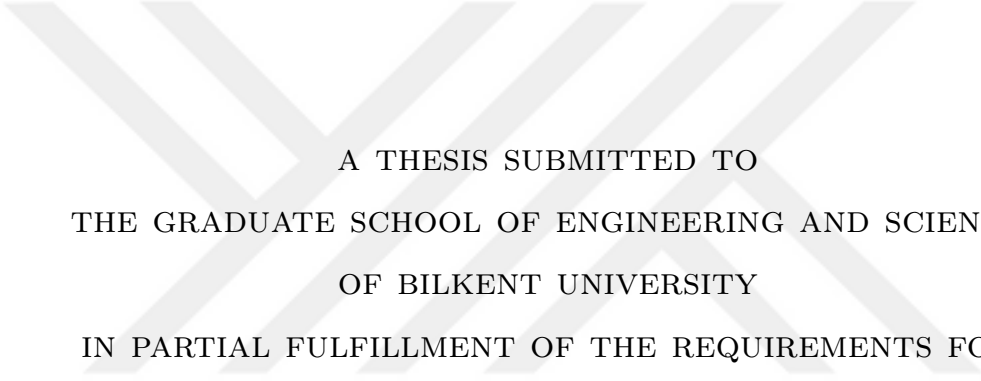


VISIBLE LIGHT POSITIONING UNDER HARDWARE IMPAIRMENTS



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By
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Visible Light Positioning under Hardware Impairments

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July 2024

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.



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ABSTRACT

VISIBLE LIGHT POSITIONING UNDER HARDWARE IMPAIRMENTS

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In this thesis, we examine two distinct research problems in visible light positioning (VLP). Specifically, we explore the impact of lacking knowledge about luminous flux degradation in light-emitting diodes (LEDs) and mismatched orientations for the elements of intelligent reflecting surfaces (IRSs) on the performance of VLP systems.

In the first part of the thesis, the position estimation problem based on received power measurements is investigated for visible light systems in the presence of luminous flux degradation of LEDs. When the receiver is unaware of this degradation and performs position estimation accordingly, there exists a mismatch between the true model and the assumed model. For this scenario, the misspecified Cramér-Rao bound (MCRB) and the mismatched maximum likelihood (MML) estimator are derived to quantify the performance loss due to this model mismatch. Also, the Cramér-Rao lower bound (CRB) and the maximum likelihood (ML) estimator are derived when the receiver knows the degradation formula for the LEDs but does not know the decay rate parameter in that formula. In addition, in the presence of full knowledge about the degradation formula and the decay rate parameters, the CRB and the ML estimator are obtained to specify the best achievable performance. By evaluating the theoretical limits and the estimators in these three scenarios, we reveal the effects of the information about the LED degradation model and the decay rate parameters on position estimation performance. It is shown that the model mismatch can result in significant degradation in localization performance at high signal-to-noise ratios, which can be compensated by conducting joint position and decay rate parameter estimation.

Accurate localization can be performed in visible light systems in non-line-of-sight (NLOS) scenarios by utilizing IRSs, which are commonly in the form of mirror arrays with adjustable orientations. When signals transmitted from LEDs are reflected from IRSs and collected by a receiver, the position of the receiver can be estimated based on power measurements by utilizing the known parameters of the LEDs and IRSs. Since the orientation vectors of IRS elements (mirrors) cannot be adjusted perfectly in practice, it is important to evaluate

the effects of mismatches between desired and true orientations of IRS elements. To this aim, in the second part of the thesis, we derive the MCRB and the MML estimator for specifying the estimation performance and the lower bound in the presence of mismatches in IRS orientations. We also provide comparisons with the conventional ML estimator and the CRB in absence of orientation mismatches for quantifying the effects of mismatches. It is shown that orientation mismatches can result in significant degradation in localization accuracy at high signal-to-noise ratios.

Keywords: Visible light, luminous flux, LED degradation, intelligent reflecting surfaces, reconfigurable intelligent surfaces, estimation, localization.

ÖZET

DONANIM HATALARI ALTINDA GÖRÜNÜR IŞIK KONUMLANDIRMA

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Bu tezde, görünür ışık konumlandırma (GIK) alanındaki iki farklı araştırma problemini incelemekteyiz. Özel olarak, ışık yayan diyotlardaki (LED’lerdeki) ışık akısı bozulması hakkındaki bilgi eksikliğinin ve akıllı yansıtıcı yüzeylerin (AYY’lerin) elemanları için uyumsuz yönelimlerin GIK sistemlerinin performansı üzerindeki etkisini araştırmaktayız.

Tezin ilk bölümünde LED’lerin ışık akısı bozulması durumunda, görünür ışık sistemleri için alınan güç ölçümlerine dayalı konum kestirim problemi incelenmektedir. Alıcı bu bozulmanın farkında olmadığı ve konum kestirimini buna göre yaptığı ve, gerçek model ile varsayılan model arasında bir uyumsuzluk söz konusudur. Bu senaryo için, yanlış belirlenmiş Cramér-Rao sınırı (YBCRAS) ve uyumsuz maksimum olabilirlik (UMO) kestiricisi, bu model uyumsuzluğundan kaynaklanan performans kaybını ölçmek için türetilmektedir. Ayrıca, Cramér-Rao alt sınırı (CRAS) ve maksimum olabilirlik (MO) kestiricisi, alıcının LED’lerin bozulma formülünü bildiği ancak bu formüldeki bozulma oranı parametresini bilmediği durumlarda türetilmektedir. Ek olarak, bozulma formülü ve bozulma hızı parametreleri hakkında tam bilgi mevcut olduğunda, elde edilebilecek en iyi performansı belirlemek için CRAS ve MO kestiricisi elde edilmektedir. Bu üç senaryodaki kuramsal alt limitleri ve kestiricileri değerlendirerek, LED bozulma modeli ve bozulma hızı parametreleri hakkındaki bilgilerin konum kestirim performansı üzerindeki etkilerini ortaya çıkarmaktayız. Model uyumsuzluğunun, yüksek sinyal-gürültü oranlarında konumlandırma performansında önemli bir bozulmaya yol açabileceği ve bu durumun, konum ve bozulma oranı parametrelerinin birlikte kestirilmesiyle telafi edilebileceği gösterilmektedir.

Görüş hattı dışı (NLOS) senaryolarda görünür ışık sistemlerinde, genellikle ayarlanabilir yönelimlere sahip ayna dizileri biçiminde olan AYY’ler kullanılarak hassas konumlandırma gerçekleştirilebilmektedir. LED’lerden iletilen sinyaller AYY’lerden yansıtılıp bir alıcı tarafından toplandığında, LED’lerin ve AYY’lerin bilinen parametreleri kullanılarak

güç ölçümlerine dayalı olarak alıcının konumu kestirilebilmektedir. AYY elemanlarının (aynaların) yönelim vektörleri pratikte mükemmel şekilde ayarlanamayacağından, AYY elemanlarının istenen ve gerçek yönelimleri arasındaki uyumsuzlukların etkilerinin değerlendirilmesi önemlidir. Bu amaçla tezin ikinci bölümünde, AYY yönelimlerindeki uyumsuzlukların varlığında kestirim performansını ve alt sınırı belirlemek için YBCRAS ve UMO kestiricisi türetilmektedir. Uyumsuzlukların etkilerini ölçmek için yönelim uyumsuzlukları olmadığında geleneksel MO kestiricisi ve CRAS ile karşılaştırmalar da sunulmaktadır. Yönelim uyumsuzluklarının, yüksek sinyal-gürültü oranlarında konumlandırma hassasiyetinde önemli bir bozulmaya neden olabileceği gösterilmektedir.



Anahtar sözcükler: Görünür ışık, ışık akısı, LED bozulması, akıllı yansıtıcı yüzeyler, yeniden yapılandırılabilir akıllı yüzeyler, kestirim, konumlandırma.

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Chapter 1

Introduction

Accurate positioning is crucial in a wide range of indoor applications such as robot navigation, location-aware services, object tracking, and autonomous vehicles [1–6]. Despite extensive research spanning decades on radio frequency (RF) based indoor positioning systems, there remains a lack of a system that is affordable, accurate, and universally accessible. This inherent challenge in RF-based systems stems from multipath propagation complicating the determination of distance or direction related parameters reliably [6–8]. Visible light positioning (VLP), leveraging the widespread use of light-emitting diodes (LEDs) as indoor lighting sources, has showcased its potential for accurate indoor localization ([9–11] and references therein). Via VLP systems, accurate localization can be achieved through the effective utilization of existing LED lighting infrastructures, resulting in minimal deployment requirements and energy costs.

LED-based indoor VLP systems offer a wide array of diverse applications. One of such important applications is asset tracking in health facilities. Positioning services in a hospital can be utilized for monitoring the location of a wheelchair or a portable medical apparatus [6]. VLP systems can also be implemented within a factory setting to navigate robots and oversee inventory storage. In addition, VLP systems can be deployed to provide navigational services to people in airports, shopping centers and public spaces such as stadia, museums, and theaters [12].

This thesis addresses two primary research problems within VLP. The first pertains to VLP under luminous flux degradation of LEDs [13], examined extensively in Chapter 2. The

second concerns VLP using IRSs under mismatched orientations [14], explored in detail in Chapter 3. Through investigating these issues, the research aims to enhance the positioning accuracy performance of VLP systems. Subsequent sections encompass a comprehensive literature review and outline the main contributions of this thesis.

1.1 Visible Light Positioning under Luminous Flux Degradation of LEDs

1.1.1 Literature Review

In the literature, various position estimation algorithms have been devised, and thorough examinations of theoretical accuracy limits have been conducted for VLP systems. The developed algorithms are based on either a single parameter type such as received signal strength (RSS) [15–21], time of arrival (TOA) [22], angle of arrival (AOA) [23], and time-difference-of-arrival (TDOA) [24], or a combination of multiple parameters [25–27]. In practical VLP systems, the use of RSS-based positioning is widespread thanks to its cost-effective hardware implementation with high positioning accuracy [9]. Hence, it is commonly preferred over TOA-based techniques, which require synchronization and accurate time delay estimation [18, 19]. In [25], both RSS and AOA information is utilized in a visible light communication (VLC) system with multiple LED transmitters and a single receiver for three-dimensional (3D) localization and the derived Cramér-Rao lower bound (CRB) based on the RSS information encompasses 3D positioning across various deployment scenarios. RSS and AOA are also utilized in a 3D indoor positioning approach using a single LED array and multiple VLC receivers [26]. In [15] and [16], two-dimensional VLP utilizing RSS measurements is studied assuming a fixed receiver height. This is achieved through an aperture-based receiver, and the CRB is derived to evaluate the performance of positioning algorithms based on the considered settings. Also, in [17], the authors investigate the impact of non-line-of-sight (NLOS) signal components, signal-to-noise ratio (SNR), transmission distance, and the number of LED sources on performance of RSS-based VLP systems, and obtain the closed-form CRB expressions for both location and orientation estimation. In [22], a TOA-based CRB is derived to assess the performance of the designed maximum likelihood (ML) estimators employing direct and two-step positioning techniques in synchronous and asynchronous

VLP networks. In [20], simultaneous position and orientation estimation using multiple LED transmitters and multiple VLC receivers is investigated for RSS-based VLP systems. In [24], an indoor positioning system based on TDOA using visible light signals from LEDs is studied, where transmitting ID information alongside the pilot signals from the LEDs is not required. Additionally, [23] examines the CRB on the AOA based indoor 3D visible light positioning, which utilizes an aperture-based receiver alongside non-directional LEDs.

In addition to VLP algorithms that are based on visible light propagation models (e.g., [15–27]), there also exist data driven (learning) approaches to VLP such as [28–44]. In [28], the authors propose an indoor real-time 3D VLP system, employing fingerprinting and the extreme learning machine approach to attain precise positioning, enhanced resistance to interference, and reliable real-time performance. The VLP system initially splits the large indoor positioning environment into regular VLP kernels, thereby decreasing the size of the fingerprint database and leading to a significant exponential reduction in training time. In [30], the authors introduce deep artificial neural network (ANN) models to jointly estimate the 3D position and orientation of a randomly located VLC-receiver, considering random orientation and unknown emitting power. Leveraging RSS-based fingerprinting, the approach involves creating a dataset with instantaneous received SNR, coupled with corresponding 3D position and orientation angles. Also, the study in [29] introduces a technique called maximum received signal strength recognition (MRR) and a hybrid algorithm named weighted optimum k -nearest neighbors (WOKNN), which is a combination of optimum k -NN (OKNN) and weighted k -NN (WKNN). It is employed to estimate the 2D location of a VLC-receiver using RSS measurements. Although data driven VLP algorithms can provide position estimation in the absence of statistical models, they require collection of training data from the environment, which may be costly and inappropriate for certain applications. In this manuscript, we focus on a model based VLP approach.

Most of the studies in the VLP literature assume constant power outputs for LED transmitters. However, in practice, luminous flux of LEDs (hence, LED transmit powers) degrade gradually over time [45]. Although there exist a few studies such as [46–48] that take into account the impact of LED transmitter power uncertainty on the accuracy of a VLP system, there is no work in the literature that investigates the effects of luminous flux degradation of LEDs on the accuracy of VLP systems.

In Chapter 2, we investigate VLP by considering the luminous flux degradation of LEDs.

In particular, we quantify the deterioration in localization performance resulting from the mismatch between an ideal model with no decay in power outputs over time and a practical model with certain decay rates by conducting a misspecified Cramér-Rao bound (MCRB) analysis [49]. We also derive CRBs and position estimators in the presence of model knowledge related to luminous flux degradation of LEDs. Although an MCRB analysis is conducted for localization in [50], where the authors investigate reconfigurable intelligent surface (RIS) assisted near-field localization for a single-antenna user equipment and single-antenna base station in the presence of line-of-sight blockage with a practical RIS amplitude model, there exist no studies in the literature that derive the MCRB for VLP systems by considering the power decays of LED transmitters.

1.1.2 Contributions

In Chapter 2, the problem of VLP under luminous flux degradation of LEDs considering three distinct scenarios is investigated. The primary contributions and novel aspects of this chapter can be outlined as follows:

- Position estimation problems in visible light systems under luminous flux degradation of LEDs are formulated for the first time in the literature.
- A mismatched maximum likelihood (MML) estimator is derived for the scenario where the VLC receiver is unaware of the power degradation of the LED transmitters and the corresponding MCRB and lower bound (LB) expressions are derived and used as benchmarks to assess localization performance in the presence of model mismatch.
- ML estimators and corresponding CRBs are derived when the VLC receiver knows the power decay formulas for the LEDs by considering two different scenarios with known and unknown decay rate parameters.

1.2 Visible Light Positioning with Intelligent Reflecting Surfaces under Mismatched Orientations

1.2.1 Literature Review

More recently, the emergence of intelligent reflecting surface (IRS), also referred to as reconfigurable intelligent surface (RIS), has drawn significant attention due to its considerable advantages in widening wireless network coverage (specifically for RF systems), enhancing wireless communication rates, and reducing deployment and energy costs [51, 52]. An IRS is a planar surface made up of numerous low-cost reflecting elements, each capable of independently altering the amplitude and/or phase of an incident signal [53]. In [54], the authors propose an IRS-based method to improve spectrum and energy efficiency while reducing the implementation cost of wireless communication systems. Also, the authors of [55] investigate the joint design of the beamformer at the transmitter and the IRS phase shifts for an IRS-assisted wireless communication system, and demonstrate that IRSs hold significant potential for creating high-speed and energy-efficient communication networks.

There are also a number of studies in the literature that explore IRS-assisted VLC systems [56–69]. Conventionally, communication performance of VLC systems heavily depends on the presence of line-of-sight (LoS) paths. IRS, capable of re-configuring the wireless propagation channel, mitigates the LoS requirement in VLC systems. In [56], two intelligent reflection systems using programmable metasurfaces and mirrors to concentrate incident light onto the VLC receiver are proposed, and the detector plane irradiance expressions for both systems to evaluate their aiming and focusing performance are derived. The authors in [61] investigate a VLC system incorporating an IRS mirror array to address LoS blockage issues caused by the random orientation of the user device, self-blockage, and external obstructions. In [57], the authors employ optical IRSs to relax the LoS constraint in vehicle-to-vehicle (V2V) VLC systems and derive a closed-form expression for the number of IRS elements needed to achieve a desired energy or spectral efficiency. The authors in [58] derive an IRS-aided parallel multiple-input-multiple-output (MIMO) VLC channel model and demonstrate that incorporating IRS into such a system significantly enhances its asymptotic capacity. Also, in [60], the authors derive asymptotic capacity expressions for optical IRS-assisted MIMO VLC under various emission power constraints in the high-SNR regime and

propose two algorithms to maximize these capacities by jointly optimizing the alignment of optical IRS elements and the transmitted power. In [59], the authors investigate a downlink energy efficiency maximization problem in an IRS-VLC system by formulating a resource allocation problem based on achievable system sum rate versus total power consumption, under user data and transmit power constraints. In [67], the authors investigate a nonorthogonal multiple access (NOMA)-based multiple-input–single-output (MISO) VLC system enhanced by optical intelligent reflecting surfaces (OIRSs), analyzing both Line of Sight and OIRS-reflected channel gains, and optimizing active and passive beamforming to maximize the sum data rate using a block coordinate descent algorithm. The authors in [66] investigate physical layer security in IRS-aided VLC, transforming the secrecy rate maximization problem into an assignment problem, and proposing an iterative Kuhn-Munkres algorithm to optimize it, with simulations verifying improved secrecy rates and approximation performance. Additionally, the authors in [63] propose a novel indoor VLC system that is jointly assisted by a mirror array-based RIS in the channel and an liquid crystal (LC)-based RIS at the receiver, formulating and solving a rate maximization problem that optimizes the angles of the mirror array and the refractive index of the LC-based receiver.

While a significant amount of research is focused on IRS-assisted RF communications, there are also a number of studies in the literature that examine the utilization of IRSs in RF-based positioning systems [70–74]. In [71], the authors investigate the utilization of IRSs for user positioning and introduce an angle-domain method to acquire angle of arrival (AoA) data for the direct path between users and IRSs, subsequently employing AoA positioning at the access point (AP) for user localization. In [73], the authors investigate the target localization problem utilizing semi-passive IRS and apply the MUSIC algorithm to estimate the target direction. Also, in [72], a maximum likelihood (ML)-based localization method utilizing multiple single-receive RF IRSs, which employs beamspace orthogonal matching pursuit (OMP) for AoA estimation and least squares line intersection is presented. In [70], the authors investigate a downlink positioning problem aided by IRSs, analyze it through Fisher information and propose a two-step scheme to optimize IRSs to improve positioning performance. Additionally, in [74], an IRS-aided multi-user positioning protocol is proposed, and the optimization problem for positioning is formulated and solved using a configuration optimization algorithm.

To the best of our knowledge, there exist only two studies in the literature on IRS-assisted visible light positioning (VLP) [75, 76]. The authors in [75] propose an algorithm based on

sparse Bayesian learning for solving the localization problem in IRS-assisted indoor VLP, deriving the Fisher information matrix, position estimation bound (PEB), and RMSE expressions for NLOS scenarios. In [76], the authors formulate an RSS-based position estimation problem in a VLP system utilizing IRSs. An ML estimator is derived utilizing signals from both the LoS and IRS elements, along with the corresponding CRB, and an algorithm is proposed to optimally adjust the IRS elements to maximize the received power. However, this study assumes a perfect alignment between the desired orientations of IRS elements and their actual orientations, an ideal scenario rarely achieved in practical settings due to inaccuracies in gyroscope readings and noise.

