which is the main result of this section. This result comes with a price of an additional condition on the nonlinear potential $\mathcal{F}$. However, as we will see by an example in the next section, this condition is not so severe. In the following theorem C will connote a positive number, and C^0 will connote a positive number depending on the initial data.

Theorem 2.12. *Let $u(t)$ be a global strong solution to the problem (2.1.1). Suppose that the operators $P, Q, A, F(\cdot)$ satisfy the conditions of Theorem 2.1 together with (2.1.19), (2.1.20) and (2.1.21). Suppose further that $F(\cdot)$ satisfies*

$$\|P^{-\frac{1}{2}} F(u(t))\| \leq M(\|P^{\frac{1}{2}} u(t)\|)(1 + \tilde{L}_1 + \mathcal{A}(u(t)) + \mathcal{F}(u(t))) (\mathcal{A}(u(t)) + \tilde{L}_1)^{\frac{1}{2}}, \quad (2.1.43)$$

where $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing function, and $\tilde{L}_1$ is any number that satisfies

$$\tilde{L}_1 \geq \max\{0, L_1\}, \quad (2.1.44)$$

with L_1 as in the condition (2.1.5) on $\mathcal{A}$. Then,

$$\sup_{t \geq 0} \left(\|P^{\frac{1}{2}} u_t(t)\|^2 + \mathcal{A}(u(t)) \right) \leq C^0.$$

Proof. First observe that, due to the definition (2.1.44) of $\tilde{L}_1$ and the condition (2.1.5) on $\mathcal{A}$, all the terms in the condition (2.1.43) are nonnegative. Set

$$h(t) := \frac{1}{2} \|P^{\frac{1}{2}} u_t(t)\|^2 + \mathcal{A}(u(t)) + \tilde{L}_1 \geq 0.$$

Using the definition of energy (2.1.9) and the energy inequality (2.1.10) we see that

$$h'(t) + \|Q^{\frac{1}{2}} u_t(t)\|^2 = \frac{d}{dt} \mathcal{F}(u(t)) = (F(u(t)), u_t(t)).$$

Applying the Cauchy-Schwarz inequality to the term on the right hand side, and dropping the damping term we obtain

$$h'(t) \leq \|P^{-\frac{1}{2}} F(u(t))\| \|P^{\frac{1}{2}} u_t(t)\|.$$

Hence, by the assumption (2.1.43) and the Young's inequality we have

$$\begin{aligned} h'(t) &\leq M(\|P^{\frac{1}{2}} u(t)\|)(1 + \tilde{L}_1 + \mathcal{A}(u(t)) + \mathcal{F}(u(t))) (\mathcal{A}(u(t)) + \tilde{L}_1)^{\frac{1}{2}} \|P^{\frac{1}{2}} u_t(t)\| \\ &\leq M(\|P^{\frac{1}{2}} u(t)\|)(1 + \tilde{L}_1 + \mathcal{A}(u(t)) + \mathcal{F}(u(t))) h(t). \end{aligned}$$

Note that the assumptions of Proposition 2.10 hold. Consequently, from the statement of Proposition 2.10, and the fact that M is nondecreasing we deduce that

$$M(\|P^{\frac{1}{2}} u(t)\|) \leq C^0, \quad (2.1.45)$$

Thus, above inequality reduces to

$$h'(t) \leq C^0(1 + \tilde{L}_1 + \mathcal{A}(u(t)) + \mathcal{F}(u(t))) h(t).$$

It follows from this inequality that for any $t \geq 0$, $\tau \in [0, 1]$ we have

$$\begin{aligned} h(t + \tau) &\leq h(t) e^{C^0 \int_t^{t+\tau} (1 + \tilde{L}_1 + \mathcal{A}(u(s)) + \mathcal{F}(u(s))) ds} \\ &\leq h(t) e^{C^0 \int_t^{t+1} (1 + \tilde{L}_1 + \mathcal{A}(u(s)) + \mathcal{F}(u(s))) ds}. \end{aligned}$$

It is easy to see that, the Proposition 2.7 (with $s = 1$), which is valid under current assumptions, implies that

$$\int_t^{t+1} (1 + \tilde{L}_1 + \mathcal{A}(u(t)) + \mathcal{F}(u(s))) ds \leq C^0.$$

Hence, we deduce from the previous inequality that

$$h(t + \tau) \leq C^0 h(t) \quad \forall t \geq 0, \forall \tau \in [0, 1],$$

or equivalently

$$h(t) \leq C^0 h(t - \tau) \quad \forall t \geq 1, \forall \tau \in [0, 1].$$

The former inequality in particular implies

$$\sup_{t \in [0, 1]} h(t) \leq C^0.$$

Integrating the latter over $\tau \in [0, 1]$ one gets

$$h(t) \leq C^0 \int_{t-1}^t h(s) ds \quad \forall t \geq 1.$$

By Remark 2.8 after Proposition 2.7

$$\int_{t-1}^t h(s) ds \leq C^0.$$

Thus, we conclude

$$\sup_{t \geq 1} h(t) \leq C^0.$$

The result of the theorem follows. □

Remark 2.13. The statement of Theorem 2.12 is valid also for the solutions to the undamped problem (2.1.38). In fact, the only difference in the proof is that, instead of using the result of Proposition 2.10 when obtaining the estimate (2.1.45), one needs to use Proposition 2.11, which tells the same thing as Proposition 2.10, but for the undamped equation (2.1.38).

2.2 Concrete examples

In this section we apply the abstract results obtained in the previous section to several concrete problems. In all of the examples considered below the role of the abstract Hilbert

space $\mathcal{H}$ will be played by $L^2(\Omega)$ with standard inner-product, where Ω is the domain on which the problem is considered. Before proceeding let us give some remarks that apply to all of the examples.

Remark 2.14. The condition (2.1.7) on the nonlinear potential $\mathcal{F}$ is usually found in the literature with $R_0 = 0$. The corresponding typical example is $F(u) = |u|^q u$, where $q > 0$ is a real number, and the condition (2.1.7) is then verified with $\alpha = \frac{q}{4} > 0$. Allowing R_0 to be nonzero in (2.1.7), we can now handle nonlinearities of the form $F(u) = |u|^q u + p(u)$, where $q > 0$ and p is a polynomial of degree strictly less than $q + 1$. Note also that the assumption (2.1.6) is fulfilled with $\mathcal{F}(u) : \int_0^u F(s) ds$.

Remark 2.15. In all of the examples of initial-boundary value problems presented below we have chosen the Dirichlet boundary conditions for convenience. However, all of the results apply to the case with the Neumann or mixed (for equations 4th-order in space for example) boundary conditions as well.

Remark 2.16. Although in this section we only give example problems for damped equations, it is important to emphasize that, by Remark 2.8 and Remark 2.9 similar results hold also for the solutions to the problems for the corresponding undamped equations.

Remark 2.17. Note that the Theorem 2.1 does not assume any of the embedding inequalities (2.1.19)-(2.1.21). It follows that, the statement of this theorem, after the apparent modifications, holds also for the Cauchy initial value problems for all the equations that we consider in subsections §2.2.1 and §2.2.2, where instead we have usually worked with (except in §2.2.1.2) the initial boundary value problems on bounded domains. In fact, in order not to add more to the already existing repetitive structure of these subsections, we have totally ignored the Theorem 2.1, except its first part which characterizes the initial data with negative ‘energy’, and we will not mention more of it after this remark. We will also skip by just saying that, the assumptions (second part) of the Theorem 2.1 are compatible with the existence of initial data with arbitrarily positive initial energies for each of the equations that we consider here. Indeed, one can actually employ exactly the same strategy as used at the end of §2.2.1.1 to derive blowing-up initial data with prescribed arbitrarily large positive energy.

In what follows all solutions are functions of space and time, and this dependence will most of the time be suppressed in writing.

2.2.1 Semilinear problems

Before proceeding with the application of the results of §2.1 to concrete semilinear problems, we give here by a remark, the content of which applies to all of the semilinear problems considered in this subsection.

Remark 2.18. By semilinear problems we mean that the only nonlinearity in the abstract problem (2.1.1) of the previous section is the operator F , and the possibly nonlinear operator A of the previous section specializes to become a positive selfadjoint linear operator, like the operators P and Q . Consequently, $\mathcal{A}(u)$ becomes $\frac{1}{2}\|A^{\frac{1}{2}}u\|^2$, and A satisfies the conditions (2.1.3)-(2.1.5) with the constants $\beta = R_1 = L_1 = 0$. Moreover, the condition (2.1.7) on F holds now for any $\alpha > 0$. Finally, since $L_1 = 0$, it is enough to take $\tilde{L}_1 \geq 0$ in the condition (2.1.43) of the Theorem 2.12.

Below, in all of the examples of this subsection, the discussion will assume the content of Remark 2.18 without any reference.

2.2.1.1 Semilinear structurally damped wave equations

Our first example is the following Dirichlet initial-boundary value problem for a family of semilinear structurally damped wave equations:

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^\theta u_t = |u|^q u \mathbf{1}_\omega(x), & x \in \Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t)|_{\partial\Omega} = 0, & t \geq 0, \end{cases} \quad (2.2.1)$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a domain with smooth boundary $\partial\Omega$, ω is an open subset of Ω , $\mathbf{1}_\omega$ is the characteristic function of ω , and the damping exponent $\theta \in [0, 1]$ and $q > 0$ are given numbers.

When $n \geq 3$, the local well-posedness of this problem is known only for the exponents $q \in (0, \frac{2}{n-2}]$; and we have local well-posedness for all $q > 0$, when $n = 1, 2$.

Theorem 2.19. *Suppose $q \geq 0$, if $n = 1, 2$, or $q \in (0, \frac{2}{n-2}]$, if $n \geq 3$. Then, for each initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists $T > 0$ such that the problem (2.2.1) admits a unique local solution*

$$u(t) \in \mathcal{E}_T := C(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$$

in the sense of (2.1.8). Moreover, this solution verifies the energy inequality (2.1.10), where the energy is as in (2.1.9).

It is easy to see that the problem (2.2.1) is of the same form as the abstract problem (2.1.1) with $P = I$, $Q = (-\Delta_0)^\theta$, $A = -\Delta_0$, and $F(u) = |u|^q u \mathbf{1}_\omega$, $-\Delta_0$ is the Laplacian operator under Dirichlet boundary conditions defined in $L^2(\Omega)$, and I is the identity operator. Since Ω is bounded and we have Dirichlet boundary conditions, by the Poincaré inequality we

see that all of the embedding inequalities (2.1.19)-(2.1.21) hold with coefficients $d_1 = 2\lambda_1^{-1}$, $d_2 = 2\lambda_1^{\theta-1}$, $d_3 = \lambda_1^{-\theta}$ and $D_1 = D_2 = 0$, where $\lambda_1 > 0$ is the smallest eigenvalue of the operator Δ_0 . Moreover, we have

$$(F(u), u) = \int_{\omega} |u|^{q+2} dx = (q+2)\mathcal{F}(u),$$

from which we see that the condition (2.1.7) is verified with $\alpha = \frac{q}{4} > 0$ and $R_0 = 0$. Consequently, we deduce that all results up to the Theorem 2.12 of the previous section hold for the solutions to the problem (2.2.1), of course after proper translation, in particular all of the finite time blow-up results.

Let us now see that the condition (2.1.43) of Theorem 2.12 is also fulfilled for the admissible values of the exponent q , i.e. $q \in (0, \frac{2}{n-2}]$ if $n \geq 3$, and $q > 0$ if $n = 1, 2$. First, suppose that $n \geq 3$. Since $q \in (0, \frac{2}{n-2}]$, $2 < 2(q+1) \leq \frac{2n}{n-2}$. Hence, by the L^p -embedding and the Sobolev embedding inequalities we have

$$\begin{aligned} \|P^{-\frac{1}{2}}F(u)\| &= \| |u|^q u \mathbf{1}_{\omega} \| = \|u\|_{L^{2(q+1)}(\omega)}^{q+1} \\ &\leq C \|u\|_{L^{\frac{2n}{n-2}}(\omega)}^{q+1} \leq C \|u\|_{L^{\frac{2n}{n-2}}(\Omega)}^{q+1} \leq C \|\nabla u\|^{q+1} = C \|A^{\frac{1}{2}}u\|^{q+1}. \end{aligned}$$

Note that $q+1 \in (1, \frac{n}{n-2}] \subset (1, 3)$, when $n \geq 3$. Consequently, we may estimate the right hand side of the last inequality from above using Young's inequality, and deduce

$$\|P^{-\frac{1}{2}}F(u)\| \leq C \|A^{\frac{1}{2}}u\| \left(1 + \|A^{\frac{1}{2}}u\|^2\right),$$

from which we immediately conclude that, when $n \geq 3$, F fulfills the condition (2.1.43) of Theorem 2.12 for all permissible values of the exponent q . Next, assume that $n = 1, 2$, and let $q > 0$ be a given arbitrary number. If $q \leq 2$, then we may proceed as the case $n \geq 3$, so suppose $q > 2$. This time using the L^p -interpolation inequality together with the Sobolev embedding inequality we may obtain for any $p \geq 2(q+1)$

$$\begin{aligned} \|P^{-\frac{1}{2}}F(u)\| &= \| |u|^q u \mathbf{1}_{\omega} \| = \|u\|_{L^{2(q+1)}(\omega)}^{q+1} \\ &\leq \|u\|_{L^2(\omega)}^{(1-\delta)(q+1)} \|u\|_{L^p(\omega)}^{\delta(q+1)} \leq \|u\|_{L^2(\omega)}^{(1-\delta)(q+1)} \|u\|_{L^p(\Omega)}^{\delta(q+1)} \leq C \|u\|_{L^2(\Omega)}^{(1-\delta)(q+1)} \|\nabla u\|^{\delta(q+1)}, \end{aligned}$$

where $\delta = \frac{2(q+1)-2}{p-2} = \frac{2q}{p-2}$. Note that $\delta \in (0, 1]$, when $p \in [2(q+1), \infty)$, and the correspondence is a bijection. Also, since we have assumed $q > 2$, $q+1 > 3$, and consequently we may choose an appropriate value of $p \in [2(q+1), \infty)$ such that $\delta(q+1) = 3$. That value is $p = \frac{2}{3}q(q+1) + 2$, which belongs to $(2(q+1), \infty)$ when $q > 2$. After this choice the previous inequality reads

$$\|P^{-\frac{1}{2}}F(u)\| \leq C \|u\|^{q-2} \|\nabla u\|^3 = M(\|P^{\frac{1}{2}}u\|) \|A^{\frac{1}{2}}u\|^3,$$

where $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing function. From this inequality it is straightforward to conclude that, when $n = 1, 2$, F verifies the condition (2.1.43) of Theorem 2.12 for all $q > 0$.

So, we have shown that all of the abstract theorems of the previous section are valid for the solutions to the problem (2.2.1) given in Theorem 2.19. The following theorem summarizes what we have achieved.

Theorem 2.20. *Assume that all conditions of Theorem 2.19 are satisfied. Let $u(t)$ be the unique solution to the problem (2.2.1). Suppose the initial data satisfy one of the following conditions:*

1.

$$E(0) < 0,$$

or

2.

$$E(0) \geq 0,$$

and

(a)

$$2(u_0, u_1) + \|(-\Delta)^{\frac{\theta}{2}} u_0\|^2 > 2q(2+q)\lambda_1^{1-\theta} \left(1 + \sqrt{1 + \frac{2q\lambda_1^{1-2\theta}}{4+q}} \right) E(0),$$

or

(b)

$$2(u_0, u_1) > 2\sqrt{2}\|u_0\|E(0)^{\frac{1}{2}} + \left(\sqrt{\frac{4+q}{2+q}} - 1 \right) \|(-\Delta)^{\frac{\theta}{2}} u_0\|^2,$$

where

$$E(0) := \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{q+2} \int_{\Omega} |u_0|^{q+2} dx.$$

Then, there exists $t_1 > 0$ such that

$$\|u(t)\|^2 + \int_0^t \|(-\Delta)^{\frac{\theta}{2}} u(s)\|^2 ds \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

Moreover, any global solution $(u(t), u_t(t))$ belongs to $L^\infty(\mathbb{R}^+; H_0^1(\Omega) \times L^2(\Omega))$.

Let us now compare the conditions 2.(a) and 2.(b). The first thing to observe is that the condition 2.(a) depends on λ_1 , the first eigenvalue of the operator Δ_0 , which in turn depends

on the domain Ω , while the condition 2.(b) is invariant to changes in domain. Observe that for each $\theta \in [0, 1]$ the right hand side of the condition 2.(a) grows indefinitely as λ_1 tends to zero. On the other hand, for $\theta \in [0, 1)$, the right hand side of 2.(a) tends to zero as λ_1 increases to infinity, however when $\theta = 1$ it converges bounded away from zero. Now, roughly speaking, λ_1 grows indefinitely as the domain ‘shrinks’, and tends to zero as the domain ‘grows’. Consequently, it is safe to say that for each $\theta \in [0, 1]$ the condition 2.(b) is a better one compared to the condition 2.(a) for small λ_1 , that is for ‘large’ domains. Similarly, for each $\theta \in [0, 1)$ we can say that the opposite is true. Another difference one can easily see favors the condition 2.(a), which allows initial data with nonnegative energy and such that the inner-product (u_0, u_1) is negative. However, such a thing is not premitted by the condition 2.(b).

Let us now give a quantitative example in 1-dimension, which illustrates the feature of the condition 2.(a) of the Theorem 2.20 mentioned at the end of the previous paragraph. Let $\omega = \Omega = [0, \pi]$, $\theta = 1$ and $q = 2$ in . It is well-known that, the normalized (in $L^2(0, \pi)$) eigenfunctions and the eigenvalues of the operator $-\Delta_0$ are

$$e_k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(kx), \quad \lambda_k = k^2, \quad k \geq 1.$$

Both of the initial datum we take as multiples of some eigenfunction, that is we assume

$$u_0(x) = a_0 e_{k_0}(x), \quad u_1(x) = a_1 e_{k_0}(x), \quad x \in [0, \pi],$$

where the coefficients $a_0, a_1 \in \mathbb{R}$ and index $k_0 \geq 1$ will be chosen shortly. First, with a little manipulation of trigonometric identities we easily calculate

$$\int_0^\pi u_0^4(x) dx = \frac{4a_0^4}{\pi^2} \int_0^\pi \sin^4(k_0 x) dx = \frac{3}{2\pi} a_0^4.$$

Consequently, the initial energy has the value

$$E(0) = \frac{a_1^2}{2} + \frac{k_0^2 a_0^2}{2} - \frac{3}{8\pi} a_0^4.$$

Hence, the condition 2.(a) reads

$$2a_0 a_1 + k_0^2 a_0^2 > 16 \left(1 + \sqrt{\frac{2}{3k_0^2}}\right) \left[\frac{a_1^2}{2} + \frac{k_0^2 a_0^2}{2} - \frac{3}{8\pi} a_0^4\right].$$

Firstly, to have $(u_0, u_1) < 0$, the corresponding coefficients a_0 and a_1 must have different signs. In view of this, we set $a_1 = -\eta a_0$ with $a_0, \eta > 0$. Now, the condition 2.(a) becomes

$$(k_0^2 - 2\eta)a_0^2 > 16(1 + \sqrt{\frac{2}{3k_0^2}}) \left[\frac{\eta^2 + k_0^2}{2} a_0^2 - \frac{3}{8\pi} a_0^4 \right]. \quad (2.2.2)$$

Note that, the term in the square brackets on the right hand side of this inequality is exactly the initial energy. So, in order to satisfy this inequality with positive initial energy, we must choose $k_0^2 > 2\eta$. Now, observe that the function defined on $(0, +\infty)$ by

$$g(r) := C_1 r - C_2 r^2, \quad r > 0,$$

verifies $g(r) > 0$, if and only if $r \in (0, \frac{C_1}{C_2})$. Consequently, the initial energy is positive, if and only if

$$0 < a_0^2 < \frac{4\pi}{3}(\eta^2 + k_0^2). \quad (2.2.3)$$

Also, by the same argument we see that the condition (2.2.2) holds, if and only if

$$0 < a_0^2 < \frac{4\pi}{3}(\eta^2 + k_0^2) - \frac{\pi}{6(1 + \sqrt{\frac{2}{3k_0^2}})}(k_0^2 - 2\eta). \quad (2.2.4)$$

From these last two statements it follows that, provided that we choose $k_0^2 > 2\eta$, if we choose a_0 according to (2.2.4), both the positivity of the initial energy, and the condition 2.(a) of the Theorem 2.20 are realized. Moreover, since the function g defined above attains its maximum at the point $r_0 = \frac{C_1}{2C_2}$ with $g(r_0) = \frac{C_1^2}{4C_2}$, we deduce that the initial energy takes its maximum value at

$$a_0^2 = \frac{2\pi}{3}(\eta^2 + k_0^2), \quad (2.2.5)$$

with the value

$$E(0) = \frac{\pi}{6}(\eta^2 + k_0^2)^2. \quad (2.2.6)$$

It is easy to see that the value of a_0 given by (2.2.5) belongs to the interval specified by the condition (2.2.4). Thus, we choose a_0 as in (2.2.5). Remembering we need to choose $k_0^2 > 2\eta$, we see from (2.2.6) that, by choosing $k_0 \geq 1$ and $\eta > 0$ large, we may make the initial energy as positive as we desire. Since we also want to supplement this example with a simulation, we wish not to choose k_0 , the frequency of the initial data, too high, and accordingly we choose $k_0 = 3$. Also, choosing $\eta = 4$ we make sure that $k_0^2 > 2\eta$. Finally, with all the choices above we conclude that, the initial data given by

$$u_0(x) = \frac{50\sqrt{2\pi}}{3} \sin(3x), \quad u_1(x) = -\frac{200\sqrt{2\pi}}{3} \sin(3x), \quad x \in [0, \pi], \quad (2.2.7)$$

have the initial energy

$$E(0) = \frac{625\pi}{6} > 0, \quad (2.2.8)$$

satisfy

$$(u_0, u_1) = -\frac{10^4\pi^2}{9} < 0,$$

and also verify the condition 2.(a) of the Theorem 2.20, so that, the corresponding solution to the problem (2.2.1) with aforementioned values of the parameters blows up in finite time in accordance with the statement of that theorem. The outputs of the simulation, that was carried out for this example with the initial data as in (2.2.7), are presented in Figure 4.2 in Chapter §4.

2.2.1.2 Semilinear damped Klein-Gordon equations

Let us consider the following Cauchy problem for a semilinear damped Klein-Gordon equation

$$\begin{cases} u_{tt} - \Delta u + mu - a\Delta u_t + bu_t = |u|^q u, & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (2.2.9)$$

where $a, b \geq 0$ and $m, q > 0$ are given numbers, $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$ are given functions, $n \geq 1$.

Similar to the initial-boundary value problem (2.2.1) for the structurally damped wave equations, the local well-posedness of this problem is known only for the exponents $q \in [0, \frac{2}{n-2}]$, when $n \geq 3$; and for all $q > 0$, when $n = 1, 2$. We have the following theorem.

Theorem 2.21. *Suppose $q \geq 0$, if $n = 1, 2$, or $q \in [0, \frac{2}{n-2}]$, if $n \geq 3$. Then, for each initial data $(u_0, u_1) \in H_0^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, there exists $T > 0$ such that the problem (2.2.9) admits a unique local solution*

$$u(t) \in \mathcal{E}_T := C(0, T; H_0^1(\mathbb{R}^n)) \cap C^1(0, T; L^2(\mathbb{R}^n)) \cap H^1(0, T; H_0^1(\mathbb{R}^n))$$

in the sense of (2.1.8). Moreover, this solution verifies the energy inequality (2.1.10), where the energy is as in (2.1.9).

This equation can be written in the form (2.1.1) with $P = I$, $A = -\Delta_0 + mI$, $Q = -a\Delta_0 + bI$ and $F(u) = |u|^q u$. It is easy to see that, since $m > 0$, the embedding inequalities (2.1.19) and (2.1.20) are satisfied for all $a, b \geq 0$, with coefficients $d_1 = 2m$, $d_2 = 2\max\{a, b\}$ and $D_1 = D_2 = 0$, and the inequality (2.1.21) holds with $d_3 = b$ provided that $b > 0$. Moreover, the condition (2.1.43) of Theorem 2.12 is also satisfied: If $n = 1, 2$, the argument is same

as in §2.2.1.1 (with $\omega = \Omega = \mathbb{R}^n$). If $n \geq 3$, since $2 < 2(q+1) \leq \frac{2n}{n-2}$, we have by the L^p -interpolation inequality and the Sobolev embedding inequalities

$$\begin{aligned} \|P^{-\frac{1}{2}}F(u)\| &= \| |u|^q u \| = \|u\|_{L^{2(q+1)}}^{q+1} \leq \|u\|^{(1-\delta)(q+1)} \|u\|_{L^{\frac{2n}{n-2}}}^{\delta(q+1)} \\ &\leq C \|u\|^{(1-\delta)(q+1)} \|\nabla u\|^{\delta(q+1)} \leq M(\|P^{\frac{1}{2}}u\|) \|A^{\frac{1}{2}}u\|^{\delta(q+1)}, \end{aligned} \quad (2.2.10)$$

where $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing function, and

$$\delta = \frac{2(q+1)-2}{\frac{2n}{n-2}-2} = \frac{q(n-2)}{2} \in (0, 1].$$

Observe that, δ increases as q increases, and consequently $\delta(q+1)$ takes its maximum value when $q = \frac{2}{n-2}$ and $\delta = 1$. Therefore, since $n \geq 3$, we see that

$$\delta(q+1) \leq \frac{2}{n-2} + 1 \leq 3.$$

Hence, an easy application of the Young inequality in (2.2.10) shows that F verifies the condition (2.1.43) of Theorem 2.12, when $n \geq 3$, as well.

In light of the previous paragraph, we have the following theorem as a consequence of the abstract results of §2.1.

