AUCTION DESIGN AND OPTIMAL ALLOCATION BY LINEAR PROGRAMMING

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By
Halil İbrahim Bayrak
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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For the sale of a single object through an auction, we assume discrete type space for agents and make use of linear programming to find optimal mechanism design for a risk-neutral seller. First, we show that the celebrated incentive compatible mechanism, second price auction, is not optimal. We find a slightly different optimal mechanism referred to as "discrete second price auction". Second we consider the problem of allocation with costly inspection. We obtain the optimal solution in the form of a favored-agent mechanism by the Greedy Algorithm. Moreover, we relax the common prior assumption and maximize the worst-case utility of an ambiguity averse seller for the two problems mentioned above. While the problem does not yield a useful optimal mechanism in general, optimal solutions for some special cases are obtained.

Keywords: Linear programming, Auction design, Costly verification, Ambiguity aversion.

ÖZET

DOĞRUSAL PROGRAMLAMA İLE İHALE TASARIMI VE EN İYİ ATAMA

Halil İbrahim Bayrak
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Ağustos, 2015

Bu çalışmada, bir ürünün ihalesi için, alıcıların ayrık değerlere sahip oldukları varsayılarak doğrusal programlama ile riske duyarsız satıcı için en iyi ihale tasarımı bulunmuştur. Ünlü teşvik uyumlu tasarım, en iyi ikinci fiyatla ihalenin en iyi çözüm olmadığı görülmüştür. En iyi tasarımın ondan biraz farklı olan "ayrık en iyi ikinci fiyatla ihale" olduğu gösterilmiştir. İkinci olarak, maliyetli denetleme ile atama problemine bakılmıştır. Açgözlü algoritma kullanılarak, en iyi sonucun ayrıcalıklı katılımcı tasarımı olduğu gösterilmiştir. Ayrıca ortak olasılık dağılımı varsayımı gevşetilip, yukarıdaki iki problemde belirsizlikten kaçınan satıcı için en kötü durumdaki fayda enbüyüklenmiştir. Bu problemin genel olarak en iyi çözümü olmasa da bazı özel durumlar için en iyi tasarım bulunmuştur.

Anahtar sözcükler: Doğrusal programlama, İhale tasarımı, Maliyetli denetleme, Belirsizlikten kaçınma.

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Chapter 1

Introduction

Mechanism design is a subsection of game theory. We can simply define the interest of this topic as design of games between agents who wish to claim a set of goods. A game will require agents to take an action upon which the good will be allocated according to the design of the game. In the literature, agents are assumed to be rational so they will use the design of the game to maximize their utility. In other words, participants will have strategies and we need to consider their optimal strategy to design the game which results in an allocation of our choice. The authority who has the good may have different kinds of objectives. He may want to sell it at a good price or he may want to give it to the agent who values it most. Whatever the objective may be, undertaking a game allows the owner of the good to deduce information about the agents when their valuations of the good are unknown.

Our work on mechanism design is highly motivated by Vohra [1]. In his paper [1] and book [2], he used linear programming to derive the celebrated results from mechanism design literature. As he stated in [1], we will also use duality theorem and elementary results from linear programming and we will show more detailed characterizations of the optimal mechanisms.

As a further step, we relax the common prior assumption from the literature, which means that agents will not take their valuations with respect to a single distribution. We assume that agents still take valuations from a distribution but we do not know it exactly. This setting will introduce ambiguity into our framework. The owner of the good is assumed to be ambiguity averse so he wishes to maximize his worst case objective. We assume that buyers are ambiguity neutral.

The rest of this thesis includes Chapter 2 where we will review the mechanism design literature and give concepts that will be used later. In Chapter 3, we look at the problem of designing a mechanism with monetary transfers in order to maximize the expected revenue. This problem has been already solved by Myerson [3] as second price auction with reserve price in continuous type spaces. Vohra [1] derived a similar solution using linear programming in discrete type space. We will first show that the optimal mechanism in discrete type space for the Myerson framework is not exactly the second price auction. Chapter 4 deals with allocation with inspection problem which was first considered and solved by Ben-Porath, Dekel and Lipman [4]. For this mechanism we do not utilize monetary transfers but we can inspect an agent's report. Optimal mechanism was derived in [4] for a continuous type space and asymmetric agents. For this problem, Vohra [1] shows some results about the optimal mechanism when agents are symmetric and type space is discrete. However, there are some unanswered issues that we will settle here utilizing linear programming. In Chapter 5, we relax the common prior assumption and consider these two problems in the shoes of an ambiguity averse seller. We will maximize the worst case objective of the seller and give the structure of the optimal mechanism under some assumptions. Finally we will have conclusions in Chapter 6.

Chapter 2

Literature Review

Second price auction is an important part of the literature when we consider the mechanisms with monetary transfers. These mechanisms allow the designer to charge agents. In the second price auction, the highest bid obtains the good and pays the amount of the second highest bid.

Vickrey is known as the introducer of the second price auction to academics. In his paper [5], he analyzed the Dutch auction and the second price auction and showed the ways in which second price auction is superior to Dutch auction. In Dutch auction, we begin with a high asking price and lower it until some agent is willing to pay the announced price. Vickrey first studied the Dutch auction and showed that it is inefficient for securing an optimum allocation. Then he showed that shifting to second price auction provides an over-all gain in most of the cases. Secondly, in Dutch auction, since a bidder should have some strategy based on the information he has about other bidders' type, any failure to misinterpret in this matter would increase the chances that optimum allocation will not be achieved. However, the second price auction proves to be strategy proof and overcomes this disadvantage. Then a modified Dutch auction is presented, which is equivalent to the second price auction's procedure. Vickrey also pointed out some disadvantages of these auctions in practice. One can also consult his findings on these auctions where there are multiple items to be auctioned.

Later Myerson [3] showed that in fact, second price auction with reserve price is the optimal mechanism to maximize owner's expected revenue for the sale of a single good. In the process of proving this result, he uses The Revelation Principle which says that for any feasible mechanism there exists a direct mechanism which gives the same utilities to the participants. Direct mechanisms are strategy proof. In other words, it is a dominant strategy for any agent to report their true type. We shall also utilize the revelation principle and restrict our feasible mechanisms to direct mechanisms without loss of generality.

In [3], Myerson made another important contribution, the Revenue Equivalence Theorem. It says that expected revenue of the seller is determined by the allocation rule and utilities of agents when they bid their lowest type. Therefore, if two auctions that charge agents differently have the same mentioned attributes, then they both yield the same expected revenue.

When we look at the problem of mechanism design with monetary transfers, we will closely follow Vohra [1] in which optimality of second price auction with reserve price is derived using linear programming. Use of this mathematical tool forces Vohra to work on discrete type space for agents. Although Vohra does not discuss it, this deviation from the framework of Myerson [3] also affects the optimal way of charging the agents as we will show in Chapter 3.

Before we continue with the literature in allocation with costly verification, it will prove to be useful discussing the mechanism and its optimal solution in advance. As the designer, we need to specify: (1) the allocation rule that will decide who the good will be awarded to, (2) inspection rule that says which agent's report should be inspected with what probability. In the optimal solution, which is a favored-agent mechanism, we have an optimal cut-off and optimal choice of a favorite agent. While other agents are allocated and inspected if they are the highest bid over the cut-off, favorite agent is allocated and not inspected if other agents fall under the cut-off.

Ben-Porath, Dekel and Lipman [4] were the first to consider mechanism design with costly inspection. They take out the monetary transfers and assume that agents are unsymmetrical. They state that revelation principle also applies here so that they focus on direct mechanisms. It is shown that optimal auction is in the form of favored-agent mechanism.

Using linear programming, Vohra [1] also shows that the optimal mechanism for the inspection problem has a threshold above which the highest bidder is awarded and inspected. Since he assumes that agents are symmetric, under that threshold, randomizing the good and no inspection is optimal. This solution under the symmetry assumption is also pointed out by Ben-Porath, Dekel and Lipman [4]. In [1], analysis of the optimal mechanism is incomplete since Vohra does not give any qualitative information about the threshold. In Chapter 4, we will use linear programming and give the optimal threshold. We will also consider the case when inspection cost is different for each agent and end up with a discrete analog of the favored-agent mechanism from [4].

For Chapter 5, we will relax an assumption from [1], common prior assumption which we also have for other chapters. This assumption means that all agents take valuations with respect to a single commonly known distribution. We will look at the problem of mechanisms with monetary transfers and with costly inspection when the valuation distribution is unknown. This setting introduces ambiguity to our framework. We assume that the seller is ambiguity averse and buyers are ambiguity neutral.

This relaxation is also considered by Bose, Ozdenoren and Pape [6]. We will follow their work in the following sense: (1) they follow Gilboa and Schmeidler and model the ambiguity aversion using the maxmin expected utility model, (2) they focus on direct mechanisms. They state that revelation principle holds in their setting which also covers our setting. Interested reader can also refer to [7] which also has ambiguity averse seller and ambiguity neutral buyers and says that revelation principle holds. In [6], it is shown that in the set of optimal mechanisms, there exists a full insurance mechanism.

Chapter 3

Mechanism Design with Monetary Transfers for Discrete Valuations

In this chapter, we will first go through the work of Vohra [1] where he derives the results of Myerson [3] via linear programming in a discrete type space setting. However, we will see that working on a discrete type space for buyers results in a deviation from the second price auction contrary to a claim by Vohra. We shall show a modification where the second price auction is optimal.

3.1 Review of Optimization and Mechanism Design

There is a risk neutral seller with a single good and n risk neutral buyers that have non-negative private valuation for the good. These private valuations will be addressed as "agents' types" and valuation of the seller is assumed to be 0. Vohra defines $T = \{1, 2, \ldots, m\}$ as the type space whose discrete form will allow him to use linear programming. The common prior assumption means that types of agents are independent draws from a commonly known distribution F. He also assumes that the density function of F satisfies $f_i > 0$ for all $i \in T$. Another important element of this setting is the Revelation Principle through which he focuses on direct mechanisms only.

Before problem formulation, let us give the notation. We use $t \in T^n$ to denote a profile vector. The symbols a and p are defined to be allocation and payment rule, respectively. The symmetry assumption allows focusing on one agent, say agent 1. We use $a_i(i, t^{-1})$ for the allocation to agent 1, and $p_i(i, t^{-1})$ is the payment made by agent 1 when she reports her type as $i \in T$ and all other agents report t^{-1}. We use $\pi(t^{-1})$ for the probability of agents having types that give rise to the profile t^{-1}. Number of agents with type i in profile t is shown by $n_i(t)$. Interim (expected) allocation and payment are denoted accordingly when all agents other than 1 report their type truthfully:

$$A(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1})\pi(t^{-1}),$$
$$P(i) = \sum_{t^{-1} \in T^{n-1}} p_i(i, t^{-1})\pi(t^{-1}).$$

The objective of the problem is to maximize seller's expected revenue, and we face the following constrained maximization problem:

$$\begin{aligned}
\max_{P,A,a} \quad & \sum_{i \in T} f_i P(i) & & \\
\text{s.t.} \quad & iA(i) - P(i) \geq iA(j) - P(j) & \forall i, j \in T & \quad (1) \\
& iA(i) - P(i) \geq 0 & \forall i \in T & \quad (2) \\
& A(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1})\pi(t^{-1}) & \forall i \in T & \quad (3) \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 & \forall t \in T^n & \quad (4) \\
& a_i(t) \geq 0 & \forall i \in T, \forall t \in T^n. & \quad (5)
\end{aligned}$$

Obviously, constraints (4) and (5) ensure that only one good is allocated for each profile and no agent receives a negative amount while constraint (3) only relates interim allocation variables to allocation rule variables. Constraint (2) expresses the individual rationality. The left hand side is expectation of the agent's utility when she reports her type truthfully. We force it to be non-negative. Otherwise, for a risk neutral agent, participating to our mechanism would not be rational. Finally, constraint (1) is called the Bayes-Nash Incentive Compatibility (BNIC). With this restriction, we will ensure that reporting a different type than the true one will result in an expected utility which is less than or equal to the case when the type is truthfully reported. It is clear that we are only interested in the mechanisms in which the optimal strategy is to report truthfully. Such mechanisms are also called strategy-proof and it simplifies the quest for the optimal mechanism.

Next, Vohra [1] looks into the system of constraints (1) and (2). Let us rewrite them as:

$$iA(i) - iA(j) \geq P(i) - P(j) \qquad \forall i, j \in T, \qquad (1)$$
$$iA(i) \geq P(i) \qquad \forall i \in T. \qquad (2)$$

Introducing a vertex for each type and an arc between every type (i, j) of length $iA(i) - iA(j)$, we will obtain the following network.

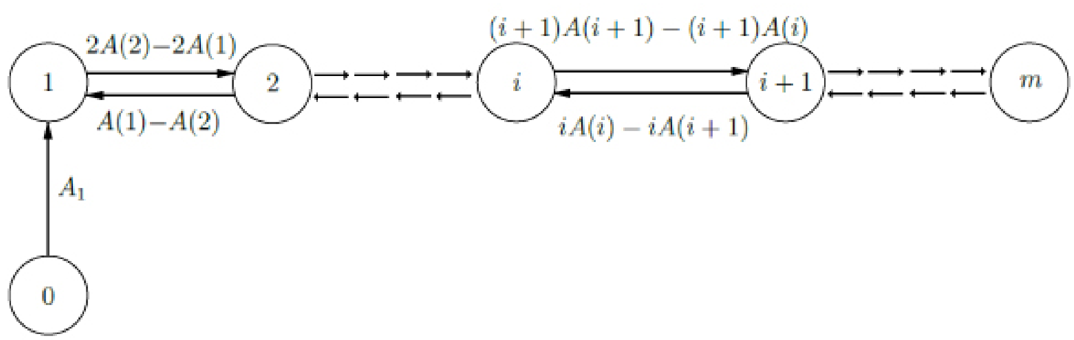

Figure 3.1: Network of types

Let us further consider the length of the cycle $i \to i+1 \to i$:

$$\begin{gathered}(i+1)A(i+1) - (i+1)A(i) + iA(i) - iA(i+1) \geq P(i+1) - P(i) + P(i) - P(i+1), \\ A(i+1) - A(i) \geq 0.\end{gathered} \qquad (3.1)$$

Theorem 3.1. *The system (1) is feasible if and only if interim allocations are monotonic. If $i \leq j$, then $A(i) \leq A(j)$.*

The idea behind the proof of Theorem 3.1 is utilizing the duality theorem of linear programming. As it is pointed out in [1], system (1) is the dual of shortest path problem on the network 3.1. Duality theorem states that if the dual of a problem is unbounded than the primal is infeasible. Therefore, in the network we need a bounded solution which will be satisfied only if the network has no negative cycle length. As we showed in the equation 3.1, this is equivalent to monotonicity of interim allocations.