In Chapter 3, we investigate the problem of IRS-assisted VLP, particularly focusing on a scenario where there is a mismatch between the actual orientations of the IRS elements and the desired orientations assumed by the VLC receiver. To this aim, we employ a misspecified Cramér-Rao bound (MCRB) analysis [49] to quantify the deterioration in positioning performance due to the mismatch between the true orientations of the IRS elements and the assumed orientations at the receiver.

1.2.2 Contributions

Chapter 3 analyzes the problem of VLP with IRSs under mismatched orientations. The main contributions and novel aspects of this chapter can be summarized as follows:

- The problem of IRS-assisted VLP under mismatched orientations is formulated for the first time in the literature.
- A mismatched maximum likelihood (MML) estimator is derived and the corresponding MCRB and lower bound (LB) expressions are obtained, serving as benchmarks for evaluating localization performance.
- A comparative analysis is performed between the MML estimator and the maximum likelihood (ML) estimator (and also between the MCRB and the CRB) to reveal the effects of orientation mismatches on localization performance.

1.3 Organization of the Thesis

The structure of this thesis is outlined as follows. In Chapter 2, we examine the problem of VLP under luminous flux degradation of LEDs considering 3 scenarios: model mismatch, model knowledge and full knowledge. In Chapter 3, we investigate the problem of VLP with IRSs under mismatched orientations, contrasting these findings with results obtained under ideal conditions (absence of mismatched orientations) in our numerical results. Chapter 4 presents the concluding remarks and future research work.



Chapter 2

Visible Light Positioning under Luminous Flux Degradation of LEDs

In this chapter, we investigate the received power-based position estimation problem in visible light communication systems, considering the luminous flux degradation of LEDs. We begin by detailing the system model and problem formulation, specifying the three scenarios we consider. Then, we explore the scenario where the VLC receiver is unaware of the power degradation of the LEDs, deriving the MCRB, the LB, and the MML estimator. Next, we consider the scenario where the VLC receiver knows the degradation formula but does not know the decay rate parameter. For this scenario, we derive the CRB and ML estimator for the joint estimation of position and decay rate parameter. Subsequently, we focus on the scenario where there is full knowledge of both the degradation model and the decay rate parameter. We derive the CRB and the ML estimator to establish a performance benchmark. Finally, numerical examples are presented and the performance of the VLP system under these three scenarios are compared, and concluding remarks are made. We demonstrate that at high SNR, best performance is achieved when the VLC receiver possesses full knowledge. Conversely, the worst performance occurs under conditions of model mismatch. However, the performance degradation observed at high SNRs due to this mismatch can be significantly mitigated by simultaneously estimating both the position and the decay rate parameter.

2.1 System Model and Problem Description

We investigate a VLP system with N_L LED transmitters at known locations represented by $\mathbf{l}_i$ for $i \in \{1, \dots, N_L\}$ and a VLC receiver at an unknown location $\mathbf{x}$. The VLC receiver is to estimate its location based on signals coming from the LED transmitters. For this purpose, it collects power measurements from the LED transmitters, which are given by [77]

$$P_{R,i} = R_p P_{T,i} h_i(\mathbf{x}) + \eta_i \quad (2.1)$$

for $i = 1, \dots, N_L$. In (2.1), $P_{R,i}$ is the received power at the VLC receiver due to the signal from the i th LED transmitter, R_p represents the responsivity of the photo-detector (PD) at the VLC receiver, $P_{T,i}$ is the transmit power of the i th LED transmitter, $h_i(\mathbf{x})$ denotes the channel coefficient between the i th LED transmitter and the VLC receiver, and η_i is zero-mean Gaussian noise with variance σ_i^2 , which is independent of η_j for all $j \neq i$ [77]. As noted from (2.1), it is assumed that the signals coming from distinct LED transmitters are processed separately and their power levels are measured individually at the VLC receiver. This is based on the use of a certain type of multiple access protocol, such as time-division or frequency-division multiple access [78–80]. In addition, the set of possible locations for the VLC receiver is denoted by $\mathcal{L}$ (e.g., all possible locations in a given room), that is, $\mathbf{x} \in \mathcal{L}$, and it is assumed that no prior statistical information is available about $\mathbf{x}$ [81].

As in [9, 25, 82], a line-of-sight scenario is considered between each LED transmitter and the VLC receiver, and the channel coefficients in (2.1) are expressed as

$$h_i(\mathbf{x}) = \frac{(m_i + 1) A_R [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{m_i} (\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R}{2\pi \|\mathbf{x} - \mathbf{l}_i\|^{m_i+3}} \quad (2.2)$$

where m_i is the Lambertian order for the i th LED transmitter, A_R is the area of the PD at the VLC receiver, and $\mathbf{n}_R$ and $\mathbf{n}_T^i$ represent the orientation vectors of the VLC receiver and the i th LED transmitter, respectively [83]. The VLC receiver is assumed to have the knowledge of the parameters A_R , R_p , $\mathbf{n}_R$, m_i , $\mathbf{l}_i$, and $\mathbf{n}_T^i$, and σ_i^2 [22, 25]. For instance, the orientation of the VLC receiver ($\mathbf{n}_R$) can be determined by a gyroscope, as demonstrated in various experimental studies [19, 84–86]. Additionally, the LED parameters (m_i , $\mathbf{l}_i$, and $\mathbf{n}_T^i$) can be reported to the VLC receiver via visible light communications [22].

In practice, luminous flux of an LED gradually depreciates over time, which is commonly modeled by an exponential decay formula as in [45, Sec. 5.2.4]. As the luminous flux is

directly proportional to the transmitted optical power from an LED through luminous efficacy [87], the transmitted power of the i th LED in (2.1) can be modeled as

$$P_{T,i} = P_{T,i}^0 e^{-\alpha_i t} \quad (2.3)$$

for $i \in \{1, \dots, N_L\}$, where $P_{T,i}^0$ represents the initial power level of the i th LED, t is the operating time in hours, and $\alpha_i > 0$ is the decay rate for the i th LED [45, Sec. 5.2.4]. It is assumed that the initial power levels of the LEDs and the operating times are known. (The operating times are taken to be the same without loss of generality.)

Our aim is to analyze the effects of luminous flux degradation of LEDs on visible light positioning based on power measurements. To this aim, we consider three different scenarios:

Scenario 1: The VLC receiver is unaware of the degradation in LEDs, and assumes that the transmit powers are given by $P_{T,i}^0$ for $i = 1, \dots, N_L$. (This is equivalent to assuming that there is no power decay in the LEDs; i.e., $\alpha_i = 0$ in (2.3) for all $i \in \{1, \dots, N_L\}$.)

Scenario 2: The VLC receiver knows the depreciation formula in (2.3); however, it does not know the decay rates, α_i 's.

Scenario 3: The VLC receiver knows the depreciation formula in (2.3) and the decay rates, α_i 's.

In the following sections, we derive theoretical limits on position estimation accuracy and obtain estimators that asymptotically achieve those theoretical limits in these three scenarios. By comparing the performance of the estimators and the theoretical limits, the effects of degradation in LEDs can be quantified for various scenarios.

2.2 Scenario 1: Model Mismatch

In Scenario 1, the VLC receiver assumes that $P_{T,i} = P_{T,i}^0$ for $i = 1, \dots, N_L$, that is, it is unaware of the degradation in the LEDs. Based on this assumption and the N_L measurements modeled by (2.1), the likelihood function for the position of the VLC receiver can be stated

as

$$\tilde{p}(\mathbf{P}_R | \mathbf{x}) = \left(\prod_{i=1}^{N_L} \frac{1}{\sqrt{2\pi\sigma_i^2}} \right) e^{-\sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 R_p h_i(\mathbf{x}))^2}{2\sigma_i^2}} \quad (2.4)$$

where $\mathbf{P}_R$ represents a vector consisting of the power measurements; i.e., $\mathbf{P}_R = [P_{R,1} \cdots P_{R,N_L}]^T$. The expression in (2.4) can be referred to as the *misspecified parametric PDF for $\mathbf{x}$* since the transmit powers are not specified accurately.

For comparison purposes, we can specify the true model, which takes into account the luminous flux degradation of the LEDs with time. In this case, the VLC receiver would perfectly know the depreciation formula and the decay rates of the LEDs. Hence, the *true PDF* can be expressed as

$$p(\mathbf{P}_R) = \left(\prod_{i=1}^{N_L} \frac{1}{\sqrt{2\pi\sigma_i^2}} \right) e^{-\sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\bar{\mathbf{x}}))^2}{2\sigma_i^2}} \quad (2.5)$$

where $\bar{\mathbf{x}}$ denotes the true position vector of the VLC receiver.

In order to reveal the degradation caused by the use of the misspecified model in (2.4) instead of the true model in (2.5), we first derive a position estimator and a theoretical limit on estimation accuracy in the presence of model mismatch.

2.2.1 MML Estimator

The MML estimator corresponds to the parameter that maximizes the misspecified parametric PDF [49]. Therefore, in Scenario 1, the MML estimator can be expressed as

$$\hat{\mathbf{x}}_{\text{MML}}(\mathbf{P}_R) = \arg \max_{\mathbf{x} \in \mathcal{L}} \tilde{p}(\mathbf{P}_R | \mathbf{x}). \quad (2.6)$$

where $\mathcal{L}$ denotes the set of possible locations of the VLC receiver. From (2.4) and (2.6), the MML estimator based on the received signal power measurements $\mathbf{P}_R$ in (2.1) can be expressed as

$$\hat{\mathbf{x}}_{\text{MML}}(\mathbf{P}_R) = \arg \min_{\mathbf{x} \in \mathcal{L}} \sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 R_p h_i(\mathbf{x}))^2}{2\sigma_i^2}. \quad (2.7)$$

It can be shown that under suitable regularity conditions, the MML estimator is asymptotically unbiased and its error covariance matrix is equal to the MCRB, which is derived in

the next section [49]. It is noted that (2.7) constitutes a nonlinear least squares problem. It is straightforward to derive the gradient of the objective function in (2.7), leveraging the closed-form expression for the first-order derivative of the channel coefficient $h_i(\mathbf{x})$, as presented in **Appendix A**. Algorithms such as the gradient descent method, Gauss-Newton method, and Levenberg-Marquardt algorithm can be used to solve it numerically [88]. In addition, multiple gradient descents with a few initial points can be performed not to get stuck with local minima.

2.2.2 MCRB

To derive the theoretical limit on position estimation accuracy in Scenario 1, we first introduce the pseudo-true parameter [49], which is the parameter that minimizes the Kullback-Leibler (KL) divergence between the true PDF $p(\mathbf{P}_R)$ and the misspecified parametric PDF $\tilde{p}(\mathbf{P}_R | \mathbf{x})$. The pseudo-true parameter can be obtained as follows:

$$\mathbf{x}_0 = \arg \min_{\mathbf{x} \in \mathbb{R}^3} D(p(\mathbf{P}_R) \| \tilde{p}(\mathbf{P}_R | \mathbf{x})) \quad (2.8)$$

where $D(p(\mathbf{P}_R) \| \tilde{p}(\mathbf{P}_R | \mathbf{x}))$ denotes the KL divergence between the PDFs $p(\mathbf{P}_R)$ and $\tilde{p}(\mathbf{P}_R | \mathbf{x})$. From the definition of the KL divergence, the objective function in (2.8) can be stated as

$$\begin{aligned} D(p(\mathbf{P}_R) \| \tilde{p}(\mathbf{P}_R | \mathbf{x})) &= \int \log \left(\frac{p(\mathbf{P}_R)}{\tilde{p}(\mathbf{P}_R | \mathbf{x})} \right) p(\mathbf{P}_R) d\mathbf{P}_R \\ &= \mathbb{E}_p \left\{ \log \left(\frac{p(\mathbf{P}_R)}{\tilde{p}(\mathbf{P}_R | \mathbf{x})} \right) \right\}. \end{aligned} \quad (2.9)$$

Let $\hat{\mathbf{x}}(\mathbf{P}_R)$ denote a misspecified-unbiased (MS-unbiased) estimator of the true position, $\bar{\mathbf{x}}$. That is, the expected value of the estimator $\hat{\mathbf{x}}(\mathbf{P}_R)$ under the true model is equal to $\mathbf{x}_0$ [50]. The MCRB is a lower bound on the covariance matrix of any MS-unbiased estimator $\hat{\mathbf{x}}(\mathbf{P}_R)$ of $\bar{\mathbf{x}}$ and is given by [49], [89]

$$\mathbb{E}_p \{ (\hat{\mathbf{x}}(\mathbf{P}_R) - \mathbf{x}_0)(\hat{\mathbf{x}}(\mathbf{P}_R) - \mathbf{x}_0)^T \} \succeq \text{MCRB}(\mathbf{x}_0) \quad (2.10)$$

where $\mathbb{E}_p \{ \cdot \}$ denotes the expectation under the true model $p(\mathbf{P}_R)$, and

$$\text{MCRB}(\mathbf{x}_0) \triangleq \mathbf{A}_{\mathbf{x}_0}^{-1} \mathbf{B}_{\mathbf{x}_0} \mathbf{A}_{\mathbf{x}_0}^{-1}. \quad (2.11)$$

The (m, n) th elements of the matrices $\mathbf{A}_{\mathbf{x}_0}$ and $\mathbf{B}_{\mathbf{x}_0}$ in (2.11) are calculated as

$$[\mathbf{A}_{\mathbf{x}_0}]_{mn} = \mathbb{E}_p \left\{ \frac{\partial^2 \log \tilde{p}(\mathbf{P}_R | \mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} \bigg|_{\mathbf{x}=\mathbf{x}_0} \right\}, \quad (2.12)$$

$$[\mathbf{B}_{\mathbf{x}_0}]_{mn} = \mathbb{E}_p \left\{ \frac{\partial \log \tilde{p}(\mathbf{P}_R | \mathbf{x})}{\partial \mathbf{x}(m)} \frac{\partial \log \tilde{p}(\mathbf{P}_R | \mathbf{x})}{\partial \mathbf{x}(n)} \bigg|_{\mathbf{x}=\mathbf{x}_0} \right\} \quad (2.13)$$

for $1 \leq m, n \leq 3$ with $\mathbf{x}(n)$ denoting the n th element of $\mathbf{x}$. As we are typically not interested in the value of the pseudo-true parameter, the MCRB is employed to establish a lower bound (LB) for any MS-unbiased estimator with respect to the true parameter value [49, 50]. Namely, the error covariance matrix for any MS-unbiased estimator $\hat{\mathbf{x}}(\mathbf{P}_R)$ is lower bounded as follows:

$$\mathbb{E}_p \{ (\hat{\mathbf{x}}(\mathbf{P}_R) - \bar{\mathbf{x}})(\hat{\mathbf{x}}(\mathbf{P}_R) - \bar{\mathbf{x}})^T \} \succeq \text{LB}(\mathbf{x}_0) \quad (2.14)$$

where

$$\text{LB}(\mathbf{x}_0) \triangleq \text{MCRB}(\mathbf{x}_0) + (\bar{\mathbf{x}} - \mathbf{x}_0)(\bar{\mathbf{x}} - \mathbf{x}_0)^T. \quad (2.15)$$

In (2.15), the last term is a bias term and it is independent of the signal-to-noise ratio (SNR). As a result, as the SNR goes to infinity, the MCRB goes to zero and the bias term becomes tight with the error covariance matrix of the MS-unbiased estimator [50].

To derive the MCRB, hence, the LB in (2.15), the following derivations are performed:

2.2.2.1 Determination of Pseudo-True Parameter

To derive the MCRB related to the estimation of the VLC receiver position under the model mismatch, we first need to calculate the pseudo-true parameter $\mathbf{x}_0$ in (2.8). Specifically, we should find the value of $\mathbf{x}$ that minimizes the KL divergence between the PDFs $p(\mathbf{P}_R)$ in (2.4) and $\tilde{p}(\mathbf{P}_R | \mathbf{x})$ in (2.5). Based on (2.4) and (2.5), the expression in (2.9) can be calculated as follows:

$$\begin{aligned} D(p(\mathbf{P}_R) || \tilde{p}(\mathbf{P}_R | \mathbf{x})) &= \mathbb{E}_p \left\{ \sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 R_p h_i(\mathbf{x}))^2}{2\sigma_i^2} - \sum_{j=1}^{N_L} \frac{(P_{R,j} - P_{T,j}^0 e^{-\alpha_j t} R_p h_j(\bar{\mathbf{x}}))^2}{2\sigma_j^2} \right\} \\ &= \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\bar{\mathbf{x}}) - P_{T,i}^0 R_p h_i(\mathbf{x}))^2}{2\sigma_i^2}. \end{aligned} \quad (2.16)$$

From (2.8) and (2.16), the value of $\mathbf{x}$ that minimizes the KL divergence can be obtained as follows:

$$\mathbf{x}_0 = \arg \min_{\mathbf{x} \in \mathbb{R}^3} \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\bar{\mathbf{x}}) - P_{T,i}^0 R_p h_i(\mathbf{x}))^2}{2\sigma_i^2} \quad (2.17)$$

2.2.2.2 Derivation of MCRB

Once the pseudo-true parameter is obtained, we compute the matrices $\mathbf{A}_{\mathbf{x}_0}$ in (2.12) and $\mathbf{B}_{\mathbf{x}_0}$ in (2.13) for calculating the MCRB from (2.11). Based on the expressions in (2.4) and (2.5), the second-order derivatives in (2.12) can be obtained as follows:

$$\begin{aligned} \frac{\partial^2 \log \tilde{p}(\mathbf{P}_R | \mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} &= \sum_{i=1}^{N_L} \frac{(P_{R,i} P_{T,i}^0 R_p)}{\sigma_i^2} \frac{\partial^2 h_i(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} \\ &\quad - \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 R_p)^2}{\sigma_i^2} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(n)} \\ &\quad - \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 R_p)^2}{\sigma_i^2} h_i(\mathbf{x}) \frac{\partial^2 h_i(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)}. \end{aligned} \quad (2.18)$$

Then, by evaluating the expression in (2.18) at $\mathbf{x} = \mathbf{x}_0$, the elements of matrix $\mathbf{A}_{\mathbf{x}_0}$ in (2.12) can be derived as

$$\begin{aligned} [\mathbf{A}_{\mathbf{x}_0}]_{mn} &= \sum_{i=1}^{N_L} \frac{\mathbb{E}_p\{P_{R,i}\} P_{T,i}^0 R_p}{\sigma_i^2} \frac{\partial^2 h_i(\mathbf{x}_0)}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} \\ &\quad - \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 R_p)^2}{\sigma_i^2} \frac{\partial h_i(\mathbf{x}_0)}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x}_0)}{\partial \mathbf{x}(n)} \\ &\quad - \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 R_p)^2}{\sigma_i^2} h_i(\mathbf{x}_0) \frac{\partial^2 h_i(\mathbf{x}_0)}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} \end{aligned} \quad (2.19)$$

where $\mathbb{E}_p\{P_{R,i}\} = P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\bar{\mathbf{x}})$.