Theorem 2.22. *Assume that all conditions of Theorem 2.21 are satisfied. Let $u(t)$ be the unique solution to the problem (2.2.9). Suppose the initial data satisfy one of the following conditions:*

1.

$$E(0) < 0,$$

or

2.

$$E(0) \geq 0,$$

and

(a)

$$2(u_0, u_1) + a\|\nabla u_0\|^2 + b\|u_0\|^2 > \frac{2q(2+q)}{\max\{a, b\}} \left(1 + \sqrt{1 + \frac{2qm}{(4+q)(\max\{a, b\})^2}} \right) E(0),$$

or

(b)

$$2(u_0, u_1) > 2\sqrt{2}\|u_0\|E(0)^{\frac{1}{2}} + \left(\sqrt{\frac{4+q}{2+q}} - 1 \right) (a\|\nabla u_0\|^2 + b\|u_0\|^2),$$

where

$$E(0) := \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{q+2} \int_{\mathbb{R}^n} |u_0|^{q+2} dx.$$

Then, there exists $t_1 > 0$ such that

$$\|u(t)\|^2 + \int_0^t (a\|\nabla u(s)\|^2 + b\|u(s)\|^2) ds \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

Moreover, if $b \neq 0$, then any global solution $(u(t), u_t(t))$ belongs to $L^\infty(\mathbb{R}^+; H_0^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$.

2.2.1.3 Semilinear damped beam and plate equations

Let us consider the following initial-boundary value problem

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u - a\Delta u_t + bu_t = |u|^q u, & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), & x \in \Omega \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, & t \geq 0, \end{cases} \quad (2.2.11)$$

where $a, b \geq 0$ and $q > 0$ are given numbers. In 1-dimension this equation is referred to as the beam equation, and in 2-dimensions it is called the plate equations, as it either appears as a model for vibrating beams or plates, which are 1- and 2-dimensional objects, respectively.

One can easily show by means of the Sobolev embedding inequality and the Galerkin method that, for initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ the problem (2.2.11) admits a unique solution in the same phase space provided that the exponent of the nonlinearity verifies $q \in [0, \frac{n+4}{n-4}]$, when $n \geq 5$, and for all $q > 0$, when $1 \leq n \leq 4$. We have the following local well-posedness theorem.

Theorem 2.23. *Suppose $q \geq 0$, if $1 \leq n \leq 4$, or $q \in [0, \frac{n+4}{n-4}]$, if $n \geq 5$. Then, for each initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, there exists $T > 0$ such that the problem (2.2.11) admits a unique local solution*

$$u(t) \in \mathcal{E}_T := C(0, T; H_0^2(\Omega)) \cap C^1(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^1(\Omega))$$

in the sense of (2.1.8). Moreover, this solution verifies the energy inequality (2.1.10), where the energy is as in (2.1.9).

This equation can be written in the form (2.1.1) with $P = I$, $A = \Delta_0^2 - \Delta_0$, $Q = -a\Delta_0 + bI$ and $F(u) = |u|^q u$. It is easy to see by the Poincaré inequality that, the embedding inequalities (2.1.19) and (2.1.20) are satisfied for all $a, b \geq 0$, with coefficients $d_1 = 2(\lambda_1^2 + \lambda_1)^{-1}$, $d_2 =$

$2\lambda_1^{-1} \max\{a, b\}$ and $D_1 = D_2 = 0$, and the inequality (2.1.21) holds with $d_3 = a\lambda_1^{-1} + b$ provided that $a + b > 0$.

Next, let us see that the condition (2.1.43) of Theorem 2.12 is also satisfied. First, if $n \geq 5$, since $2 < 2(q+1) \leq \frac{4n}{n-4}$, we have by the L^p -interpolation inequality and the Sobolev embedding inequalities

$$\begin{aligned} \|P^{-\frac{1}{2}}F(u)\| &= \| |u|^q u \| = \|u\|_{L^{2(q+1)}}^{q+1} \leq \|u\|^{(1-\delta)(q+1)} \|u\|_{L^{\frac{4n}{n-4}}}^{\delta(q+1)} \\ &\leq C \|u\|^{(1-\delta)(q+1)} \|\Delta u\|^{\delta(q+1)} \leq M(\|P^{\frac{1}{2}}u\|) \|A^{\frac{1}{2}}u\|^{\delta(q+1)}, \end{aligned} \quad (2.2.12)$$

where $M: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing function, and

$$\delta = \frac{2(q+1) - 2}{\frac{4n}{n-4} - 2} = \frac{q(n-4)}{n+4} \in (0, 1].$$

To proceed from here as before by utilizing the Young inequality in (2.2.12), and verify the condition (2.1.43) of Theorem 2.12, we need that

$$\delta(q+1) \leq 3 \Leftrightarrow q(q+1) \leq 3 \frac{n+4}{n-4},$$

which is true for the admissible values of q , if and only if

$$\begin{cases} 0 < q \leq \frac{1}{2} + \sqrt{\frac{1}{4} + 3 \frac{n+4}{n-4}} < \frac{n+4}{n-4}, & \text{when } 5 \leq n \leq 11, \\ 0 < q \leq \frac{n+4}{n-4}, & \text{when } 12 \leq n. \end{cases} \quad (2.2.13)$$

Finally, if $1 \leq n \leq 4$, and $q > 0$ is arbitrary, we may again use exactly the same approach of §2.2.1.1, and deduce that the condition (2.1.43) of Theorem 2.12 is valid: When $0 < q \leq 2$ we estimate as in the case $n \geq 5$, which is succesful, and when $q > 2$ use the same estimate as was used in §2.2.1.1 for the case $n = 1, 2$ (with $\omega = \Omega$).

A final note is that, although the Theorem 2.12 requires the embedding inequality (2.1.21), which necessitates that $a + b > 0$, by the Remark 2.13 after that theorem we see that the Theorem 2.12 also holds for the case $a + b = 0$, the undamped equation. Thus, we deduce from the abstract results of §2.1 the following theorem.

Theorem 2.24. *Assume that all conditions of Theorem 2.23 are satisfied. Let $u(t)$ be the unique solution to the problem (2.2.11). Suppose the initial data satisfy one of the following conditions:*

1.

$$E(0) < 0,$$

or

2.

$$E(0) \geq 0,$$

and

(a)

$$2(u_0, u_1) + a\|\nabla u_0\|^2 + b\|u_0\|^2 > \frac{2q(2+q)\lambda_1}{\max\{a, b\}} \left(1 + \sqrt{1 + \frac{2q\lambda_1^2}{(4+q)(\lambda_1^2 + \lambda_1)(\max\{a, b\})^2}} \right) E(0),$$

or

(b)

$$2(u_0, u_1) > 2\sqrt{2}\|u_0\|E(0)^{\frac{1}{2}} + \left(\sqrt{\frac{4+q}{2+q}} - 1 \right) (a\|\nabla u_0\|^2 + b\|u_0\|^2),$$

where

$$E(0) := \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{q+2} \int_{\mathbb{R}^n} |u_0|^{q+2} dx.$$

Then, there exists $t_1 > 0$ such that

$$\|u(t)\|^2 + \int_0^t (a\|\nabla u(s)\|^2 + b\|u(s)\|^2) ds \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

Moreover, any global solution $(u(t), u_t(t))$ belongs to $L^\infty(\mathbb{R}^+; H_0^2(\Omega) \times L^2(\Omega))$.

2.2.1.4 Semilinear damped Boussinesq type equations

Consider the following initial-boundary value problem for a semilinear Boussinesq type equation with damping

$$\begin{cases} u_{tt} + \Delta^2 u - \nu \Delta u - a \Delta u_t + b u_t = -\Delta(|u|^q u), & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t)|_{\partial\Omega} = 0, & t \geq 0, \end{cases} \quad (2.2.14)$$

where $a, b, \nu \geq 0$ and $q > 0$ are given numbers, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$.

One can prove the following local well-posedness of the solutions to this problem:

Theorem 2.25. *Suppose $q \geq 0$, if $n \leq 4$, or $q \in [0, \frac{4}{n-4}]$, if $n \geq 5$. Then, for each initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, there exists $T > 0$ such that the problem (2.2.9) admits a unique*

local solution

$$u(t) \in \mathcal{E}_T := C(0, T; H_0^2(\Omega)) \cap C^1(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^1(\Omega))$$

in the sense of (2.1.8). Moreover, this solution verifies the energy inequality (2.1.10), where the energy is as in (2.1.9).

If we apply $(-\Delta_0)^{-1}$ to both sides of the equation, we see that the equation takes the form of (2.1.1) with $P = (-\Delta_0)^{-1}$, $A = -\Delta_0 + \nu I$, $Q = aI + b(-\Delta_0)^{-1}$ and $F(u) = |u|^q u$. Since Ω is bounded, thanks to the Poincaré inequality we see that all of the embedding inequalities (2.1.19)-(2.1.21) hold true, with numbers $d_1 = 2(\lambda_1^{-2} + \nu^{-1}\lambda_1^{-1})$, $d_2 = 2\lambda_1^{-1} \max\{a, \frac{b}{\nu}\}$, $d_3 = a^{-1}\lambda_1^{-1} + b^{-1}$ and $D_1 = D_2 = 0$. Consequently, we deduce that all of the abstract blow-up theorems of §2.1 are applicable to the problem (2.2.14). We have the following theorem:

Theorem 2.26. *Assume that all conditions of Theorem 2.25 are satisfied. Let $u(t)$ be a local solution to the problem (2.2.14). Suppose that the initial data satisfy one of the following conditions:*

1.

$$E(0) < 0,$$

or

2.

$$E(0) \geq 0,$$

and

(a)

$$2((-\Delta)^{-1}u_0, u_1) + a\|u_0\|^2 + b\|u_0\|_{-1}^2 > \frac{2q(2+q)\lambda_1}{\max\{a, \frac{b}{\nu}\}} \left(1 + \sqrt{1 + \frac{2q(1 + \nu^{-1}\lambda_1)}{(4+q)(\max\{a, \frac{b}{\nu}\})^2}} \right) E(0),$$

or

(b)

$$2((-\Delta)^{-1}u_0, u_1) > 2\sqrt{2}\|u_0\|E(0)^{\frac{1}{2}} + \left(\sqrt{\frac{4+q}{2+q}} - 1 \right) (a\|u_0\|^2 + b\|u_0\|_{-1}^2),$$

where

$$E(0) := \frac{1}{2}\|u_1\|_{-1}^2 + \frac{1}{2}\|\nabla u_0\|^2 + \frac{\nu}{2}\|u_0\|^2 - \frac{1}{q+2} \int_{\Omega} |u_0|^{q+2} dx.$$

Then, there exists $t_1 > 0$ such that

$$\|u(t)\|_{-1}^2 + \int_0^t (a\|u(s)\|^2 + b\|u(s)\|_{-1}^2) ds \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

2.2.2 Quasilinear problems

2.2.2.1 Quasilinear structurally damped wave equations

We are interested in the following Dirichlet initial-boundary value problem for a family of quasilinear structurally damped wave equations:

$$\begin{cases} u_{tt} - \nabla [(\nu_0 + \nu|\nabla u|^m) \nabla u] + a(-\Delta)^\theta u_t = |u|^q u, & x \in \Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t)|_{\partial\Omega} = 0, & t \geq 0, \end{cases} \quad (2.2.15)$$

where $\nu_0 \in \mathbb{R}$, $a, \nu > 0$, the damping exponent $\theta \in [0, 1]$ and the exponents $q > m > 0$ are given numbers, and $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a domain with smooth boundary $\partial\Omega$.

When $n > m + 2$, the local well-posedness of this problem is known only for the exponents $q \in [0, \frac{n(m+2)}{2(n-m-2)} - 1]$; and we have local well-posedness for all $q > 0$, when $n \leq m + 2$.

Theorem 2.27. *Suppose $q \geq 0$, if $n \leq m + 2$, or $q \in [0, \frac{n(m+2)}{2(n-m-2)} - 1]$, if $n > m + 2$. Then, for each initial data $(u_0, u_1) \in W_0^{1,m+2}(\Omega) \times L^2(\Omega)$, there exists $T > 0$ such that the problem (2.2.15) admits a unique local solution*

$$u(t) \in \mathcal{E}_T := C(0, T; W_0^{1,m+2}(\Omega)) \cap C^1(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^\theta(0m))$$

in the sense of (2.1.8). Moreover, this solution verifies the energy inequality (2.1.10), where the energy is as in (2.1.9).

Observe that the problem (2.2.15) is of the same form as the abstract problem (2.1.1) with $P = I$, $Q = a(-\Delta_0)^\theta$, $A(u) = -\nu_0 \Delta_0 u + \nu \nabla \cdot (|\nabla u|^m \nabla u)$, and $F(u) = |u|^q u$. In this case, we have

$$\mathcal{A}(u) = \frac{\nu_0}{2} \|\nabla u\|^2 + \frac{\nu}{m+2} \|\nabla u\|_{L^{m+2}}^{m+2}, \quad (2.2.16)$$

which verifies the condition (2.1.3). It is also easy to see that, if $\nu_0 \geq 0$, then A verifies the condition (2.1.5) with $L_1 = 0$, and the condition (2.1.4) with $R_1 = 0$ and $\beta = \frac{m}{4} < \frac{q}{4} = \alpha$, which also implies that F satisfies the condition (2.1.7) (see Remark 2.14). In fact for this

value of β we have

$$(A(u), u) - 2(1 + 2\beta)\mathcal{A}(u) = -\frac{m}{2}\nu_0\|\nabla u\|^2 \leq 0,$$

for each $u \in W_0^{1,m+2}(\Omega)$. Now, suppose $\nu_0 < 0$. Since $q > m$, we may choose $\beta \in (\frac{m}{4}, \frac{q}{4})$. Such a choice, in particular, implies that $\beta < \alpha$, which is required by the condition (2.1.7) on F . After this choice we have

$$(A(u), u) - 2(1 + 2\beta)\mathcal{A}(u) = -2\beta\nu_0\|\nabla u\|^2 - \frac{(4\beta - m)\nu}{m+2}\|\nabla u\|_{L^{m+2}}^{m+2}. \quad (2.2.17)$$

Since Ω is bounded, by consecutive applications of the L^p -embedding inequality and the Young- ε inequality we have for any $\varepsilon > 0$

$$\begin{aligned} -2\beta\nu_0\|\nabla u\|^2 &\leq 2\beta|\nu_0||\Omega|^{\frac{m}{m+2}}\|\nabla u\|_{L^{m+2}}^2 \\ &\leq 2\beta|\nu_0|\left(\frac{m|\Omega|}{(\varepsilon q)^{\frac{2}{m}}(m+2)} + \varepsilon\|\nabla u\|_{L^{m+2}}^{m+2}\right). \end{aligned}$$

Choosing

$$\varepsilon = \frac{(4\beta - m)\nu}{2\beta|\nu_0|(m+2)}$$

in the estimate above, and plugging it into the equality (2.2.17), we deduce that A satisfies for any $\beta \in (\frac{m}{4}, \frac{q}{4})$

$$(A(u), u) - 2(1 + 2\beta)\mathcal{A}(u) \leq R_1, \quad \forall u \in W_0^{1,m+2}(\Omega),$$

with

$$R_1 = \frac{(2\beta|\nu_0|)^{\frac{m+2}{m}}m|\Omega|}{((4\beta - m)\nu q)^{\frac{2}{m}}(m+2)^{\frac{m-2}{m}}}, \quad (2.2.18)$$

which is the assumption (2.1.4). A similar estimate to that used above shows also that A satisfies its final assumption (2.1.5), that is

$$-L_1 \leq \mathcal{A}(u), \quad \forall u \in W_0^{1,m+2}(\Omega),$$

with

$$L_1 = \frac{|\nu_0|^{\frac{m+2}{m}}m|\Omega|}{2(2\nu q)^{\frac{2}{m}}(m+2)^{\frac{m-2}{m}}}. \quad (2.2.19)$$

Indeed, looking at (2.2.16) we see that, by a similar use of the L^p -embedding and Young- ε inequalities, we may kill the negative quadratic term ($\nu_0 < 0$) under the positive term at the expense of the positive constant L_1 .

Let us now investigate the embedding inequalities (2.1.19)-(2.1.21) for the operators at

hand. We only consider the harder case $\nu_0 < 0$. We may, yet again, use the L^p -embedding and Young- ε inequalities in the identity (2.2.16); this time instead of only killing the negative quadratic term, we produce a positive one at the expense of a larger constant:

$$\mathcal{A}(u) + \tilde{D} \geq \frac{|\nu_0|}{2} \|\nabla u\|^2,$$

where

$$\tilde{D} = \frac{(|\nu_0|)^{\frac{m+2}{m}} m |\Omega|}{(\nu q)^{\frac{2}{m}} (m+2)^{\frac{m-2}{m}}} = 2^{\frac{m+2}{m}} L_1 \geq L_1. \quad (2.2.20)$$

Consequently, we see by the Poincaré inequality that all of the embedding inequalities hold with constants $d_1 = 2|\nu_0|^{-1} \lambda_1^{-1}$, $D_1 = 2|\nu_0|^{-1} \lambda_1^{-1} \tilde{D}$, $d_2 = 2a|\nu_0|^{-1} \lambda_1^{\theta-1}$, $D_2 = 2a|\nu_0|^{-1} \lambda_1^{\theta-1} \tilde{D}$, $d_3 = a^{-1} \lambda^{-\theta}$.

Thus, we see that all assumptions on the operators required by the blow-up theorems, Theorem 2.1, Theorem 2.3 and Theorem 2.4, are satisfied, and consequently, their statements are valid for the solutions to the quasilinear problem (2.2.15). We have the following theorem.

Theorem 2.28. *Assume that all conditions of Theorem 2.27 are satisfied. Let $u(t)$ be the unique solution to the problem (2.2.15). Suppose the initial data satisfy one of the following conditions:*

1.

$$B_0 + (q - m)L_1 < 0,$$

or

2.

$$B_0 + (q - m)L_1 \geq 0,$$

and

(a)

$$2(u_0, u_1) + a \|(-\Delta)^{\frac{\theta}{2}} u_0\|^2 > \kappa B_0 + \left(\frac{2}{\kappa |\nu_0| \lambda_1 (4 + q)} + 2a |\nu_0|^{-1} \lambda_1^{\theta-1} \right) \tilde{D},$$

or

(b)

$$2(u_0, u_1) > 2\sqrt{2} \|u_0\| B_0^{\frac{1}{2}} + \left(\sqrt{\frac{4+q}{2+q}} - 1 \right) \|(-\Delta)^{\frac{\theta}{2}} u_0\|^2,$$

where

$$B_0 := 2(q + 2)E(0) + 2R_1,$$

$$E(0) := \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{q+2} \int_{\omega} |u_0|^{q+2} dx,$$

$$\kappa = \frac{2q(2+q)|\nu_0|\lambda_1^{1-\theta}}{a} \left(1 + \sqrt{1 + \frac{2q|\nu_0|\lambda_1^{1-2\theta}}{(4+q)a^2}} \right),$$

and R_1 , L_1 , $\tilde{D}$ are as in (2.2.18), (2.2.19), (2.2.20), respectively. Then, there exists $t_1 > 0$ such that

$$\|u(t)\|^2 + \int_0^t \|(-\Delta)^{\frac{\theta}{2}} u(s)\|^2 ds \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

Let us stress here that, the preceeding theorem is given for the worst case scenario, that is for $\nu_0 < 0$. The conditions on the initial data can be improved upon for the case $\nu_0 \geq 0$. For example, one can take $L_1 = R_1 = 0$, and also improve the constants that appear in the embedding inequalities (2.1.19) and (2.1.20).

Next, let us focus on the condition (2.1.43) of Theorem 2.12, and try to see for which of the permissible values of the exponents q, m indicated in Theorem 2.27, this condition holds. Let us here note that, by the previous discussions, the only remaining ingredient of the Theorem 2.12 is this condition. To avoid separate cases we assume the more difficult case $\nu_0 < 0$. First, let us note that, similar to the previous estimates, applying the L^p -embedding and Young- ε inequalities, we may deduce from (2.2.16) that

$$\mathcal{A}(u) + \tilde{L}_1 \geq \frac{\nu}{2(m+2)} \|\nabla u\|_{L^{m+2}}^{m+2}, \quad \forall u \in W_0^{1,m+2}(\Omega), \quad (2.2.21)$$

where

$$\tilde{L}_1 = \frac{|\nu_0|^{\frac{m+2}{m}} m |\Omega|}{2(\nu q)^{\frac{2}{m}} (m+2)^{\frac{m-2}{m}}}.$$

Note that $\tilde{L}_1 = 2^{\frac{2}{m}} L_1 \geq L_1$, where L_1 is as in (2.2.19), which is necessary for the condition (2.1.43) of Theorem 2.12. Now, assume that $n > m+2$, and $q \in (0, \frac{n(m+2)}{2(n-m-2)} - 1]$. Then, $2 < 2(q+1) \leq \frac{n(m+2)}{n-m-2}$. Hence, after applying the L^p -embedding and the Sobolev embedding inequalities, thanks to the estimate (2.2.21) we have

$$\begin{aligned} \|P^{-\frac{1}{2}} F(u)\| &= \| |u|^q u \| = \|u\|_{L^{2(q+1)}}^{q+1} \\ &\leq C \|u\|_{L^{\frac{n(m+2)}{n-m-2}}}^{q+1} \leq C \|\nabla u\|_{L^{m+2}}^{q+1} \leq C (\mathcal{A}(u) + \tilde{L}_1)^{\frac{q+1}{m+2}}. \end{aligned}$$

To verify the condition (2.1.43) from this inequality, as was done in the semilinear examples, we use the Young inequality, for which we need to restrict the exponent to satisfy $\frac{q+1}{m+2} \leq \frac{3}{2}$, that is $q \leq \frac{3}{2}(m+2) - 1$. To see that, this really is a restriction, observe that for the permissible values q , $n > m+2$, we have $\frac{q+1}{m+2} \in (\frac{1}{m+2}, \frac{n}{2(n-m-2)}]$, and this interval can be

made arbitrarily large by an appropriate choice of $m > 0$ for $n \geq 3$. Also, note that, as $m > 0$, the condition $\frac{q+1}{m+2} \leq \frac{3}{2}$ holds whenever $q \leq 2$.

Finally, suppose $n \leq m$, and let $q > 0$ be any given number. If $q \leq 2$, we may proceed as the preceeding paragraph; see the last sentence of that paragraph. So, assume $q > 2$. Using the L^p -interpolation inequality together with the Sobolev embedding inequality we obtain for any $p \geq 2(q+1)$

$$\begin{aligned} \|P^{-\frac{1}{2}}F(u)\| &= \| |u|^q u \| = \|u\|_{L^{2(q+1)}}^{q+1} \\ &\leq \|u\|^{(1-\delta)(q+1)} \|u\|_{L^p(\Omega)}^{\delta(q+1)} \leq C \|u\|^{(1-\delta)(q+1)} \|\nabla u\|_{L^{m+2}(\Omega)}^{\delta(q+1)}, \end{aligned}$$

where $\delta = \frac{2(q+1)-2}{p-2} = \frac{2q}{p-2}$. Hence, in view of the estimate (2.2.21) we get

$$\|P^{-\frac{1}{2}}F(u)\| \leq M(\|P^{\frac{1}{2}}u\|) (\mathcal{A}(u) + \tilde{L}_1)^{\frac{\delta(q+1)}{m+2}}.$$

Choosing, as in the semilinear case, $p = \frac{2}{3}q(q+1) + 2$, which, when $q > 2$, belongs to the interval $(2(q+1), \infty)$, we have $\delta(q+1) = 3$. Since $m > 0$, this assures that $\frac{\delta(q+1)}{m+2} \leq \frac{3}{2}$, and we conclude with the help of the Young inequality that, in this case the condition (2.1.43) of the Theorem 2.12 is fulfilled for all of the permissible values of the exponent q .

The discussion of the previous two paragraphs gives us the following theorem on the uniform boundedness in time of the global solutions to the problem (2.2.15).

Theorem 2.29. *Assume that all conditions of Theorem 2.27 are satisfied, and further assume $q \leq \frac{3}{2}(m+2) - 1$ holds, if $n > m+2$. Then, any global solution $(u(t), u_t(t))$ to the problem (2.2.15) belongs to $L^\infty(\mathbb{R}^+; W_0^{1,m+2}(\Omega) \times L^2(\Omega))$.*

2.2.2.2 Kirchhoff type equations

As the final example let us consider the following initial-boundary value problem for a damped Kirchhoff type equation,

$$\begin{cases} u_{tt} - M(\|\nabla u\|)\Delta u + bu_t = |u|^q u, & x \in \Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t)|_{\partial\Omega} = 0, & t \geq 0, \end{cases} \quad (2.2.22)$$

where $b \geq 0$ and $q > 0$ are given numbers, and $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$$M(s) = \nu s^m + \nu_0, \quad \forall s \in \mathbb{R}^+,$$

with $0 < m < q$, $\nu > 0$ and $\nu_0 \in \mathbb{R}$.

Theorem 2.30. *Suppose $q \geq 0$, if $n = 1, 2$, or $q \in [0, \frac{2}{n-2}]$, if $n \geq 3$. Then, for each initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists $T > 0$ such that the problem (2.2.1) admits a unique local solution*

$$u(t) \in \mathcal{E}_T := C(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$$

in the sense of (2.1.8). Moreover, this solution verifies the energy inequality (2.1.10), where the energy is as in (2.1.9).