Now set $A(0)$ and $P(0)$ to 0 and calculate the shortest path from node 0 to i:

$$\sum_{k=1}^{i} kA(k) - kA(k-1) \geq P(i). \tag{3.2}$$

It turns out that the network 3.1 does not satisfy the triangle inequality and all the shortest paths between all the nodes are present in the figure. The left hand side of the inequality 3.2 is the tightest upper bound for each $P(i)$. Then in order to maximize expected revenue set each $P(i)$ to its upper bound:

$$\begin{aligned} \sum_{i\in T} f_i P(i) &= \sum_{i\in T} f_i \sum_{k=1}^{i} kA(k) - kA(k-1) = \sum_{i\in T} f_i (iA(i) - \sum_{k=1}^{i} A(k-1)) \\ &= \sum_{i\in T} f_i iA(i) - (1-F(i))A(i) = \sum_{i\in T} f_i \nu(i) A(i), \end{aligned}$$

where $\nu(i) = i - \frac{1-F(i)}{f_i}$, hazard function is substracted from the agent's type. If hazard function is monotone then $\nu(i)$ is non-decreasing in i. Now the formulation is:

$$\begin{aligned} \max_{A,a} \quad & \sum_{i\in T} f_i \nu(i) A(i) \\ \text{s.t.} \quad & 0 \leq A(1) \leq \cdots \leq A(m) \\ & A(i) = \sum_{t^{-1}\in T^{n-1}} a_i(i, t^{-1})\pi(t^{-1}) && \forall i \in T && (3) \\ & \sum_{i\in T} n_i(t) a_i(t) \leq 1 && \forall t \in T^n && (4) \\ & a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n. && (5) \end{aligned}$$

After this point, Vohra [1] takes out $a_i(t)$ variables in order to end up with a polymatroid optimization problem.

Theorem 3.2. ***(Border's Theorem)*** *The expected allocation $A(i)$ is feasible if and only if*

$$n \sum_{i \in S} f_i A(i) \leq 1 - (\sum_{i \notin S} f_i)^n \qquad \forall S \subseteq T. \tag{3.3}$$

Interested reader can be directed to Border [8] for the proof. In [1], it is pointed out that system of constraints (3), (4) and (5) is actually a transportation problem. If we apply the maxflow-mincut characterization, we can see that the system is feasible if and only if the inequalities 3.3 are satisfied.

Now the formulation becomes:

$$\begin{aligned} \max_A \quad & \sum_{i \in T} f_i \nu(i) A(i) \\ \text{s.t.} \quad & 0 \leq A(1) \leq \cdots \leq A(m) \\ & n \sum_{i \in S} f_i A(i) \leq 1 - (\sum_{i \notin S} f_i)^n \qquad \forall S \subseteq T. \end{aligned}$$

Then, Vohra [1] defines the function $G(S)$ which is non-decreasing, non-negative and submodular. He sets $x_i = f_i A(i)$ for all $i \in T$.

$$\begin{aligned} \max_A \quad & \sum_{i \in T} \nu(i) x_i & (OPTb) \\ \text{s.t.} \quad & 0 \leq \frac{x_1}{f_1} \leq \cdots \leq \frac{x_m}{f_m} \\ & \sum_{i \in S} x_i \leq G(S) = \frac{1 - (\sum_{i \notin S} f_i)^n}{n} \qquad \forall S \subseteq T. \end{aligned}$$

Ignoring the monotonicity constraint, this becomes a polymatroid optimization problem which can be solved optimally by the Greedy Algorithm. Under the monotone $\nu(i)$ function assumption, the optimal solution is as follows: (i^* is the lowest type such that $\nu(i^*) \geq 0$)

$$A(i) = \begin{cases} \frac{F(i)^n - F(i-1)^n}{f_i n} & \text{if } i \geq i^* \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that this solution for $A(i)$ satisfies the monotonicity constraint. Therefore it is optimal for $OPTb$.

3.2 Optimal Payment Rule

According to Vohra [1] the second price auction with reserve achieves the same interim allocation and expected payment values as the optimal solution of $OPTb$. However, we will see that this is not the case. Recall the optimal solution for the expected payments:

$$P(i) = \sum_{k=1}^{i} kA(k) - kA(k-1) = iA(i) - \sum_{k=1}^{i} A(k-1).$$

Let us define the underlying payment rule as $p_i(i, t^{-1})$ which is the payment that agent 1 makes when he reports i and others report t^{-1}. Next, we show that following setting is consistent with the expected payments:

$$p_i(i, t^{-1}) = ia_i(i, t^{-1}) - \sum_{k=1}^{i} a_{k-1}(k-1, t^{-1}), \tag{3.4}$$

$$\sum_{t^{-1} \in T^{n-1}} p_i(i, t^{-1})\pi(t^{-1}) = \sum_{t^{-1} \in T^{n-1}} \big(ia_i(i, t^{-1}) - \sum_{k=1}^{i} a_{k-1}(k-1, t^{-1})\big)\pi(t^{-1}) = P(i).$$

Consider the allocation rule in which we will allocate the good to the highest bidder if he is over the reserve price i^*. In [1], it is showed that this allocation rule is consistent with the optimal interim allocations. However, when we use this allocation rule to find the optimal price rule from the equation 3.4, we find a payment scheme different from the second price auction. From now on we will call this mechanism "discrete second price auction" where the second highest bid is defined as $s = \max_{k \in t^{-1}} k$. The allocation and payment formulae are as follows:

$$a_i^*(i, t^{-1}) = \begin{cases} \frac{1}{n_i(t)} & \text{if } i \geq \max_{k \in t^{-1}} k \geq i^* \\ 0 & \text{otherwise} \end{cases}$$

$$p_i^*(i, t^{-1}) = \begin{cases} \frac{i}{n_i(t)} & \text{if } i = \max_{k \in t^{-1}} k \geq i^* \\ s + \frac{n_s(t^{-1})}{n_s(t^{-1})+1} & \text{if } i > s = \max_{k \in t^{-1}} k \geq i^* \\ i^* & \text{if } i \geq i^* > \max_{k \in t^{-1}} k \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.1. *For this example, we have two agents and our type set is* $T = \{1, 2, \ldots, 10\}$. *Consider the allocation rule where the reserve price is* 6. *Then we can calculate the payment of winner in profile* $(8, 2)$ *as* 6 *since the winner would not change if he bid* 7 *or* 6. *However, for the profile* $(8, 6)$ *payment is different from the second price auction:*

$$p_8(8, 6) = 8 \cdot a_8(8, 6) - a_7(7, 6) - a_6(6, 6) = 8 - 1 - 0.5 = 6.5.$$

In [9], Edelman, Ostrovsky and Schwarz investigate the Generalized Second Price Auction which is used in keyword auctions for online advertising. They point out that winner pay the second highest bid plus a minimum increment. This is consistent with our optimal solution, discrete second price auction. If we think about the type set of agents in terms of money, it is clear that it cannot be continuous.

Actually, this difference has resulted from the assumption that the type set is discrete. The second price auction defined in Myerson does not have the case where some agents share the good because this case has zero probability when the type set is continuous. However, we have this case with positive probability, and the seller can improve his expected revenue by exploiting the sole winner. After all, the sole winner does not share the good with the second highest bidders. Also realize that as the number of the agents who are willing to pay the second highest bid increases, payment of the sole winner increases, and converges to the type just above the second highest bid.

Now consider another setting where we allow buyers to make bids only in the multiples of q where $0 < q < 1$. Again we will work on a finite set so we set M as the highest bid such that $M = m \cdot q$. Our type set is the same as before, $T = \{1, 2, \ldots, m\}$, but this time, type i actually refers to bid $i \cdot q$. We will have the following changes in the formulation:

$$
\begin{aligned}
\max_{P,A,a} \quad & q\sum_{i\in T} f_i P(i) \\
\text{s.t.} \quad & qiA(i) - qP(i) \geq qiA(j) - qP(j) && \forall i,j \in T && (1)\\
& qiA(i) - qP(i) \geq 0 && \forall i \in T && (2)\\
& A(i) = \sum_{t^{-1}\in T^{n-1}} a_i(i,t^{-1})\pi(t^{-1}) && \forall i \in T && (3)\\
& \sum_{i\in T} n_i(t)a_i(t) \leq 1 && \forall t \in T^n && (4)\\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n. && (5)
\end{aligned}
$$

In the constraints (1) and (2), q cancels out, and since it is a scalar we can also remove it from the objective. Having the same assumptions on distribution F, the optimal solution will be the same with a slight difference in the payment rule:

$$
p_i(i,t^{-1}) = \begin{cases} q\cdot \frac{i}{n_i(t)} & \text{if } i = \max\limits_{k\in t^{-1}} k \geq i^* \\ q\cdot s + q\cdot \frac{n_s(t^{-1})}{n_s(t^{-1})+1} & \text{if } i > s = \max\limits_{k\in t^{-1}} k \geq i^* \\ q\cdot i^* & \text{if } i \geq i^* > \max\limits_{k\in t^{-1}} k \\ 0 & \text{otherwise.} \end{cases}
$$

Next we want to show that taking the limit of this payment rule as q approaches to 0 will give us the second price auction but before that we need to rewrite the payment rule in a different way since as q changes our type set will also change. Take a j value which is a multiple of q and less than M so that buyers can have it as their bid. Then there exists a type $j' \in T$ such that $j = j' \cdot q$ (define s and i^* in the same way):

$$
p_{j'}(j',t^{-1}) = \begin{cases} q\cdot \frac{j}{q}\cdot \frac{1}{n_{j'}(t)} & \text{if } j' = \max\limits_{k\in t^{-1}} k \geq i^* \\ q\cdot \frac{s}{q} + q\cdot \frac{n_{s'}(t^{-1})}{n_{s'}(t^{-1})+1} & \text{if } j' > s' = \max\limits_{k\in t^{-1}} k \geq i^* \\ q\cdot \frac{i^*}{q} & \text{if } j' \geq i^{*'} > \max\limits_{k\in t^{-1}} k \\ 0 & \text{otherwise.} \end{cases}
$$

As q approaches to 0, type set should also change in order to satisfy $j = j' \cdot q$ for any positive $j \leq M$. It is clear that the only point that made our optimal payment rule different from the second price auction is now gone. Therefore when we take the limit, we get exactly the Myerson's [3] second price auction in continuous types. Then we can say that the optimal solution Vohra [1] reached by using linear programming is consistent with the previous results.

Note that this setting is also considered by Harris and Raviv [10] when there are two agents. Because of an assumption they have for the seller's valuation, their optimal payments scheme is same as our result with reserve price zero. They also state that when the difference between two subsequent types approaches to zero optimal mechanism becomes the second price auction.

Chapter 4

Allocation with Costly Inspection for Discrete Valuations

Imagine that you are the head of an organization which has different kinds of departments. You want to allocate a resource to the department which can put it to its best use to profit the organization. All departments privately know that they can derive some profit from the resource and they all desire to claim the resource to be credited for the profit. It is not logical to use monetary transfers in this problem since what your departments pay lowers the profit that they can achieve for the organization. Therefore as Ben-Porath, Dekel and Lipman [4] propose, you will be given another tool, inspection. We can describe the usage of this tool in a game structure as follows: you first ask departments to report the contribution that they can make if they claim the resource. Then, according to allocation rule of the game, you choose the winner and according to inspection rule you inspect the accuracy of its report. Inspection is error free but it carries some cost. Therefore, you would be better off if you skip the inspection in some cases. Your objective is to find the optimal mechanism consisting of the allocation and inspection rules to maximize the organization's total welfare. In other words, you want to make an efficient allocation and minimize the inspection cost at the same time.

For this chapter, we will look into two versions of this problem. In the first one, agents are symmetric as in Vohra [1]. Our work is different from [1] in two ways: (1) we work on a different model, (2) we give the exact optimal mechanism. Then we shall look into the case where agents have different costs of inspection. Note that our results are discrete type space analogs of Ben-Porath, Dekel and Lipman's [4] results in continuous type space.

4.1 Symmetric Agents

We can use the exact same passage from the previous chapter to define our environment. There is a risk neutral seller with a single good and n risk neutral buyers that have non-negative private valuation for the good. These private valuations will be addressed as agents' types and valuation of the seller is assumed to be 0. Define $T = \{1, 2, \ldots, m\}$ as the type space whose discrete form will allow us to use linear programming. We have the common prior assumption which means that types of agents are independent draws from a commonly known distribution F. We also assume that density function of F satisfies $f_i > 0$ for all $i \in T$. Another important setting is the Revelation Principle through which we focus on direct mechanisms only.

We first take a look at the formulation from Vohra [1]:

$$\begin{aligned} \max_{A,c} \quad & \sum_{i \in T} f_i i A(i) - K \sum_{i \in T} f_i A(i) \big(1 - c(i)\big) & \\ \text{s.t.} \quad & iA(i) \geq iA(j)c(j) & \forall i, j \in T \\ & 0 \leq c(i) \leq 1 & \forall i \in T \\ & n \sum_{i \in \mathcal{S}} f_i A(i) \leq 1 - (\sum_{i \notin \mathcal{S}} f_i)^n & \forall \mathcal{S} \subseteq T. \end{aligned}$$

where $1 - c(i)$ is the probability of inspecting type i conditional on the allocation to type i. Inspection cost is K. Vohra states that type 1 will never inspected since it is not reasonable to inspect the lowest type. Then $c(i) \leq \frac{A(1)}{A(i)}$ can be deducted from the first constraint. Set $c(i) = \min_{i \in T}\{1, \frac{A(1)}{A(i)}\}$ which will be equal to

$\frac{A(1)}{A(i)}$ if $A(1)$ is the minimum interim allocation. Checking the objective, we have:

$$\sum_{i\in T} f_i i A(i) - K\sum_{i\in T} f_i A(i)\big(1-\frac{A(1)}{A(i)}\big) = \sum_{i\in T} f_i(i-K)A(i) - KA(1).$$

Although we will work on a different formulation, we will end up having this objective at some point.