Similarly, via (2.4) and (2.5), the first-order derivatives in (2.13) are calculated as

$$\frac{\partial \log \tilde{p}(\mathbf{P}_R | \mathbf{x})}{\partial \mathbf{x}(m)} = \sum_{i=1}^{N_L} P_{T,i}^0 R_p \frac{(P_{R,i} - P_{T,i}^0 R_p h_i(\mathbf{x}))}{\sigma_i^2} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(m)}. \quad (2.20)$$

Then, based on (2.20), the elements of matrix $\mathbf{B}_{\mathbf{x}_0}$ in (2.13) can be obtained as follows:

$$\begin{aligned}
[\mathbf{B}_{\mathbf{x}_0}]_{mn} &= \mathbb{E}_p \left\{ \sum_{i=1}^{N_L} \left(P_{T,i}^0 R_p \frac{(P_{R,i} - P_{T,i}^0 R_p h_i(\mathbf{x}_0))}{\sigma_i^2} \frac{\partial h_i(\mathbf{x}_0)}{\partial \mathbf{x}(m)} \right) \right. \\
&\quad \times \left. \sum_{j=1}^{N_L} \left(P_{T,j}^0 R_p \frac{(P_{R,j} - P_{T,j}^0 R_p h_j(\mathbf{x}_0))}{\sigma_j^2} \frac{\partial h_j(\mathbf{x}_0)}{\partial \mathbf{x}(n)} \right) \right\} \\
&= \sum_{i=1}^{N_L} \left(\left(\frac{P_{T,i}^0 R_p}{\sigma_i^2} \right)^2 \mathbb{E}_p \{ (P_{R,i} - P_{T,i}^0 R_p h_i(\mathbf{x}_0))^2 \} \frac{\partial h_i(\mathbf{x}_0)}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x}_0)}{\partial \mathbf{x}(n)} \right) \\
&\quad + \sum_{i=1}^{N_L} \sum_{j=1, j \neq i}^{N_L} \left(\frac{P_{T,i}^0 P_{T,j}^0 R_p^2}{\sigma_i^2 \sigma_j^2} (P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\bar{\mathbf{x}}) - P_{T,i}^0 R_p h_i(\mathbf{x}_0)) \right. \\
&\quad \times \left. (P_{T,j}^0 e^{-\alpha_j t} R_p h_j(\bar{\mathbf{x}}) - P_{T,j}^0 R_p h_j(\mathbf{x}_0)) \frac{\partial h_i(\mathbf{x}_0)}{\partial \mathbf{x}(m)} \frac{\partial h_j(\mathbf{x}_0)}{\partial \mathbf{x}(n)} \right) \quad (2.21)
\end{aligned}$$

where $\mathbb{E}_p \{ (P_{R,i} - P_{T,i}^0 R_p h_i(\mathbf{x}_0))^2 \} = \sigma_i^2 + (P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\bar{\mathbf{x}}) - P_{T,i}^0 R_p h_i(\mathbf{x}_0))^2$.

The first and second order derivatives of the channel coefficient $h_i(\cdot)$ appearing in (2.19) and (2.21) can be derived explicitly in terms of the system parameters, as presented in **Appendix A**. Overall, based on (2.17), (2.19), and (2.21), the MCRB in (2.11) can be calculated, which also leads to the derivation of the LB in (2.15). Via this theoretical limit, we can evaluate the effects of omitting the power decays of LEDs.

2.3 Scenario 2: Model Knowledge

In Scenario 2, the VLC receiver knows the formula in (2.3) but it does not know the decay rates, i.e., the α_i parameters. Typically, the same types of LEDs are employed in a given network. Hence, the decay rates can be taken to be the same in practical cases; i.e., $\alpha_i = \alpha$ for $i = 1, \dots, N_L$. (If all the decay rates were different and unknown, accurate localization might not be possible as there would exist $N_L + 3$ unknown parameters with no specific relations and N_L measurements. Specifically, all the power measurements would be scaled by unknown and different factors (the $e^{-\alpha_i t}$ terms in (2.3)), making localization an ill-posed estimation problem.¹) In this scenario, the likelihood function for the unknown position

¹In such a case, one could simply ignore the power decays in the LEDs and assume that $P_{T,i} = P_{T,i}^0$ for $i = 1, \dots, N_L$, as in Scenario 1.

vector $\mathbf{x}$ and the common decay parameter α can be obtained from (2.1) and (2.3) as

$$p(\mathbf{P}_R | \mathbf{x}, \alpha) = \left(\prod_{i=1}^{N_L} \frac{1}{\sqrt{2\pi\sigma_i^2}} \right) e^{-\sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 e^{-\alpha t} R_p h_i(\mathbf{x}))^2}{2\sigma_i^2}} \quad (2.22)$$

In order to evaluate the effects of the unknown decay rate of transmit powers on localization accuracy, we aim to derive the ML estimator and the theoretical limit on position estimation errors, namely, the CRB, in Scenario 2.

The ML estimator for the position of the VLC receiver in Scenario 2 can be stated as [90]

$$(\hat{\mathbf{x}}, \hat{\alpha}) = \arg \max_{\mathbf{x} \in \mathcal{L}, \alpha > 0} p(\mathbf{P}_R | \mathbf{x}, \alpha) \quad (2.23)$$

where $\mathcal{L}$ denotes the set of possible locations of the VLC receiver. Based on (2.22), the ML estimator in (2.23) can be derived, after some manipulation, as

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathcal{L}} \sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 e^{-\hat{\alpha}(\mathbf{x})t} R_p h_i(\mathbf{x}))^2}{2\sigma_i^2} \quad (2.24)$$

where $\hat{\alpha}(\mathbf{x})$ is given by

$$\hat{\alpha}(\mathbf{x}) = \begin{cases} -\frac{1}{t} \log g(\mathbf{x}), & \text{if } 0 < g(\mathbf{x}) < 1 \\ 0, & \text{otherwise} \end{cases} \quad (2.25)$$

with $g(\mathbf{x})$ being defined as

$$g(\mathbf{x}) \triangleq \frac{\sum_{i=1}^{N_L} \frac{P_{R,i} P_{T,i}^0 h_i(\mathbf{x})}{\sigma_i^2}}{R_p \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 h_i(\mathbf{x}))^2}{\sigma_i^2}}. \quad (2.26)$$

It is noted that the four dimensional search in (2.23) can be reduced to three dimensions in (2.24) based on the expression in (2.25). Hence, the ML estimator in Scenario 2 and the MML estimator in Scenario 1 (see (2.7)) have similar levels of complexity.

To derive the CRB, we first define the vector of unknown parameters Θ , which consists of the position of the VLC receiver position, $\mathbf{x}$, and the decay rate parameter, α , namely, $\Theta \triangleq [\mathbf{x}^T \alpha]^T = [\mathbf{x}(1) \mathbf{x}(2) \mathbf{x}(3) \alpha]^T$. Then, by denoting the likelihood function in (2.22) as $p(\mathbf{P}_R | \Theta)$, the Fisher information matrix (FIM) can be expressed as [90]

$$\mathbf{I}(\Theta) = \mathbb{E} \left\{ (\nabla_{\Theta} \log p(\mathbf{P}_R | \Theta)) (\nabla_{\Theta} \log p(\mathbf{P}_R | \Theta))^T \right\} \quad (2.27)$$

where ∇_{Θ} denotes the gradient operator, which yields a four-dimensional column vector in this scenario. Based on the FIM, the CRB on the position estimation error can be stated as follows [90]:

$$\mathbb{E} \{ \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \} \geq \text{trace} \{ [\mathbf{I}(\Theta)^{-1}]_{1:3,1:3} \} \triangleq \text{CRB}_{\mathbf{x}} \quad (2.28)$$

where $\hat{\mathbf{x}}$ denotes any unbiased estimator of $\mathbf{x}$ in Scenario 2, and $[\mathbf{I}(\Theta)^{-1}]_{1:3,1:3}$ corresponds to the upper 3×3 block of the inverse FIM. After some manipulation, the elements of the FIM in (2.27) can be computed from (2.22) as

$$[\mathbf{I}(\Theta)]_{m,n} = R_P^2 \sum_{i=1}^{N_L} \left(\frac{(P_{T,i}^0)^2}{\sigma_i^2} \frac{\partial(e^{-\alpha t} h_i(\mathbf{x}))}{\partial \Theta(m)} \frac{\partial(e^{-\alpha t} h_i(\mathbf{x}))}{\partial \Theta(n)} \right) \quad (2.29)$$

for $m, n \in \{1, 2, 3, 4\}$, where $\Theta(n)$ denotes the n th element of Θ . In particular, $[\mathbf{I}(\Theta)]_{m,n}$ can be derived explicitly as follows:

$$[\mathbf{I}(\Theta)]_{m,n} = R_P^2 \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 e^{-\alpha t})^2}{\sigma_i^2} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(n)} \quad \text{for } m, n \in \{1, 2, 3\}, \quad (2.30)$$

$$[\mathbf{I}(\Theta)]_{4,n} = -t R_P^2 \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 e^{-\alpha t})^2 h_i(\mathbf{x})}{\sigma_i^2} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(n)} \quad \text{for } n \in \{1, 2, 3\}, \quad (2.31)$$

$$[\mathbf{I}(\Theta)]_{m,4} = -t R_P^2 \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 e^{-\alpha t})^2 h_i(\mathbf{x})}{\sigma_i^2} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(m)} \quad \text{for } m \in \{1, 2, 3\}, \quad (2.32)$$

$$[\mathbf{I}(\Theta)]_{4,4} = (t R_P e^{-\alpha t})^2 \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 h_i(\mathbf{x}))^2}{\sigma_i^2}. \quad (2.33)$$

In (2.30)–(2.33), the partial derivatives of the channel coefficient $h_i(\mathbf{x})$ can be evaluated based on the expressions in **Appendix A**. Consequently, the elements of the FIM, hence, the CRB in (2.28), can be calculated, which specifies the estimation accuracy limit when the VLC receiver is aware of the power decay model but does not know the decay rate.²

²Although the sign of the decay rate parameter ($\alpha > 0$) is taken into account in the ML estimator in Scenario 2, the corresponding CRB is calculated for a generic α . However, at high SNRs, the CRB still provides a valid bound on the performance of the ML estimator in Scenario 2, as investigated in Section 2.5.

2.4 Scenario 3: Full Knowledge

In Scenario 3, the VLC receiver knows the formula in (2.3) and the decay rate parameters, α_i 's. In other words, Scenario 3 corresponds to the scenario with perfect knowledge and provides a performance bound for the other scenarios. Based on (2.1) and (2.3), the likelihood function in Scenario 3 is given by

$$p(\mathbf{P}_R | \mathbf{x}) = \left(\prod_{i=1}^{N_L} \frac{1}{\sqrt{2\pi\sigma_i^2}} \right) e^{-\sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\mathbf{x}))^2}{2\sigma_i^2}}. \quad (2.34)$$

Accordingly, the ML estimator is expressed as

$$\hat{\mathbf{x}} = \arg \max_{\mathbf{x} \in \mathcal{L}} p(\mathbf{P}_R | \mathbf{x}) \quad (2.35)$$

which can be specified, based on (2.34), as follows:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathcal{L}} \sum_{i=1}^{N_L} \frac{(P_{R,i} - P_{T,i}^0 e^{-\alpha_i t} R_p h_i(\mathbf{x}))^2}{2\sigma_i^2}. \quad (2.36)$$

Similar to the ML estimator in Scenario 2 (see (2.24) and (2.25)), the ML estimator in (2.36) also requires a three-dimensional search. However, it does not need to estimate the decay rate during the search (cf. (2.25)) as it is assumed to be known in Scenario 3.

To evaluate the CRB in Scenario 3, we first calculate the FIM, which is defined as [90]

$$\mathbf{I}(\mathbf{x}) = \mathbb{E} \left\{ (\nabla_{\mathbf{x}} \log p(\mathbf{P}_R | \mathbf{x})) (\nabla_{\mathbf{x}} \log p(\mathbf{P}_R | \mathbf{x}))^T \right\} \quad (2.37)$$

with $\nabla_{\mathbf{x}}$ denoting the gradient operator with respect to $\mathbf{x}$. Based on the FIM, the CRB on the position estimation error is specified as [90]

$$\mathbb{E} \{ \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \} \geq \text{trace} \{ \mathbf{I}(\mathbf{x})^{-1} \} \triangleq \text{CRB} \quad (2.38)$$

where $\hat{\mathbf{x}}$ denotes an unbiased estimate of $\mathbf{x}$ in Scenario 3. After some manipulation, the elements of the FIM in (2.37) can be derived from (2.34) as

$$[\mathbf{I}(\mathbf{x})]_{m,n} = R_p^2 \sum_{i=1}^{N_L} \frac{(P_{T,i}^0 e^{-\alpha_i t})^2}{\sigma_i^2} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(n)} \quad (2.39)$$

for $m, n \in \{1, 2, 3\}$. The FIM can be evaluated via (2.39) and the partial derivative expressions in **Appendix A**, leading to the evaluation of the CRB in (2.38).

Remark: Based on the MCRB and the LB expressions in Scenario 1 and the CRB expressions in Scenario 2 and Scenario 3, the effects of omitting the power decay of LEDs, considering the decay model, and knowing the decay rate parameters can be evaluated in various situations. Since these theoretical limits are achieved by the MML and the ML estimators at high SNRs, they can be used to assess the localization performance and to design system parameters in high SNR conditions.

In Fig. 2.1, a flowchart of the proposed algorithms is presented considering the three scenarios.

2.5 Simulation Results

In this section, we conduct simulations to investigate the performance of the proposed approaches and to evaluate the theoretical limits. Similar to [81], we consider a room with dimensions of $4 \times 4 \times 3$ meters (width, depth, and height, respectively) and 9 LEDs (i.e., $N_L = 9$) placed at the following coordinates in meters: $\{(-1, 1, 3), (0, 1, 3), (1, 1, 3), (-1, 0, 3), (0, 0, 3), (1, 0, 3), (-1, -1, 3), (0, -1, 3), (1, -1, 3)\}$. These LED positions are selected to ensure a symmetrical coverage of the room, with the center of the room floor being represented by the coordinates $(0, 0, 0)$. The orientations of the LEDs are defined as $\mathbf{n}_T^i = [0, 0, -1]^T$ for $i = 1, \dots, N_L$, indicating that they all point downwards. Also, the Lambertian orders of the LEDs, m_i 's, are set to 1. The responsivity of the PD is set to $R_p = 1$, and the area of the PD is taken as $A_R = 1 \text{ cm}^2$. The orientation of the receiver is defined as $\mathbf{n}_R = [0, 0, 1]^T$, indicating that it points upwards. Furthermore, we assume that the noise variances are equal, denoted as $\sigma_i^2 = \sigma^2$ for $i = 1, \dots, N_L$.³

To evaluate the performance of the MML estimator under Scenario 1 and the ML estimators under Scenario 2 and Scenario 3, the VLC receiver is placed at $\mathbf{x} = (0.5, 0.5, 0.85)$ meters, the initial power levels of the LEDs are set to be identical as $P_{T,i}^0 = 10 \text{ W}$, and the decay rates of the LEDs are assumed to be the same and taken as 10^{-5} based on the decay rate calculations in [45, Sec. 7]. In Fig. 2.2, the RMSE performance of the MML estimator in Scenario 1 and the corresponding MCRB and LB are plotted versus $10 \log_{10}(1/\sigma^2)$ for two different time intervals (10000 and 50000 hours). It can be observed from the figure

³The σ^2 values considered in the simulations are compatible with the practical values reported in [91].

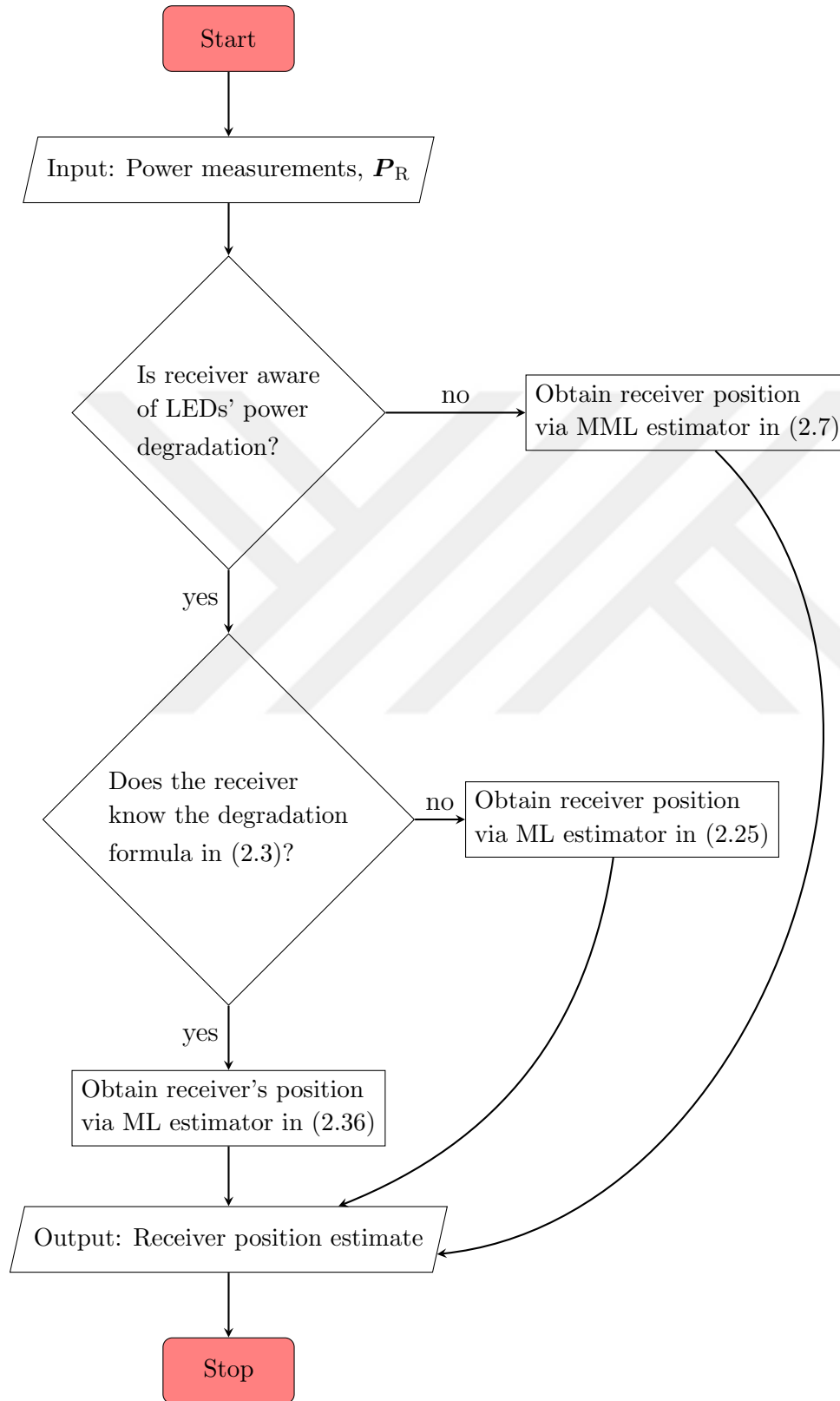


Figure 2.1: A flow chart of the algorithms based on the three scenarios.