This equation can be written in the form of the abstract equation (2.1.1) with $P = I$, $Q = bI$, $A(u) = M(\|\nabla u\|)\Delta_0 u$ and $F(u) = |u|^q u$. Correspondingly, we have

$$\mathcal{A}(u) = \frac{\nu}{m+2} \|\nabla u\|^{m+2} + \frac{\nu_0}{2} \|\nabla u\|^2,$$

and it is easily seen that the condition (2.1.3) is verified. Note that, this form of $\mathcal{A}$ is very similar to that of the previous example of §2.2.2.1; the only difference is now we have L^2 -norm of the gradient instead of the L^{m+2} -norm. Consequently, the derivation of all of the conditions (2.1.4) and (2.1.5) for this operator $\mathcal{A}$ is similar to that of the previous example, however, this time the Young- ε inequality by itself is sufficient, that is we do not need the L^p -embedding inequality. As a consequence of this last fact, all that changes in the calculations is that, we no more have dependence on the size of the domain. More precisely, in the worst case ($\nu_0 < 0$), $\mathcal{A}$ satisfies the conditions (2.1.4) and (2.1.5) with numbers

$$R_1 = \frac{(2\beta|\nu_0|)^{\frac{m+2}{m}} m}{((4\beta - m)\nu q)^{\frac{2}{m}} (m+2)^{\frac{m-2}{m}}}, \quad (2.2.23)$$

where $\beta \in (\frac{m}{4}, \frac{q}{4})$ is arbitrary to choose, and

$$L_1 = \frac{|\nu_0|^{\frac{m+2}{m}} m}{2(2\nu q)^{\frac{2}{m}} (m+2)^{\frac{m-2}{m}}}. \quad (2.2.24)$$

One has a similar story for the embedding inequalities (2.1.19) and (2.1.20); they can be derived similar to the calculations of §2.2.2.1. Paying attention to the fact that the damping operator Q is different than that of §2.2.2.1, we see that all of the embedding inequalities are valid for the operators at hand, and the corresponding constants in the embeddings are $d_1 = 2|\nu_0|^{-1}\lambda_1^{-1}$, $D_1 = 2|\nu_0|^{-1}\lambda_1^{-1}\tilde{D}$, $d_2 = 2b|\nu_0|^{-1}\lambda_1^{-1}$, $D_2 = 2b|\nu_0|^{-1}\lambda_1^{-1}\tilde{D}$, $d_3 = b^{-1}$, where

$$\tilde{D} = \frac{(|\nu_0|)^{\frac{m+2}{m}} m}{(\nu q)^{\frac{2}{m}} (m+2)^{\frac{m-2}{m}}} = 2^{\frac{m+2}{m}} L_1 \geq L_1. \quad (2.2.25)$$

Finally, let us show that the condition (2.1.43) of the Theorem 2.12 is true for all values of the exponents q , which are allowed by the local well-posedness theorem, Theorem 2.30. Assume $\nu_0 < 0$. Similar to the estimate (2.2.21) we have

$$\mathcal{A}(u) + \tilde{L}_1 \geq \frac{\nu}{2(m+2)} \|\nabla u\|^{m+2}, \quad \forall u \in H_0^1(\Omega), \quad (2.2.26)$$

where

$$\tilde{L}_1 = \frac{|\nu_0|^{\frac{m+2}{m}} m}{2(\nu q)^{\frac{2}{m}} (m+2)^{\frac{m-2}{m}}}.$$

Again, we have $\tilde{L}_1 = 2^{\frac{2}{m}} L_1 \geq L_1$, where L_1 is as in (2.2.24), which is necessary for the condition (2.1.43) of Theorem 2.12. The case $n = 1, 2$ is exactly the same as the case $n \leq m+2$ of §2.2.2.1, so we omit it. Suppose $n \geq 3$, and $q \in [0, \frac{2}{n-2}]$. Then, $2(q+1) \in (2, \frac{2n}{n-2}]$, and consequently, by the L^p -embedding and the Sobolev embedding inequalities, we have thanks to the estimate (2.2.26)

$$\begin{aligned} \|P^{-\frac{1}{2}} F(u)\| &= \| |u|^q u \| = \|u\|_{L^{2(q+1)}}^{q+1} \\ &\leq C \|u\|_{L^{\frac{2n}{n-2}}}^{q+1} \leq C \|\nabla u\|^{q+1} \leq C (\mathcal{A}(u) + \tilde{L}_1)^{\frac{q+1}{m+2}}. \end{aligned}$$

Now, since $q \leq \frac{2}{n-2}$ and $m > 0$, we have

$$\frac{q+1}{m+2} \leq \frac{n}{(n-2)(m+2)} \leq \frac{1}{2} \frac{n}{(n-2)} = \frac{1}{2} \left(1 + \frac{2}{n-2}\right) \leq \frac{3}{2}$$

when $n \geq 3$. Thus, with the help of the Young inequality we easily see that the condition (2.1.43) of the Theorem 2.12 holds.

To conclude, we see that all of the assumptions on the operators required by the abstract blow-up theorems, Theorem 2.1, Theorem 2.3 and Theorem 2.4, and the abstract uniform boundedness theorem, Theorem 2.12 are satisfied, and therefore, their statements are valid for the solutions to the Kirchhof type problem (2.2.22). We have the following theorem.

Theorem 2.31. *Assume that all conditions of Theorem 2.30 are satisfied. Let $u(t)$ be the unique solution to the problem (2.2.22). Suppose the initial data satisfy one of the following conditions:*

1.

$$B_0 + (q-m)L_1 < 0,$$

or

2.

$$B_0 + (q-m)L_1 \geq 0,$$

and

(a)

$$2(u_0, u_1) + b\|u_0\|^2 > \kappa B_0 + \left(\frac{2}{\kappa|\nu_0|\lambda_1(4+q)} + 2b|\nu_0|^{-1}\lambda_1^{-1} \right) \tilde{D},$$

or

(b)

$$2(u_0, u_1) > 2\sqrt{2}\|u_0\|B_0^{\frac{1}{2}} + \left(\sqrt{\frac{4+q}{2+q}} - 1 \right) \|(-\Delta)^{\frac{\theta}{2}}u_0\|^2,$$

where

$$\begin{aligned} B_0 &:= 2(q+2)E(0) + 2R_1, \\ E(0) &:= \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{q+2} \int_{\omega} |u_0|^{q+2} dx, \\ \kappa &= \frac{2q(2+q)|\nu_0|\lambda_1^1}{b} \left(1 + \sqrt{1 + \frac{2q|\nu_0|\lambda_1^1}{(4+q)b^2}} \right), \end{aligned}$$

and R_1 , L_1 , $\tilde{D}$ are as in (2.2.23), (2.2.24), (2.2.25), respectively. Then, there exists $t_1 > 0$ such that

$$\|u(t)\|^2 + \int_0^t \|(-\Delta)^{\frac{\theta}{2}}u(s)\|^2 ds \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

Moreover, any global solution $(u(t), u_t(t))$ to the problem (2.2.22) belongs to $L^\infty(\mathbb{R}^+; H_0^1(\Omega) \times L^2(\Omega))$.

2.3 Quasilinear strongly damped wave equations with convection

In this section we consider the following initial-boundary value problem:

$$u_{tt} - \nabla \left[(a_0 + a|\nabla u|^{m-2}) \nabla u \right] - b\Delta u_t = g(x, t, u, \nabla u) + |u|^{p-2}u, \quad x \in \Omega, t > 0, \quad (2.3.1)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, t > 0, \quad (2.3.2)$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (2.3.3)$$

where

$$p > m \geq 2, a_0, b \geq 0, a > 0 \quad (2.3.4)$$

are given numbers, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. We assume that

$$|g(x, t, u, \mathbf{p})| \leq M_1(|\mathbf{p}|^{m/2} + |u|^{p/2}), \forall (x, t) \in \Omega \times \mathbb{R}^+, u \in \mathbb{R}, \mathbf{p} \in \mathbb{R}^n \quad (2.3.5)$$

and $M_1 > 0$ is some positive number.

This problem doesn't fit in the abstract form of §2.1, in that it contains the convective term g . The following two theorems establish sufficient conditions on the initial data that ensure finite time blow-up of solutions to the problem (2.3.1)-(2.3.3), and they are the main content of the paper [2]. The first theorem deals with the case of nonpositive initial energies:

Theorem 2.32. *Assume that $u(t)$ is a local solution to the problem (2.3.1)-(2.3.3), and suppose that the conditions (2.3.4) and (2.3.5) are satisfied. If the initial functions satisfy the conditions*

$$E_0 := \frac{1}{2}\|u_1\|^2 + \frac{a_0 + \lambda b}{2}\|\nabla u_0\|^2 + \frac{a}{m}\|\nabla u_0\|_m^m + \frac{\lambda^2 - \lambda}{2}\|u_0\|^2 - \frac{1}{p}\|u_0\|_p^p \leq 0, \quad (2.3.6)$$

$$(u_1, u_0) > \frac{b}{2} \left(\sqrt{\frac{p+m+4}{p+m-4}} - 1 \right) \|\nabla u_0\|^2 + (\lambda + 1) \sqrt{\frac{p+m+4}{p+m-4}} \|u_0\|^2, \quad (2.3.7)$$

where

$$\lambda := 2M_1 \sqrt{\frac{\frac{m}{a} + p}{(p-m)(p+m-4)}},$$

then there exists

$$t_1 \leq \frac{2}{\sqrt{(p+m+4)(p+m-4)}} \ln \frac{(u_1, u_0) + \frac{b}{2} \left(\sqrt{\frac{p+m+4}{p+m-4}} - 1 \right) \|\nabla u_0\|^2 + (\lambda + 1) \sqrt{\frac{p+m+4}{p+m-4}} \|u_0\|^2}{(u_1, u_0) - \frac{b}{2} \left(\sqrt{\frac{p+m+4}{p+m-4}} - 1 \right) \|\nabla u_0\|^2 + (\lambda + 1) \sqrt{\frac{p+m+4}{p+m-4}} \|u_0\|^2}$$

such that

$$\|\nabla u(t)\|^2 \rightarrow \infty \text{ as } t \rightarrow t_1^-.$$

Proof. For $\lambda > 0$ if we make the change of variables

$$u(x, t) = v(x, t)e^{\lambda t}, \quad x \in \Omega, t \geq 0, \quad (\text{CoV})$$

in (2.3.1) and divide the resulting equation by $e^{\lambda t}$:

$$\begin{aligned} v_{tt} - \nabla \left[\left(a_0 + \lambda b + ae^{\lambda(m-2)t} |\nabla v|^{m-2} \right) \nabla v \right] + \lambda^2 v + 2\lambda v_t - b\Delta v_t = \\ e^{-\lambda t} g(x, t, e^{\lambda t} v, e^{\lambda t} \nabla v) + e^{\lambda(p-2)t} |v|^{p-2} v. \end{aligned} \quad (2.3.8)$$

Similarly, we have

$$v|_{\Omega} = 0, \quad (2.3.9)$$

and

$$v(x, 0) = u_0(x), v_t(x, 0) = -\lambda u_0(x) + u_1(x), \quad x \in \Omega. \quad (2.3.10)$$

We note that the blow up of v is equivalent to the blow up of u . So, now on we will work on v , and (2.3.8), which v satisfies. Let us multiply (2.3.8) by v_t in $L^2(\Omega)$. By the boundary condition (2.3.9) we get

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \|v_t\|^2 + \frac{a_0 + \lambda b}{2} \|\nabla v\|^2 + \frac{a}{m} e^{\lambda(m-2)t} \|\nabla v\|_m^m + \frac{\lambda^2}{2} \|v\|^2 - \frac{1}{p} e^{\lambda(p-2)t} \|v\|_p^p \right] \\ & + 2\lambda \|v_t\|^2 + b \|\nabla v_t\|^2 - \frac{a\lambda(m-2)}{m} e^{\lambda(m-2)t} \|\nabla v\|_m^m + \frac{\lambda(p-2)}{p} e^{\lambda(p-2)t} \|v\|_p^p \\ & = e^{-\lambda t} (a(x, t), e^{\lambda t} v, e^{\lambda t} \nabla v, v_t)_{L^2(\Omega)}. \end{aligned} \quad (2.3.11)$$

Using the assumption (2.3.5) and the ε -inequality we estimate the last term above in the following way.

$$\begin{aligned} & e^{-\lambda t} (g(x, t), e^{\lambda t} v, e^{\lambda t} \nabla v, v_t) \\ & \leq \varepsilon_1 e^{\lambda(m-2)t} \|\nabla v\|_m^m + \varepsilon_0 e^{\lambda(p-2)t} \|v\|_p^p + \frac{M_1^2}{2} \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_0} \right) \|v_t\|^2. \end{aligned}$$

We define

$$\begin{aligned} H(t) &:= \frac{1}{p} e^{\lambda(p-2)t} \|v\|_p^p - \frac{1}{2} \|v_t\|^2 - \frac{a_0 + \lambda b}{2} \|\nabla v\|^2 \\ & \quad - \frac{a}{m} e^{\lambda(m-2)t} \|\nabla v\|_m^m - \frac{\lambda^2}{2} \|v\|^2, \end{aligned} \quad (2.3.12)$$

which is the negative of the energy, and note that $H(0) = -E_0$ defined in the statement of the theorem. Employing the last inequality, and using this notation, we obtain from (2.3.11) the following inequality

$$\begin{aligned} \frac{d}{dt} H(t) &\geq [\lambda(p-2) - \varepsilon_0 p] H(t) + \left\{ \frac{1}{2} [\lambda(p-2) - \varepsilon_0 p] + 2\lambda \right\} \|v_t\|^2 + \frac{\lambda^2}{2} [\lambda(p-2) - \varepsilon_0 p] \|v\|^2 + \\ & b \|\nabla v_t\|^2 + \frac{1}{2} [\lambda(p-2) - \varepsilon_0 p] (a_0 + b\lambda) \|\nabla v\|^2 + \frac{a}{m} [\lambda(p-2) - \varepsilon_0 p - \lambda(m-2)] e^{\lambda(m-2)t} \|\nabla v\|_m^m + \\ & \varepsilon_0 e^{\lambda(p-2)t} \|v\|_p^p - \varepsilon_0 e^{\lambda(p-2)t} \|v\|_p^p - \varepsilon_1 e^{\lambda(m-2)t} \|\nabla v\|_m^m - \frac{M_1^2}{4} \left[\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_1} \right] \|v_t\|^2. \end{aligned} \quad (2.3.13)$$

Choosing $\varepsilon_0 = \frac{\lambda(p-m)}{2p}$ and $\varepsilon_1 = \frac{a\lambda(p-m)}{2pm}$ we obtain from (2.3.13) the inequality

$$\frac{d}{dt}H(t) \geq \frac{\lambda}{2}(p+m-4)H(t) + \left[\frac{\lambda}{4}(p+m-4) + 2\lambda - \frac{2p(m+a)}{m\lambda(p-m)} \right] \|v_t\|^2 + b\|\nabla v_t\|^2. \quad (2.3.14)$$

Due to the choice

$$\lambda \geq 2M_1 \sqrt{\frac{(\frac{m}{a} + p)}{(p-m)(p+m-4)}}$$

we obtain from (2.3.14)

$$\frac{d}{dt}H(t) \geq \frac{\lambda}{2}(p+m-4)H(t) + 2\lambda\|v_t\|^2 + b\|\nabla v_t\|^2. \quad (2.3.15)$$

Since by (2.3.6) $H(0) \geq 0$, we obtain from (2.3.15) that

$$H(t) \geq e^{\frac{\lambda}{2}(p+m-4)t} H(0) \geq 0.$$

Therefore integrating (2.3.15) we get

$$H(t) \geq 2\lambda \int_0^t \|v_\tau(\tau)\|^2 d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 d\tau. \quad (2.3.16)$$

Let us consider the function

$$\Psi(t) = \|v(t)\|^2 + 2\lambda \int_0^t \|v(\tau)\|^2 d\tau + b \int_0^t \|\nabla v(\tau)\|^2 d\tau + C_0, \quad (2.3.17)$$

where v is a local solution of the problem (2.3.8)-(2.3.10) and C_0 is a positive parameter to be chosen below. It is easy to see that

$$\Psi'(t) = 2(v, v_t) + 4\lambda \int_0^t (v_\tau, v) d\tau + 2b \int_0^t (\nabla v_\tau, \nabla v) d\tau + 2\lambda\|u_0\|^2 + b\|\nabla u_0\|^2 \quad (2.3.18)$$

and

$$\Psi''(t) = 2\|v_t\|^2 + 2(v, v_{tt} + 2\lambda v_t - b\Delta v_t).$$

By using the equation (2.3.8) we obtain from the last equality

$$\begin{aligned} \Psi''(t) = 2\|v_t\|^2 + 2\left(v, \nabla \left[\left(a_0 + \lambda b + ae^{\lambda(m-2)t} |\nabla v|^{m-2} \right) \nabla v \right] \right) - \\ 2\lambda^2 \|v\|^2 + 2e^{-\lambda t} (v, g(x, t, e^{\lambda t} v, e^{\lambda t} \nabla v)) + 2e^{\lambda(p-2)t} \|v\|_p^p. \end{aligned} \quad (2.3.19)$$

Due to the condition (2.3.5) we have

$$2e^{-\lambda t} |(v, g(x, t, e^{\lambda t} v, e^{\lambda t} \nabla v))| \leq \nu_1 e^{\lambda(m-2)t} \|\nabla v\|_m^m + \nu_0 e^{\lambda(p-2)t} \|v\|_p^p + \frac{M_1^2}{2} \left(\frac{1}{\nu_1} + \frac{1}{\nu_0} \right) \|v\|^2,$$

where ν_0 and ν_1 are positive parameters we are going to chose below.

By using this inequality we infer from (2.3.19) the following estimate from below for $\Psi''(t)$

$$\begin{aligned} \Psi''(t) \geq 2\|v_t\|^2 - \left[2\lambda^2 + \frac{M_1^2}{2} \left(\frac{1}{\nu_1} + \frac{1}{\nu_0} \right) \right] \|v\|^2 - 2(a_0 + \lambda b) \|\nabla v\|^2 + \\ (2 - \nu_0) e^{\lambda(p-2)t} \|v\|_p^p - (2a + \nu_1) e^{\lambda(m-2)t} \|v\|_m^m. \end{aligned}$$

By choosing $\nu_0 = \frac{p-m}{p}$ and $\nu_1 = \frac{a}{m}(p-m)$ we get

$$\begin{aligned} \Psi''(t) \geq 2\|v_t\|^2 - \left[2\lambda^2 + \frac{M_1^2(\frac{m}{a} + p)}{2(p-m)} \right] \|v\|^2 - 2(a_0 + \lambda b) \|\nabla v\|^2 + \\ \frac{p+m}{p} e^{\lambda(p-2)t} \|v\|_p^p - \frac{a(p+m)}{m} e^{\lambda(m-2)t} \|v\|_m^m \\ = (p+m)H(t) + \left(2 + \frac{p+m}{2} \right) \|v_t\|^2 + \left(\frac{(p+m)(a_0 + \lambda b)}{2} - 2(a_0 + \lambda b) \right) \|\nabla v\|^2 + \\ \left[\lambda^2 \left(\frac{p+m}{2} - 2 \right) - \frac{M_1^2(\frac{m}{a} + p)}{2(p-m)} \right] \|v\|^2 + \left(\frac{a(p+m)}{m} - \frac{a(p+m)}{m} \right) e^{\lambda(m-2)t} \|v\|_m^m. \quad (2.3.20) \end{aligned}$$

Since

$$\lambda \geq 2M_1 \sqrt{\frac{(\frac{m}{a} + p)}{(p-m)(p+m-4)}}$$

we deduce from (2.3.20) the estimate

$$\Psi''(t) \geq (p+m)H(t) + \left(2 + \frac{p+m}{2} \right) \|v_t\|^2. \quad (2.3.21)$$

As $p+m \geq 2 + \frac{p+m}{2}$ and $H(t) \geq 0$ by using the estimate (2.3.16) in the last inequality we get

$$\Psi''(t) \geq 4(1+\alpha) \left[\|v_t\|^2 + 2\lambda \int_0^t \|v_\tau(\tau)\|^2 d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 d\tau \right], \quad (2.3.22)$$

where $\alpha = \frac{p+m-4}{8}$. As $\Psi(t) \geq 0$ it follows from (2.3.22) that

$$\begin{aligned} \Psi''(t)\Psi(t) - (1+\alpha) [\Psi'(t)] \geq \\ 4(1+\alpha) \left[\|v_t\|^2 + 2\lambda \int_0^t \|v_\tau(\tau)\|^2 d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 d\tau + C_0 \right] \Psi(t) - \\ 4(1+\alpha) \left[(v, v_t) + 2\lambda \int_0^t (v_\tau, v) d\tau + b \int_0^t (\nabla v_\tau, \nabla v) d\tau + \lambda \|u_0\|^2 + \frac{b}{2} \|\nabla u_0\|^2 \right]^2 - \\ 4(1+\alpha) C_0 \Psi(t). \end{aligned}$$

We choose now

$$C_0 = \lambda \|u_0\|^2 + \frac{b}{2} \|\nabla u_0\|^2.$$

By the Cauchy- Schwartz inequality we deduce the following inequality

$$\Psi''(t)\Psi(t) - (1 + \alpha) [\Psi'(t)]^2 \geq -4(1 + \alpha)C_0\Psi(t) \geq -4(1 + \alpha)\Psi^2(t). \quad (2.3.23)$$

Thus, $\Psi(t)$ satisfies the main differential inequality of Lemma 1.1 with $C_1 = 0$, and $C_2 = 4(1 + \alpha)$. The other conditions of the lemma are fulfilled thanks to the assumptions (2.3.7). Thus, Lemma 1.1 holds, and the statement of Theorem 2.32 follows from the statement of the lemma, and the Poincaré inequality. □

Now, we address the case when the initial energy E_0 , as defined in the condition (2.3.6) of the previous theorem, is positive.

Theorem 2.33. *Let $u(t)$ be a local solution to the problem (2.3.1)-(2.3.3) with initial data (u_0, u_1) . Suppose that the conditions (2.3.4) and (2.3.5) are satisfied. If the initial data obey*

$$E_0 > 0,$$

and

$$(u_0, u_1) > \frac{(\lambda + 1)}{2} \kappa \|u_0\|^2 + \frac{b}{4} (\kappa - 2) \|\nabla u_0\|^2 + \frac{1}{2} (\kappa - d_1) E_0, \quad (2.3.24)$$

where

$$\kappa = \sqrt{d_1^2 + \frac{8(p + m)}{p + m - 4}}, \quad (2.3.25)$$

then there exists $t_1 > 0$ such that

$$\|\nabla u(t)\|^2 \rightarrow \infty \text{ as } t \rightarrow t_1^-.$$

Proof. Assume $E_0 = -H(0) > 0$. For the sake of short notation let $d_1 := \frac{\lambda}{2}(p + m - 4)$. For this case the inequality (2.3.15) derived in the previous section still holds. From (2.3.15) we deduce

$$\begin{aligned} H(t) &\geq H(0)e^{d_1 t} + 2\lambda \int_0^t \|v_\tau(\tau)\|^2 e^{d_1(t-\tau)} d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 e^{d_1(t-\tau)} d\tau \\ &\geq -E_0 e^{d_1 t} + 2\lambda \int_0^t \|v_\tau(\tau)\|^2 d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 d\tau. \end{aligned} \quad (2.3.26)$$

This time let us consider the function

$$\Psi(t) = \|v(t)\|^2 + 2\lambda \int_0^t \|v(\tau)\|^2 d\tau + b \int_0^t \|\nabla v(\tau)\|^2 d\tau + C_0 + \beta e^{d_1 t}, \quad (2.3.27)$$

where β, C_0 are positive constants which we will choose later. We have

$$\Psi'(t) = 2(v, v_t) + 4\lambda \int_0^t (v_\tau, v) d\tau + 2b \int_0^t (\nabla v_\tau, \nabla v) d\tau + 2\lambda \|u_0\|^2 + b \|\nabla u_0\|^2 + d_1 \beta e^{d_1 t}, \quad (2.3.28)$$

and

$$\Psi''(t) = 2\|v_t\|^2 + 2(v, v_{tt} + 2\lambda v_t - b\Delta v_t) + d_1^2 \beta e^{d_1 t}. \quad (2.3.29)$$

Doing exactly the same calculations as in the previous theorem we arrive at the following version of (2.3.21)

$$\Psi''(t) \geq (p+m)H(t) + \left(2 + \frac{p+m}{2}\right)\|v_t\|^2 + d_1^2 \beta e^{d_1 t}.$$

Utilizing the energy estimate (2.3.26) in this inequality we obtain

$$\begin{aligned} \Psi''(t) &\geq 4(1+\alpha) \left[\|v_t\|^2 + 2\lambda \int_0^t \|v_\tau(\tau)\|^2 d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 d\tau \right] + (d_1^2 \beta - E_0(p+m)) e^{d_1 t} \\ &= 4(1+\alpha) \left[\|v_t\|^2 + 2\lambda \int_0^t \|v_\tau(\tau)\|^2 d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 d\tau + C_0 + \frac{d_1^2 \beta}{4} e^{d_1 t} \right] \\ &\quad - 4(1+\alpha) C_0 - \left(\alpha d_1^2 + \frac{E_0(p+m)}{\beta} \right) \beta e^{d_1 t}, \end{aligned} \quad (2.3.30)$$

where as before $\alpha = \frac{p+m-4}{8}$. Since $\Psi(t) \geq 0$, it follows from (2.3.30) that

$$\begin{aligned} \Psi''(t)\Psi(t) - (1+\alpha) [\Psi'(t)]^2 &\geq \\ &4(1+\alpha) \left[\|v_t\|^2 + 2\lambda \int_0^t \|v_\tau(\tau)\|^2 d\tau + b \int_0^t \|\nabla v_\tau(\tau)\|^2 d\tau + C_0 + \frac{d_1^2 \beta}{4} e^{d_1 t} \right] \Psi(t) \\ &- 4(1+\alpha) \left[(v, v_t) + 2\lambda \int_0^t (v_\tau, v) d\tau + b \int_0^t (\nabla v_\tau, \nabla v) d\tau + \lambda \|u_0\|^2 + \frac{b}{2} \|\nabla u_0\|^2 + \frac{d_1 \beta}{2} e^{d_1 t} \right]^2 \\ &- \left[4(1+\alpha) C_0 + \left(\alpha d_1^2 + \frac{E_0(p+m)}{\beta} \right) \beta e^{d_1 t} \right] \Psi(t). \end{aligned}$$

We choose

$$C_0 = \lambda \|u_0\|^2 + \frac{b}{2} \|\nabla u_0\|^2,$$

and

$$\beta = E_0.$$

With this choice of C_0 we have by the Cauchy-Schwarz inequality that the sum of the first two terms on the right of above inequality is nonnegative. Also observe that, since $p, m \geq 2$

$$4(1+\alpha) = \frac{p+m+4}{2} \leq p+m \leq \alpha d_1^2 + p+m.$$

In view of these, we deduce from the inequality above the following one:

$$\Psi''(t)\Psi(t) - (1 + \alpha) [\Psi'(t)] \geq -(\alpha d_1^2 + p + m)(C_0 + \beta e^{d_1 t})\Psi(t) \geq -(\alpha d_1^2 + p + m)\Psi^2(t).$$

Hence, we see that the main inequality of the Lemma 1.2 is satisfied. Straightforward calculations show that, the condition (2.3.24) makes sure that all the other assumptions of this lemma are verified by the initial data (u_0, u_1) . Thus, as before, the conclusion of the theorem follows from the statement of the lemma, and the Poincaré inequality. $\square$

Remark 2.34. It is not trivial if there are any initial functions $(u_0, u_1) \in W^{1,m}(\Omega) \cap L^p(\Omega) \times L^2(\Omega)$ satisfying the initial condition (2.3.24). Here, our aim is to show that, in fact for any positive number $K > 0$ there are infinitely many initial data, for which the initial energy verifies

$$E_0 = K, \quad (2.3.31)$$

and the condition (2.3.24) is fulfilled.