We define different variables from our previous chapter. Inspection cost will be denoted by K. We no longer have payment variables but we have inspection. Define $s_i(i,t^{-1})$ to be probability that agent 1 receives the good without inspection when he reports i and other agents report t^{-1}. Interim (expected) allocation is defined in the same way, and now we also have expected inspection skipping variable S_i when agents other than 1 report their type truthfully:

$$\begin{aligned} A(i) &= \sum_{t^{-1}\in T^{n-1}} a_i(i,t^{-1})\pi(t^{-1}),\\ S(i) &= \sum_{t^{-1}\in T^{n-1}} s_i(i,t^{-1})\pi(t^{-1}). \end{aligned}$$

The objective of the problem is to maximize expected utility from the allocation with inspection in addition to the savings from not inspecting:

$$\begin{aligned} \max_{S,A,s,a} \quad & \sum_{i\in T} f_i(i-K)A(i) + K\sum_{i\in T} f_i S(i) \\ \text{s.t.} \quad & iA(i) \geq iS(j) && \forall i,j\in T && (1)\\ & A(i) = \sum_{t^{-1}\in T^{n-1}} a_i(i,t^{-1})\pi(t^{-1}) && \forall i\in T && (2)\\ & S(i) = \sum_{t^{-1}\in T^{n-1}} s_i(i,t^{-1})\pi(t^{-1}) && \forall i\in T && (3)\\ & a_i(t) \geq s_i(t) && \forall i\in T, \forall t\in T^n && (4)\\ & \sum_{i\in T} n_i(t)a_i(t) \leq 1 && \forall t\in T^n && (5)\\ & a_i(t)\geq 0 && \forall i\in T, \forall t\in T^n && (6)\\ & s_i(t)\geq 0 && \forall i\in T, \forall t\in T^n. && (7) \end{aligned}$$

We do not allow a negative amount of allocation as in (6) and we only have one good to allocate as in (5). By constraints (4) and (7), it is obvious that the probability of getting the object without any inspection should be between 0 and the actual allocation value. Constraints (2) and (3) simply provides that expected variables are consistent with the underlying mechanism.

We can say that constraint (1) is the incentive compatibility constraint in which the left hand side is the utility of agent if he reports his type truthfully. There is no effect of inspection since he will keep his object if we find out that he was truthful. However in the right hand side, the utility of lying is affected by the inspection. Seller will reclaim the good if the type was misreported. Therefore, agent can expect a utility from lying only if the seller choose to skip the inspection.

Let us simplify the formulation. Ignore the $s_i(i, t^{-1})$ variables for the time being and focus on the expected skipping variable. In this way we can leave out the constraints (3), (4) and (7). Besides, notice that in the constraint (1), multiplier i cancels out. Also apply Borders Theorem 3.3 to constraints (2), (5) and (6). So we obtain:

$$\begin{aligned}
\max_{S,A} \quad & \sum_{i\in T} f_i(i-K)A(i) + K\sum_{i\in T} f_i S(i) & \\
\text{s.t.} \quad & A(i) \geq S(j) & \forall i,j \in T \\
& n\sum_{i\in\mathcal{S}} f_i A(i) \leq 1-(\sum_{i\notin\mathcal{S}} f_i)^n & \forall \mathcal{S}\subseteq T.
\end{aligned}$$

Then we can make the conclusion that incentive compatibility constraint is satisfied if we have $\min_{i\in T} A(i) \geq S(j)$ for all j in T. Since there is no other constraint on $S(j)$ and it has a positive multiplier in the objective, we set it to its upper bound, $\min_{i\in T} A(i) = S(j)$.

Proposition 4.1. *In the optimal solution, we will have* $\min_{i\in T} A(i) = A(1)$.

Proof. Assume the contrary that $\min_{i\in T} A^*(i) = A^*(l)$ where $l \in T$ and $l \neq 1$. Since we know the minimum interim allocation $A^*(l)$, we also know that for other variables there are only two possible cases. For $i \in T\setminus\{l\}$, we have either

$A^*(i) > A^*(l)$ or $A^*(i) = A^*(l)$. Let us define two subsets for our type set:

$$L = \{k \in T \mid A^*(k) = A^*(l)\},$$
$$U = \{k \in T \mid A^*(k) > A^*(l)\}.$$

We will make the following rearrangements in the formulation. We set $A^*(l) = S(j)$ for all j in T, which is the optimal solution. Besides, for all k in L, denote $A^*(k)$ by $A^*(l)$ which are equal in the optimal solution. This gives the problem:

$$\begin{aligned} \max_{A} \quad & \sum_{i \in U} f_i(i-K)A(i) + \sum_{i \in L} f_i(i-K)A(l) + KA(l) & \\ \text{s.t.} \quad & n \sum_{i \in \mathcal{S}} f_i A(i) \leq 1 - (\sum_{i \notin \mathcal{S}} f_i)^n & \forall \mathcal{S} \subseteq T. \end{aligned}$$

As a final rearrangement, set $z_i = f_i A(i)$ for all i in U and set $z_l = A(l) \sum_{i \in L} f_i$. Also define $f'_i = f_i$ when i in U and $f'_l = \sum_{i \in L} f_i$. Since we have a submodular function in our constraints we can find the optimal solution by the Greedy Algorithm. We start from the variable with highest coefficient and set it to its tightest upper bound:

$$\begin{aligned} \max_{z} \quad & \sum_{i \in U}(i-K)z_i + \Big(\frac{\sum_{i \in L} f_i i + (1 - \sum_{i \in L} f_i)K}{\sum_{i \in L} f_i}\Big) z_l & OPTi \\ \text{s.t.} \quad & n \sum_{i \in S} z_i \leq 1 - (\sum_{i \notin S} f'_i)^n & \forall \mathcal{S} \subseteq U \cup \{l\}. \end{aligned}$$

Notice that since we have $A^*(1) > A^*(l)$, then we should have $1 \in U$. This means that coefficient of z_1 should be greater than coefficient of z_l:

$$1 - K > \mathrm{E}\left[i \mid i \in L\right] + \frac{(1 - \sum_{i \in L} f_i)K}{\sum_{i \in L} f_i}. \tag{4.1}$$

This is obviously not possible since $1 < \mathrm{E}\left[i \mid i \in L\right]$. This is a contradiction. □

Actually we can describe better the optimal solution from inequality 4.1. Any type i is in the set U, if and only if $i - K$ is bigger than the right side of 4.1. This tells us that if i is in U then any $j \in T$ bigger than i is also in U. In the optimal solution we have a cut-off i^* in the following sense: any type greater than or equal to the cut-off is in U and all other types are in the set L. Define a function $Q(i) : T \setminus \{1\} \rightarrow \mathbb{R}$ in order to find this optimal cut-off:

$$Q(i) = i - \mathrm{E}\left[j \mid j < i\right] - \frac{K}{\sum_{j<i} f_j} = i - \Big(\frac{\sum_{j<i} f_j j + K}{\sum_{j<i} f_j}\Big).$$

Proposition 4.2. *There is a unique i^* such that the function $Q(i)$ is positive when $i \geq i^*$ and negative otherwise.*

Proof. Consider $Q(i+1)$:

$$\begin{aligned} Q(i+1) \quad &= i+1-\Big(\frac{\sum_{j<i+1} f_j j + K}{F(i)}\Big) = i+1-\Big(\frac{\sum_{j<i} f_j j + f_i i + K}{F(i)}\Big) \\ &= i+1-\Big(\frac{f_i i}{F(i)} + \frac{\sum_{j<i} f_j j + K}{F(i)}\Big) = 1 + \frac{F(i-1)i}{F(i)} - \frac{F(i-1)(i-Q(i))}{F(i)} \\ &= 1 + \frac{F(i-1)Q(i)}{F(i)}. \end{aligned}$$

In the end, we have:

$$Q(i+1) = 1 + \frac{F(i-1)Q(i)}{F(i)}, \qquad Q(i) = \frac{F(i)(Q(i+1)-1)}{F(i-1)}.$$

This means that if $Q(i) \geq 0$, then $Q(i+1)$ is also positive. If $Q(i+1) < 0$, then $Q(i)$ is also negative.

In conclusion, if we have $i^* \in T \setminus \{1\}$ where $Q(i^*) \geq 0$ and $Q(i^*-1) < 0$, then for the types above i^*, $Q(i)$ is positive, and it is negative otherwise. □

Note that for the continuous type space, we should find the type where $Q(i) = 0$. This result is consistent with the results from [4].

We now use the Greedy Algorithm to find the optimal interim allocations of the formulation $OPTi$. Obviously, above the cut-off, coefficient $i-K$ is monotone. Setting variable z_i to its tightest constraint will give the same optimal solution as optimal interim allocations from Chapter 3.1. For the variables that are under the cut-off, this time we will have positive interim allocations. Consider the constraint when $S = T$ which is the tightest constraint on z_1 in order to find the optimal solution:

$$z_1 + \sum_{i=i^*}^{m} \frac{F(i)^n - F(i-1)^n}{n} \leq \frac{1}{n} \quad \Rightarrow \quad z_1 \leq \frac{1}{n} - \frac{1-F(i^*-1)^n}{n} = \frac{F(i^*-1)^n}{n}.$$

Therefore we can give the optimal solution for the interim allocations whose monotonicity is easy to prove:

$$A(i) = \begin{cases} \frac{F(i)^n - F(i-1)^n}{f_i n} & \text{if } i \geq i^* \\ \frac{F(i^*-1)^{n-1}}{n} & \text{otherwise.} \end{cases}$$

We know that allocating the good to the highest bidder is an allocation rule consistent with this solution when the maximum bid is over the cut-off. Otherwise it is optimal to equally distribute the good to all agents, or in other words, choose the winner with lottery where all agents have $\frac{1}{n}$ winning probability:

$$A(1) = \frac{1}{n} \cdot P(\text{other agents bid under the cut-off}) = \frac{F(i^* - 1)^{n-1}}{n}.$$

In order to completely characterize an optimal mechanism, we should also define an optimal inspection skipping rule. We know that $A(1) = S(j)$ for all j in T, which allows us to set $a_1(1, t^1) = s_j(j, t^{-1})$ for all j in T and t^{-1} in T^{n-1}. It means that we can skip allocation with probability $\frac{1}{n}$ if all other agents bid under the cut-off. Note that optimal inspection skipping rule is only dependent on the profile of other agents:

$$a_i^*(i, t^{-1}) = \begin{cases} \frac{1}{n_i(t)} & \text{if } i \geq \max\limits_{k \in t^{-1}} k \text{ and } i \geq i^* \\ \frac{1}{n} & \text{if } \max\limits_{k \in t^{-1}} k < i^* \text{ and } i < i^* \\ 0 & \text{otherwise,} \end{cases}$$

$$s_i^*(i, t^{-1}) = \begin{cases} \frac{1}{n} & \text{if } \max\limits_{k \in t^{-1}} k < i^* \\ 0 & \text{otherwise.} \end{cases}$$

In this mechanism, we allocate the good to the maximum bid if it is above the cut-off and we randomize the allocation if everyone bid under the cut-off. The inspection certainly takes place if there is more than one bid over the cut-off. If there was one bid over the cut-off, then we skip the inspection with probability $\frac{1}{n}$. If there is no bid above the cut-off, we allocate everyone $\frac{1}{n}$ and skip the inspection for all with probability $\frac{1}{n}$. Since there is no inspection skipping when there is no allocation, conditional probability says: do not inspect at all.

We gave the exact structure of the optimal mechanism. In [1], Vohra states that there exists a cut-off, above which we award the object to the highest bidder and inspect with positive probability. Under the cut-off, randomize the object and do not inspect. However, he does not give the optimal cut-off or the optimal inspection probability.

Our result is consistent with the results of Ben-Porath, Dekel and Lipman [4]. They do not assume symmetric agents so that as the optimal mechanism, they have a favored-agent mechanism which we will also derive in the next section. In this mechanism, there is a favourite agent who gets the good whenever no agent bids over their cut-off. Choice of favourite agent is dependent on a function which has distribution f and inspection cost K as its inputs. Note that this function is the continuous version of $Q(i)$ from Proposition 4.2. Since we assume that these inputs are same for all agents, it is optimal to choose any of them as favourite agent or to allocate them equally.

4.2 When Inspection Cost is Different for all Agents

The symmetry assumption is not applicable in this case so we need to make changes in the formulation. First of all, we should consider the sum of utilities for all agents, and also each agent should have different incentive compatibility constraints. Besides, expectation-based variables should be extended to represent all agents. In this case, we have the formulation:

$$
\begin{aligned}
\max_{S,A,s,a} \quad & \sum_{i \in N} \Big(\sum_{j \in T} f_j (j - K_i) A_i(j) + K_i \sum_{j \in T} f_j S_i(j) \Big) & & \\
\text{s.t.} \quad & A_i(j) \geq S_i(j') & \forall i \in N, j, j' \in T \quad & (1) \\
& A_i(j) = \sum_{t^{-i} \in T^{n-i}} a_i(j, t^{-i}) \pi(t^{-i}) & \forall i \in N, \forall j \in T \quad & (2) \\
& S_i(j) = \sum_{t^{-i} \in T^{n-i}} s_i(j, t^{-i}) \pi(t^{-i}) & \forall i \in N, \forall j \in T \quad & (3) \\
& a_i(t) \geq s_i(t) & \forall i \in T, \forall t \in T^n \quad & (4) \\
& \sum_{i \in N} a_i(t) \leq 1 & \forall t \in T^n \quad & (5) \\
& a_i(t) \geq 0 & \forall i \in N, \forall t \in T^n \quad & (6) \\
& s_i(t) \geq 0 & \forall i \in N, \forall t \in T^n. \quad & (7)
\end{aligned}
$$

We use K_i for the inspection cost for agent $i \in N$ where N is the set of agents. We use $a_i(j, t^{-i})$ to represent the allocation to agent i when she bids j and all other agents form the profile $t^{-i} \in T^{n-i}$ where T^{n-i} is the set of profiles that can take place for agents other than i, and $s_i(j, t^{-i})$ is defined in a similar way. We needed to write some constraints for all agents, and in the constraint (5), we sum allocations for profile t over agents not types.

Let us ignore the inspection skipping rule again and get rid of constraints (3), (4) and (7). This results in:

$$
\begin{aligned}
\max_{S,A,a} \quad & \sum_{i \in N} \Big(\sum_{j \in T} f_j (j - K_i) A_i(j) + K_i \sum_{j \in T} f_j S_i(j) \Big) & & \\
\text{s.t.} \quad & A_i(j) \geq S_i(j') & \forall i \in N, \forall j, j' \in T \quad & (1) \\
& A_i(j) = \sum_{t^{-i} \in T^{n-i}} a_i(j, t^{-i}) \pi(t^{-i}) & \forall i \in N, \forall j \in T \quad & (2) \\
& \sum_{i \in N} a_i(t) \leq 1 & \forall t \in T^n \quad & (5) \\
& a_i(t) \geq 0 & \forall i \in N, \forall t \in T^n. \quad & (6)
\end{aligned}
$$

We will use again Border's Theorem [8] but in this case feasibility condition changes a bit.