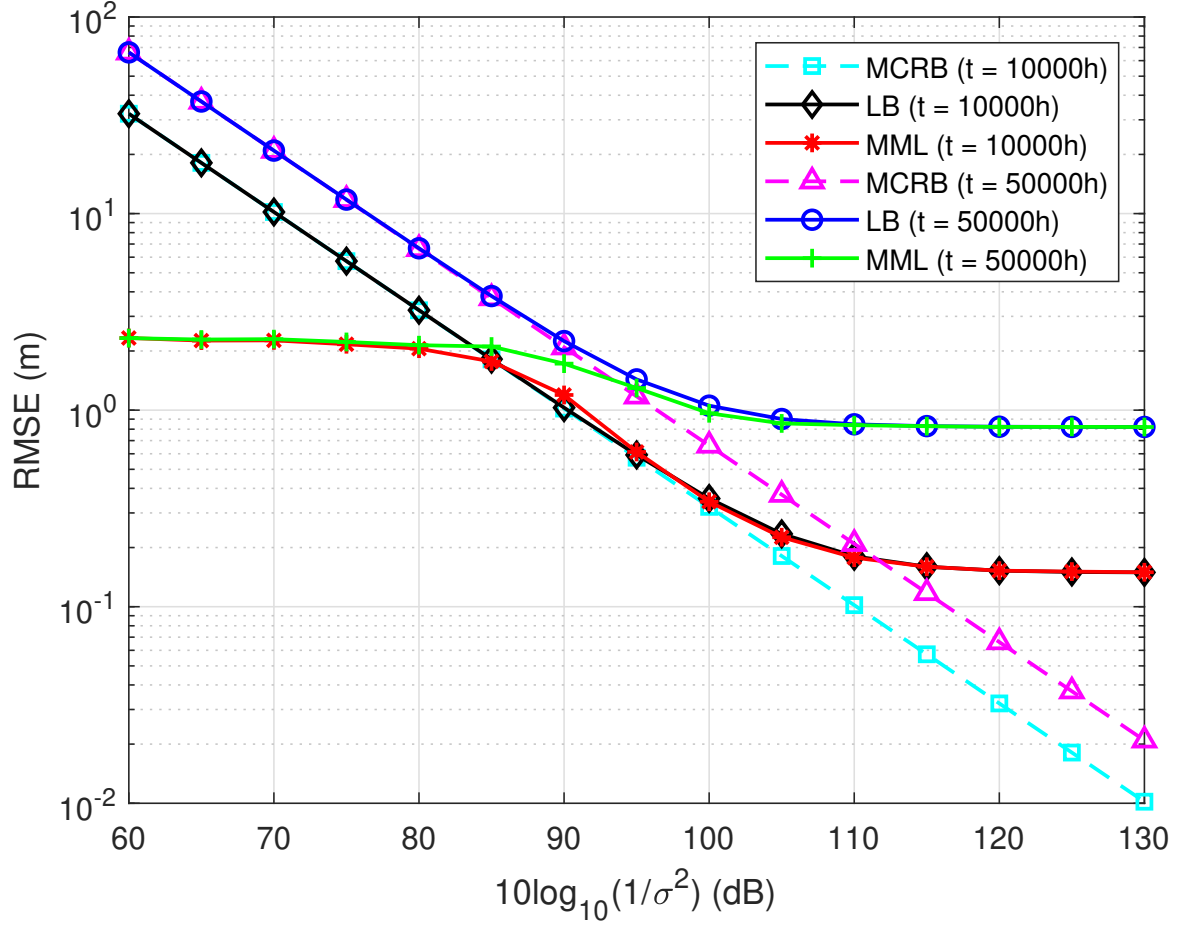


Figure 2.2: Performance of the MML estimator and the corresponding MCRB and LB versus $10 \log_{10}(1/\sigma^2)$ in Scenario 1 for LED decay rates of $\alpha = 10^{-5}$ and time intervals of $t \in \{10000h, 50000h\}$ when the VLC receiver is positioned at $(0.5, 0, 5, 0.85)$ m.

that for both time intervals, the LB plateaus at low noise variances and the MCRB goes to zero, as expected. Additionally, it is worth noting that the MCRB and LB values at $t = 10000$ h are lower than those at $t = 50000$ h. This is due to the gradual reduction in power output of the LED transmitters over time. Furthermore, it can also be observed that, rather than the RMSE of the MML estimator decreasing to zero at low noise variances, it tends to converge to the plateaued LB. This is due to the fact that, at low noise variances, estimation errors are primarily due to the model mismatch and remain consistent even as noise variances decrease. However, it can be noticed that the RMSE of the MML estimator is lower than the established bounds at high noise levels. This can be attributed to the fact that the location of the VLC receiver is restricted to a room with dimensions $4\text{ m} \times 4\text{ m} \times 3\text{ m}$, and the MML estimator in (2.7) performs its search within this specific volume; that is, $\mathcal{L} = [-2, 2] \times [-2, 2] \times [0, 3]$ in (2.7). Calculations made within this volume based on random estimates for the position of the VLC receiver lead to an RMSE of around 2.1 meters, which closely matches the estimation error of the estimators under conditions of high noise levels.

In Fig. 2.3, the RMSE performance of the ML estimator in Scenario 2 and the corresponding CRB are depicted versus $10 \log_{10}(1/\sigma^2)$ for $t = 10000$ h and $t = 50000$ h. Similar to Fig. 2.2, it is observed that the RMSEs of the ML estimators for both time intervals are lower than the corresponding CRBs at high levels of noise variances, which is primarily due to the spatial constraints within which the ML estimator in (2.24) and (2.25) performs its search. At lower levels of noise variances, the RMSEs of the ML estimators converge to the CRBs for both time intervals. Additionally, it is important to highlight that while the RMSE values of the ML estimator approach the lower bound for both time intervals at very low noise levels, the ML estimator actually exhibits better RMSE performance at $t = 10000$ h compared to the established lower bound, especially under conditions of moderate noise levels. This can be attributed to the constraint imposed on the decay rate estimates within the ML estimator. Namely, while the decay rate parameter is considered as a real number in the CRB calculations, it is searched among positive real numbers in the ML estimator in (2.23) (equivalently, in (2.24) and (2.25)). Furthermore, as in Scenario 1, the performance of the ML estimator for $t = 10000$ h is better than that for $t = 50000$ h, which is due to the power degradation over time, as discussed previously.

In Fig. 2.4, we illustrate the RMSE performance of the ML estimator and the established theoretical lower bound in Scenario 3. The estimator performance is plotted as a function of $10 \log_{10}(1/\sigma^2)$ for two different time intervals. Similar to Figs. 2.2 and 2.3, the RMSEs of the

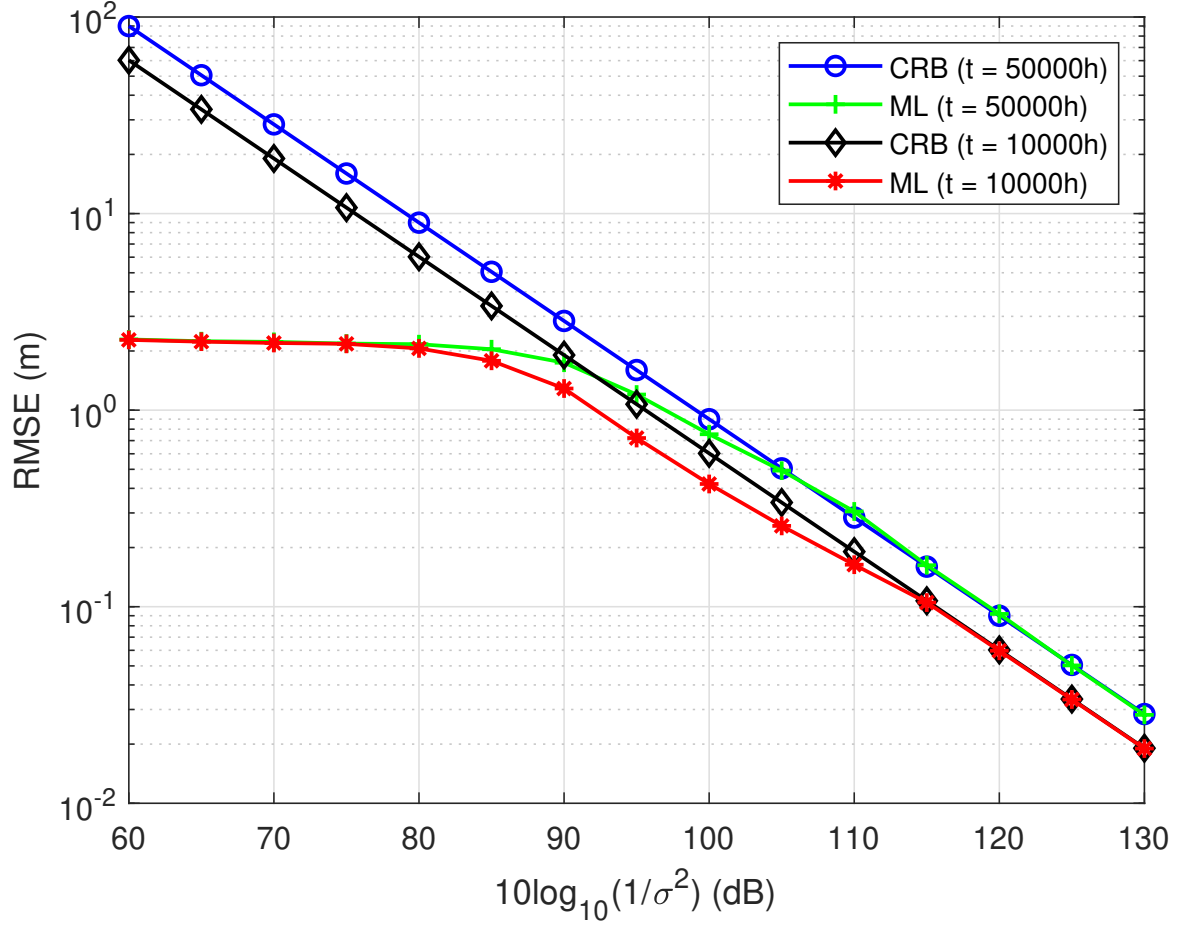


Figure 2.3: Performance of the ML estimator and the corresponding CRB versus $10 \log_{10}(1/\sigma^2)$ in Scenario 2 for LED decay rates of $\alpha = 10^{-5}$ and time intervals of $t \in \{10000 h, 50000 h\}$ when the VLC receiver is positioned at $(0.5, 0, 5, 0.85)$ m.

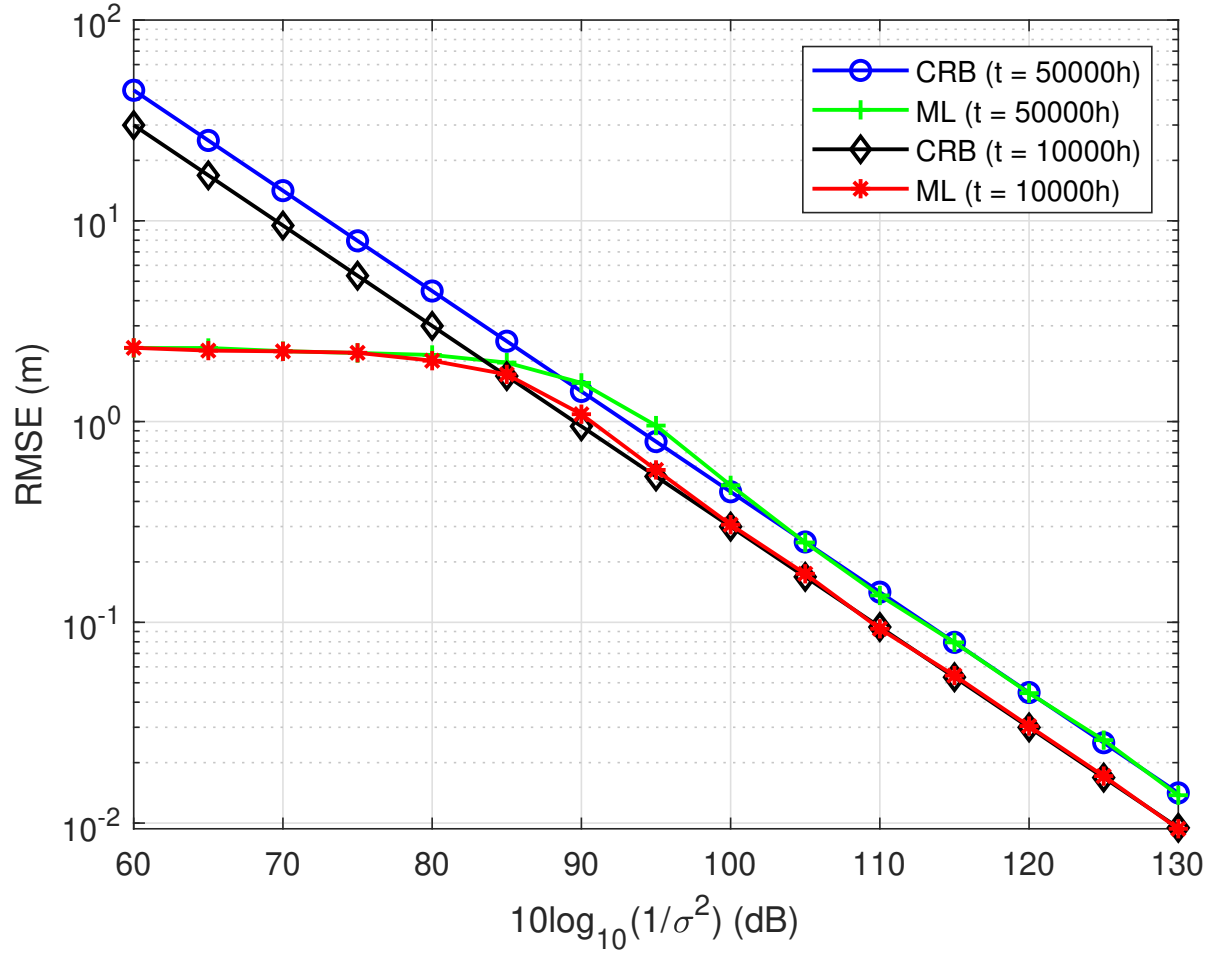


Figure 2.4: Performance of the ML estimator and the corresponding CRB versus $10\log_{10}(1/\sigma^2)$ in Scenario 3 for LED decay rates of $\alpha = 10^{-5}$ and time intervals of $t \in \{10000h, 50000h\}$ when the VLC receiver is positioned at $(0.5, 0, 5, 0.85)$ m.

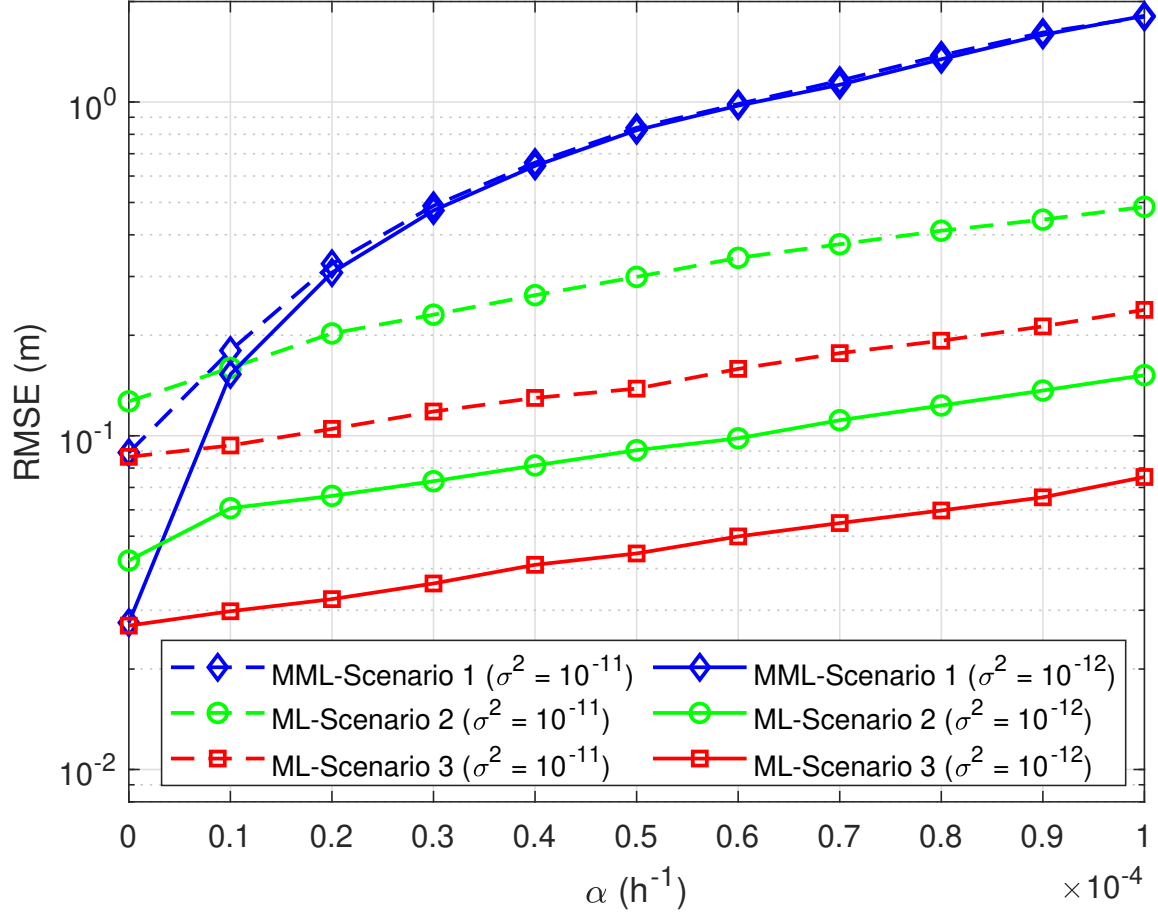


Figure 2.5: RMSE performance of all the estimators versus decay rate, α , for $t = 10000$ h and the noise levels of $\sigma^2 = 10^{-11}$ and $\sigma^2 = 10^{-12}$ when the VLC receiver is positioned at (0.5, 0, 5, 0.85) m.

ML estimators for both time intervals are lower than the corresponding CRBs at high levels of noise variances due to the consideration of the room dimensions in the ML estimator in (2.36). Also, at lower levels of noise variances, the RMSEs of the ML estimators converge to the lower bounds for both time intervals. Again, as in Scenario 1 and Scenario 2, the performance of the ML estimator at $t = 10000$ h is better than that at $t = 50000$ h, which can be attributed to the gradual deterioration in power over time, as previously explained.

In order to compare the estimators in Scenarios 1, 2, and 3, we present in Fig. 2.5 the RMSE performances of the estimators as a function of the decay rate α of the LEDs considering noise variance of $\sigma^2 = 10^{-11}$ and $\sigma^2 = 10^{-12}$. It is observed that the MML estimator under Scenario 1 and the ML estimator under Scenario 3 attain the same RMSE values for $\alpha = 0$ at both noise levels. This is due to the fact that when $\alpha = 0$, there is no mismatch

between the assumed model and the true model; hence, Scenario 1 and Scenario 3 coincide. However, the ML estimator under Scenario 2 performs worse since it is unaware of the α value and tries estimating it. Also, it can be observed that as the decay rate α increases, that is, as the mismatch between the true and the assumed models increases, the RMSE performance of the MML estimator deteriorates significantly in comparison to the RMSE performances of the ML estimators in Scenario 2 and Scenario 3. Again, it can be observed that the ML estimator under Scenario 3 always outperforms that in Scenario 2 since the α value is known under Scenario 3 and unknown under Scenario 2. Additionally, it is worth noting that the RMSE values of the MML estimator does not improve significantly with decreasing noise variance as α increases (as the model mismatch becomes the main source of error). Furthermore, it can be noted that the difference in RMSE values between the ML estimator under Scenario 3 and the MML estimator under Scenario 1 is greater at a noise variance of $\sigma^2 = 10^{-12}$ compared to $\sigma^2 = 10^{-11}$. To be more precise, the sensitivity to the model mismatch increases with decreasing noise variance as it is more pronounced at $\sigma^2 = 10^{-12}$ than at $\sigma^2 = 10^{-11}$. This demonstrates that being unaware of the true power decay pattern of the LEDs can be a significant inhibiting factor for accurate visible light positioning at low noise levels.