Let $K > 0$ be given, and take any $\hat{u}_0 \in W^{1,m}(\Omega) \cap L^p(\Omega)$ such that $\|\hat{u}_0\| = 1$. We set

$$u_0 = \eta_0 \hat{u}_0,$$

and

$$u_1 = \eta_1 u_0,$$

where $\eta_0, \eta_1 > 0$ are numbers to be determined later. Also, we define the positive numbers h_i , $i = 1, 2, 3$, by the relations

$$h_1 := \|\nabla \hat{u}_0\|^2, \quad h_2 := \|\nabla \hat{u}_0\|_m^m, \quad h_3 := \|\hat{u}_0\|_p^p.$$

Consequently, we have

$$\|u_0\|^2 = \eta_0^2, \quad \|\nabla u_0\|^2 = h_1 \eta_0^2, \quad \|\nabla u_0\|_m^m = h_2 \eta_0^m, \quad \|u_0\|_p^p = h_3 \eta_0^p,$$

and in view of these identities the conditions (2.3.24) and (2.3.31) become equivalent to

$$f_1(\eta_0, \eta_1) := \eta_1 - \frac{(\lambda + 1)}{2} \kappa - \frac{b}{4} (\kappa - 2) h_1 - \frac{1}{2} (\kappa - d_1) \frac{E_0}{\eta_0^2} > 0, \quad (2.3.32)$$

and

$$f_2(\eta_0, \eta_1) := \frac{1}{2} \eta_1^2 + \frac{a_0 + \lambda b}{2} h_1 + \frac{a}{m} h_2 \eta_0^{m-2} + \frac{\lambda^2 - \lambda}{2} - \frac{1}{p} h_3 \eta_0^{p-2} - \frac{E_0}{\eta_0^2} = 0. \quad (2.3.33)$$

So, we want to find η_0, η_1 for which (2.3.32), (2.3.33) are satisfied. First, choose η_1 large enough so that $f_1(1, \eta_1) > 0$ and $f_2(1, \eta_1) > 0$. Note that, as f_1 increases as η_0 increases, for this choice of η_1 we have $f_1(\eta_0, \eta_1) > 0$ for any $\eta_0 \geq 1$. Moreover, for fixed η_1 , since $p > m \geq 2$, $f_2(\eta_0, \eta_2) \rightarrow -\infty$ as $\eta_0 \rightarrow +\infty$. Consequently, because f_2 is continuous in its arguments and $f_2(1, \eta_1) > 0$, we can find $\eta_0 > 1$ such that $f_2(\eta_0, \eta_2) = 0$. Choosing η_0 as such we deduce that the pair (η_0, η_1) verify (2.3.32) and (2.3.33). Thus, we conclude that, for any given number $K > 0$ and any function $\hat{u}_0 \in W^{1,m}(\Omega) \cap L^p(\Omega)$ with $\|\hat{u}_0\| = 1$, there are numbers $\eta_0, \eta_1 > 0$ such that the initial functions defined by

$$(u_0, u_1) := (\eta_0 \hat{u}_0, \eta_1 \eta_0 \hat{u}_0)$$

satisfy the condition (2.3.24) of Theorem 2.33, and have the property that the value of the corresponding initial energy is K .

Remark 2.35. Observe that, we allow the case $b = 0$, which corresponds to the undamped wave equations with convection. Moreover, although Theorem 2.32 and Theorem 2.33 above are proven for the case with strong damping, that is we have the term $-b\Delta u_t$ in (2.4.1), it can easily be seen that the proof is the same if instead we have the more general term $b(-\Delta)^\theta$, where $\theta \in [0, 1]$. In other words, after apparent modifications similar versions of Theorem 2.32 and Theorem 2.33 are valid also for the solutions to the initial-boundary value problem to the quasilinear structurally damped wave equations.

2.4 Quasilinear strongly damped Boussinesq type equations with convection

We consider the following initial-boundary value problem for a damped Boussinesq type equation with a convective term

$$u_{tt} + \Delta^2 u - \nu \Delta u - a \Delta u_t + b u_t + \gamma(u) \cdot \nabla u = -\Delta f(u), \quad x \in \Omega, \quad t > 0, \quad (2.4.1)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \quad (2.4.2)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (2.4.3)$$

where $\nu, b \in \mathbb{R}$, $a \geq 0$ are given numbers, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. The nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to satisfy for all $s \in \mathbb{R}$ the conditions

$$f(s)s - (2 + \beta)F(s) \geq -R_0 \quad (2.4.4)$$

for some $\beta > 0$, $R_0 \geq 0$, where

$$F(s) = \int_0^s f(\xi) d\xi,$$

and

$$f'(s) \geq c_1 |s|^{p-2} - c_2, \text{ with } f(0) = 0, \quad p > 2. \quad (2.4.5)$$

The assumption we impose on the nonlinearity $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ in the convective term is that the inequality

$$|\Gamma(s)|^2 \leq c_3 |s|^{2q} + c_4, \quad (2.4.6)$$

holds for all $s \in \mathbb{R}$ and some number $q \geq 0$, where

$$\Gamma(s) = \int_0^s \gamma(\xi) d\xi.$$

The following theorem is the main result of this section, and the sufficient condition provided therein holds true independent of the sign of the initial energy. In what follows $\|\cdot\|_s$ will denote the $H^s(\Omega)$ -norm, $s \in \mathbb{R}$.

Theorem 2.36. *Let u be a local solution to the problem (2.4.1)-(2.4.3), and suppose f satisfies the assumptions (2.4.4), (2.4.5), and γ satisfies (2.4.6) with $q \in [0, \frac{p}{2}]$. Suppose that the initial data satisfy*

$$\begin{aligned} & \left((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1 \right) \\ & > A_1 \|u_0\|_{-1}^2 + A_2 \|u_0\|^2 + \left(\frac{2a}{(4+\beta)c_2\beta^2\lambda} \right)^{\frac{1}{2}} (2C_B + (4+\beta)E_\lambda(u_0, u_1 - \lambda u_0)) \end{aligned} \quad (2.4.7)$$

where

$$\begin{aligned} A_1 &= \left(\frac{2(4+\beta)c_2\lambda}{a} \right)^{\frac{1}{2}} + \left(\frac{(8+\beta)(2\lambda+b)}{\beta} \right)^{\frac{1}{2}} - \frac{2\lambda+b}{2}, \\ A_2 &= \left(\frac{(8+\beta)a}{\beta} \right)^{\frac{1}{2}} + \lambda - \frac{a}{2}, \end{aligned}$$

$E_\lambda(v, v_t)$ is given by (2.4.13), and C_B, λ are as described in (2.4.26), (2.4.24), respectively. Then,

$$\|u(t)\| \rightarrow \infty \quad \text{as } t \rightarrow t_1^-$$

for some $t_1 > 0$.

Proof. To obtain the blow-up of the solution u to the system (2.4.1)-(2.4.3) we investigate the equivalent problem of blow-up of the function

$$v(x, t) = e^{-\lambda t} u(x, t), \quad (2.4.8)$$

where we choose $\lambda > 0$ big later in the sequel. It follows that v is the solution to the following problem

$$v_{tt} + \Delta^2 v - (\nu + \lambda a) \Delta v + (\lambda^2 + b\lambda)v - a \Delta v_t + (2\lambda + b)v_t + \gamma(e^{\lambda t}v) \cdot \nabla v = -e^{-\lambda t} \Delta f(e^{\lambda t}v), \quad (2.4.9)$$

$$v|_{\partial\Omega} = 0, \quad (2.4.10)$$

$$v(x, 0) = u_0(x), \quad v_t(x, 0) = u_1(x) - \lambda u_0(x). \quad (2.4.11)$$

Let us apply $(-\Delta)^{-1}$ to the equation (2.4.9). Multiplying the resulting equality by v_t in $L^2(\Omega)$ we get

$$\begin{aligned} \frac{d}{dt} E_\lambda(t) + \lambda e^{-2\lambda t} \left[(f(e^{\lambda t}v(t)), e^{\lambda t}v(t)) - 2(F(e^{\lambda t}v(t)), 1) \right] \\ + (2\lambda + b) \|v_t(t)\|_{-1}^2 + a \|v_t(t)\|^2 = -((- \Delta)^{-1} [\gamma(e^{\lambda t}v(t)) \cdot \nabla v(t)], v_t(t)), \end{aligned} \quad (2.4.12)$$

where

$$\begin{aligned} E_\lambda(t) = \frac{1}{2} \|v_t(t)\|_{-1}^2 + \frac{1}{2} \|v(t)\|_1^2 + \frac{a\lambda + \nu}{2} \|v(t)\|^2 + \frac{\lambda^2 + b\lambda}{2} \|v(t)\|_{-1}^2 \\ - e^{-2\lambda t} (F(e^{\lambda t}v(t)), 1). \end{aligned} \quad (2.4.13)$$

Using the assumption (2.4.4) on f we obtain

$$\begin{aligned} \frac{d}{dt} E_\lambda(t) + \beta \lambda e^{-2\lambda t} (F(e^{\lambda t}v(t)), 1) - R_0 |\Omega| \lambda e^{-2\lambda t} \\ + (2\lambda + b) \|v_t(t)\|_{-1}^2 + a \|v_t(t)\|^2 \leq -((- \Delta)^{-1} [\gamma(e^{\lambda t}v(t)) \cdot \nabla v(t)], v_t(t)). \end{aligned} \quad (2.4.14)$$

Next, the growth assumption (2.4.5) implies

$$F(s) \geq \frac{c_1}{p(p-1)} |s|^p - c_2 s^2, \quad \text{for all } s \in \mathbb{R}.$$

Thus, we deduce from the inequality (2.4.14) the following inequality

$$\begin{aligned} \frac{d}{dt} E_\lambda(t) + \frac{c_1 \beta \lambda}{p(p-1)} e^{(p-2)\lambda t} \|v(t)\|_{L^p}^p - c_2 \beta \lambda \|v(t)\|^2 \\ - R_0 |\Omega| \lambda e^{-2\lambda t} + (2\lambda + b) \|v_t(t)\|_{-1}^2 + a \|v_t(t)\|^2 \\ \leq -((- \Delta)^{-1} [\gamma(e^{\lambda t}v(t)) \cdot \nabla v(t)], v_t(t)). \end{aligned} \quad (2.4.15)$$

The term on the right hand side of (2.4.15) can be estimated in the following way. Note that

$$\gamma_i(e^{\lambda t}v) v_{x_i} = e^{-\lambda t} \frac{\partial}{\partial x_i} \Gamma_i(e^{\lambda t}v), \quad 1 \leq i \leq n.$$

Consequently, after integration by parts by the triangular inequality we have

$$\begin{aligned} |((-\Delta)^{-1} [\gamma(e^{\lambda t} v(t)) \cdot \nabla v(t)], v_t(t))| &= e^{-\lambda t} \left| \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} \Gamma_i(e^{\lambda t} v(t)), (-\Delta)^{-1} v_t(t) \right) \right| \\ &\leq e^{-\lambda t} \sum_{i=1}^n \left| \left(\Gamma_i(e^{\lambda t} v(t)), \frac{\partial}{\partial x_i} (-\Delta)^{-1} v_t(t) \right) \right|. \end{aligned}$$

Utilizing the Cauchy-Schwarz and Young- ε inequalities consequently in each term of the sum we get

$$\begin{aligned} |((-\Delta)^{-1} [\gamma(e^{\lambda t} v(t)) \cdot \nabla v(t)], v_t(t))| &\leq e^{-\lambda t} \sum_{i=1}^n \|\Gamma_i(e^{\lambda t} v(t))\| \left\| \frac{\partial}{\partial x_i} (-\Delta)^{-1} v_t(t) \right\| \\ &\leq \frac{e^{-2\lambda t}}{4\varepsilon} \sum_{i=1}^n \|\Gamma_i(e^{\lambda t} v(t))\|^2 + \varepsilon \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} (-\Delta)^{-1} v_t(t) \right\|^2 \\ &= \frac{e^{-2\lambda t}}{4\varepsilon} \|\Gamma(e^{\lambda t} v(t))\|^2 + \varepsilon \|\nabla (-\Delta)^{-1} v_t(t)\|^2 \\ &= \frac{e^{-2\lambda t}}{4\varepsilon} \|\Gamma(e^{\lambda t} v(t))\|^2 + \varepsilon \|v_t(t)\|_{-1}^2. \end{aligned} \tag{2.4.16}$$

Moreover, by the assumption (2.4.6) on Γ we deduce that

$$e^{-2\lambda t} \|\Gamma(e^{\lambda t} v(t))\|^2 \leq c_3 e^{2(q-1)\lambda t} \|v(t)\|_{L^{2q}}^{2q} + c_4 |\Omega| e^{-2\lambda t}.$$

Note that by assumption $2q \in [0, p]$. If $q = 0$, we simply have

$$e^{-2\lambda t} \|\Gamma(e^{\lambda t} v(t))\|^2 \leq (c_3 + c_4) |\Omega| e^{-2\lambda t},$$

which is easier to handle, and we omit the proof of this case to avoid separate cases. When $0 < 2q \leq p$, with the help of the L^p -embedding and the Young inequalities, we proceed from the last inequality as

$$\begin{aligned} e^{-2\lambda t} \|\Gamma(e^{\lambda t} v(t))\|^2 &\leq c_3 |\Omega|^{\frac{p-2q}{p}} e^{2(q-1)\lambda t} \|v(t)\|_{L^p}^{2q} + c_4 |\Omega| e^{-2\lambda t} \\ &= c_3 e^{-2\lambda t} |\Omega|^{\frac{p-2q}{p}} e^{2q\lambda t} \|v(t)\|_{L^p}^{2q} + c_4 |\Omega| e^{-2\lambda t} \\ &\leq \frac{2qc_3}{p} e^{(p-2)\lambda t} \|v(t)\|_{L^p}^p + \left(\frac{(p-2q)c_3}{p} + c_4 \right) |\Omega| e^{-2\lambda t}. \end{aligned}$$

Employing this estimate in the inequality (2.4.16), and setting $\varepsilon = 1$ we have

$$\begin{aligned} |((-\Delta)^{-1} [\gamma(e^{\lambda t} v(t)) \cdot \nabla v(t)], v_t(t))| &\leq \frac{qc_3}{2p} e^{(p-2)\lambda t} \|v(t)\|_{L^p}^p + \left(\frac{(p-2q)c_3}{4p} + \frac{c_4}{4} \right) |\Omega| e^{-2\lambda t} + \|v_t(t)\|_{-1}^2. \end{aligned} \tag{2.4.17}$$

Finally, utilizing the resulting estimate in the inequality (2.4.15) we arrive at

$$\begin{aligned} \frac{d}{dt} E_\lambda(t) + \left(\frac{c_1 \beta \lambda}{p(p-1)} - \frac{qc_3}{2p} \right) e^{(p-2)\lambda t} \|v(t)\|_{L^p}^p + (2\lambda + b - 1) \|v_t(t)\|_{-1}^2 + a \|v_t(t)\|^2 \\ \leq c_2 \beta \lambda \|v(t)\|^2 + \left(R_0 \lambda + \frac{c_4}{4} + \frac{(p-2q)c_3}{4p} \right) |\Omega| e^{-2\lambda t}. \end{aligned}$$

Integrating this last inequality over $(0, t)$ we obtain the following estimate for $E_\lambda(t)$

$$\begin{aligned} E_\lambda(t) \leq E_\lambda(0) - \left(\frac{c_1 \beta \lambda}{p(p-1)} - \frac{qc_3}{2p} \right) \int_0^t e^{(p-2)\lambda \tau} \|v(\tau)\|_{L^p}^p d\tau \\ - (2\lambda + b - 1) \int_0^t \|v_t(\tau)\|_{-1}^2 d\tau - a \int_0^t \|v_t(\tau)\|^2 d\tau + c_2 \beta \lambda \int_0^t \|v(\tau)\|^2 d\tau \\ + \left(\frac{R_0}{2} + \frac{c_4}{8\lambda} + \frac{(p-2q)c_3}{8p\lambda} \right) |\Omega| (1 - e^{-2\lambda t}). \end{aligned} \quad (2.4.18)$$

Next, let us apply $(-\Delta)^{-1}$ to the equation (2.4.9), and multiply the resulting equality by v in $L^2(\Omega)$. We obtain the following equality

$$\begin{aligned} \frac{d}{dt} \left\{ \left((-\Delta)^{-\frac{1}{2}} v(t), (-\Delta)^{-\frac{1}{2}} v_t(t) \right) + \frac{2\lambda + b}{2} \|v(t)\|_{-1}^2 + \frac{a}{2} \|v(t)\|^2 \right\} \\ = \|v_t(t)\|_{-1}^2 - \|v(t)\|_1^2 - (a\lambda + \nu) \|v(t)\|^2 \\ - \left((-\Delta)^{-1} [\gamma(e^{\lambda t} v(t)) \cdot \nabla v(t)], v(t) \right) + e^{-2\lambda t} (f(e^{\lambda t} v(t)), e^{\lambda t} v(t)). \end{aligned} \quad (2.4.19)$$

To this end let us introduce the following useful nonnegative function

$$\Psi(t) := \frac{1}{2} \|v(t)\|_{-1}^2 + \frac{2\lambda + b}{2} \int_0^t \|v(\tau)\|_{-1}^2 d\tau + \frac{a}{2} \int_0^t \|v(\tau)\|^2 d\tau + B_0, \quad (2.4.20)$$

where $B_0 \geq 0$ is to be determined later. Then,

$$\Psi'(t) = \left((-\Delta)^{-\frac{1}{2}} v(t), (-\Delta)^{-\frac{1}{2}} v_t(t) \right) + \frac{2\lambda + b}{2} \|v(t)\|_{-1}^2 + \frac{a}{2} \|v(t)\|^2, \quad (2.4.21)$$

or equivalently

$$\begin{aligned} \Psi'(t) = \left((-\Delta)^{-\frac{1}{2}} v(t), (-\Delta)^{-\frac{1}{2}} v_t(t) \right) + (2\lambda + b) \int_0^t \left((-\Delta)^{-\frac{1}{2}} v_t(\tau), (-\Delta)^{-\frac{1}{2}} v(\tau) \right) d\tau \\ + a \int_0^t (v_t(\tau), v(\tau)) d\tau + \frac{2\lambda + b}{2} \|v(0)\|_{-1}^2 + \frac{a}{2} \|v(0)\|^2. \end{aligned} \quad (2.4.22)$$

Due to (2.4.21) we may rewrite (2.4.19) as

$$\begin{aligned} \Psi''(t) = \|v_t(t)\|_{-1}^2 - \|v(t)\|_1^2 - (a\lambda + \nu) \|v(t)\|^2 + e^{-2\lambda t} (f(e^{\lambda t} v(t)), e^{\lambda t} v(t)) \\ - \left((-\Delta)^{-1} [\gamma(e^{\lambda t} v(t)) \cdot \nabla v(t)], v(t) \right). \end{aligned}$$

In view of the assumption (2.4.4) on f we deduce from the last equality

$$\begin{aligned}\Psi''(t) \geq & \|v_t(t)\|_{-1}^2 - \|v(t)\|_1^2 - (a\lambda + \nu)\|v(t)\|^2 + (2 + \beta)e^{-2\lambda t} (F(e^{\lambda t}v(t)), 1) \\ & - R_0|\Omega|e^{-2\lambda t} - ((-\Delta)^{-1} [\gamma(e^{\lambda t}v(t)) \cdot \nabla v(t)], v(t)).\end{aligned}$$

Utilizing the definition of the energy (2.4.13) in the inequality above we obtain

$$\begin{aligned}\Psi''(t) \geq & (2 + \frac{\beta}{4})\|v_t(t)\|_{-1}^2 + \frac{\beta}{4}\|v(t)\|_1^2 + \frac{\beta}{4}(a\lambda + \nu)\|v(t)\|^2 \\ & + (1 + \frac{\beta}{4})(\lambda^2 + b\lambda)\|v(t)\|_{-1}^2 + \frac{\beta}{2}e^{-2\lambda t} (F(e^{\lambda t}v(t)), 1) - (2 + \frac{\beta}{2})E_\lambda(t) \\ & - R_0|\Omega|e^{-2\lambda t} - ((-\Delta)^{-1} [\gamma(e^{\lambda t}v(t)) \cdot \nabla v(t)], v(t)).\end{aligned}$$

Thanks to the growth assumption (2.4.5) on f from the last inequality we may deduce the following inequality

$$\begin{aligned}\Psi''(t) \geq & (2 + \frac{\beta}{4})\|v_t(t)\|_{-1}^2 + \left(\frac{\beta}{4}(a\lambda + \nu) - \frac{c_2\beta}{2}\right)\|v(t)\|^2 \\ & + \frac{c_1\beta}{2p(p-1)}e^{(p-2)\lambda t}\|v(t)\|_{L^p}^p - (2 + \frac{\beta}{2})E_\lambda(t) \\ & - R_0|\Omega|e^{-2\lambda t} - ((-\Delta)^{-1} [\gamma(e^{\lambda t}v(t)) \cdot \nabla v(t)], v(t)).\end{aligned}\tag{2.4.23}$$

We estimate the last term on the right hand side of the inequality above similar as before, see the inequalities (2.4.16) and (2.4.17), but replace $\varepsilon > 0$ by $\frac{1}{\varepsilon}$ in (2.4.16). As an analog of the estimate (2.4.17) we have

$$\begin{aligned}& |((-\Delta)^{-1} [\gamma(e^{\lambda t}v(t)) \cdot \nabla v(t)], v(t))| \\ & \leq \varepsilon \frac{qc_3}{2p}e^{(p-2)\lambda t}\|v(t)\|_{L^p}^p + \varepsilon \left(\frac{(p-2q)c_3}{4p} + \frac{c_4}{4}\right)|\Omega|e^{-2\lambda t} + \frac{1}{\varepsilon}\|v(t)\|_{-1}^2 \\ & \leq \varepsilon \frac{qc_3}{2p}e^{(p-2)\lambda t}\|v(t)\|_{L^p}^p + \varepsilon \left(\frac{(p-2q)c_3}{4p} + \frac{c_4}{4}\right)|\Omega|e^{-2\lambda t} + \frac{1}{\varepsilon\lambda_1}\|v(t)\|^2,\end{aligned}$$

where in the last line we have used the Poincaré inequality. Making use of this inequality inside the inequality (2.4.23) we find

$$\begin{aligned}\Psi''(t) \geq & (2 + \frac{\beta}{4})\|v_t(t)\|_{-1}^2 + \left(\frac{\beta}{4}(a\lambda + \nu) - \frac{c_2\beta}{2} - \frac{1}{\varepsilon\lambda_1}\right)\|v(t)\|^2 \\ & + \left(\frac{c_1\beta}{2p(p-1)} - \varepsilon \frac{qc_3}{2p}\right)e^{(p-2)\lambda t}\|v(t)\|_{L^p}^p \\ & - (2 + \frac{\beta}{2})E_\lambda(t) - \left(R_0 + \varepsilon \left(\frac{(p-2q)c_3}{4p} + \frac{c_4}{4}\right)\right)|\Omega|e^{-2\lambda t}.\end{aligned}$$

Finally, we use the estimate (2.4.18) for $E_\lambda(t)$ to arrive at the following estimate from below for $\Psi''(t)$

$$\begin{aligned}
\Psi''(t) \geq & (2 + \frac{\beta}{4}) \|v_t(t)\|_{-1}^2 + \left(\frac{\beta}{4}(a\lambda + \nu) - \frac{c_2\beta}{2} - \frac{1}{\varepsilon\lambda_1} \right) \|v(t)\|^2 \\
& + \left(\frac{c_1\beta}{2p(p-1)} - \varepsilon \frac{qc_3}{2p} \right) e^{(p-2)\lambda t} \|v(t)\|_{L^p}^p \\
& + (2 + \frac{\beta}{2}) \left(\frac{c_1\beta\lambda}{p(p-1)} - \frac{qc_3}{2p} \right) \int_0^t e^{(p-2)\lambda\tau} \|v(\tau)\|_{L^p}^p d\tau \\
& + (2 + \frac{\beta}{2})(2\lambda + b - 1) \int_0^t \|v_t(\tau)\|_{-1}^2 d\tau + (2 + \frac{\beta}{2})a \int_0^t \|v_t(\tau)\|^2 d\tau \\
& - (2 + \frac{\beta}{2}) \left(\frac{R_0}{2} + \frac{c_4}{8\lambda} + \frac{(p-2q)c_3}{8p\lambda} \right) (1 - e^{-2\lambda t}) \\
& - \left(R_0 + \varepsilon \left(\frac{(p-2q)c_3}{4p} + \frac{c_4}{4} \right) \right) |\Omega| e^{-2\lambda t} \\
& - (2 + \frac{\beta}{2}) E_\lambda(0) - (2 + \frac{\beta}{2}) c_2 \beta \lambda \int_0^t \|v(\tau)\|^2 d\tau.
\end{aligned}$$

In the inequality above we first choose $\varepsilon > 0$ small enough so that

$$\frac{c_1\beta}{2p(p-1)} - \varepsilon \frac{qc_3}{2p} \geq 0.$$

Once ε is fixed we choose λ large enough so that

$$\begin{aligned}
2\lambda + b - 1 & \geq 0, \\
\frac{\beta}{4}(a\lambda + \nu) - \frac{c_2\beta}{2} - \frac{1}{\varepsilon\lambda_1} & \geq 0, \\
\frac{c_1\beta\lambda}{p(p-1)} - \frac{qc_3}{2p} & \geq 0.
\end{aligned} \tag{2.4.24}$$