Theorem 4.1. ***(Border's Theorem)*** *The expected allocation $A_i(j)$ is feasible if and only if*

$$\sum_{i \in N} \sum_{j \in S_i} f_j A_i(j) \leq 1 - \prod_{i \in N} (\sum_{j \notin S_i} f_j) \qquad \forall \mathcal{S}_i \subseteq T. \tag{4.2}$$

Proof. Same logic in Vohra [1] also applies here. We will show that the system of constraints (2), (5) and (6) is actually a transportation problem in the following manner:

$$\begin{aligned} & f_j A_i(j) = \sum_{t^{-i} \in T^{n-i}} a_i(j, t^{-i}) \pi(t) && \forall i \in N, \forall j \in T && (2) \\ & \sum_{i \in N} a_i(t) \pi(t) \leq \pi(t) && \forall t \in T^n && (5) \\ & a_i(t) \pi(t) \geq 0 && \forall i \in N, \forall t \in T^n. && (6) \end{aligned}$$

Now define $x_i(j, t^{-i}) = a_i(j, t^{-i}) \pi(t)$ and realize that $x_i(t)$ variable can be considered as transportation from profile $t = (j, t^{-i})$ to type j of agent i:

$$\begin{aligned} & \sum_{t^{-i} \in T^{n-i}} x_i(j, t^{-i}) = f_j A_i(j) && \forall i \in N, \forall j \in T && (2) \\ & \sum_{i \in N} x_i(t) \leq \pi(t) && \forall t \in T^n && (5) \\ & x_i(t) \geq 0 && \forall i \in N, \forall t \in T^n. && (6) \end{aligned}$$

Constraint (2) makes sure that type j of agent i gets its demand $f_j A_i(j)$ from the profiles where agent i bids j. In constraint (5), we see that every profile t has a supply of $\pi(t)$ and cannot exceed it. By (6), it is ensured that transportation is in non-negative values.

From maxflow-mincut theorem, we know that this system is feasible if and only if total demand of subsets $\mathcal{S}_i$ is less than or equal to total supply of profiles connected to them for all possible subsets. Left hand side of equation 4.2 is obviously the demand of subsets $\mathcal{S}_i$. Supply of profiles that are connected to these types is simply the probability that at least one type from some $\mathcal{S}_i$ is reported:

$$\begin{aligned} P(\text{at least one type from some } \mathcal{S}_i \text{ is reported}) &= 1 - P(\text{no types from any } \mathcal{S}_i \text{ is reported}), \\ &= 1 - \prod_{i \in N} (\sum_{j \notin S_i} f_j). \end{aligned}$$

$\square$

Let us look at the formulation now:

$$\begin{aligned}
\max_{S,A} \quad & \sum_{i\in N}\Big(\sum_{j\in T} f_j(j-K_i)A_i(j) + K_i\sum_{j\in T} f_j S(j)\Big) & & \\
\text{s.t.} \quad & A_i(j) \geq S_i(j') & \forall i\in N, \forall j, j'\in T \qquad & (1)\\
& \sum_{i\in N}\sum_{j\in S_i} f_j A_i(j) \leq 1 - \prod_{i\in N}(\sum_{j\notin S_i} f_j) & \forall \mathcal{S}_i \subseteq T_i. \qquad & (8)
\end{aligned}$$

Next we will show that the right hand side of constraint (8) is again submodular. Recall the definition: a function F is submodular if and only if it satisfies the following inequality for any subsets A, B of T such that $A \subset B$ and for any $v \in T$:

$$F(A\cup\{v\}) - F(A) \geq F(B\cup\{v\}) - F(B). \tag{4.3}$$

We need to define a function in order to check submodularity. Although all agents take valuations from the same type set, define $T_i = T$ for each agent $i \in N$. Any type in set T_i will have index i so that we know if a type is for a specific agent. For example, 1_i denotes the type 1 for agent i. Now consider the subsets of T_i and their union as $\mathcal{S} = \cup_{i\in N}\mathcal{S}_i$. Since we have indexes for types, set $\mathcal{S}$ might contain several type j which represents several agents' valuations.

Proposition 4.3. *$G'(\mathcal{S})$ is a non-decreasing, non-negative submodular function:*

$$G'(\mathcal{S}) = 1 - \prod_{i\in N}(\sum_{j_i\notin S} f_j).$$

Proof. Actually we already proved that $G'(\mathcal{S})$ is non-decreasing and non-negative, when we showed that it is actually the probability that at least one type from $\mathcal{S}$ is reported. In order to look at the submodularity, consider left hand side of the inequality 4.3 for our function $G'(\mathcal{S})$:

$$\begin{aligned} G'(A \cup \{v\}) - G'(A) \quad &= 1 - \prod_{i \in N}\Big(\sum_{j_i \notin A \cup \{v\}} f_j\Big) - 1 + \prod_{i \in N}\Big(\sum_{j_i \notin A} f_j\Big) \\ &= \prod_{i \in N}\Big(\sum_{j_i \notin A \cup \{v\}} f_j\Big)(1 + f_v) - \prod_{i \in N}\Big(\sum_{j_i \notin A \cup \{v\}} f_j\Big) \\ &= \prod_{i \in N}\Big(\sum_{j_i \notin A \cup \{v\}} f_j\Big)(f_v). \end{aligned}$$

Since this is the probability that no agent report the types in set A times an agent reports type v, we know that this probability is larger than the same probability for set B when $A \subset B$. Therefore we have proved submodularity by showing that the following inequality holds for any subsets A, B of $\cup_{i \in N} T_i$ such that $A \subset B$ and for any $v \in T$:

$$G'(A \cup \{v\}) - G'(A) \geq G'(B \cup \{v\}) - G'(B).$$

□

Again ignoring the constraint (1) and setting each expected inspection skipping variable to minimum interim allocation, we end up with a polymatroid optimization problem. Then we can apply the Greedy Algorithm to find the optimal interim allocations.

As in Proposition 4.1, minimum interim allocation will be $A_i(1)$ for each agent i in N. Define the following sets for each agent where $\min\limits_{j \in T} A_i^*(j) = A_i^*(1)$ for each i:

$$\begin{aligned} L_i &= \{j \in T \mid A_i^*(j) = A_i^*(1)\}, \\ U_i &= \{j \in T \mid A_i^*(j) > A_i^*(1)\}. \end{aligned}$$

Since $j - K_i$ is monotone for all agents, type set will be divided to two subsets where types above a cut-off t_i^* will be in U_i and others will be in set L_i. Note that we will probably have different optimal cut-offs for each agent. We will look into optimal values of these cut-offs when we apply the Greedy Algorithm.

Set $A_i^*(1) = S_i(j)$ for all j in T. Also for all j in L_i, denote $A_i^*(j)$ by $A_i^*(1)$ which are equal in the optimal solution. Then set $z_i(j) = f_j A_i(j)$ for all j in U_i

and for all i in N. For the rest of the types, set $z_i(1) = A_i(1)\sum_{j\in L_i} f_j$ for all agents. Also define $f_i(j) = f_j$ when j is in U_i and $f_i(1) = \sum_{j\in L_i} f_j$ for all i in N. Use the Greedy Algorithm to find the optimal solution:

$$\begin{aligned}
\max_{z} \quad & \sum_{i\in N}\Bigg(\sum_{j\in U_i}(j-K_i)z_i(j) + \Big(\frac{\sum_{j\in L_i} f_j j + (1-F_{t_i^*-1})K_i}{F_{t_i^*-1}}\Big)z_i(1)\Bigg) && OPTk \\
\text{s.t.} \quad & \sum_{i\in N}\sum_{j\in S_i} z_i(j) \le 1 - \prod_{i\in N}(\sum_{j\notin S_i} f_i(j)) && \forall \mathcal{S}_i \subseteq U_i \cup \{1\}.
\end{aligned}$$

Assume that inspection costs are different for all agents. Obviously highest coefficient will be $m - K_p$ where p is the agent with the minimum inspection cost. Set $z_p(m)$ to its tightest upper bound:

$$z_p(m) \le 1 - F(m-1), \text{ where } \mathcal{S}_p = \{m\},\, \mathcal{S}_i = \emptyset \text{ for } i \in N \setminus \{p\}.$$

Then we have $A_p(m) = 1$. Suppose that second highest coefficient is again from agent p, which is $(m-1-K_p)$. Look at the following constraint for $\mathcal{S}_p = \{m, m-1\}$ and $\mathcal{S}_i = \emptyset$ for $i \in N \setminus \{p\}$:

$$z_p(m) + z_p(m-1) \le 1 - F(m-2) \quad \Rightarrow \quad z_p(m-1) \le 1 - F(m-2) - f_m = f_{m-1}.$$

Optimal solution is $A_p(m-1) = 1$ again. It means that as long as agent p bids above $m-1$, he will claim the good for sure. Let us say that third highest coefficient is from another agent, say $q \in N$. Of course the variable will be $z_q(m)$ which is the highest for that agent. We should look at the constraint when $\mathcal{S}_p = \{m, m-1\}$, $\mathcal{S}_q = \{m\}$ and $\mathcal{S}_i = \emptyset$ for $i \in N \setminus \{p, q\}$:

$$z_p(m) + z_p(m-1) + z_q(m) \le 1 - F(m-2)F(m-1),$$

$$z_q(m) \le 1 - F(m-2)F(m-1) - f_m - f_{m-1} = f_m F(m-2).$$

Therefore $A_q(m) = F(m-2)$ which is the probability that agent p does not bid above $m-2$. It is obvious that the optimal interim allocation value for $A_i(j)$ is the probability that $j - K_i$ is the highest value among the agents when j is in the set U_i for agent i:

$$A_i(j) = P(j - K_i = \max_{n\in N, b\in t} b - K_n).$$

This interim allocation is consistent with the allocation rule where the seller allocates the good to the highest $j - K_i$ as long as $j \geq t_i^*$. Now we find the optimal cut-offs for each agent.

After finding the optimal solutions for all types in U_i for all agents, we are left with variables $z_i(l)$. Because of the Greedy Algorithm, we should go with the variable that has the highest coefficient. Assume there is an agent r in N with the highest coefficient:

$$\Big(\frac{\sum_{j\in L_r} f_j j + (1 - F_{t_r^*-1})K_r}{F_{t_r^*-1}}\Big) = \max_{n\in N}\Big(\frac{\sum_{b\in L_n} f_b b + (1 - F_{t_n^*-1})K_n}{F_{t_n^*-1}}\Big).$$

Then consider the following constraint to find the optimal solution for $z_r(l)$ where $\mathcal{S}_r = T$ and $\mathcal{S}_i = \{m, m-1 \ldots t_i^*\}$ for $i \in N \setminus \{r\}$:

$$\sum_{i\in N}\sum_{j=t_i^*}^{m} z_i(j) + z_r(l) \leq 1 \quad \Rightarrow \quad z_r(l) \leq 1 - P(\text{someone bids above } t_i^*),$$

$$z_r(l) \leq P(\text{everyone bids under } t_i^*) = \prod_{i\in N} F(t_i^* - 1).$$

Therefore the optimal solution is $A_r(l) = \prod_{i\in N\setminus\{r\}} F(t_i^* - 1)$, which is consistent with allocating the good to the agent r every time there is no bid above the cut-offs. As this result also suggests, all other variables $z_i(l)$ are equal to zero for agents other than r. It is easy to check that for the next variable and its tightest constraint, right hand side is equal to zero.

Actually we also found the optimal cut-off values for each agent. Consider agent r. We know that coefficients for $z_i(j)$ satisfies the following inequality when $j \in U_r$:

$$j - K_r > \Big(\frac{\sum_{j\in L_r} f_j j + (1 - F_{t_r^*-1})K_r}{F_{t_r^*-1}}\Big),$$

$$j > \Big(\frac{\sum_{j\in L_r} f_j j + K_r}{F_{t_r^*-1}}\Big).$$

From proposition 4.2, we know that there is a unique cut-off t_r^*. In fact, the inequality above should hold for other agents according to the Greedy Algorithm, since their types in sets U_i are improved before $z_r(l)$:

$$j - K_i > \Big(\frac{\sum_{j\in L_r} f_j j + (1 - F_{t_r^*-1})K_r}{F_{t_r^*-1}}\Big),$$

$$j > \Big(\frac{\sum_{j\in L_r} f_j j + (1-F_{t_r^*-1})K_r}{F_{t_r^*-1}}\Big) + K_i.$$

It can be easily shown that cut-offs for agents other than r are also unique by altering the function in Proposition 4.2. Just subtract K_r and add K_i, which are constants.

Proposition 4.4. *Agent r who gets the good if there is no bid above the cut-offs t_i^* is the agent with the highest inspection cost.*

Proof. Recall that we chose the agent r since he had the highest coefficient among variables $z_i(l)$. Define function $R(i,K) : T\setminus\{1\}\times \mathbb{R} \to \mathbb{R}$ similar to the function from the proposition 5.2:

$$R(i,K) = \Big(\frac{\sum_{j=1}^{i-1} f_j j + (1-F_{i-1})K}{F_{i-1}}\Big).$$

Obviously, this function is increasing as K increases. However, this is not enough to prove our proposition since as K increases the optimal cut-off may also increase. Lets consider two different costs A, B and their corresponding optimal cut-offs $a+1, b+1$ respectively. Assume that $B > A$ which will result in $b+1 > a+1$. Look at the following difference:

$$R(b+1,B) - R(a+1,A) = \Big(\frac{\sum_{j=1}^{b} f_j j + (1-F_b)B}{F_b}\Big) - \Big(\frac{\sum_{j=1}^{a} f_j j + (1-F_a)A}{F_a}\Big)$$

$$= \frac{F_a\big(\sum_{j=1}^{a} f_j j + (F_b - F_a)(a+1) + \sum_{j=1}^{b-a-1} f_j j + (1-F_b)B\big) - F_b\big(\sum_{j=1}^{a} f_j j + (1-F_a)A\big)}{F_b F_a}$$

$$= \frac{\sum_{j=1}^{b-a-1} f_j j}{F_b} + \frac{(F_b - F_a)\big(F_a(a+1) - \sum_{j=1}^{a} f_j j - A\big) + F_a(B-A)}{F_b F_a} - (B-A)$$

$$= \frac{\sum_{j=1}^{b-a} f_j j}{F_b} + \frac{(F_b - F_a)\big(a+1-A-R(a+1,A)\big)}{F_b} + \frac{(1-F_b)(B-A)}{F_b} \geq 0.$$

The first and the last term are obviously non-negative. The second term is also non-negative since $a+1-A-R(a+1,A)$ is equal to $Q(a+1)$ when cost K is A. As it is shown in Proposition 5.2, if $a+1$ is the optimal cut-off than $Q(a+1)$ should be non-negative. This concludes that as inspection cost increases function $R(i,K)$ also increases. □

The optimal solution turns out to be a "favored-agent mechanism" where the agent with the highest cost is the favourite. Let us set $a_i(1_i, t^1) = s_i(j_i, t^{-1})$ for all i in N, j in T and t^{-1} in T^{n-1} and look at the optimal mechanism (we again use notation j_i as the type j of agent i):

$$a_i^*(j_i, t^{-1}) = \begin{cases} 1 & \text{if } j_i - K_i = \max\limits_{b_n \in t^{-1}, n \in N} b_n - K_n \text{ and } j_i \geq t_i^* \\ 1 & \text{if } \max\limits_{b_n \in t^{-1}} b_n < t_n^* \text{ for all } n \in N \text{ and } K_i = \max\limits_{n \in N} K_n \\ 0 & \text{otherwise,} \end{cases}$$

$$s_i^*(j_i, t^{-1}) = \begin{cases} 1 & \text{if } \max\limits_{b_n \in t^{-1}} b_n < t_n^* \text{ for all } n \in N \text{ and } K_i = \max\limits_{n \in N} K_n \\ 0 & \text{otherwise.} \end{cases}$$

Here we allocate the good to the highest $j_i - K_i$ value if that j_i is above the optimal cut-off for t_i. Realize that we do not necessarily allocate the good to the highest bidder. Lower bids from an agent with lower inspection cost can also get the good. If everyone bids under their respective cut-off, then we allocate the good to our favourite agent.