To further compare the estimators in Scenarios 1, 2, and 3, we illustrate in Fig. 2.6 the RMSE performances of the estimators as a function of $10\log_{10}(1/\sigma^2)$. Similar to the observations made in Fig. 2.2, Fig. 2.3, and Fig. 2.4, at high noise levels, the estimators exhibit the same RMSE performance for the two time intervals. This is as a result of the spatial constraints of the room within which estimators perform their search. Also, similar to the observations made in the previous figures, the RMSE performance of the estimators at $t = 20000 h$ is better than their corresponding RMSE performance at $t = 60000 h$, which is due to the power degradation of the LEDs over time, as discussed previously. It can be noted that the best estimates are obtained by the ML estimator under Scenario 3 for $t = 20000 h$, particularly under moderate to high SNRs. Conversely, the worst estimates are made by the MML estimator under Scenario 1 for $t = 60000 h$. This is due to the fact the power degradation formula and the decay rate of the LEDs are completely known by the VLC receiver under Scenario 3 and for $t = 20000 h$, there is less degradation in the power of the LEDs compared to $t = 60000 h$. However, under Scenario 1, the estimates are calculated without the knowledge of the power degradation formula and the decay rate of the LEDs, coupled with greater LED power deterioration at $t = 60000 h$. Under very high SNRs, the ML estimator in Scenario 3 outperforms both the ML and MML estimators in

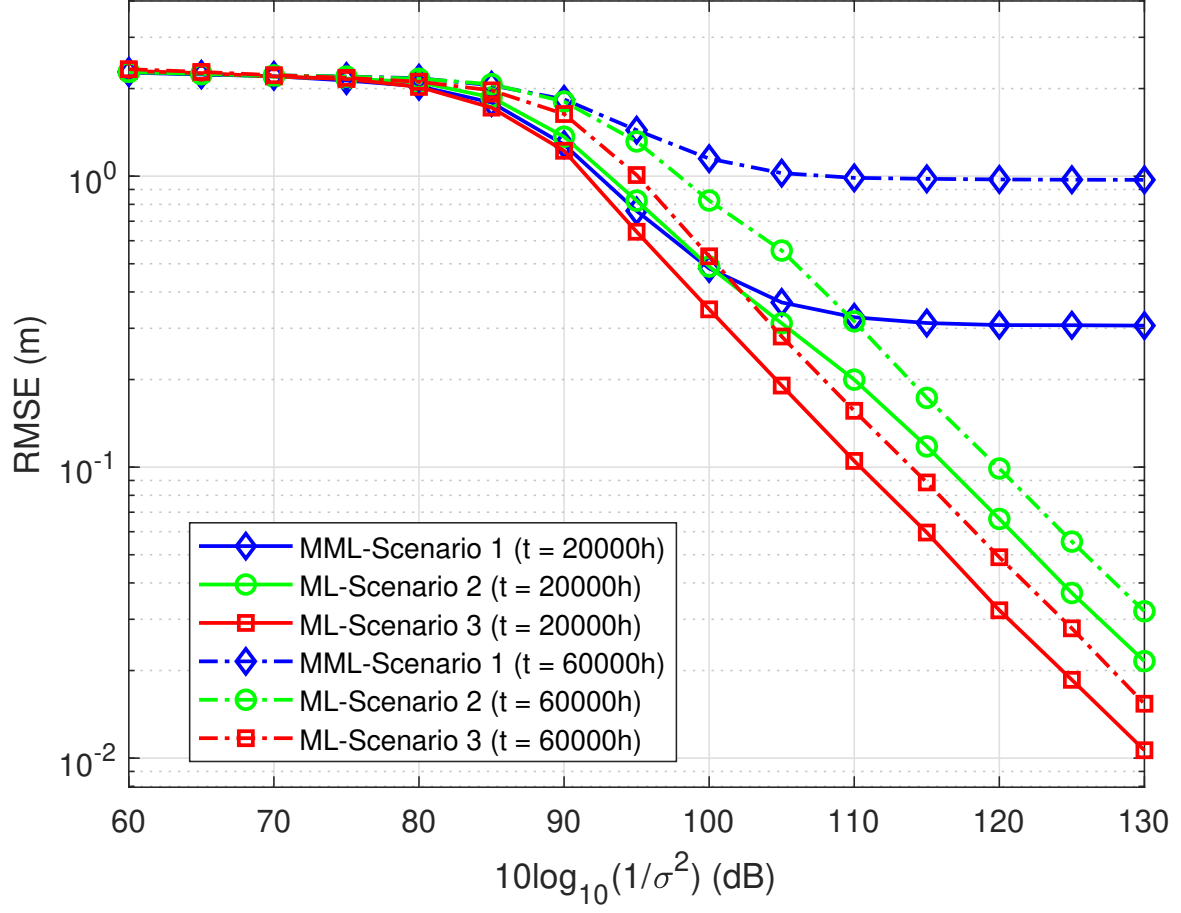


Figure 2.6: RMSE performance of all the estimators versus $10\log_{10}(1/\sigma^2)$ for LED decay rates of $\alpha = 10^{-5}$ and time intervals of $t \in \{20000h, 60000h\}$ when the VLC receiver is positioned at $(0.5, 0, 5, 0.85)$ m.

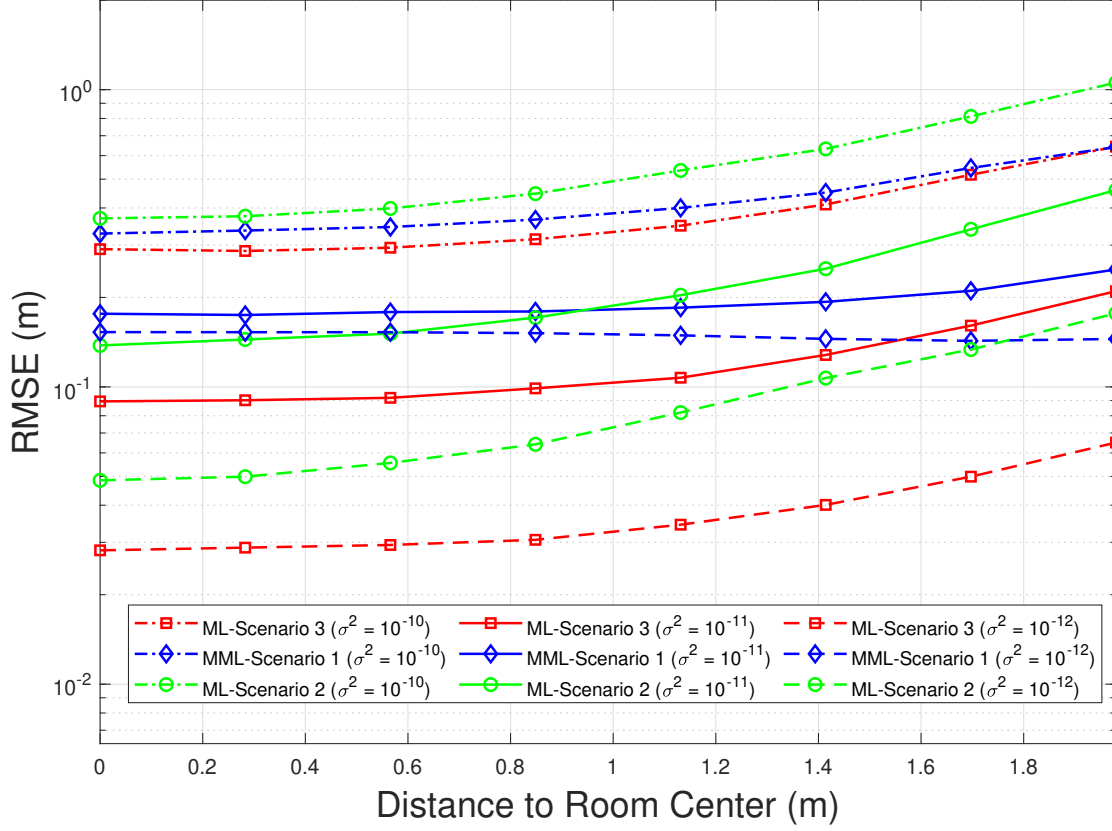


Figure 2.7: RMSE performance of the estimators versus distance of the VLC receiver to the room center at a fixed height for LED decay rate of $\alpha = 10^{-5}$, time interval of $t = 10000$ h, and noise variances of $\sigma^2 \in \{10^{-12}, 10^{-11}, 10^{-10}\}$ when the VLC receiver moves on the straight line from $(0, 0, 0.85)$ m towards $(1.4, 1.4, 0.85)$ m.

Scenarios 2 and 1, respectively, for both time intervals, with the ML estimator in Scenario 2 also outperforming the MML estimator in Scenario 1. This is due to the full knowledge of the power degradation formula and the decay rate of the LEDs in Scenario 3, the knowledge of only the the power degradation formula in Scenario 2 and the lack of knowledge of the power degradation formula and the decay rate of the LEDs in Scenario 1.

In Fig. 2.7, we plot the RMSE performances of the estimators under the three scenarios versus the distance of the VLC receiver to the room center at a fixed height when the VLC receiver moves on the straight path from $(0, 0, 0.85)$ m towards $(1.4, 1.4, 0.85)$ m by considering noise variances of $\sigma^2 = 10^{-12}$, $\sigma^2 = 10^{-11}$ and $\sigma^2 = 10^{-10}$. It is noted from the figure that as the VLC receiver moves away from the center, the RMSEs of the estimators tend to increase for all noise levels due to decreasing power levels at the VLC receiver. An

exception to this is observed for $\sigma^2 = 10^{-12}$ in Scenario 1 as the model mismatch becomes the main source of error for low noise variances, and the RMSE value stays almost the same. In addition, the RMSE values of the ML estimator under Scenario 3 consistently remain lower than those of the estimators in Scenario 1 and Scenario 2 for each value of the noise variance. Moreover, it is worth noting that the ML estimator under Scenario 2 achieves lower RMSE values than the MML estimator under Scenario 1 over a wider range of distances as the variance decreases. This is mainly due to the fact that the effects of the model mismatch in Scenario 1 become more significant at low noise variances (as observed in Fig. 2.2). It is also noted that as the VLC receiver moves from (0, 0, 0.85) m to (1.4, 1.4, 0.85) m, the power measurements generally reduce and the ML estimator in Scenario 2 cannot estimate the decay rate and the position accurately; hence, is outperformed by the MML estimator in Scenario 1 after a distance of 0.94 m for $\sigma^2 = 10^{-11}$ and 1.77 m for $\sigma^2 = 10^{-12}$. In particular, since both the position and the decay rate are jointly estimated by the ML estimator in Scenario 2, its performance has the highest sensitivity to the changes in the received power levels (equivalently, to the SNRs), as noted from Fig. 2.7.

2.6 Concluding Remarks

We have investigated the received power based position estimation problem in visible light systems in the presence of luminous flux degradation of LEDs. We have considered three different scenarios with varying levels of knowledge about the LED degradation. When the VLC receiver is unaware of the degradation and performs position estimation under that assumption, there exists a mismatch between the true model and the assumed model. In this scenario (Scenario 1), we have derived the MCRB and the MML estimator to quantify the performance loss due to this model mismatch. Then, we have assumed that the VLC receiver knows the power decay formula for the LEDs but does not know the decay rate. In this scenario (Scenario 2), we derived the CRB and the ML estimator. In the final scenario (Scenario 3), we have considered the presence of full knowledge (both the model and the decay rate parameters) at the VLC receiver, and obtained the CRB and the ML estimator. (Among the three scenarios, Scenario 2 has a practical significance since the joint estimation of the VLC receiver position and the unknown decay rate parameter is performed in that scenario.) By comparing the theoretical limits and the estimators in these three scenarios, we have revealed the effects of the knowledge of the LED degradation model and the decay rate

parameter on position estimation performance. In particular, the model mismatch results in significant performance loss at high SNRs, hence, it prevents achieving very low estimation errors. With the knowledge of the degradation model, this performance loss at high SNRs can significantly be reduced by performing joint estimation of the position and the decay rate parameter. Of course, when the decay rate parameter is also known beforehand, the best position estimation accuracy is achieved by performing position estimation based on the exactly known model.



Chapter 3

Visible Light Positioning with Intelligent Reflecting Surfaces under Mismatched Orientations

In this chapter, we formulate and study a position estimation problem based on received power measurements for a VLP system employing an IRS with a mismatch between the true and assumed orientations of the IRS elements. We first describe extensively the system model. Secondly, we derive the MCRB, LB and MML estimator for specifying the estimation performance of a VLP system in the presence of mismatched orientations. Next, we make comparisons with the ML estimator and the CRB in absence of orientation mismatches to quantify the impact of mismatches. Our analysis demonstrates that at high SNRs, mismatched orientations can lead to significant performance degradation.

3.1 System Model

Consider a VLP system with N_L LED transmitters at known locations denoted by $\mathbf{l}_i \in \mathbb{R}^3$ for $i \in \{1, \dots, N_L\}$ and a VLC receiver at an unknown location $\mathbf{x} \in \mathbb{R}^3$. The orientation vectors of the i th LED transmitter and the VLC receiver are denoted by $\mathbf{n}_i \in \mathbb{R}^3$ and $\mathbf{n}_R \in \mathbb{R}^3$, respectively. Also, there is an IRS with N_I flat surfaces (small mirrors), with surface areas

denoted by $S_1, \dots, S_{N_I}$ having orientation vectors $\tilde{\mathbf{n}}_1, \dots, \tilde{\mathbf{n}}_{N_I}$ and locations $\tilde{\mathbf{l}}_1, \dots, \tilde{\mathbf{l}}_{N_I}$. It is assumed that each IRS element surface causes glossy reflections (Phong's model) [56, 92] with the reflectance coefficient remaining constant across each element, where ρ_k denotes the reflectance coefficient for the k th IRS element. It is assumed that both ρ_k 's and S_k 's are assumed to be known for $k \in \{1, 2, \dots, N_I\}$ [76].

The VLC receiver aims to determine its own position using signals received from the LED transmitters. To this aim, it gathers power measurement from the LED transmitters. The power measurement received by the VLC receiver due to the signal emitted by the i th LED transmitter is expressed as [76]

$$P_{\text{RX},i} = P_{\text{TX},i} H_i^{\text{LOS}}(\mathbf{x}) + P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k) + \eta_i \quad (3.1)$$

for $i = 1, \dots, N_L$, where $P_{\text{RX},i}$ is the received power at the VLC receiver due to the signal from the i th LED transmitter, $P_{\text{TX},i}$ is the transmit power of the i th LED transmitter, $H_i^{\text{LOS}}(\mathbf{x})$ denotes the line-of-sight (LOS) channel coefficient between the i th LED transmitter and the VLC receiver, $dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k)$ denotes the reflected channel coefficient of the path between the i th LED transmitter and the VLC receiver, which makes a single reflection from an infinitesimally small area around $\tilde{\mathbf{l}}_k$ at the k th IRS element, and η_i is zero-mean Gaussian noise with a variance of σ_i^2 , which is independent of η_j for all $j \neq i$. As indicated in (3.1), it is presumed that the signals emitted by different LED transmitters undergo separate processing, and their individual power levels are measured at the VLC receiver. This assumption relies on employing a specific form of multiple access protocol, such as time-division or frequency-division multiple access [78].

The LOS channel gain in (3.1) can be calculated as

$$H_i^{\text{LOS}}(\mathbf{x}) = \frac{(m_i + 1) A_R [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{m_i} (\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R}{2\pi \|\mathbf{x} - \mathbf{l}_i\|^{m_i+3}} \quad (3.2)$$

and the reflected channel gain in (3.1) can be expressed as follows [76]:

$$\begin{aligned} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k) = & \frac{(m_i + 1) [(\tilde{\mathbf{l}}_k - \mathbf{l}_i)^T \mathbf{n}_i]^{m_i} ((\mathbf{l}_i - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k)}{4\pi^2 \|\mathbf{l}_i - \tilde{\mathbf{l}}_k\|^{m_i+3} \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} (A_R \rho_k ((\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R) dS_k \\ & \times \left(2r_k \frac{(\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|} + (1 - r_k)(\mu_k + 1) \cos(\beta_{ik} - \alpha_{ik})^{\mu_k} \right), \end{aligned} \quad (3.3)$$

where m_i is the Lambertian order for the i th LED transmitter, A_R is the area of the PD at the VLC receiver, and α_{ik} and β_{ik} denote, respectively, the incidence angle for the direct path between the i th LED transmitter and the reflective point $\tilde{\mathbf{l}}_k$ at the k th reflecting surface, and the irradiance angle for the direct path between the reflective point $\tilde{\mathbf{l}}_k$ at the k th reflecting surface and the VLC receiver, μ_k represents the directivity of reflection and $r_k \in [0, 1]$ denotes the proportion of diffuse component by the IRS. We define the total channel gain due to the LOS component and the reflections from the N_I IRS elements from the i th LED to the VLC receiver as

$$h_i(\mathbf{x}, \tilde{\mathbf{n}}_k) \triangleq H_i^{\text{LOS}}(\mathbf{x}) + \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k). \quad (3.4)$$

In practical implementations, the orientation vectors of the IRS elements cannot be adjusted perfectly and deviate from the true (desired) orientations due to device imperfections and noise (inaccuracy) in gyroscope readings. Consequently, our objective is to investigate the effects of these mismatched orientation vectors of IRS elements on the accuracy of visible light positioning based on power measurements.

3.2 Estimation under Mismatched Orientations: MML and MCRB

Let $\tilde{\mathbf{n}}_k^m$ for $k = 1, \dots, N_I$ denote the mismatched orientation vectors of the IRS elements and let $\tilde{\mathbf{n}}_k$ for $k = 1, \dots, N_I$ represent the true orientation vectors. In particular, the visible light system aims to set the orientations vectors of the IRS elements to $\tilde{\mathbf{n}}_k^m$ for $k = 1, \dots, N_I$. However, due to imperfections, their actual values become $\tilde{\mathbf{n}}_k$ for $k = 1, \dots, N_I$, which are unknown to the visible light system. Since the VLC receiver only has the knowledge of the mismatched orientations vectors (does not know the true orientation vectors) in this setting, the likelihood function for the position of the VLC receiver is expressed, based on (3.1), as

$$\tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x}) = \left(\prod_{i=1}^{N_L} \frac{1}{\sqrt{2\pi\sigma_i^2}} \right) e^{-\sum_{i=1}^{N_L} \frac{\left(P_{\text{RX},i} - P_{\text{TX},i} H_i^{\text{LOS}}(\mathbf{x}) - P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k^m) \right)^2}{2\sigma_i^2}} \quad (3.5)$$

where $\mathbf{P}_{\text{RX}}$ represents a vector consisting of the power measurements; i.e., $\mathbf{P}_{\text{RX}} = [P_{\text{RX},1} \cdots P_{\text{RX},N_L}]^T$. The expression in (3.5) can be referred to as the *misspecified parametric*

PDF for $\mathbf{x}$, given that the reflected power measurements are inaccurately specified due the mismatched orientations [13].

For the purpose of comparison, we also define the true model, incorporating the true orientation vectors of the IRS elements, $\tilde{\mathbf{n}}_k$ for $k = 1, \dots, N_I$. Hence, the true PDF can be formulated as follows:

$$p(\mathbf{P}_{\text{RX}}) = \left(\prod_{i=1}^{N_L} \frac{1}{\sqrt{2\pi\sigma_i^2}} \right) e^{-\sum_{i=1}^{N_L} \frac{\left(P_{\text{RX},i} - P_{\text{TX},i} H_i^{\text{LOS}}(\bar{\mathbf{x}}) - P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\bar{\mathbf{x}}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k) \right)^2}{2\sigma_i^2}} \quad (3.6)$$

where $\bar{\mathbf{x}}$ represents the true position vector of the VLC receiver.

3.2.1 MML Estimator

Given the misspecified parametric PDF in (3.5), MML estimator, defined as the parameter that maximizes this PDF, can be expressed as [13, 49]:

$$\hat{\mathbf{x}}_{\text{MML}}(\mathbf{P}_{\text{RX}}) = \arg \max_{\mathbf{x} \in \mathcal{L}} \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x}) \quad (3.7)$$

where $\mathcal{L}$ denotes the set of possible locations of the VLC receiver. Based on (3.5) and (3.7), the MML estimator derived from the received signal power measurements $\mathbf{P}_{\text{RX}}$ in (3.1) is given by

$$\hat{\mathbf{x}}_{\text{MML}}(\mathbf{P}_{\text{RX}}) = \arg \min_{\mathbf{x} \in \mathcal{L}} \sum_{i=1}^{N_L} \frac{\left(P_{\text{RX},i} - P_{\text{TX},i} H_i^{\text{LOS}}(\mathbf{x}) - P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k) \right)^2}{\sigma_i^2}. \quad (3.8)$$

Under appropriate regularity conditions, it is established that the MML estimator, $\hat{\mathbf{x}}_{\text{MML}}(\mathbf{P}_{\text{RX}})$, is asymptotically unbiased, and its error covariance matrix equals the MCRB($\mathbf{x}_0$), which is derived in the subsequent section [49].