Ater these choices we reach to the following inequality

$$\begin{aligned}
\Psi''(t) \geq & (2 + \frac{\beta}{4}) \|v_t(t)\|_{-1}^2 + (2 + \frac{\beta}{2})(2\lambda + b) \int_0^t \|v_t(\tau)\|_{-1}^2 d\tau \\
& + (2 + \frac{\beta}{4})a \int_0^t \|v_t(\tau)\|^2 d\tau - C_B \\
& - (2 + \frac{\beta}{2}) E_\lambda(0) - (2 + \frac{\beta}{2}) c_2 \beta \lambda \int_0^t \|v(\tau)\|^2 d\tau,
\end{aligned} \tag{2.4.25}$$

where C_B is a positive constant such that

$$\begin{aligned}
C_B \geq & (2 + \frac{\beta}{2}) \left(\frac{R_0}{2} + \frac{c_4}{8\lambda} + \frac{(p-2q)c_3}{8p\lambda} \right) (1 - e^{-2\lambda t}) \\
& + \left(R_0 + \varepsilon \left(\frac{(p-2q)c_3}{4p} + \frac{c_4}{4} \right) \right) |\Omega| e^{-2\lambda t}
\end{aligned} \tag{2.4.26}$$

for all $t \geq 0$. We add $\frac{(4+\beta)c_2\beta\lambda}{a}\Psi(t)$ to both sides of (2.4.25). Noting that by definition (2.4.20) of $\Psi(t)$

$$\frac{(4+\beta)c_2\beta\lambda}{a}\Psi(t) - (2 + \frac{\beta}{2})c_2\beta\lambda \int_0^t \|v(\tau)\|^2 d\tau \geq \frac{(4+\beta)c_2\beta\lambda B_0}{a},$$

we deduce from (2.4.25)

$$\begin{aligned} \Psi''(t) + \frac{(4+\beta)c_2\beta\lambda}{a}\Psi(t) &\geq (2 + \frac{\beta}{4})\|v_t(t)\|_{-1}^2 + (2 + \frac{\beta}{2})(2\lambda + b) \int_0^t \|v_t(\tau)\|_{-1}^2 d\tau \\ &\quad + (2 + \frac{\beta}{4})a \int_0^t \|v_t(\tau)\|^2 d\tau - C_B \\ &\quad - (2 + \frac{\beta}{2})E_\lambda(0) + \frac{(4+\beta)c_2\beta\lambda B_0}{a}. \end{aligned} \quad (2.4.27)$$

Now, by successive applications of the Cauchy-Schwarz inequality to (2.4.22) we estimate $\Psi'(t)$ in the following way

$$\begin{aligned} \Psi'(t) &\leq \|v(t)\|_{-1}\|v_t(t)\|_{-1} + (2\lambda + b) \left(\int_0^t \|v(\tau)\|_{-1}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|v_t(\tau)\|_{-1}^2 d\tau \right)^{\frac{1}{2}} \\ &\quad + a \left(\int_0^t \|v(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|v_t(\tau)\|^2 d\tau \right)^{\frac{1}{2}} + \frac{2\lambda + b}{2}\|v(0)\|_{-1}^2 + \frac{a}{2}\|v(0)\|^2 \\ &\leq \left[\frac{1}{2}\|v(t)\|_{-1}^2 + \frac{2\lambda + b}{2} \int_0^t \|v(\tau)\|_{-1}^2 d\tau + \frac{a}{2} \int_0^t \|v(\tau)\|^2 d\tau + B_0 \right]^{\frac{1}{2}} \\ &\quad \left[2\|v_t(t)\|_{-1}^2 + 2(2\lambda + b) \int_0^t \|v(\tau)\|^2 d\tau + 2a \int_0^t \|v_t(\tau)\|^2 d\tau + B_1 \right]^{\frac{1}{2}} \\ &= [\Psi(t)]^{\frac{1}{2}} \left[2\|v_t(t)\|_{-1}^2 + 2(2\lambda + b) \int_0^t \|v(\tau)\|^2 d\tau + 2a \int_0^t \|v_t(\tau)\|^2 d\tau + B_1 \right]^{\frac{1}{2}}, \end{aligned}$$

where

$$B_1 = \frac{2\lambda + b}{B_0}\|v(0)\|_{-1}^4 + \frac{a}{B_0}\|v(0)\|^4. \quad (2.4.28)$$

Therefore, in light of the inequality (2.4.27) we deduce from the inequality above

$$\begin{aligned} (1 + \frac{\beta}{8})[\Psi'(t)]^2 &\leq \Psi(t) \left[(2 + \frac{\beta}{4})\|v_t(t)\|_{-1}^2 + (2 + \frac{\beta}{4})(2\lambda + b) \int_0^t \|v(\tau)\|^2 d\tau \right. \\ &\quad \left. + (2 + \frac{\beta}{4})a \int_0^t \|v_t(\tau)\|^2 d\tau + (1 + \frac{\beta}{8})B_1 \right] \\ &\leq \Psi(t) \left[\Psi''(t) + \frac{(4+\beta)c_2\beta\lambda}{a}\Psi(t) + B_2 \right], \end{aligned} \quad (2.4.29)$$

where

$$B_2 = (1 + \frac{\beta}{8})B_1 + C_B + (2 + \frac{\beta}{2})E_\lambda(0) - \frac{(4+\beta)c_2\beta\lambda B_0}{a}. \quad (2.4.30)$$

It is easy to see from (2.4.28), (2.4.30) that for any choice of $B_0 \geq A_0$, where

$$A_0 = \frac{a}{2(4+\beta)c_2\beta\lambda} \left[C_B + \left(2 + \frac{\beta}{2}\right)E_\lambda(0) + \left(\left(C_B + \left(2 + \frac{\beta}{2}\right)E_\lambda(0)\right)^2 + \frac{(4+\frac{\beta}{2})(4+\beta)c_2\beta\lambda}{a} ((2\lambda+b)\|v(0)\|_{-1}^4 + a\|v(0)\|^4) \right)^{\frac{1}{2}} \right]$$

we have $B_2 \leq 0$. Since

$$\begin{aligned} & \left(\left(C_B + \left(2 + \frac{\beta}{2}\right)E_\lambda(0)\right)^2 + \frac{(4+\frac{\beta}{2})(4+\beta)c_2\beta\lambda}{a} \cdot ((2\lambda+b)\|v(0)\|_{-1}^4 + a\|v(0)\|^4) \right)^{\frac{1}{2}} \\ & \leq C_B + \left(2 + \frac{\beta}{2}\right)E_\lambda(0) + \left(\frac{(4+\frac{\beta}{2})(4+\beta)c_2\beta\lambda(2\lambda+b)}{a} \right)^{\frac{1}{2}} \|v(0)\|_{-1}^2 \\ & + \left((4+\frac{\beta}{2})(4+\beta)c_2\beta\lambda \right)^{\frac{1}{2}} \|v(0)\|^2, \end{aligned}$$

the choice

$$\begin{aligned} B_0 = \frac{a}{2(4+\beta)c_2\beta\lambda} & \left[2C_B + (4+\beta)E_\lambda(0) + \left(\frac{(4+\frac{\beta}{2})(4+\beta)c_2\beta\lambda(2\lambda+b)}{a} \right)^{\frac{1}{2}} \|v(0)\|_{-1}^2 \right. \\ & \left. + \left((4+\frac{\beta}{2})(4+\beta)c_2\beta\lambda \right)^{\frac{1}{2}} \|v(0)\|^2 \right] \end{aligned} \quad (2.4.31)$$

makes $B_2 \leq 0$. Thus, we conclude from (2.4.29) that the solution $v(t)$ to the problem (2.4.9)-(2.4.11) satisfies for all $t \geq 0$

$$\left(1 + \frac{\beta}{8}\right) [\Psi'(t)]^2 \leq \Psi''(t)\Psi(t) + \frac{(4+\beta)c_2\beta\lambda}{a} \Psi^2(t), \quad (2.4.32)$$

which is the main differential inequality of the Lemma 1.2 with

$$\alpha = \frac{\beta}{8}, \quad C_1 = 0, \quad C_2 = \frac{(4+\beta)c_2\beta\lambda}{a}.$$

Thus, the conclusions of Lemma 1.2 hold for $\Psi(t)$ provided that

$$\Psi'(0) > \left(\frac{8(4+\beta)c_2\lambda}{a} \right)^{\frac{1}{2}} \Psi(0),$$

or equivalently

$$\begin{aligned}
& \left((-\Delta)^{-\frac{1}{2}} v(0), (-\Delta)^{-\frac{1}{2}} v_t(0) \right) \\
& > \left[\left(\frac{2(4+\beta)c_2\lambda}{a} \right)^{\frac{1}{2}} - \frac{2\lambda+b}{2} \right] \|v(0)\|_{-1}^2 - \frac{a}{2} \|v(0)\|^2 + \left(\frac{8(4+\beta)c_2\lambda}{a} \right)^{\frac{1}{2}} B_0 \\
& = \left[\left(\frac{2(4+\beta)c_2\lambda}{a} \right)^{\frac{1}{2}} + \left(\frac{(8+\beta)(2\lambda+b)}{\beta} \right)^{\frac{1}{2}} - \frac{2\lambda+b}{2} \right] \|v(0)\|_{-1}^2 \\
& \quad + \left[\left(\frac{(8+\beta)a}{\beta} \right)^{\frac{1}{2}} - \frac{a}{2} \right] \|v(0)\|^2 + \left(\frac{2a}{(4+\beta)c_2\beta^2\lambda} \right)^{\frac{1}{2}} (2C_B + (4+\beta)E_\lambda(0)).
\end{aligned}$$

The assumption (2.4.7) of the theorem is equivalent to this one, and is derived from this inequality via (2.4.11). Hence, we deduce that, under the given assumptions on the initial functions, $\Psi(t)$ blows up in finite time, which by the Poincaré inequality implies the statement of the theorem. □

Remark 2.37. Observe that, we allow the case $a = 0$, which corresponds to the undamped Boussinesq type equations with convection. Moreover, although Theorem 2.36 above is proven for the case with strong damping, that is we have the term $-a\Delta u_t$ in (2.4.1), it can easily be seen that the proof is the same if instead we have the more general term $a(-\Delta)^\theta$, where $\theta \in [0, 1]$. In other words, after apparent modifications a similar version of Theorem 2.36 holds also for the solutions to the initial-boundary value problem to the quasilinear structurally damped Boussinesq type equations.

2.5 The potential well theory

In this section we present the concept of a potential well in the special case of the strongly damped wave equations

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) = |u(x, t)|^q u(x, t), & x \in \Omega, \ t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, \ t \geq 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (2.5.1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $q > 0$ is a given number. The potential well theory was first developed in [35] for the Dirichlet initial-boundary value problem for the semilinear wave equations, and later it was also established for many other equations and types of problems.

The local well-posedness of the problem (2.5.1) in the space $\mathcal{E} := H_0^1(\Omega) \times L^2(\Omega)$ is classical when $0 < q \leq \frac{2}{n-2}$, which we will now on assume to hold. Consequently, $H_0^1(\Omega) \subset L^{q+2}(\Omega)$, and we have the Sobolev embedding inequality

$$\|\phi\|_{L^{q+2}} \leq \kappa_s \|\nabla \phi\|, \quad \forall \phi \in H_0^1(\Omega), \quad (2.5.2)$$

where $\kappa_s > 0$ is the best (smallest) constant for which (2.5.2) holds. To this end, for a given local solution $(u(t), u_t(t)) \in C(0, T; \mathcal{E})$, let us introduce the total potential energy

$$J(u(t)) := \frac{1}{2} \|\nabla u(t)\|^2 - \frac{1}{q+2} \|u(t)\|_{L^{q+2}}^{q+2}, \quad (2.5.3)$$

and the total energy

$$E(u(t), u_t(t)) := \frac{1}{2} \|u_t(t)\|^2 + J(u(t)). \quad (2.5.4)$$

Also, let us define the functional

$$I(u(t)) := \|\nabla u(t)\|^2 - \|u(t)\|_{L^{q+2}}^{q+2}, \quad (2.5.5)$$

and the sets

$$\mathcal{U} := \{\phi \in H_0^1(\Omega) : I(\phi) < 0\}, \quad (2.5.6)$$

$$\mathcal{S} := \{\phi \in H_0^1(\Omega) : I(\phi) > 0\}, \quad (2.5.7)$$

and

$$\mathcal{N} := \{\phi \in H_0^1(\Omega) : I(\phi) = 0\} / \{0\}. \quad (2.5.8)$$

By the implicit function theorem, the set $\mathcal{N}$ is a submanifold of $H_0^1(\Omega)$ of codimension 1, and is referred to as the Nehari manifold in the literature.

Furthermore, let us define for $\phi \in H_0^1(\Omega)/\{0\}$ the functional

$$K_s(\phi) := \frac{\|\phi\|_{L^{q+2}}}{\|\nabla \phi\|}. \quad (2.5.9)$$

Note that by the Sobolev embedding inequality (2.5.2) we necessarily have

$$0 < K_s(\phi) \leq \kappa_s,$$

for any $\phi \in H_0^1(\Omega)/\{0\}$. Now, consider the function $h : (0, \kappa_s] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ given by

$$h(\kappa, r) := \frac{1}{2} r^2 - \frac{\kappa^{q+2}}{q+2} r^{q+2}. \quad (2.5.10)$$

Observe that, by the definitions (2.5.9), (2.5.3) and (2.5.5) we have

$$J(u(t)) = h(K_s(u(t)), \|\nabla u(t)\|), \quad (2.5.11)$$

and

$$I(u(t)) = \frac{\partial h}{\partial r}(K_s(u(t)), \|\nabla u(t)\|) \|\nabla u(t)\|, \quad (2.5.12)$$

whenever $u(t) \neq 0$.

In the following lemma we present some properties of the function h that are important for our investigation.

Lemma 2.38. *Let $h : (0, \kappa_s] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be as in (2.5.10), and let $0 \leq \delta < d$, where*

$$d := \frac{q}{2(q+2)} \kappa_s^{-\frac{2(q+2)}{q}}. \quad (2.5.13)$$

Then, the set $h^{-1}((-\infty, \delta]) \subset (0, \kappa_s] \times \mathbb{R}^+$ is the union of two disjoint nonempty connected sets, i.e. $h^{-1}((-\infty, \delta]) = A_\delta \cup B_\delta$, such that $A_\delta \subset (0, \kappa_s] \times [0, r_*]$ and $B_\delta \subset (0, \kappa_s] \times [r^*, \infty)$ with $r_* < r_0 < r^*$, where

$$r_0 = \kappa_s^{-\frac{q+2}{q}}. \quad (2.5.14)$$

Moreover, there exists a positive constant $C_\delta > 0$ such that

$$\frac{\partial h}{\partial r}(\kappa, r)r \Big|_{A_\delta} \geq C_\delta r^2, \quad (2.5.15)$$

and

$$\sup_{(\kappa, r) \in B_\delta} \frac{\partial h}{\partial r}(\kappa, r)r = \frac{\partial h}{\partial r}(\kappa_s, r^*)r^* < 0. \quad (2.5.16)$$

Proof. First note that

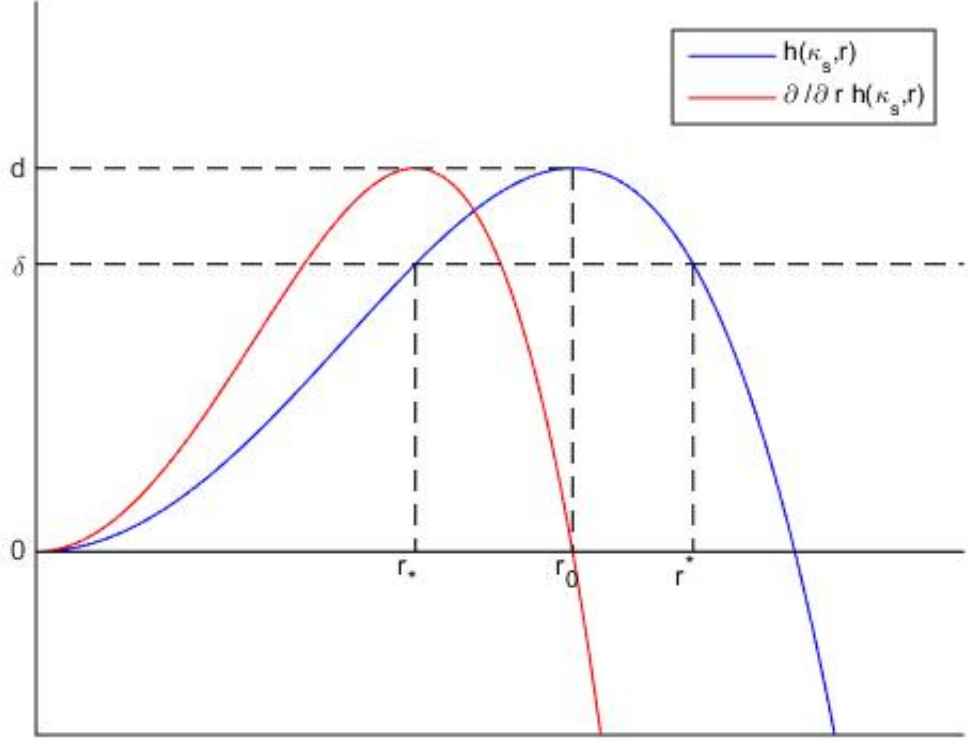
$$h(\kappa_s, r) = \frac{1}{2}r^2 - \frac{\kappa_s^{q+2}}{q+2}r^{q+2} \leq \frac{1}{2}r^2 - \frac{\kappa^{q+2}}{q+2}r^{q+2} = h(\kappa, r)$$

for each $\kappa \in (0, \kappa_s]$ and $r \in \mathbb{R}^+$. Moreover, the function $h(\kappa_s, \cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}$ takes its global maximum at $r = r_0$ with the value $h(\kappa_s, r_0) = d$, where d, r_0 are as in (2.5.13), (2.5.14); is increasing when $r \in (0, r_0)$ and decreasing when $r \in (r_0, \infty)$, see Figure 2.1. It follows that, given $0 \leq \delta < d$,

$$(h(\kappa_s, \cdot))^{-1}((-\infty, \delta]) = [0, r_*] \cup [r^*, \infty),$$

where $\{r_*, r^*\} = (h(\kappa_s, \cdot))^{-1}(\{\delta\})$ and $r_* < r_0 < r^*$. Since $h(\kappa_s, \cdot) \leq h(\kappa, \cdot)$ for each $\kappa \in (0, \kappa_s]$, we have

$$(h(\kappa, \cdot))^{-1}((-\infty, \delta]) \subset (h(\kappa_s, \cdot))^{-1}((-\infty, \delta]) = [0, r_*] \cup [r^*, \infty).$$

FIGURE 2.1: The graphs of the functions $h(\kappa_s, r)$ and $\frac{\partial h}{\partial r}(\kappa_s, r)$.

Hence, we see that the first part of the statement of the lemma holds with

$$A_\delta = \bigcup_{\kappa \in (0, \kappa_s]} \{\kappa\} \times (h(\kappa, \cdot))^{-1}((-\infty, \delta]) \cap [0, r_*] \subset (0, \kappa_s] \times [0, r_*],$$

and

$$B_\delta = \bigcup_{\kappa \in (0, \kappa_s]} \{\kappa\} \times (h(\kappa, \cdot))^{-1}((-\infty, \delta]) \cap [r^*, \infty) \subset (0, \kappa_s] \times [r^*, \infty).$$

Next, the inequalities

$$\inf_{(\kappa, r) \in A_\delta} \frac{\partial h}{\partial r}(\kappa, r)r \geq 0, \quad (2.5.17)$$

$$\sup_{(\kappa, r) \in B_\delta} \frac{\partial h}{\partial r}(\kappa, r)r \leq 0, \quad (2.5.18)$$

and the fact that $(\kappa_s, r^*) \in B_\delta$ with $\frac{\partial h}{\partial r}(\kappa_s, r^*)r^* < 0$ follow easily from the definitions of A_δ and B_δ and the fact that the functions $h(\kappa, \cdot)$, $\kappa \in (0, \kappa_s]$, have the same basic shape, namely they satisfy $h(\kappa, 0) = 0$, monotonely increase to a maximum value greater than δ , and then monotonely decrease to negative infinity.

We want to improve upon the inequalities (2.5.17) and (2.5.18) to obtain the estimates (2.5.15) and (2.5.16), respectively. Let us first start by showing (2.5.15). Observe that

$$\frac{\partial h}{\partial r}(\kappa_s, r) = r - \kappa_s^{q+2} r^{q+1} \leq r - \kappa^{q+2} r^{q+1} = \frac{\partial h}{\partial r}(\kappa, r), \quad \forall (\kappa, r) \in (0, \kappa_s] \times \mathbb{R}^+.$$

Thus, to prove (2.5.15) it suffices to show the existence of a $C_\delta > 0$ such that

$$\left. \frac{\partial h}{\partial r}(\kappa_s, r) \right|_{\{r: (\kappa_s, r) \in A_\delta\}} \geq C_\delta r. \quad (2.5.19)$$

Note that, with our previous notation we have

$$\{r : (\kappa_s, r) \in A_\delta\} = [0, r_*], \quad (2.5.20)$$

and $r_* > 0$ provided that $\delta > 0$, which we may assume without loss of generality, as in that case $A_\delta = (0, \kappa_s] \times \{0\}$ and (2.5.15) holds trivially. In this case we necessarily have

$$\delta_* := \frac{\partial h}{\partial r}(\kappa_s, r_*) > 0, \quad (2.5.21)$$

where the number δ_* depends on δ and decreases to zero as δ increases to d . Now, since $q > 0$, the function defined by

$$g(r) := \kappa_s^{q+2} r^{q+1} \quad (2.5.22)$$

is convex for all $r \geq 0$, and consequently we have for all $r \in [0, r_*]$

$$g(r) \leq \frac{g(r_*) - g(0)}{r_*} r + g(0) = \frac{g(r_*)}{r_*} r.$$

Hence, from the definitions (2.5.10) and (2.5.22) of h and g , respectively, we conclude that

$$\frac{\partial h}{\partial r}(\kappa_s, r) = r - g(r) \geq \left(1 - \frac{g(r_*)}{r_*}\right) r, \quad \forall r \in [0, r_*].$$

Thus, in view of (2.5.20) we see that the estimate (2.5.19) holds with

$$C_\delta = 1 - \frac{g(r_*)}{r_*}.$$

All that remains is to show that $C_\delta > 0$, which follows immediately from (2.5.21), as we have

$$C_\delta = 1 - \frac{g(r_*)}{r_*} = \frac{1}{r_*} \frac{\partial h}{\partial r}(\kappa_s, r_*) = \frac{\delta_*}{r_*} > 0.$$

This proves the estimate (2.5.15).

Next, let us prove the estimate (2.5.16). Another common feature of the functions $h(\kappa_s, \cdot)$, $\kappa \in (0, \kappa_s]$, is that they all start convex, but then become concave to remain so afterwards,

and the inflection point comes before the maximum point. This, in particular, implies that

$$\frac{\partial^2 h}{\partial r^2}(\kappa, r) \leq 0, \quad \forall (\kappa, r) \in B_\delta,$$

and we deduce for each $\kappa \in (0, \kappa_s]$ that

$$\sup_{\{r: (\kappa, r) \in B_\delta\}} \frac{\partial h}{\partial r}(\kappa, r) = \sup_{[r^*(\kappa), \infty)} \frac{\partial h}{\partial r}(\kappa, r) = \frac{\partial h}{\partial r}(\kappa, r^*(\kappa)) < 0, \quad (2.5.23)$$

where

$$r^*(\kappa) := \inf \{r : (\kappa, r) \in B_\delta \cap \{\kappa\} \times [r^*, \infty)\}.$$

From (2.5.23) we easily deduce

$$\sup_{\{r: (\kappa, r) \in B_\delta\}} \frac{\partial h}{\partial r}(\kappa, r)r = \frac{\partial h}{\partial r}(\kappa, r^*(\kappa))r^*(\kappa) < 0, \quad \forall \kappa \in (0, \kappa_s]. \quad (2.5.24)$$

Note also that for all $\kappa \in (0, \kappa_s]$ we have

$$r^*(\kappa) \geq r^*(\kappa_s) = r^*, \quad (2.5.25)$$

and also

$$h(\kappa, r^*(\kappa)) = \delta. \quad (2.5.26)$$

Now, by definition for all $(\kappa, r) \in (0, \kappa_s] \times \mathbb{R}^+$ we have

$$\frac{\partial h}{\partial r}(\kappa, r)r = r^2 - \kappa^{q+2}r^{q+2} = (q+2)h(\kappa, r) - \frac{q}{2}r^2.$$

Hence, thanks to the inequality (2.5.25) and the identity (2.5.26), we deduce from the last equality

$$\begin{aligned} \frac{\partial h}{\partial r}(\kappa, r^*(\kappa))r^*(\kappa) &= (q+2)(\kappa, r^*(\kappa)) - \frac{q}{2}[r^*(\kappa)]^2 = (q+2)(\kappa, r^*) - \frac{q}{2}[r^*(\kappa)]^2 \\ &\leq (q+2)(\kappa, r^*) - \frac{q}{2}[r^*]^2 = \frac{\partial h}{\partial r}(\kappa, r^*)r^*, \end{aligned}$$

for all $\kappa \in (0, \kappa_s]$. Finally combining this inequality with the inequality (2.5.24) we easily deduce (2.5.16), and complete the proof of the lemma. □

Remark 2.39. It is easy to see that, if in the Lemma 2.38 we take $\delta = 0$, then $A_\delta = (0, \kappa_s] \times \{0\}$. Also, the statement of the lemma can be extended to the case $\delta < 0$, in which case one has $A_\delta = \emptyset$, and all statements concerning B_δ remain the same.

The first thing to observe in light of the Lemma 2.38 is that, the set

$$E_d := \{(\phi, \varphi) \in \mathcal{E} : E(\phi, \varphi) < d\}, \quad (2.5.27)$$

where d is as in (2.5.13), is the disjoint union of two disconnected sets, namely

$$E_d = \mathcal{S}_d \cup \mathcal{U}_d, \quad (2.5.28)$$

where

$$\mathcal{S}_d := E_d \cap \{(\phi, \varphi) \in \mathcal{E} : \phi \in \mathcal{S} \cup \{0\}\}, \quad (2.5.29)$$

$$\mathcal{U}_d := E_d \cap \{(\phi, \varphi) \in \mathcal{E} : \phi \in \mathcal{U}\}, \quad (2.5.30)$$

and $\mathcal{S}, \mathcal{U}$ are as in (2.5.7) and (2.5.6), respectively. Here, also note that, by the Remark 2.39, $\mathcal{S}_d = \emptyset$ if $d < 0$. Indeed, suppose $(\phi, \varphi) \in E_d$, i.e. $E(\phi, \varphi) < d$. From the identity (2.5.4) we see that

$$E(\phi, \varphi) \geq J(\phi), \quad \forall (\phi, \varphi) \in \mathcal{E}.$$

Hence, $J(\phi) < d$ and in view of the identity (2.5.11) we deduce from the Lemma 2.38 that, $(K_s(\phi), \|\nabla\phi\|) \in (0, \kappa_s] \times \mathbb{R}^+$ belongs either to A_δ or to B_δ . Consequently, by the identity (2.5.12) and the inequalities (2.5.15) and (2.5.16) (see also the Remark 2.39) we conclude that either

$$I(\phi) \geq C_0 \|\nabla\phi\|^2, \quad (2.5.31)$$

or

$$I(\phi) \leq -C_1, \quad (2.5.32)$$

where $C_0, C_1 > 0$ are constants depending on the number $E(\phi, \varphi)$. This translates to either $\phi \in \mathcal{S} \cup \{0\}$ or $\phi \in \mathcal{U}$, and since by assumption $(\phi, \varphi) \in E_d$, we conclude that, either $(\phi, \varphi) \in \mathcal{S}_d$ or $(\phi, \varphi) \in \mathcal{U}_d$. This gives one side of the inclusion for (2.5.28), and the other side trivially follows from the definitions (2.5.29) and (2.5.30). The disconnectedness of these sets follows from the statement of the lemma. Indeed, we have that,

$$(\phi, \varphi) \in \mathcal{S}_d \Rightarrow \|\nabla\phi\| \leq r_*, \quad (2.5.33)$$

and

$$(\phi, \varphi) \in \mathcal{U}_d \Rightarrow \|\nabla\phi\| \geq r^*, \quad (2.5.34)$$

where $r_* < r^*$ are the inverse images of the number $E(\phi, \varphi)$ under the map $h(\kappa_s, \cdot)$ and h is given by (2.5.10).