Inspection will take place whenever an agent other than the favourite claims the good. Favourite agent will not be inspected unless there is another agent who bids over his cut-off.

This optimal mechanism is consistent with the favored-agent mechanism of Ben-Porath, Dekel and Lipman [4] derived for continuous type spaces using different methods. In their problem environment, agents take their valuations according to different distributions so that it is not optimal to choose the agent with highest inspection cost as the favourite agent every time. However, they state that as the inspection cost of agent i increases he is more likely to be the favourite agent. We assume common prior assumption so that our favourite agent is not affected by the distribution. Note that we still use a similar function $R(i, K)$ that align with [4] to find the optimal cut-offs.

Chapter 5

Ambiguity Averse Seller

In optimal auction mechanism literature, it is a common assumption that bidders' valuations are independently drawn from a unique distribution. We now relax this common prior assumption such that neither the seller nor the buyers know the valuation distribution of other players. We will investigate the form of optimal mechanism for an ambiguity averse seller. Neither the seller nor the buyers know the valuation distribution of other players. We have n ambiguity neutral buyers. Type distribution is a random outcome from a set of distributions $\mathcal{P}$ which is finite and known by all players. Here we look into the same problems from previous chapters and continue to make use of linear programming.

First we consider the mechanism design problem with monetary transfers. We will maximize seller's worst case expected revenue and see that some of Vohra's results from [1] also apply in this problem. In the second section, we try to maximize worst case social welfare for mechanism design with costly inspection problem. For both of the problems, we give the optimal mechanism for some special cases. Besides, we propose an algorithm that gives the optimal solution for some complicated cases.

5.1 Mechanism with Monetary Transfers

The environment of our problem is similar to Chapter 3 except the following differences. We have a single ambiguity averse seller and n ambiguity neutral buyers. Type distribution is a random outcome from a set of distributions $\mathcal{P}$ which is finite and known by all players.

Now we have $\pi_f(t^{-1})$ as the probability of agents having types that give rise to the profile t^{-1} where $f \in \mathcal{P}$. Interim (expected) allocation and payment variables are denoted accordingly:

$$A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) \quad \forall f \in \mathcal{P}$$
$$P_f(i) = \sum_{t^{-1} \in T^{n-1}} p_i(i, t^{-1}) \pi_f(t^{-1}) \quad \forall f \in \mathcal{P}.$$

Since our seller is ambiguity averse, the objective of our problem is to maximize the seller's worst case expected revenue. We focus on direct mechanisms as in Bose, Ozdenoren and Pape [6] and face the following constrained maximization problem:

$$\begin{aligned}
\max_{P,a} \quad & \left\{ \min_{f \in \mathcal{P}} \sum_{i \in T} f_i P_f(i) \right\} && && \\
\text{s.t.} \quad & iA_f(i) - P_f(i) \geq iA_f(j) - P_f(j) && \forall i, j \in T && \forall f \in \mathcal{P} \quad (1) \\
& iA_f(i) - P_f(i) \geq 0 && \forall i \in T && \forall f \in \mathcal{P} \quad (2) \\
& A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) && \forall i \in T && \forall f \in \mathcal{P} \quad (3) \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 && \forall t \in T^n && \quad (4) \\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n. && \quad (5)
\end{aligned}$$

We just extended our problem for each f in $\mathcal{P}$. Since buyers are ambiguity neutral, it is enough to have constraint (1) in order to achieve a direct mechanism.

At this point, we can take the same step as in Chapter 3 where we associated the system of constraints (1) and (2) with a shortest path problem as in Figure 3.1. In this case there will be a different network for each f in $\mathcal{P}$ and each system can be feasible if and only if respected interim allocations are monotonic. See Theorem 3.1. Since there is no other limit on the expected payments, it is only reasonable to set them to their tightest upper bound:

$$
\begin{aligned}
\max_{A,a} \quad & \left\{ \min_{f \in P} \sum_{i \in T} f_i \nu_f(i) A_f(i) \right\} \\
\text{s.t.} \quad & 0 \le A_f(1) \le \cdots \le A_f(m) && && \forall f \in P \\
& A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) && \forall i \in T && \forall f \in P && (3) \\
& \sum_{i \in T} n_i(t) a_i(t) \le 1 && \forall t \in T^n && && (4) \\
& a_i(t) \ge 0 && \forall i \in T, \forall t \in T^n, && && (5)
\end{aligned}
$$

where $\nu_f(i) = i - \frac{1-F(i)}{f_i}$ is defined in the same way but it is different for all f in $\mathcal{P}$.

Our next step now differs from Chapter 3. We cannot take out the allocation rule variables since it is not possible to apply Border's theorem for each f in $\mathcal{P}$. Therefore we will take out the interim allocation variables and concentrate on the underlying allocation rule:

$$
\begin{aligned}
\max_{a} \quad & \left\{ \min_{f \in P} \sum_{i \in T} f_i \nu_f(i) \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) \right\} \\
\text{s.t.} \quad & 0 \le \sum_{t^{-1} \in T^{n-1}} a_1(1, t^{-1}) \pi_f(t^{-1}) \le \cdots \le \sum_{t^{-1} \in T^{n-1}} a_m(m, t^{-1}) \pi_f(t^{-1}) && \forall f \in P \\
& \sum_{i \in T} n_i(t) a_i(t) \le 1 && \forall t \in T^n \\
& a_i(t) \ge 0 && \forall i \in T, \forall t \in T^n.
\end{aligned}
$$

Introducing a new variable, z, we can linearize this problem. Below, the final form of the formulation can be found:

$$
\begin{aligned}
\max_{a,z} \quad & z && (OPTk) \\
\text{s.t.} \quad & z \leq \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_f(i) a_i(i, t^{-1}) \pi_f(i, t^{-1}) && \forall f \in \\
& 0 \leq \sum_{t^{-1} \in T^{n-1}} a_1(1, t^{-1}) \pi_f(t^{-1}) \leq \cdots \leq \sum_{t^{-1} \in T^{n-1}} a_m(m, t^{-1}) \pi_f(t^{-1}) && \forall f \in \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 && \forall t \in \\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in
\end{aligned}
$$

5.1.1 The Solution Approach

Throughout this section, we assume that $\nu_f(i)$ is monotonic in i for all f in $\mathcal{P}$. In order to solve the problem $OPTk$, we will look into $\nu_f(i)$ values and see that the knapsack problem helps us to pin down the optimal allocation rule for some cases. For the remaining cases, we will make a further assumption and propose an algorithm which gives the optimal solution.

Consider the following graphic in Figure 5.1 for $m = 10$ where for all f in $\mathcal{P}$, we have $\nu_f(i^* - 1) < 0$ and $\nu_f(i^*) \geq 0$. It is obvious that the discrete second price auction with reserve price i^* is optimal for this case.

Theorem 5.1. *If for all $f \in \mathcal{P}$, $\nu_f(i)$ starts taking non-negative values from the same type i^*, then the optimal solution is discrete second price auction with reserve price i^*.*

Proof. Consider the constraints that bound the variable z:

$$z \leq \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_f(i) a_i(i, t^{-1}) \pi_f(i, t^{-1}) \quad \forall f \in P.$$

Given that the condition on the $\nu_f(i)$ value holds, we know that the function in the right hand side reaches its maximum value if the discrete second price auction with reserve price i^* is applied. See the results in Section 3.1. Since this is true

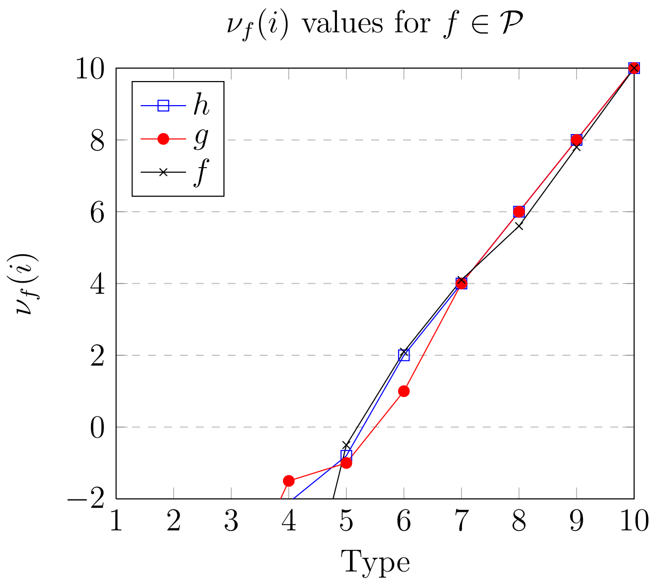

Figure 5.1: $\nu_f(i)$ is positive for $i \geq i^* = 6$

for any $f \in \mathcal{P}$, the following value for z cannot be improved, so it is optimal:

$$z = \min_{f \in \mathcal{P}} \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_f(i) a_i^*(i, t^{-1}) \pi_f(i, t^{-1}),$$

where $a_i^*(i, t^{-1})$ is the discrete second price auction with reserve price i^*. $\square$

Of course in most cases, $\nu_f(i)$ values will start to take non-negative values in different types for some $f \in \mathcal{P}$ as in Figure 5.2. Since we know the $\nu_f(i)$ values, we also know the optimal reserve prices for each f. Assume that for $f \in \mathcal{P}$, i_f^* is the optimal reserve price which satisfies $\nu_f(i_f^* - 1) < 0$ and $\nu_f(i_f^*) \geq 0$.

Theorem 5.2. *If the optimal reserve price for $h \in \mathcal{P}$ is i_h^* and the following condition holds where $a_i^*(i, t^{-1})$ is the discrete second price auction with reserve price i_h^*:*

$$\sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_h(i) a_i^*(i, t^{-1}) \pi_h(i, t^{-1}) = \min_{f \in \mathcal{P}} \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_f(i) a_i^*(i, t^{-1}) \pi_f(i, t^{-1}),$$

then discrete second price auction with reserve price i_h^* is the optimal solution.

Proof. Consider the constraints that bound the variable z:

$$z \leq \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_f(i) a_i(i, t^{-1}) \pi_f(i, t^{-1}) \quad \forall f \in P.$$

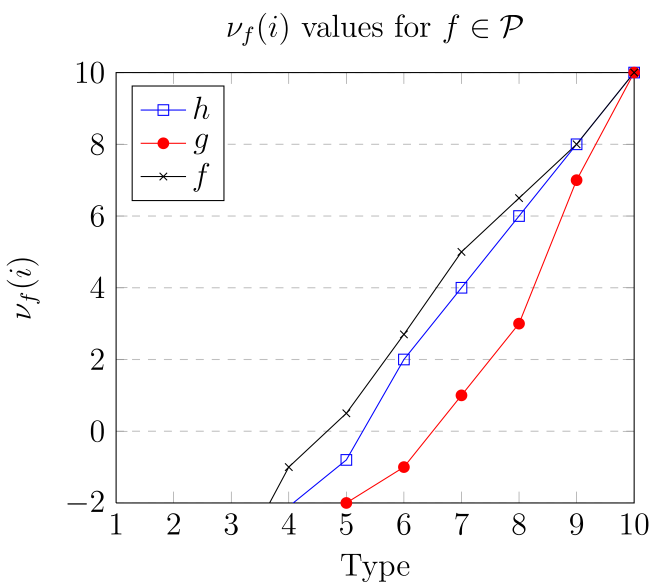

Figure 5.2: $\nu_f(i)$ turns positive in different i values

For distribution h, we know that the function in the right hand side reaches its maximum value if the discrete second price auction with reserve price i_h^* is applied. See the results in Section 3.1. When it has the minimum value as given, the following value for z cannot be improved so it is optimal:

$$z = \min_{f \in \mathcal{P}} \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_f(i) a_i^*(i, t^{-1}) \pi_f(i, t^{-1}),$$

where $a_i^*(i, t^{-1})$ is the discrete second price auction with reserve price i_h^*. $\square$

Using the knapsack problem, we can give more general characterizations for the optimal allocation rule.

Proposition 5.1. *Assuming $f_i \nu_f(i)$ is monotone for all $f \in \mathcal{P}$, the optimal allocation rule for the following problem fulfills the condition $a_k(k, t^{-1}) \geq a_j(j, t^{-1})$*

for all $t^{-1} \in T^{n-1}$ *and* $k, j \in T$ *where* $k \geq j$*:*

$$\begin{aligned}
\max_{a,z} \quad & z && OPTk' \\
s.t. \quad & z \leq \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} f_i \nu_f(i) a_i(i, t^{-1}) \pi_f(t^{-1}) && \forall f \in \mathcal{P} \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 && \forall t \in T^n \\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n.
\end{aligned}$$

Proof. Assume to the contrary that there exists k in T and $t^{-1} \in T^{n-1}$ such that $a_k(k, t^{-1}) < a_{k-1}(k-1, t^{-1})$. Define $t = (k, t^{-1})$ and $t' = (k-1, t^{-1})$. We can find l in T such that $n_l(t)$, $n_l(t') > 0$.

Note that for the case $k - 1 < l$, we can increase $a_l(t')$ and decrease $a_{k-1}(t')$ so that allocation rule stays feasible, and we have an improved worst case expected revenue. For $k - 1 \geq l$, if $a_l(t) > 0$ then we can increase $a_k(t)$ and decrease $a_l(t)$ in order to improve.