3.2.2 MCRB

To obtain the theoretical limit on the accuracy of position estimation using the MML estimator, we first define the pseudo-true parameter [49], which is the parameter that minimizes

the Kullback-Leibler (KL) divergence between the misspecified parametric PDF $\tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})$ in (3.5) and the true PDF $p(\mathbf{P}_{\text{RX}})$ in (3.6). Namely, the pseudo-true parameter is given by

$$\mathbf{x}_0 = \arg \min_{\mathbf{x} \in \mathbb{R}^3} D(p(\mathbf{P}_{\text{RX}}) \| \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})) \quad (3.9)$$

where $D(p(\mathbf{P}_{\text{RX}}) \| \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x}))$ represents the KL divergence between the PDFs $p(\mathbf{P}_{\text{RX}})$ and $\tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})$. The objective function in (3.9) can be expressed based on the definition of the KL divergence as follows:

$$\begin{aligned} D(p(\mathbf{P}_{\text{RX}}) \| \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})) &= \int \log \left(\frac{p(\mathbf{P}_{\text{RX}})}{\tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})} \right) p(\mathbf{P}_{\text{RX}}) d\mathbf{P}_{\text{RX}} \\ &= \mathbb{E}_p \left\{ \log \left(\frac{p(\mathbf{P}_{\text{RX}})}{\tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})} \right) \right\}. \end{aligned} \quad (3.10)$$

We let $\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}})$ represent a misspecified-unbiased (MS-unbiased) estimator of the true position, $\bar{\mathbf{x}}$. That is, the expected value of the estimator $\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}})$ under the true model is equal to $\mathbf{x}_0$ [50]. The MCRB is a lower bound on the covariance matrix of any MS-unbiased estimator $\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}})$ of $\bar{\mathbf{x}}$ and is expressed as [49, 89]:

$$\mathbb{E}_p \{ (\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}}) - \mathbf{x}_0)(\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}}) - \mathbf{x}_0)^T \} \succeq \text{MCRB}(\mathbf{x}_0) \quad (3.11)$$

where $\mathbb{E}_p \{ \cdot \}$ represents the expectation under the true model $p(\mathbf{P}_{\text{RX}})$ in (3.6), and

$$\text{MCRB}(\mathbf{x}_0) \triangleq \mathbf{A}_{\mathbf{x}_0}^{-1} \mathbf{B}_{\mathbf{x}_0} \mathbf{A}_{\mathbf{x}_0}^{-1}. \quad (3.12)$$

The elements corresponding to the (m, n) th entries of matrices $\mathbf{A}_{\mathbf{x}_0}$ and $\mathbf{B}_{\mathbf{x}_0}$ in (3.12) are computed as follows:

$$[\mathbf{A}_{\mathbf{x}_0}]_{mn} = \mathbb{E}_p \left\{ \left. \frac{\partial^2 \log \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} \right|_{\mathbf{x}=\mathbf{x}_0} \right\}, \quad (3.13)$$

$$[\mathbf{B}_{\mathbf{x}_0}]_{mn} = \mathbb{E}_p \left\{ \left. \frac{\partial \log \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})}{\partial \mathbf{x}(m)} \frac{\partial \log \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})}{\partial \mathbf{x}(n)} \right|_{\mathbf{x}=\mathbf{x}_0} \right\} \quad (3.14)$$

for $1 \leq m, n \leq 3$, where $\mathbf{x}(n)$ represents the n th element of $\mathbf{x}$. Given that the value of the pseudo-true parameter typically is not of interest, the MCRB is utilized to establish a lower bound (LB) for any MS-unbiased estimator with respect to the true parameter value [49]. Specifically, the error covariance matrix for any MS-unbiased estimator $\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}})$ is lower bounded as follows [13]:

$$\mathbb{E}_p \{ (\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}}) - \bar{\mathbf{x}})(\hat{\mathbf{x}}(\mathbf{P}_{\text{RX}}) - \bar{\mathbf{x}})^T \} \succeq \text{LB}(\mathbf{x}_0) \quad (3.15)$$

where

$$\text{LB}(\mathbf{x}_0) \triangleq \text{MCRB}(\mathbf{x}_0) + (\bar{\mathbf{x}} - \mathbf{x}_0)(\bar{\mathbf{x}} - \mathbf{x}_0)^T. \quad (3.16)$$

The last term in (3.16) represents a bias component that is independent of the SNR. Consequently, as the SNR goes to infinity, the MCRB tends toward zero, and the bias component closely approaches the error covariance matrix of the MS-unbiased estimator.

To obtain the MCRB, and consequently the LB in (3.16), the following derivations are carried out.

3.2.2.1 Derivation of the Pseudo-True Parameter

We first obtain the pseudo-true parameter $\mathbf{x}_0$, as defined in (3.9), as it is essential for computing the MCRB associated with estimating the position of the VLC receiver in the presence of mismatched orientations. This involves finding the value of $\mathbf{x}$ that minimizes the KL divergence between the PDFs $\tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})$ in (3.5) and $p(\mathbf{P}_{\text{RX}})$ in (3.6). Utilizing equations (3.5) and (3.6), the expression in (3.10) can be computed as follows:

$$\begin{aligned} D(p(\mathbf{P}_{\text{RX}}) \| \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})) &= \mathbb{E}_p \left\{ - \sum_{i=1}^{N_L} \frac{\left(P_{\text{RX},i} - P_{\text{TX},i} H_i^{\text{LOS}}(\bar{\mathbf{x}}) - P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\bar{\mathbf{x}}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k) \right)^2}{2\sigma_i^2} \right. \\ &\quad \left. + \sum_{i=1}^{N_L} \frac{\left(P_{\text{RX},i} - P_{\text{TX},i} H_i^{\text{LOS}}(\mathbf{x}) - P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k^m) \right)^2}{2\sigma_i^2} \right\} \\ &= \sum_{i=1}^{N_L} \frac{(\mathbb{E}_p(P_{\text{RX},i}) - \mathbb{E}_{\tilde{p}}(P_{\text{RX},i}))^2}{2\sigma_i^2} \end{aligned} \quad (3.17)$$

where $\mathbb{E}_p(P_{\text{RX},i}) = P_{\text{TX},i} H_i^{\text{LOS}}(\bar{\mathbf{x}}) + P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\bar{\mathbf{x}}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k)$ and $\mathbb{E}_{\tilde{p}}(P_{\text{RX},i}) = P_{\text{TX},i} H_i^{\text{LOS}}(\mathbf{x}) + P_{\text{TX},i} \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k^m)$. Based on (3.10) and (3.17), we can find the value of $\mathbf{x}$ that minimizes the Kullback-Leibler (KL) divergence as

$$\mathbf{x}_0 = \arg \min_{\mathbf{x} \in \mathbb{R}^3} \sum_{i=1}^{N_L} \frac{(\mathbb{E}_p(P_{\text{RX},i}) - \mathbb{E}_{\tilde{p}}(P_{\text{RX},i}))^2}{\sigma_i^2}. \quad (3.18)$$

3.2.2.2 MCRB Derivation

After obtaining the pseudo-true parameter, the matrices $\mathbf{A}_{\mathbf{x}_0}$ in (3.13) and $\mathbf{B}_{\mathbf{x}_0}$ in (3.14) are computed to calculate the MCRB outlined in (3.12). The second-order derivatives in (3.13)

can be obtained from (3.5) as follows:

$$\begin{aligned} \frac{\partial^2 \log \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} &= \sum_{i=1}^{N_L} \frac{(P_{\text{RX},i} P_{\text{TX},i})}{\sigma_i^2} \frac{\partial^2 h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} - \sum_{i=1}^{N_L} \frac{(P_{\text{TX},i})^2}{\sigma_i^2} \frac{\partial h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n)} \\ &\quad - \sum_{i=1}^{N_L} \frac{(P_{\text{TX},i})^2}{\sigma_i^2} h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m) \frac{\partial^2 h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} \end{aligned} \quad (3.19)$$

where $h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m) = H_i^{\text{LOS}}(\mathbf{x}) + \sum_{k=1}^{N_I} \int_{S_k} dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k^m)$. Then, by evaluating (3.19) at $\mathbf{x} = \mathbf{x}_0$, the elements of matrix $\mathbf{A}_{\mathbf{x}_0}$ in (3.13) can be derived as

$$\begin{aligned} [\mathbf{A}_{\mathbf{x}_0}]_{mn} &= \sum_{i=1}^{N_L} \frac{\mathbb{E}_p\{P_{\text{RX},i}\} P_{\text{TX},i}}{\sigma_i^2} \frac{\partial^2 h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} - \sum_{i=1}^{N_L} \frac{(P_{\text{TX},i})^2}{\sigma_i^2} \frac{\partial h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n)} \\ &\quad - \sum_{i=1}^{N_L} \frac{(P_{\text{TX},i})^2}{\sigma_i^2} h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m) \frac{\partial^2 h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)}. \end{aligned} \quad (3.20)$$

The first-order derivatives in (3.14) can be obtained from (3.5) as follows:

$$\frac{\partial \log \tilde{p}(\mathbf{P}_{\text{RX}} | \mathbf{x})}{\partial \mathbf{x}(m)} = \sum_{i=1}^{N_L} \frac{P_{\text{TX},i} (P_{\text{RX},i} - P_{\text{TX},i} h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m))}{\sigma_i^2} \frac{\partial h_i(\mathbf{x}, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(m)}. \quad (3.21)$$

Also, the elements of matrix $\mathbf{B}_{\mathbf{x}_0}$ in (3.14) are obtained utilizing (3.21) as follows:

$$\begin{aligned} [\mathbf{B}_{\mathbf{x}_0}]_{mn} &= \mathbb{E}_p \left\{ \sum_{i=1}^{N_L} P_{\text{TX},i} \frac{(P_{\text{RX},i} - P_{\text{TX},i} h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m))}{\sigma_i^2} \frac{\partial h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(m)} \right. \\ &\quad \times \left. \sum_{j=1}^{N_L} P_{\text{TX},j} \frac{(P_{\text{RX},j} - P_{\text{TX},j} h_j(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m))}{\sigma_j^2} \frac{\partial h_j(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n)} \right\} \\ &= \sum_{i=1}^{N_L} \left(\frac{P_{\text{TX},i}}{\sigma_i^2} \right)^2 \mathbb{E}_p \{ (P_{\text{RX},i} - P_{\text{TX},i} h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m))^2 \} \frac{\partial h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(m)} \frac{\partial h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n)} \\ &\quad + \sum_{i=1}^{N_L} \sum_{j=1, j \neq i}^{N_L} \frac{P_{\text{TX},i} P_{\text{TX},j}}{\sigma_i^2 \sigma_j^2} (P_{\text{TX},i} h_i(\bar{\mathbf{x}}, \tilde{\mathbf{n}}_k) - P_{\text{TX},i} h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)) \\ &\quad \times (P_{\text{TX},j} h_j(\bar{\mathbf{x}}, \tilde{\mathbf{n}}_k) - P_{\text{TX},j} h_j(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)) \frac{\partial h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(m)} \frac{\partial h_j(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m)}{\partial \mathbf{x}(n)} \end{aligned} \quad (3.22)$$

where $\mathbb{E}_p \{ (P_{\text{RX},i} - P_{\text{TX},i} h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m))^2 \} = \sigma_i^2 + (P_{\text{TX},i} h_i(\bar{\mathbf{x}}, \tilde{\mathbf{n}}_k) - P_{\text{TX},i} h_i(\mathbf{x}_0, \tilde{\mathbf{n}}_k^m))^2$.

The first and second order derivatives of the channel coefficient $h_i(\cdot)$, as indicated in (3.20) and (3.23), can be explicitly obtained with respect to the system parameters, as detailed in **Appendix B**.

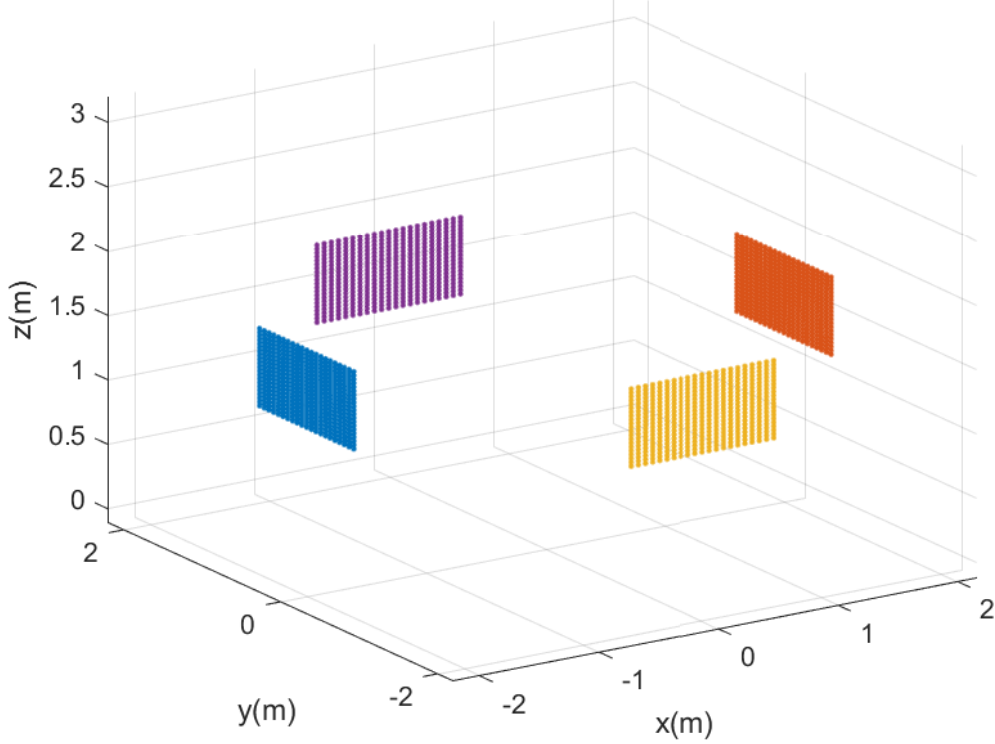


Figure 3.1: A depiction of the IRS elements on the walls of the room.

Remark: Based on the derived MCRB and LB expressions, we can evaluate the impact of mismatched orientations of IRS elements; namely, the effects of imperfections in IRS orientation adjustment. For comparison purposes, we can examine the maximum likelihood (ML) estimator and the corresponding Cramér-Rao bound (CRB), as derived without orientation mismatches in [76, Sec. II]. Since the theoretical limits are achieved by the MML and ML estimators at high SNRs, they serve as benchmarks for assessing localization performance and for designing system parameters under high SNR conditions.

3.3 Numerical Results

In this section, we conduct simulations to investigate the effects of mismatched orientations of IRS elements on localization accuracy by evaluating the proposed estimator and the theoretical bound. As in [76], we consider a room with dimensions of $4 \times 4 \times 3$ meters (width, depth, and height, respectively) and 4 LEDs ($N_L = 4$) placed at the following

coordinates in meters: $(-1, 1, 3), (1, 1, 3), (1, -1, 3), (-1, -1, 3)$. These LED positions are selected to ensure a symmetrical coverage of the room, with the center of the room floor being represented by the coordinates $(0, 0, 0)$. The orientations of the LEDs are defined as $\mathbf{n}_i = [0, 0, -1]^T$ for $i = 1, \dots, N_L$, indicating that they all point downwards. The transmit powers of the LEDs, $P_{\text{TX},i}$, are all set to 5 W and the area of the PD is taken as $A_R = 1 \text{ cm}^2$. The orientation vector of the receiver is defined as $\mathbf{n}_R = [0, 0, 1]^T$, indicating that it points upwards. Furthermore, we assume that the noise variances are equal, denoted as $\sigma_i^2 = \sigma^2$ for all i (see (3.1)). As depicted in Fig. 3.1, each of the four walls has $N_I/4$ IRS units arranged in a $\sqrt{N_I/4} \times \sqrt{N_I/4}$ grid. In the simulations, N_I is set to 1764, resulting in 21×21 IRS elements per wall [76]. Each IRS element has a rectangular shape with a width w_u of 4 cm and a height h_u of 2 cm, giving each IRS element an area S_k of 8 cm^2 . The IRS units are spaced $w_d = 2 \text{ cm}$ apart horizontally and $h_u = 1 \text{ cm}$ apart vertically to prevent inter-element blockage, as detailed in Fig. 3 of [76]. The IRS units are positioned to center both horizontally and vertically on the wall, aligning the center of the two-dimensional IRS array with the center of the wall. In the simulations, the fraction of diffuse component, r_k , is set to 0 and the directivity of reflection, μ_k , is set to 5, and the reflectance coefficient of the IRS elements is set to $\rho_k = 0.95 \forall k$. In addition, the mismatched orientations (the imperfect orientation information at the VLC receiver) are set for all k 's as $\tilde{\mathbf{n}}_k^m = [1 \ 0 \ 0]^T$ ($\theta_1 = \pi/2, \phi_1 = 0$), $\tilde{\mathbf{n}}_k^m = [-1 \ 0 \ 0]^T$ ($\theta_2 = \pi/2, \phi_2 = \pi$), $\tilde{\mathbf{n}}_k^m = [0 \ 1 \ 0]^T$ ($\theta_3 = \pi/2, \phi_3 = \pi/2$) and $\tilde{\mathbf{n}}_k^m = [0 \ -1 \ 0]^T$ ($\theta_4 = \pi/2, \phi_4 = -\pi/2$) for the IRS elements located on the wall at $x = -2, x = 2, y = -2$, and $y = 2$, respectively. On the other hand, the true orientations are obtained as

$$\tilde{\mathbf{n}}_k = [\cos(\phi_i^t) \sin(\theta_i^t) \ \sin(\phi_i^t) \sin(\theta_i^t) \ \cos(\theta_i^t)]^T \quad (3.24)$$

with

$$\theta_i^t = \theta_i + \epsilon_i, \quad (3.25)$$

$$\phi_i^t = \phi_i + \epsilon_i \quad (3.26)$$

for $i \in \{1, 2, 3, 4\}$, where ϵ_i denotes a uniform random variable over the interval $[-k, k]$. This setting corresponds to a scenario where the visible light system aims to set the orientation vectors of the IRS elements to be perpendicular to the walls; however, due to imperfections, the true orientations are not perfectly perpendicular as specified above.