The sets $\mathcal{S}_d$ and $\mathcal{U}_d$ are invariant under the flow of the semigroup generated by the problem (2.5.1), and that is the content of the following proposition.

Proposition 2.40. *Let $(u(t), u_t(t)) \in C(0, T; \mathcal{E})$ be the unique solution to the problem (2.5.1), where the initial data verify $E(u_0, u_1) < d$. Then, either $(u_0, u_1) \in \mathcal{S}_d$ or $(u_0, u_1) \in \mathcal{U}_d$, where $\mathcal{S}_d$, $\mathcal{U}_d$ are given by (2.5.29) and (2.5.30), respectively. In the former case $(u(t), u_t(t)) \in \mathcal{S}_d$ for all $t \geq 0$, and there exists a constant $C_0 > 0$, such that*

$$I(u(t)) \geq C_0 \|\nabla u(t)\|^2, \quad \forall t \geq 0. \quad (2.5.35)$$

In the latter case $(u(t), u_t(t)) \in \mathcal{U}_d$ for all $t \geq 0$, and there exists a constant $C_1 > 0$, such that

$$I(u(t)) \leq -C_1, \quad \forall t \geq 0. \quad (2.5.36)$$

Proof. First, let $(u_0, u_1) \in \mathcal{E}$ be such that $0 \leq E(u_0, u_1) < d$. The first part of the proposition immediately follows from (2.5.28). Next, upon multiplying the equation (2.5.1) by u_t and integrating over time and space one arrives at the energy identity

$$E(u(t), u_t(t)) = E(u_0, u_1) - \int_0^t \|\nabla u_t(s)\|^2 ds. \quad (2.5.37)$$

From this identity and (2.5.4) one easily deduces that

$$E(u(t), u_t(t)) \leq E(u_0, u_1) < d, \quad \forall t \geq 0. \quad (2.5.38)$$

Thus, it follows from (2.5.28), (2.5.33) and (2.5.34) that, for each $t \geq 0$ either $(u(t), u_t(t)) \in \mathcal{S}_d$ or $(u(t), u_t(t)) \in \mathcal{U}_d$, and in the former case $0 \leq \|\nabla u(t)\| \leq r_*$, while in the latter case $\|\nabla u(t)\| \geq r^*$, where $r_* < r^*$ are the inverse images of the number $E(u_0, u_1)$ under the map $h(\kappa_s, \cdot)$ and h is given by (2.5.10). Thus, by the fact that $(u(t), u_t(t)) \in C(0, T; \mathcal{E})$, we deduce that, either $(u(t), u_t(t)) \in \mathcal{S}_d$ for all $t \geq 0$, or $(u(t), u_t(t)) \in \mathcal{U}_d$ for all $t \geq 0$. Moreover, the validity of the estimates (2.5.35) and (2.5.36) follows from the same arguments, which were used in deriving the inequalities (2.5.31) and (2.5.32). Finally, if $E(u_0, u_1) < 0$, then we necessarily have $(u_0, u_1) \in \mathcal{U}_d$, and by the same arguments we have that, $(u(t), u_t(t)) \in \mathcal{U}_d$ for all $t \geq 0$ and the estimate (2.5.36) is valid.

□

Now, we are in position to give the main theorems of this section. The first theorem is concerned with the solutions to (2.5.1) with initial data that belong to stable set $\mathcal{S}_d$. The discussion so far already establishes the global existence of such solutions. However,

we can prove a stronger result, namely that all solutions starting from $\mathcal{S}_d$ decay to zero exponentially.

Theorem 2.41. *Let $(u(t), u_t(t)) \in C(0, T; \mathcal{E})$ be the unique solution to the problem (2.5.1), and suppose the initial data verify $(u_0, u_1) \in \mathcal{S}_d$, where $\mathcal{S}_d$ is defined as in (2.5.29). Then, $\mathcal{E}$ -norm of the solution decays to zero with an exponential rate.*

Proof. By the Proposition 2.40 we have $(u(t), u_t(t)) \in \mathcal{S}_d$ for all $t \geq 0$, and hence we have

$$I(u(t)) \geq 0, \quad \forall t \geq 0. \quad (2.5.39)$$

First, if we multiply the equation (2.5.1) by $u_t(t)$ in $L^2(\Omega)$ we obtain the energy relation

$$\frac{d}{dt}E(u(t), u_t(t)) = -\|\nabla u_t(t)\|^2. \quad (2.5.40)$$

Next, let us define the nonnegative functional

$$\Psi(t) := \frac{1}{2}\|u(t)\|^2 + \frac{1}{2} \int_0^t \|\nabla u(s)\|^2 ds. \quad (2.5.41)$$

Then, we have

$$\Psi'(t) = (u(t), u_t(t)) + \frac{1}{2}\|\nabla u(t)\|^2, \quad (2.5.42)$$

and

$$\Psi''(t) = (u(t), u_{tt}(t) - \Delta u_t(t)) + \|u_t(t)\|^2. \quad (2.5.43)$$

Utilizing the equation (2.5.1) in the identity (2.5.43) we arrive at

$$\begin{aligned} \Psi''(t) &= \|u_t(t)\|^2 - \|\nabla u(t)\|^2 + \|u(t)\|_{L^{q+2}}^{q+2} \\ &= \|u_t(t)\|^2 - I(u(t)). \end{aligned} \quad (2.5.44)$$

Let us multiply this identity with a small number $\varepsilon > 0$, which we will choose later, and add it with the energy relation (2.5.40). We obtain

$$\frac{d}{dt}H_\varepsilon(t) = -\|\nabla u_t(t)\|^2 + \varepsilon\|u_t(t)\|^2 - \varepsilon I(u(t)), \quad (2.5.45)$$

where

$$H_\varepsilon(t) := E(u(t), u_t(t)) + \varepsilon\Psi'(t). \quad (2.5.46)$$

Observe that by the definitions (2.5.3)-(2.5.5) and the inequality (2.5.39) we have

$$\begin{aligned}
\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 &\geq E(u(t), u_t(t)) \\
&= \frac{1}{2}\|u_t(t)\|^2 + \frac{q}{2(q+2)}\|\nabla u(t)\|^2 + \frac{1}{q+2}I(u(t)) \\
&\geq \frac{1}{2}\|u_t(t)\|^2 + \frac{q}{2(q+2)}\|\nabla u(t)\|^2,
\end{aligned} \tag{2.5.47}$$

that is $E(u(t), u_t(t)) \geq 0$ for all $t \geq 0$, and is equivalent to $\|(u(t), u_t(t))\|_{\mathcal{E}}^2$. Moreover, by the Cauchy-Schwarz and the Poincaré inequalities we see from (2.5.42) that

$$|\Psi'(t)| \leq \frac{1}{2}\|u_t(t)\|^2 + \frac{\lambda_1^{-1} + 1}{2}\|\nabla u(t)\|^2.$$

Hence, in view of this inequality and the inequality (2.5.47) we easily deduce from the definition (2.5.46) that, for sufficiently small $\varepsilon > 0$ $H_\varepsilon(t)$ is equivalent to $\|(u(t), u_t(t))\|_{\mathcal{E}}^2$, and therefore to $E(u(t), u_t(t))$ as well.

Turning back to the equation (2.5.45), by the fact that $u(t) \in \mathcal{S}_d$ for all $t \geq 0$, due to the Proposition 2.40 we have that, there exists a constant $C_0 > 0$ such that

$$I(u(t)) \geq C_0\|\nabla u(t)\|^2, \quad \forall t \geq 0.$$

Employing this in the inequality (2.5.45), thanks to the Poincaré inequality we arrive at

$$\begin{aligned}
\frac{d}{dt}H_\varepsilon(t) &\leq -(\lambda_1 - \varepsilon)\|u_t(t)\|^2 - \varepsilon C_0\|\nabla u(t)\|^2 + \frac{2}{q+2}\varepsilon C_0\|u(t)\|_{L^{q+2}}^{q+2} \\
&= -2\varepsilon C_0 E(u(t), u_t(t)) - (\lambda_1 - \varepsilon(C_0 + 1))\|u_t(t)\|^2.
\end{aligned}$$

Hence, choosing $\varepsilon > 0$ small enough we may obtain that

$$\frac{d}{dt}H_\varepsilon(t) \leq -2\varepsilon C_0 E(u(t), u_t(t)) \leq CH_\varepsilon(t), \quad \forall t \geq 0,$$

for some number $C > 0$, where the last inequality is due to the equivalence of $H_\varepsilon(t)$ and $E(u(t), u_t(t))$. Finally, the exponential decay of all the quantities $H_\varepsilon(t)$, $E(u(t), u_t(t))$ and $\|(u(t), u_t(t))\|_{\mathcal{E}}$ to zero follow by the Gronwall inequality and the fact that all of these quantities are equivalent. This finishes the proof of the theorem. □

In the next theorem, we show that, all solutions to the problem (2.5.1), that originate from the unstable set $\mathcal{U}_d \subset \mathcal{E}$, must blow up in finite time.

Theorem 2.42. *Let $(u(t), u_t(t)) \in C(0, T; \mathcal{E})$ be the unique solution to the problem (2.5.1), and suppose the initial data verify $(u_0, u_1) \in \mathcal{U}_d$, where $\mathcal{U}_d$ is defined as in (2.5.30). Then, there exists $t_1 > 0$ such that*

$$\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds \rightarrow \infty, \quad \text{as } t \nearrow t_1^-.$$

Proof. By the Proposition (2.40) we have that, $(u(t), u_t(t)) \in \mathcal{U}_d$ for all $t \geq 0$, and there exists a constant $C_1 > 0$ such that

$$I(u(t)) \leq -C_1, \quad \forall t \geq 0. \quad (2.5.48)$$

Similar to the proof of the Theorem 2.41 consider the functional

$$\Psi(t) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \int_0^t \|\nabla u(s)\|^2 ds + \Psi_0, \quad (2.5.49)$$

where $\Psi_0 > 0$ is a constant, which we will fix later. Then,

$$\begin{aligned} \Psi'(t) &= (u(t), u_t(t)) + \frac{1}{2} \|\nabla u(t)\|^2 \\ &= (u(t), u_t(t)) + \int_0^t (\nabla u(s), \nabla u_t(s)) ds + \frac{1}{2} \|\nabla u_0\|^2, \end{aligned} \quad (2.5.50)$$

and $\Psi''(t)$ is same as in the proof of the Theorem 2.41 and is given by (2.5.43). Consequently, by the same calculations as in the Theorem 2.41 we see that, the identity (2.5.44) is valid, and utilizing the inequality (2.5.48) in that identity we deduce

$$\Psi''(t) \geq C_1 > 0, \quad \forall t \geq 0,$$

that is $\Psi''(t)$ is uniformly convex. It follows that

$$\Psi(t) \rightarrow +\infty, \quad \text{and} \quad \Psi'(t) \rightarrow +\infty, \quad \text{as } t \rightarrow \infty. \quad (2.5.51)$$

Moreover, as

$$I(u(t)) = (q+2)E(u(t), u_t(t)) - \frac{q+2}{2} \|u_t(t)\|^2 - \frac{q}{2} \|\nabla u(t)\|^2,$$

we may rewrite the identity (2.5.44) as

$$\Psi''(t) = \frac{q+4}{2} \|u_t(t)\|^2 + \frac{q}{2} \|\nabla u(t)\|^2 - (q+2)E(u(t), u_t(t)).$$

Employing the energy identity (2.5.37) in the equality above we arrive at

$$\begin{aligned}\Psi''(t) &= \frac{q+4}{2}\|u_t(t)\|^2 + \frac{q}{2}\|\nabla u(t)\|^2 + (q+2)\int_0^t \|\nabla u_t(s)\|^2 ds - (q+2)E(u_0, u_1) \\ &= (q+2)\left[\|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds\right] \\ &\quad + \frac{q}{2}(\|u_t(t)\|^2 + \|\nabla u(t)\|^2) - (q+2)E(u_0, u_1).\end{aligned}\tag{2.5.52}$$

Next, using the second line of the identity (2.5.50), by an application of the Cauchy-Schwarz inequality we get the estimate

$$\begin{aligned}[\Psi'(t)]^2 &= \left[(u(t), u_t(t)) + \int_0^t (\nabla u(s), \nabla u_t(s)) ds + \frac{1}{2}\|\nabla u_0\|^2\right]^2 \\ &\leq \left[\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds + \frac{1}{2}\|\nabla u_0\|^2\right] \left[\|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + \frac{1}{2}\|\nabla u_0\|^2\right].\end{aligned}$$

Let us choose $\Psi_0 = \frac{1}{4}\|\nabla u_0\|^2$, which is strictly positive because by assumption $(u_0, u_1) \in \mathcal{U}_d$, and hence $u_0 \neq 0$. Then, in view of the definition (2.5.49) of $\Psi(t)$, we see that the last inequality yields

$$[\Psi'(t)]^2 \leq 2\Psi(t) \left[\|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + \frac{1}{2}\|\nabla u_0\|^2\right].$$

If we multiply the equality (2.5.52) by $\Psi(t)$ and utilize the previous inequality we find for all $t \geq 0$

$$\begin{aligned}\Psi''(t)\Psi(t) &\geq \left(1 + \frac{q}{2}\right)[\Psi'(t)]^2 \\ &\quad + \left(\frac{q}{2}(\|u_t(t)\|^2 + \|\nabla u(t)\|^2) - (q+2)\left[E(u_0, u_1) - \frac{1}{2}\|\nabla u_0\|^2\right]\right)\Psi(t).\end{aligned}\tag{2.5.53}$$

An easy application of the Cauchy-Schwarz and the Poincaré inequalities to (2.5.50) shows that

$$\Psi'(t) \leq C(\|u_t(t)\|^2 + \|\nabla u(t)\|^2)$$

for some constant $C > 0$. Thus, by (2.5.51) we see that there exists $t_0 \geq 0$ such that,

$$\Psi(t_0) > 0, \quad \Psi'(t_0) > 0,$$

and the term on the second line of the inequality (2.5.53) is nonnegative for all $t \geq t_0$, so that we have

$$\Psi''(t)\Psi(t) \geq \left(1 + \frac{q}{2}\right)[\Psi'(t)]^2, \quad \forall t \geq t_0.$$

Hence, all assumptions of the Lemma 1.1 are fulfilled for $t \geq t_0$, and the statement of the theorem follows from the statement of the lemma and the definition (2.5.49) of $\Psi(t)$.

□

Remark 2.43. The parameter d defined in (2.5.13) is referred to as the depth of the potential well, where the potential well $P_w \subset H_0^1(\Omega)$ is defined in terms of the total potential energy $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$P_w := \mathcal{S} \cap \phi \in H_0^1(\Omega) : J(\phi) < d = \pi_1(\mathcal{S}_d), \quad (2.5.54)$$

where $\pi_1 : H_0^1(\Omega) \times L^2(\Omega) \rightarrow H_0^1(\Omega)$ is the projection operator onto the first coordinate. It is easy to see by inspection that, an alternative characterization for this quantity is

$$d = \min \|\nabla \phi\| = 1 \max_{\lambda \geq 0} J(\lambda \phi).$$

Let $\phi_0 \in H_0^1(\Omega)$ be such that it satisfies the Sobolev inequality (2.5.2) with the best constant $\kappa_s > 0$ and is such that

$$J(\phi_0) = d.$$

The existence of such a function is well-known. Then, we necessarily have

$$I(\phi_0) = 0,$$

that is ϕ_0 belongs to the Nehari manifold $\mathcal{N}$ defined in (2.5.8). Moreover, ϕ_0 is a local minimizer of the functional J restricted to the Nehari manifold, while the global minimizer is the trivial function.

Now, it is easy to see that, the steady state solutions to the problem (2.5.1) necessarily lie in the Nehari manifold $\mathcal{N}$. Then, by the variational arguments one easily sees that the steady state solutions are exactly those that locally minimize the functional J restricted to the Nehari manifold. It follows that, the only steady state solutions to the problem (2.5.1) are the trivial function, which is a stable steady state, and functions of the form of ϕ_0 described in the previous paragraph, which are unstable as they are a saddle point for the functional J . In fact, from the Theorem 2.41 and the Theorem 2.42 it follows that, for any $0 < \alpha < 1$, $\beta > 1$ the solutions to the problem (2.5.1) with initial data $(\alpha\phi_0, 0)$ and $(\beta\phi_0, 0)$, respectively decay to zero and blow up in finite time.

We conclude this section by providing an example of initial data that belong to $\mathcal{S}_d$, mainly for purposes of computer simulation, see Figure 4.4 and Figure 4.5 in §4, and we do so for

the following 1-dimensional version of the problem (2.5.1)

$$\begin{cases} u_{tt}(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) - \frac{\partial^2}{\partial x^2} u_t(x, t) = u^3(x, t), & x \in (0, \pi), \quad t > 0, \\ u(0, t) = u(\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in [0, \pi]. \end{cases} \quad (2.5.55)$$

We would like to find an unstable steady state solution to this problem, see the Remark 2.43, however finding the function that minimizes constant in the Sobolev embedding (2.5.2) is too big of a task. For this reason we will choose $u_0(x)$ to be a constant multiple of one of the eigenfunctions of the Laplacian operator $-\frac{\partial^2}{\partial x^2}$ under Dirichlet boundary conditions, namely for some $n_0 \geq 1$, and $a_0 > 0$ we will choose

$$u_0(x) = a_0 \sin(n_0 x), \quad x \in [0, \pi], \quad (2.5.56)$$

and minimize the Sobolev constant among these functions. It is easy to calculate that

$$\int_0^\pi \left(\frac{\partial}{\partial x} \sin(nx) \right)^2 dx = \frac{\pi n^2}{2}, \quad \text{and} \quad \int_0^\pi \sin^4(nx) dx = \frac{3\pi}{8}, \quad \forall n \geq 1. \quad (2.5.57)$$

Consequently, we have

$$K_s(\sin(nx)) = \frac{\left(\frac{3\pi}{8}\right)^{\frac{1}{4}}}{\left(\frac{\pi n^2}{2}\right)^{\frac{1}{2}}} = \left(\frac{3}{2\pi}\right)^{\frac{1}{4}} \frac{1}{n}, \quad n \geq 1,$$

where K_s is defined in (2.5.9). Note that

$$\sup_{n \geq 1} K_s(\sin(nx)) = K_s(\sin(x)) = \left(\frac{3}{2\pi}\right)^{\frac{1}{4}},$$

and according to this we choose $n_0 = 1$ in (2.5.56). After these choices, in view of (2.5.57) we have

$$J(u_0) = E(u_0, u_1) = \frac{1}{2} \left\| \frac{\partial}{\partial x} u_0 \right\|^2 - \frac{1}{4} \|u_0\|_{L^4}^4 dx = \frac{a_0^2 \pi}{4} - \frac{3a_0^4 \pi}{32}, \quad (2.5.58)$$

and

$$I(u_0) = \left\| \frac{\partial}{\partial x} u_0 \right\|^2 - \|u_0\|_{L^4}^4 dx = \frac{a_0^2 \pi}{2} - \frac{3a_0^4 \pi}{8}. \quad (2.5.59)$$

We are interested in the value of $a_0 > 0$ such that $I(u_0) = 0$, and that value is

$$a_0 = \frac{2\sqrt{3}}{3}. \quad (2.5.60)$$

Thus, we hope that the initial functions given by

$$(u_0(x), u_1(x)) = \left(\frac{2\sqrt{3}}{3} \sin(x), 0\right) =: (\hat{\phi}_0, 0) \quad (2.5.61)$$

are close to the steady state solution, and in this direction in §4 we present simulations for the problem (2.5.55) with these initial data, see the Figure 4.5, and also with initial data

$$(u_0(x), u_1(x)) = (0.99\hat{\phi}_0, 0), \quad (2.5.62)$$

which belongs to the stable set $\mathcal{S}_d$, see the Figure 4.4. Finally, let us note that, from the relations (2.5.58), (2.5.60) and (2.5.61) we can calculate the following upper bound for the depth of the potential well of the problem (2.5.1):

$$d \leq J(\hat{\phi}_0) = \frac{\pi}{6}. \quad (2.5.63)$$

Chapter 3

Stable Equations

3.1 Semilinear structurally damped wave equations

In this section we consider the following initial-boundary value problem for a structurally damped wave equation in three dimensions

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) + \gamma(-\Delta)^\theta u_t(x, t) + f(u) = g(x), & x \in \Omega, \ t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, \ t \geq 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (3.1.1)$$

where $\gamma > 0$, $\theta \in [0, 1]$ are given numbers, $\Omega \subset \mathbb{R}^3$ is a bounded domain, and $g \in L^2(\Omega)$ is a given function. The nonlinearity f is assumed to satisfy the growth condition

$$-C_1 + a_1|s|^q \leq f'(s) \leq C_2 + a_2|s|^q, \quad \forall s \in \mathbb{R}, \quad (3.1.2)$$

with some constants $C_i, a_i \geq 0$, $i = 1, 2$, and $q \geq 0$. The typical example of such a nonlinearity is $f(u) = |u|^q u + \lambda u$ with $\lambda \in \mathbb{R}$.

Let us note here that the sign of the nonlinearity f is the opposite to that of the unstable structurally damped wave equation considered in §2.2.1.1. This change in the sign alters the behaviour of the corresponding solution dramatically. Indeed, it does so in many different ways. For example, the local well-posedness of the problem in §2.2.1.1 can be given a positive answer only when $q \in [0, \frac{2}{n-2}]$ ($q \in [0, 2]$ in 3-dimensions), while, as the Theorem 3.1 states, one can show the local well-posedness of the problem (3.1.1) for any $q \geq 0$, if $\theta \geq \frac{3}{4}$. Moreover, all solutions to the problem (3.1.1) are global in time, even better they possess an absorbing ball in the energy phase space, which is in high contrast with the fact

that, the system in §2.2.1.1 trajectories in the energy phase space that blow up in finite time.

Classically, the global existence of solutions to the problem (3.1.1) is known for all $q \geq 0$. However, in 3-dimensions for example, the uniqueness of the solutions was only established up to a so-called ‘critical exponent’, that is when $q \in [0, 4]$, which is independent of $\theta \in [0, 1]$. However, a recent breakthrough was made by the paper [22], which extends the uniqueness results to the cases

$$\begin{aligned} q &\in [0, \infty), \text{ when } \theta \in [\frac{3}{4}, 1], \\ q &\in [0, \frac{8\theta}{3-4\theta}], \text{ when } \theta \in [\frac{1}{2}, \frac{3}{4}). \end{aligned} \quad (3.1.3)$$

Thus, when $\theta = \frac{1}{2}$, we have well-posedness for $q \in [0, 4]$, that is up to the critical exponent, and in fact, this is true for all $\theta \in [0, \frac{1}{2}]$, see [39] and [40]. This result shows that, the strength of the damping plays a significant role in the local well-posedness theory of the problem (3.1.1), but the effects only start to appear when θ exceeds $\frac{1}{2}$. This phenomenon is deeply related to the fact that, the linear version of the problem (3.1.1), i.e. $f = 0$, exhibits hyperbolic behaviour when $\theta \in [0, \frac{1}{2})$, whereas it becomes parabolic when $\theta \in (\frac{1}{2}, 1]$, and the behaviour of the critical case $\theta = \frac{1}{2}$ depends on the damping constant γ .

For the sake of completeness let us present here the aforementioned the local well-posedness result due to [22].

Theorem 3.1. *Suppose that f satisfies the growth condition (3.1.2), $\theta \in [\frac{1}{2}, 1]$, and q verifies (3.1.3). Then, for each initial data*

$$(u_0, u_1) \in \mathcal{E} := H_0^1(\Omega) \cap L^{q+2}(\Omega) \times L^2(\Omega), \quad (3.1.4)$$

the problem (3.1.1) admits a unique global solution

$$u(t) \in \mathcal{E}_T := L^\infty(\mathbb{R}^+, H_0^1(\Omega) \cap L^{q+2}(\Omega)) \cap W^{1,\infty}(\mathbb{R}^+, L^2(\Omega))$$

in the sense of (2.1.8) of §2.1. Moreover, this solution verifies the energy inequality (2.1.10), where the energy is as in (2.1.9).

Note that, this theorem states more than just the global well-posedness of solutions to (3.1.1). It also says that, for any initial data $(u_0, u_1) \in \mathcal{E}$ there is a constant $C > 0$, depending on the $\mathcal{E}$ -norm of the initial data, such that, one has

$$\|u(t)\|_1 + \|u(t)\|_{L^{q+2}} + \|u_t(t)\| \leq C, \quad \forall t \geq 0.$$

This fact will be used later in the sequel.

It is also important to note that, we don't know, for this range of exponents q , if the solution evolves continuously in time in the phase space $\mathcal{E}$, but we only know that the solution (u, u_t) belongs to $L^\infty(\mathbb{R}^+, \mathcal{E})$. As will be noted several times, the absence of the knowledge of such a continuous evolution will be costly. The absence stems from the method of the proof of the Theorem 3.1, as it proves the uniqueness in the wider space

$$\mathcal{E}_\theta := H_0^\theta(\Omega) \times H^{-\theta}(\Omega), \quad (3.1.5)$$

which implies the uniqueness in $\mathcal{E}$, since $\mathcal{E}_\theta \subset \mathcal{E}$. As a by-product of the method, from the proof one can infer the continuous dependence on time and initial data of the evolution in the space $\mathcal{E}_\theta$, but not in $\mathcal{E}$.