If none of the conditions mentioned above holds, we have $n_l(t') = 0$ for $k-1 < l$ and $a_l(t) = 0$ for $k - 1 \geq l$. Therefore we know that $n_k(t) = 1$ and $a_k(t) < 1$ because of the following fact:

$$0 \leq a_k(k, t^{-1}) < a_{k-1}(k-1, t^{-1}) \leq n_{k-1}(t') a_{k-1}(t') \leq 1.$$

The same fact allows us to find a small $\alpha \in \mathbb{R}$ to construct a new feasible solution in the following way:

$$a^*_{k-1}(t') = a_{k-1}(t') - \alpha, \qquad a^*_k(t) = a_k(t) + \alpha \cdot n_{k-1}(t').$$

Keeping all other variables identical, we can write down the change in the constraints bounding z in the following way:

$$- f_{k-1} \nu_f(k-1) n_{k-1}(t') \alpha \pi_f(t'^{-1}) + f_k \nu_f(k) n_{k-1}(t') \alpha \pi_f(t^{-1}) \quad \forall f \in \mathcal{P}.$$

Since we assumed $f_i \nu_f(i)$ to be monotone, this expression is non-negative for all f in $\mathcal{P}$. Therefore we have a feasible solution with an improved objective and a contradiction. $\square$

Proposition 5.2. *Assuming $f_i\nu_f(i)$ to be monotone for all $f \in \mathcal{P}$, optimal allocation rule for $OPTk'$ satisfies $A_f(k) \geq A_f(j)$ for all $k, j \in T$ and $f \in \mathcal{P}$ where $k \geq j$.*

Proof. Recall the equation for interim allocations:

$$A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1})\pi_f(t^{-1}) \quad \forall f \in \mathcal{P}.$$

We know that $a_k(k, t^{-1}) \geq a_{k-1}(k-1, t^{-1})$ for all $t^{-1} \in T^{n-1}$ from Proposition 5.1. Therefore, interim allocations should be monotone also:

$$\begin{aligned} \sum_{t^{-1} \in T^{n-1}} a_k(k, t^{-1})\pi_f(t^{-1}) \geq \sum_{t^{-1} \in T^{n-1}} a_{k-1}(k-1, t^{-1})\pi_f(t^{-1}) \quad & \forall f \in \mathcal{P}, \\ A_f(k) \geq A_f(k-1) \quad & \forall f \in \mathcal{P}. \end{aligned}$$

□

Proposition 5.3. *Under the assumption of monotone $f_i\nu_f(i)$ for all f in $\mathcal{P}$, optimal allocation rule satisfies $a_k(t) \geq a_j(t)$ for all $k, j \in T$ such that $n_k(t)$, $n_j(t)$ are positive and $k \geq j$.*

Proof. Assume the contrary that there exists k in T such that in the optimal solution for $OPTk'$, we have $a_k(t) < a_{k-1}(t)$. In this case we can increase $a_k(t)$ and decrease $a_{k-1}(t)$ so that allocation rule stays feasible and we have a improved worst case expected revenue. This is a contradiction. □

Now we focus on the case where there are two agents and the type distribution set is equal to $\mathcal{P} = \{f, g\}$. We will describe an algorithm below, referred to as Algorithm 1.

Theorem 5.3. *Algorithm 1 gives the exact solution when $\nu_f(i)f_i$ and $\nu_g(i)g_i$ are non-decreasing in $i \in T$.*

Proof. Firstly we will show that solution of Algorithm 1 is feasible. Assume that the solution a^* from the algorithm yields interim allocations that are not

Algorithm 1 Optimal mechanism for two agents and two distributions

Initialize: $\triangleright$ We assume $i^*_f > i^*_g$

$$i^* \leftarrow \max_{f \in \mathcal{P}} \{i^*_f\} = i^*_f$$

$$a^*_i(i,j) \leftarrow \begin{cases} 1 & \text{if } i \geq i^* \text{ and } i > j \\ 0.5 & \text{if } i \geq i^* \text{ and } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$obj_f \leftarrow \sum_{i \in T} \sum_{j \in T} f_i \nu_f(i) a^*_i(i,j) f_j$$

$$obj_g \leftarrow \sum_{i \in T} \sum_{j \in T} g_i \nu_g(i) a^*_i(i,j) g_j$$

$$S \leftarrow \{(i,j) \in T \mid i^*_f > i \geq i^*_g \text{ and } i \geq j\} \text{ and } \Gamma : S \to \mathbb{R}$$

$$\Gamma(i,j) \leftarrow \frac{\nu_g(i) g_i g_j}{\nu_f(i) f_i f_j}$$

while S is not empty and $obj_f > obj_g$ **do**
  $\Gamma(k,l) \leftarrow \min_{(i,j) \in S} \Gamma(i,j)$
  if $k \neq l$ **then**
    if $obj_f + \nu_f(i) f_k f_g > obj_f + \nu_f(i) f_k f_g$ **then**
      $a^*_k(k,l) \leftarrow 1$
      $obj_f \leftarrow \sum_{i \in T} \sum_{j \in T} f_i \nu_f(i) a^*_i(i,j) f_j$
      $obj_g \leftarrow \sum_{i \in T} \sum_{j \in T} g_i \nu_g(i) a^*_i(i,j) g_j$
    else
      $a^*_k(k,l) \leftarrow \frac{obj_f - obj_g}{\nu_f(k) f_k f_l - \nu_g(k) g_k g_l}$
      Stop at the current solution
    end if
  end if
  if $k = l$ **then**
    if $obj_f + \nu_f(i) f_k f_g > obj_f + \nu_f(i) f_k f_g$ **then**
      $a^*_k(k,l) \leftarrow 0.5$
      $obj_f \leftarrow \sum_{i \in T} \sum_{j \in T} f_i \nu_f(i) a^*_i(i,j) f_j$
      $obj_g \leftarrow \sum_{i \in T} \sum_{j \in T} g_i \nu_g(i) a^*_i(i,j) g_j$
    else
      $a^*_k(k,l) \leftarrow \frac{obj_f - obj_g}{\nu_f(k) f_k f_l - \nu_g(k) g_k g_l}$
      Stop at the current solution
    end if
  end if
  Exclude (k,l) from S
end while
Stop at the current solution

monotone. Then there exists $i \in T$ such that one of the following inequalities hold:

$$\sum_{j=1}^{m} a^*_{i-1}(i-1,j)f_j > \sum_{j=1}^{m} a^*_i(i,j)f_j, \qquad \sum_{j=1}^{m} a^*_{i-1}(i-1,j)g_j > \sum_{j=1}^{m} a^*_i(i,j)g_j.$$

Since f_j and g_j are positive for all j, we can conclude that there exists $k \in T$ such that $a^*_{i-1}(i-1,k) > a^*_i(i,k)$.

Consider $\Gamma(i,k)$ and $\Gamma(i-1,k)$. We know that $v_g(i)g_i g_k \geq v_g(i-1)g_{i-1}g_k \geq 0$ and $0 \geq v_f(i)f_i f_k \geq v_f(i-1)f_{i-1}f_k$. Therefore, we should have $\Gamma(i,k) \geq \Gamma(i-1,k)$. Algorithm 1 increased $a^*(i,k)$ before $a^*(i-1,k)$.

Case 1: $a^*(i,k) = 1$. Contradiction because of $a^*(i-1,k) > a^*(i,k)$

Case 2: $a^*(i,k) = 0.5$. Then $i = k$ and $i-1 < k$. Contradiction because Algorithm 1 only increases those $a(i,j)$ variables where $i \geq j$.

Case 3: $a^*(i,k) = \frac{obj_f - obj_g}{\nu_f(i')f_{i'}f_{j'} - \nu_g(i')g_{i'}g_{j'}}$. Then Algorithm 1 stopped here and did not increase $a^*(i-1,k)$. Contradiction.

Case 4: $a^*(i,k) = 0$. Then Algorithm 1 stopped before $a^*(i,k)$. Contradiction.

Realize that these are the only values that Algorithm 1 can assign to variable $a^*(i,k)$. Therefore the solution yields monotone interim allocations. Besides, Algorithm 1 always assigns values between 1 and 0. We can conclude that a^* is feasible.

Now assume that there exists an allocation rule $a' \neq a^*$ such that it is feasible and gives $z' > z^*$. Let us consider the constraints on z for distribution f:

$$z \leq \mu_f + obj_f \qquad \forall f \in \mathcal{P},$$

where

$$\sum_{j \in T} \sum_{i=1}^{i^*_g - 1} \nu_f(i) a_i(i,j) f(i) f(j) = 0$$

$$\sum_{j \in T} \sum_{i=i^*_g}^{i^*_f - 1} \nu_f(i) a_i(i,j) f(i) f(j) = \mu_f$$

$$\sum_{j \in T} \sum_{i=i^*_f}^{m} \nu_f(i) a_i(i,j) f(i) f(j) = obj_f.$$

We know that $\nu_f(i)$ function is negative for both distributions when i is less than $i_g^* - 1$. Therefore we can say that no allocation in the profiles with maximum bid of $i_g^* - 1$ or less is the optimum solution. When $\nu_f(i)$ is positive for all f, by Proposition 5.3 we can say that allocating the good to highest bid is the optimal solution which yields obj_f value for those profiles. The following inequalities should hold in order for a' to yield greater worst case expected revenue:

$$\mu'_f + obj_f > \mu^*_f + obj_f,$$
$$\mu'_g + obj_g > \mu^*_g + obj_g.$$

Then $\mu'_f > \mu^*_f$ and $\mu'_g > \mu^*_g$ should be satisfied. This means that there exists $k \in [i_g^*, i_f^* - 1]$ and $j < k$ such that $a_k^*(k,j) < a'_k(k,j)$ since $\nu_g(i) \geq 0$ for all $i \in [i_g^*, i_f^* - 1]$:

$$\begin{aligned} \nu_g(k)a_k^*(k,j)g(k)g(j) < \nu_g(k)a'_k(k,j)g(k)g(j), \\ \nu_f(k)a_k^*(k,j)f(k)f(j) > \nu_f(k)a'_k(k,j)f(k)f(j). \end{aligned} \tag{5.1}$$

Since $\nu_f(i) < 0$ for all $i \in [i_g^*, i_f^* - 1]$, inequalities 5.1 imply that there exists $q \in [i_g^*, i_f^* - 1]$ and $r < q$ such that $a_q^*(q,r) > a'_q(q,r)$ and so that we can have:

$$\begin{aligned} \nu_g(k)a_k^*(k,j)g(k)g(j) + \nu_g(q)a_q^*(q,r)g(q)g(r) < \nu_g(k)a'_k(k,j)g(k)g(j) \\ + \nu_g(q)a'_q(q,r)g(q)g(r), \end{aligned}$$

$$\begin{aligned} \nu_f(k)a_k^*(k,j)f(k)f(j) + \nu_f(q)a_q^*(q,r)f(q)f(r) < \nu_f(k)a'_k(k,j)f(k)f(j) \\ + \nu_f(q)a'_q(q,r)f(q)f(r). \end{aligned}$$

Rearranging these inequalities, we get:

$$\nu_g(q)g(q)g(r)\big(a_q^*(q,r) - a'_q(q,r)\big) < \nu_g(k)g(k)g(j)\big(a'_k(k,j) - a_k^*(k,j)\big),$$
$$\nu_f(q)f(q)f(r)\big(a_q^*(q,r) - a'_q(q,r)\big) < \nu_f(k)f(k)f(j)\big(a'_k(k,j) - a_k^*(k,j)\big).$$

Knowing that the differences between allocation rule variables are positive and $\nu_f(i) < 0$ for all $i \in [i_g^*, i_f^* - 1]$ divide both sides with each other to get:

$$\frac{\nu_g(q)g(q)g(r)}{\nu_f(q)f(q)f(r)} < \frac{\nu_g(k)g(k)g(j)}{\nu_f(k)f(k)f(j)},$$

meaning that Algorithm 1 should have chosen to increase $a_k(k,j)$ before $a_q(q,r)$. Case 1: $a_k^*(k,j) = 1$. Contradiction because of $a'_k(k,j) > a_k^*(k,j)$.

Case 2: $a_k^*(k,j) = 0.5$. Then $j = k$. Then $a'_k(k,j) = a'_k(j,k)$ should also hold since agents are symmetric. Assumption $a'_k(k,j) > a_k^*(k,j) = 0.5$ leads to $2a'_k(k,j) > 1$. Contradiction, $a'_k(k,j)$ is not feasible.
Case 3: $a_k^*(k,j) = \frac{obj_f - obj_g}{\nu_f(k) f_k f_j - \nu_g(k) g_k g_j}$. Then Algorithm 1 stopped here and did not increase $a_q^*(q,r)$. Contradiction.
Case 4: $a_k^*(k,j) = 0$. Then Algorithm 1 stopped before $a_q^*(q,r)$. Contradiction.
There is no other case. Solution given by the Algorithm 1 is optimal. $\square$

Simply put, Algorithm 1 finds profiles and allocation probabilities which increase the worst expected revenue most and decrease the other expected revenue least. Note that it assigns the good to the highest bidder in that profile.

We still use the same equation 3.4 in order to pin down the optimal payment rule. However, we can no longer claim that the optimal mechanism is in the form of discrete second price auction with reserve if Theorems 5.1 and 5.2 fail to hold.

Example 5.1. *Consider the following allocation rule which is given by algorithm 1. We have two agents and our type set is $T = \{1, 2, \ldots 10\}$.*

$$a_i^*(i,j) = \begin{cases} 1 & \text{if } i \geq 6 \text{ and } i > j \\ 0.5 & \text{if } i \geq 6 \text{ and } i = j \\ 0.7 & \text{if } i = 5 \text{ and } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

Realize that there is nothing interesting in the payment of profile $(5, 2)$, which will be 3.5. However things get complicated for profile $(i, 2)$ when i is 6 or more. Lets calculate the payment for the profile $(6, 2)$:

$$p_6(6,2) = 6 \cdot a_6(6,2) - a_5(5,2) = 6 - 0.7 = 5.3.$$

This is strange since in the profile $(6, 3)$ winner still has to pay 6. In this new mechanism the bid of the defeated agent affects the payment.

We do not claim that this payment rule is the unique optimal solution. However, we can say that the all-pay auction from Vohra [1] is not optimal in this framework. The reason is simply having more than one expected payment for each type.

If we relax the assumption of monotone $\nu_f(i)f_i$, even if we have monotone $\nu_f(i)$, we may end up with an allocation rule where $a_k(k,j) < a_{k-1}(k-1,j)$. Realize that this may result in a payment rule where seller has to make a payment to winning agent of profile (k,j) for no allocation when $a_k(k,j) = 0$.