To evaluate the performance of the MML estimator and to compare it against the ML estimator that is implemented assuming perfectly set (i.e., perpendicular) IRS orientations,

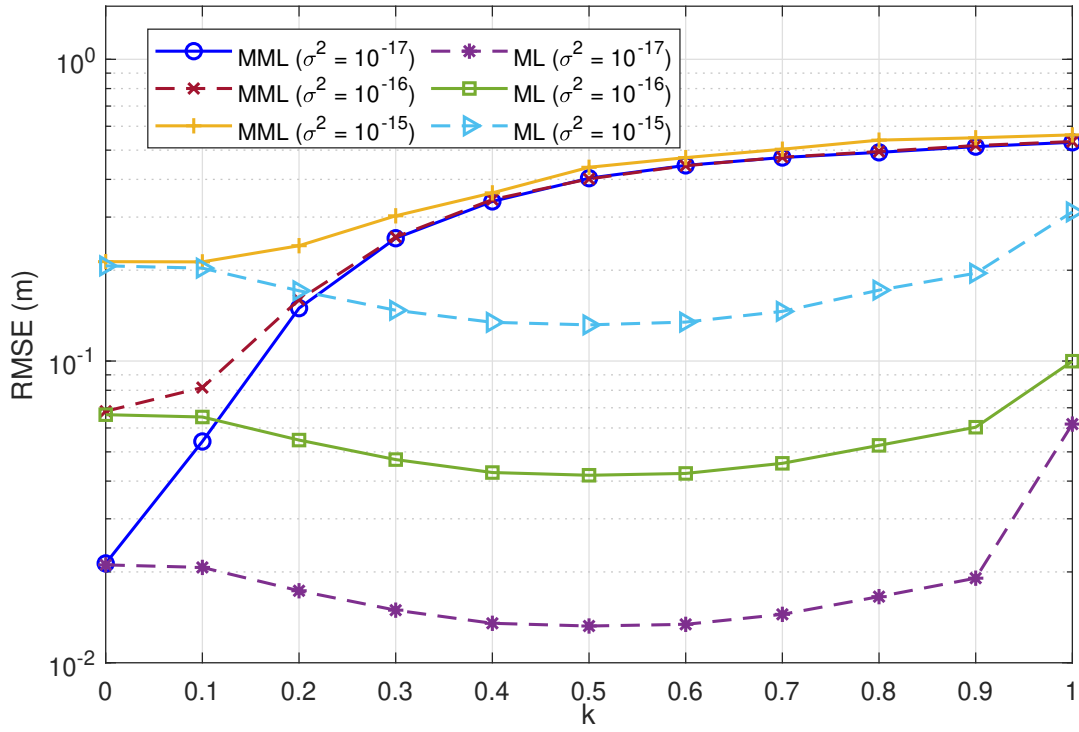


Figure 3.2: Performance of the MML estimator versus k for $\sigma^2 \in \{10^{-15}, 10^{-16}, 10^{-17}\}$ in NLOS scenario when the VLC receiver is positioned at (0.5, 0.5, 0.85) meters.

the VLC receiver is placed at $\mathbf{x} = (0.5, 0.5, 0.85)$ m. In Fig. 3.2, we plot the root mean-squared error (RMSE) performance of the MML estimator and the ML estimator versus k for three different noise levels: 10^{-15} , 10^{-16} , and 10^{-17} . For each k value, the simulation is run 500 times to calculate the RMSE. It can be observed that the RMSE values of the MML estimator for all noise levels tend to rise as k increases. This is due to the fact that the mismatch between the true IRS element orientations and the assumed orientations increases with k , consequently leading to degraded performance of the MML estimator. Additionally, the MML estimator exhibits its worst RMSE performance at $\sigma^2 = 10^{-15}$ compared to its performance at $\sigma^2 = 10^{-16}$ and $\sigma^2 = 10^{-17}$, due to the higher noise content in the received power measurements. The RMSE values of the MML estimator for all three noise levels when $k = 0$ coincides with the true model as there is no mismatch between the actual and assumed orientation vectors of the IRS elements. It is observed that as k increases, the RMSE performance does not improve significantly with decreasing noise variance as the mismatch between the assumed and true orientations of the IRS elements becomes the primary source of error. Moreover, it is worth noting that the RMSE performance of the ML estimator does not follow a monotone pattern with respect to k . The optimal IRS orientations depend on the actual position of the VLC receiver; hence, RMSE values of the ML estimator vary, being high for certain k values and low for others. On average, the optimal orientations of the IRS elements are achieved at around $k = 0.5$ when the VLC receiver is located at $(0.5, 0.5, 0.85)$ m. The RMSE values tend to rise for k values greater or less than 0.5. This outcome is consistent with the findings in [76], which developed an algorithm for optimally adjusting the IRS orientations and demonstrated that IRS perpendicularity is not optimal for the position $(0.5, 0.5, 0.85)$. Furthermore, unlike the MML estimator, with increasing k values, the RMSE performance of the ML estimator improves significantly with decreasing noise levels as it is not affected by the orientation mismatches.

In Fig. 3.3, we plot the RMSE performance of the MML estimator and the theoretical bounds versus $10 \log_{10}(1/\sigma^2)$ for different k values of 0.5 and 1. The orientations of the IRS elements are generated using MATLAB with a seed value of 1. It can be observed that for both values of k , the LBs plateau at low noise variances and the MCRBs approach zero, as anticipated. Additionally, it is worth noting that the MCRB and LB values at $k = 0.5$ are lower than those at $k = 1$. This is due the greater mismatch of orientation vectors at $k = 1$ compared to $k = 0.5$. It can also be observed that the RMSE of the MML estimator tends to converge to the LB at low noise variances for both values of k . This occurs because, at low noise variances, estimation errors are mainly attributed to orientation mismatches and

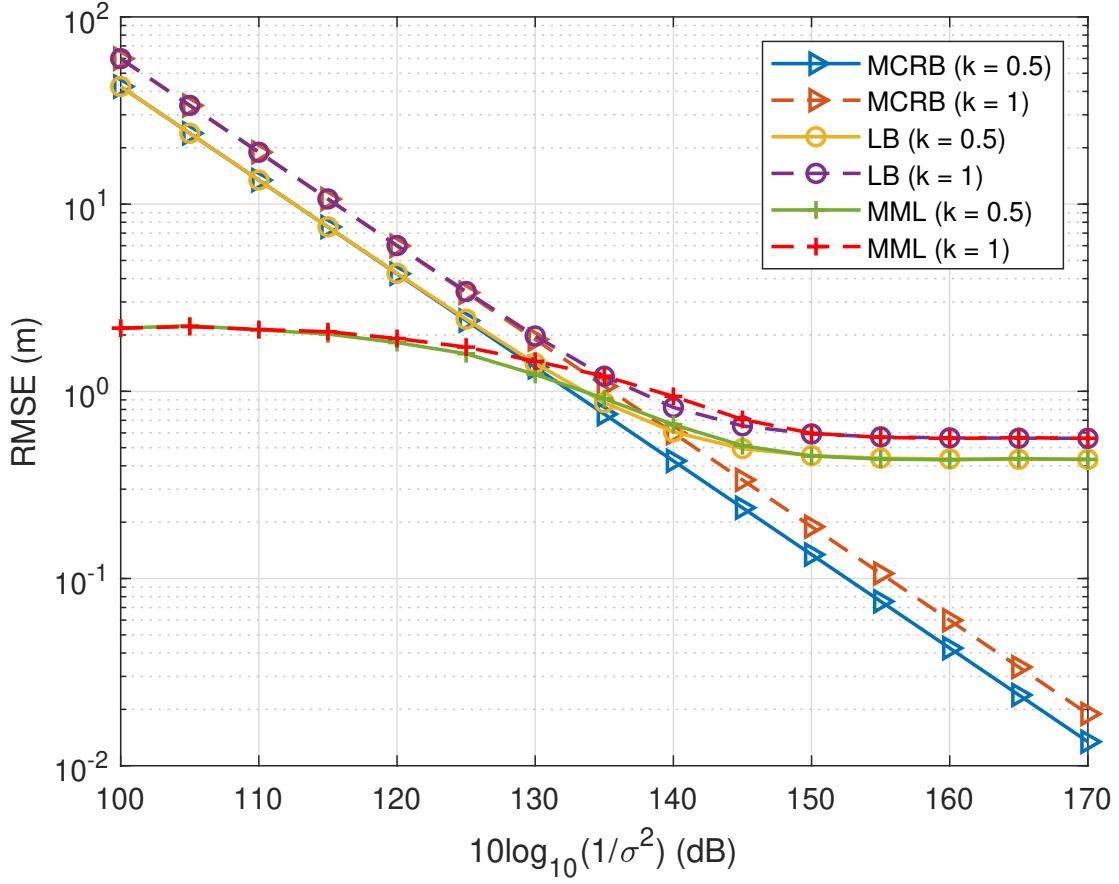


Figure 3.3: RMSE performance of the MML estimator and the theoretical limits versus $10 \log_{10}(1/\sigma^2)$ for $k \in \{0.5, 1\}$ in NLOS scenario when the VLC receiver is positioned at (0.5, 0.5, 0.85) meters.

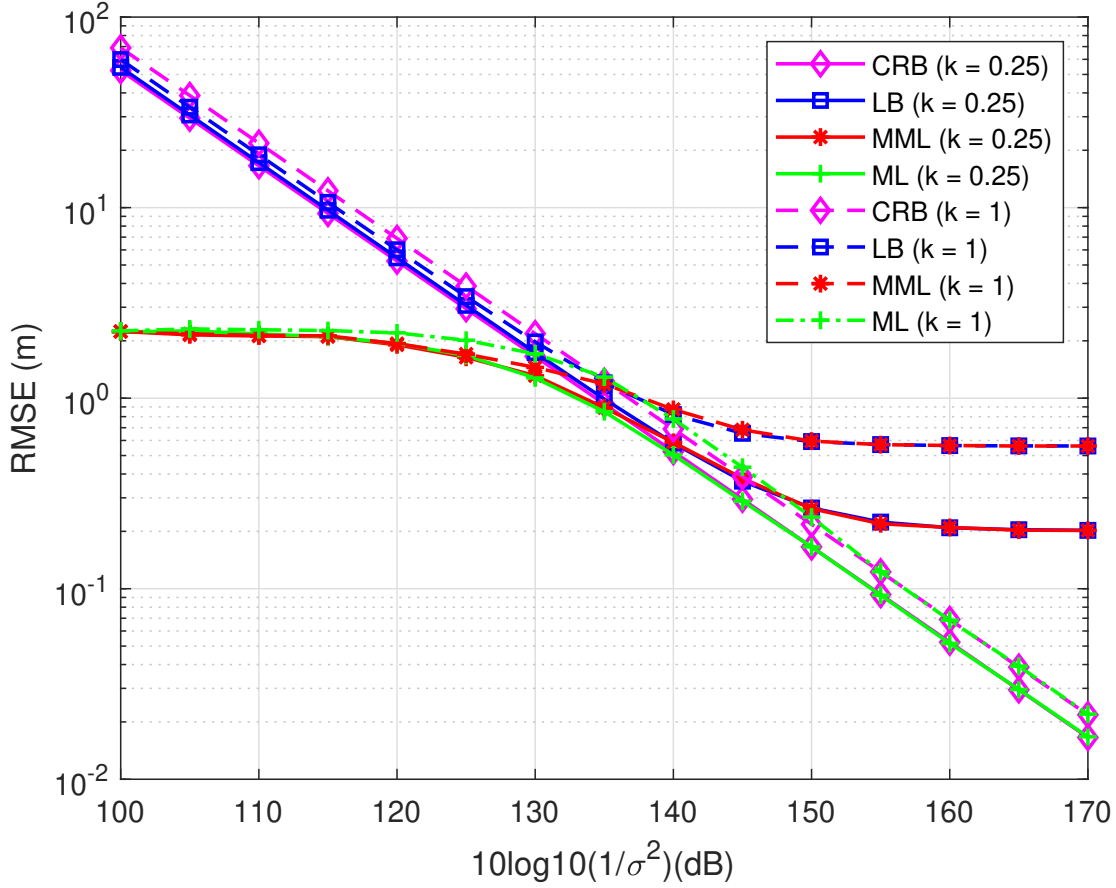


Figure 3.4: RMSE performance of the MML estimator and the theoretical limits versus $10 \log_{10}(1/\sigma^2)$ for $k \in \{0.25, 1\}$ in NLOS scenario when the VLC receiver is positioned at (0.5, 0.5, 0.85) meters.

remain consistent even as noise variance decreases. However, at high noise variances, the RMSE of the MML estimator is observed to be lower than the established bounds. This can be attributed to the fact that the location of the VLC receiver is restricted to a room with dimensions $4 \times 4 \times 3$ meters, and the MML estimator performs search within this region. The calculations made within this region relying on random estimates for the position of the VLC receiver yield an RMSE of approximately 2.1 meters, closely aligning with the estimation error of the estimators under high noise level conditions.

In Fig. 3.4, we plot the RMSE performance of the MML estimator and ML estimator, and their corresponding theoretical bounds versus $10 \log_{10}(1/\sigma^2)$ for $k \in \{0.25, 1\}$, with the orientations of the IRS elements generated using MATLAB and a seed of 1. As anticipated, the LBs plateau at low noise variances, and the MCRBs and CRBs approach zero for both values of k . It can also be observed that the RMSE values of the MML estimator and ML

estimator for both values of k tend to converge to the LB and the CRB, respectively, at low noise variances. The RMSE values of MML estimator when $k = 0.25$ are lower than those for $k = 1$ due to the increased orientation mismatch. However, for the ML estimator, the RMSE values are lower when $k = 0.25$ than when $k = 1$ due to fact that the generated orientations when $k = 0.25$ are closer to optimal for the VLC receiver location than when $k = 1$. Similar to what has been observed in Fig. 3.3, at high noise variances, the RMSEs of the MML estimator and ML estimator are observed to be lower than their established bounds due to the restriction on the search region for the estimators, as previously discussed. Additionally, it can be observed at very low noise variances that the ML estimator outperforms the MML estimator for both values of k , as the orientation mismatches become the primary source of error under the conditions of low noise variances.

3.4 Concluding Remarks

A position estimation problem has been formulated based on received power measurements for an IRS-assisted VLP system with a mismatch between the true and assumed orientations of IRS elements. The MML estimator, MCRB and LB have been derived to quantify the degradation in performance due to orientation mismatches. To assess the performance degradation of the MML estimator arising from orientation mismatches, a comparative analysis has been conducted against the ML estimator and the CRB under the scenario of no orientation mismatches. By comparing the MML and the LB with the ML estimator and the CRB, we have demonstrated the impact of orientation mismatches on position estimation performance. Specifically, orientation mismatches lead to a significant performance loss at high SNRs, thus hindering the achievement of very low estimation errors.

Chapter 4

Conclusion and Future Work

In the first part of this thesis (Chapter 2), we have investigated the position estimation problem in VLP systems using power measurements, accounting for the luminous flux degradation of LEDs. We have analyzed three distinct scenarios: the first is model mismatch, the second is model knowledge, and the third is full knowledge. In the first scenario, the VLC receiver is unaware of the power degradation of the LEDs and conducts position estimation under this inaccurate assumption, leading to a mismatch between the true model and the assumed model. We have derived the MML estimator, the MCRB, and subsequently the LB to evaluate the performance loss caused by this model mismatch. In the second scenario, we have considered a situation where the VLC receiver is aware of the power degradation of the LEDs and knows the power degradation formula. However, it lacks the knowledge of the specific decay rate of the LEDs. In this case, we have derived the ML estimator and the CRB to evaluate the VLP system performance in such a scenario. In the final scenario, the VLC receiver is fully informed about the power degradation of the LEDs. It not only knows the power degradation formula but also the decay rate of the LEDs. For this comprehensive scenario, we have derived the ML estimator and the CRB, referring to it as the full knowledge scenario. Numerical results have been presented to compare the theoretical limits and the estimators in the three scenarios. We have demonstrated the effects of the knowledge of the LED degradation model and the decay rate parameter on the accuracy of position estimation. We have shown that the model mismatch leads to substantial performance loss at high SNRs, preventing the achievement of very low estimation errors. However, knowing the degradation model can significantly reduce this performance loss at high SNRs by jointly

estimating the position and the decay rate parameter. Naturally, when the decay rate parameter is also known in advance, the highest position estimation accuracy is attained by performing position estimation based on the full knowledge of the model, as it possesses all the necessary information about the LED power degradation characteristics.

Several potential directions for future research exist within this field. Firstly, it is important to note that our study assumes perfect orientation information at the VLC receiver, e.g., from a gyroscope. However, there can be inaccuracies in the orientation information in practice. Future work could focus on taking these inaccuracies into account. Additionally, our current study has focused on one-shot position estimation without considering dynamic scenarios. In future work, tracking algorithms can be investigated for dynamic scenarios. Furthermore, another promising avenue for future research involves accounting for variations in the decay rates of the LEDs, which could impact the accuracy and reliability of the positioning system. Expanding research to incorporate these factors would contribute to a more robust and practical implementation of VLP systems.

In the second part of this thesis (Chapter 3), we have examined the effects of orientation mismatches of IRS elements on the position estimation accuracy of a VLP system utilizing an IRS. We have derived the MCRB, the LB and the MML estimator to quantify the performance loss due to the mismatched orientations. We have also conducted a comparative analysis with the MML estimator and the CRB serving as the benchmarks for performance evaluation. Our findings have shown that orientation mismatches can lead to a notable decline in performance, particularly under the condition of high SNR.

Also, numerous promising avenues for future research are present in this area. These include investigating novel methods to mitigate the impact of orientation mismatches in IRS-assisted VLP systems, potentially through adaptive algorithms that can dynamically adjust to varying orientation conditions. Research into optimizing IRS element adjustments to address orientation mismatches is also crucial. Considering dynamic tracking algorithms suitable for real-time applications represents another promising area of investigation. Additionally, studying the impact of LED power degradation in VLP systems using IRS in NLOS scenarios is essential for advancing the field.

Appendix A

First and Second Derivatives of Channel Coefficient in Eqn. (2.2)

The first-order partial derivatives of the channel coefficient $h_i(\mathbf{x})$ with respect to the position parameter $\mathbf{x}$ can be calculated via (2.2) as follows:

$$\frac{\partial h_i(\mathbf{x})}{\partial \mathbf{x}(n)} = \frac{(m_i + 1)A_R}{2\pi} \left(\frac{1}{v(\mathbf{x})} \frac{\partial u(\mathbf{x})}{\partial \mathbf{x}(n)} - \frac{u(\mathbf{x})}{v^2(\mathbf{x})} \frac{\partial v(\mathbf{x})}{\partial \mathbf{x}(n)} \right) \quad (\text{A.1})$$

where

$$u(\mathbf{x}) = [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{m_i} (\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R, \quad (\text{A.2})$$

$$v(\mathbf{x}) = \|\mathbf{x} - \mathbf{l}_i\|^{m_i+3}, \quad (\text{A.3})$$

$$\begin{aligned} \frac{\partial u(\mathbf{x})}{\partial \mathbf{x}(n)} &= m_i((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{m_i-1} \\ &\quad \times \mathbf{n}_T^i(n) - [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{m_i} \mathbf{n}_R(n), \end{aligned} \quad (\text{A.4})$$

$$\frac{\partial v(\mathbf{x})}{\partial \mathbf{x}(n)} = (m_i + 3)(\mathbf{x}(n) - \mathbf{l}_i(n)) \|\mathbf{x} - \mathbf{l}_i\|^{m_i+1}. \quad (\text{A.5})$$

In addition, the second-order partial derivatives of the channel coefficient with respect to the position parameter can be calculated from (A.1) as follows:

$$\begin{aligned} \frac{\partial^2 h_i(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} &= \frac{(m_i + 1)A_R}{2\pi} \left(\frac{1}{v(\mathbf{x})} \frac{\partial^2 u(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} - \frac{1}{v^2(\mathbf{x})} \frac{\partial v(\mathbf{x})}{\partial \mathbf{x}(m)} \frac{\partial u(\mathbf{x})}{\partial \mathbf{x}(n)} \right. \\ &\quad \left. - \frac{1}{q(\mathbf{x})} \frac{\partial b_n(\mathbf{x})}{\partial \mathbf{x}(m)} + \frac{b_n(\mathbf{x})}{q^2(\mathbf{x})} \frac{\partial q(\mathbf{x})}{\partial \mathbf{x}(m)} \right) \end{aligned} \quad (\text{A.6})$$

where

$$\begin{aligned} \frac{\partial^2 u(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} = & \mathfrak{m}_i(\mathfrak{m}_i - 1)((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{\mathfrak{m}_i - 2} \mathbf{n}_T^i(n) \mathbf{n}_T^i(m) \\ & - \mathfrak{m}_i [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{\mathfrak{m}_i - 1} (\mathbf{n}_T^i(n) \mathbf{n}_R(m) + \mathbf{n}_T^i(m) \mathbf{n}_R(n)), \end{aligned} \quad (\text{A.7})$$

$$b_n(\mathbf{x}) \triangleq (\mathfrak{m}_i + 3)(\mathbf{x}(n) - \mathbf{l}_i(n)) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{\mathfrak{m}_i} ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R), \quad (\text{A.8})$$

$$q(\mathbf{x}) \triangleq \|\mathbf{x} - \mathbf{l}_i\|^{\mathfrak{m}_i + 5}, \quad (\text{A.9})$$

$$\frac{\partial b_n(\mathbf{x})}{\partial \mathbf{x}(m)} = \begin{cases} (\mathfrak{m}_i + 3)(\mathbf{x}(n) - \mathbf{l}_i(n)) [\mathfrak{m}_i \mathbf{n}_T(m) ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i)^{\mathfrak{m}_i - 1} \\ - \mathbf{n}_R(m) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i)^{\mathfrak{m}_i}], & \text{if } m \neq n \\ (\mathfrak{m}_i + 3) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i]^{\mathfrak{m}_i} ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) + (\mathfrak{m}_i + 3)(\mathbf{x}(m) - \mathbf{l}_i(m)) \\ \times [\mathfrak{m}_i \mathbf{n}_T(m) ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i)^{\mathfrak{m}_i - 1} - \mathbf{n}_R(m) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_T^i)^{\mathfrak{m}_i}], & \text{if } m = n \end{cases} \quad (\text{A.10})$$

$$\frac{\partial q(\mathbf{x})}{\partial \mathbf{x}(m)} = (\mathfrak{m}_i + 5)(\mathbf{x}(m) - \mathbf{l}_i(m)) \|\mathbf{x} - \mathbf{l}_i\|^{\mathfrak{m}_i + 3}. \quad (\text{A.11})$$

Although the first-order derivative of $h_i(\mathbf{x})$ was presented in [22, Eqn. (14)], the second-order derivatives given by (A.6)–(A.11) were not available in the literature.