We are interested in the asymptotic behaviour of the solutions to the problem (3.1.1). A well-known fact about this problem is that, for all $q > 0$, it possesses an absorbing ball in the phase space $\mathcal{E}$, which means that, there exists a fixed ball in $\mathcal{E}$, centered around zero, such that, any solution (u, u_t) starting from an initial pair of functions in $\mathcal{E}$ eventually enters this ball, and stays in it. After such a result, it is common and straightforward, to apply the machinery of attractor theory, and construct a finite dimensional global attractor, which by definition is a finite dimensional compact subset of $\mathcal{E}$, that is strictly invariant under the flow of the evolution, and 'attracts' the whole space $\mathcal{E}$. This task is undertaken in [22] for the case $\theta \in [\frac{1}{2}, 1]$, and we will not discuss it here. Nevertheless, it is worth noting that, in [22] the authors were only able to construct a weak attractor, one that, instead of being compact in $\mathcal{E}$, is merely compact in $\mathcal{E}_\theta$. The reason behind this is the absence of the continuity in time of the evolution in the space $\mathcal{E}$, which one has in the weaker space $\mathcal{E}_\theta$, as was mentioned above. See [27] or [4] for an exposition of the attractor theory.

Before proceeding further some preparatiion is necessary. For $N \geq 1$, P_N will denote the projection operator in $L^2(\Omega)$ onto the subspace generated by the first N eigenfunctions of the Laplacian operator $-\Delta_0$, and we set $Q_N = I - P_N$. For $s \in \mathbb{R}$ one has by the Parseval's identity

$$\|u\|_s^2 = \|(-\Delta)^{\frac{s}{2}} u\|^2 = \sum_i \lambda_i^s (u, w_i)^2, \quad \forall u \in H^s(\Omega),$$

from which it is easy to deduce the following Poincaré type inequality, true for all $s \geq r \in \mathbb{R}$, and $N \geq 1$:

$$\|Q_N u\|_s^2 \geq \lambda_{N+1}^{s-r} \|Q_N u\|_r^2, \quad \forall u \in H_0^s(\Omega). \quad (3.1.6)$$

In the sequel we will also make use of the well-known coercivity inequality

$$d_1(|u|^p + |v|^p)|u - v|^2 \leq (|u|^p u - |v|^p v)(u - v) \leq d_2(|u|^p + |v|^p)|u - v|^2, \quad p \geq 1, \quad (3.1.7)$$

which holds for any $u, v \in \mathbb{R}$, where $d_1, d_2 > 0$ are constants depending only on p .

3.1.1 Finite number of determining modes

Let us first give a definition.

Definition 3.2. The problem (3.1.1) is said to have a *finite number of determining modes* in $\mathcal{E}$, if there exists $N \geq 1$ such that, for any two solutions u, v to (3.1.1), that enjoy the property

$$\|P_N w(t)\|_{\mathcal{E}} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

it also necessarily holds true that

$$\|w(t)\|_{\mathcal{E}} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $w = u - v$.

Here, P_N denotes the projection operator in $L^2(\Omega)$ onto the subspace generated by the first N eigenfunctions of the Laplacian operator under Dirichlet boundary conditions, Δ_0 . More vaguely, one can say that, if a dynamical system in an infinite dimensional space possesses a finite number of determining modes, then the long time dynamics will resemble that of a system evolving in a finite dimensional space. It is also worth noting that, as the range of P_N is a finite dimensional space, and all norms on a finite dimensional vector space are equivalent, the $\mathcal{E}$ -norm that appears in the assumption part of the Definition 3.2 can be replaced by any other norm.

The existence of a finite number of determining modes was classically first observed for the heat equation. Later, same phenomenon was discovered in the dynamics of other dissipative systems, and other notions such as determining nodes, volume elements and functionals were introduced, see [19], [10], [8], [21], [16] and the references therein. In [8] for example, the authors prove the existence of finitely many determining modes in the space $\mathcal{E}$ for the same problem as (3.1.1) with $\theta = 1$, but only when the nonlinearity grows subcritically, i.e. when $q \in [0, 4]$ in 3-dimensions. This result is contemporary with the old well-posedness result, and assumes same range of exponents. However, so far no one has tried to establish the existence of finite number of determining modes with the new range of exponents, which is provided by the breakthrough paper [22].

In the rest of these notes, our aim is to establish the existence of finitely many determining modes for this extended range of exponents, and also provide an upper bound for this number. However, the lacking continuity in time of the solution in the energy phase space

$\mathcal{E}$ for this case renders this task out of our reach. Instead, we can only prove the existence of finitely many ‘weakly’ determining modes, which is the content of the main result of this chapter.

Theorem 3.3. *Let the nonlinearity f satisfy (3.1.2) with q as in (3.1.3). Then, there exists $N \geq 1$ such that for any two weak solutions u, v of (3.1.1) with initial data in $\mathcal{E}$ that satisfy*

$$(P_N(w)(t), P_N(w_t)(t)) \rightarrow 0 \text{ in } \mathcal{E} \text{ as } t \rightarrow +\infty, \quad (3.1.8)$$

it holds true that

$$\int_t^{t+1} [\|w(\tau)\|_1^2 + \|w_t(\tau)\|^2 + \|w(\tau)\|_{q+2}^{q+2}] d\tau \rightarrow 0 \quad (3.1.9)$$

and

$$(w(t), w_t(t)) \rightarrow 0 \text{ in } \mathcal{E}_0 \quad (3.1.10)$$

as $t \rightarrow +\infty$, where $w = u - v$.

Proof. Let u, v be two solutions to the problem (3.1.1) with initial data $(u_0, u_1), (v_0, v_1) \in \mathcal{E}$. Then, the function $w = u - v$ is the solution to the following problem,

$$w_{tt} - \Delta w + \gamma(-\Delta)^\theta w_t + (f(u) - f(v)) = 0, \quad x \in \Omega, \quad t > 0, \quad (3.1.11)$$

$$w(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \quad (3.1.12)$$

$$w(x, 0) = u_0(x) - v_0(x), \quad w_t(x, 0) = u_1(x) - v_1(x), \quad x \in \Omega. \quad (3.1.13)$$

Upon multiplying (3.1.11) by $Q_N w + \frac{2}{\gamma}(-\Delta)^{-\theta} Q_N w_t$ in $L^2(\Omega)$, and integrating by parts we arrive at

$$\begin{aligned} \frac{d}{dt} H_N(t) + \|Q_N w_t\|^2 + \|Q_N w\|_1^2 + (f(u) - f(v), w) \\ = (f(u) - f(v), P_N w) - \frac{2}{\gamma} (f(u) - f(v), (-\Delta)^{-\theta} Q_N w_t), \end{aligned} \quad (3.1.14)$$

where

$$H_N(t) = (Q_N w_t, Q_N w) + \frac{\gamma}{2} \|Q_N w\|_\theta^2 + \frac{1}{\gamma} \|Q_N w_t\|_{-\theta}^2 + \frac{1}{\gamma} \|Q_N w\|_{1-\theta}^2. \quad (3.1.15)$$

The left hand side of the assumption (3.1.2) implies that

$$(f(u) - f(v), w) \geq -C_1 \|w\|^2 + \frac{a_1}{q+1} (|u|^q u - |v|^q v, w).$$

Employing the inequality (3.1.7) in the above inequality we find

$$\begin{aligned}
(f(u) - f(v), w) &\geq -C_1 \|w\|^2 + \frac{a_1 d_1}{q+1} (|u|^q + |v|^q, w^2) \\
&= -C_1 \|P_N w\|^2 - C_1 \|Q_N w\|^2 + \frac{a_1 d_1}{q+1} (|u|^q + |v|^q, w^2) \\
&\geq -C_1 \|P_N w\|^2 - \frac{C_1}{\lambda_{N+1}} \|Q_N w\|_1^2 + \frac{a_1 d_1}{q+1} (|u|^q + |v|^q, w^2),
\end{aligned} \tag{3.1.16}$$

where the last inequality is due to the Poincare inequality (3.1.6). To estimate the first term on the right hand side of (3.1.14) we note that the right hand side of (3.1.2) implies

$$|f(u) - f(v)| \leq C_2 |w| + \frac{a_2}{q+1} (|u|^{q+1} + |v|^{q+1}).$$

Using this together with the Cauchy-Schwarz and the Hölder's inequality with $\frac{q+1}{q+2} + \frac{1}{q+2} = 1$ we find

$$\begin{aligned}
|(f(u) - f(v), P_N w)| &\leq \left(C_2 |w| + \frac{a_2}{q+1} (|u|^{q+1} + |v|^{q+1}), P_N w \right) \\
&\leq C_2 (\|u\| + \|v\|) \|P_N w\| + \frac{a_2}{q+1} (\|u\|_{L^{q+2}}^{q+1} + \|v\|_{L^{q+2}}^{q+1}) \|P_N w\|_{L^{q+2}}.
\end{aligned} \tag{3.1.17}$$

Let us first handle the last term on the right hand side of (3.1.14). To this end let us note that $(-\Delta)^{-\theta} Q_N w_t \in L^\infty(0, T; H_0^{2\theta}(\Omega))$. Also observe that, since by the Sobolev embedding theorem in 3-dimensions $H_0^s(\Omega)$ is continuously embedded into $L^{\frac{6}{3-2s}}(\Omega)$, we deduce from the assumption (3.1.3) that, for any $\theta \in [\frac{1}{2}, 1]$ we can find $s_0 < 2\theta$ such that, $H_0^{s_0}(\Omega)$ is continuously embedded into $L^{q+2}(\Omega)$. Moreover, we have for some $\kappa_s > 0$ the embedding inequality

$$\|(-\Delta)^{-\theta} Q_N w_t\|_{L^{q+2}}^2 \leq \kappa_s \|(-\Delta)^{-\theta} Q_N w_t\|_{s_0}^2 = \kappa_s \|Q_N w_t\|_{s_0-2\theta}^2. \tag{3.1.18}$$

This inequality together with the assumption (3.1.2), the Hölder's inequality with $\frac{1}{p+2} + \frac{p+1}{p+2} = 1$, and the Young- ε inequality gives us

$$\begin{aligned}
|(f(u) - f(v), (-\Delta)^{-\theta} Q_N w_t)| &\leq \left(C_2 |w| + \frac{a_2}{q+1} \| |u|^q u - |v|^q v \|, |(-\Delta)^{-\theta} Q_N w_t| \right) \\
&\leq C_2 \|Q_N w_t\|_{-2\theta} \|w\| + \frac{a_2}{q+1} \| |u|^q u - |v|^q v \|_{L^{\frac{q+2}{q+1}}} \|(-\Delta)^{-\theta} Q_N w_t\|_{L^{q+2}} \\
&\leq \frac{C_2^2}{4} \|Q_N w_t\|_{-2\theta}^2 + \|w\|^2 + \frac{\kappa_s a_2^2}{4\varepsilon(q+1)^2} \|Q_N w_t\|_{s_0-2\theta}^2 + \varepsilon \| |u|^q u - |v|^q v \|_{L^{\frac{q+2}{q+1}}}^2.
\end{aligned} \tag{3.1.19}$$

Since $s_0 - 2\theta < 0$, by an application of the Poincare inequality (3.1.6) to the inequality (3.1.19) we obtain

$$\begin{aligned} & |(f(u) - f(v), (-\Delta)^{-\theta} Q_N w_t)| \\ & \leq \|P_N w\|^2 + \frac{1}{\lambda_{N+1}} \|Q_N w\|_1^2 + \left(\frac{C_2^2}{4\lambda_{N+1}^{2\theta}} + \frac{\kappa_s a_2^2}{4\varepsilon(q+1)^2 \lambda_{N+1}^{2\theta-s_0}} \right) \|Q_N w_t\|^2 \\ & \quad + \varepsilon \| |u|^q u - |v|^q v \|_{L^{\frac{q+2}{q+1}}}^2. \end{aligned} \quad (3.1.20)$$

To estimate the last term in the inequality (3.1.20) we sequentially use the coercivity inequality (3.1.7), the Hölder inequality with $\frac{q+2}{2(q+1)} + \frac{q}{2(q+1)} = 1$, and the triangular inequality:

$$\begin{aligned} \| |u|^q u - |v|^q v \|_{L^{\frac{q+2}{q+1}}}^2 & \leq d_2^2 \| (|u|^q + |v|^q) |w| \|_{L^{\frac{q+2}{q+1}}}^2 \\ & = d_2^2 \left(\int_{\Omega} (|u|^q + |v|^q)^{\frac{q+2}{2(q+1)}} (|u|^q + |v|^q)^{\frac{q+2}{2(q+1)}} |w|^{\frac{q+2}{q+1}} dx \right)^{\frac{2(q+1)}{q+2}} \\ & \leq d_2^2 \| |u|^q + |v|^q \|_{L^{\frac{q+2}{q}}} (|u|^q + |v|^q, w^2) \\ & \leq d_2^2 (\|u\|_{L^{q+2}}^q + \|v\|_{L^{q+2}}^q) (|u|^q + |v|^q, w^2). \end{aligned}$$

Inserting this into (3.1.20) we arrive at

$$\begin{aligned} & |(f(u) - f(v), (-\Delta)^{-\theta} Q_N w_t)| \\ & \leq \|P_N w\|^2 + \frac{1}{\lambda_{N+1}} \|Q_N w\|_1^2 + \left(\frac{C_2^2}{4\lambda_{N+1}^{2\theta}} + \frac{\kappa_s a_2^2}{4\varepsilon(q+1)^2 \lambda_{N+1}^{2\theta-s_0}} \right) \|Q_N w_t\|^2 \\ & \quad + \varepsilon d_2^2 (\|u\|_{L^{q+2}}^q + \|v\|_{L^{q+2}}^q) (|u|^q + |v|^q, w^2). \end{aligned} \quad (3.1.21)$$

Finally, we insert the estimates (3.1.16), (3.1.17), and (3.1.21) into the equality (3.1.14). In doing so we obtain

$$\begin{aligned} & \frac{d}{dt} H_N(t) + A_1 \|Q_N w_t\|^2 + A_2 \|Q_N w\|_1^2 + A_3 (|u|^q + |v|^q, w^2) \\ & \leq C_2 (\|u\| + \|v\|) \|P_N w\| + \frac{a_2}{q+1} (\|u\|_{L^{q+2}}^{q+1} + \|v\|_{L^{q+2}}^{q+1}) \|P_N w\|_{L^{q+2}} \\ & \quad + \left(\frac{2}{\gamma} + C_1 \right) \|P_N w\|^2, \end{aligned} \quad (3.1.22)$$

where

$$\begin{aligned} A_1 & = 1 - \frac{2}{\gamma} \left(\frac{C_2^2}{4\lambda_{N+1}^{2\theta}} + \frac{\kappa_s a_2^2}{4\varepsilon(q+1)^2 \lambda_{N+1}^{2\theta-s_0}} \right), \\ A_2 & = 1 - \frac{C_1}{\lambda_{N+1}} - \frac{2}{\gamma \lambda_{N+1}}, \end{aligned}$$

and

$$A_3 = \frac{a_1 d_1}{q+1} - \frac{2\varepsilon d_2^2}{\gamma} (\|u\|_{L^{q+2}}^q + \|v\|_{L^{q+2}}^q).$$

Since the solutions u, v are bounded in $\mathcal{E}$, we can choose ε such that $A_3 > 0$ for all time. Once ε is fixed, since $\lambda_i \rightarrow +\infty$ monotonically as $i \rightarrow +\infty$, and $2\theta - s_0 > 0$, we can find $N \geq 1$ large enough such that both $A_1 > 0$, and $A_2 > 0$. Thus, the inequality (3.1.22) reduces to

$$\frac{d}{dt} H_N(t) + C J_N(t) \leq h_N(t) \quad , t \geq 0, \quad (3.1.23)$$

where

$$J_N(t) = \|Q_N w_t(t)\|^2 + \|Q_N w(t)\|_1^2 + (|u(t)|^q + |v(t)|^q, w^2(t)), \quad (3.1.24)$$

and

$$h_N(t) = C_2 (\|u\| + \|v\|) \|P_N w\| + \frac{a_2}{q+1} (\|u\|_{L^{q+2}}^{q+1} + \|v\|_{L^{q+2}}^{q+1}) \|P_N w\|_{L^{q+2}} + \left(\frac{2}{\gamma} + C_1\right) \|P_N w\|^2.$$

On the other hand, a simple application of the Cauchy-Schwarz inequality, and the Young's inequality with ε in the definition (3.1.15) of $H_N(t)$ reveals that $H_N(t)$ is equivalent to the $\mathcal{E}_0$ -norm of $w(t)$, which is controlled by $J_N(t)$. Hence, it follows from (3.1.23) that

$$\frac{d}{dt} H_N(t) + C H_N(t) \leq h_N(t)$$

holds for all $t \geq 0$. Furthermore, note that by the boundedness of the solutions u, v in the phase space $\mathcal{E}$, and the assumption (3.1.8) of the theorem it follows that $h_N(t) \rightarrow 0$ as $t \rightarrow +\infty$. Therefore, by Gronwall's inequality we deduce from the last inequality that $H_N(t) \rightarrow 0$ as $t \rightarrow +\infty$. This implies the statement (3.1.10) since $H_N(t)$ controls the square of the $\mathcal{E}_0$ -norm of (w, w_t) . Indeed, by the Cauchy-Schwarz and Young- ε inequalities for any $\varepsilon > 0$ we have

$$\begin{aligned} (Q_N w_t, Q_N w) &= ((-\Delta)^{\frac{\theta}{2}} Q_N w, (-\Delta)^{-\frac{\theta}{2}} Q_N w_t) \leq \|Q_N w\|_{\theta} \|Q_N w_t\|_{-\theta} \\ &\leq \varepsilon \|Q_N w\|_{\theta}^2 + \frac{1}{4\varepsilon} \|Q_N w_t\|_{-\theta}^2. \end{aligned}$$

Choosing $\varepsilon = \frac{\gamma}{2\sqrt{2}}$, and using this estimate in the definition of H_N , (3.1.15), we see that

$$\begin{aligned} H_N(t) &= (Q_N w_t, Q_N w) + \frac{\gamma}{2} \|Q_N w\|_{\theta}^2 + \frac{1}{\gamma} \|Q_N w_t\|_{-\theta}^2 + \frac{1}{\gamma} \|Q_N w\|_{1-\theta}^2 \\ &\geq \frac{1}{\sqrt{2}} \|Q_N w\|_{\theta}^2 + \frac{1}{\sqrt{2}} \|Q_N w_t\|_{-\theta}^2 = \frac{1}{\sqrt{2}} \|(w, w_t)\|_{\mathcal{E}}^2. \end{aligned}$$

Furthermore, since $H_N(t)$ and $h_N(t)$ tend to zero as time increases, integrating (3.1.23) over $(t, t+1)$ we see that

$$\int_t^{t+1} J_N(\tau) d\tau \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Finally, from the definition of $J_N(t)$, (3.1.24), and the assumption (3.1.8), it is easy to see that

$$\int_t^{t+1} J(\tau) d\tau \rightarrow 0 \quad \text{as } t \rightarrow +\infty, \quad (3.1.25)$$

where

$$J(t) = \|w_t(t)\|^2 + \|w(t)\|_1^2 + (|u(t)|^q + |v(t)|^q, w^2(t)). \quad (3.1.26)$$

This finishes the proof of the theorem.

□

Chapter 4

Simulations and Numerical Analysis

In this chapter we would like to present and justify some numerical simulations carried out for the initial-boundary value problem for the 1-dimensional structurally damped wave equation

$$\begin{cases} u_{tt} - \partial_x^2 u + 2\gamma(-\partial_x^2)^\theta u_t = f(u), & x \in [0, \pi], \ t > 0, \\ u(0, t) = u(\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \end{cases} \quad (4.0.1)$$

where we will typically choose the nonlinearity in the form $f(u) = \pm|u|^q u$, $q > 0$ arbitrary. Thanks to the Sobolev embedding inequality, for such nonlinearities in 1-dimension we have the estimate

$$\|f(u) - f(v)\|^2 \leq C \int_0^\pi (|u|^q + |v|^q)^2 (u - v)^2 dx \leq C(\|u\|_1^q + \|v\|_1^q)^2 \|u - v\|^2. \quad (4.0.2)$$

Moreover, essentially by the same embedding relations, one can prove the local well-posedness in the phase space $\mathcal{E} := H_0^1(0, \pi) \times L^2(0, \pi)$, together which one also obtains the continuous dependence of the solution on the initial data and the continuity of the evolution in $\mathcal{E}$. The proof of these facts, respectively contain the derivation of the following inequalities for the solutions: For any $T < T_{max}$ one has

$$\|u(t) - v(t)\|_1 \leq C_1(t) \|u(0) - v(0)\|_1, \quad t \in [0, T], \quad (4.0.3)$$

where $C_1 : [0, T] \rightarrow \mathbb{R}^+$ is a bounded function with $C_1(0) = 1$, and

$$\|u(t) - u(0)\|_1 \leq C_2(\|u_0\|_1) M(t), \quad t \in [0, T], \quad (4.0.4)$$

where $C_2 : [0, T] \rightarrow \mathbb{R}^+$ is a bounded function and $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous monotone nondecreasing function with $M(0) = 0$. We will need these facts later.

4.1 The linear equation and a representation formula

Before proceeding with the exposition of our numerical scheme, let us first analyze the linear problem

$$\begin{cases} u_{tt} - \partial_x^2 u + 2\gamma(-\partial_x^2)^\theta u_t = h, & x \in [0, \pi], \ t > 0, \\ u(0, t) = u(\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \end{cases} \quad (4.1.1)$$

where $h \in L^1(0, T; L^2(0, \pi))$ is a given function, and the initial data belong to $\mathcal{E}$ as before. Note that the eigenvalues and the eigenfunctions, orthonormal in $L^2(0, \pi)$, of the Dirichlet operator $-\partial_x^2$ are given by

$$\lambda_n = n^2, \quad e_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx), \quad n \geq 1,$$

respectively. Thus, it is easy to see that, upon multiplying the linear equation by e_n , $n \geq 1$, and integrating by parts, we reach to a countable family of ODE's for the Fourier modes of the solution, $u_n(t) := (u(t), e_n)$. Solving these ODE's and utilizing the formula

$$u(t) = \sum_{n \geq 1} u_n(t) e_n,$$

we find an explicit representation formula for the solution to the linear problem (4.1.1), which, for convenience, may be written in the operator form as

$$\begin{aligned} u(t) = e^{\mathbf{A}t} & \left[\cosh(\sqrt{\mathbf{B}}t) u_0 + \frac{\sinh(\sqrt{\mathbf{B}}t)}{\sqrt{\mathbf{B}}} (u_1 - \mathbf{A}u_0) \right] \\ & + \int_0^t e^{\mathbf{A}(t-s)} \frac{\sinh(\sqrt{\mathbf{B}}(t-s))}{\sqrt{\mathbf{B}}} h(s) ds, \end{aligned} \quad (4.1.2)$$

where the operators $\mathbf{A}, \mathbf{B}$ are the real and (possibly) imaginary parts of the roots of the characteristic polynomial for the linear equation

$$\mathbf{R}_{1,2} = -\gamma(-\partial_x^2)^\theta \pm \sqrt{\gamma^2(-\partial_x^2)^{2\theta} + \partial_x^2} =: \mathbf{A} \pm \sqrt{\mathbf{B}}. \quad (4.1.3)$$

From this identity we can see that the real part of the roots of the characteristic polynomial of all the ODE's we have solved are negative, and the root of the characteristic polynomial for the n^{th} ODE has a nonzero imaginary part if and only if $\gamma^2 n^{2\theta} - n^2 < 0$. It follows that

almost all Fourier modes of the solution oscillate provided that $\theta \in [0, \frac{1}{2})$, and almost all of them purely decay provided that $\theta \in (\frac{1}{2}, 1]$. When $\theta = \frac{1}{2}$, then all modes either oscillate or purely decay, depending on whether $0 < \gamma < 1$ or $\gamma \geq 1$. In other words, the strength of the damping determines whether the linear equation is hyperbolic or parabolic.

For the future ease of presentation of ideas we rewrite the representation formula (4.1.2) in the following form:

$$u(t) = L_1(u_0, u_1, t) + \int_0^t L_2(h(s), t-s) ds. \quad (4.1.4)$$

Let us now say a few words on the operators L_1 and L_2 . First note that, the operator L_2 is continuous from $H_0^1(0, \pi)$ into $L^2(0, \pi)$ uniformly for any $t \geq 0$, that is there exists $C > 0$ independent of $t > 0$ such that

$$\|L_2(\varphi, t)\|_1 \leq C\|\varphi\|, \quad (4.1.5)$$

for all $\varphi \in L^2(0, \pi)$ and $t \geq 0$. Indeed, we have by the Parseval's identity

$$\begin{aligned} \|L_2(\varphi, t)\|_1^2 &= \sum_{n \geq 1} n^2 e^{-2\gamma n^{2\theta} t} \frac{|\sinh(\sqrt{\gamma^2 n^{4\theta} - n^2} t)|^2}{|\gamma^2 n^{4\theta} - n^2|} (\varphi, e_n)^2 \\ &\leq C \sum_{n \geq 1} (\varphi, e_n)^2 = C\|\varphi\|^2, \end{aligned}$$

where the inequality is due to the fact that, for each $\theta \in [0, 1]$ one can find $C > 0$, which does not depend on $n \geq 1$, such that

$$\frac{n^2}{|\gamma^2 n^{4\theta} - n^2|} \leq C, \quad \forall n \geq 1.$$

Let us here note that, although this inequality seems implausible when the denominator becomes zero, in which case we call that mode a degenerate one, this is not the case, because the expression for the degenerate modes is not given by the expression (4.1.2). The justification of this inequality for the degenerate modes is an easy exercise, and is not detailed here for the sake of a fluent exposition.

Next, again by the Parseval's identity we have for the operator L_1

$$\begin{aligned} \|L_1(\varphi, \psi, t)\|_1^2 &\leq C(\|\varphi\|_1^2 + \|L_2(\psi - \mathbf{A}\varphi, t)\|_1^2) \\ &\leq C(\|\varphi\|_1^2 + \|\psi\|^2 + \|L_2(\mathbf{A}\varphi, t)\|^2), \end{aligned}$$

which holds for any $(\varphi, \psi) \in H_0^1(0, \pi) \times L^2(0, \pi)$ and $t \geq 0$. The last term on the right hand side of the previous inequality can be estimated similar to before as

$$\|L_2(\mathbf{A}\varphi, t)\|^2 \leq C\|\varphi\|_1^2, \quad t \geq 0,$$

this time thanks to the fact that for any $\theta \in [0, 1]$ one can find $C > 0$ such that

$$\frac{n^{2\theta}}{|\gamma^2 n^{4\theta} - n^2|} \leq C$$

for each $n \geq 1$. Again, we do not include the degenerate case into our calculations. Consequently, we easily arrive at the estimate

$$\|L_1(\varphi, \psi, t)\|_1 \leq C(\|\varphi\|_1 + \|\psi\|), \quad (4.1.6)$$

which holds for any $(\varphi, \psi) \in H_0^1(0, \pi) \times L^2(0, \pi)$ and $t \geq 0$.