5.2 Mechanism with Costly Inspection

Notation is similar to first section of Chapter 4. Note that we will assume symmetry among the agents so that they will have the same inspection cost. Again we focus on direct mechanisms as in [6]. Let us look at the formulation where our objective is to maximize the worst case social welfare:

$$
\begin{aligned}
\max_{S,A,s,a} \quad & \left\{ \min_{f\in\mathcal{P}} \sum_{i\in T} f_i(i-K)A_f(i) + K\sum_{i\in T} f_i S_f(i) \right\} \\
\text{s.t.} \quad & A_f(i) \geq S_f(j) && \forall i,j \in T, \forall f \in \mathcal{P} \quad (1) \\
& A_f(i) = \sum_{t^{-1}\in T^{n-1}} a_i(i,t^{-1})\pi_f(t^{-1}) && \forall i \in T, \forall f \in \mathcal{P} \quad (2) \\
& S_f(i) = \sum_{t^{-1}\in T^{n-1}} s_i(i,t^{-1})\pi_f(t^{-1}) && \forall i \in T, \forall f \in \mathcal{P} \quad (3) \\
& a_i(t) \geq s_i(t) && \forall i \in T, \forall t \in T^n \quad (4) \\
& \sum_{i\in T} n_i(t)a_i(t) \leq 1 && \forall t \in T^n \quad (5) \\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n \quad (6) \\
& s_i(t) \geq 0 && \forall i \in T, \forall t \in T^n. \quad (7)
\end{aligned}
$$

Drop $s_i(t)$ variables and introduce variable z in order to linearize the formulation:

$$
\begin{aligned}
\max_{S,A,a} \quad & z \\
\text{s.t.} \quad & z \leq \sum_{i\in T} f_i(i-K)A_f(i) + K\sum_{i\in T} f_i S_f(i) && \forall f \in \mathcal{P} && (8)\\
& A_f(i) \geq S_f(j) && \forall i,j \in T, \forall f \in \mathcal{P} && (1)\\
& A_f(i) = \sum_{t^{-1}\in T^{n-1}} a_i(i,t^{-1})\pi_f(t^{-1}) && \forall i \in T, \forall f \in \mathcal{P} && (2)\\
& \sum_{i\in T} n_i(t)a_i(t) \leq 1 && \forall t \in T^n && (5)\\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n. && (6)
\end{aligned}
$$

5.2.1 The Solution Approach

The reasoning behind Theorem 5.1 and Theorem 5.2 also applies for mechanism with costly inspection. Adapting the first theorem, it is obvious that if the optimal mechanism for an $f \in \mathcal{P}$, when it is the common prior, has the same cut-off as all other distributions in $\mathcal{P}$, then the same mechanism is optimal for the ambiguity averse seller. For Theorem 5.2, again consider the optimal mechanism for f when it is the common prior. If distribution f attains worst-case when constraint (8) is checked, then this solution is optimal. Proofs of Theorems 5.1 and 5.2 can be altered with a little effort to prove these arguments.

We will now show that if there is a distribution in $\mathcal{P}$ which is first order stochastically dominated by all other distributions, we can solve the problem considering only that dominated distribution. Recall that f first order stochastically dominates g if and only if we have $F(i) \leq G(i)$ for all i and with a strict inequality at some i. This means that probability distribution for f is more concentrated on highest types.

Theorem 5.4. *If $g \in \mathcal{P}$ is first order stochastically dominated by all $f \in \mathcal{P}\setminus\{g\}$, then the following is the optimal mechanism where t_g^* is the optimal cut-off for* *distribution g:*

$$a_i^*(i,t^{-1}) = \begin{cases} \frac{1}{n_i(t)} & \textit{if } i \geq \underset{k\in t^{-1}}{max}\, k \textit{ and } i \geq t_g^* \\ \frac{1}{n} & \textit{if } \underset{k\in t^{-1}}{max}\, k < t_g^* \textit{ and } i < t_g^* \\ 0 & \textit{otherwise,} \end{cases}$$

$$s_i^*(i,t^{-1}) = \begin{cases} \frac{1}{n} & \textit{if } \underset{k\in t^{-1}}{max}\, k < t_g^* \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Firstly, this mechanism is feasible for all $f \in \mathcal{P}$. Note that it is the optimal solution if we have g as the unique prior. Therefore, if we show that $g \in \mathcal{P}$ is the worst case then we complete the proof because the given mechanism yields the maximum utility for g and it cannot be improved.

Let us look at the constraint (8) with expected allocations obtained from this mechanism. This is for all f in $\mathcal{P}$:

$$z \leq \sum_{i=t_g^*}^{m}(i-K)\frac{F(i)^n - F(i-1)^n}{n} + \sum_{i=1}^{t_g^*-1} f_i(i-K)\frac{F(t_g^*-1)^{n-1}}{n} + K\frac{F(t_g^*-1)^{n-1}}{n}.$$

Define $\nu_f = \frac{\sum_{i=1}^{t_g^*-1} f_i i + K}{F(t_g^*-1)}$. We rearrange the right hand side:

$$\begin{aligned}
&\sum_{i=t_g^*}^{m}(i-K)\frac{F(i)^n - F(i-1)^n}{n} + \sum_{i=1}^{t_g^*-1} f_i i\frac{F(t_g^*-1)^{n-1}}{n} + (1-F(t_g^*-1))K\frac{F(t_g^*-1)^{n-1}}{n} \\
&= \sum_{i=t_g^*}^{m}(i-K)\frac{F(i)^n - F(i-1)^n}{n} + (\nu_f - K)\frac{F(t_g^*-1)^n}{n} \\
&= (m-K)\frac{1^n - F(m-1)^n}{n} + \ldots (t_g^*-K)\frac{F(t_g^*)^n - F(t_g^*-1)^n}{n} + (\nu_f - K)\frac{F(t_g^*-1)^n}{n} \\
&= (m-K) - \sum_{i=t_g^*}^{m-1}\frac{F(i)^n}{n} - (t_g^* - \nu_f)\frac{F(t_g^*-1)^n}{n}.
\end{aligned} \tag{5.2}$$

Now we can compare this value for g and any f that first order stochastically dominates g. If we have $\nu_f \geq \nu_g$ then the following is obvious since we have

$F(i) \leq G(i)$ for all i in T and $F(j) < G(j)$ for some j in T. The right hand side of constraint (8) for f is bigger than the right hand side of constraint (8) for g:

$$-\sum_{i=t_g^*}^{m-1} \frac{G(i)^n}{n} - (t_g^* - \nu_g)\frac{G(t_g^*-1)^n}{n} < -\sum_{i=t_g^*}^{m-1} \frac{F(i)^n}{n} - (t_g^* - \nu_f)\frac{F(t_g^*-1)^n}{n}. \tag{5.3}$$

When $\nu_f < \nu_g$ we look into two cases. First one is if $F(t_g^*-1) = G(t_g^*-1)$. Look at ν values:

$$\frac{\sum_{i=1}^{t_g^*-1} f_i i + K}{F(t_g^*-1)} < \frac{\sum_{i=1}^{t_g^*-1} g_i i + K}{G(t_g^*-1)} \quad \Rightarrow \quad \frac{\sum_{i=1}^{t_g^*-1} f_i i}{F(t_g^*-1)} < \frac{\sum_{i=1}^{t_g^*-1} g_i i}{G(t_g^*-1)},$$

$$\sum_{i=1}^{t_g^*-1} \frac{f_i}{F(t_g^*-1)} i < \sum_{i=1}^{t_g^*-1} \frac{g_i}{G(t_g^*-1)} i.$$

Define $\frac{f_i}{F(t_g^*-1)}$ as a new distribution which will also dominate $\frac{g_i}{G(t_g^*-1)}$ or they will be exactly the same. In both cases, this expectation violates the assumption that f first order stochastically dominates g. We cannot have $\nu_f < \nu_g$ and $F(t_g^*-1) = G(t_g^*-1)$ at the same time.

Consider the case when $\nu_f < \nu_g$ and $F(t_g^*-1) < G(t_g^*-1)$. Again look at the ν values:

$$\frac{\sum_{i=1}^{t_g^*-1} f_i i + K}{F(t_g^*-1)} < \frac{\sum_{i=1}^{t_g^*-1} g_i i + K}{G(t_g^*-1)}.$$

For this to be true the following inequality should hold:

$$K < \frac{F(t_g^*-1)\sum_{i=1}^{t_g^*-1} g_i i - G(t_g^*-1)\sum_{i=1}^{t_g^*-1} f_i i}{G(t_g^*-1) - F(t_g^*-1)}. \tag{5.4}$$

We will use this inequality in comparing constraints (8) of distributions. We wanted to show the following where denominator n is canceled out from inequality 5.3:

$$-\sum_{i=t_g^*}^{m-1} G(i)^n - (t_g^* - \nu_g)G(t_g^*-1)^n < -\sum_{i=t_g^*}^{m-1} F(i)^n - (t_g^* - \nu_f)F(t_g^*-1)^n,$$

$$\nu_g G(t_g^*-1)^n - \nu_f F(t_g^*-1)^n - t_g^*(G(t_g^*-1)^n - F(t_g^*-1)^n) < \sum_{i=t_g^*}^{m-1} G(i)^n - F(i)^n.$$

We know that the right hand side is non-negative. Consider the left hand side which is equal to following if we replace ν_f with $\frac{\sum_{i=1}^{t_g^*-1} f_i i+K}{F(t_g^*-1)}$ and ν_g with its respected value:

$$G(t_g^*-1)^{n-1}\sum_{i=1}^{t_g^*-1} g_i i - F(t_g^*-1)^{n-1}\sum_{i=1}^{t_g^*-1} f_i i + (G(t_g^*-1)^{n-1} - F(t_g^*-1)^{n-1})K \\ - t_g^*(G(t_g^*-1)^n - F(t_g^*-1)^n).$$

Since we know that K is bounded from inequality 5.4, the expression above is less than the following: (we got this result by using the bound of K and making some simplifications.)

$$(G(t_g^*-1)^n - F(t_g^*-1)^n)\frac{\sum_{i=1}^{t_g^*-1}(g_i - f_i)(i - t_g^* - 1)}{G(t_g^*-1) - F(t_g^*-1)}.$$

Recall that we are looking in the case where $F(t_g^*-1) < G(t_g^*-1)$. Therefore, $G(t_g^*-1)^n - F(t_g^*-1)^n$ and $G(t_g^*-1) - F(t_g^*-1)$ are positive. Let us look at $\sum_{i=1}^{t_g^*-1}(g_i - f_i)(i - t_g^*)$:

$$\begin{aligned}\sum_{i=1}^{t_g^*-1}(g_i - f_i)(i - t_g^*) = \quad & \big(G(1) - F(1)\big)(1 - t_g^*) \\ & + \big(G(2) - F(2) - (G(1) - F(1))\big)(2 - t_g^*) + \ldots \\ & + \big(G(t_g^*) - F(t_g^*) - (G(t_g^*-1) - F(t_g^*-1))\big)(t_g^* - 1 - t_g^*), \\ & = -\sum_{i=1}^{t_g^*-1} G(i) - F(i) < 0.\end{aligned}$$

This bound turns out to be negative. Therefore left hand side we investigated is definitely less than the non-negative right hand side $\sum_{i=t_g^*}^{m-1} G(i)^n - F(i)^n$. It means that inequality 5.3 is satisfied for this case also. There is no any other case so we proved that dominated distribution g is the worst case and the given mechanism is optimal. □

For the case with two agents and $\mathcal{P} = \{f, g\}$, we propose an algorithm below (Algorithm 2) which gives the optimal mechanism. This algorithm works if the distributions are collinear up to a point.

Theorem 5.5. *Algorithm 2 gives the exact solution when $\alpha f_i = g_i$ holds for all $i \in T$ such that $i < \max_{f \in \mathcal{P}} \nu_f^*$, α is a positive scalar and ν_f^i is given by the following:*

$$\nu_f^i = \frac{\sum_{j=1}^{i-1} f_j j + K}{F(i-1)}. \tag{5.5}$$

Proof. If the algorithm stopped at the first **if clause** where $\nu^* = \nu_h^*$, then we have a mechanism which is the optimal solution to the worst case distribution $h \in \mathcal{P}$ (adapt Theorem 5.2). Therefore, given mechanism is optimal for maximizing the worst case utility.

The plan for the rest of proof is as follows: we will first show that allocation rule from the second **if clause** is feasible. Then we will prove its optimality.

For the second **if clause**, without getting in the **while clause** it is not possible to satisfy $\nu^* < \nu_h^*$. In the **while clause**, ν^* at least gets the value of ν_h^* which gives the optimum cut-off of the minimum objective of that time and then the mechanism changes, which means that the condition $\nu^* < \nu_h^*$ is satisfied only if the minimum obj_h also changes after the change of mechanism. Then we can assume that last update on ν^* changes the worst case g to f or vice versa. Therefore, using the rearranged constraint (8) in 5.2, the following holds. Assume that obj_f is the worst case for the allocation with cut-off i^*:

$$-\sum_{i \geq i^*+1}^{m-1} \frac{F(i)^2}{2} - (i^* + 1 - \nu_f^{i^*+1}) \frac{F(i^*)^2}{2} > -\sum_{i \geq i^*+1}^{m-1} \frac{G(i)^2}{2} - (i^* + 1 - \nu_g^{i^*+1}) \frac{G(i^*)^2}{2},$$

$$-\sum_{i \geq i^*}^{m-1} \frac{F(i)^2}{2} - (i^* - \nu_f^{i^*}) \frac{F(i^*-1)^2}{2} < -\sum_{i \geq i^*}^{m-1} \frac{G(i)^2}{2} - (i^* - \nu_g^{i^*}) \frac{G(i^*-1)^2}{2}.$$

First inequality is the relation between constraint (8)'s value for each distribution when allocation with cut-off i^*+1 is undertaken. Since our algorithm did not give this mechanism as a solution then the distribution g achieving the worst-case can be improved. In other words, $\nu^* > \nu_h^*$ is satisfied. Since g attains the worst-case we have $\nu_h^* = \nu_g^*$. After an update in **while clause**, we have $\nu^* \leq i^*$ and ν^* is not less than ν_g^*. Therefore our worst case is not g anymore but f so that the second "if" condition is satisfied. Second inequality above shows this relation when we

Algorithm 2 Optimal mechanism for two agents, $\mathcal{P} = \{f, g\}$ and $\alpha f_i = g_i$

Initialize:

$$\nu^* \leftarrow \max_{f \in \mathcal{P}} \{\nu_f^*\}$$

$$a_i(i,j) = \begin{cases} \frac{1}{n_i(t)} & \text{if } i \geq j \text{ and } i \geq \nu^* \\ \frac{1}{2} & \text{if } j < \nu^* \text{ and } i < \nu^* \\ 0 & \text{otherwise} \end{cases}$$

$$obj_f \leftarrow \sum_{i \in T} f_i(i-K)A_f(i) + KA_f(1)$$

$$obj_g \leftarrow \sum_{i \in T} g_i(i-K)A_g(i) + KA_g(1)$$

▷ Assume that $obj_h = \min_{f \in \mathcal{P}} obj_f$

while $\nu^* > \nu_h^*$ **do**

  $\nu^* \leftarrow \max\{\nu^* - 1, \nu_h^*\}$

$$a_i(i,j) = \begin{cases} \frac{1}{n_i(t)} & \text{if } i \geq j \text{ and } i \geq \nu^* \\ \frac{1}{2} & \text{if } j < \nu^* \text{ and } i < \nu^* \\ 0 & \text{otherwise} \end{cases}$$

$$obj_f \leftarrow \sum_{i \in T} f_i(i-K)A_f(i) + KA_f(1)$$

$$obj_g \leftarrow \sum_{i \in T} g_i(i-K)A_g(i) + KA_g(1)$$

end while

if $\nu^* = \nu_h^*$ **then**

  Stop at the current solution

end if

if $\nu^* < \nu_h^*$ **then**

  $i^* \leftarrow \min_{j \in T, j \geq \nu^*} j$

  $x \leftarrow \frac{obj_f - obj_g}{g_{i^*}G(i^*-1)(\nu_g^{i^*} - i^*) - f_{i^*}F(i^*-1)(\nu_f^{i^*} - i^*)}$

$$a_i(i,j) = \begin{cases} \frac{1}{n_i(t)} & \text{if } i \geq j \text{ and } i > i^* \\ \frac{1}{2} & \text{if } i = j = i^* \\ 1 - x & \text{if } i = i^* \text{ and } j < i \\ 0 + x & \text{if } i < j \text{ and } j = i^* \\ \frac{1}{2} & \max_{k \in t} k < i^* \\ 0 & \text{otherwise} \end{cases}$$

  Stop at the current solution

end if

have allocation with cut-off i^*. Note that $\nu_g^{i^*+1}$ and $\nu_f^{i^*}$ values are given by the equation 5.5.