Appendix B

First and Second Derivatives of Channel Coefficient in Eqn. (3.4)

The first-order derivatives of the channel coefficient $h_i(\mathbf{x}, \tilde{\mathbf{n}}_k)$ appearing in (3.20) and (3.23) with respect to the position parameter $\mathbf{x}$ can be calculated using (3.4) as follows:

$$\frac{\partial h_i(\mathbf{x}, \tilde{\mathbf{n}}_k)}{\partial \mathbf{x}(m)} = \frac{\partial H_i^{\text{LOS}}(\mathbf{x})}{\partial \mathbf{x}(m)} + \frac{\sum_{k=1}^{N_i} \int_{S_k} \partial d H_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k)}{\partial \mathbf{x}(m)} \quad (\text{B.1})$$

where

$$\frac{\partial H_i^{\text{LOS}}(\mathbf{x})}{\partial \mathbf{x}(m)} = \frac{(\mathbf{m}_i + 1) A_R}{2\pi} \left(\frac{1}{v(\mathbf{x})} \frac{\partial u(\mathbf{x})}{\partial \mathbf{x}(n)} - \frac{u(\mathbf{x})}{v^2(\mathbf{x})} \frac{\partial v(\mathbf{x})}{\partial \mathbf{x}(n)} \right) \quad (\text{B.2})$$

with

$$u(\mathbf{x}) = [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{\mathbf{m}_i} (\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R, \quad (\text{B.3})$$

$$v(\mathbf{x}) = \|\mathbf{x} - \mathbf{l}_i\|^{\mathbf{m}_i+3}, \quad (\text{B.4})$$

$$\frac{\partial u(\mathbf{x})}{\partial \mathbf{x}(n)} = \mathbf{m}_i ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{\mathbf{m}_i-1} \mathbf{n}_i(n) - [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{\mathbf{m}_i} \mathbf{n}_R(n), \quad (\text{B.5})$$

$$\frac{\partial v(\mathbf{x})}{\partial \mathbf{x}(n)} = (\mathbf{m}_i + 3)(\mathbf{x}(n) - \mathbf{l}_i(n)) \|\mathbf{x} - \mathbf{l}_i\|^{\mathbf{m}_i+1}, \quad (\text{B.6})$$

and

$$\begin{aligned}
\frac{\partial dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k)}{\partial \mathbf{x}(m)} &= \frac{(\mathbf{m}_i + 1) \left[(\tilde{\mathbf{l}}_k - \mathbf{l}_i)^T \mathbf{n}_i \right]^{\mathbf{m}_i} ((\mathbf{l}_i - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k) \rho_k dS_k}{4\pi^2 \|\mathbf{l}_i - \tilde{\mathbf{l}}_k\|^{\mathbf{m}_i+3}} \\
&\times \left\{ 2r_k \left[\frac{\tilde{\mathbf{n}}_k(m) (\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R - \mathbf{n}_R(m) (\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^4} - \frac{4(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)) (\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k (\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^6} \right] + \right. \\
&(1 - r_k)(\mu_k + 1) \left[(\cos(\beta_{ik} - \alpha_{ik}))^{\mu_k} (-\mathbf{n}_R(m) \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-3} - 3(\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-5} (\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))) + \right. \\
&\left. \left. \mu_k \cos(\beta_{ik} - \alpha_{ik})^{\mu_k-1} \frac{(\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} \frac{\partial \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(m)} \right] \right\} \quad (\text{B.7})
\end{aligned}$$

with

$$\cos(\beta_{ik} - \alpha_{ik}) = \cos(\alpha_{ik}) \left(((\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k) \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-1} \right) + \sin(\alpha_{ik}) \left(\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-1} \right), \quad (\text{B.8})$$

$$\begin{aligned}
\frac{\partial \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(m)} &= \cos(\alpha_{ik}) \left(\tilde{\mathbf{n}}_k(m) \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-1} - (\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-3} (\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)) \right) \\
&+ \sin(\alpha_{ik}) \left(\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|^{-1} \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-1} g(m) - \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^{-3} (\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)) \|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \right), \quad (\text{B.9})
\end{aligned}$$

$$\cos(\alpha_{ik}) = ((\mathbf{l}_i - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k) \|\mathbf{l}_i - \tilde{\mathbf{l}}_k\|^{-1}, \quad (\text{B.10})$$

$$\sin(\alpha_{ik}) = \|(\mathbf{l}_i - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{l}_i - \tilde{\mathbf{l}}_k\|^{-1}, \quad (\text{B.11})$$

$$\begin{aligned}
g(m) &= \tilde{\mathbf{n}}_k(z(m+1)) [(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)) \tilde{\mathbf{n}}_k(z(m+1)) - (\mathbf{x}(z(m+1)) - \tilde{\mathbf{l}}_k(z(m+1))) \tilde{\mathbf{n}}_k(m)] \\
&- \tilde{\mathbf{n}}_k(z(m+2)) [(\mathbf{x}(z(m+2)) - \tilde{\mathbf{l}}_k(z(m+2))) \tilde{\mathbf{n}}_k(m) - (\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)) \tilde{\mathbf{n}}_k(z(m+2))], \quad (\text{B.12})
\end{aligned}$$

$$z(m) = \begin{cases} m, & \text{if } m \leq 3 \\ m - 3, & \text{otherwise} \end{cases}. \quad (\text{B.13})$$

In addition, the second-order partial derivatives of the channel coefficient with respect to the position parameter can be derived from (B.1) as follows:

$$\frac{\partial^2 h_i(\mathbf{x}, \tilde{\mathbf{n}}_k)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} = \frac{\partial^2 H_i^{\text{LOS}}(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} + \frac{\sum_{k=1}^{N_I} \int_{S_k} \partial^2 dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k, \tilde{\mathbf{n}}_k)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} \quad (\text{B.14})$$

where $\frac{\partial^2 H_i^{\text{LOS}}(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)}$ is defined by

$$\begin{aligned} \frac{\partial^2 H_i^{\text{LOS}}(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} = & \frac{(m_i + 1)A_R}{2\pi} \left(\frac{1}{v(\mathbf{x})} \frac{\partial^2 u(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} - \frac{1}{v^2(\mathbf{x})} \frac{\partial v(\mathbf{x})}{\partial \mathbf{x}(m)} \frac{\partial u(\mathbf{x})}{\partial \mathbf{x}(n)} \right. \\ & \left. - \frac{1}{q(\mathbf{x})} \frac{\partial b_n(\mathbf{x})}{\partial \mathbf{x}(m)} + \frac{b_n(\mathbf{x})}{q^2(\mathbf{x})} \frac{\partial q(\mathbf{x})}{\partial \mathbf{x}(m)} \right) \end{aligned} \quad (\text{B.15})$$

with $\frac{\partial^2 u(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)}$, $b_n(\mathbf{x})$, $q(\mathbf{x})$, $\frac{\partial b_n(\mathbf{x})}{\partial \mathbf{x}(m)}$, and $\frac{\partial q(\mathbf{x})}{\partial \mathbf{x}(m)}$ being given as follows:

$$\begin{aligned} \frac{\partial^2 u(\mathbf{x})}{\partial \mathbf{x}(m) \partial \mathbf{x}(n)} = & m_i(m_i - 1)((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{m_i-2} \mathbf{n}_i(n) \mathbf{n}_i(m) \\ & - m_i [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{m_i-1} (\mathbf{n}_i(n) \mathbf{n}_R(m) + \mathbf{n}_i(m) \mathbf{n}_R(n)) \end{aligned} \quad (\text{B.16})$$

$$b_n(\mathbf{x}) \triangleq (m_i + 3)(\mathbf{x}(n) - \mathbf{l}_i(n)) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{m_i} ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) \quad (\text{B.17})$$

$$q(\mathbf{x}) \triangleq \|\mathbf{x} - \mathbf{l}_i\|^{m_i+5} \quad (\text{B.18})$$

$$\frac{\partial b_n(\mathbf{x})}{\partial \mathbf{x}(m)} = \begin{cases} (m_i + 3)(\mathbf{x}(n) - \mathbf{l}_i(n)) [m_i \mathbf{n}_T(m) ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i)^{m_i-1} \\ - \mathbf{n}_R(m) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i)^{m_i}], & \text{if } m \neq n \\ (m_i + 3) [(\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i]^{m_i} ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) + (m_i + 3)(\mathbf{x}(m) - \mathbf{l}_i(m)) \\ \times [m_i \mathbf{n}_T(m) ((\mathbf{l}_i - \mathbf{x})^T \mathbf{n}_R) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i)^{m_i-1} - \mathbf{n}_R(m) ((\mathbf{x} - \mathbf{l}_i)^T \mathbf{n}_i)^{m_i}], & \text{if } m = n \end{cases} \quad (\text{B.19})$$

$$\frac{\partial q(\mathbf{x})}{\partial \mathbf{x}(m)} = (m_i + 5)(\mathbf{x}(m) - \mathbf{l}_i(m)) \|\mathbf{x} - \mathbf{l}_i\|^{m_i+3}. \quad (\text{B.20})$$

Also, $\frac{\partial^2 dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k \tilde{\mathbf{n}}_k)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)}$ in (B.14) is defined as

$$\begin{aligned} \frac{\partial^2 dH_{i,k}^{\text{ref}}(\mathbf{x}, \tilde{\mathbf{l}}_k \tilde{\mathbf{n}}_k)}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} = & c \left(\frac{\partial^2 f_1(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} (f_2(\mathbf{x}) + f_3(\mathbf{x})) + \frac{\partial f_1(\mathbf{x})}{\partial \mathbf{x}(m)} \left(\frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}(n)} + \frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}(n)} \right) \right. \\ & \left. + \frac{\partial f_1(\mathbf{x})}{\partial \mathbf{x}(n)} \left(\frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}(m)} + \frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}(m)} \right) + f_1(\mathbf{x}) \left(\frac{\partial^2 f_2(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} + \frac{\partial^2 f_3(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} \right) \right) \end{aligned} \quad (\text{B.21})$$

with c , $f_1(\mathbf{x})$, $f_2(\mathbf{x})$, and $f_3(\mathbf{x})$ being expressed as

$$c = \frac{(m_i + 1)A_R \rho_k dS_k [(\tilde{\mathbf{l}}_k - \mathbf{l}_i)^T \mathbf{n}_i]^{m_i} ((\mathbf{l}_i - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k)}{4\pi^2 \|\mathbf{l}_i - \tilde{\mathbf{l}}_k\|^{m_i+3}}, \quad (\text{B.22})$$

$$f_1(\mathbf{x}) = \frac{((\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3}, \quad (\text{B.23})$$

$$f_2(\mathbf{x}) = 2r_k \frac{(\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|}, \quad (\text{B.24})$$

$$f_3(\mathbf{x}) = (1 - r_k)(\mu_k + 1) \cos(\beta_{ik} - \alpha_{ik})^{\mu_k}. \quad (\text{B.25})$$

The first-order partial derivatives of $f_1(\mathbf{x})$, $f_2(\mathbf{x})$, and $f_3(\mathbf{x})$ in (B.23)–(B.25) with respect to $\mathbf{x}$ are obtained as follows:

$$\frac{\partial f_1(\mathbf{x})}{\partial \mathbf{x}(m)} = \frac{-\mathbf{n}_R(m)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} - \frac{3(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))((\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5}, \quad (\text{B.26})$$

$$\frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}(m)} = 2r_k \left(\frac{\tilde{\mathbf{n}}_k(m)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|} - \frac{(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))((\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} \right), \quad (\text{B.27})$$

$$\frac{\partial f_3(\mathbf{x})}{\partial \mathbf{x}(m)} = \mu_k(1 - r_k)(\mu_k + 1) \cos(\beta_{ik} - \alpha_{ik})^{\mu_k-1} \frac{\partial \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(m)}. \quad (\text{B.28})$$

Moreover, the second-order derivatives of $f_1(\mathbf{x})$, $f_2(\mathbf{x})$, and $f_3(\mathbf{x})$ in (B.23)–(B.25) with respect to $\mathbf{x}$ are also obtained as follows:

$$\begin{aligned} \frac{\partial^2 f_1(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} &= \frac{3\mathbf{n}_R(m)(\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n))}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5} + 3(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)) \left[\frac{\mathbf{n}_R(n)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5} \right. \\ &\quad \left. + \frac{5(\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n))((\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^7} \right], \end{aligned} \quad (\text{B.29})$$

$$\frac{\partial^2 f_2(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} = -2r_k \left(\frac{\tilde{\mathbf{n}}_k(m)(\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n)) + (\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))\tilde{\mathbf{n}}_k(n)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} \right) \quad (\text{B.30})$$

$$- \frac{3(\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n))(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))((\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5}, \quad (\text{B.31})$$

$$\begin{aligned} \frac{\partial^2 f_3(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} &= \mu_k(1 - r_k)(\mu_k + 1) \left((\mu_k - 1) \cos(\beta_{ik} - \alpha_{ik})^{\mu_k-2} \frac{\partial \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(m)} \frac{\partial \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(n)} \right. \\ &\quad \left. + \cos(\beta_{ik} - \alpha_{ik})^{\mu_k-1} \frac{\partial^2 \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} \right), \end{aligned} \quad (\text{B.32})$$

with

$$\frac{\partial^2 \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} = \frac{\cos(\alpha_{ik})}{2r_k} \frac{\partial^2 f_2(\mathbf{x})}{\partial \mathbf{x}(n) \partial \mathbf{x}(m)} + \sin(\alpha_{ik}) \times \begin{cases} A & \text{if } n = z(m+1) \\ B & \text{if } n = z(m+2) \end{cases} \quad (\text{B.33})$$

where the values of A and B in (B.33) are as stated follows:

$$A = \frac{-\tilde{\mathbf{n}}_k(z(m+1))\tilde{\mathbf{n}}_k(m)}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{x} - \tilde{\mathbf{l}}_k\|} - \frac{g(n)g(m)}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|^3 \|\mathbf{x} - \tilde{\mathbf{l}}_k\|} \\ - \frac{g(n)((\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))) + g(m)((\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n)))}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} \\ + \frac{3(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))(\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n))\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5}, \quad (\text{B.34})$$

$$B = \frac{-\tilde{\mathbf{n}}_k(z(m+2))\tilde{\mathbf{n}}_k(m)}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{x} - \tilde{\mathbf{l}}_k\|} - \frac{g(n)g(m)}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|^3 \|\mathbf{x} - \tilde{\mathbf{l}}_k\|} \\ - \frac{g(n)((\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))) + g(m)((\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n)))}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} \\ + \frac{3(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))(\mathbf{x}(n) - \tilde{\mathbf{l}}_k(n))\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5}. \quad (\text{B.35})$$

When $m = n$, (B.29), (B.31), and (B.32) respectively become

$$\frac{\partial^2 f_1(\mathbf{x})}{\partial \mathbf{x}^2(m)} = \frac{3\mathbf{n}_R(m)(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5} - \frac{3((\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R) - \mathbf{n}_R(m)(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5} \\ + \frac{15(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))^2((\tilde{\mathbf{l}}_k - \mathbf{x})^T \mathbf{n}_R)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^7}, \quad (\text{B.36})$$

$$\frac{\partial^2 f_2(\mathbf{x})}{\partial \mathbf{x}^2(m)} = 2r_k \left(\frac{-\tilde{\mathbf{n}}_k(m)(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)) - (\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k - (\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))\tilde{\mathbf{n}}_k(m)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} \right. \\ \left. + \frac{3(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))^2((\mathbf{x} - \tilde{\mathbf{l}}_k)^T \tilde{\mathbf{n}}_k)}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5} \right), \quad (\text{B.37})$$

$$\frac{\partial^2 f_3(\mathbf{x})}{\partial \mathbf{x}^2(m)} = \mu_k(1 - r_k)(\mu_k + 1) \left[(\mu_k - 1) \cos(\beta_{ik} - \alpha_{ik})^{\mu_k - 2} \left(\frac{\partial \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}(m)} \right)^2 \right. \\ \left. + \cos(\beta_{ik} - \alpha_{ik})^{\mu_k - 1} \frac{\partial^2 \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}^2(m)} \right], \quad (\text{B.38})$$

with

$$\begin{aligned}
\frac{\partial^2 \cos(\beta_{ik} - \alpha_{ik})}{\partial \mathbf{x}^2(m)} &= \frac{\cos(\alpha_{ik})}{2r_k} \frac{\partial^2 f_2(\mathbf{x})}{\partial \mathbf{x}^2(m)} + \sin(\alpha_{ik}) \left(\frac{\tilde{\mathbf{n}}_k^2(z(m+1)) + \tilde{\mathbf{n}}_k^2(z(m+2))}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{x} - \tilde{\mathbf{l}}_k\|} \right. \\
&\quad - \frac{g^2(m)}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|^3 \|\mathbf{x} - \tilde{\mathbf{l}}_k\|} - \frac{g(m)(1 + (\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m)))}{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\| \|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} \\
&\quad \left. - \frac{\|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^3} + \frac{3(\mathbf{x}(m) - \tilde{\mathbf{l}}_k(m))^2 \|(\mathbf{x} - \tilde{\mathbf{l}}_k) \times \tilde{\mathbf{n}}_k\|}{\|\mathbf{x} - \tilde{\mathbf{l}}_k\|^5} \right). \quad (\text{B.39})
\end{aligned}$$

While the first-order derivative of the channel coefficient $h_i(\mathbf{x}, \tilde{\mathbf{n}}_k)$ was provided in [76, Eqn. (12)], the second-order derivatives as specified in (B.14)-(B.35) have not been available in the existing literature.

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