4.2 The numerical scheme

The idea of our numerical scheme comes from the representation formula (4.1.4). In fact, from this formula follows the following implicit relation for the solutions of the nonlinear problem (4.0.1):

$$u(t) = L_1(u_0, u_1, t) + \int_0^t L_2(f(u(s)), t-s) ds. \quad (4.2.1)$$

Now, fix $0 < T < T_{max}$, which denotes the total duration of our simulation. Suppose that, we have a mesh of size $(N_x + 1) \times (N_t + 1)$, that covers the rectangle $[0, \pi] \times [0, T]$, and let $\delta = \frac{T}{N_t}$ denote the time increment of our numerical scheme. For given $N \geq 1$, if P_N denotes the linear projection operator onto the first N Fourier modes, then we define the approximating numerical solution $\tilde{u}$ by the iterative formula

$$\tilde{u}((k+1)\delta) = L_1(P_N \tilde{u}(k\delta), P_N \tilde{u}_t(k\delta), \delta) + \int_0^\delta L_2(P_N f(P_N \tilde{u}(k\delta)), \delta-s) ds, \quad k \geq 0, \quad (4.2.2)$$

with the initial assignment $\tilde{u}(0) = P_N u_0$ and $\tilde{u}_t(0) = P_N u_1$. Typically one takes N such that $P_N u_0$ and $P_N u_1$ contain almost all of the L^2 -mass of u_0 and u_1 , respectively. Also, to minimize the errors in numerical integration one needs to take the space dimension of the mesh a couple of times bigger than N .

This algorithm is a one-step algorithm, in that each step utilizes in its calculations the data of the previous step only. This has the disadvantage of an increase in the error accumulation, which makes it harder to justify the convergence of the scheme, but on the other hand, has the advantage of speed. In fact, the code prepared with this algorithm has the execution time proportional to $N_x \times N_t \times N$, which is the best possible one can hope for from an algorithm, that outputs a matrix of size $N_x \times N_t$, and uses the Fourier decomposition method with N modes.

4.3 Error analysis

Primarily, let us define an auxillary solution $\hat{u}$. At the first step, for a small time increment $\delta > 0$, consider the linear problem which is obtained by replacing the nonlinearity $f(u(t))$ with the constant term $f(u_0)$. For $t \in [0, \delta]$, let $\hat{u}(t)$ be equal to the solution of this linear problem with initial data (u_0, u_1) . At the $(k+1)^{th}$ step, $k \geq 1$, plug $f(\hat{u}(k\delta))$ to the equation, and for $t \in [k\delta, (k+1)\delta]$, set $\hat{u}(t)$ to be the solution to the linear problem with initial data $(\hat{u}(k\delta), \hat{u}_t(k\delta))$. In other words we set

$$\hat{u}(k\delta + \tau) = L_1(\hat{u}(k\delta), \hat{u}_t(k\delta), \tau) + \int_0^\tau L_2(f(\hat{u}(k\delta)), \tau - s) ds, \quad k \geq 0, \quad \tau \in [0, \delta], \quad (4.3.1)$$

with the initial assignment $\hat{u}(0) = u_0$ and $\hat{u}_t(0) = u_1$.

To demonstrate the convergence of our approximate solution $\tilde{u}$ to the real solution u , we need to investigate the convergence of the auxillary solution $\hat{u}$ to u , and the convergence of the approximate solution $\tilde{u}$ to the auxillary solution $\hat{u}$. The justification of the latter approximation is more or less well-known, so we focus our attention on the former.

Let us fix $T > 0$ such that the local solution exists upto this time. We will let N_t increase, and correspondingly, δ will decrease towards zero, by which we hope to show that the error in the approximation accumulated in time T also decreases to zero. As the first step we estimate the initial error. In view of the estimate (4.1.5), from the identities (4.2.1) and (4.3.1) we deduce that

$$\begin{aligned} \|u(\delta) - \hat{u}(\delta)\|_1 &= \left\| \int_0^\delta L_2(f(u(s)) - f(u_0), \delta - s) ds \right\|_1 \\ &\leq C \int_0^\delta \|f(u(s)) - f(u_0)\| ds. \end{aligned} \quad (4.3.2)$$

In view of the estimates (4.0.2), (4.0.4) and the embedding inequality

$$\|u\| \leq \|u\|_1,$$

we conclude from the inequality (4.3.2) that

$$\begin{aligned} \|u(\delta) - \hat{u}(\delta)\|_1 &\leq C \int_0^\delta (\|u(s)\|_1^q + \|u_0\|_1^q) \|u(s) - u_0\|_1 ds \\ &\leq C_2 (\|u\|_{L^\infty(0, T; H_0^1)}) \delta M(\delta), \end{aligned} \quad (4.3.3)$$

where again C_2 is a bounded function, and M is monotone nondecreasing with $M(0) = 0$. Next, we want estimate the error at time $(k+1)\delta$ in terms of the error at time $k\delta$, $1 \leq k \leq N_t - 1$. For this purpose we introduce a new solution, denoted by $u_{k\delta}$, which starts its evolution at $t = k\delta$ with $u_{k\delta}(k\delta) = \hat{u}(k\delta)$, but follows the nonlinear dynamics. By the same considerations as before we have

$$\|u_{k\delta}((k+1)\delta) - \hat{u}((k+1)\delta)\|_1 \leq C_2(\|u\|_{L^\infty(0,T;H_0^1)})\delta M(\delta). \quad (4.3.4)$$

Moreover, by the continuous dependence on the initial data for the nonlinear dynamics, the inequality (4.0.3), we see that

$$\|u((k+1)\delta) - u_{k\delta}((k+1)\delta)\|_1 \leq C\|u(k\delta) - u_{k\delta}(k\delta)\|_1 = C\|u(k\delta) - \hat{u}(k\delta)\|_1. \quad (4.3.5)$$

Thus, combining the estimates (4.3.4) and (4.3.5) by the help of the triangular inequality we find the following iterative inequalities valid for $1 \leq k \leq N_t - 1$

$$\|u((k+1)\delta) - \hat{u}((k+1)\delta)\|_1 \leq C\|u(k\delta) - \hat{u}(k\delta)\|_1 + C_2(\|u\|_{L^\infty(0,T;H_0^1)})\delta M(\delta). \quad (4.3.6)$$

Plugging the estimate (4.3.3) for the first step into the inequality (4.3.6) for $k = 1$, and then plugging the resulting estimate into the inequality (4.3.6) for $k = 2$, and continuing in this manner until we plug the final estimate into the inequality (4.3.6) for $k = N_t - 1$, we easily arrive at the inequality

$$\|u(T) - \hat{u}(T)\|_1 \leq N_t C_2(\|u\|_{L^\infty(0,T;H_0^1)})\delta M(\delta) = C_2(\|u\|_{L^\infty(0,T;H_0^1)})TM(\delta).$$

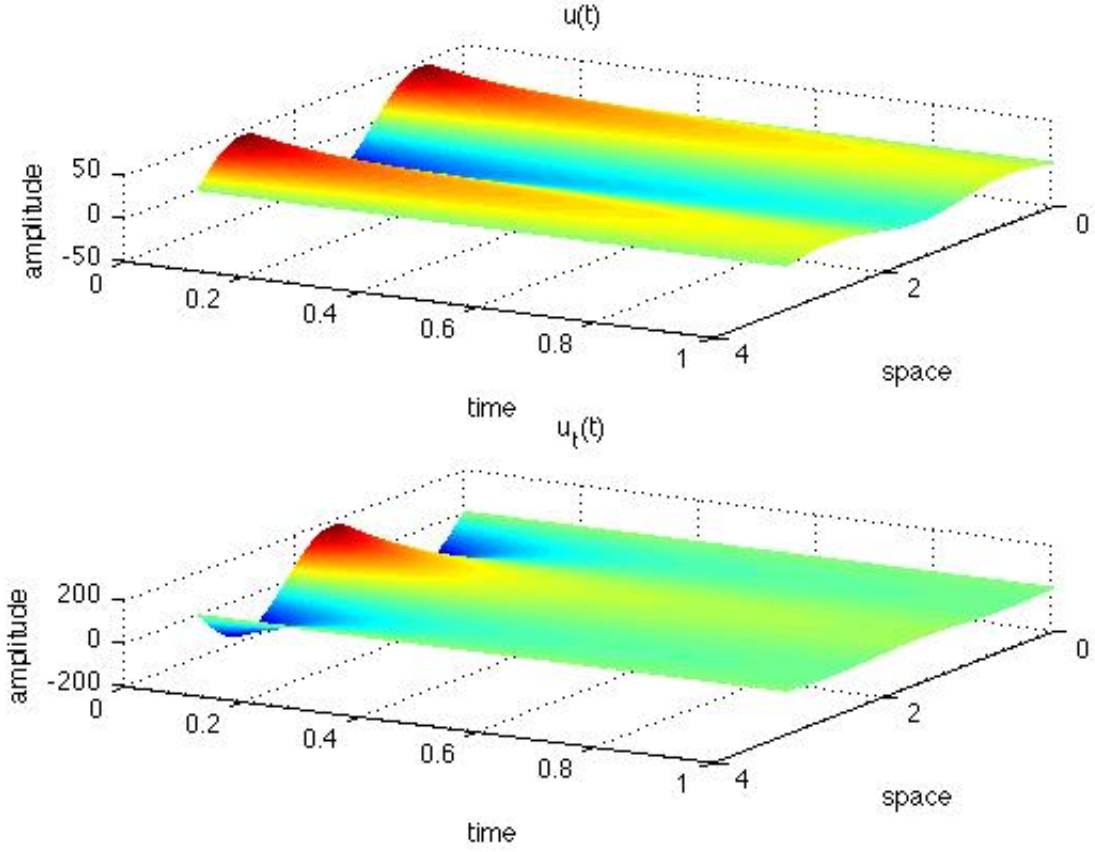
Finally, by the Sobolev embedding inequality we deduce the estimate

$$\|u(T) - \hat{u}(T)\|_{L^\infty} \leq C_2(\|u\|_{L^\infty(0,T;H_0^1)})TM(\delta),$$

which in view of the facts that M is continuous with $M(0) = 0$, C_2 is bounded and the assumption that the solution to the nonlinear problem locally exists in $[0, T]$, so that it belongs to $L^\infty(0, T; H_0^1)$, implies that, choosing $\delta > 0$ small enough we may bring the exact nonlinear solution u and the auxillary solution $\hat{u}$ as close as we want at time T with respect to the supremum norm.

4.4 Outputs of simulations

In this section we present the outputs of the simulations, which we have carried out for the linear, stable and unstable versions of the problem (4.0.1), see Figure 4.1 - Figure 4.5. In

FIGURE 4.1: Linear dynamics of a strongly damped wave equation, $f(u) = 0$.

all cases, in order to compare, we consider the same problem, namely

$$\begin{cases} u_{tt}(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) - \frac{\partial^2}{\partial x^2} u_t(x, t) = \pm u^3(x, t), & x \in (0, \pi), \quad t > 0, \\ u(0, t) = u(\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in [0, \pi]. \end{cases} \quad (4.4.1)$$

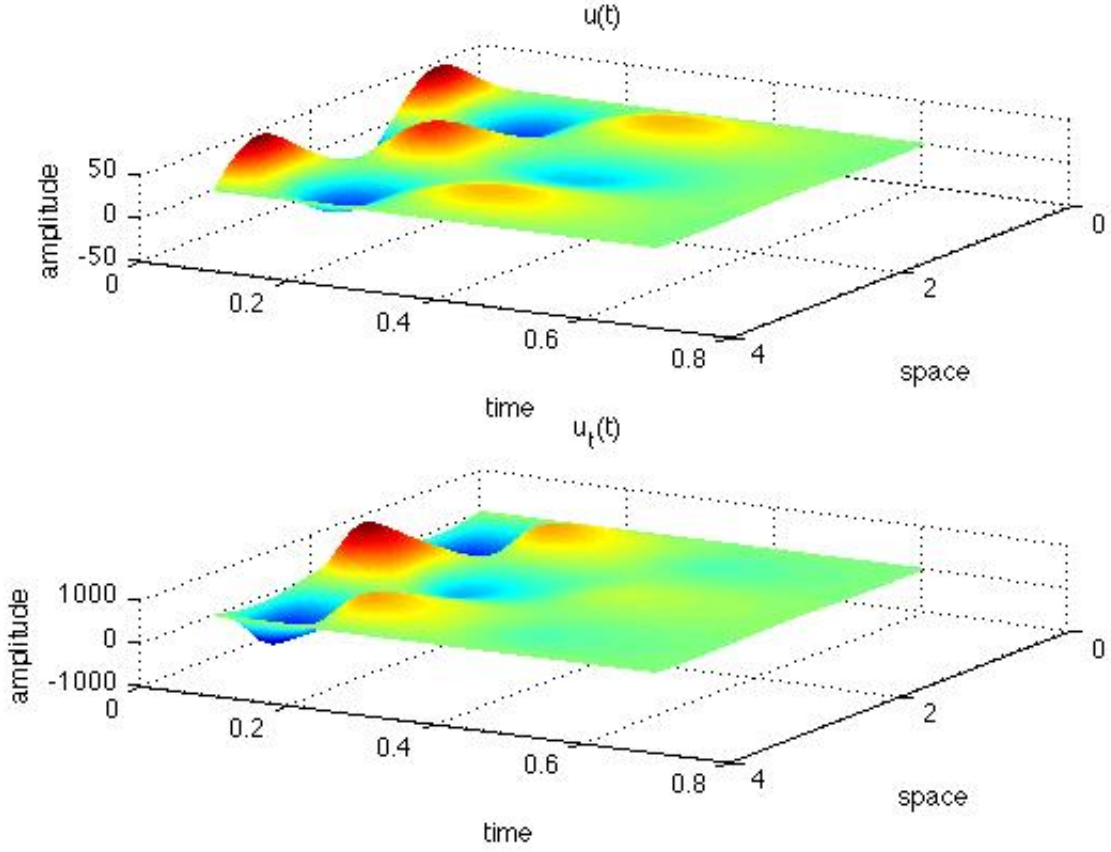
Note that, when we have the plus sign in the nonlinearity, this problem is the same problem as considered in §2.5, where the potential well theory was introduced.

The first three figures all have the same initial data, namely

$$u_0(x) = \frac{50\sqrt{2\pi}}{3} \sin(3x), \quad u_1(x) = -\frac{200\sqrt{2\pi}}{3} \sin(3x).$$

These functions are the initial functions derived in §2.2.1.1, which have high positive initial energy, namely, see (2.2.8)

$$E(u_0, u_1) = 625 \frac{\pi}{6},$$

FIGURE 4.2: Stable nonlinear dynamics of a strongly damped wave equation, $f(u) = -u^3$.

satisfy

$$\int_0^\pi u_0(x)u_1(x)dx = -\frac{20000\pi}{3} < 0,$$

and the corresponding solution blows up in finite time (in the case of unstable problem). Note that, this value of the initial energy is well above the depth of the potential well, which in (2.5.63) in §2.5 was estimated from above as

$$d \leq \frac{\pi}{6}.$$

The Figure 4.1 shows the output of the simulation carried out for the linear equation. The Figure 4.2 and the Figure 4.3 show the results for the stable and unstable nonlinear problems, respectively. Note that, for the unstable problem the effect of negative inner-product of the initial data may be observed as a transient period, where the solution u ‘waits’ for its time derivative to evolve into the positive inner-product region, and then the blow-up occurs in a rather short time. Also, there is an unexpected observation in the case of the unstable problem. Note that, we have chosen the initial data to be even symmetric with respect to the point $x = \frac{\pi}{2}$. It is also clear that the change of variable

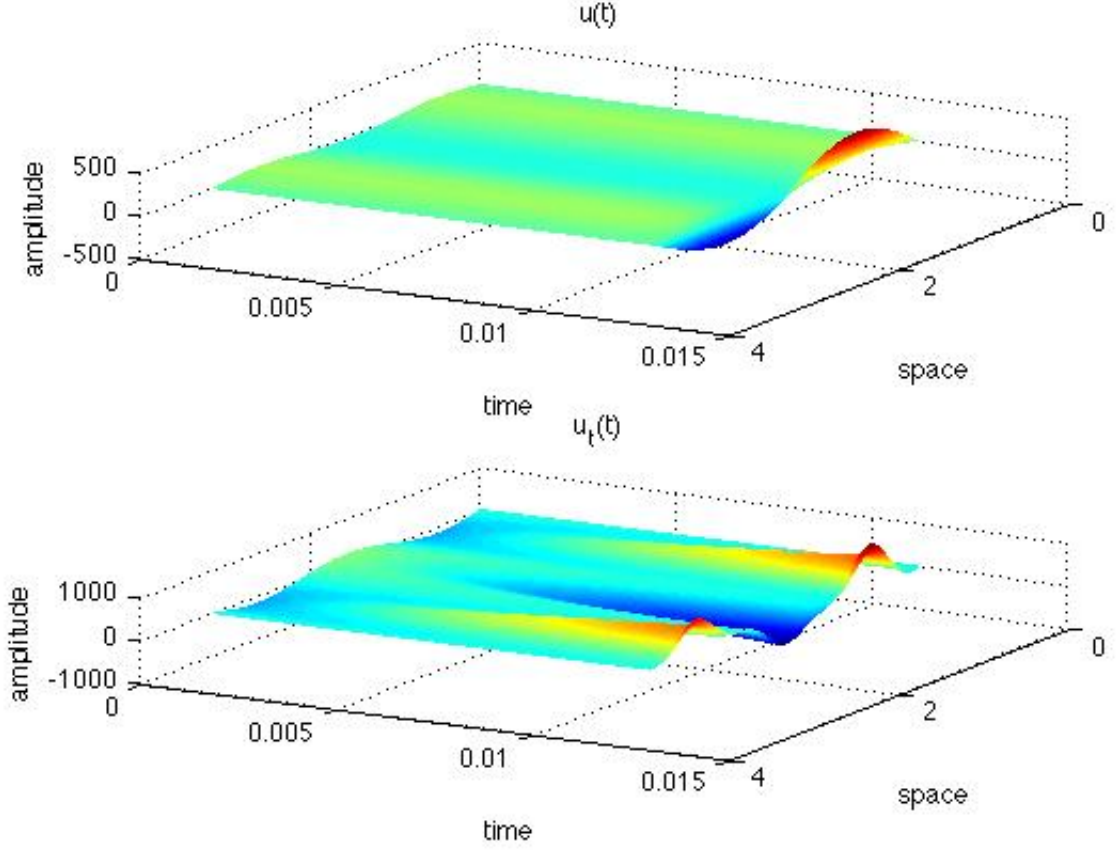


FIGURE 4.3: Unstable nonlinear dynamics of a strongly damped wave equation with blow-up,
 $f(u) = u^3$.

$\hat{x} = \pi - x$ does not change the equation. It follows that the corresponding solution must remain even symmetric around $x = \frac{\pi}{2}$. However, as it can be seen from Figure 4.3, this symmetry breaks shortly before the blow-up. Another interesting observation is the fact that the stable nonlinearity helps the solution decay faster compared to the linear solution, and it also introduces oscillations, which were nonexistent in the linear dynamics.

The last two figures are the outputs of the simulation for the unstable problem with initial data described as in the end of the section §2.5. Namely, we set, see the Figure 4.5,

$$(u_0(x), u_1(x)) = \left(\frac{2\sqrt{3}}{3} \sin(x), 0 \right) =: (\hat{\phi}_0, 0),$$

which corresponds to the initial data that was hoped to be close to the unstable steady state solution, and we take

$$(u_0(x), u_1(x)) = (0.99\hat{\phi}_0, 0),$$

see the Figure 4.4, which is just inside the stable set $\mathcal{S}_d$ introduced in §2.5. The outputs

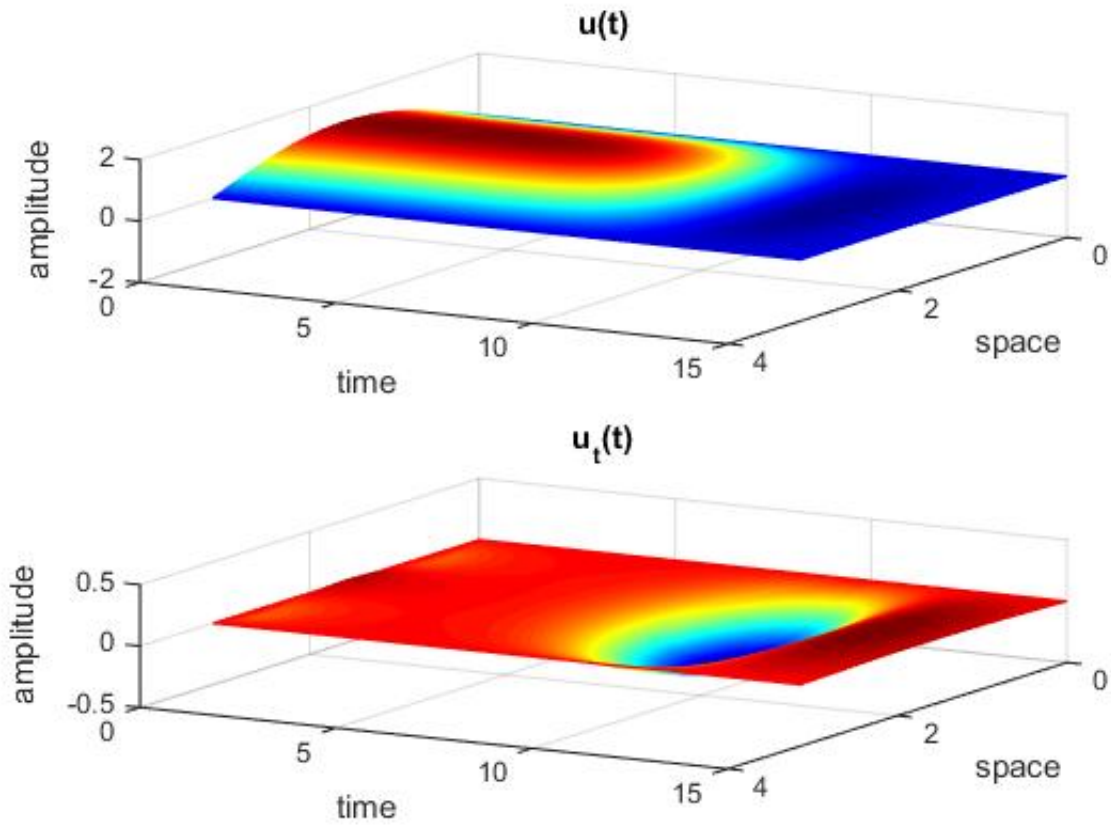


FIGURE 4.4: Unstable nonlinear dynamics of a strongly damped wave equation with initial data inside the stable set $\mathcal{S}_d$.

verify the theoretical results obtained, that is, the function $\hat{\phi}_0$ is close to the solution of the time independent problem, and the solutions starting from the stable set $\mathcal{S}_d$ decay to zero exponentially.

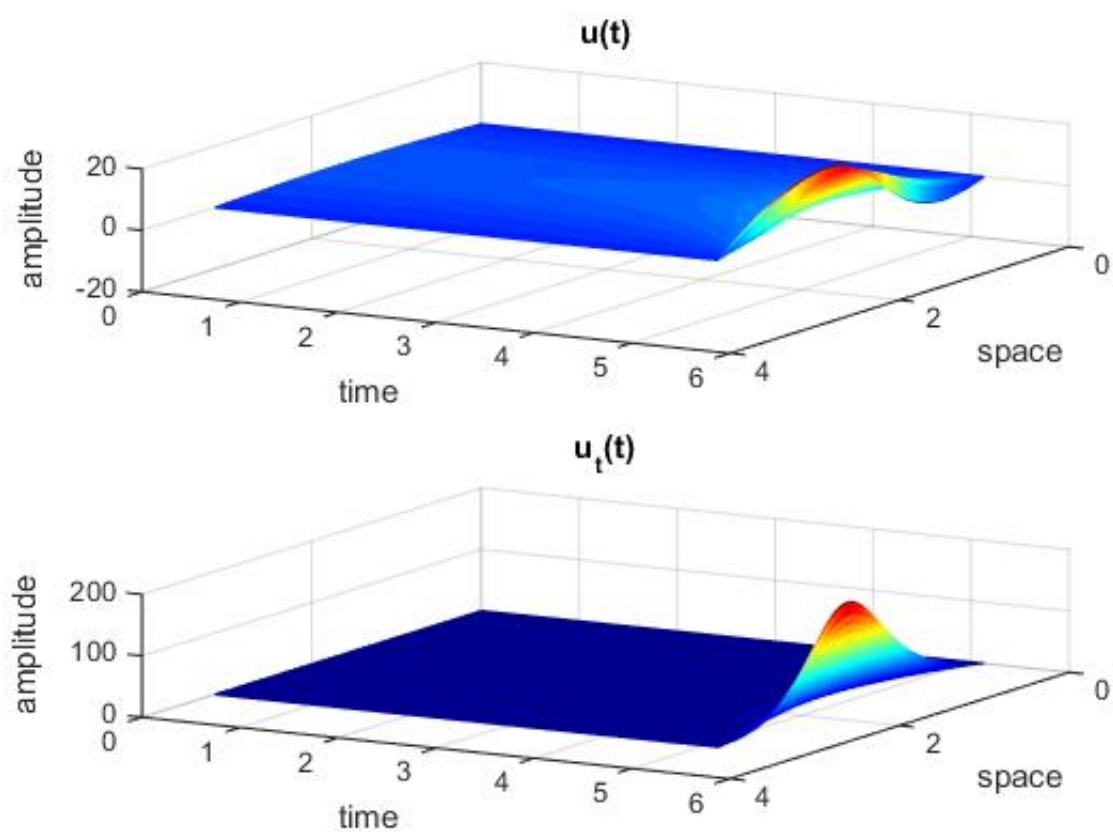


FIGURE 4.5: Unstable nonlinear dynamics of a strongly damped wave equation close to a unstable steady state solution.

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