Realize that first inequality above can be achieved by the algorithm's allocation when $x = 0.5$ and second inequality is given by the same allocation when $x = 0$. Next we will show that the value given to x results in $obj_f^* = obj_g^*$. Because of linearity of these functions we can say that $0 \leq x \leq 0.5$ so that given allocation rule is feasible. Let

$$A_g^*(i) = \begin{cases} \frac{G(i)^2 - G(i-1)^2}{2g_i} & \text{if } i \geq i^* + 1 \\ \frac{G(i^*)^2 - G(i^*-1)^2}{2g_{i^*}} - \sum_{j=1}^{i^*-1} g_j x & \text{if } i = i^* \\ \frac{G(i^*-1)}{2} + g_{i^*} x & \text{otherwise.} \end{cases}$$

This interim allocation is consistent with the allocation given by the algorithm. In order to check, set $x = 0.5$ and $x = 0$. $A_g^*(i^*)$ gets the values $\frac{G(i^*)}{2}$ and $\frac{G(i^*)^2 - G(i^*-1)^2}{2g_{i^*}}$, respectively. Let us look at the obj_g^* resulted from the algorithm's allocation: (We will also use the last given values to obj_g and obj_f from the algorithm.)

$$\begin{aligned} obj_g^* = \; & \frac{(m-K)}{2} - \sum_{i \geq i^*+1}^{m-1} \frac{G(i)^2}{2} - (i^* + 1 - K)\frac{G(i^*)^2}{2} + g_{i^*}(i^* - K)\Big(\frac{G(i^*)^2 - G(i^*-1)^2}{2g_{i^*}} \\ & - G(i^*-1)x\Big) + G(i^*-1)(\nu_g^{i^*} - K)\Big(\frac{G(i^*-1)}{2} + g_{i^*}x\Big) \\ = \; & \frac{(m-K)}{2} - \sum_{i \geq i^*+1}^{m-1} \frac{G(i)^2}{2} - (i^* + 1 - K)\frac{G(i^*)^2}{2} + g_{i^*}(i^* - K)\frac{G(i^*)^2 - G(i^*-1)^2}{2g_{i^*}} \\ & + G(i^*-1)(\nu_g^{i^*} - K)\frac{G(i^*-1)}{2} + x\Big(g_{i^*}G(i^*-1)(\nu_g^{i^*} - i^*)\Big) \\ = \; & \frac{(m-K)}{2} - \sum_{i \geq i^*}^{m-1} \frac{G(i)^2}{2} - (i^* - \nu_g^{i^*})\frac{G(i^*-1)^2}{2} + x\Big(g_{i^*}G(i^*-1)(\nu_g^{i^*} - i^*)\Big) \\ = \; & obj_g + x\Big(g_{i^*}G(i^*-1)(\nu_g^{i^*} - i^*)\Big). \end{aligned}$$

This equality is also true for obj_f^*. Let us find the value of x when $obj_g^* = obj_f^*$:

$$obj_g + x\Big(g_{i^*}G(i^*-1)(\nu_g^{i^*} - i^*)\Big) = obj_f + x\Big(f_{i^*}F(i^*-1)(\nu_f^{i^*} - i^*)\Big),$$

$$x\Big(g_{i^*}G(i^*-1)(\nu_g^{i^*} - i^*) - f_{i^*}F(i^*-1)(\nu_f^{i^*} - i^*)\Big) = obj_f - obj_g,$$

$$x = \frac{obj_f - obj_g}{g_{i^*}G(i^*-1)(\nu_g^{i^*} - i^*) - f_{i^*}F(i^*-1)(\nu_f^{i^*} - i^*)}.$$

This is the exact value given to x. Therefore, we have a feasible allocation and $obj_f^* = obj_g^*$.

In order to prove optimality let us assume that there exists a feasible allocation $a'(t)$ such that $obj_g' > obj_g^*$ and $obj_f' > obj_f^*$. Assume that $\nu_g^* = \min_{f \in \mathcal{P}} \{\nu_f^*\}$. Consider the $z_i^{g*} = g_i A_g^*(i)$ variables for the algorithm's allocation:

$$z_i^g = \begin{cases} \frac{G(i)^2 - G(i-1)^2}{2} & \text{if } i \geq i^* + 1 \\ \frac{G(i^*)^2 - G(i^*-1)^2}{2} - g_{i^*} \sum_{j=1}^{i^*-1} g_j x & \text{if } i = i^* \\ g_i \frac{G(i^*-1)}{2} + g_i g_{i^*} x & \text{if } i \leq i^* - 1. \end{cases}$$

Firstly, recall that $i^* \geq j_g^*$. The following inequality is true since the solution given by the algorithm is equal to solution given by the Greedy Algorithm for $i \geq i^* + 1$:

$$\sum_{i \geq i^*+1}^{m} (i - K) z_i^{g*} \geq \sum_{i \geq i^*+1}^{m} (i - K) z_i^{g'}, \tag{5.6}$$

where $z_i^{g'}$ is calculated according to $a'(t)$ variables. Assume that this holds with strict inequality. Then we have a type $d \in [i^* + 1, m]$ such that $z_d^{g*} > z_d^{g'}$. In order to have $obj_g' > obj_g^*$, there should be $j \in [1, d]$ such that $z_j^{g*} < z_j^{g'}$. Let us look at the constraint with submodular function when g is the common prior and $\mathcal{S} = T$:

$$\sum_{i \in T \setminus \{j,d\}} z_i^{g*} + z_d^{g'} + z_j^{g'} \leq \frac{1}{n}.$$

We know that this constraint is binding for the algorithm's solution. Therefore

$$z_d^{g'} + z_j^{g'} \leq z_d^{g*} + z_j^{g*},$$

$$z_j^{g'} - z_j^{g*} \leq z_d^{g*} - z_d^{g'}.$$

It means that increase in variable $z_j^{g'}$ cannot be larger than the decrease in $z_d^{g'}$. Since $d - K \geq j - K$, it is not possible to have $obj_g' > obj_g^*$. Then our assumption about the inequality 5.6 is wrong. We have $z_i^{g*} = z_i^{g'}$ for $i \geq i^* + 1$.

Next we will consider $z_i^{f*} = f_i A_f^*(i)$ variables for the algorithm's allocation. We already know that $z_i^{f*} = z_i^{f'}$ for $i \geq i^* + 1$. In order to have $obj_f' > obj_f^*$, there should be $j \in [1, i^*]$ such that $z_j^{f*} < z_j^{f'}$. Then we have a type $d \in [1, i^*]$ such that $z_d^{f*} > z_d^{f'}$. Realize that if we choose d to be less than i^*, then $A_f(d)$ will became the minimum interim allocation. Therefore, the expected inspection variables will also decrease and coefficient of $A_f(d)$ will be $d - K + \frac{K}{f_d}$. We will show that in such a case coefficient of $z_1^{f'}$ is larger than coefficient of $z_{i^*}^{f'}$.

Consider the function $Q(i)$ from Proposition 4.2 for distribution f. We know that $Q(i^*) = i^* - \nu_f^{i^*} < 0$ since i^* is less than the optimal cut-off for f, ν_f^*. From the proof of Proposition 4.2:

$$Q(i^* - 1) = \frac{F(i^* - 1)(Q(i^*) - 1)}{F(i^* - 2)} < Q(i^*) - 1.$$

Since this inequality is true for all $i \in T$ such that $i < \nu_f^*$:

$$Q(2) + i^* - 2 < \cdots < Q(i^* - 2) + 2 < Q(i^* - 1) + 1 < Q(i^*) < 0.$$

We know that $Q(2) = 2 - \nu_f^2$ and $\nu_f^2 = 1 + \frac{K}{f_1}$. Therefore coefficient of $z_1^{f'}$ is larger than coefficient of $z_{i^*}^{f'}$:

$$(i^* - K) - (1 + \frac{K}{f_1} - K) < 0.$$

We cannot have $j = i^*$ and $d = 1$. Only possible assignment of types to j and d are as follows: $j = l$ and $d = i^*$ where l refers to the case when all interim allocations below $i^* - 1$ are equal to each other. Look at the definition of set L in Proposition 4.1. The reason is that coefficient of $z_l^{f'}$ is larger than coefficient of $z_{i^*}^{f'}$. However this way of increasing obj'_f will decrease obj'_g since the coefficient of $z_l^{g'}$ is less than coefficient of $z_{i^*}^{g'}$. Recall that i^* is larger than the optimal cut-off for g, ν_g^*. This is a contradiction. □

Optimal solution in the algorithm allocates the good to the highest bid if it is strictly above cut-off i^*. If the highest bid is exactly i^*, then we decide the allocation after a lottery where the highest bidder has a bigger chance of winning. Under the cut-off, we allocate the good randomly.

We made an assumption of collinearity in order to end up with a simple mechanism with monotone interim allocations. The only important calculation done by

Algorithm 2 is finding the optimal cut-off and the optimal allocation probability in that cut-off that maximizes the worst case expected revenue. We again deduce the inspection skipping rule from equality $a_1(1,t^1) = s_j(j,t^{-1})$ for all j in T and t^{-1} in T^{n-1}.

We now look at an example to see the form of optimal solution if we do not make any of the previous assumptions.

Example 5.2. *We have two agents and $\mathcal{P} = \{f, g\}$. Inspection cost is $K = 0.1$, type set is $T = \{1, 2, \dots, 10\}$ and density functions are as follows:*

$$f_i = \begin{cases} 0.1001 & \textit{if } i = 1 \\ 0.0001 & \textit{if } i = 2 \\ 0.0001 & \textit{if } i = 3 \\ 0.0001 & \textit{if } i = 4 \\ 0.0001 & \textit{if } i = 5 \\ 0.0001 & \textit{if } i = 6 \\ 0.4001 & \textit{if } i = 7 \\ 0.4991 & \textit{if } i = 8 \\ 0.0001 & \textit{if } i = 9 \\ 0.0001 & \textit{if } i = 10, \end{cases} \qquad g_i = \begin{cases} 0.005 & \textit{if } i = 1 \\ 0.005 & \textit{if } i = 2 \\ 0.005 & \textit{if } i = 3 \\ 0.005 & \textit{if } i = 4 \\ 0.005 & \textit{if } i = 5 \\ 0.33 & \textit{if } i = 6 \\ 0.27 & \textit{if } i = 7 \\ 0.27 & \textit{if } i = 8 \\ 0.1 & \textit{if } i = 9 \\ 0.005 & \textit{if } i = 10. \end{cases}$$

Optimal solution for interim allocation variables are:

$$A_f(i) = \begin{cases} \frac{F(i)^2 - F(i-1)^2}{2f_i} & \textit{if } i \geq 7 \\ 0.058 & \textit{if } i = 6 \\ 0.1 & \textit{if } i = 5 \\ 0.1 & \textit{if } i = 4 \\ 0.1 & \textit{if } i = 3 \\ 0.05 & \textit{if } i = 2 \\ 0.05 & \textit{if } i = 1, \end{cases} \qquad A_g(i) = \begin{cases} \frac{G(i)^2 - G(i-1)^2}{2g_i} & \textit{if } i \geq 7 \\ 0.18 & \textit{if } i = 6 \\ 0.144 & \textit{if } i = 5 \\ 0.144 & \textit{if } i = 4 \\ 0.144 & \textit{if } i = 3 \\ 0.144 & \textit{if } i = 2 \\ 0.144 & \textit{if } i = 1. \end{cases}$$

Optimal solution for expected inspection skipping variables are:

$$S_f(i) = 0.05 \quad \forall i \in T, \qquad S_g(i) = \begin{cases} 0.144 & \textit{if } i \geq 6 \ \textit{ or } i = 1 \\ 0.141 & \textit{if } i = 5 \\ 0.136 & \textit{if } i = 4 \\ 0.131 & \textit{if } i = 3 \\ 0.129 & \textit{if } i = 2. \end{cases}$$

The distributions from the example above have the following optimal cut-offs when they are the common prior, $j_f^* = 2, j_g^* = 7$. First of all, optimal inspection skipping for g is not always equal to $A_g(1)$. Secondly, underlying allocation rule is designed in a way that expected allocation for type 6 breaks the monotonicity for distribution f but not for g. Therefore, violation of an assumption that we made might require an optimal mechanism that has monotonicity violating expected allocations and complex inspection skipping rule.

Chapter 6

Conclusion

In this thesis, we first focused on solving two different mechanism design problems via linear programming. For mechanism with monetary transfers, we showed that the second price auction with reserve is not optimal when valuations of agents comes from a discrete set. We can do better by using discrete second price auction with reserve which is actually optimal under the monotone hazard function assumption. For mechanism with costly inspection, we revisited the results of Ben-Porath, Dekel and Lipman [4]. The linear model for this problem simplifies to a polymatroid optimization problem which can be solved by the Greedy Algorithm. This simple procedure helped us to find the optimal choice of cut-off and favourite agent for the optimal favored-agent mechanism.

Moreover, we relaxed the common prior assumption and introduced ambiguity of the prior distribution into the framework. Again we used linear programming to maximize the worst case utility of an ambiguity averse seller when buyers were ambiguity neutral. As a result, we can say that both of the mechanisms require some restrictions on the prior set so that we can obtain an optimal solution with nice properties such as monotonicity. We specify several assumptions and the optimal mechanism under these restrictions. Besides, we proposed an algorithm for each problem to find the optimal solution under some other assumption.

We did not restrict the buyers to bid the valuations from the type set but we assumed that they have those valuations. As a further research, one can look into optimal mechanisms when the seller restricts the set of reports and buyers have continuous type set. Note that there is a literature on the effect of this restriction by the seller to the auction types.

Mechanisms for ambiguity averse seller can be studied to design optimal mechanisms for a larger class of prior sets. One can also consider the framework when the buyers are also ambiguity averse.

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